Dynamical Systems with Semi-Markovian Perturbations and Their Use in Structural Reliability

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Abstract The aim of this chapter is to present dynamical systems evolving in continuous-time and perturbed by semi-Markov processes (SMP). We investigate both probabilistic modeling and statistical estimation of such models. This work was initially developed in order to study cracking problems for the confinement device in nuclear power plants, where a jump Markov process was used as the perturbing process. The new key element here is the use of SMPs instead of Markov ones for the randomization of the system. Several numerical illustrations in reliability are investigated, accompanied with guidelines for a practical numerical implementation.

1 Introduction

In many industrial applications, structures may suffer degradations induced by the corresponding operating conditions. Degradations may be induced by thermal cyclic loadings, mechanical loadings, seismic activity, neutron irradiation, thermal sever transients, etc., which may lead to the failure of the structure.

Mechanisms that cause failures are complex due to their interdependencies and their different physical time-scales. Besides, these degradation mechanisms cannot always be described through deterministic models. Thus, a stochastic approach is often required. As a motivating example, we rely on the widely studied engineering issue referred to as the "crack-growth" problem [22, 27]: in structural mechanics, the main degradation process that leads to fatigue aging is due to the propagation of small defects into cracks in structures subject to small yet cyclic loadings. Many

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industrial fields are concerned, such as aeronautics, nuclear plants, automobile or bridge building among others. Qualitatively, the process remains the same whatever the material considered (e.g., aluminum in aeronautics, steel for confinement devices or pressure vessels, and concrete for bridges). The modeling methodology could be tackled with the mathematical tools provided in this work.

Even in well-controlled lab experiments supervised with cutting edge technology [20], crack-growth remains a very unstable phenomenon: deterministic models have been provided from structural mechanics, e.g., through computationally intensive finite-elements analysis. Yet, it is now acknowledged that probabilistic modeling are required to handle such degradation processes. Beyond the uncertainty propagation approaches offered by the probabilistic mechanics point of view, many authors rather suggested to completely randomize the modeling through a description relying on stochastic processes and dynamical systems (the pioneers in that domain being, to our knowledge, [19, 26]). The present chapter clearly enters this framework, drawing inspiration from [1, 9, 13, 15, 21] among many others.

As such, the stochastic models developed here do not necessarily aim to provide an exact physical representation of the phenomenon. We rather suggest to describe the evolution of an observable variable that characterizes the degradation process well. Hence, a structure is said to "fail" when its level of degradation exceeds a given threshold. The time evolution of the observable degradation process is described by a positive-valued stochastic process $Z = (Z_t, t \geq 0)$ governed by a first order stochastic differential system:

$$\dot{Z}_t = C(Z_t, X_t), \quad Z_0 = z, \tag{1}$$

where $\dot{Z}_t \doteq \mathrm{d}Z_t/\mathrm{d}t$ stands for the first order derivative of Z_t, C is a positive function, and $z > 0$ is the starting point of Z. The process $X = (X_t, t \geq 0)$ is a pure jump process with a countable state space. This model reflects the following physical point of view: the level of degradation Z increases on continuous sample paths; yet, its evolution shifts at discrete instants of time due to random shocks with random intensities induced by the operating conditions. These changes are modeled by the jump process X.

In the case where X is a jump Markov process, the coupled process $(Z, X) = (Z_t, X_t, t \geq 0)$ with state space $\mathbb{R}_+ \times E$ owns a well-characterized infinitesimal generator. Such a modeling belongs to the wider family of stochastic processes referred to as *Piecewise Deterministic Markov Processes*. These hybrid processes are an alternative to diffusion processes [7, 8, 12]. They virtually give a representation of many stochastic process being the mixture of deterministic motions and random jumps. A schematic view of three sample paths of the system defined in (1) are given in Fig. 1, when (Z, X) is observed from the starting point $t_0 = 0$ up to the random time τ when Z reaches an absorbing point Δ.

The purpose of this work is to model the perturbing process X by a semi-Markov process (SMP) and to derive the basic analysis for the associated dynamical system. We insist on the opportunity of considering X to be a SMP rather than a Markov

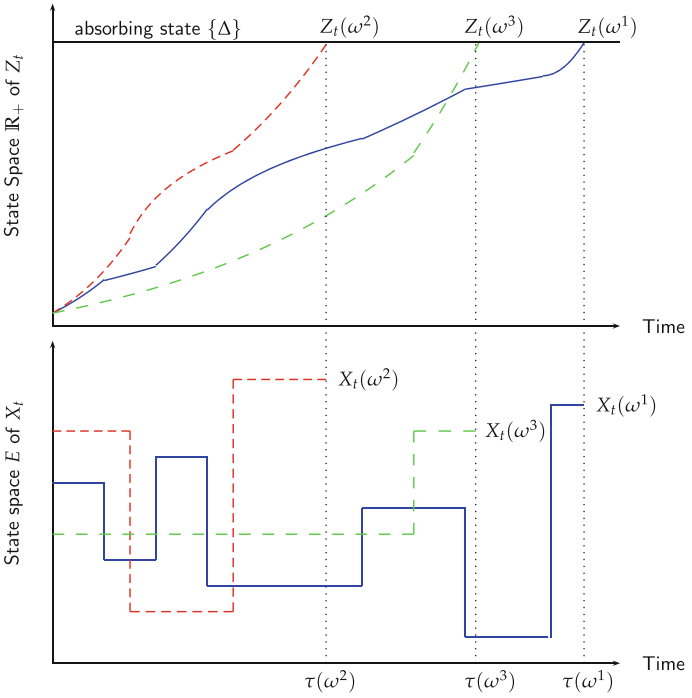

Fig. 1 Modeling degradation paths

process: the more flexible is the randomizing process X, the broader is the model and the wider is the range of its application.

The main contributions described in this chapter are of two kinds:

(1) First, we investigate the probabilistic characterization of the dynamical system (1) when X is semi-Markovian through Markov renewal theory, which allows to calculate the reliability function understood in the following sense: if a threshold Δ define an absorbing state of the system or, equivalently, a failure boundary for the degradation process, the failure time τ is

$$\tau = \inf\{t \geq 0 : Z_t \geq \Delta\},$$

and the associated reliability function turns to

$$R(t) = \mathbb{P}(Z_t < \Delta).$$

Interpreting (Z, X) as an extended SMP, we build a solvable Markov Renewal Equation (MRE) for the associated transition function, then deriving a closed-form. Still, this Markov renewal formulation required numerical resolution:

we propose a detailed guidelines to compute the reliability and give numerical example. This issue is addressed in Sect. 3.

(2) Second, we study the statistical inference of the system, that is, the estimation of the deterministic parameters of function C as well as the estimation of the SMP X. The degradation process being the only process whose paths can be collected during laboratory measurements, we only dispose of some sample paths of Z, observed before the system fails, and defined on the random time interval $[0, \tau]$. From these paths, we develop (1) a method to estimate the parameters of the function C, through an asymptotic analysis of the system (1) followed by a classical regression analysis; (2) a method to estimate the paths of X (as well as its state space E), since samples of X are not directly observed; (3) the construction of the likelihood function associated with the semi-Markov kernel of X and an approached maximum likelihood estimator for the kernel. This is developed in Sect. 4.

Meanwhile, let us start by an introductory section devoted to Markov renewal processes (MRP) theory.

2 Semi-Markov Processes: Background

This section recalls a few basics on SMPs. A larger view can be found for instance in [6, 11, 12, 17, 21, 24], yet the material provided here should hopefully be sufficient for the understanding of the main results developed throughout this chapter.

2.1 Notations and Settings

Consider an infinite countable set, say E, and an E-valued pure jump stochastic process $X = (X_t)_{t\in\mathbb{R}_+}$. Let $0 = S_0 \leq S_1 \leq ... \leq S_n \leq S_{n+1} \leq ...$ be the jump times of X, and $J_0, J_1, J_2, \ldots$ the successively visited states of X. Note that S_0 may also take positive values. Let $\mathbb{N}$ be the set of non-negative integers. Then, X is connected to (J_n, S_n) through

$$X_t = J_n, \quad if \quad S_n \leq t < S_{n+1}, \quad t \geq 0 \qquad \text{and} \quad J_n = X_{S_n}, \quad n \geq 0.$$

Definition 2.1. The stochastic process $(J_n, S_n)_{n\in\mathbb{N}}$ is said to be a Markov renewal process (MRP), with state space E, if it satisfies, a.s., the following equality

$$\begin{aligned}\mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \leq t \mid J_0, \ldots, J_n; S_1, \ldots, S_n)\\ = \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \leq t \mid J_n)\end{aligned}$$

for all $j \in E$, all $t \geq 0$, and all $n \in \mathbb{N}$. In this case, X is called a SMP.

