Chapter 10
Stochastic Models for Environmental Stress Screening

There are different ways of improving reliability characteristics of manufactured items. The most common methodology adopted in industry is burn-in, which is a method of 'elimination' of initial failures (infant mortality). As was mentioned previously, the 'sufficient condition' for employing the *traditional* burn-in is the initially decreasing failure rate. For example, when a population of items is heterogeneous, and therefore consists of subpopulations with ordered failure (hazard) rates, it obviously contains weaker (with larger failure rates) subpopulations. As the weakest populations are dying out first, the failure rate of this population is often initially decreasing and burn-in can be effectively applied.

It should be noted that not all populations of engineering items that contain 'weaker' items to be eliminated exhibit this shape of the failure rate. For example, the 'weakness' of some manufactured items can result from the latent defects that can create *additional* failure modes. The failure rate in this case is not necessarily decreasing (see Example 10.1), and therefore traditional burn-in should not be applied. However, by applying the short-time excessive stress, the weaker items in the population with increasing failure rate can be eliminated by the environmental stress screening (ESS), and therefore the reliability characteristics of the population of items that have successfully passed the ESS test can still improve. This is the *crucial distinction of this operation from burn-in.* In fact, the formal difference between the ESS and burn-in has not been clearly defined in the literature. In our discussions, we understand the ESS as the method of elimination of items with additional (nonconventional) failure modes, whereas burn-in targets elimination of weaker items with conventional failure modes and it is effective *only* when the population failure rate is initially decreasing. Another important distinction of the proposed model from burn-in is that the ESS can also create new defects in items that were previously defect-free.

Numerous stochastic models of burn-in have been intensively studied in the literature during the last decades. Although some practical engineering approaches to the ESS modeling were reported (e.g., [2, 4]), to the authors' best knowledge, there has been little research dealing with adequately advanced stochastic modeling and analysis of the ESS.

M. Finkelstein and J. H. Cha, *Stochastic Modeling for Reliability*,
Springer Series in Reliability Engineering, DOI: 10.1007/978-1-4471-5028-2_10,

In this chapter, we develop a stochastic model for the ESS, analyze its effect on the population characteristics of the screened items and describe related optimization problems. We assume that, due to substandard materials of faulty manufacturing process, some of the manufactured items are susceptible to additional cause of failure (failure mode), i.e., shocks (such as electrical or mechanical shocks). We define the ESS as a procedure of applying a shock of the controlled magnitude, i.e., a short-time excessive stress. In practice, for example, a shock can be understood as a short-time electric impulse. For the ESS to be effective, the corresponding magnitude should be reasonably larger than the magnitude of shocks that occur in field usage.

Our modeling is within the framework of the general shock models. We will consider two different types of ESS models in this chapter. In the first model, the failure of an item occurs when the magnitude of the stress (shock) exceeds its strength. The larger magnitude of the ESS shock (within 'physical limits') implies the better reliability characteristics of survived items in field usage but at the same time, the larger cost of the ESS as more items with defects are discarded. An important feature of our model is that we assume that the item during field usage is exposed to the point process of environmental shocks of an ordinary, not excessive magnitude. These shocks can obviously destroy only defective items that have passed the ESS or were induced by the ESS. In the second model, an external shock can either destroy an item with a given probability or increase the 'size of the defect' by a random amount. We also analyze the effect of the ESS on the population characteristics of the screened items and discuss related optimization problems. We will extensively use the general stress–strength model described in Sect. 9.4.1.

10.1 Stress–Strength Type ESS Model

10.1.1 Stochastic Model for ESS

The description and assumptions of our model are as follows. During the manufacturing process, the items with the failure rate $r(t)$ and the corresponding lifetime T_N (which is only due to 'normal' failure mode) and also the defective items with the lifetime T_D are produced. Let the proportion of the nondefective items be π and that of the defective items be $1 - \pi$.

The defective items, in addition to the normal failure mode of the nondefective items, are characterized by a new additional failure mode. In this chapter, we assume that this additional failure mode describes susceptibility to external shocks. For example, consider the case when the normal (nondefective) items, in accordance with specifications, should not be susceptible to electrical or mechanical shocks. However, due to substandard materials or a faulty manufacturing process, some of the produced items are susceptible to these shocks [4]. For instance, during the manufacturing process, the items can be exposed to a strong electric shock and this shock may result in some defective items which are even sensitive to electrical shocks of a 'normal' magnitude, whereas the nondefective items are not sensitive to it [3]. Another example is when a small crack in a material of the defective item is sensitive to mechanical impulses (e.g., vibration) in field use, which eventually can result in its failure. Thus, we assume that shocks of a 'normal' magnitude also occur in field operation, and therefore the defective items can fail due to this failure mode. On the other hand, the nondefective items do not fail from external shocks of this type in field operation as they do not have the corresponding failure mode.

In accordance with our description, the survival function of T_N is

$$P(T_N > t) = \exp\{-\int_0^t r(u)\mathrm{d}u\}.$$

Let the two failure modes of the defective items be independent. Then, the corresponding survival function is given by the competing risks model (series system):

$$P(T_D > t) = \exp\{-\int_0^t r(u)\mathrm{d}u\} \cdot P(T_E > t), \tag{10.1}$$

where T_E is the lifetime that accounts only for the external shock failure mode.

Suppose that during the field operation, the external shocks occur in accordance with the NHPP $\{N(t), t \geq 0\}$ with rate $\lambda(t)$. Denote by S_i the magnitude (stress) of the ith shock and assume that $S_i, i = 1, 2, \ldots$ are i.i.d. random variables with the common Cdf $M(s) = \Pr(S_i \leq s)$ $(\bar{M}(s) \equiv 1 - M(s))$ and the corresponding pdf $m(s)$. The defective item is characterized by its random strength U, i.e., the resistance ability to external shocks. Here, the strength is understood as the *'maximum stress level that the defective item can survive'*. The corresponding Cdf, Sf, pdf, and FR of U are denoted by $G_U(u)$, $\bar{G}_U(u)$, $g_U(u)$ and $r_U(u)$, respectively. For each $i = 1, 2, \ldots$, the operable system survives if $S_i \leq U$ and fails if $S_i > U$, '*independently of everything else*'. Then, in accordance with Theorem 9.10, Eq. (10.1) reads now

$$P(T_D > t) = \exp\{-\int_0^t r(u)\mathrm{d}u\} \cdot \exp\{-\int_0^t p(u)\lambda(u)\mathrm{d}u\}, \tag{10.2}$$

where

$$p(t) \equiv \frac{\int_0^\infty \int_0^v \exp\{-\overline{M}(r)\int_0^t \lambda(x)\mathrm{d}x\} \cdot g_U(r)dr\, m(v)\mathrm{d}v}{\int_0^\infty \exp\{-\bar{M}(r)\int_0^t \lambda(x)\mathrm{d}x\} g_U(r)dr}. \tag{10.3}$$

From (10.2), we see that the lifetimes of the nondefective and defective items are obviously stochastically ordered: $T_D <_{fr} T_N$, where "$<_{fr}$"denotes, as usual, the failure (hazard) rate ordering of two random variables.

