Warranty/Maintenance: On Modeling Non-zero Rectification Times

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Abstract This chapter revisits modelling of warranty/maintenance costs under the assumption that both, the warranty repairs and the maintenance actions, require non-negligible completion time. We provide an intuition on this topic by summarising our previous results, as well as the published work of other authors. We closely examine a case study that provides an excellent motivation for extending the research in this area. Also, again assuming non-negligible repair and maintenance times, we propose a simulation model for the expected warranty costs that integrates the concepts of reliability improvement and warranty. We conclude with a discussion on new directions for future research.

1 Introduction

A product warranty is an agreement offered by a producer to a consumer to repair or replace a faulty item, or to partially or fully reimburse the consumer in the event of a product failure. From the buyer's viewpoint, the product warranty assures free (partially or fully) of charge replacement or repair of a faulty product. It also provides information on the reliability and quality of the product. On the other hand, from the producer's viewpoint, the product warranty plays a protectional as well as a promotional role.

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Maintenance is an operation that involves fixing the product should it become faulty or out of order. It also includes performing routine actions which keep the product in working condition (a scheduled maintenance) or prevent operational problems from occurring (a preventive maintenance). Overall, maintenance consists of all actions that aim to retain or restore the product (or system) to a state in which it can perform the functions it is designed for.

In most published work on warranty and maintenance, the warranty repair times (and the maintenance times) are assumed to be negligible, i.e. the expected duration of the repair is small (negligible) compared to the expected lifetime of the product. And, yes, in many cases this is a reasonable assumption. But there are situations where the length of the repair (or the duration of the maintenance action) impacts significantly the operational cost. For example, if the maintenance is performed on an assembly line, which produces the main components of a system, the whole production process might be affected, e.g. put on hold, and it could lead to significant losses. If a taxi driver has to wait a couple of weeks until his car (taxi) undergoes a warranty repair, his loss of income could be quite high.

Why is it important to study models with non-negligible warranty repairs or maintenance times? First, the warranty period is a finite interval of time and the total repair time could be a significant portion of it. The total length of the repair time could be of importance in the warranty contract. Moreover, lengthy repairs/maintenance actions may lead to high penalty costs that have to be taken into account in the cost–benefit analysis. Therefore, taking into account the length and the type of the warranty repairs and the duration of the maintenance action is an important component in the warranty/maintenance cost modelling.

In this chapter, some results (see [2, 3]) regarding the evaluation of the expected warranty cost under non-renewing and renewing free replacement warranty policies over the warranty period and over the product life cycle are summarised. We allow for non-zero warranty repair time and assign costs, which are dependent on the length of the repair. Moreover, we review the advances in this area of modelling presented in [9] and [4]. We provide an insight into the importance of this type of modelling by summarising a case study presented in [6]. Lastly, again assuming non-negligible repair and maintenance times, we propose a simulation model for the expected warranty costs that integrates the concepts of reliability improvement and warranty. We conclude with a discussion on new directions for future research.

The outline of this chapter is as follows: In Sect. 2 we recall some basic warranty/maintenance terminology. In Sects. 3 and 4 we summarise the models for non-zero warranty repairs under non-renewing and renewing warranty policies. Sections 5 and 6 review two maintenance models with non-zero maintenance times. A case study is summarised in Sect. 7. In Sect. 8 we propose a new simulation model and Sect. 9 concludes this chapter.

2 Miscellaneous

This section provides the terminology used in warranty and maintenance analysis, that we need for our write-up.

2.1 Warranty Policy

The typical warranty coverage used in the industry can be classified as follows:

- Non-renewing warranty: The expenses associated with the failure of the product during the warranty period of length T are covered (fully or partially) by the warranter.
- Renewing warranty: The expenses associated with the failure of the product during the warranty period of length T are covered (fully or partially) by the warranter. In addition, after each warranty repair, the repaired item is warranted anew for a period T.

2.2 Maintenance Policy

The two classical models mostly studied in the maintenance literature are:

- Block-based model—In this model, a preventive maintenance action is performed periodically over a fixed time interval τ, i.e. at calendar times $\tau, 2\tau, 3\tau, \ldots,$ a maintenance action is invoked. The block-based policy is proposed for a calendar-time-based maintenance model. At failure, the corrective maintenance is carried out.
- Age-based model—In this model, a preventive maintenance action is performed as soon as the product (system) reaches a pre-specified age κ. In addition, corrective maintenance is executed at failure.

2.3 Degree of Repair

In our presentation we consider different types of repairs. Pham and Wang [5] classified repairs according to the degree to which they restore the product. They propose the following classification:

- **Improved Repair:** A repair brings the product to a state *better* than when it was initially purchased. This is equivalent to the replacement of the faulty item by a new and improved item.

- **Perfect (or Complete) Repair:** A repair completely resets the performance of the product so that upon restart the product operates as a new one. This type of repair is equivalent to a replacement of the faulty item by a new one, identical to the original.
- **Imperfect Repair:** A repair contributes to some noticeable improvement of the product. It effectively sets back the clock for the repaired item. After the repair the performance and expected lifetime of the item are as they were at an earlier age.
- **Minimal Repair:** A repair has no impact on the performance of the item. The repair brings the product from a 'down' to an 'up' state without affecting its performance.
- **Worse Repair:** A repair contributes to some noticeable worsening of the product. It effectively sets forward the clock for the repaired item. After the repair the performance of the item is as it would have been at a later age.
- **Worst Repair:** A repair accidentally leads to the product's destruction.

In the following two sections, we summarise our results on modelling non-zero warranty repair times (as given in [2, 3]) based on the alternating renewal process.

3 Non-Renewing Warranty: Non-zero Repair Times

This section is concerned with the non-renewing warranty and incorporates non-zero warranty repair times.

3.1 The Model

We consider the following model: At the beginning the item is in operating ('on') condition for a time X_1. Then the repair ('off') condition starts and the item remains in it for a time Y_1. After the repair completion, the item is operative for a time X_2, which is followed by Y_2 long repair and so on.

The time between two consecutive returns of the virtual age, $V(t)$, of the item to 0 forms a renewal cycle, see Fig. 1. We suppose that both sequences of random

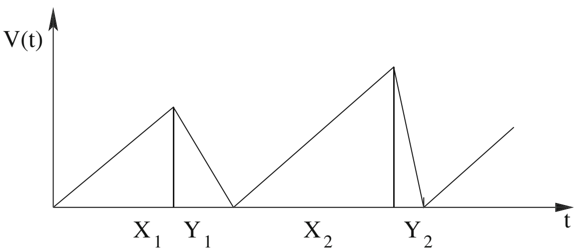


Fig. 1 The virtual age of the item

variables $\{X_i\}_1^\infty$ and $\{Y_i\}_1^\infty$ are independent and identically distributed. Moreover, we assume that X_i and Y_i are independent for $i = 1, 2, \ldots$. The above model is the well-known model of alternating renewal process [7] and we use it to model the warranty cost.

We assume that the cost of the ith repair is random and with form $C_i = A + \delta Y_i$, where A, and δ are known constants.

Moreover, we suppose that the cost is incurred at the end of the renewal cycle. Also, if the warranty coverage expires during a repair period, the corresponding repair is completed and its cost is fully incurred by the warranter. In this case we have a **complete renewal cycle**. If the warranty expires during an operating period, the cost of the following repair is not included in the total costs and we have an **incomplete renewal cycle**.

Life cycle of a product is defined as a time while the product is still usable and contemporary. It is assumed that during the life cycle, after the expiration of the warranty period for the initially purchased item, at the time of the first off warranty failure, the consumer purchases an identical item to the initial one with the same warranty coverage. We will assume that a life cycle can end only at off warranty time. The latter assumption is reasonable because the length of the life cycle is mainly determined by the consumer.

We aim to evaluate: (1) the warranty expenses under non-renewing free replacement warranty of duration T and (2) the expected total warranty costs over the life cycle L of the item. To achieve these goals, as a preliminary, we obtain some results regarding the alternating renewal process.

3.2 Alternating Renewal Process in Finite Horizon

Consider the length of a renewal cycle $X + Y$ with the cumulative distribution function (cdf) F_{X+Y}. Consider the alternating renewal process with "on" time distribution F_X and "off" time distribution F_Y. Denoting

$$S_n = \sum_{i=1}^{n} (X_i + Y_i) \quad \text{and} \quad S_0 = 0$$

it follows that S_n is the time of the completion of the nth repair and corresponding

$$N(t) = \max \{n : S_n \le t\}$$

is the number of complete renewal cycles before time t (cf. [7]). Denote by $m_{X+Y}(y) = E(N(t))$ the corresponding renewal function. It is known (cf. [7]) that

$$P\,(\text{on at } t) = \bar{F}_X(t) + \int_0^t \bar{F}_X(t-y)\, dm_{X+Y}(y), \tag{1}$$

which is the probability of having operating item at time t. It is easy to see that $P\,(\text{off at } t) = 1 - P\,(\text{on at}\, t)$ is equivalent to

$$\begin{aligned} P\,(\text{off at } t) = &\int_0^t \bar{F}_Y(t-u)\, dF_X(u) \\ &+ \int_0^t \int_0^{t-u} \bar{F}_Y(t-u-v)\, dF_X(v)\, dm_{X+Y}(u). \end{aligned} \tag{2}$$

Theorem 3.1

$$P(S_{N(T)} \le t \,|\, \text{on at } T) = \frac{\bar{F}_X(T) + \int_0^t \bar{F}_X(T-u) dm_{X+Y}(u)}{\bar{F}_X(T) + \int_0^T \bar{F}_X(T-u) dm_{X+Y}(u)}, \quad 0 \le t \le T \tag{3}$$