Remark 2.1. We assume that the above probability is independent of n and S_n, and in this case the MRP is called *time homogeneous*. Only time-homogeneous MRP are considered in the sequel.

The MRP $(J_n, S_n)_{n\in\mathbb{N}}$ is determined by the *initial distribution* α, with $\alpha(i) = \mathbb{P}(J_0 = i)$, $i \in E$ and by the transition kernel

$$Q_{ij}(t) := \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \le t \mid J_n = i),$$

called the *semi-Markov kernel* of X. The process (J_n) is a Markov chain with state space E and transition probabilities $p_{ij} := Q_{ij}(\infty) := \lim_{t\to\infty} Q_{ij}(t)$, called the embedded Markov chain (EMC) of X. It is worth noticing that here $Q_{ii}(t) \equiv 0$, for all $i \in E$, but in general we can consider semi-Markov kernels by dropping this hypothesis.

An important point is the following decomposition of the semi-Markov kernel

$$Q_{ij}(t) := \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \le t \mid J_n = i) = p_{ij}F_{ij}(t), \quad t \ge 0, \quad i, j \in E,$$

where p_{ij} is the transition kernel of the EMC (J_n), and $F_{ij}(t) := \mathbb{P}(S_{n+1} - S_n \le t \mid J_n = i, J_{n+1} = j)$ is the conditional distribution function of the sojourn time in the state i given that the next visited state is j, (with $j \ne i$). Let us also, define the distribution function $H_i(t) := \sum_{j\in E} Q_{ij}(t)$ and its mean value m_i, which is the mean sojourn time of X in state i. In general, Q_{ij} is a subdistribution, i.e., $Q_{ij}(\infty) \le 1$, hence H_i is a distribution function, $H_i(\infty) = 1$, and $Q_{ij}(0-) = H_i(0-) = 0$.

Remark 2.2. A special case of semi-Markov processes is the one where $F_{ij}(\cdot)$ does not depend on j, i.e., $F_{ij}(t) \equiv F_i(t) \equiv H_i(t)$, and

$$Q_{ij}(t) = p_{ij}F_i(t).$$

Any general semi-Markov process can be transformed into one of this kind (see, e.g., [17]).

Example 2.1. A Markov process with state space $E = \mathbb{N}$ and generating matrix $\mathbf{A} = (a_{ij})_{i,j\in E}$ is a special semi-Markov process with semi-Markov kernel

$$Q_{ij}(t) = \frac{a_{ij}}{a_i}(1 - e^{-a_i t}), \quad i \ne j, \quad a_i \ne 0,$$

where $a_i := -a_{ii}$, $i \in E$, and $Q_{ij}(t) = 0$, if $i = j$ or $a_i = 0$. In this case, the transition function of the EMC is $p_{ij} = a_{ij}/a_i$ and we recover an exponential distribution for the conditional distribution function of the sojourn time such as $F_i(t) = 1 - \exp(-a_i t)$, with $t \ge 0$.

A usual restriction that fits practical applications is to assume a *regularity* condition for the SMP of interest. To specify this condition, we introduce the counting process $(N(t), t \ge 0)$ which counts the number of jumps of X in the time interval

$(0, t]$, by $N(t) := \sup\{n \geq 0 : S_n \leq t\}$. Also, define $N_i(t)$ to be the number of visits of X to state $i \in E$ in the time interval $(0, t]$. That is to say,

$$N_i(t) := \sum_{n=0}^{N(t)} \mathbf{1}_{\{J_n = i\}} = \sum_{n=0}^{\infty} \mathbf{1}_{\{J_n = i, S_n \leq t\}}.$$

If we consider the (eventually delayed) renewal process $(S_n^i)_{n\geq 0}$ of successive times of visits to state i, then $N_i(t)$ is the counting process of renewals. Now, a SMP X is said to be regular if

$$\mathbb{P}_i(N(t) < \infty) = 1,$$

for any $t \geq 0$ and any $i \in E$.

For regular SMPs we have $S_n < S_{n+1}$, for any $n \in \mathbb{N}$, and $S_n \to \infty$. In the sequel, we are concerned with regular SMPs.

Let us also have a brief discussion about the nature of the different states of an MRP. An MRP is irreducible, if, and only if, its EMC (J_n) is irreducible. A state i is recurrent (transient) in the MRP, if, and only if, it is recurrent (transient) in the EMC. For an irreducible finite MRP, a state i is positive recurrent in the MRP, if, and only if, it is recurrent in the EMC and if for all $j \in E$, $m_j < \infty$. If the EMC of an MRP is irreducible and recurrent, then all the states are positive-recurrent, if, and only if, $m := \nu m := \sum_i \nu_i m_i < \infty$, and null-recurrent, if, and only if, $m = \infty$ [where ν is the stationary probability of EMC (J_n)]. A state i is said to be periodic with period $a > 0$ if $G_{ii}(\cdot)$ (the distribution function of the random variable $S_2^i - S_1^i$) is discrete concentrated on $\{ka : k \in \mathbb{N}\}$. Such a distribution is also said to be periodic. In the opposite case it is called aperiodic. Note that the term *period* has a completely different meaning from the corresponding one of the classical Markov chain theory.

2.2 Markov Renewal Equation

An essential tool in semi-Markov theory is the MRE which can be solved using the so-called *Markov renewal function*. To unveil this function, we first need to introduce the convolution in the Stieljes-sense.

For $\phi(i, t),\ i \in E, t \geq 0$ a real-valued measurable function, the convolution of ϕ by Q is defined by

$$Q * \phi(i, t) := \sum_{k\in E} \int_0^t Q_{ik}(ds)\phi(k, t - s).$$

Now, consider the n-fold convolution of Q by itself. For any $i, j \in E$,

$$Q_{ij}^{(n)}(t) = \begin{cases} \sum_{k\in E} \int_0^t Q_{ik}(ds) Q_{kj}^{(n-1)}(t-s) & n \geq 2, \\ Q_{ij}(t) & n = 1, \\ \delta_{ij} \mathbf{1}_{\{t\geq 0\}} & n = 0, \end{cases}$$

where δ_{ij} is the Kronecker delta, that is to say, $\delta_{ij} = 1$ if $i = j$, 0 otherwise.

It is easy to prove (e.g., by induction) the following fundamental equality

$$Q_{ij}^{(n)}(t) = \mathbb{P}_i(J_n = j, S_n \leq t),$$

where, as usual, $\mathbb{P}_i(\cdot)$ means $\mathbb{P}(\cdot \mid J_0 = i)$, and $\mathbb{E}_i$ is the corresponding expectation. The Markov renewal function $\psi_{ij}(t),\ i, j \in E, t \geq 0$ is defined by

$$\begin{aligned} \psi_{ij}(t) &:= \mathbb{E}_i[N_j(t)] = \mathbb{E}_i \sum_{n=0}^{\infty} \mathbf{1}_{\{J_n=j, S_n\leq t\}} \\ &= \sum_{n=0}^{\infty} \mathbb{P}_i(J_n = j, S_n \leq t) = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t). \end{aligned}$$

In matrix form, this writes

$$\psi(t) = (I(t) - Q(t))^{(-1)} = \sum_{n=0}^{\infty} Q^{(n)}(t).$$

This can also be written as

$$\psi(t) = I(t) + Q * \psi(t), \tag{2}$$

where $I(t) = I$ (the identity matrix), if $t \geq 0$ and $I(t) = 0$, if $t < 0$.

Equation (2) is a special case of what is called a MRE. A general MRE is one of the following form:

$$\Theta(t) = g(t) + Q * \Theta(t), \tag{3}$$

where $\Theta(t) = (\Theta_{ij}(t))_{i,j\in E}$, $g(t) = (g_{ij}(t))_{i,j\in E}$ are matrix-valued measurable functions, with $\Theta_{ij}(t) = L_{ij}(t) = 0$ for $t < 0$. The function $g(t)$ is a given while $\Theta(t)$ is unknown.

The following Theorem bring some results about existence and unicity of a solution to MRE as (3).

