A Trinity of the Borcherds Φ-Function

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Abstract We discuss a trinity, i.e., three distinct expressions, of the Borcherds Φ-function on the analogy of the trinity of the Dedekind η-function.

1 Introduction—A Trinity of Dedekind η-Function

The Dedekind η-function is the holomorphic function on the complex upper half-plane $\mathfrak{H}$ defined as the infinite product

$$\eta(\tau) := q^{1/24} \prod_{n>0} \bigl(1 - q^n\bigr),$$

where $q := e^{2\pi i\tau}$. It is classical that $\eta(\tau)^{24}$ is a modular form for $SL_2(\mathbf{Z})$ of weight 12 vanishing at $+i\infty$ and this property characterizes the Dedekind η-function up to a constant.

Let us recall the trinity of the Dedekind η-function. Besides the definition as above, the Dedekind η-function admits at least two other distinct expressions, one analytic and the other algebro-geometric. Precisely speaking, we consider the Petersson norm

$$\bigl\|\eta(\tau)\bigr\| := (\Im\tau)^{1/4} \bigl|\eta(\tau)\bigr|$$

rather than the Dedekind η-function itself.

Let us explain an analytic counterpart of the Dedekind η-function. For $\tau \in \mathfrak{H}$, let E_τ be the elliptic curve defined by

$$E_\tau := \mathbf{C}/\mathbf{Z} + \tau\mathbf{Z},$$

which is equipped with the flat Kähler metric of normalized volume 1

$$g_\tau := dz \otimes d\bar{z}/\Im\tau.$$

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The Laplacian of (E_τ, g_τ) is the differential operator defined as

$$\square_\tau := -\Im\tau\,\frac{\partial^2}{\partial z\partial\bar{z}} = -\frac{\Im\tau}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

The set of eigenvalues of $\square_\tau$ is given by $\{\pi^2|m\tau+n|^2/\Im\tau\}_{(m,n)\in\mathbf{Z}^2}$ and hence the spectral zeta function of $\square_\tau$ is defined as

$$\zeta_\tau(s) := \sum_{(m,n)\neq(0,0)}\left(\frac{\Im\tau}{\pi^2\,|m\tau+n|^2}\right)^s.$$

It is classical that $\zeta_\tau(s)$ converges absolutely when $\Re s > 1$ and extends to a meromorphic function on $\mathbf{C}$. Moreover, $\zeta_\tau(s)$ is holomorphic at $s=0$. The value

$$\det{}^*\square_\tau := \exp\bigl(-\zeta_\tau'(0)\bigr)$$

is called the (regularized) determinant of $\square_\tau$ on the analogy of the identity for finite dimensional, non-degenerate, Hermitian matrices

$$\log\det H = -\frac{d}{ds}\Big|_{s=0}\operatorname{Tr} H^{-s}.$$

By Ray-Singer [29], the classical Kronecker limit formula can be stated as follows in this setting:

Theorem 1 *The following equality holds*

$$\det{}^*\square_\tau = 4\bigl\|\eta(\tau)\bigr\|^4.$$

Let us explain an algebro-geometric counterpart of the Dedekind η-function. Let $M_{m,n}(K)$ be the set of $m\times n$-matrices with entries in $K\subset\mathbf{C}$. Recall that every elliptic curve is expressed as the complete intersection of two quadrics of $\mathbf{P}^3$

$$E_A := \left\{[x]\in\mathbf{P}^3;\ \begin{array}{l} f_1(x) = a_{11}x_1^2 + a_{12}x_2^2 + a_{13}x_3^2 + a_{14}x_4^2 = 0 \\ f_2(x) = a_{21}x_1^2 + a_{22}x_2^2 + a_{23}x_3^2 + a_{24}x_4^2 = 0 \end{array}\right\},$$

where $A = (a_{ij}) = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) \in M_{2,4}(\mathbf{C})$. For $A\in M_{2,4}(\mathbf{C})$ and $1\le i<j\le 4$, we define

$$\Delta_{ij}(A) := \det(\mathbf{a}_i, \mathbf{a}_j).$$

Since the value $\|\eta(\tau)\|$ depends only on the isomorphism class of the elliptic curve E_τ, it makes sense to set $\|\eta(E_\tau)\| := \|\eta(\tau)\|$.