Proof:

$$\begin{aligned} &P(S_{N(T)} \le t \,|\, \text{on at } T) P(\text{on at } \ T) \\ &\quad = P(\text{on at } \ T \mid S_{N(T)} = 0) P(S_{N(T)} = 0) \\ &\qquad + \int_0^t P(\text{on at } T \mid S_{N(T)} = u)\, dF_{S_{N(T)}}(u) \\ &\quad = \frac{P(X_1 + Y_1 > T, X_1 > T)}{P(X_1 + Y_1 > T)} P(X_1 + Y_1 > T) \\ &\qquad + \int_0^t P(\text{on at } T \mid X_n + Y_n > T-u)\, \bar{F}_{X+Y}(T-u)\, dm_{X+Y}(u) \\ &\quad = \bar{F}_X(T) + \int_0^t P(X_n > T-u \,|\, X_n + Y_n > T-u)\, \bar{F}_{X+Y}(T-u)\, dm_{X+Y}(u) \\ &\quad = \bar{F}_X(T) + \int_0^t \frac{\bar{F}_X(T-u)}{\bar{F}_{X+Y}(T-u)} \bar{F}_{X+Y}(T-u)\, dm_{X+Y}(u) \\ &\quad = \bar{F}_X(T) + \int_0^t \bar{F}_X(T-u)\, dm_{X+Y}(u) \end{aligned}$$

Therefore, using (1), the proof is completed. □

Corollary 3.2

$$P(S_{N(T)} = 0 \,|\, \text{on at } T) = \frac{\bar{F}_X(T)}{\bar{F}_X(T) + \int_0^T \bar{F}_X(T-u)\, dm_{X+Y}(u)} \tag{4}$$

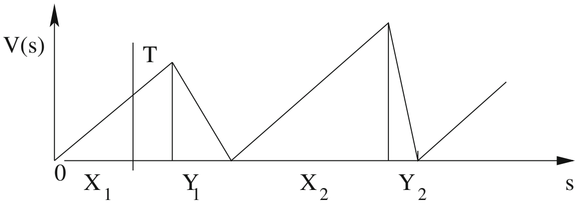


Fig. 2 $S_{N(T)} = 0$

Theorem 3.3 For $T \leq t$,

$$P(S_{N(T)} + X_{N(T)+1} \leq t \mid \text{on at } T)$$
$$= \frac{\bar{F}_X(T) - \bar{F}_X(t)}{\bar{F}_X(T) + \int_0^T \bar{F}_X(T-u)\, dm_{X+Y}(u)}$$
$$+ \frac{\int_0^T \left(\bar{F}_X(T-u) - \bar{F}_X(t-u)\right) dm_{X+Y}(u)}{\bar{F}_X(T) + \int_0^T \bar{F}_X(T-u)\, dm_{X+Y}(u)}$$

The proof is similar to that of Theorem 3.1, hence it is omitted.

We sketch another proof of Theorem 3.3 by utilising the multiplication rule and the total probability rule. Namely, by conditioning on $S_{N(T)}$, we consider the following two cases:

1. $S_{N(T)} = 0$, (**Fig.** 2).
   The following events are equivalent.

$$\{S_{N(T)} + X_{N(T)+1} \leq t,\ S_{N(T)} = 0, \quad \text{on at } T\} \Longleftrightarrow \{T < X \leq t\}.$$

   The probability of the latter is equal to

$$F_X(t) - F_X(T). \tag{5}$$

2. $S_{N(T)} = w \neq 0$, (**Fig.** 3).
   The following events are equivalent $\{S_{N(T)} + X_{N(T)+1} \leq t,\ S_{N(T)} = w \neq 0, \quad \text{on at } T\} \Leftrightarrow$ {there is a renewal before T} say at time w with probability $dm_{X+Y}(w)$ and $\{T - w < X < t - w\}$, which will occur with probability $F_X(t-w) - F_X(T-w)$. The probability of the second event is

$$\int_0^T (F_X(t-w) - F_X(T-w))\, dm_{X+Y}(w) \tag{6}$$

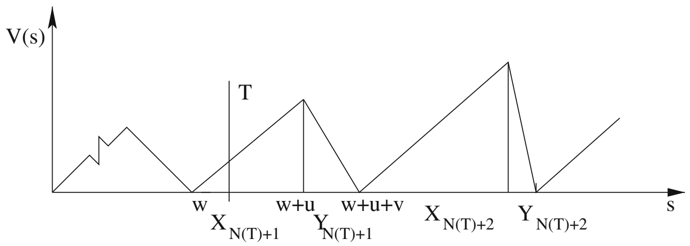


Fig. 3 $S_{N(T)} = w \neq 0$

Adding (5) and (6) evaluates $P(S_{N(T)} \leq t,\ \text{on at } T)$. Then, the result of the theorem follows from the multiplication rule and (1). □

Theorem 3.4

$$P(S_{N(T)+1} + X_{N(T)+2} \leq t \mid \text{off at } T)$$
$$= \frac{1}{P(\text{off at } T)} \times \left(\int_0^T \int_{T-u}^{t-u} F_X(t-u-v)\, dF_Y(v)\, dF_X(u) \right.$$
$$\left. + \int_0^T \int_0^{T-w} \int_{T-w-u}^{t-w-u} F_X(t-w-u-v)\, dF_Y(v)\, dF_X(u)\, dm_{X+Y}(w) \right) \tag{7}$$

The proof of this theorem is similar to that given in Theorem 3.3, hence it is omitted.

3.3 Warranty Cost Analysis

Here we derive the expected warranty cost for non-renewing warranty over warranty period of length T and over the life cycle of length L. By assumption, the random variables C_i are independent and identically distributed and $E(C) = A + \delta E(Y)$.

3.3.1 Expected Costs Over (0, *T*)

Denote by $C(t)$ the total warranty cost accumulated up to time t. We have to distinguish two cases: first the warranty expires during an "off" time, then the total cost is accumulated over $N(T) + 1$ complete renewal cycles. Second, the warranty expires during an "on" time, so that only $N(T)$ complete renewal cycles contribute to the cost. Then

$$C(T) = \begin{cases} \sum_{i=1}^{N(T)} C_i, & \text{if the item is "on" at time T} \\ \sum_{i=1}^{N(T)+1} C_i, & \text{if the item is "off" at time T} \end{cases} \tag{8}$$

and the following result holds:

Theorem 3.5

$$E(C(T)) = (m_{X+Y}(T) + 1)\, E(C) - E(C_{N(t)+1}| \text{ on at } T) P(\text{on at } T).$$

Proof: Using that $N(t) + 1$ is a stopping time for the sequence $\{C_i\}_1^\infty$ and Wald's equation (see [7]) we have

$$\begin{aligned} E(C(T)) &= E\left(\sum_{i=1}^{N(T)} C_i|\text{on at } T\right) P(\text{on at } T) + E\left(\sum_{i=1}^{N(T)+1} C_i|\text{off at } T\right) P(\text{off at } T) \\ &= E\left(\sum_{i=1}^{N(T)+1} C_i - C_{N(T)+1}|\text{on at } T\right) P(\text{ on at } T) \\ &\quad + E\left(\sum_{i=1}^{N(T)+1} C_i|\text{off at } T\right) P(\text{off at } T) \\ &= E\left(\sum_{i=1}^{N(T)+1} C_i|\text{on at } T\right) P(\text{on at } T) + E\left(\sum_{i=1}^{N(T)+1} C_i|\text{off at } T\right) P(\text{off at } T) \\ &\quad - E(C_{N(T)+1}|\text{on at } T) P(\text{on at } T) \\ &= E\left(\sum_{i=1}^{N(T)+1} C_i\right) - E(C_{N(T)+1}|\text{on at } T)\, P(\text{on at } T) \\ &= (m_{X+Y}(T) + 1)\, E(C) - E(C_{N(t)+1}|\text{on at } T)\, P(\text{on at } T) \end{aligned}$$

□

We need to find $E(C_{N(t)+1} \mid \text{on at } T)\ P(\text{on at } T)$. The latter probability is given by (1). Since $C_{N(T)+1} = A + \delta Y_{N(T)+1}$, we need to evaluate $E(Y_{N(t)+1} \mid \text{on at } T)$. The following result holds:

Theorem 3.6

$$E(C(T)) = (A + \delta E(Y))\, (m_{X+Y}(T) + P(\text{off at } T))$$

The following lemma will be needed for the proof of the theorem:

Lemma 3.7

$$E(Y_{N(t)+1} \mid \text{on at } T) = E(Y)$$

Proof: By conditioning on $S_{N(T)}$, and using Theorem 3.1 and Corollary 3.2 we obtain

$$\begin{aligned}
&E(Y_{N(t)+1} | \text{on at } T) \\
&\quad = E(Y_{N(t)+1} | S_{N(T)} = 0, \quad \text{on at } T)\, P(S_{N(T)} = 0 | \text{on at } T) \\
&\qquad + \int_0^T E(Y_{N(t)+1} | S_{N(T)} = s, \ \text{on at } T)\, dP(S_{N(T)} \le s | \text{on at } T) \\
&\quad = E(Y_1 \mid X_1 > T)\, \frac{\bar{F}_X(T)}{\bar{F}_X(T) + \int_0^T \bar{F}_X(T-u)\, dm_{X+Y}(u)} \\
&\qquad + \int_0^T E(Y_n \mid X_n > T - s) \frac{\bar{F}_X(T-s)\, dm_{X+Y}(s)}{\bar{F}_X(T) + \int_0^T \bar{F}_X(T-u)\, dm_{X+Y}(u)} \\
&\quad = \int_0^T E(Y)\, dF_{S_{N(T)} \mid \text{on at } T}\ (s) = E(Y)
\end{aligned}$$

Using Lemma 3.7 it is easy to complete the proof of Theorem 3.6.

Proof: Indeed

$$\begin{aligned}
E(C(T)) &= E\left(\sum_{n=1}^{N(T)+1} C_i\right) - E(C_{N(T)+1} \mid \text{on at } T)\, P(\text{on at } T) \\
&= (m_{X+Y}(T) + 1)\, E(C) - E(C)\, P(\text{on at } T) \\
&= E(C)\, (m_{X+Y}(T) + P(\text{off at } T)).
\end{aligned}$$

□

3.3.2 Expected Costs Over (0, L)

Now we will focus on the evaluation of the expected warranty costs over the life cycle of an item. Let us consider the time between two consecutive purchases made by the consumer. Denote this time by ξ. It is a positive continuous random variable such that:

$$\xi = \begin{cases} S_{N(T)} + X_{N(T)+1}, & \text{if the item is "on" at time T} \\ S_{N(T)+1} + X_{N(T)+2}, & \text{if the item is "off" at time T} \end{cases}$$

Then, the expected costs over $(0, L)$ are expressed in terms of ξ in the following way:

$$E(C(L)) = E(N^*(L) + 1)\, E(C(T)),$$

where $N^*(t)$ is a renewal process with interevent time equal to ξ. Denote by $m^*_\xi(t)$ the renewal function of $N^*(t)$. Then

$$E(C(L)) = (m^*_\xi(L) + 1) E(C(T)). \tag{9}$$

In what follows we derive the distribution of the interevent time ξ.