Theorem 2.1. *(Markov Renewal Theorem [25]) Let* **B** *be the space of all locally bounded, on* $\mathbb{R}_+$*, matrix functions* $\Theta(t)$*, i.e.,* $\|\Theta(t)\| = \sup_{i,j} \left|\Theta_{i,j}(t)\right|$ *is bounded on sets* $[0, \xi]$*, for every* $\xi \in \mathbb{R}_+$*. Also, denote by* $\overline{H}_i(t) := 1 - H_i(t)$*. Let the following conditions be fulfilled:*

(1) *The EMC* (J_n) *is ergodic, i.e., irreducible and positive-recurrent, with stationary probability* $\nu = (\nu_i, i \in E)$.
(2) *The mean sojourn time in every state is finite, i.e., for every* $i \in E$,

$$m_i := \int_0^\infty \overline{H}_i(t)dt < \infty, \quad \text{and} \quad m := \sum_{i \in E} \nu_i m_i > 0.$$

(3) *The distribution functions* $H_i(t)$, $i \in E$, *are nonperiodic.*
(4) *The functions* $L_{ij}(t)$, $t \geq 0$, *are direct Riemann integrable, i.e., they satisfy the following two conditions, for any* $i, j \in E$*:*

$$\sum_{n \geq 0} \sup_{n \leq t \leq n+1} \left| L_{ij}(t) \right| < \infty,$$

and

$$\lim_{\Delta \downarrow 0} \left\{ \Delta \sum_{n \geq 0} \left[\sup_{n\Delta \leq t \leq (n+1)\Delta} L_{ij}(t) - \inf_{n\Delta \leq t \leq (n+1)\Delta} L_{ij}(t) \right] \right\} = 0.$$

Then Eq. (3) has a unique solution $\Theta = \psi * L(t)$ *belonging to* $\mathbf{B}$, *and*

$$\lim_{t \to \infty} \Theta_{ij}(t) = \frac{1}{m} \sum_{\ell \in E} \nu_\ell \int_0^\infty L_{\ell j}(t)dt. \tag{4}$$

Finally, we unveil another very important function to characterize the process, namely, the semi-Markov transition function

$$P_{ij}(t) := \mathbb{P}(X_t = j \mid X_0 = i), \quad i, j \in E, t \geq 0,$$

which is the conditional marginal law of the process. It can be shown that P verifies a particular MRE, which will be essential in the development of our probability assessments in the next section.

Proposition 2.1. *The transition function* $P(t) = (P_{ij}(t))$ *satisfies the following MRE*

$$P(t) = I(t) - H(t) + Q * P(t),$$

which, under Conditions (1–3) of Theorem 2.1, has the unique solution

$$P(t) = \psi * (I(t) - H(t)),$$

and, for any $i, j \in E$,

$$\lim_{t \to \infty} P_{ji}(t) = \nu_i m_i / m =: \pi_i.$$

Here $H(t) = diag(H_i(t))$ *is a diagonal matrix.*

It is worth noticing that, in general, the stationary distribution π of the SMP X is not equal to the stationary distribution ν of the EMC (J_n). Nevertheless, we have $\pi = \nu$ when, for example, m_i is independent of $i \in E$.

3 A Dynamical Differential System for Structural Reliability Study

We now turn back to the main motivation of the chapter, that is, investigating the following differential system:

$$\dot{Z}_t = C(Z_t, X_t), \qquad Z_0 = z. \tag{5}$$

To ensure that (5) owns a unique solution, we set the usual regularity assumption for C, that is, $C : \mathbb{R}_+ \times E \longrightarrow \mathbb{R}_+$ is measurable and Lipschitz w.r.t. the first argument, uniformly on the second.

We also set some restrictions for the reliability study of (5). Looking toward the description of what is understood here as the degradation process, the following assumptions naturally rise from physical considerations:

- the level of degradation is positive and increases across time;
- the failure domain is defined by a threshold $\Delta \in \mathbb{R}_+^* = (0, \infty)$.

These assumptions require that the function $C : (x, i) \to C(x, i)$ is strictly positive for all $x \in \mathbb{R}_+$, $i \in E$. Moreover, we set $\Delta > z > 0$ to ensure that the system does not starts in a failure state.

Now, to be specific with the reliability analysis of (5), we define $U = [z, \Delta)$ the set of working states with $0 < z < \Delta$ and $D = [\Delta, \infty)$ the set of down states. Assuming a nonreparable system and thanks to the continuous, increasing evolution of Z, failure occurs as soon as point Δ is reached: this point is an absorbing state of the system. The failure time can thus be written as a function of the coupled process (Z, X):

$$\tau = \inf\{t \geq 0 : Z_t \in D\} \equiv \inf\{t \geq 0 : (Z_t, X_t) \in D \times E\}. \tag{6}$$

The reliability and the cumulative distribution function (CDF) of τ turn to

$$R(t) = \mathbb{P}((Z_t, X_t) \in U \times E) = 1 - F_\tau(t). \tag{7}$$

In the remaining of this section, we interpret (Z, X) as an *extended* MRP. We then derive a solvable MRE whose solution is the transition function of the (Z_t, X_t). Then, reliability (7) has a closed-form which can be computed numerically. A numerical illustration is investigated that confirmed our theoretical results and that hopefully bring some insights on the understanding of the semi-Markov kernel associated with (Z, X).

3.1 The Coupled Process as an Extended Markov Renewal Process

In a previous work [4], we considered the system (5) with X a jump Markov process. Here, a more general assumption is made regarding the nature of the perturbing process: we set $X = (X_t, t \geq 0)$ a SMP with finite state space E, which describes random variations in the environment of $Z = (Z_t, t \geq 0)$. The pure jump process X is defined by its semi-Markov kernel

$$Q_{ij}(t) = \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \leq t | J_n = i), \tag{8}$$

where $i, j \in E$ and $t \geq 0$. As from the previous section, the process $(J_n, S_n, n \in \mathbb{N})$ is the embedded MRP of the SMP X, where $(S_n, n \in \mathbb{N})$ is the random sequence describing the jump times. The random sequence $J_n = X_{S_n}$ is the EMC with transition probabilities $(p_{ij})_{i,j \in E}$, such as $p_{ij} = Q_{ij}(\infty)$. We also put $\alpha_i = \mathbb{P}(X_0 = i)$ the initial distribution of X. Besides, we consider with no loss of generality that the conditional CDF of the sojourn time does not depend on the arrival point j as in Remark 2.2, that is, $F_{ij}(t) \equiv F_i(t)$. The semi-Markov kernel of X thus writes $Q_{ij}(t) = p_{ij} F_i(t)$.

From now, we start to be specific to the couple (Z, X) defined by (5): for any $t < S_1$, we denote by $\varphi_{z,i}(t)$ the deterministic function describing the solution to (5), when $X_0 = i$. Hence, $\varphi_{z,i}(t)$ is the solution before the first jump time of X, conditionally on the starting value $(Z_0, X_0) = (z, i)$. Note that we assume that Z_0 and X_0 are independent.

We are finally ready to associate to (Z, X) the "extended" MRP $(\zeta_n, J_n, S_n, n \in \mathbb{N})$, by extending the "standard" MRP (J_n, S_n) with a third component as follows:

$$\zeta_n = Z_{S_n}, \qquad J_n = X_{S_n}, \qquad n \in \mathbb{N}.$$

As for a usual MRP, we may introduce the appropriate mathematical tools. Thenceforth, consider the semi-Markov kernel associated with the triplet (ζ_n, J_n, S_n): it is denoted by L and defined, for $t > 0$, by

$$L_{ij}(z, B, t) := \mathbb{P}_{z,i}(\zeta_1 \in B, J_1 = j, S_1 - S_0 \leq t), \tag{9}$$

where B is a subset of $\mathscr{B}$, the Borel $\sigma-$field of $\mathbb{R}_+$ and $\mathbb{P}_{z,i}(\cdot) := \mathbb{P}(\cdot | Z_0 = z, X_0 = i)$. The Stieltjes-convolution of L with a measurable function ϕ on the space $\mathbb{R}_+ \times E$, denoted by "$*$", is defined by

$$(L * \phi)_{ij}(z, t) = \sum_{k \in E} \int_{\mathbb{R}_+} \int_0^t L_{ik}(z, \mathrm{d}y, \mathrm{d}s) \phi_{kj}(y, t - s),$$

for $i, j \in E$ and $z > 0$. In the same way, the n-fold convolutions of the semi-Markov kernel L are defined recursively. For $n = 0, 1,$

$$L_{ij}^{(0)}(z, B, t) = \mathbb{1}_{\{i=j\}}\mathbb{1}_B(z)\mathbb{1}_{\mathbb{R}_+}(t), \qquad L_{ij}^{(1)}(z, B, t) = L_{ij}(z, B, t),$$

where $\mathbb{1}_B(x)$ is the indicator function, i.e., $\mathbb{1}_B(x) = 1$ if $x \in B$, 0 otherwise. For $n \geq 2$, the n-fold convolution turns to

$$L_{ij}^{(n)}(z, B, t) := (L * L^{(n-1)})_{ij}(z, B, t).$$

The Markov renewal function Ψ of the triplet is

$$\Psi_{ij}(z, B, t) = \sum_{n\geq 0} L_{ij}^{(n)}(z, B, t).$$

In the case at hand, we consider that (ζ_n, J_n, S_n) is a *normal* MRP, that is, $\Psi_{ij}(z, B, t) < \infty$ for any fixed $t > 0, z > 0, B \in \mathscr{B}$ and $i, j \in E$, which implies also that the SMP Z is regular.