Theorem 2 *With the same notation as above, the following equality holds*

$$2^8\bigl\|\eta(E_A)\bigr\|^{24} = \prod_{1\le i<j\le 4}\bigl|\Delta_{ij}(A)\bigr|^2\cdot\left(\frac{2\sqrt{-1}}{\pi^2}\int_{E_A}\alpha_A\wedge\overline{\alpha}_A\right)^6.$$

Here $\alpha_A \in H^0(E_A, \Omega^1_{E_A})$ is defined as the residue of f_1, f_2, i.e.,

$$\alpha_A := \Xi|_{E_A},$$

where Ξ is a meromorphic 1-form on $\mathbf{P}^3$ satisfying the equation

$$df_1 \wedge df_2 \wedge \Xi = \sum_{i=1}^{4} (-1)^{i-1} x_i dx_1 \wedge dx_{i-1} \wedge dx_{i+1} \wedge dx_4.$$

For $A = (a_{ij}) \in M_{2,4}(\mathbf{C})$, one can associate another elliptic curve

$$C_A := \left\{(x, y) \in \mathbf{C}^2;\ y^2 = 4(a_{11}x + a_{21})(a_{12}x + a_{22})(a_{13}x + a_{23})(a_{14}x + a_{24})\right\}.$$

Namely, C_A is the double covering of $\mathbf{P}^1$ with 4 branch points $(a_{11} : -a_{21})$, $(a_{12} : -a_{22})$, $(a_{13} : -a_{23})$, $(a_{14} : -a_{24})$. If $a_{11} = 0$ and $a_{12} = 1$, then C_A is an elliptic curve expressed by the Weierstrass equation. It is not difficult to see $C_A \cong E_A$ and

$$2^8 \left\| \eta(C_A) \right\|^{24} = \prod_{1 \le i < j \le 4} \left| \Delta_{ij}(A) \right|^2 \cdot \left(\frac{\sqrt{-1}}{2\pi^2} \int_{C_A} \frac{dx}{y} \wedge \overline{\frac{dx}{y}} \right)^6.$$

(We shall study an analogue of E_A and C_A for $K3$ surfaces later.)

Theorem 2 is easily verified when E_A is the projective embedding of E_τ by the linear system $|4\Theta|$. In this situation, the equations of E_A are the linear relations between the theta functions $\theta_{a,b}(z, \tau)$ $(a, b \in \{0, \frac{1}{2}\})$. General case of Theorem 2 follows from this special case by the invariance of the expression in Theorem 2 under the action of $GL_2(\mathbf{C}) \times (\mathbf{C}^*)^4$. See [16] for the details.

In this survey, we explain a generalization of the trinity of the Dedekind η-function as above to that of the Borcherds Φ-function. For this, we make the following replacements:

- elliptic curves $\Longrightarrow$ Enriques surfaces
- determinant of Laplacian $\Longrightarrow$ analytic torsion
- $\prod_{1 \le i < j \le 4} \Delta_{ij}(A) \Longrightarrow$ resultant of three quadratic forms in three variables

For the analytic aspect of the Borcherds Φ-function, our explanation is based on [34, 36], while for the algebro-geometric aspect of the Borcherds Φ-function, our explanation is based on [16]. In this survey, we will not give proofs. We refer the reader to these papers for the details.

2 Borcherds Φ-Function

In this section, we recall the Borcherds Φ-function.

2.1 Domains of Type IV and Its Realization as a Tube Domain

A free $\mathbf{Z}$-module of finite rank equipped with a non-degenerate, integral, symmetric bilinear form is called a lattice. The automorphism group of a lattice L is denoted by $O(L)$. For a lattice $L = (\mathbf{Z}^r, \langle\cdot,\cdot\rangle_L)$ and $k \in \mathbf{Q}$, we set $L(k) := (\mathbf{Z}^r, k\langle\cdot,\cdot\rangle_L)$. We define $\mathbb{U} := (\mathbf{Z}^2, \binom{0\,1}{1\,0})$. There exists a unique positive-definite, even, unimodular lattice of rank 8, up to an isometry. This lattice is denoted by $\mathbb{E}_8$.