Denote the population lifetime by T. As it consists of defective and nondefective items with given proportions, the corresponding survival function is the following mixture

$$\begin{aligned}\bar{F}(t) &\equiv P(T > t) \\ &= \pi \exp\{-\int_0^t r(u)\mathrm{d}u\} + (1-\pi)\exp\{-\int_0^t r(u)\mathrm{d}u\} \cdot \exp\{-\int_0^t p(u)\lambda(u)\mathrm{d}u\}.\end{aligned} \tag{10.4}$$

Thus, (10.4) defines the survival function of an item in field usage that is chosen at random from the population of manufactured items.

In what follows, we will describe the impact of the ESS on the population structure and on the corresponding population lifetime distribution. Therefore, we must define first the ESS that we consider in this chapter.

ESS Process

During the ESS, all items are exposed to a single shock with the fixed magnitude s. If the strength of a defective item is larger than s then it survives; otherwise it fails. Depending on the magnitude s, a proportion of nondefective items, $\rho(s)$, $0 \le \rho(s) < 1$, *becomes defective, where* $\rho(s)$ *is an increasing function of its argument. The items failed during the ESS are discarded and only the survived items are put into the field operation.*

Thus the ESS, in principle, can induce defects. Furthermore, as those with induced defects but not failed are not identifiable, they are also put into the field operation.

Recall that shock's magnitudes in field operation are i.i.d. random variables. We assume that the corresponding mean is substantially smaller than the magnitude of stress allowed for the ESS (otherwise there is no reason to perform the ESS). Therefore, the shocks in field operation can hardly 'produce' defective items out of nondefective ones (or this effect is negligible). On the other hand, these shocks can still destroy the defective item with a given strength.

Denote the population lifetime after the ESS with magnitude s by T_{ESS}.

Theorem 10.1 *Under the given assumptions, the population distribution and the corresponding failure rate (after the ESS) are*

$$\begin{aligned}\bar{F}_E(t,s) = P(T_{ESS} > t) = \exp\{-\int_0^t r(u)\mathrm{d}u\} \cdot \frac{(1-\rho(s))\pi}{(1-\rho(s))\pi + \rho(s)\pi + \bar{G}_U(s)(1-\pi)} \\ + \exp\{-\int_0^t r(u)\mathrm{d}u\} \cdot \exp\{-\int_0^t p(u)\lambda(u)\mathrm{d}u\} \cdot \frac{\rho(s)\pi}{(1-\rho(s))\pi + \rho(s)\pi + \bar{G}_U(s)(1-\pi)} \\ + \exp\{-\int_0^t r(u)\mathrm{d}u\} \cdot \exp\{-\int_0^t p(s,u)\lambda(u)\mathrm{d}u\} \cdot \frac{\bar{G}_U(s)(1-\pi)}{(1-\rho(s))\pi + \rho(s)\pi + \bar{G}_U(s)(1-\pi)},\end{aligned} \tag{10.5}$$

and

$$\begin{aligned}\lambda_E(t,s) = r(t) \cdot \frac{\pi(1)\bar{F}_1(t)}{\sum_{i=1}^{3}\pi(i)\bar{F}_i(t)} + [r(t) + p(t)\lambda(t)] \cdot \frac{\pi(2)\bar{F}_2(t)}{\sum_{i=1}^{3}\pi(i)\bar{F}_i(t)} \\ + [r(t) + p(s,t)\lambda(t)] \cdot \frac{\pi(3)\bar{F}_3(t)}{\sum_{i=1}^{3}\pi(i)\bar{F}_i(t)},\end{aligned} \tag{10.6}$$

respectively.

Proof Observe that there are now three subpopulations after the ESS and we can define the corresponding frailty variable Z:

(i) the subpopulation with nondefective items ($Z = 1$); (ii) the subpopulation with defective items which were originally nondefective ($Z = 2$); (iii) the subpopulation with defective items which were originally defective but have survived the ESS ($Z = 3$). Then, in accordance with our notation, the distribution of Z is given by

$$\begin{aligned}\pi(1) \equiv P(Z = 1) = \frac{(1-\rho(s))\pi}{(1-\rho(s))\pi + \rho(s)\pi + \bar{G}_U(s)(1-\pi)}, \\ \pi(2) \equiv P(Z = 2) = \frac{\rho(s)\pi}{(1-\rho(s))\pi + \rho(s)\pi + \bar{G}_U(s)(1-\pi)}, \\ \pi(3) \equiv P(Z = 3) = \frac{\bar{G}_U(s)(1-\pi)}{(1-\rho(s))\pi + \rho(s)\pi + \bar{G}_U(s)(1-\pi)}.\end{aligned}$$

Therefore,

$$\bar{F}_1(t) \equiv P(T_{ESS} > t | Z = 1) = \exp\{-\int_0^t r(u)\mathrm{d}u\}$$

and

$$\bar{F}_2(t) \equiv P(T_{ESS} > t|Z = 2) = \exp\{-\int_0^t r(u)\mathrm{d}u\} \cdot \exp\{-\int_0^t p(u)\lambda(u)\mathrm{d}u\},$$

where $p(t)$ is given by (10.3).

Derivation of $P(T_{ESS} > t|Z = 3)$ is not so straightforward. Indeed, it should be taken into account that when we apply a shock of the controlled magnitude s during the ESS, this means that the strength of the defective item that had passed it is larger than s and, therefore, the distribution of the *remaining strength* U_s (given that the strength is larger than s) is

$$G_U(u|s) \equiv P(U \leq u|U > s) = 1 - \bar{G}(u)/\bar{G}(s),\ u > s.$$

Thus, the function $p(t)$ in (10.3) should be modified to

$$\begin{aligned} p(s,t) &= \frac{\int_0^\infty \int_0^v \exp\{-\bar{M}(r)\int_0^t \lambda(x)\mathrm{d}x\} \cdot g_U(r|s)dr\, m(v)\mathrm{d}v}{\int_0^\infty \exp\{-\bar{M}(r)\int_0^t \lambda(x)\mathrm{d}x\} g_U(r|s)dr} \\ &= \frac{\int_s^\infty \int_s^v \exp\{-\bar{M}(r)\int_0^t \lambda(x)\mathrm{d}x\} \cdot g_U(r)dr\, m(v)\mathrm{d}v}{\int_s^\infty \exp\{-\bar{M}(r)\int_0^t \lambda(x)\mathrm{d}x\} g_U(r)dr}, \end{aligned} \tag{10.7}$$

where, $g_U(u|s)$ is the corresponding pdf of $G_U(u|s)$, which is given by

$$g_U(u|s) = \begin{cases} 0, & if\ \ u \leq s \\ \frac{g_U(u)}{\bar{G}_U(s)}, & if\ \ u > s \end{cases}.$$

Finally,

$$\bar{F}_3(t) \equiv P(T_{ESS} > t|Z = 3) = \exp\{-\int_0^t r(u)\mathrm{d}u\} \cdot \exp\{-\int_0^t p(s,u)\lambda(u)\mathrm{d}u\}.$$

Therefore, Eqs. (10.5) and (10.6) hold.