For the process (ζ_n, J_n, S_n), a MRE has the following form

$$\Theta_{ij}(z, B, t) = g_{ij}(z, B, t) + (L * \Theta)_{ij}(z, B, t), \tag{10}$$

where $g_{ij}, i, j \in E$ are known functions and $\Theta_{ij}, i, j \in E$ are the unknown functions. The solution to (10), thanks to the results of the previous section, is

$$\Theta_{ij}(z, B, t) = (\Psi * g)_{ij}(z, B, t). \tag{11}$$

3.2 The Transition Function

Consider the transition function P of the couple process (Z, X), defined by

$$P_{ij}(z, B, t) := \mathbb{P}_{z,i}(Z_t \in B, X_t = j), \qquad i, j \in E, B \in \mathscr{B}. \tag{12}$$

We aim at building a MRE suitable for P. For this purpose, we first need a closed-form expression for L. This is achieved in the following Lemma.

Lemma 3.1. *The semi-Markov kernel L of the extended MRP (ζ_n, J_n, S_n) satisfies, for $i \neq j$,*

$$L_{ij}(z, B, \mathrm{d}t) = \delta_{\varphi_{z,i}(t)}(B)Q_{ij}(\mathrm{d}t),$$

where $\delta_x(B)$ is the Dirac distribution, equal to 1 if $x \in B$, 0 otherwise. When $i = j$, we have $L_{ii}(\cdot, \cdot, \cdot) = 0$.

Proof. Conditioning on definition (9), and by definition (8) of Q, we get,

$$L_{ij}(z, B, \mathrm{d}t) = Q_{ij}(\mathrm{d}t) \times \mathbb{P}_{z,i}(\zeta_1 \in B | J_1 = j, S_1 = t).$$

Then, Z_t is fully characterized by $\varphi_{z,i}(t)$ before the first jump time S_1, thus $\mathbb{P}_{z,i}(\zeta_1 \in B | J_1 = j, S_1 = t) = \mathbb{P}_{z,i}(Z_t \in B) = \delta_{\varphi_{z,i}(t)}(B)$, and the result follows. □

Note that, by considering the decomposition $Q_{ij}(t) = p_{ij} F_i(t)$, Lemma 3.1 implies that

$$L_{ij}(z, B, \mathrm{d}t) = \delta_{\varphi_{z,i}(t)}(B) p_{ij} f_i(t) \mathrm{d}t,$$

where $f_i(t) = \mathrm{d}F_i(t)/\mathrm{d}t$ is the conditional probability density function of the sojourn time.

Example 3.1. Consider the special case of X a jump Markov process as defined in the Example 2.1. Then,

$$L_{ij}(z, B, \mathrm{d}t) = a_{ij} e^{-a_i t} \delta_{\varphi_{z,i}(t)}(B) \mathrm{d}t.$$

We may now proceed to the result on the transition function of the coupled process (Z, X).

Proposition 3.1. *The transition function P satisfies the MRE*

$$P_{ij}(z, B, t) = g_{ij}(z, B, t) + (L * P)_{ij}(z, B, t),$$

whose unique solution is $P_{ij}(z, B, t) = (\Psi * g)_{ij}(z, B, t)$, with

$$g_{ij}(t) = [1 - F_i(t)] \mathbb{1}_B(\varphi_{z,i}(t)) \mathbb{1}_{\{i=j\}}. \tag{13}$$

Proof. From (12), it holds that

$$P_{ij}(z, B, t) = \underbrace{\mathbb{P}_{z,i}(Z_t \in B, X_t = j, S_1 > t)}_{P_1} + \underbrace{\mathbb{P}_{z,i}(Z_t \in B, X_t = j, S_1 \leq t)}_{P_2}.$$

Before the first jump, $X_t = X_0$ and Z_t evolves according to $\varphi_{z,i}(t)$. Thus, we easily see that $P_1 = [1 - F_i(t)] \mathbb{1}_B(\varphi_t(z, i)) \mathbb{1}_{\{i=j\}}$. From Total Probability Theorem, P_2 turns to

$$P_2 = \sum_{\substack{k \in E \\ k \neq i}} \int_0^t \mathbb{P}_{z,i}(Z_t \in B, X_t = j | J_1 = k, S_1 = s) \mathbb{P}_{z,i}(J_1 = k, S_1 \in \mathrm{d}s).$$

By definition (8), $\mathbb{P}_{z,i}(J_1 = k, S_1 \in \mathrm{d}s) = Q_{ik}(\mathrm{d}s)$. Noticing that $\mathbb{P}_{z,i}(Z_t \in B, X_t = j | J_1 = k, S_1 = s) = P_{kj}(\varphi_{z,i}(s), B, t - s)$, then P_2 is fully known. Thus, with L given as in Lemma (3.1), expression $P_1 + P_2$ turns to

$$P_{ij}(z, B, t) = [1 - F_i(t)]\, \mathbb{1}_B(\varphi_{z,i}(t))\mathbb{1}_{\{i=j\}} + \sum_{k \in E} \int_{\mathbb{R}_+} \int_0^t L_{ik}(z, \mathrm{d}y, \mathrm{d}s) P_{kj}(y, B, t - s).$$

This last equation is of the general form of (10), with g equaling (13). Since the first term into the right-hand side is bounded, its solution is given by (11) and is unique. □

3.3 Application to Reliability Calculus

Enjoying a closed-form for the transition function P, this section intends to show its implication for reliability calculus. Recall that $U = [z, \Delta)$ is the set of working states and $D = [\Delta, \infty)$ is the set of down states. The reliability function is easily expressed as a function of the transition function P of the couple:

$$R(t) = \mathbb{P}((Z_t, X_t) \in U \times E) = \sum_{i,j \in E} \alpha_i P_{ij}(z, U, t).$$

Through Proposition 3.1, P is known. Hence R (as well as F_τ) is fully characterized:

$$R(t) = 1 - F_\tau(t) = \sum_{i,j \in E} \alpha_i \times (\Psi * g)_{ij}(z, U, t).$$

The computation of R thus requires Ψ, determined by summing the n-fold convolutions of the kernel L, which is the essential block of the whole process. We have a closer look to this quantity in the next paragraph.

3.3.1 Insights on the Semi-Markov Kernel

The kernel L can be calculated at a given time point $t > 0$ for the Borel subset U of working state, by integrating the expression given in Lemma 3.1. To this hand, we introduce the quantity

$$t_{z,i}(\Delta) = \inf \left\{ t \geq 0 : \varphi_{z,i}(t) \geq \Delta \right\},$$

which represent the (deterministic) time for the system to enter D when no jump is observed, and when the system starts from $(Z_0, X_0) = (z, i)$. Then, the kernel is easily seen to equals

$$L_{ij}(z, U, t) = p_{ij} \int_0^t f_i(s) \mathbb{1}_U(\varphi_{z,i}(s)) \mathrm{d}s = p_{ij} F_i \left(\min \left\{ t, t_{z,i}(\Delta) \right\} \right).$$

Conversely, the same kind of computation holds when considering the set of down states D, and we have

$$L_{ij}(z, D, t) = p_{ij} \left(F_i(t) - F_i(t_{z,i}(\Delta)) \right) \mathbb{1}_{\{t > t_{z,i}(\Delta)\}}.$$

To illustrate this, let us consider again the special case of a Markov jump process.

Example 3.2. Assume X is a jump Markov process as defined in Example 2.1. Then, we have the following closed-form for the kernel L when considering subset U and D:

$$L_{ij}(z, U, t) = \frac{a_{ij}}{a_i} \left(1 - e^{-a_i \min(t_{z,i}(\Delta), t)} \right),$$

and

$$L_{ij}(z, D, t) = \frac{a_{ij}}{a_i} \left(e^{-a_i t} - e^{-a_i t_{z,i}(\Delta)} \right) \mathbb{1}_{\{t > t_{z,i}(\Delta)\}}.$$

These expressions pave the way for the numerical implementation that leads to the evaluation of the reliability, as detailed in the next paragraph.