Let Λ be a lattice of signature $(2, b^-)$. We define an open manifold Ω_Λ of dimension b^- as

$$\Omega_\Lambda := \left\{[Z] \in \mathbf{P}(\Lambda \otimes \mathbf{C});\ \langle Z, Z\rangle_\Lambda = 0,\ \langle Z, \bar{Z}\rangle_\Lambda > 0\right\}.$$

Then Ω_Λ is the set of maximal positive-definite subspaces of $\Lambda \otimes \mathbf{R}$ and is isomorphic to $SO(2, b^-)/SO(2) \times SO(b^-)$. Hence each connected component of Ω_Λ is isomorphic to a symmetric bounded domain of type IV of dimension b^-.

Assume that there exists $k \in \mathbf{Z}_{>0}$ and a lattice of signature $(1, b^- - 1)$ such that $\Lambda = \mathbb{U}(k) \oplus L$. Let $\{\mathbf{e}, \mathbf{f}\}$ be a basis of $\mathbb{U}(k)$ with $\mathbf{e}^2 = \mathbf{f}^2 = 0$, $\mathbf{e} \cdot \mathbf{f} = k$. We set $\mathbf{v} := \mathbf{e} \in \mathbb{U}(k)$ and $\mathbf{v}' := \mathbf{f}/k \in \mathbb{U}(k)^\vee$. Then we have an isomorphism of complex manifolds $L \otimes \mathbf{R} + i\mathcal{C}_L \cong \Omega_\Lambda$ given by the map

$$L \otimes \mathbf{R} + i\mathcal{C}_L \ni z \to Z = \left[\mathbf{v} - \frac{\langle z, z\rangle_L}{2}\mathbf{v}' + z\right] \in \Omega_\Lambda.$$

Here $\mathcal{C}_L := \{x \in L \otimes \mathbf{R};\ \langle x, x\rangle_L > 0\}$ is the positive cone of L. Since L is Lorentzian and hence $\mathcal{C}_L$ consists of two connected components, we choose one of them, say $\mathcal{C}_L^+$. Write Ω_Λ^+ for the component of Ω_Λ corresponding to $L \otimes \mathbf{R} + i\mathcal{C}_L^+$. Then we have the decomposition $\Omega_\Lambda = \Omega_\Lambda^+ \amalg \overline{\Omega_\Lambda^+}$. The subgroup of $O(\Lambda)$ preserving the connected components Ω_Λ^+, $\overline{\Omega_\Lambda^+}$ is denoted by $O^+(\Lambda)$. Clearly, $[O(\Lambda) : O^+(\Lambda)] = 2$.

2.2 Automorphic Forms over Domains of Type IV

Let us recall the notion of automorphic forms over Ω_Λ^+. There are several mutually equivalent definitions.

2.2.1 Automorphic Form as a Multicanonical Form on Ω_Λ^+

Let $\mathcal{L}$ be the tautological line bundle on Ω_Λ^+:

$$\mathcal{L} := \mathcal{O}_{\mathbf{P}(\Lambda\otimes\mathbf{C})}(-1)|_{\Omega_\Lambda^+} \subset \Omega_\Lambda^+ \times (\Lambda \otimes \mathbf{C}).$$

The natural action of $O^+(\Lambda)$ on $\Omega_\Lambda^+ \times (\Lambda \otimes \mathbf{C})$ induces the $O^+(\Lambda)$-action on $\mathcal{L}$. A holomorphic section $f \in H^0(\Omega_\Lambda^+, \mathcal{L}^k)$ is called an automorphic form for $\Gamma \subset O^+(\Lambda)$ of weight k with character χ if

$$f(\gamma Z) = \chi(\gamma)\, \gamma f(Z)$$

for all $Z \in \Omega_\Lambda^+$ and $\gamma \in \Gamma$, where $\chi : \Gamma \to \mathbf{C}^*$ is a finite character.

2.2.2 Automorphic Form as a Homogeneous Function on the Cone over $\boldsymbol{\Omega_\Lambda^+}$

Let $C_{\Omega_\Lambda^+}$ be the cone over Ω_Λ^+ obtained from $\mathcal{L}$ by contracting the zero section. Then a holomorphic function $F \in \mathcal{O}(C_{\Omega_\Lambda^+})$ is called an automorphic form on Ω_Λ^+ for $\Gamma \subset O^+(\Lambda)$ of weight k with character χ if

$$F\big(\gamma(\zeta)\big) = \chi(\gamma)\, F(\zeta), \qquad F(\lambda\, \zeta) = \lambda^{-k}\, F(\zeta)$$

for all $\zeta \in C_{\Omega_\Lambda^+}$, $\gamma \in \Gamma$ and $\lambda \in \mathbf{C}^*$.