Theorem 3.8

$$\begin{aligned}
P(\xi \le t) = {} & \bar{F}_X(T) - \bar{F}_X(t) \\
& + \int_0^T (\bar{F}_X(T-u) - \bar{F}_X(t-u))\, dm_{X+Y}(u) \\
& + \int_0^T \int_{T-u}^{t-u} F_X(t-u-v)\, dF_Y(v)\, dF_X(u) \\
& + \int_0^T \int_0^{T-w} \int_{T-w-u}^{t-w-u} F_X(t-w-u-v)\, dF_Y(v)\, dF_X(u)\, dm_{X+Y}(w)
\end{aligned}$$

Proof:

$$\begin{aligned}
P(\xi \le t) = {} & P(S_{N(T)} + X_{N(T)+1} \le t \mid \text{on at } T) P(\text{on at } T) \\
& + P(S_{N(T)+1} + X_{N(T)+2} \le t \mid \text{off at } T)\, P(\text{off at } T)
\end{aligned}$$

Applying Theorems 3.3 and 3.4 and using (1) and (2) completes the proof. □

3.4 Example

As an illustration of the ideas we will consider an example assuming that the lifetime of the item and the repair time are exponentially distributed random variables with parameters λ and μ.

3.4.1 Expected Costs Over $(0, T)$

In order to evaluate the expected warranty costs over $(0, T)$, we need to find the corresponding renewal function. Using Laplace transforms it can be shown that the renewal function for the renewal process with interevent time $X + Y$ is

Table 1 Expected warranty cost over a warranty period, $\mu = 92$

	T					
λ	0.5	1.0	1.5	2.0	2.5	3.0
2	2.958815	5.916262	8.873708	11.831155	14.788602	17.746049
3	4.392487	8.781961	13.171434	17.560908	21.950382	26.339855

Table 2 Expected warranty cost over a warranty period, $\mu = 122$

	T					
λ	0.5	1.0	1.5	2.0	2.5	3.0
2	2.968527	5.936269	8.904011	11.871752	14.839494	17.807236
3	4.417737	8.833737	13.249737	17.665737	22.081737	26.497737

$$m_{X+Y}(t) = \frac{\lambda\mu}{\lambda+\mu}\left(t - \frac{1}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)t}\right)\right).$$

Using (2), we get $P(\text{off at } t) = \frac{\lambda}{\lambda+\mu}(1 - e^{-(\lambda+\mu)T})$. Then, the expected warranty cost for non-renewing free replacement warranty policy with duration T is equal to

$$E(C(T)) = \left(A + \frac{\delta}{\mu}\right)\left(\frac{\lambda}{\lambda+\mu}\right)\left(\mu T + \frac{\lambda}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)T}\right)\right). \tag{10}$$

For selected values of T and λ and for $A = 3$ and $\delta = 2$, numerical values for the expected warranty cost are calculated and summarised in Tables 1 and 2.

The comparison between the two tables shows that it is better to have a longer average repair time (4 days for Table 1 against 3 days for Table 2). A possible reason for this result is the fact that for the fixed values of T and λ the value of μ will reflect on the number of renewal cycles per warranty period. Indeed, larger values of μ will increase the number of renewal cycles within the warranty period, which will increase the value of the expected warranty cost over $(0, T)$. Providing that the penalty cost δ is not too high, this is a reasonable strategy. On the other hand if δ is high and low expected warranty costs are targeted, it will require a reduction of the average repair time.

3.4.2 Expected Costs Over $(0, L)$

Even in this simple case of exponential lifetime and exponential repair time we encounter difficulties in evaluating the expected warranty cost over the life cycle of the item. The standard approach of finding the renewal function of the renewal process generated by the random variable ξ led to an expression with a limited value. The attempt to use MAPLE or MATHEMATICA to simplify the result was also not very successful. Hence, we used a numerical procedure. Based on the ideas of Xie [10], a

Table 3 Expected warranty cost over a life cycle, $\lambda = 2$, $\mu = 122$

	T					
L	0.5	1.0	1.5	2.0	2.5	3.0
3	10.75976	15.28904	17.36461	22.13644	24.21849	17.80759
5	16.69591	23.07293	26.60188	30.76411	29.57897	35.28823
7	22.66831	30.99828	35.74898	39.24450	43.15709	46.17403
10	31.53661	42.86935	49.25831	54.04590	57.49978	59.12158
15	46.36995	62.65530	71.51108	77.50511	82.34271	86.33264

Table 4 Expected warranty cost over a life cycle, $\lambda = 6$, $\mu = 122$

	T					
L	0.5	1.0	1.5	2.0	2.5	3.0
3	43.36922	51.46913	51.76007	68.92752	84.11078	51.75959
5	69.29022	83.65021	92.53079	102.91867	86.26324	103.51294
7	95.74140	112.94833	125.45428	135.87711	129.39141	154.36811
10	133.98574	156.33294	169.55625	172.45343	172.52479	203.51039
15	198.68439	230.64985	249.49196	255.03050	258.75241	258.7822

renewal equation solver has been written by Dr Richard Arnold in programming language `R`. The solver evaluates the renewal function under known cdf (in closed form), known pdf, or data for the renewal points. The last option is an extension of [10].

Using (9) and assuming $A = 3$ and $\delta = 2$, the expected warranty cost for selected values of L were evaluated. The comparison between Tables 3 and 4, with the given values of λ and μ, shows that the improvement of the reliability and quality of the product, reflecting on the increase of its average operating time, will highly reduce the expected warranty cost.

4 Renewing Warranty: Non-zero Repair Times

The alternating renewal process described in Sect. 3.2 is also assumed here. However, now we consider renewing warranty policy with perfect warranty repairs. Again, the cost of the ith repair is assumed to be $C_i = A + \delta Y_i$, and the random variables C_i are independent and identically distributed and their expected value is $A + \delta E(Y)$ (Fig. 4).

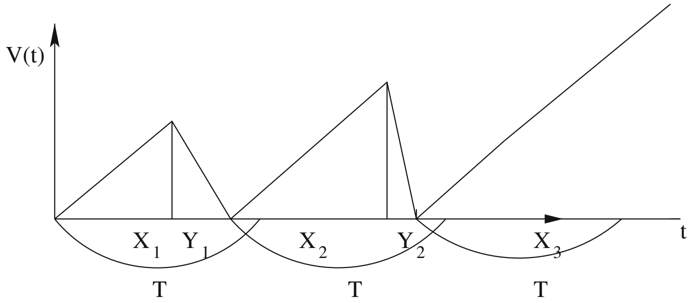


Fig. 4 Renewing warranty

4.1 The Model

We model the functioning of the item as in Sect. 3.1. Taking into account the impact of the renewing warranty, we adjust the model as follows: at the end of the repair time, the item is warranted anew for a period of length T, i.e. after each repair the item is assumed to be as good as new. If the warranty period ends during an operating period, the cost of the following repair is not incurred by the warranter and the warranty coverage expires. Here we will distinguish between warranty coverage W_T, which is a random variable, and warranty period, which is a predetermined constant T.

4.2 Warranty Cost Analysis

Here we derive the expected warranty cost for renewing warranty over warranty period of length T and over the life cycle of length L.

4.2.1 Expected Cost Under Renewing Warranty Coverage

Due to the mechanism of the renewing warranty, W_T is equal to:

$$W_T = \begin{cases} T, & \text{if } X_1 > T \\ T + \sum_{i=1}^{n}(X_i + Y_i), & \text{if } X_1 \le T, \cdots, X_n \le T, X_{n+1} > T \text{ for some } n. \end{cases}$$

Then, the warranty cost $C(W_T)$ over the warranty coverage is a random variable and its distribution is:

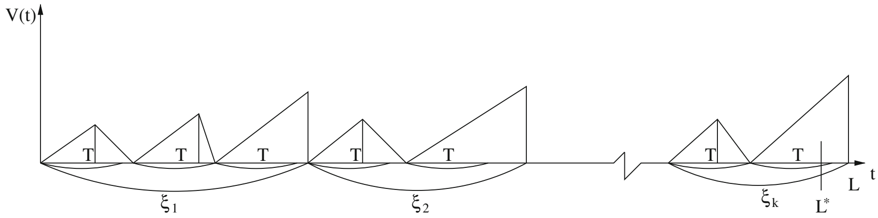


Fig. 5 Life cycle over (0, L)

$$C(W_T) = \begin{cases} 0, & \text{with } 1 - F_X(T) \\ C_1, & \text{with } F_X(T)(1 - F_X(T)) \\ \vdots & \\ \sum_{i=1}^{n} C_i, & \text{with } F_X(T)^n(1 - F_X(T)). \\ \vdots & \end{cases} \tag{11}$$

Thus, $C(W_T)$ has a geometric distribution with parameter $(1 - F_X(T))$ and

$$E(C(W_T)) = \frac{F_X(T)}{1 - F_X(T)} \quad (A + \delta E(Y)). \tag{12}$$

Therefore, provided that the distributions of X and Y are known, using (12), the expected cost under renewing warranty coverage can easily be evaluated. Otherwise, (12) should be used with appropriate estimations of $F_X(T)$ and $E(Y)$.

4.2.2 Expected Costs Under Renewing Warranty Coverage Over Life Cycle

Let L^* be a prespecified time during which a product is considered to be contemporary and competitive with similar products in the market. Let L be the time of the first off warranty failure of the product after L^*. Then, we call (0, L) the life cycle of the item. The idea of life cycle and the relationship between L and L^* are represented in Fig. 5.