3.3.2 Numerical Implementation

The numerical calculation of R successively requires the kernel L, the n-fold convolutions $L^{(n)}$ for each $n \geq 0$, the Markov renewal function Ψ built upon the $L^{(n)}$ and the transition function P, by a convolution between g and Ψ. Since convolution products are time-consuming, any simplification would mean a great time-saving. By Lemma 3.1, the n-fold convolution of L turns to

$$L_{ij}^{(n)}(z, B, t) = \sum_{\substack{k \in E \\ k \neq i}} p_{ik} \int_0^t f_i(s) L_{kj}^{(n-1)}(\varphi_{z,i}(s), B, t - s) \mathrm{d}s, \tag{14}$$

hence removing the integral on $\mathbb{R}_+$, thanks to the Dirac distribution. Since our main interest is the reliability, we compute P just for the subset $B \equiv U$, that is,

$$P_{ij}(z, U, t) = \int_U \int_0^t \Psi_{ij}(z, \mathrm{d}y, \mathrm{d}s) f_j(t - s) \mathbb{1}_U(\varphi_{y,j}(t - s)). \tag{15}$$

Indeed, the sum on E has been removed thanks to the structure of g. Furthermore, the integration on $y \in \mathbb{R}_+$ is limited on U since $\mathbb{1}_U(\varphi_{y,j}(t - s))$ is zero elsewhere.

Now, these functions have to be properly discretized to achieve the numerical computation. In the following, a function with an upper index "#" means its discretized version. This discretization must be operated on both intervals $U = [z, \Delta)$ and $[0, t]$, thus we set two numerical partitions

$$\left\{z = y_0 < y_1 < \cdots < y_\ell < \cdots < y_L = \Delta^-\right\}$$
$$\text{and } \{0 = t_0 < t_1 < \cdots < t_m < \cdots < t_M = t\}.$$

Both L and M, being the respective numbers of discretization steps for $[z, \Delta)$ and $[0, t]$, have to be sufficiently large. When $L, M \to \infty$ each numerical function tends to the associated "true" one. For instance, when $L, M \to \infty$, then $L^\# \to L$ uniformly w.r.t a given matrix norm, for example, $||L|| = \max_{i,j} L_{ij}(z, y, t)$ with t, z, B fixed. Hence, the discrete (numerical) version of (15) is

$$P_{ij}^\#(z, U, t) = \sum_{y_\ell \in [z,\Delta)} \sum_{t_m \in (0,t]} \Delta_{yt} \Psi_{ij}^\#(z, y_\ell, t_m) f_j(t - t_m) \mathbb{1}_{\varphi_{y_\ell, j}(t-t_m)}(U),$$

where $\Delta_{yt} \Psi_{ij}^\#(z, y_\ell, t_m)$ is the only unknown, which stands for the numerical evaluation of $\Psi(z, \mathrm{d}y, \mathrm{d}s)$ in (15). It can be evaluated through

$$\Delta_{yt} \Psi_{ij}^\#(z, y_\ell, t_m) = \sum_{n \geq 0} \Delta_{yt} L^{\#(n)}(z, y_\ell, t_m).$$

The difference $\Delta_{yt} L^{\#(n)}$ is calculated by finite differences on y and t:

$$\begin{aligned}\Delta_{yt} L^{\#(n)}(z, y_\ell, t_m) = &[L^{\#(n)}(z, y_\ell, t_m) - L^{\#(n)}(z, y_{\ell-1}, t_m)] \\ &- [L^{\#(n)}(z, y_\ell, t_{m-1}) - L^{\#(n)}(z, y_{\ell-1}, t_{m-1})].\end{aligned}$$

Each element in $L^{\#(n)}$ is obtained by the discretized version of (14):

$$L_{ij}^{\#(n)}(z, y_\ell, t_m) = \sum_{\substack{k \in E \\ k \neq i}} p_{ik} \sum_{t_m \in (0,t]} f_i(t_m) L_{kj}^{\#(n-1)}(\varphi_{z,i}(t_m), y_\ell, t - t_m) \Delta t_m,$$

with $\Delta t_m = t_m - t_{m-1}$, the time-step discretization. Finally, we point out that the sum on the n-fold convolutions of the kernel in the evaluation of $\Psi^\#$ is truncated from the rank n^*, provided that $||L^{\#(n^*)}|| < \varepsilon$. We put ε a small real number, chosen closed to the machine precision. Note that the integer n^* is finite since $L_{ij}^{\#(n)}(z, y, t) \xrightarrow[n \to \infty]{} 0$ for a normal MRP with fixed values of $i, j \in E, t > 0, z > 0$ and $y \in [z, \Delta]$.

3.4 Numerical Illustration

As an illustration to the results and the methodology presented along this section, we suggest to study the process Z governed by

$$\dot{Z}_t = aZ_t \times c(X_t), \qquad Z_0 = z, \tag{16}$$

with $a = 0.01$, $z = 1$, $\Delta = 10$. To fix the idea, we set X a five-state jump *Markov* process with $E = \{1, 2, 3, 4, 5\}$ and a matrix generator given by

$$\mathbf{A} = \begin{pmatrix} -0.2 & 0.16 & 0 & 0.04 & 0 \\ 0.12 & -0.2 & 0.08 & 0 & 0 \\ 0.14 & 0 & -0.2 & 0 & 0.06 \\ 0 & 0.07 & 0 & -0.1 & 0.03 \\ 0 & 0 & 0.05 & 0.05 & -0.1 \end{pmatrix}$$

The initial law is $\alpha = (1/4\ 1/2\ 1/4\ 0\ 0)$.Finally, the function $c : \{1, 2, 3, 4, 5\} \to \{0.5, 1, 1.5, 2, 4\}$ is a one-to-one mapping introduced to "control" the randomizing process X. Note that the multiplicative form of system (16) is reminiscent of stochastic crack-growth modeling and is suitable to describing a wide family of degradation processes.

Before we carry on reliability computations, we suggest to get a better insight into the semi-Markov kernel L of (Z, X) as defined in system (16): X being Markovian, the expressions of L on subsets U and D exactly match the Example 3.2 and can be straightforwardly computed. Rather than plotting L, consider the functions $H_i(z, B, t) = \sum_{j \in E} L_{ij}(z, B, t)$ and the CDF

$$\mathbf{H}_i(t) := \mathbb{P}_i(S_1 \leq t) = H_i(z, U, t) + H_i(z, D, t).$$

The function $\mathbf{H}_i$ is the CDF of the sojourn time for the jump process X to be in the state $X_0 = i$. The function $H_i(z, B, \cdot)$ is a sub distribution: when $B \equiv U$, $H_i(z, U, t)$ represents the probability for the system, starting from (z, i), to remain in a safe state when X is jumping for the first time. Similarly, $H_i(z, U, t)$ describes the probability for the system to be in a failure state when the first jump occurs. These remarks are illustrated in Fig. 2, representing $H_i(z, B, \cdot)$ for $i = 1, 5$, respectively on U and D. The function $H_1(z, D, \cdot)$ is approximately zero, meaning that $H_1(z, U, \cdot) \cong \mathbf{H}_1(\cdot)$ is a CDF. Conversely, starting from $(z, 5)$, this probability is strictly greater than zero. As a matter of fact, state 5 for X_t corresponds to a "shock" inducing a strong multiplicative change to the system (16): Z increases a lot faster to the absorbing point Δ. Also remark that we graphically establish that $\mathbf{H}_5(t) = H_5(z, U, \cdot) + H_5(z, D, \cdot)$ is a CDF.

Let us now evaluate the reliability of system described by Eq. (16) through a Markov renewal argument. To do this, the numerical resolution of the MRE is performed with $M = L = 100$ points of discretization.