2.2.3 Automorphic Form as a Function on $\boldsymbol{\Omega_\Lambda^+}$

Let $\ell \in \Lambda \otimes \mathbf{R}$ be such that $\langle \ell, \ell \rangle \geq 0$. Observe that

$$\sigma_\ell(Z) := \frac{Z}{\langle \ell, Z \rangle}, \quad Z \in \Omega_\Lambda^+$$

is a nowhere vanishing holomorphic section of $\mathcal{L}$. Via the assignment $f \mapsto f/\sigma_\ell^k$, we can define automorphic forms as follows: A holomorphic function $F(Z) \in \mathcal{O}(\Omega_\Lambda^+)$ is an automorphic form for Γ of weight k with character χ if for all $Z \in \Omega_\Lambda^+$ and $\gamma \in \Gamma$,

$$F(\gamma Z) = \chi(\gamma) \left(\frac{\langle \ell, \gamma Z \rangle}{\langle \ell, Z \rangle} \right)^k F(Z).$$

The choice of ℓ corresponds to the choice of a hyperplane at infinity of $\mathbf{P}(\Lambda \otimes \mathbf{C})$.

2.2.4 Automorphic Form as a Function on $\boldsymbol{L \otimes \mathbf{R} + i\mathcal{C}_L^+}$

We have the $O^+(\Lambda)$-action on the tube domain $L \otimes \mathbf{R} + i\mathcal{C}_L^+$ via the identification $\Omega_\Lambda^+ \cong L \otimes \mathbf{R} + i\mathcal{C}_L^+$. Write $J(\gamma, y)$ for the Jacobian determinant of $\gamma \in O^+(\Lambda) \subset \mathrm{Aut}(L \otimes \mathbf{R} + i\mathcal{C}_L^+)$. By the relation between the canonical line bundle of Ω_Λ^+ and $\mathcal{L}$, there is a holomorphic function $j(\gamma, z)$ with

$$j(\gamma, z)^{\dim \Omega_\Lambda} = J(\gamma, z).$$

A holomorphic function $F(z) \in \mathcal{O}(L \otimes \mathbf{R} + i\mathcal{C}_L^+)$ is an automorphic form for Γ of weight k with character χ if for all $z \in L \otimes \mathbf{R} + i\mathcal{C}_L^+$ and $\gamma \in \Gamma$,

$$F(\gamma \cdot z) = \chi(\gamma)\, j(\gamma, z)^k\, F(z).$$

2.3 Borcherds Φ-Function

Define the Enriques lattice $\boldsymbol{\Lambda}$ as

$$\boldsymbol{\Lambda} := \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(-2).$$

Then $\boldsymbol{\Lambda}$ is an even lattice of signature (2, 10). We define the discriminant divisor of $\Omega_{\boldsymbol{\Lambda}}$ by

$$\mathcal{D}_{\boldsymbol{\Lambda}} := \sum_{d \in \boldsymbol{\Lambda}/\pm 1,\, d^2 = -2} d^{\perp},$$

where $d^{\perp} := \{[Z] \in \Omega_{\boldsymbol{\Lambda}}^+;\ \langle d, Z \rangle = 0\}$. Define $\{c(n)\}$ by the generating series:

$$\sum_{n \in \mathbf{Z}} c(n)\, q^n = \eta(\tau)^{-8} \eta(2\tau)^8 \eta(4\tau)^{-8}.$$

2.3.1 Borcherds Φ-Function at the Level 1 Cusp

Let $\mathbf{v}$ be a primitive isotropic vector of $\mathbb{U} \subset \boldsymbol{\Lambda}$ and set $L_1 := \mathbf{v}^{\perp}/\mathbf{v} \cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$. Then $L_1 \otimes \mathbf{R} + i\,\mathcal{C}_{L_1}^+ \cong \Omega_{\boldsymbol{\Lambda}}^+$.