$\square$

We will now discuss the effect of the ESS on the quality of the population after the screening by comparing $\bar{F}_E(t,s)$ with the survival function without screening, $F(t)$ defined by Eq. (10.4). As the ESS in our model can create defective items, theoretically in some cases this operation may have a negative effect on the population of items.

Definition 10.1 The severity (stress) level s is said to be *inadmissible* under the survival function criterion if

$$\bar{F}(t) \geq \bar{F}_E(t,s), \text{ for all } t > 0.$$

Otherwise, the severity (stress) level s is said to be *admissible.*

Obviously, the inadmissible severity levels should not be considered in the ESS practice as reliability of items in field use is worse than that without the ESS in this case. Note that the condition for the 'admissibility' in Definition 10.1 means that $\bar{F}(t) < \bar{F}_E(t,s)$ for some $t > 0$ and not for all $t > 0$. However, for obvious practical reasons, we are mostly interested in the latter case. The following definition addresses this setting.

Definition 10.2 The severity (stress) level s is said to be *positively admissible* under the survival function criterion if

$$\bar{F}(t) < \bar{F}_E(t,s), \text{ for all } t > 0.$$

Theorem 10.2 (i) *If* $\rho(s) < (1-\pi)G_U(s)$, *then this severity level s is positively admissible under the survival function criterion.*

(ii) *If* $\rho(s)\pi > \pi(1-\pi) + (1-\pi)^2\bar{G}_U(s)$, *then this severity level s is inadmissible under the survival function criterion.*

Proof Denote for convenience, $\lambda_1(t) \equiv r(t)$; $\lambda_2(t) \equiv r(t) + p(t)\lambda(t)$; $\lambda_3(t) \equiv r(t) + p(s,t)\lambda(t)$. Note that Eq. (10.7) can be written in a compact and a meaningful way as

$$p(s,t) = \int_0^\infty I(v \in [s,\infty))h(s,t,v)m(v)\mathrm{d}v,$$

where

$$h(s,t,v) \equiv \frac{\int_s^v \exp\{-\bar{M}(r)\int_0^t \lambda(x)\mathrm{d}x\} \cdot g_U(r)dr}{\int_s^\infty \exp\{-\bar{M}(r)\int_0^t \lambda(x)\mathrm{d}x\}g_U(r)dr}.$$

and $I(\cdot)$ is the corresponding indicator. Observe that, for all fixed t and v,

$$\frac{\partial}{\partial s}h(s,t,v) = \frac{1}{\left(\int\limits_s^\infty \exp\{-\bar{M}(r)\int\limits_0^t \lambda(x)\mathrm{d}x\}g_U(r)dr\right)^2}$$
$$\times\left[-\exp\{-\bar{M}(s)\int\limits_0^t \lambda(x)\mathrm{d}x\}g_U(s)\cdot\int\limits_s^\infty \exp\{-\bar{M}(r)\int\limits_0^t \lambda(x)\mathrm{d}x\}g_U(r)dr\right.$$
$$\left.+\exp\{-\bar{M}(s)\int\limits_0^t \lambda(x)\mathrm{d}x\}g_U(s)\cdot\int\limits_s^v \exp\{-\bar{M}(r)\int\limits_0^t \lambda(x)\mathrm{d}x\}g_U(r)dr\right] < 0,$$

for all $s > 0$. Therefore, the function $p(s,t)$ is strictly decreasing in s for each fixed t. This implies that $p(s,t) < p(t)$, for all $t > 0$ and $s > 0$. Thus we have the following failure rate ordering:

$$\lambda_1(t) < \lambda_3(t) < \lambda_2(t), \quad \text{for all } t > 0, \tag{10.12}$$

and accordingly,

$$\bar{F}_1(t) > \bar{F}_3(t) > \bar{F}_2(t), \quad \text{for all } t > 0,$$

where $\bar{F}_i(t) \equiv \exp\{-\int_0^t \lambda_i(u)\mathrm{d}u\}, \ i = 1,2,3$. Observe that, in accordance with (10.4),

$$\bar{F}(t) = \pi\bar{F}_1(t) + (1-\pi)\bar{F}_2(t),$$

whereas in accordance with (10.5),

$$\bar{F}_E(t,s) = \pi(1)\bar{F}_1(t) + \pi(2)\bar{F}_2(t) + \pi(3)\bar{F}_3(t).$$

Therefore, if $\pi(2) + \pi(3) < 1 - \pi$, or equivalently, $\pi(1) > \pi$, then

$$\begin{aligned}\bar{F}_E(t,s) - \bar{F}(t) &> (\pi(1)-\pi)\bar{F}_1(t) + [\pi(2)\bar{F}_2(t) + \pi(3)\bar{F}_2(t) - (1-\pi)\bar{F}_2(t)] \\ &= (\pi(1)-\pi)\bar{F}_1(t) - (\pi(1)-\pi)\bar{F}_2(t) > 0,\end{aligned}$$

for all $t > 0$. The condition $\pi(2) + \pi(3) < 1 - \pi$ is equivalent to $\rho(s) < (1-\pi)G_U(s)$. This completes the proof of (i).

By a similar reasoning, if $\pi(2) > 1 - \pi$, or equivalently, $\rho(s)\pi > \pi(1-\pi) + (1-\pi)^2\,\bar{G}_U(s)$, then the severity level s is inadmissible under the survival function criterion.

□

Remark 10.1

(i) The conditions in Theorem 10.2 do not imply the admissibility/inadmissibility of the corresponding severity level under the failure rate criterion. That is, the condition $\pi(2)+\pi(3)<1-\pi$ does not imply $\lambda_T(t)>\lambda_E(t,s)$, for all $t>0$, where $\lambda_T(t)$ is the failure rate which corresponds to $\bar{F}(t)$ defined in (10.4).

(ii) The failure rate ordering (10.8) will be important for our further reasoning. This ordering implies that the quality of defective items improves after the ESS, but they are still obviously 'worse' than the nondefective items.