In what follows we evaluate the expected warranty costs over (0, L), where the value of L^* is known. Let us consider the continuous positive random variable, ξ, representing the time between two consecutive product purchases. By definition,

$$\xi = \begin{cases} X_1 & \text{if } X_1 > T \\ \sum_{i=1}^{n}(X_i + Y_i) + X_{n+1} & \text{if } X_1 \leq T, \ldots, X_n \leq T, X_{n+1} > T \text{ for some } n. \end{cases}$$

Then, the expected costs over (0, L), denoted by $E(C(L))$, are expressed in terms of ξ in the following way:

$$E(C(L)) = (m^*_\xi(L) + 1)\, E(C(W_T)),$$

where $m^*_\xi(t)$ is the renewal function of the renewal process generated by ξ.

Now, let us introduce the age parameter for ξ denoted by τ, i.e. τ is the time origin where ξ is measured from. We will derive the probability density function (pdf) of ξ, $g_\xi(\tau, t)$, given τ. The following theorem holds:

Theorem 4.9 *The pdf, $g_\xi(\tau, t)$, satisfies the following integral equation:*

$$g_\xi(\tau, t) = \begin{cases} f_X(t) + \int_0^{t-T} \int_0^{t-T-u} g_\xi \\ \quad (\tau + u + v, t - u - v) f_Y(v) f_X(u)\, dv du & \text{if } T < t < 2T \\ f_X(t) + \int_0^{T} \int_0^{t-T-u} g_\xi \\ \quad (\tau + u + v, t - u - v) f_Y(v) f_X(u)\, dv du & \text{if } t \geq 2T. \end{cases} \tag{13}$$

Proof: Using the definition of pdf, namely,

$$g_\xi(\tau, t)\Delta t \approx P(\xi \in (t, t + \Delta t))$$

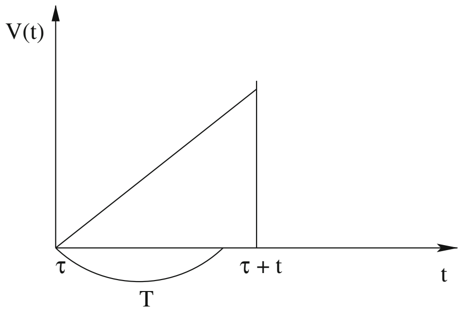
for small Δt, we will condition on X_1. There are two scenarios under which the event $\{\xi \in (t, t + \Delta t)\}$ can occur. Their pictorial representations are given in Figs. 6 and 7.

- Scenario 1 (Fig. 6)
  $X_1 > T$, thus $\xi = X_1$. Then $\{\xi \in (t, t + \Delta t)\} \equiv \{X_1 \in (t, t + \Delta t)\}$.
- Scenario 2 (Fig. 7)
  $X_1 \leq T$.

Here our main idea is to find a relationship between $g_\xi(\tau, t)$ and $g_\xi(\tau + s, t - s)$, where s is the point of the first warranty renewal. We need to consider two cases:

1. $T < t < 2T$.
   Due to the definition of ξ, $0 < s < t - T$. (Recall that ξ will terminate only if $X_k > T$ for some k.) Then,

Fig. 6 $X_1 > T$



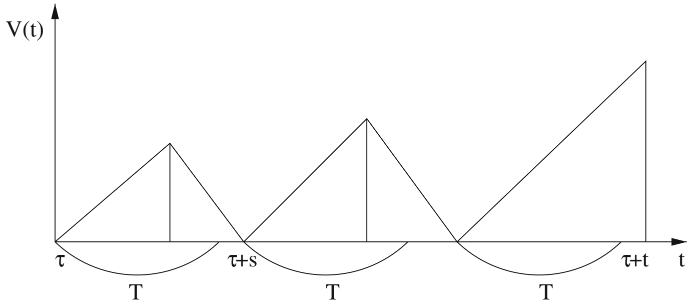
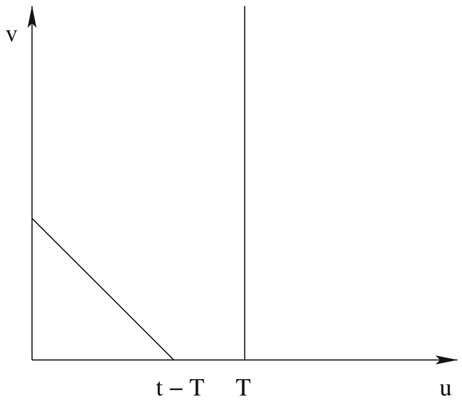


Fig. 7 $X_1 < T$

Fig. 8 $T < t < 2T$, $t - T < T$



$$0 < s < t - T < 2T - T = T \Rightarrow 0 < s < T$$

and (see Fig. 8)

$$\{X_1 + Y_1 \in (s, s + \Delta s)\} \subset \{X_1 \in (u, u + \Delta u)\} \text{ for any } 0 < u \leq s < T.$$

Hence, using Scenario 1, we obtain

$$g_\xi(\tau, t) = f_X(t) + \int_0^{t-T} \int_0^{t-T-u} g_\xi(\tau + u + v, t - u - v) f_Y(v) f_X(u)\, dv\, du.$$

2. $t > 2T$.
   Again, $0 < s < t - T$, but now $t - T > T$, (see Fig. 9), $\{\xi \in (t, t + \Delta t)\}$ is equivalent to the event: there is a failure at time u (measured from the origin τ) and $u < T$ (i.e. failure within the warranty period) which occurs with probability $f_X(u)du$, and repair lasting v, which occurs with probability $f_Y(v)dv$ and $\{\xi \in (t - u - v, t - u - v + \Delta(t - u - v))\}$ with initial age $\tau + u + v$, which occurs with probability $g_\xi(\tau + u + v, t - u - v)\Delta(t - u - v)$. Then, taking into account

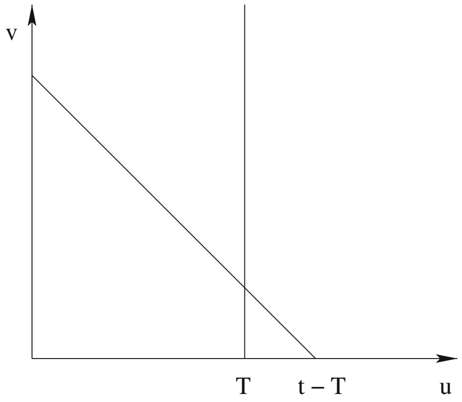


Fig. 9 $t - T > T$

Scenario 2, we have

$$g_\xi(\tau, t) = f_X(t) + \int_0^T \int_0^{t-T-u} g_\xi(\tau + u + v, t - u - v) f_Y(v) f_X(u)\, dv\, du,$$

which completes the proof of the theorem.

4.2.3 Numerical Procedure for Calculating the pdf of ξ

Let us denote $S = X_1 + Y_1$. It is easy to notice that:

1. The support of S is $(0, t - T)$ for any t.
2. The differences in the limits of integration in (13) are due to the restriction $X_1 < T$.

Equation (13) can be rewritten in terms of S in the following way:

$$g_\xi(\tau, t) = f_X(t) + \int_0^{t-T} g_\xi(\tau + s, t - s) f_T(s) ds, \quad t > T, \tau \geq 0 \tag{14}$$

where $f_T(s) = \int_0^{T \wedge s} f_X(u) f_Y(s - u) du$. The sub-density $f_T(s)$ reflects the comments at the beginning of this section.

Let us consider a grid of step h in the two-dimensional plane (τ, t). Let NL be the number of points on both τ and t axes. Note that, for convenience, the count of the points starts from 1.

Using the definition of Riemann–Stiltjes integral, $g_\xi(\tau, t)$ can be approximated by:

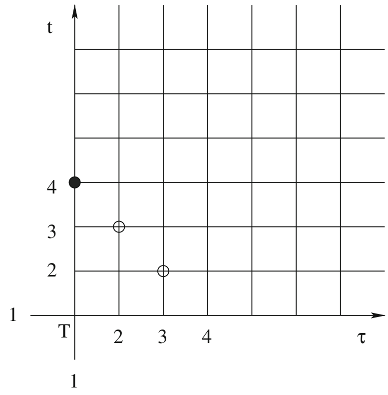


Fig. 10 Grid in (τ, t) plane

$$\begin{cases} g_{i1} = 0, \quad i = 1, 2, \cdots, NL; \\ g_{i2} = f_X(T + h); \\ g_{1j} = f_X(T + (j-1)h) + h \sum_{k=1}^{j-2} g_{1+k,j-k} f_T(kh), \quad j = 3, 4, \ldots, NL; \\ g_{ij} = g_{1j}, \quad i = 2, 3, \ldots, NL. \end{cases}$$

The reasoning for this algorithm is the following: The sum approximating the integration in (14) consists of only values of $g_\xi(\cdot, \cdot)$ calculated over the diagonals of the grid, i.e. if g_{ij} is to be calculated, then the previous values of $g_\xi(\dot{,}\cdot)$ needed are only those $g_{i+k,j-k}$ for $k = 1, \ldots, j-k$. These values are calculated at points located on the diagonal consisting of (i, j). In each step of the procedure, once g_{1j} is evaluated, the remaining values $g_{ij}, i = 2, 3, \ldots, NL$ are assigned to equal to g_{1j}. This is because the meaning of the first parameter is the age and the distribution of ξ is independent of the age. The parameter τ was introduced only for convenience in an attempt to simplify the notations and the reasoning of the computational procedure (Fig. 10).