Definition 1 The Borcherds Φ-function is the formal Fourier series on the tube domain $L_1 \otimes \mathbf{R} + i\mathcal{C}_{L_1}^+$ defined as

$$\Phi_1(z) := \prod_{\lambda \in L_1 \cap \overline{\mathcal{C}_{L_1}^+} \setminus \{0\}} \left(\frac{1 - e^{\pi i \langle \lambda, z \rangle}}{1 + e^{\pi i \langle \lambda, z \rangle}} \right)^{c(\lambda^2/2)}.$$

2.3.2 Borcherds Φ-Function at the Level 2 Cusp

Let $\mathbf{v}$ be a primitive isotropic vector of $\mathbb{U}(2) \subset \boldsymbol{\Lambda}$ and set $L_2 = \mathbf{v}^{\perp}/\mathbf{v} \cong \mathbb{U} \oplus \mathbb{E}_8(2)$. Then $L_2 \otimes \mathbf{R} + i\,\mathcal{C}_{L_2}^+ \cong \Omega_{\boldsymbol{\Lambda}}^+$.

Definition 2 The Borcherds Φ-function is the formal Fourier series on the tube domain $L_2 \otimes \mathbf{R} + i\,\mathcal{C}^+_{L_2}$ defined as

$$\Phi_2(z) := 2^8 e^{2\pi i \langle \rho, z\rangle} \prod_{\lambda \in L_2,\, \langle \lambda, \rho\rangle > 0 \,\text{or}\, \lambda \in \mathbf{N}\rho} \left(1 - e^{2\pi i \langle \lambda, z\rangle}\right)^{(-1)^{\langle \rho - \rho', \lambda\rangle} c(\lambda^2/2)},$$

where $\rho = ((0,1),0)$, $\rho' = ((1,0),0) \in L_2$.

Theorem 3 (Borcherds [8, 9]) *For $j = 1, 2$, the formal Fourier series $\Phi_j(z)$ as above converges absolutely for $z \in L_j \otimes \mathbf{R} + i\,\mathcal{C}^+_{L_j}$ with $\Im z \gg 0$ and extends to an automorphic form on $L_j \otimes \mathbf{R} + i\,\mathcal{C}^+_{L_j}$ for $O^+(\boldsymbol{\Lambda})$ of weight* 4*. Regarded as holomorphic functions on $\Omega^+_{\boldsymbol{\Lambda}}$, one has the equality up to a constant of modulus* 1

$$\Phi_1 = \Phi_2.$$

In what follows, we write $\Phi(z)$ for $\Phi_1(z)$ and $\Phi_2(z)$.

Definition 3 The Petersson norm of Φ is the C^∞ function on $L_j \otimes \mathbf{R} + i\,\mathcal{C}^+_{L_j}$ defined as

$$\left\|\Phi(z)\right\|^2 := \langle \Im z, \Im z\rangle^4 \left|\Phi_j(z)\right|^2.$$

Since the Petersson norm $\|\Phi(z)\|$ is $O^+(\boldsymbol{\Lambda})$-invariant, we regard $\|\Phi(z)\|$ as a function on the orthogonal modular variety $\Omega^+_{\boldsymbol{\Lambda}}/O^+(\boldsymbol{\Lambda})$.

By [9, Th. 13.3], $\log\|\Phi\|$ is defined as the finite part of the divergent integral:

$$-4\log\left\|\Phi(Z)\right\| - 8\left(\Gamma'(1) + \log(2\pi)\right) = \mathrm{Pf}\int_{SL_2(\mathbf{Z})\backslash\mathfrak{H}} F(\tau)\cdot\overline{\Theta_{\boldsymbol{\Lambda}}(\tau, Z)}\, y\,\frac{dx\,dy}{y^2},$$

where $F(\tau)$ is a certain vector-valued elliptic modular form for $Mp_2(\mathbf{Z})$ (cf. [36, Def. 7.6] with $\Lambda = \boldsymbol{\Lambda}$) and $\Theta_{\boldsymbol{\Lambda}}(\tau, Z)$ is the Siegel theta function [9] of the Enriques lattice $\boldsymbol{\Lambda}$. Then the expressions $\Phi_1(z)$ and $\Phi_2(z)$ are obtained by computing the above integral at the level 1 cusp and the level 2 cusp, respectively. For the necessity of the constant 2^8 in $\Phi_2(z)$, see [9, Th. 13.3 (5)] and [36, Eq. (7.9)].