Remark 10.2 The effect of applying two consecutive shocks with severity s during the ESS can be also considered. After this type of the ESS, we also have three subpopulations with failure rates $\lambda_1(t)=r(t)$, $\lambda_2(t)=r(t)+p(t)\lambda(t)$ and $\lambda_3(t)=r(t)+p(s,t)\lambda(t)$ and the corresponding proportions

$$\pi^{(2)}(1)=\frac{(1-\rho(s))\pi(1)}{(1-\rho(s))\pi(1)+\rho(s)\pi(1)+[\pi(3)+\bar{G}_U(s)\pi(2)]},$$
$$\pi^{(2)}(2)=\frac{\rho(s)\pi(1)}{(1-\rho(s))\pi(1)+\rho(s)\pi(1)+[\pi(3)+\bar{G}_U(s)\pi(2)]},$$
$$\pi^{(2)}(3)=\frac{\pi(3)+\bar{G}_U(s)\pi(2)}{(1-\rho(s))\pi(1)+\rho(s)\pi(1)+[\pi(3)+\bar{G}_U(s)\pi(2)]}.$$

10.1.2 Optimal Severity

In this subsection, we will consider the problem of determining the optimal severity level (magnitude) of the ESS. Let τ be the mission time of an item in the field operation. If it does not fail during this time, then the mission is considered to be successful. Thus, the probability of the mission success needs to be maximized and we should find the optimal severity level s^* that satisfies

$$\bar{F}_E(\tau,s^*)=\max_{s>0}\bar{F}_E(\tau,s).$$

Alternatively, let $MRL(s)$ be the mean time to failure of an item in the field operation as a function of s, i.e., $MRL(s)\equiv\int_0^\infty \bar{F}_E(t,s)\mathrm{d}t$. Then, the optimal severity level which maximizes the mean time to failure should be obtained as

$$MRL(s^*)=\max_{s>0}MRL(s).$$

For defining the optimal severity, we should consider the admissible severity class rather than the positively admissible class as we have to take into account all admissible severity levels. It is often more convenient to describe the dual inadmissible class. The following theorem provides the upper bound for the optimal severity level that maximizes the mission success probability or mean time to failure in field usage.

Theorem 10.3 *Suppose that* $\rho(\infty) \equiv \lim_{s\to\infty} \rho(s) > (1-\pi)$ *and let*

$$s_0 \equiv \inf_{s \geq 0}\{s:\ \rho(s)\pi > \pi(1-\pi) + (1-\pi)^2\bar{G}_U(s)\}.$$

Then the severity levels in (s_0, ∞) *are inadmissible. Therefore,* s_0 *is the upper bound for the optimal severity level.*

Proof From Theorem 10.2, the condition for inadmissibility is

$$\rho(s)\pi > \pi(1-\pi) + (1-\pi)^2\bar{G}_U(s).$$

Here, the function $\rho(s)\pi$ is increasing from 0 to $\rho(\infty)\pi$, whereas the function $\pi(1-\pi) + (1-\pi)^2\bar{G}_U(s)$ decreases from $(1-\pi)$ to $\pi(1-\pi)$. Thus, if $\rho(\infty)\pi > \pi(1-\pi)$, or equivalently, $\rho(\infty) > (1-\pi)$, then there exists $s_0 \in (0,\infty)$ such that the severity levels in (s_0,∞) are inadmissible. Therefore, s_0 is the *upper bound* for the optimal severity.

□

Remark 10.3 It is reasonable to assume that in practice, $\lim_{s\to\infty} \rho(s) = 1$ and that the proportion of the defective items $(1-\pi)$ is relatively small. Therefore, the condition $\rho(\infty) > (1-\pi)$ can be satisfied in almost all practical cases.

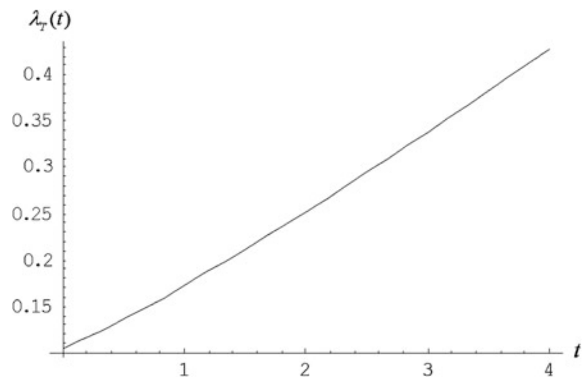
Example 10.1 Let $r(t) = 0.1t,\ t \geq 0,\ \lambda(t) = 1,\ t \geq 0,\ m(s) = 3\exp\{-3s\}$, $s \geq 0,\ g_U(u) = 4u\exp\{-2u^2\},\ u \geq 0,\ \pi = 0.7,\ \tau = 4.0$ and

$$\rho(s) = \begin{cases} 0, & 0 \leq s < 1, \\ 1 - \exp\{-0.05(s-1)\}, & s \geq 1 \end{cases}.$$

Note that the failure rate of the population distribution before the ESS, which is obtained based on (10.4), is given by Fig. 10.1.

Therefore, as $\lambda_T(t)$ is increasing, the burn-in procedure *should not be applied to this population.* On the other hand, as $\rho(s)$ is strictly increasing for $s \geq 1$, there exists a unique solution of the equation

Fig. 10.1 The graph of $\lambda_T(t)$



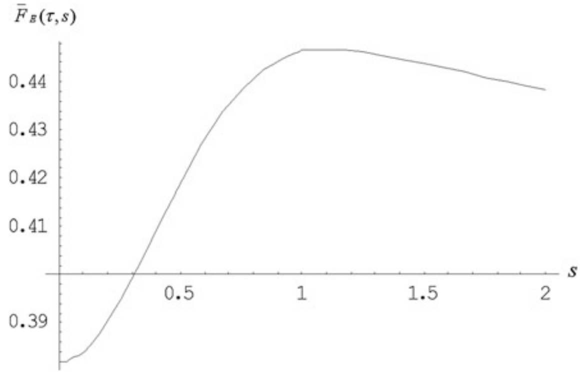


Fig. 10.2 The graph of $\bar{F}_E(\tau, s)$

$$\rho(s)\pi = \pi(1-\pi) + (1-\pi)^2\bar{G}_U(s),$$

which is the upper bound for the optimal severity level. Therefore, the ESS as a method of elimination of defective items is justified in this case. Solving this equation numerically results in $s_0 \approx 8.13$. Therefore, it is now sufficient to search for the optimal severity level in the interval $[0,\ 8.13]$. The graph of $\bar{F}_E(\tau, s)$ is presented in Fig. 10.2. The optimal severity level in this case is $s^* \approx 1.08$ and the maximum probability of the mission success is $\bar{F}_E(\tau; s^*) \approx 0.447$.

Based on the foregoing results, we can consider now certain cost structures for determining the cost-based optimal severity level. As previously, an item is chosen at random from our initial population and is exposed to a shock of magnitude s during the ESS. If it survives, it is put into the field operation, otherwise the failed item is discarded and the new one is chosen from the population, etc. This procedure is repeated until the first survived item is obtained. Let c_{sr} be the shop replacement cost (actually, it is the cost of a new item) and c_s be the cost for conducting the ESS. Let $c_1(s)$, as a function of s, be the expected cost for eventually obtaining a component which has survived the ESS. Then

$$c_1(s) = \frac{c_s + c_{sr}[1 - \{\pi + (1-\pi)\bar{G}_U(s)\}]}{\pi + (1-\pi)\bar{G}_U(s)},$$

where $1/[\pi + (1-\pi)\bar{G}_U(s)]$ is the total number of trials until the first 'success'.