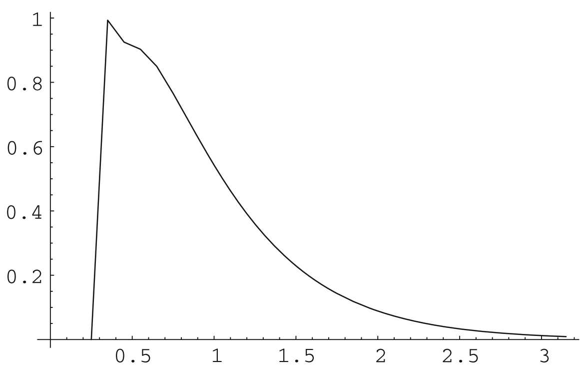
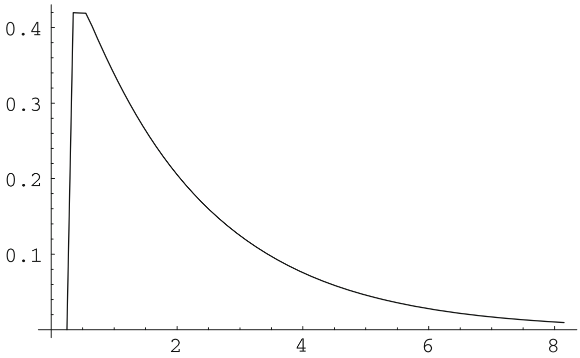


Fig. 11 $\lambda = 2, \mu = 10, T = 0.25$

Fig. 12 $\lambda = 0.5, \mu = 52, T = 0.25$



We used MATHEMATICA to write a code for the numerical procedure for obtaining the density $g_\xi(t)$. The graphs of the pdf of ξ show that the distribution of ξ is skewed to the right. It is unimodal and the value of its maximum decreases when T increases. For fixed value of T, the shape of the density depends on λ, μ (Figs. 11–12). For more details on $g_\xi(t)$, see [3].

4.2.4 Estimating the Renewal Function of ξ

Our next step is to use the suggested numerical procedure to evaluate the renewal function generated by ξ. Again, we used the renewal equation solver written by Dr. Richard Arnold.

4.3 Example

As an illustration we consider an example assuming that the lifetime of the item and the repair time are exponentially distributed random variables with parameters λ and μ. The same procedure is valid for general distribution of the operating time X and the repair time Y.

4.3.1 Expected Costs Over $(0, W_T)$

Using (12) for selected values of T and λ and for $A = 3$ and $\delta = 2$, numerical values for the expected warranty cost were calculated and summarised in the following two tables. We measure the time in years.

In the first row of Table 5 $\lambda = 2$, which means that on average there is a failure of the product every 6 months. The length of the following repair, again on average, is set to be equal to approximately four days. As expected the expected warranty cost is an increasing function of the length of the warranty period. The second row represents the values of the expected warranty cost for $\lambda = 4/3$, which means that on average, there is a failure of the product every nine months. The values of the remaining parameters are kept the same as for the first row. By comparing the rows in Table 5, it is easy to see that the shorter mean operating time leads to higher expected warranty cost, which is in agreement with our intuition. These conclusions are similar to the ones we have reached for non-renewing warranty in [2].

In Table 6, the length of the repair, again on average, is set to be equal to approximately 1 week. The same as for Table 5 comments apply for Table 6 which is: the expected warranty cost is an increasing function of the length of the warranty period T, i.e. row-wise increasing values of the expected warranty cost and it is a decreasing function of the mean operating time, i.e. column-wise increasing values of the expected warranty cost.

The comparison between Tables 5 and 6 shows that it is better to have a shorter average repair time (four days for Table 5 against 1 week for Table 6). This conclusion is opposite to the one we reached in [2]. This is due to the differences between renewing and non-renewing warranty policies. The comparison between Tables 6 and 7, (all parameters are kept the same, only the value of δ is different) does not lead to surprising conclusions. The expected warranty cost is an increasing function of δ. In [2], the expected warranty cost over a warranty period as a function of δ had a maximum. This is because the warranty coverage W_T is a random variable, against fixed length T of the warranty in a non-renewing scenario.

4.3.2 Expected Costs Over $(0, L)$

Using the computational procedure for (13) and assuming $A = 3$ and $\delta = 2$, the expected warranty cost for selected values of L were evaluated. The comparison between these values shows that the improvement of the reliability and quality of the product, reflecting on the increase of its average operating time, will highly reduce the expected warranty cost. For numerical results and graphical summary, the reader is referred to [3].

5 Maintenance: Non-zero Periodic Preventive Repairs

Next, we summarise the results presented in Wang and Zhang [9]. The authors consider a simple deteriorating system. After a failure the system is replaced at a high

Table 5 Expected warranty cost over a warranty period, $\mu = 122$, $\delta = 2$

	T			
λ	1/12	1/4	1/2	1
2	0.547054	1.9568	5.18301	19.2719
4/3	0.354484	1.19332	2.85874	8.4268
1	0.262137	0.856732	1.9568	5.18301

Table 6 Expected warranty cost over a warranty period, $\mu = 52, \delta = 2$

λ	T 1/12	1/4	1/2	1
2	0.551057	1.97111	5.22093	19.4129
4/3	0.357077	1.20205	2.87965	8.48845
1	0.264055	0.863	1.97111	5.22093

Table 7 Expected warranty cost over a warranty period, $\mu = 52$ and $\delta = 22$

λ	T 1/12	1/4	1/2	1
2	0.620811	2.22062	5.88181	21.8702
4/3	0.402277	1.35421	3.24417	9.56294
1	0.297479	0.972241	2.22062	5.88181

cost. To extend the operating lifetime and to reduce the operating cost, at the time the system lifetime reaches a constant level B, the system could be repaired preventively, through an imperfect repair. The following scenario is considered: the successive operating times of the system after preventive repair form a stochastically decreasing geometric process, while the consecutive non-zero preventive repair times of the system form a stochastically increasing geometric process. The objective of this study is to determine an optimal bivariate replacement policy such that the average cost rate (the long-run average cost per unit time) is minimised.

5.1 The Model

The model is constructed under the following assumptions:

- At the beginning, a new system with preventive repairs (PR) is installed. At some point of time the system will be replaced by a new one and the replacement time is negligible.
- The PR will be adopted as soon as the operating time of the system reaches level B, and the PR is imperfect. Henceforth, the following notations will be used:
  - X_n - the operating time of the system after the $(n-1)$th PR with cdf $F_n(t) = F(a^{n-1}t), a \geq 1$; $EX_1 = \lambda$. $\{X_n\}, n = 1, 2, 3, \ldots$ form a stochastically decreasing geometric process with ratio a.
  - Y_n - the repair time of the system in the nth cycle with cdf $G_n(t) = G(b^{n-1}t)$, $0 < b \leq 1$; $\mu = EY_1$. $\{Y_n\}, n = 1, 2, 3, \ldots$ form a stochastically increasing geometric process with ratio b.
  - $\{X_n\}$ and $\{Y_n\}, n = 1, 2, 3, \ldots$ are independent.
  - A bivariate replacement maintenance policy (B, N) is adopted, i.e. B is a fixed period of time between consecutive PR and N is the number of PR's before the system is replaced. In other words, if the system is free of failure until the

$(N+1)^{st}$ PR, then it is replaced instead of performing a PR. At failure the system brings failure cost η and it is instantaneously replaced.
- The time between two consecutive system replacements $\tau_{\{.\}}$ is called a renewal cycle. $\{\tau_1, \tau_2, \tau_3, \ldots\}$ form a renewal process, where τ_1 is the time to first replacement.

5.2 The Average Cost Rate

All results and their derivations follow the presentation in [9]. Before the main result regarding the average cost rate $C(B, N)$ is provided, we need a list of preliminary results, so as to facilitate the understanding and utilisation of the main result.

1. We start with the distribution of M, the number of PR before system replacement. It is easy to see that:

$$P(M=0) = F(B) \quad \text{and} \quad P(M=k) = \prod_{i=0}^{k-1} \bar{F}(a^i B) F(a^k B). \tag{15}$$

2. The system total operating time $T(B, N)$ before renewal can be expressed as follows:

$$T(B, N) = \begin{cases} MB + \{X_{M+1} | X_{M+1} \le B\}, & \text{if } M \le N \\ (N+1)B, & \text{if } M > N. \end{cases} \tag{16}$$

3. The total PR time $S(B, N)$ in a renewal cycle is:

$$S(B, N) = \begin{cases} Y_1 + Y_2 + \ldots + Y_M, & \text{if } M \le N \\ Y_1 + Y_2 + \ldots + Y_N, & \text{if } M > N. \end{cases} \tag{17}$$

4. The total cost function $\Phi(B, N)$ in a renewal cycle is given by:

$$\begin{aligned} \Phi(B, N) = & \left(-c_w (MB + X_{M+1} | X_{M+1} \le B) + c_r \sum_{k=1}^{M} Y_k + \eta \right) I_{\{M \le N\}} \\ & + \left(-c_w (N+1)B + c_r \sum_{k=1}^{N} Y_k \right) I_{\{M > N\}} + c, \end{aligned} \tag{18}$$

where $I_{\{.\}}$ is an indicator function, c_w is the system's working reward rate, c_r is the system's PR cost rate, c is the system's replacement cost, η is the system's invalidation cost.

Thus, now having the expressions (16), (17) and (18) and noticing that

$$E[X_k|X_k \le B] = \frac{1}{F(a^{k-1}B)} \int_0^B x dF(a^{k-1}x),$$

allow to derive the expectations of the above random variables as follows:

1. $$E[T(B,N)] = \int_0^B x dF(x) + \sum_{k=1}^{N} \left[kB + \frac{1}{F(a^k B)} \int_0^B x dF(a^k x) \right] \times \prod_{i=0}^{k-1} \bar{F}(a^i B) F(a^k B) + (N+1) B \prod_{i=0}^{N} \bar{F}(a^i B); \tag{19}$$

2. $$E[S(B,N)] = \sum_{k=1}^{N} \left[F(a^k B) \prod_{i=0}^{k-1} \bar{F}(a^i B) \sum_{i=1}^{k} \frac{\mu}{b^{i-1}} \right] + \sum_{i=1}^{N} \frac{\mu}{b^{i-1}} \prod_{k=0}^{N} \bar{F}(a^k B); \tag{20}$$

3. $$E[\Phi(B,N)] = -c_w \sum_{k=1}^{N} \left[kB + \frac{1}{F(a^k B)} \int_0^B x dF(a^k x) \right] \times (-c_w) \left[\prod_{i=0}^{k-1} \bar{F}(a^i B) F(a^k B) + \int_0^B x dF(x) \right] + F(b) E\eta + \sum_{k=1}^{N} \left[c_r \sum_{i=1}^{k} \frac{\mu}{b^{i-1}} + E\eta \right] \times \prod_{i=0}^{k-1} \bar{F}(a^i B) F(a^k B) - c_w (N+1) B \times \prod_{i=0}^{N} \bar{F}(a^i B) + c_r \sum_{i=1}^{N} \frac{\mu}{b^{i-1}} \prod_{i=0}^{N} \bar{F}(a^i B) + c. \tag{21}$$