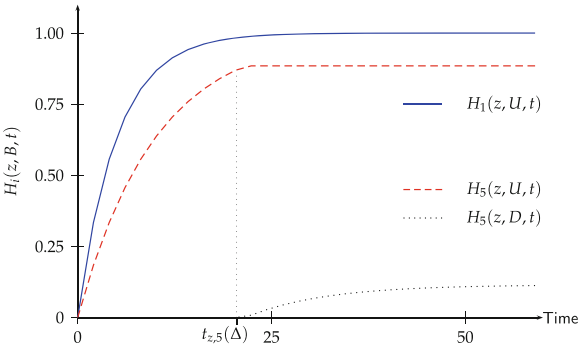

Fig. 2 Function $H_i(z, B, \cdot)$ associated with the semi-Markov kernel

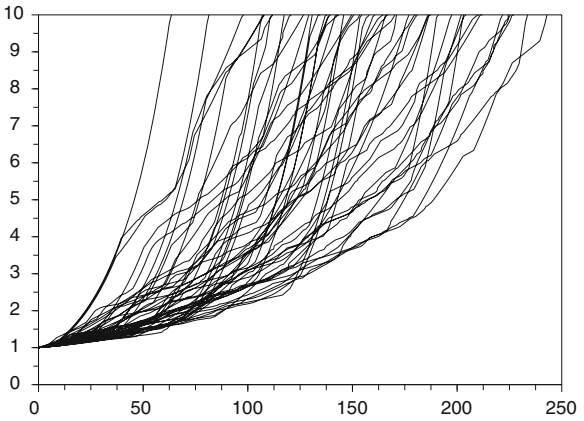

Fig. 3 50 randomly simulated paths of Z_t

As a comparison, we compute the reliability thanks to the usual Monte-Carlo method, which consists in simulating a large number of paths of Z and counting when the state $\{\Delta\}$ is reached or not. This principle is illustrated in Fig. 3 for $K = 50$ trajectories. By the way, these trajectories helps to catch the nature of this particular numerical illustration.

We use the empirical estimator computed on $K = 50,000$ paths $(Z_t^k)_{k=1,\ldots,K}$ simulated through Monte-Carlo techniques, that is $\widehat{R}(t) = \frac{1}{K}\sum_{k=1}^{K} \mathbb{1}_{\{Z_t^k < \Delta\}}$. This estimator is compared with the direct calculus of R through the MRE developed here. Results can be found on Fig. 4 where the Monte-Carlo estimator is used as a reference for sanity-check of the validity of both theoretical results and numerical implementation. One can take note of the very good similarity between the reliability curves obtained via the two methods.

Moreover we represent in Fig. 5 an evaluation of f_τ, the probability density function of the failure time. With the very same kind of argument that we devel-

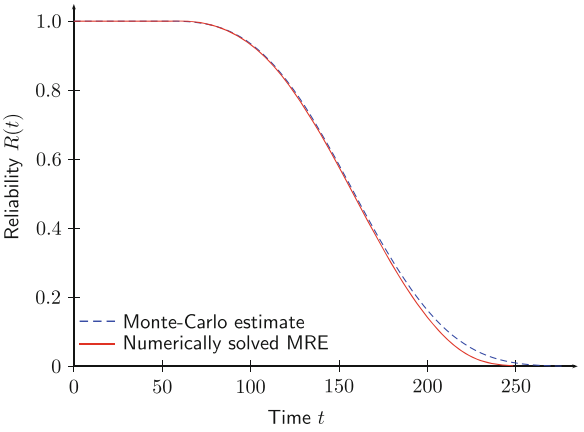

Fig. 4 Comparing the Monte-Carlo estimator to the numerically solved Markov renewal equation for reliability function

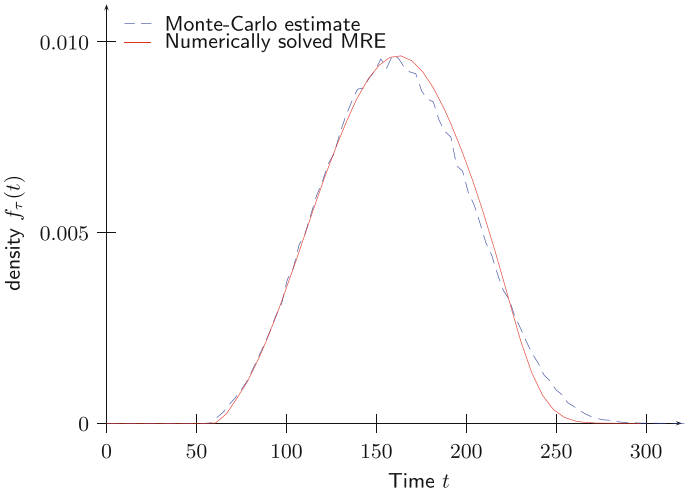

Fig. 5 Comparing the Monte-Carlo estimator to the numerically solved Markov renewal equation for the density function of the failure time

oped, f_τ can be obtained by solving a MRE or via Monte-Carlo simulation. This is done through some routine calculus starting from the basic fact that

$$f_\tau(t) = -\frac{\mathrm{d}}{\mathrm{d}t} R(t).$$

Again, we acknowledge the very good correlation between our proposal and the Monte-Carlo estimate. Moreover, we observe that our proposal is smoother as compared to the Monte-Carlo approach, which would require a huge number of simulation to get a similar result.

4 Statistical Inference

This section addresses the estimation issue related to the system described by Eq. (5) in Sect. 3. It partially follows the exposition given in [5], where we studied the simpler Markov case for X in a real data study related to crack-growth analysis. The SMP estimation developed here consists of new material. We also provide statistical methods which are more robust regarding the estimation of the trajectories of the jump process, derived from the literature related to the segmentation/clustering of piecewise constant signals.

Recall the observation scheme, as plotted in Fig. 1: the only data that can reasonably be made available from experimental feedback are recorded paths of the degradation process Z. Typically, the process is observed from a starting point z which represents the smallest level of degradation that can be characterized, until it reaches the failure threshold Δ at a random time τ. This is exactly how measurements of crack-growth are acquired in [27]. Thus, the sample training data are only composed by some K paths $\left\{Z_t^k, t = 0, \ldots, \tau^k\right\}_{k=1}^K$ where τ^k is the hitting time for the k-th path.

Basically, the most ambitious goal that we would like to aim is to successively estimate (1) the function C and (2) the randomizing SMP X_t, only by considering paths of Z with right censoring. We propose in this section a first methodological effort in that sense, which, requires some additional assumptions to carry out the inference process:

- the observed paths Z are independent and identically distributed;
- the function $C \equiv C_\theta : (z, i) \to C(z, i)$ is a known parametric function, with parameters θ remaining unknown;
- there exists a function G_θ giving X as a function of Z and its first derivative, that is, the function C_θ in the dynamical system (5) may reverse so as

$$X_t = G_\theta\left(Z_t, \dot{Z}_t\right). \tag{17}$$

The first assumption of i.i.d. paths is quite usual in statistics and well motivated in the framework of structural reliability. The second assumption (the parametric modeling of C) clearly eases the inference process. Yet we underline that it has been initially motivated by application purpose. In fact, the modeling of a particular degradation process often owns a physical framework in which scientists have an idea about the general form of C_θ, with θ the parametric adjustment which remains the only unknown. Finally, concerning the third assumption, it is acknowledged that such

a function does not always exists, yet it is required to evaluate the paths of the jumping random component. Indeed, when the stochastic process X is a linear additive or a multiplicative term in the function C, we may easily find the corresponding function G, which concerns a broad family of problems. Besides, this is truly the case for most of the stochastic crack-growth formulations that we met and that initially motivated this work [19, 22, 27].

The rest of this section splits into two parts:

(1) First, we describe the estimation of C, which relies on the Bogolyubov's averaging principle [2]. A regression analysis can be performed on the asymptotic, deterministic system so as to estimate the fixed parameter θ in function C.
(2) Second, we address the estimation of the random component, that is, the jump SMP X, which is not directly observed. Once some paths of X and its state space are recovered, we can build the likelihood function, keeping in mind that the paths are defined on randomly censored time intervals. The semi-Markov kernel Q of X are then estimated by maximizing an approached likelihood function.

We finally give a numerical application to illustrate the whole estimation scheme.

4.1 Bogolyubov's Averaging Principle

An approximation of Z is obtained by analyzing the system (5) in a series scheme as in [12], that is by studying the weak convergence, when $\varepsilon \to 0$, of

$$\frac{\mathrm{d}Z_t^\varepsilon}{\mathrm{d}t} = C_\theta(Z_t^\varepsilon, X_{t/\varepsilon}), \qquad Z_0^\varepsilon = z, \tag{18}$$

where X is assumed to be ergodic and θ are the parameters of C. In fact, the change of scale $t \to t/\varepsilon$ is performed for X in order to see the behavior of the dynamical system when the random component X just adds the information it would add after a very long time of observation of (18), since $t/\varepsilon \to \infty$ when $\varepsilon \to 0$. This so-called *averaging approximation* was first introduced by Bogolyubov [2] who showed that (18) converges weakly when $\varepsilon \to 0$ to the following deterministic system

$$\frac{\mathrm{d}\widetilde{z}_t}{\mathrm{d}t} = \overline{C}_\theta(\widetilde{z}_t), \qquad \widetilde{z}_0 = z, \tag{19}$$

with $\widetilde{z}_t$ the limit deterministic process and $\overline{C}_\theta$ a mean function defined by

$$\overline{C}_\theta(z) = \lim_{T\to\infty} \frac{1}{T} \int_0^T C_\theta(z, X_t)\mathrm{d}t, \qquad a.s.$$

An illustration of this principle is provided in Fig. 6, where $\widetilde{z}_t$ is represented among a set of sample paths of Z.