Assume that if a mission (of length τ) is successful (in field operation), then the gain K is 'earned'; otherwise a penalty C is imposed, where $K > C > 0$. Then the expected gain during the field operation is

$$c_2(s) = -K\bar{F}_E(\tau, s) + CF_E(\tau, s) = -(K + C)\bar{F}_E(\tau, s) + C \tag{10.9}$$

and the total expected cost $c(s)$ is

$$\begin{aligned} c(s) &= c_1(s) + c_2(s) \\ &= \frac{c_s + c_{sr}[1 - \{\pi + (1-\pi)\bar{G}_U(s)\}]}{\pi + (1-\pi)\bar{G}_U(s)} - (K + C)\bar{F}_E(\tau; s) + C. \end{aligned}$$

The objective is now to find the optimal severity level s^* that satisfies

$$s^* = \arg \min_{s \in [0,\infty]} c(s).$$

Similar to Theorem 10.3, if $\rho(\infty) \equiv \lim_{s\to\infty} \rho(s) > ; (1-\pi)$, then the optimal severity level which minimizes $c_2(s)$ [maximizes $\bar{F}_E(\tau, s)$, as follows from (10.9)] does not exists in the interval (s_0, ∞), where s_0 is also defined by Theorem 10.3. Furthermore, as $c_1(s)$ is strictly increasing to infinity, we can conclude that the optimal severity level s^* should exist in the interval $[0, s_0]$.

Assume now that during field operation, the gain is proportional to the mean time to failure. Therefore, the total average cost function in this case is

$$c(s) = \frac{c_s + c_{sr}[1 - (1-\pi)\bar{G}_U(s)]}{(1-\pi)\bar{G}_U(s)} - K \int_0^\infty \bar{F}_E(t; s)\mathrm{d}t.$$

By the similar arguments, the optimal severity level s^* should exist in the interval $[0, s_0]$.

10.2 ESS Model with Wear Increments

10.2.1 Stochastic Model

In this subsection, we develop a stochastic model for the shock and wear based ESS. We assume that, during the manufacturing process due to substandard materials or other faults some defective items with latent defects such as, e.g., a microcrack may be produced. Such defective items are susceptible to failure from mechanical or electrical shocks during field operation. Thus the defective items, in addition to the normal failure mode of the nondefective items, are characterized by a new *additional* failure mode. On the other hand, the nondefective items do not fail from external shocks in field operation as *they do not have the corresponding failure mode.*

Denote the lifetime of the nondefective items by T_N with the corresponding failure rate $r(t)$. In accordance with our description, obviously, the survival function of T_N is defined by

$$P(T_N > t) = \exp\{-\int_0^t r(u)\mathrm{d}u\}.$$

During the field operation, the items are subject to the nonhomogeneous Poisson process (NHPP) of 'ordinary' environmental shocks $\{N(t), t \geq 0\}$ with rate $\lambda(t)$ and arrival times $T_i, i = 1, 2, \ldots$. Let, on the ith shock, the defective item fail with probability $p(T_i)$ (critical shock), whereas with probability $q(T_i)$ it increases the 'defect size' by a random amount W_i (noncritical shock). In the following, for convenience, we will loosely use the term "wear" (or degradation) for the defect size as well. In accordance with this setting, the random accumulated wear of a defective item at time t in the field use is given by

$$W(t) = \sum_{i=0}^{N_q(t)} W_i + W_M,$$

where $N_q(t)$ is the number of noncritical shocks in $[0, t)$ and $W_M > 0$ is the initial wear (defect size of the latent defect). Let R be the random boundary of the item which follows an exponential distribution with parameter θ. The failure due to wear occurs when the accumulated wear $W(t)$ reaches R. Let T_E be the lifetime in the field use that accounts only for the external shock failure mode of defective items (*i.e.*, the lifetime without any other causes of failure). Then, as follows from Eq. (4.4) and the reasoning in Sect. 4.1.2,

$$P(T_E > t) = \exp\left\{-\int_0^t (1 - M_W(-\theta)q(x))\lambda(x)\mathrm{d}x\right\}, t \geq 0,$$

regardless of the distribution of W_M. As there are two independent failure modes for defective items—i.e., the normal failure mode described by $r(t)$ and the additional one due to external shocks, the survival function for the defective items is given by the competing risks model (a series system):

$$\begin{aligned} P(T_D > t) &= \exp\{-\int_0^t r(u)\mathrm{d}u\} \cdot P(T_E > t) \\ &= \exp\{-\int_0^t r(u)\mathrm{d}u\} \cdot \exp\left\{-\int_0^t (1 - M_W(-\theta)q(x))\lambda(x)\mathrm{d}x\right\}, t \geq 0. \end{aligned}$$

Let the proportion of the nondefective items be π and that of the defective items be $1 - \pi$, respectively. Denote the population lifetime by T. Given the structure of our population, the corresponding survival function is the mixture of survival functions for the defective and nondefective items:

$$\bar{F}(t) \equiv P(T > t) = \pi \exp\{-\int_0^t r(u)\mathrm{d}u\} + (1-\pi)\exp\{-\int_0^t r(u)\mathrm{d}u\} \cdot \exp\left\{-\int_0^t (1 - M_W(-\theta)q(x))\lambda(x)\mathrm{d}x\right\}, t \geq 0. \tag{10.10}$$

Thus, (10.10) defines the survival function in field usage of the item that is chosen at random from the population of manufactured items.

In what follows, we will describe the impact of the ESS on the population distribution. Therefore, we must describe first the ESS that we consider in this chapter.

ESS Process
During the ESS, a shock with the fixed magnitude s is applied to all items (e.g., *the mechanical shock). The defective items immediately fail with probability* $\alpha(s)$, *whereas with probability* $1-\alpha(s)$ *an additional wear with magnitude* W_s *is incurred, where* $\alpha(s)$ *is an increasing function and* W_s *is stochastically increasing with s. Furthermore, depending on the magnitude s, a proportion of nondefective items,* $\rho(s)$, $0 \leq \rho(s) < 1$, *becomes defective, where* $\rho(s)$ *is an increasing function of its argument. The failed items are discarded and only the survived items are put into field operation.*

For example, the mechanical shock during the ESS can be executed by the dropping of an item from some height (the "dropping shock"), which can be considered as the magnitude of the shock. Obviously, the assumptions for $\alpha(s)$, W_s and $\rho(s)$ are justified in this case. For instance, the larger height corresponds to the larger wear W_s.

We will now derive the population distribution in field use *after the ESS*. Denote the corresponding lifetime by T_{ESS}. In the following theorem, the distribution of T_{ESS} is obtained.