For more details on the derivation of (19), (20) and (21) the reader is referred to [9]. Now, using these results it can be shown that the average cost rate of the system $C(B,N)$ is given by:

$$C(B,N) = \frac{\text{E[costs in renewal cycle]}}{\text{E[length of a renewal cycle]}} = \frac{E[\Phi(B,N)}{E[T(B,N) + S(B,N)} = \frac{-c_w \phi_1 + c_r \phi_2 + \phi_3 E\eta + c}{\phi_1 + \phi_2}, \tag{22}$$

where

- $$\phi_1 = \int_0^B x dF(x) + \sum_{k=1}^{N}\left[kB + \frac{1}{F(a^k B)}\int_0^B x dF(a^k x)\right] \times \prod_{i=0}^{k-1}\bar{F}(a^i B)F(a^k B) + (N+1)B\prod_{i=0}^{N}\bar{F}(a^i B);$$

- $$\phi_2 = \sum_{k=1}^{N}\left[F(a^k B)\prod_{i=0}^{k-1}\bar{F}(a^i B)\sum_{i=1}^{k}\frac{\mu}{b^{i-1}}\right] + \sum_{i=1}^{N}\frac{\mu}{b^{i-1}}\prod_{k=0}^{N}\bar{F}(a^k B);$$

- $$\phi_3 = F(B) + \sum_{k=1}^{N}\left[F(a^k B)\prod_{i=0}^{k-1}\bar{F}(a^i B)\right].$$

Hence, the next step is to identify the optimal replacement policy (B^*, N^*) that minimises the average cost rate of the system $C(B, N)$ given in (22).

5.3 Example

The following example is taken from [9]. Let us assume that the distribution of the nth operating time X_n is Weibull with parameters β and α, i.e.

$$F_n(t) = 1 - e^{-\left(\frac{a^{n-1}t}{\beta}\right)^{\alpha}}, \text{for} \quad t > 0.$$

Then, the average cost rate of the system $C(B, N)$ simplifies to

$$C(B, N) = \frac{c_r \mu l_1 + l_2 E\eta - c_w l_3 + c}{l_1 + l_3},$$

where

- $$l_1 = \sum_{k=1}^{N}\sum_{i=1}^{k}\frac{1}{b^{i-1}}e^{-\sum_{i=0}^{k-1}\left(\frac{a^i B}{\beta}\right)^{\alpha}} - \sum_{k=1}^{N-1}\sum_{i=1}^{k}\frac{1}{b^{i-1}}e^{-\sum_{i=0}^{k}\left(\frac{a^i B}{\beta}\right)^{\alpha}};$$

- $$l_2 = 1 + \sum_{k=2}^{N}e^{-\sum_{i=0}^{k-1}\left(\frac{a^i B}{\beta}\right)^{\alpha}} - \sum_{k=1}^{N}e^{-\sum_{i=0}^{k}\left(\frac{a^i B}{\beta}\right)^{\alpha}};$$

- $$l_3 = \int_0^B e^{-(\frac{x}{\beta})^{\alpha}}dx + \sum_{k=1}^{N}e^{-\sum_{i=0}^{k-1}\left(\frac{a^i B}{\beta}\right)^{\alpha}}\int_0^B e^{-(\frac{a^k x}{\beta})^{\alpha}}dx.$$

Assigning specific values to the parameters, such as $a = 1.05$, $b = 0.95$, $\mu = 8$, $E\eta = 1500$, $c_r = 20$, $c_w = 50$, $c = 2000$, $\beta = 1000$ and $\alpha = 2$, and after using a numerical procedure, it is shown that the optimal strategy is $(B^*, N^*) = (380, 10)$. In other words, for a system with characteristics, as given in the example, the optimal fixed period of time between consecutive PR is $B^* = 380$ and if the system is free of failure until the 11th PR, then at the time scheduled for this PR, the system has to be replaced. This maintenance strategy assures the minimum average cost rate of the system of $C(B^*, N^*) = -47.5977$. For more comments and details on the example, please see [9].

6 Maintenance: Markovian Model for Non-zero Preventive Repair Times

In this section, we summarise the Markovian approach proposed in Fang and Liu [4] to model for non-zero preventive repair times. The main objective of this study is to design a maintenance policy (B, N), so that the steady-state profit rate of the system is maximised, with B being the interval of preventive maintenance (repairs) and N being the number of failure-free preventive repairs to system replacement. The parameters of this strategy have the same meaning as in [9]. Also, the settings considered here are close to the settings in Sect. 5, but have some specifics and we discuss these below.

6.1 The Model

The model is built-up upon the following assumptions:

- At the beginning, a new system with preventive repairs (PR) is installed. At failure the system is repaired and the repair is imperfect with non-zero repair time.
- The times between two consecutive system failures are called cycles.
- The system failure in cycle N is catastrophic and the system is replaced by a new, identical system. The replacement requires negligible time.
- The PR will be adopted as soon as the operating time of the system reaches level B, and within a cycle the PR is perfect and the PR times are i.i.d. Moreover, the imperfect failure repair affects the first lifetime of the follow-up cycle and the lifetimes within a cycle are i.i.d. Henceforth, the following notations will be used:
  - $X_i^{(n)}$ - the operating time of the system after the nth PR within the ith cycle with cdf $H_i(x)$, pdf $h_i(x)$, failure rate function $a_i(x)$, and $E[X_i^{(n)}] = \lambda_i, \quad i = 1, 2, \ldots; \quad n = 0, 1, 2, \ldots$. Moreover, $\{X_i^{(0)}\}$ form a decreasing stochastic process.

- $Z_i^{(n)}$ - the preventive repair time of the system in cycle i after n PR with cdf $F_i(z)$, pdf $f_i(z)$, hazard function $b_i(z)$, and $E[Z_i^{(n)}] = b_i, \quad i = 1, 2, \ldots; \quad n = 0, 1, 2, \ldots$. Moreover, $\{Z_i^{(0)}\}$ form an increasing stochastic process.
- Y_i - the failure repair time of the system in cycle i cdf $G_i(y)$, pdf $g_i(y)$, hazard function $\mu_i(y)$, and $E[Y_i] = \mu_i, \quad i = 1, 2, \ldots; \quad n = 0, 1, 2, \ldots$. Moreover, $\{Y_i\}$ form a monotonically increasing stochastic process.
- $\{X_i^{(n)}\}$, $\{Z_i^{(n)}\}$ and $\{Y_i\}$, $\quad i = 1, 2, \ldots; \quad n = 0, 1, 2, \ldots$ are independent.

- The working reward per unit time is C_1, failure repair cost per unit time is C_2, preventive repair cost per unit time is C_3, and the system replacement cost is C.

The state of the system is modelled as follows:

- $(i, 0, n)$ - the system is working after the nth PR in cycle i, $n = 0, 1, 2, \ldots$; $i = 1, 2, \ldots, N$.
- $(i, 1, n)$ - the system is under PR after the nth PR in cycle i, $n = 0, 1, 2, \ldots$; $i = 1, 2, \ldots, N$.
- $(i, 2)$ - the system is under failure repair in cycle i, $i = 1, 2, \ldots, (N-1)$.

The modelling is reduced to a vector Markov process, so that it allows for the derivation of the state probability density equations. For more details on the modelling and results see [4].

6.2 Steady-State PR-Replacement Policy

Next we summarise the results regarding the steady-state performance measures of the system and use them to identify the optimal steady-state maintenance strategy as described in Sect. 6.1.

The authors show that:

- the steady-state replacement frequency M_r is given by

$$M_r = \frac{1}{\sum_{i=1}^{N} \frac{\int_0^B \bar{H}(x)dx}{H_i(B)} + \sum_{i=1}^{N} \frac{b_i \bar{H}(B)}{H_i(B)} + \sum_{i=1}^{N-1} \mu_i}; \tag{23}$$

- the steady-state availability A is equal to

$$A = M_r \sum_{i=1}^{N} \frac{\int_0^B \bar{H}(x)dx}{H_i(B)}; \tag{24}$$

- the steady-state PR frequency M_1 is given by

$$M_1 = M_r \sum_{i=1}^{N} \frac{b_i \bar{H}(B)}{H_i(B)}; \tag{25}$$

- the steady-state failure repair probability P is

$$P = M_r \sum_{i=1}^{N-1} \mu_i. \tag{26}$$

Therefore, using (23), (24), (25) and (26) the steady-state average profit rate $C(B, N)$ of the system is obtained to be equal to:

$$C(B, N) = \frac{C_1 \sum_{i=1}^{N} \frac{\int_0^B \bar{H}(x)dx}{H_i(B)} - C_2 \sum_{i=1}^{N-1} \mu_i - C_3 \sum_{i=1}^{N} \frac{\bar{H}(B)}{H_i(B)} - C}{\sum_{i=1}^{N} \frac{\int_0^B \bar{H}(x)dx}{H_i(B)} + \sum_{i=1}^{N} \frac{b_i \bar{H}(B)}{H_i(B)} + \sum_{i=1}^{N-1} \mu_i}. \tag{27}$$

It is easy to see that the steady-state average profit rate $C(\infty, N)$ of the system without PR is given by

$$C(B, N) = \frac{C_1 \sum_{i=1}^{N} \lambda_i - C_2 \sum_{i=1}^{N-1} \mu_i - C}{\sum_{i=1}^{N} \lambda_i + \sum_{i=1}^{N-1} \mu_i}. \tag{28}$$

Therefore, it is worth to perform PR only if $C(\infty, N) < C(B^*, N^*)$, where (B^*, N^*) are the parameters of the optimal maintenance strategy. A possible approach in finding the parameters of the optimal strategy is first, to find B_N^* for every N, so that $C(B_N^*, N)$ reaches maximum for $N = 1, 2, 3, \ldots$ and second, find the maximum among these values to determine $C(B_N^*, N^*)$, so that (B_N^*, N^*) are the parameters of the optimal maintenance policy. For more on this approach, see [11].