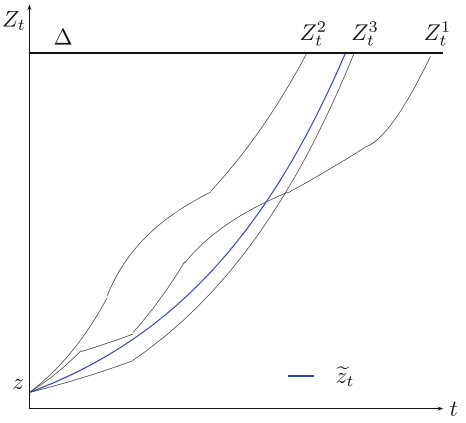

Fig. 6 Application of the Bogolyubov's averaging principle

In the particular case where X is an ergodic SMP with a stationary law π, we have

$$\overline{C}_\theta(z) = \sum_{i \in E} C_\theta(z, i)\pi_i.$$

Through this averaging technique, we have a limit deterministic system (19) associated with stochastic differential system (5). The fixed parameters θ appearing in the function C_θ are the same as the ones appearing in $\overline{C}_\theta$ but in (19) the random part was "eliminated": with the K sample paths Z_t^k, we can perform a classical regression analysis on (19) to estimate the fixed parameters θ.

4.2 The Semi-Markov Process Estimation

The SMP X is fully characterized by its kernel Q and its initial law α. Meanwhile, prior to any estimation of Q or α, some representations of the paths $\left\{X_t^k, t = 0, \ldots, \tau^k\right\}_{k=1}^K$ are needed.

4.2.1 Trajectories Estimation

Assume that there exists a function G_θ as defined in (17); hence, we may obtain a first estimation for the X_t^k's through

$$\widetilde{X}_t^k = G_\theta\left(Z_t^k, \widehat{\dot{Z}}_t^k\right), \tag{20}$$

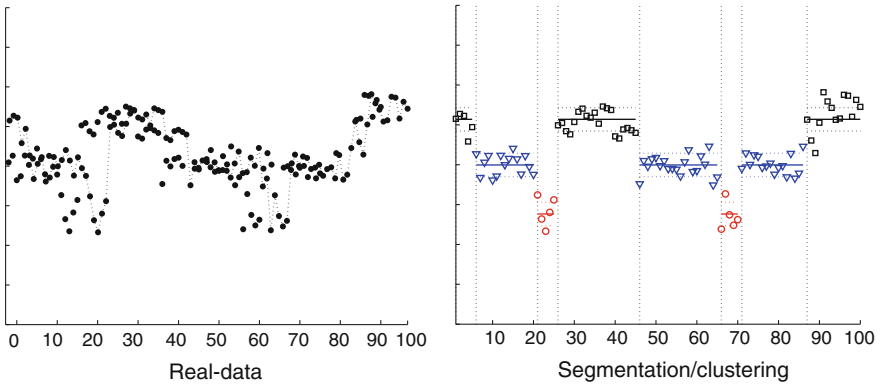

Fig. 7 Illustration of a segmentation/clustering algorithm (*Source* Picard [23])

where the derivative of Z_t can be estimated by various straightforward methods, e.g., the secant method

$$\widehat{\dot{Z}}_t^k = \frac{Z_{t+\Delta t}^k - Z_t^k}{\Delta t},$$

with Δt being the time discretization step of the data set.

Note that the parameters θ are required, whose estimation could be performed relying on the averaging principle argument just developed above. Hence, by (20), we basically extract from the trajectories of Z the "random" part that is unexplained by the averaging, deterministic process in (19). By this mean, we obtain some noisy paths taking their values in $\mathbb{R}$, in which values may be quite nearby, as illustrated in Fig. 7. Our model requires a finite state space for the underlying SMP X with some piecewise-constant shape paths, thus it is appropriate to "regroup" the values which are very close from each other to an unique state. This problem can be interpreted as the widely studied segmentation/clustering problem: basically, one wishes to perform (1) the segmentation of a signal assumed to be piecewise constant into, says, q change-points corresponding to the jumps of X and (2) the clustering of the q segments into, says, p clusters corresponding to the states of X.

The segmentation/clustering process is illustrated in Fig. 7 (from Picard [23]): the segmentation is performed on the x-axis while the clustering is performed in the y-axis.

Traditionally, this problem has been studied using hidden Markov models. This is a quite well-studied issue where the segmentation step is usually treated via dynamic programming and the clustering step may be treated through various algorithms such as the popular K-means algorithm. We adopt this naive approach for our problem (see e.g. [5]), yet we want to underline the fact that the segmentation/clustering problem received much attention recently. As a matter of fact, the treatment of huge amount of data with signal lengths up to the million of entries has been required for bioinformatics purpose. Consequently, very powerful variants and implementations

of the segmentation/clustering problem have been investigated, regarding the analysis of CGH microarray data. Authors use the traditional HMM modeling (e.g.,[10]) and also mixture modeling (e.g., [14, 23]). They provide very competitive and well integrated model selection approaches to chose both the number of segment p and the number of cluster q. Our problem of paths estimation of X is in very small dimension as compared to the problem of biological data, and can be treated very efficiently. We thus rely on these approaches to process the noisy paths $\{\widetilde{X}_t^k, k = 1 \ldots, K\}$ from (17), thus leading to K piecewise constant approximated paths $\{\widehat{X}_t^k, k = 1, \ldots, K\}$ defined on a finite state space E. The $\widehat{X}^k$s are then used for further estimations linked to the SMP X, namely for estimating its initial distribution α and its kernel Q. This issue is addressed in the following paragraph, based upon an approached maximum likelihood estimator which is equivalent to empirical estimators of the semi-Markov kernel.

4.2.2 K-Histories Empirical Estimators

For clarity purpose, we drop the "hat" on the $\widehat{X}^k$s an related quantities along this section. Note that the writing of the likelihood greatly simplifies when writing a path of X as an ordered sequence:

$$\mathscr{H}_\tau = \big((J_0, W_0), \ldots (J_{N(\tau)-1}, W_{N(\tau)-1}), (J_{N(\tau)}, U_\tau)\big)\,,$$

where

- $N(\tau)$ is the number of jumps on $[0, \tau]$,
- $J_n = X_{S_n}, n \in \mathbb{N}$ are the visited states,
- $W_n = S_{n+1} - S_n, n \in \mathbb{N}$ are the sojourn times,
- $U_\tau = \tau - S_{N(\tau)}$.

The density $f_{\mathscr{H}_\tau}$ of $\mathscr{H}_\tau$ is function of $f_\tau(t)$, the density of τ:

$$f_{\mathscr{H}_\tau}(h_t) = f_{\mathscr{H}_t}(h_t) f_\tau(t),$$

where h_t is a realization of $\mathscr{H}_\tau$.

Consider K independent MRPs $(J_n^k, S_n^k, n \geq 0), \quad k = 1, \ldots, K$, defined by the same kernel Q and initial distribution α, and K copies $\tau_k, k = 1, \ldots, K$ of τ. The same for N^k, U^k. The likelihood for the K histories writes, for t_k a realization of τ_k,

$$\mathscr{L} = \prod_{k=1}^{K} f_{\mathscr{H}_t}(h_{t_k}^k) \cdot \prod_{k=1}^{K} f_\tau(t_k).$$

As an approximation, we assume τ, $\mathscr{H}$ independent. Then, the maximization of the likelihood does not rely on the term

$$\prod_{k=1}^{K} f_\tau(t_k).$$

Hence, the approached likelihood function associated with $(\mathscr{H}^k, 1 \le k \le K)$ writes

$$\tilde{\mathscr{L}}(K) = \prod_{k=1}^{K} \alpha(J_0^k)\left(1 - \sum_{\ell \in E} Q_{J^k_{N^k(t_k)},\ell}(U^k_{t_k})\right) \times \prod_{\ell=1}^{N^k(t_k)} p_{J^k_{\ell-1},J^k_\ell} dF_{J^k_{\ell-1}J^k_\ell}(X^k_\ell),$$

where we remind the decomposition $Q_{ij}(t) = p_{ij}F_{ij}(t)$.