Theorem 10.4 *The survival function of* T_{ESS} *is given by*

$$\begin{aligned} P(T_{ESS} > t) &\equiv \bar{F}_E(t,s) \\ &= \exp\{-\int_0^t r(u)\mathrm{d}u\} \cdot \frac{(1-\rho(s))\pi}{(1-\rho(s))\pi + \rho(s)\pi + (1-\alpha(s))P(R > W_s)(1-\pi)} \\ &\quad + \exp\{-\int_0^t r(u)\mathrm{d}u\} \cdot \exp\left\{-\int_0^t (1 - M_W(-\theta)q(x))\lambda(x)\mathrm{d}x\right\} \\ &\quad \times \frac{\rho(s)\pi + (1-\alpha(s))P(R > W_s)(1-\pi)}{(1-\rho(s))\pi + \rho(s)\pi + (1-\alpha(s))P(R > sW_s)(1-\pi)}, \end{aligned}$$

and the corresponding failure rate is

$$\lambda_E(t,s) = r(t)\cdot\frac{\psi(1)\bar{F}_1(t)}{\sum_{i=1}^{2}\psi(i)\bar{F}_i(t)} + [r(t) + (1 - M_W(-\theta)q(t))\lambda(t)]\cdot\frac{\psi(2)\bar{F}_2(t)}{\sum_{i=1}^{2}\psi(i)\bar{F}_i(t)},$$

where

$$\psi(1) \equiv \frac{(1-\rho(s))\pi}{(1-\rho(s))\pi + \rho(s)\pi + (1-\alpha(s))P(R > W_s)(1-\pi)},$$

and

$$\psi(2) \equiv \frac{\rho(s)\pi + (1-\alpha(s))P(R > W_s)(1-\pi)}{(1-\rho(s))\pi + \rho(s)\pi + (1-\alpha(s))P(R > W_s)(1-\pi)}.$$

Proof Observe that there are formally three subpopulations after the ESS and we can define the corresponding frailty variable Z: (i) the subpopulation with nondefective items ($Z = 1$); (ii) the subpopulation with defective items which were originally nondefective ($Z = 2$); (iii) the subpopulation with defective items which have survived the ESS ($Z = 3$). Then, in accordance with our notation, the distribution of Z is given by

$$\pi(1) \equiv P(Z=1) = \frac{(1-\rho(s))\pi}{(1-\rho(s))\pi + \rho(s)\pi + (1-\alpha(s))P(R > W_s)(1-\pi)},$$
$$\pi(2) \equiv P(Z=2) = \frac{\rho(s)\pi}{(1-\rho(s))\pi + \rho(s)\pi + (1-\alpha(s))P(R > W_s)(1-\pi)},$$
$$\pi(3) \equiv P(Z=3) = \frac{(1-\alpha(s))P(R > W_s)(1-\pi)}{(1-\rho(s))\pi + \rho(s)\pi + (1-\alpha(s))P(R > W_s)(1-\pi)},$$

On the other hand, in field use,

$$\bar{F}_1(t) \equiv P(T_{ESS} > t|Z=1) = \exp\{-\int_0^t r(u)\mathrm{d}u\},$$
$$\bar{F}_2(t) \equiv P(T_{ESS} > t|Z=2) = \exp\{-\int_0^t r(u)\mathrm{d}u\}\cdot\exp\left\{-\int_0^t (1 - M_W(-\theta)q(x))\lambda(x)\mathrm{d}x\right\},$$
$$\bar{F}_3(t) \equiv P(T_{ESS} > t|Z=3) = \exp\{-\int_0^t r(u)\mathrm{d}u\}\cdot\exp\left\{-\int_0^t (1 - M_W(-\theta)q(x))\lambda(x)\mathrm{d}x\right\}.$$

Therefore, although there formally exist three subpopulations after the ESS, due to the exponentially distributed boundary, we actually have two subpopulations. Based on the above results, the population survival function in field use after the ESS with magnitude s is given by the following mixture

$$\begin{aligned}
\bar{F}_E(t,s) = P(T_{ESS} > t) &= \sum_{i=1}^{2} \psi(i)\bar{F}_i(t) \\
&= \exp\{-\int_0^t r(u)\mathrm{d}u\} \cdot \frac{(1-\rho(s))\pi}{(1-\rho(s))\pi + \rho(s)\pi + (1-\alpha(s))P(R > W_s)(1-\pi)} \\
&\quad + \exp\{-\int_0^t r(u)\mathrm{d}u\} \cdot \exp\left\{-\int_0^t (1 - M_W(-\theta)q(x))\lambda(x)\mathrm{d}x\right\} \\
&\quad \times \frac{\rho(s)\pi + (1-\alpha(s))P(R > W_s)(1-\pi)}{(1-\rho(s))\pi + \rho(s)\pi + (1-\alpha(s))P(R > W_s)(1-\pi)},
\end{aligned}$$

where

$$\psi(1) \equiv \frac{(1-\rho(s))\pi}{(1-\rho(s))\pi + \rho(s)\pi + (1-\alpha(s))P(R > W_s)(1-\pi)},$$

and

$$\psi(2) \equiv \frac{\rho(s)\pi + (1-\alpha(s))P(R > W_s)(1-\pi)}{(1-\rho(s))\pi + \rho(s)\pi + (1-\alpha(s))P(R > W_s)(1-\pi)}.$$

Then the corresponding failure rate is

$$\begin{aligned}
&\lambda_E(t,s) \\
&= \frac{\sum_{i=1}^{2} \psi(i) f_i(t)}{\sum_{i=1}^{2} \pi(i)\bar{F}_i(t)} = \frac{1}{\sum_{i=1}^{2} \pi(i)\bar{F}_i(t)} \left(\psi(1)\bar{F}_1(t) \cdot \frac{f_1(t)}{\bar{F}_1(t)} + \psi(2)\bar{F}_2(t) \cdot \frac{f_2(t)}{\bar{F}_2(t)}\right) \\
&= r(t) \cdot \frac{\psi(1)\bar{F}_1(t)}{\sum_{i=1}^{2} \psi(i)\bar{F}_i(t)} + [r(t) + (1 - M_W(-\theta)q(t))\lambda(t)] \cdot \frac{\psi(2)\bar{F}_2(t)}{\sum_{i=1}^{2} \psi(i)\bar{F}_i(t)}.
\end{aligned}$$

□

Therefore, due to the exponential boundary, the ESS in this case does not essentially change subpopulation distributions but only changes the subpopulation proportions.

We will discuss now the effect of the ESS on the quality of the population after the ESS by comparing $\lambda_E(t,s)$ with the failure rate without the ESS, $\lambda_T(t)$, that can be defined by Eq. (10.10). Note that as the ESS in our model can create defective items and theoretically this operation may have a negative effect on the population of items in some cases. Similar to Definitions 10.1 and 10.2:

Definition 10.3 The severity (stress) level s is said to be *inadmissible* under the failure rate function criterion if

$$\lambda_T(t) \leq \lambda_E(t,s), \text{ for all } t > 0,$$

where $\lambda_T(t)$ is the failure rate which corresponds to $\bar{F}(t)$. Otherwise, the severity (stress) level s is said to be *admissible.*

Obviously, the inadmissible severity levels should not be considered in the application of the ESS. Note that the condition for 'admissible' is that $\lambda_T(t) > \lambda_E(t,s)$, for "some $t > 0$", not "for all $t > 0$". However, for obvious practical reasons we are mostly interested in the latter case. The following definition addresses this setting.