6.3 Example

As in [4], assume that $C_1 = 4,900$, $C_2 = 2,100$, $C_3 = 20,000$ and $C = 2,200,000$. Also let $H_i(x) = 1 - e^{(0.0001 \times 1.04^{i-1} x)^2}$, for $x \geq 0$, $i = 1, 2, \ldots, N$. Moreover, $b_i = 5 \times 1.05^{i-1}$, $i = 1, 2, \ldots, N$ and $\{Y_i, i = 1, 2, \ldots, N-1\}$ is a geometric process with $\mu_i = 150 \times 1.1^{i-1}$, $i = 1, 2, \ldots, N-1$. For these parameter values, it is shown that

$$C(B, N) = 4900 - \frac{A}{B}, \tag{29}$$

where

-

$$A = \sum_{i=1}^{N} \frac{(24500 \times 1.05^{i-1} + 20000)e^{-(0.0001 \times 1.04^{i-1} B)^2}}{1 - e^{-(0.0001 \times 1.04^{i-1} B)^2}} + 10500000(1.1^{N-1} - 1) + 2200000$$

- $$B = \sum_{i=1}^{N} \frac{\int_0^B e^{-(0.0001 \times 1.04^{i-1} x)^2} dx}{1 - e^{-(0.0001 \times 1.04^{i-1} B)^2}} + \sum_{i=1}^{N} \frac{5 \times 1.05^{i-1} e^{-(0.0001 \times 1.04^{i-1} B)^2}}{1 - e^{-(0.0001 \times 1.04^{i-1} B)^2}} + 1500(1.1^{N-1} - 1).$$

By using numerical computations, the parameters of the optimal maintenance policy are found to be equal to: $B^* = 1,727.343$ and $N^* = 3$ with maximum steady-state average profit reaching $C(B^*, N^*) = 4,847.148$ per unit time. For more details and comments, see [4].

7 A Case Study: Maintenance Optimisation for Age-Based Replacement Policy

In this section, a case study of maintenance optimisation introduced by Pintelon, van Puyvelde, and Gelders [6] is summarised. An age-based replacement model, which allows for non-zero (preventive and corrective) repair times is used to determine an optimal replacement policy. This study sheds some light on problems that need to be dealt with when mathematical models are applied to solve practical problems.

7.1 Description of the Case Study

In the case study of Pintelon et.al. [6], an optimal age-based maintenance policy is sought for the bottleneck machine of a manufacturing plant of beverage cans. Cans are produced through several phases of the production lines. The bottleneck phase of the production lines is associated with the cupper by which each cup is formed with sheets of metal. The cupper capacity influences the output level of the production heavily.

The company was using a classical block replacement policy under which preventive maintenance was conducted every ten days in addition to corrective maintenance at failure. A new replacement policy is desired to efficiently maintain the equipment by incorporating the data collected by its maintenance information system and making use of mathematical models.

7.2 The Model

Here, an age-based model with a modification of non-negligible maintenance times is applied. It is referred to as an extended age-based model. The following list summarises notations and assumptions with some justification.

- At the beginning, the cupper is new. When the operation time of the cupper reaches T_a (in days), a preventive maintenance (PM) with cost p [in Belgian Franc (BF)] is carried out. In addition, at each failure (before the operation time reaches T_a) a corrective maintenance (CM) with cost c (in BF) is executed.
- The duration of CM is fixed at t_r (in days). Likewise, the duration of PM is prespecified at t_m (in days). The classical age-based model assumes negligible maintenance time, but in the settings of this case study non-zero maintenance times are appropriate. Moreover, the property of production process justifies deterministic durations of CM and PM times.
- The times between two consecutive maintenance completion times, either corrective or preventive, is said to be a cycle.
- Let T denote the time to failure of the cupper in each cycle with the cumulative distribution function $F(t)$, density function $f(t)$ and failure rate function $z(t)$.
- Single component machine: Since cupper failures are mostly caused by one component, this assumption is appropriate.
- The system has two states ("on" or "off"): The production process is required to be with high speed and high accuracy and allows for no deterioration. Hence, it is either working denoted by "on" or not working denoted by "off".
- Failure-based versus use-based maintenance: An optimal balance between the frequencies of corrective and preventive maintenance is sought.
- As-good-as-new repairs: In each maintenance action, the cupper is repaired to be as-good-as-new.
- Model approach: (1) continuous time, (2) infinite horizon, (3) stochastic model
- Continuous production process: The cupper is working continuously.
- Failure distribution is not known: The classical age-based model assumes a known failure distribution, but in this case study, it was not the case. A few appropriate failure distributions are applied for a sensitivity analysis.
- Maintenance times: No consensus was formed in regard to independence of maintenance times. Two scenarios are considered. (1) optimistic (maintenance times are independent) (2) pessimistic (maintenance times are dependent).
- Optimisation of objective functions: (1) minimisation of the long-run maintenance cost per unit time; (2) maximisation of the average availability of the cupper (this would not be an objective function for the model with negligible maintenance time).

7.3 The Extended Age-Based Policy-Objective Functions and Properties

Using the renewal reward arguments (Barlow and Hunter [1], Tijms [8]), the following objective functions are obtained in [6].

- the average availability of the cupper is equal to

$$E(\text{availability}) = \frac{E(\text{on time in a cycle})}{E(\text{cycle length})} = \frac{\int_0^{T_a} t\,dF(t) + T_a(1 - F(T_a))}{\int_0^{T_a} t\,dF(t) + t_r F(T_a) + (T_a + t_m)(1 - F(T_a))} \tag{30}$$

- the long-run maintenance cost per unit time is given by

$$E(\text{cost}) = \frac{E(\text{cost per cycle})}{E(\text{cycle length})} = \frac{cF(T_a) + p(1 - F(T_a))}{\int_0^{T_a} t\,dF(t) + t_r F(T_a) + (T_a + t_m)(1 - F(T_a))} \tag{31}$$

The former is to be maximised, while the latter is to be minimised.

Some properties of the extended age-based model given below are discussed in [6].

- If the maintenance times are negligible, then the model concerned reduces to a classical age-based model for minimising the long-run maintenance cost per unit time.
- Under the assumption that the failure time T is exponentially distributed, the optimal T_a for both objective functions become infinite, i.e. preventive maintenance is unnecessary.

7.4 Example

Numerical methods and results presented in [6] are summarised in this subsection. The preventive maintenance time is set at $t_m = 3$ h. As for the corrective maintenance time, two values are used: (1) $t_r = 10$ h (pessimistic scenario) and (2) $t_r = 5$ h (optimistic scenario). A global cost of 125,000 BF/h (approximately 4,000 dollars/h) including wages and materials is assumed. The case makes both maintenance times and global costs predictable, so using deterministic values for them is justified.

Based on the data collected by the company, the mean time between failures (MTBF) for the cupper is 12 days. For the sake of sensitivity analysis, a few different failure distributions are used with the MTBF of 12 days. The two-parameter Weibull distribution is selected as the failure distribution with $f(t) = \alpha\tau(\tau t)^{\alpha-1}e^{-(\tau t)^{\alpha}}$,

$F(t) = 1 - e^{-(\tau t)^\alpha}$ and $z(t) = \alpha\tau(\tau t)^{\alpha-1}$, where α and τ are shape and scale parameters, respectively. The managers' knowledge suggests that $\alpha = 4.0$. Using the expression for the first moment μ of T, given by

$$\mu = \frac{1}{\tau}\Gamma\left(1 + \frac{1}{\alpha}\right)$$

together with MTBF of 12 days (i.e., 1.714286 wk) and $\alpha = 4.0$, the corresponding τ can be obtained.

With the Weibull failure distribution the objective functions (30) and (31) contain integrals which cannot be evaluated analytically. If $\tau T_a < 1$, then (30) and (31) can be computed via an appropriate numerical method. Otherwise, a simulation method can be used to evaluate them. For details, see [6]

Under the pessimistic scenario $t_r = 10$ h, both objective functions (30) and (31) are optimised at $T_a = 1.190$ wk [corresponding to 8 days and a shift (8h)] with values 0.979713 and 426020 BF/wk, respectively. It is observed that as T_a increases the CM cost increases, whereas the PM cost decreases.

A comparison is made between the current model, i.e. the block-based model with preventive maintenance conducted every $T_b = 10$ days (1.428571 wk), and the extended age-based model. At optimality the former has average availability of 0.976234, while the latter has 0.979713. Contrary to the intuition, this difference may lead to a significant increase in income due to the fact that the cupper is a bottleneck machine. For details, the reader is referred to [6].

Under the optimistic scenario $t_r = 5$ h, the optimal preventive maintenance interval T_a is 1.571 wk with average availability of 0.984698 and the long-run cost of per unit time 321237 BF/wk.

A sensitivity analysis is carried out for the pessimistic scenario. In addition to the case $\alpha = 4.0$, α is set at 1, 1.5, 2.0, and 2.5 and optimal values of T_a, average availability and long-run maintenance cost per unit time are compared. It is observed that (1) the higher the shape parameter α the shorter the T_a , but the higher the optimal average availability; (2) the higher the α , the clearer the optimum; (3) it is confirmed that when $\alpha = 1$ (the exponential failure distribution), the average availability increases without bound as $T_a \to \infty$.

8 A Simulation Model

In what follows we propose a simple simulation warranty model. We extend our study [2] by assuming imperfect warranty repairs. We model the "on" times of the system using a decreasing geometric process with parameters (X_1, a) and the "off" times (the warranty repair times) by an increasing geometric process with parameters (Y_1, b). Our goal is not only to estimate the expected warranty cost over a prespecified warranty period, but also to formulate and solve an optimisation problem regarding the length of the warranty period and provide some sensitivity analysis on the results.