Remark 1 One can rewrite the expression of $\Phi(z)$ using the dual lattice of $\boldsymbol{\Lambda}$. Set $L := \mathbb{U} \oplus \mathbb{E}_8(-1)$. Since the dual lattice of $\boldsymbol{\Lambda}$ is given by $\boldsymbol{\Lambda}^\vee = \mathbb{U} \oplus L(1/2)$, we get

$$\boldsymbol{\Lambda}^\vee(2) = \mathbb{U}(2) \oplus L.$$

Then the Borcherds Φ-function can be expressed as a function on $L \otimes \mathbf{R} + i\,\mathcal{C}^+_L$

$$\Phi(z) = \prod_{\lambda \in \mathbb{L} \cap \overline{\mathcal{C}^+_L} \setminus \{0\}} \left(\frac{1 - e^{2\pi i \langle \lambda, z\rangle}}{1 + e^{2\pi i \langle \lambda, z\rangle}}\right)^{c(\lambda^2/2)} = \sum_{\lambda \in \mathbb{L} \cap \overline{\mathcal{C}^+_L},\, \lambda^2 = 0,\,\text{primitive}} \frac{\eta(\langle \lambda, z\rangle)^{16}}{\eta(2\langle \lambda, z\rangle)^8}.$$

This identity is known as the denominator identity for the fake monster superalgebra. See [9, Example 13.7] and [30] for more details about the denominator identity for the fake monster superalgebra. See [7, 8] for the Fourier expansion of $\Phi_2(z)$.

3 Enriques Surfaces and Their Moduli Space

In this section, we recall Enriques surfaces.

3.1 $K3$ Surfaces

A compact connected complex surface X is a $K3$ surface if

$$H^1(X, \mathcal{O}_X) = 0, \qquad \Omega_X^2 \cong \mathcal{O}_X.$$

It is known that the diffeomorphism type underlying a $K3$ surface is unique. In particular, the second integral cohomology group of a $K3$ surface equipped with the cup-product pairing is isometric to the $K3$-lattice

$$\mathbb{L}_{K3} := \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8(-1) \oplus \mathbb{E}_8(-1).$$

For a $K3$ surface X, an isometry of lattices $\alpha \colon H^2(X, \mathbf{Z}) \cong \mathbb{L}_{K3}$ is called a marking.

Let X be a $K3$ surface and let $\alpha \colon H^2(X, \mathbf{Z}) \cong \mathbb{L}_{K3}$ be a marking. Since Ω_X^2 is trivial, there exists a unique nowhere vanishing holomorphic 2-form η on X, up to a non-zero constant. By the Hodge decomposition, we get the natural inclusion $H^0(X, \Omega_X^2) \subset H^2(X, \mathbf{Z}) \otimes \mathbf{C}$, so that the line $\mathbf{C}\eta \in \mathbf{P}(H^2(X, \mathbf{C}))$ is uniquely determined by X. The period of (X, α) is defined as the point of $\mathbf{P}(\mathbb{L}_{K3} \otimes \mathbf{C})$ corresponding to $\mathbf{C}\eta$ via the marking α:

$$\varpi(X, \alpha) := \left[\alpha(\eta)\right] \in \Omega_{\mathbb{L}_{K3}}.$$

Here we define $\Omega_{\mathbb{L}_{K3}} = \{[Z] \in \mathbf{P}(\mathbb{L}_{K3} \otimes \mathbf{C});\ \langle Z, Z \rangle = 0, \langle Z, \bar{Z} \rangle > 0\}$ as before. Notice that $[\alpha(\eta)] \in \Omega_{\mathbb{L}_{K3}}$ by the Riemann-Hodge bilinear relations $\int_X \eta \wedge \eta = 0$ and $\int_X \eta \wedge \overline{\eta} > 0$. For $K3$ surfaces and their moduli space, see [1] for more details.