Definition 10.4 The severity (stress) level s is said to be *positively admissible* under the failure rate function criterion if

$$\lambda_T(t) > \lambda_E(t,s), \text{ for all } t > 0.$$

Theorem 10.5 *If*

$$\frac{1-\rho(s)-\pi}{(1-\pi)(1-\alpha(s))} > P(R > W_s), \tag{10.11}$$

then this severity level s is positively admissible under the failure rate function criterion. Otherwise, this severity level s is inadmissible under the failure rate function criterion.

Proof Denote for convenience, $\lambda_1(t) \equiv r(t); \lambda_2(t) \equiv r(t) + (1 - M_W(-\theta)q(t))\lambda(t)$. Clearly, we have the following failure rate ordering:

$$\lambda_1(t) < \lambda_2(t), \text{ for all } t > 0.$$

Observe that

$$\lambda_T(t) = \lambda_1(t) \cdot \frac{\pi \bar{F}_1(t)}{\pi \bar{F}_1(t) + (1-\pi)\bar{F}_2(t)} + \lambda_2(t) \cdot \frac{(1-\pi)\bar{F}_2(t)}{\pi \bar{F}_1(t) + (1-\pi)\bar{F}_2(t)},$$

and

$$\lambda_E(t,s) = \lambda_1(t) \cdot \frac{\psi(1)\bar{F}_1(t)}{\sum_{i=1}^{2} \psi(i)\bar{F}_i(t)} + \lambda_2(t) \cdot \frac{\psi(2)\bar{F}_2(t)}{\sum_{i=1}^{2} \psi(i)\bar{F}_i(t)}.$$

From this, it can be seen that both $\lambda_T(t)$ and $\lambda_E(t,s)$ are the weighted averages of $\lambda_1(t)$ and $\lambda_2(t)$ with corresponding weights, respectively. Thus, to compare $\lambda_T(t)$ and $\lambda_E(t,s)$, it is sufficient to compare the weights which corresponds to $\lambda_1(t)$, *i.e.*, if the first weight is greater, then the second one is smaller, and vice versa. Note that

$$\frac{\pi \bar{F}_1(t)}{\pi \bar{F}_1(t) + (1-\pi)\bar{F}_2(t)} = \frac{1}{1 + \frac{(1-\pi)}{\pi}\frac{\bar{F}_2(t)}{\bar{F}_1(t)}}$$

and

$$\frac{\psi(1)\bar{F}_1(t)}{\sum_{i=1}^{2}\psi(i)\bar{F}_i(t)} = \frac{1}{1 + \frac{1-\psi(1)}{\psi(1)}\frac{\bar{F}_2(t)}{\bar{F}_1(t)}}.$$

Therefore, if $\psi(1) > \pi$, i.e., if

$$\frac{(1-\rho(s))\pi}{(1-\rho(s))\pi + \rho(s)\pi + (1-\alpha(s))P(R > W_s)(1-\pi)} > \pi, \tag{10.12}$$

then $\lambda_T(t) > \lambda_E(t,s)$, for all $t > 0$. It is easy to show that the condition in (10.12) can be reduced to (10.11).

□

Remark 10.4 (i) In the ESS model considered in this section, a level s can only be positively admissible or inadmissible.

(ii) The condition (10.11) implies the admissibility/inadmissibility of the corresponding severity level under the survival function criterion, i.e., $\bar{F}(t) < \bar{F}_E(t,s)$, for all $t > 0$.

10.2.2 Optimal Severity

For further analysis, we need to describe a model for W_s as a 'function' of the shock's magnitude s. It is reasonable to assume first that if $s_1 < s_2$ then $W_{s_1} \le_{st} W_{s_2}$. Let s_b be some 'baseline severity level' (e.g., $s_b \equiv 1$), with the corresponding 'baseline distribution' of W_{s_b} denoted by $G_0(w)$. Therefore,

$$P(W_{s_b} > w) = \bar{G}_0(w),\ w \ge 0.$$

Then the assumption of the above stochastic ordering for W_s is equivalent to assuming the following accelerated life-type model [1]:

$$P(W_s > w) = \bar{G}_0(\emptyset(w,s)),\ w > 0, \tag{10.13}$$

where $\phi(w,s)$ is a function with the following properties: it is decreasing in s for each fixed w, it is increasing in w for each fixed s, $\phi(w,0) \equiv \infty$, for all $w > 0$, $\phi(0,s) \equiv 0$, $\phi(\infty,s) \equiv \infty$, for all $s > 0$. Furthermore, clearly, $\phi(w,s_b) = w$, $w \geq 0$. Therefore, (10.13) implies that if $s_1 < s_2$ then $P(W_{s_1} > w) \leq P(W_{s_2} > w)$, for all $w \geq 0$, which is, obviously, the usual stochastic ordering.

We will consider now the problem of determining the optimal severity level (magnitude) of the ESS. Let τ be the mission time of an item in field operation. If it does not fail during this time, then the mission is considered to be successful. Thus, the probability of the mission success needs to be maximized and we should find the optimal severity level s^* that satisfies

$$\bar{F}_E(\tau, s^*) = \max_{s>0} \bar{F}_E(\tau, s).$$

Alternatively, let $M(s)$ be the mean time to failure of an item in field operation as a function of s, i.e., $M(s) \equiv \int_0^\infty \bar{F}_E(t,s)\mathrm{d}t$. Then, the optimal severity level s^* which maximizes the mean time to failure should be obtained:

$$M(s^*) = \max_{s>0} M(s).$$

It is clear that, for defining s^*, we can consider only the positively admissible severity class, as the other severity levels are inadmissible. The following theorem provides the upper bound for the optimal severity level that maximizes the mission success probability or mean time to failure in field usage.

Theorem 10.6 *Suppose that* $\rho(\infty) \equiv \lim_{s\to\infty} \rho(s) > (1-\pi)$ *and let*

$$s_0 \equiv \inf_{s\geq 0}\{s : \rho(s) > (1-\pi)\}.$$

Then the severities in (s_0, ∞) *are inadmissible. Therefore,* s_0 *is the upper bound for the optimal severity level.*

Proof From Theorem 10.5, the condition for inadmissibility is

$$\frac{1-\rho(s)-\pi}{(1-\pi)(1-\alpha(s))} \leq P(R > W_s),$$

which can now be stated in detail as

$$\frac{\rho(s)-(1-\pi)\alpha(s)}{(1-\pi)(1-\alpha(s))} \geq \int_0^\infty \bar{G}_0(\phi(r,s))\theta\exp\{-\theta r\}dr. \tag{10.14}$$

The inequality in (10.14) can be restated as

$$\int_0^\infty \left(\frac{\rho(s) - (1-\pi)\alpha(s)}{(1-\pi)(1-\alpha(s))} - \bar{G}_0(\phi(r,s)) \right) \cdot \theta \exp\{-\theta r\} dr \geq 0.$$

Observe that for all $r \geq 0$ and for all fixed s,

$$\frac{\rho(s) - (1-\pi)\alpha(s)}{(1-\pi)(1-\alpha(s))} - \bar{G}_0(\phi(r,s)) \geq \frac{\rho(s) - (1-\pi)\alpha(s)}{(1-\pi)(1-\alpha(s))} - 1.$$