- **The expected warranty cost**
The evaluation of the expected warranty cost is straightforward and follows the standard approach. We assign a cost $C_i = A + cY_i$ to the "off" times, as in sect. 3.3. First, we generate the two geometric processes, the "ON" process and the "OFF" process, each with prespecified parameters. Based on the "OFF" process and the parameter values of the cost function, for a fixed value of the warranty period T, taking into account whether the warranty ends in an "on" or "off" period, the warranty cost is computed. For a fixed value of T at least 100 realisations of the "ON" and "OFF" processes are considered and the warranty cost for these realisations are averaged to obtain the expected warranty cost for the chosen value of T.
- **Optimisation problem on** T
Next, we aim to formulate and justify an optimisation problem for determining the optimal warranty period for our model. Our objective function is the probability of product's sale $P(T)$ and we aim to maximise it. We assume that $P(T)$ is an increasing function of the difference $D(T)$, which has the following representation:

$$D(T) = v \,\{\text{total "on" time in } T\} - c \,\{\text{total "off" time in } T\},$$

where $c \geq 0$ and $v \geq 0$. Of course, the probability $P(T)$ might depend on other factors, but in this study we focus only on the above difference. What could be the interpretation of the parameters v and c ? One possible interpretation is as follows: the parameter v could be thought of as the rate of customer satisfaction due to the proper product functioning, and c as the rate of customer dissatisfaction due to the product failure. Next, let

$$r = \frac{v}{c}$$

be the ratio of the two rates. Now, if the warranty expires in an "off" period, i.e. the last "off" period is included in the warranty period, and the warranty coverage consists of total of d complete cycles, our optimisation criterion becomes

$$\begin{aligned} \max \quad D(T) &= v \sum_{i=0}^{d} X_i - c \sum_{i=0}^{d} Y_i = v\,(T - \sum_{i=0}^{d} Y_i) - c \sum_{i=0}^{d} Y_i \\ &= v\,T - (v + c) \sum_{i=0}^{d} Y_i = c\,(r\,T - (1 + r) \sum_{i=0}^{d} Y_i). \end{aligned} \tag{32}$$

If the warranty expires in an "on" period, i.e. there is an incomplete cycle at the end of the warranty with d complete cycles before it, our optimisation criterion becomes:

$$\max \quad D(T) = v\,(T - \sum_{i=0}^{d} Y_i) - c\,\sum_{i=0}^{d} Y_i = v\,T - (v+c)\,\sum_{i=0}^{d} Y_i$$
$$= c\,(r\,T - (1+r)\,\sum_{i=0}^{d} Y_i). \tag{33}$$

Therefore, according to (32) and (33), the difference $D(T)$ is expressed equivalently in both cases. Of course, in the simulation we need to keep track whether the warranty expires during "on" or "off" time.
Next we present several illustrations of the model. In these illustrations the "on" times follow a geometric process with $F_1(x) = 1 - e^{-\left(\frac{t}{\beta}\right)^{\alpha}}$, for $t > 0$, i.e. the underlying distribution is Weibull with parameters $(\alpha_{on}, \beta_{on})$ and $a > 1$, and the "off" times follow a geometric process with $G_1(x)$, which is also Weibull with parameters $(\alpha_{off}, \beta_{off})$ and $0 < b < 1$. In Fig. 13, the remaining model parameters have the following values: $A = 0$, $r = 0.01$, $(\alpha_{on}, \beta_{on}) = (2, 1500)$ and $a = 1.05$, $(\alpha_{off}, \beta_{off}) = (2, 10)$ and $b = 0.95$, and the optimal value of the warranty period is $T^* = 5900$. In Fig. 14, the dependence of $D(T)$ on r is depicted for $r = 0.01$; 0.0075; 0.005, with corresponding optimal values $T^* = 5900$, 2000, 650. As expected, T^* also decreases as r decreases.
In Figs. 15 and 16, we vary the ratio r and obtain the two limiting cases $T^* = 0$ and $T^* = \infty$. As expected, when r is very small, i.e. the dissatisfaction rate is much higher that the satisfaction rate, the warranty period is zero, which will lead to $P(T) = 0$. Hence, the product has to be significantly improved before being introduced into the market. On the other hand, if r is relatively high, so that the two rates are comparable, the warranty period could be large and the probability for product sale will tend to one.

- **sensitivity analysis**
  Figures 13–16 provide an insight that the optimal value of T, if it exists, depends on the ratio $r = \frac{v}{c}$. Figure 17 depicts $D(T)$'s (and the optimal value of T) dependence on the parameter β_{off} of G_1 with the values of all of the remaining parameters as in Fig. 13. The upper curve shows $D(T)$ from Fig. 13 and the lower curve is $D(T)$ for $\beta_{off} = 15$, which leads, as expected, to a lower optimal value of $T^* = 1050$.

Fig. 13 $r = 0.01; T^* = 5900$

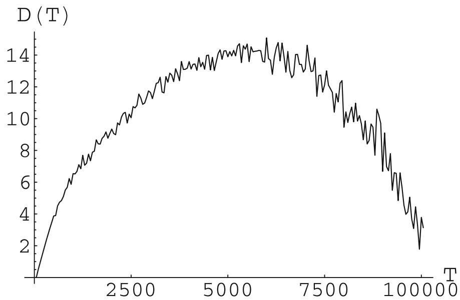


Fig. 14 $r = 0.01$; 0.0075; 0.005

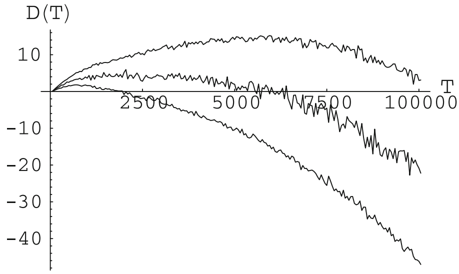


Fig. 15 $r = 0.001$; $T^{*} = 0$

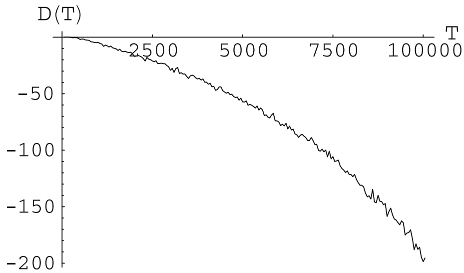


Fig. 16 $r = 0.1$; $T^{*} = \infty$

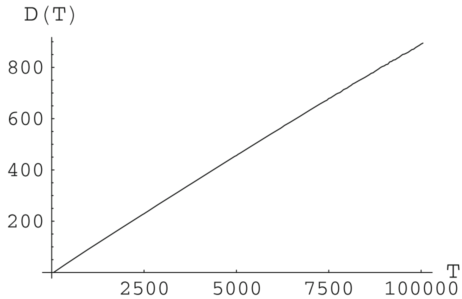
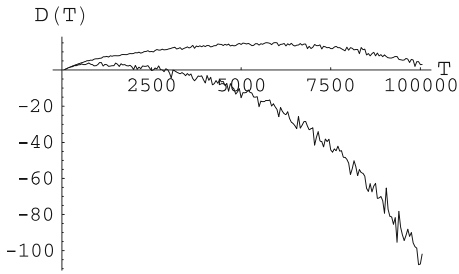


Fig. 17 $(\beta_{1;off}, \beta_{2;off}) = (10, 15)$



Lastly, Fig. 18 depicts $D(T)$'s (and the optimal value of T) dependence on the parameter b of the "off" times geometric process, keeping all remaining parameters as in Fig. 13. The upper curve is appropriately scaled curve from Fig. 13 and the

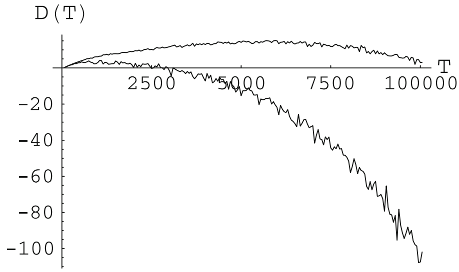


Fig. 18 $(b_1, b_2) = (1.05, 1.25)$

lower curve is $D(T)$ for $b = 0.80$, which leads, again as expected, to a lower optimal value of $T^* = 2600$.

Currently, we are working on the extension of the periodic preventive repair-replacement model presented in [9] (see Sect. 5). In this new simulation model, we introduce product warranty, and aim to solve an optimisation problem that will result in an optimal maintenance-warranty strategy with parameters (B^*, N^*, T^*). The detailed description and illustration of this model will be presented elsewhere.

9 Conclusions

In this chapter, we have reviewed several published studies with a common theme to emphasise the importance of taking into account the non-zero length of rectification actions. Our goal was to show that while modelling the product performance and related cost analysis, it is important to include in the model the non-zero times of warranty repairs, as well as the preventive/corrective maintenance repairs and the "cost" associated with them. In most situations it is acceptable to consider the repairs to be instantaneous, especially if they are not associated with high penalties, losses, or dissatisfaction. At the same time, it is well known that the harm to the producer/manufacturer's reputation due to one dissatisfied customer is much higher than the positive impact of this reputation due to a group of satisfied customers. A faulty product could lead to a high customer dissatisfaction and could have a significant negative impact on the producer's market standing. Hence, if the rate of this dissatisfaction, i.e. the "cost" of the "off" times, is taken into account, then better maintenance/warranty strategies from manufacturers' as well as customers' point of view could be designed.

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References

1. Barlow RE, Hunter LC (1960) Optimal preventive maintenance policies. Operations Research 8:90–100
2. Chukova S, Hayakawa Y (2004) Warranty cost analysis: non-zero repair time. Applied Stochastic Models in Business and Industry 20:59–71
3. Chukova S, Hayakawa Y (2004) Warranty cost analysis: renewing warranty with non-zero repair time. Int J Reliab Qual Saf Eng 1:1–20
4. Fang YT, Liu BY (2006) Preventive repair policy and replacement policy of repairable system taking non-zero preventive repair time. J Zhejiang Univ SCI A 7:207–212
5. Pham H, Wang H (1996) Imperfect maintenance. Eur J Oper Res 94:425–438
6. Pintelon LMA, Van Puyvelde FLB, Gelders LF (1995) An age-based replacement policy with non-zero repair times for a continuous production process. Int J Prod Res 3:2111–2123
7. Ross S (1996) Stochastic Processes. John Wiley & Sons
8. Tijms H (1986) Stochastic Modeling and Analysis. Wiley, New York
9. Wang GJ, Zhang YL (2006) Optimal periodic preventive repair and replacement policy assuming geometric process repair. IEEE Transactions on Reliability 55:118–122
10. Xie M (1989) On the solution of renewal-type integral equations. Communications in Statistics - Simulation 18:281–293
11. Zhang YL (1994) A bivariate optimal replacent policy for repairable systems. J Appl Probab 31(4):1123–1127