It is clear that the MLE of the initial distribution is $\widehat{\alpha}(i) = n_i/K$, where n_i is the number of trajectories starting from the state i.

The estimator of the kernel which maximized the approached likelihood is easily written by introducing the additional following statistics:

- $N_i(\tau, K)$ the number of visits in state i observed on the K censored paths:

$$N_i(\tau, K) = \sum_{k=1}^{K} \sum_{n=0}^{N^k(\tau_k)-1} \mathbb{1}_{\{J^k_n = i\}} = \sum_{n=0}^{\infty} \mathbb{1}_{\{J^k_n = i, S^k_{n+1} \le \tau_k\}},$$

- $N_{ij}(\tau, K)$ the number of transitions from state i to state j observed on the K censored paths:

$$N_{ij}(\tau, K) = \sum_{k=1}^{K} \sum_{n=0}^{N^k(\tau_k)-1} \mathbb{1}_{\{J^k_n = i, J^k_{n+1} = j\}} = \sum_{n=0}^{\infty} \mathbb{1}_{\{J^k_n = i, J^k_{n+1} = j, S^k_{n+1} \le \tau_k\}},$$

- $M_{ij}(t; \tau, K)$ the number of time the sojourn in i going to j is less than t on the K censored paths:

$$M_{ij}(t; \tau, K) = \sum_{k=1}^{K} \sum_{n=0}^{N^k(\tau_k)-1} \mathbb{1}_{\{J^k_{n+1} = j, J^k_n = i, W^k_n \le t\}}.$$

We finally get the following estimator by straightforward generalization of Moore and Pyke, provided that $F_\tau \neq \delta_0$:

$$\hat{Q}_{ij}(t; \tau, K) = \hat{p}_{ij}(\tau, K)\hat{F}_{ij}(t; \tau, K),$$

with

$$\hat{p}_{ij}(\tau, K) = \frac{N_{ij}(\tau, K)}{N_i(\tau, K)} \text{ and } \hat{F}_{ij}(t; \tau, K) = \frac{M_{ij}(t; \tau, K)}{N_{ij}(\tau, K)}.$$

In fact, the above estimators are the empirical ones.

4.3 Numerical Illustration

We now wish to provide an numerical example which integrates the whole process of estimation, as well as the probabilistic results depicted in the third section. To this aim, we study the following dynamical system, which is in the same vein as system (16):

$$\dot{Z}_t = aZ_t \times c(X_t), \qquad Z_0 = z. \tag{21}$$

As compared to (16), changes are of two kinds: first, the values of the parameters are $a = 0.02$, $z = 5$, $\Delta = 30$. Second, this is the major difference, the randomizing process X is now a three-state space SMP with $E = \{1, 2, 3\}$. The mapping c is such as $c : (1, 2, 3) \to (0.5, 1, 2)$. We also put

$$\alpha = (1/3\, 2/3\, 0)$$

the initial distribution of X. The associated semi-Markov kernel is such as $Q_{ij}(t) = p_{ij}F_{ij}(t)$, with $\mathbf{P} = (p_{ij})_{i,j\in E}$ the transition matrix and $\mathbf{F}(t) = (F_{ij}(t))_{i,j\in E}$ the distribution of sojourn times, given by

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0.9 & 0 & 0.1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{F}(t) = \begin{bmatrix} 0 & \mathscr{E}_1(t) & 0 \\ \mathscr{W}_1(t) & 0 & \mathscr{W}_2(t) \\ \mathscr{E}_2(t) & 0 & 0 \end{bmatrix}.$$

The notation $\mathscr{E}_1, \mathscr{E}_2$ stands for exponential distributions such that

$$\mathscr{E}_i(t) = 1 - \exp\{-\lambda_i t\}, \quad t \geq 0,$$

with parameters λ_1, λ_2 being respectively equal to 0.1 and 0.04. We also denote by $\mathscr{W}_1, \mathscr{W}_2$ some Weibull distributions such that

$$\mathscr{W}_i(t) = 1 - \exp\{-(t/\alpha_i)^{\beta_i}\}, \quad t \geq 0,$$

with parameters $(\alpha_i, \beta_i)_{i=1,2}$ being respectively equal to (8, 2) and (4, 0.5).

The whole estimation process sums-up as follows: denoting by $a_0 = a \times \mathbb{E}_\pi[X_t]$ with $\mathbb{E}_\pi$ the expectation regarding the stationary law π of X, the Bogolyubov's averaging principle leads to a very simple deterministic process defined by

$$\widetilde{z}_t = z \exp\{a_0 t\}.$$

Taking the log, we perform a simple least-squared analysis to estimate the parameter a_0 (see, [3] for details). Then, paths of X can be extracted, prior to the estimation of the kernel Q. Once every parameters in system (21) are known, we rely on the very same strategy as in Sect. 3 to compute the reliability, this time *with the estimated kernel* $\widehat{Q}$. The whole learning procedure takes in input some 100 paths of Z simulated

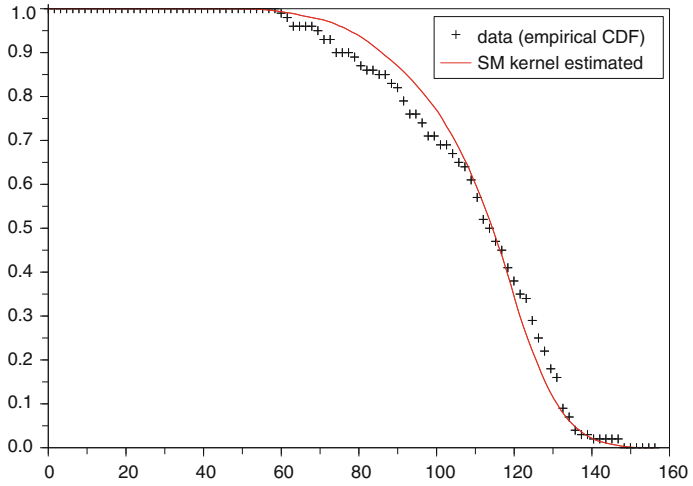

Fig. 8 Reliability evaluation and comparison to the testing set

according to (21), which consist the learning set. Some 100 other paths of Z are generated, consisting in the testing set, kept to evaluate the predictive performance of our inference strategy.

Figure 8 represents the reliability R of the system computed through the Markov renewal argument developed Sect. 3, using the estimated parameters (the kernel Q and the initial distribution law) as described in the current section. The empirical reliability which appears as an element of comparison in Fig. 8 has been computed on the test set, through

$$\widehat{R}(t) = \frac{1}{K} \sum_{k=1}^{K} \mathbb{1}_{\{Z_t^k < \Delta\}}.$$

A good fit is obtained, since the curves are quite closed: the slight discrepancy observed is likely to be due to the numerical discretization of the time interval and of the state space interval $[z, \Delta]$ of Z_t.

Remark 4.3. Note that, as future work, we plain to deeply investigate the numerical consistency of the empirical estimator of the kernel Q. It has been made for the corresponding estimator of the infinitesimal generator in the jump Markov case for X (see [3]). We also plan to take into account the dependency in τ in the likelihood maximization, since a closed-form of f_τ can be obtained through Markov renewal argument as developed in Sect. 3.

5 Concluding Remarks

Motivated by the fatigue crack-growth propagation problem, the point of view adopted in this chapter to model degradation processes does not include diffusion processes. So, we considered that the changes result from small or very small jumps. We have consequently developed a semi-Markov piecewise deterministic process as underlying model to achieve this goal.

As stated previously, this study was initially motivated and supported by the French Nuclear Power Plan Authority where we considered a Markov perturbing process. Here we considered a semi-Markov perturbing process which is much more general than the Markov one.

For a detailed modeling of reliability of SMPs the interested reader could find results in [17], for the discrete state space case, and in [16] for the general state space case. For estimation results of reliability and more general of dependability of semi-Markov systems see [18] and references therein.

Acknowledgments This chapter is dedicated to *Professor Shunji Osaki* for his 70th Birthday. It deals with semi-Markov processes and reliability which is the main area of expertise of Prof. Osaki. He has done many contributions to this field, a lot of them being included in his book "Stochastic System Reliability Modeling" (see Ref. [21] below), which is used as a basic reference by many researchers by the world including the present authors.

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