Therefore, for a fixed s, if

$$\frac{\rho(s) - (1-\pi)\alpha(s)}{(1-\pi)(1-\alpha(s))} - 1 \geq 0,$$

or equivalently, if $\rho(s) > (1-\pi)$, then for this s the condition (10.14) is satisfied, and accordingly this s is inadmissible. Note that $\rho(s)$ is increasing and, by the assumption in the theorem, $\rho(\infty) \equiv \lim_{s\to\infty} \rho(s) > (1-\pi)$. Hence, there exists $s_0 \in (0,\infty)$ such that $s_0 \equiv \inf_{s\geq 0}\{s : \rho(s) > (1-\pi)\}$ and thus the severities in (s_0, ∞) are inadmissible. Therefore, s_0 is the *upper bound* for the optimal severity. □

Remark 10.5 It would be practically reasonable to assume that $\lim_{s\to\infty} \rho(s) = 1$ and the proportion of the defective items $(1-\pi)$ is relatively small. Therefore, the condition $\rho(\infty) > (1-\pi)$ is practically satisfied in almost all cases.

Example 10.2 Suppose that $r(t) = 0.1t,\ t \geq 0,\ \lambda(t) = 1,\ t \geq 0,\ \theta = 1,\ G_0(w) = 1 - \exp\{-w\},\ w \geq 0,\ s_b = 1,\ \phi(w,s) \equiv \frac{w}{s},\ w, s > 0,\ \pi = 0.7,\ \alpha(s) = 1-\exp\{-s\},\ s \geq 0,\ \tau = 4.0$ and

$$\rho(s) = \begin{cases} 0, & 0 \leq s < 1, \\ 1 - \exp\{-0.05(s-1)\}, & s \geq 1. \end{cases}$$

Furthermore, $p(t) = 0.1,\ t \geq 0$, and the 'failure rate' for W_i's is given by $\lambda_W(w) = 3,\ w \geq 0$. In this case, $M_W(-\theta) = 3/4$ and

$$P(R > W_s) = 1 - \int_0^\infty \exp\{-\frac{r}{s}\} \cdot \exp\{-r\} dr = \frac{1}{1+s}.$$

As $\rho(s)$ is strictly increasing, there exists a unique solution of the equation

$$\rho(s) = (1-\pi),$$

and this solution is the upper bound, which is given by $s_0 = -\{\ln(0.9)/0.05\} + 1 \approx 3.11$. Therefore, it is now sufficient to search for the optimal severity level in the interval [0, 3.11]., The graph of $\bar{F}_E(\tau; s)$ is given in Fig. 10.3.

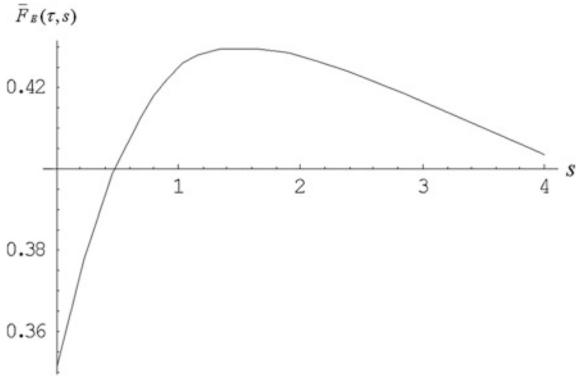


Fig. 10.3 The graph of $\bar{F}_E(\tau, s)$

The optimal severity level in this case is obtained by $s^* = 1.52$ and the maximum probability is $\bar{F}_E(\tau; s^*) \approx 0.43$.

Based on the foregoing results, we can also consider now certain cost structures for determining the optimal severity level. As previously, an item is chosen at random from our initial population and during the ESS it is exposed to a shock of magnitude s. If it survives, it is put into field operation, otherwise the failed item is discarded and a new one is chosen from the population, etc. This procedure is repeated until the first survived item is obtained. Let c_{sr} be the shop replacement cost (actually, it is the cost of a new item) and c_s be the cost for conducting the ESS. Let $c_1(s)$, as a function of s, be the expected cost for eventually obtaining a component which has survived the ESS. Then

$$c_1(s) = \frac{c_s + c_{sr}[1 - \{\pi + (1 - \alpha(s))P(R > W_s)(1 - \pi)\}]}{\pi + (1 - \alpha(s))P(R > W_s)(1 - \pi)},$$

where $1/\{\pi + (1 - \alpha(s))P(R > W_s)(1 - \pi)\}$ is the total number of trials until the first 'success'.

In field operation, assume that if the mission (of length τ) is successful, then a gain K is given; otherwise a penalty C is imposed, where $K > C > 0$. Then the expected gain during field operation (until failure) is given by

$$c_2(s) = -K\bar{F}_E(\tau; s) + CF_E(\tau; s) = -(K + C)\bar{F}_E(\tau; s) + C \tag{10.15}$$

and the total expected cost $c(s)$ is

$$\begin{aligned} c(s) &= c_1(s) + c_2(s) \\ &= \frac{c_s + c_{sr}[1 - \{\pi + (1 - \alpha(s))P(R > W_s)(1 - \pi)\}]}{\pi + (1 - \alpha(s))P(R > W_s)(1 - \pi)} - (K + C)\bar{F}_E(\tau; s) + C. \end{aligned}$$

The objective is to find the optimal severity level s^* that satisfies

$$s^* = \arg \min_{s \in [0,\infty]} c(s).$$

Similar to Theorem 10.5, if $\rho(\infty) \equiv \lim_{s\to\infty} \rho(s) > (1-\pi)$ then the optimal severity level which minimizes $c_2(s)$ (maximizes $\bar{F}_E(\tau; s)$, as follows from (10.15)) does not exist in the interval (s_0, ∞). Furthermore, $c_1(s)$ is strictly increasing to infinity. Therefore, we can conclude that the optimal severity level s^* should exist in the interval $[0, s_0]$.

Assume now that during field operation, the gain is proportional to the mean time to failure. Therefore, the total average cost function in this case is

$$c(s) = \frac{c_s + c_{sr}[1 - \{\pi + (1-\alpha(s))P(R > W_s)(1-\pi)\}]}{\pi + (1-\alpha(s))P(R > W_s)(1-\pi)} - K\int_0^\infty \bar{F}_E(t; s)\mathrm{d}t.$$

By the similar arguments, the optimal severity level s^* should exist in the interval $[0, s_0]$.

References

1. Finkelstein M (2008) Failure Rate Modelling for Reliability and Risk. Springer, London
2. Fiorentino E, Saari AE (1983) Planning of production reliability stress-screening programs. IEEE Trans Reliab 32:247–252
3. Landers TL, Malstrom RM, Schmitt N, Fant E (1994) Electronics Manufacturing Processes. Prentice Hall, New Jersey
4. Yan L, English JR (1997) Economic cost modeling of environmental-stress-screening and burn-in. IEEE Trans Reliab 46:275–282