3.2 Enriques Surfaces

A compact connected complex surface Y is an Enriques surface if

$$H^1(Y, \mathcal{O}_Y) = 0, \qquad \Omega_Y^2 \not\cong \mathcal{O}_Y, \qquad \left(\Omega_Y^2\right)^{\otimes 2} \cong \mathcal{O}_Y.$$

It is known that the universal covering of an Enriques surface is a $K3$ surface and an Enriques surface is obtained as the quotient of its universal covering by a fixed-point-free involution. Notice that a single $K3$ surface can cover many distinct Enriques surfaces (cf. [25–28] and Subsect. 5.3 below).

Let Y be an Enriques surface and let $\widetilde{Y} \to Y$ be the universal covering. Let $\iota\colon Y \to Y$ be the non-trivial covering transformation of $\widetilde{Y} \to Y$. Write $H^2(\widetilde{Y}, \mathbf{Z})_+$ and $H^2(\widetilde{Y}, \mathbf{Z})_-$ for the invariant and anti-invariant subspaces of $H^2(\widetilde{Y}, \mathbf{Z})$ with respect to the ι-action, respectively. Let $I : \mathbb{L}_{K3} \to \mathbb{L}_{K3}$ be the involution defined as

$$I(a, b, c, x, y) := (b, a, -c, y, x), \quad a, b, c \in \mathbb{U},\ x, y \in \mathbb{E}_8(-1).$$

By [13, 14], there exists a marking $\alpha\colon\ H^2(\widetilde{Y}, \mathbf{Z}) \cong \mathbb{L}_{K3}$ such that

$$\alpha \circ \iota^* \circ \alpha^{-1} = I.$$

Let $(\mathbb{L}_{K3})_+$ and $(\mathbb{L}_{K3})_-$ be the invariant and anti-invariant subspaces of $\mathbb{L}_{K3}$ with respect to the I-action, respectively. Then we have isometries of lattices

$$\alpha\big(H^2(\widetilde{Y}, \mathbf{Z})_+\big) = (\mathbb{L}_{K3})_+ \cong \mathbb{U}(2) \oplus \mathbb{E}_8(-2), \qquad \alpha\big(H^2(\widetilde{Y}, \mathbf{Z})_-\big) = (\mathbb{L}_{K3})_- \cong \boldsymbol{\Lambda}.$$

Since Y has no non-zero holomorphic 2-forms, we get $H^0(\widetilde{Y}, \Omega^2_{\widetilde{Y}}) \subset H^2(\widetilde{Y}, \mathbf{Z})_- \otimes \mathbf{C}$. Hence $\varpi(\widetilde{Y}, \alpha) \in \Omega_{\boldsymbol{\Lambda}}$ if α is a marking as above. The period of an Enriques surface $Y = \widetilde{Y}/\iota$ is defined as the period of its universal covering $\widetilde{Y}$, i.e.,

$$\varpi(Y) := \big[\varpi(\widetilde{Y}, \alpha)\big] \in \Omega^+_{\boldsymbol{\Lambda}}/O^+(\boldsymbol{\Lambda}),$$

where α is a marking satisfying $\alpha \circ \iota^* \circ \alpha^{-1} = I$ and $[\varpi(\widetilde{Y}, \alpha)]$ denotes the $O^+(\boldsymbol{\Lambda})$-orbit of $\varpi(\widetilde{Y}, \alpha)$. It is known that the isomorphism class of an Enriques surface is classified by its period:

Theorem 4 (Horikawa [13, 14]) *There exists a coarse moduli space of Enriques surfaces, denoted by $\mathcal{M}$. The period mapping induces an isomorphism between the analytic spaces*

$$\varpi : \mathcal{M} \ni [Y] \to \big[\varpi(Y)\big] \in \frac{\Omega^+_{\boldsymbol{\Lambda}} \setminus \mathcal{D}_{\boldsymbol{\Lambda}}}{O^+(\boldsymbol{\Lambda})}.$$

In what follows, we identify $\mathcal{M}$ with $(\Omega^+_{\boldsymbol{\Lambda}} \setminus \mathcal{D}_{\boldsymbol{\Lambda}})/O^+(\boldsymbol{\Lambda})$ by the map ϖ. We refer the reader to [1] for more details about Enriques surfaces and their moduli space. By Theorem 4, the period mapping for Enriques surfaces omit the discriminant locus. The Borcherds Φ-function characterize exactly the discriminant locus $\mathcal{D}_{\boldsymbol{\Lambda}}$.

Theorem 5 (Borcherds [8]) *The Borcherds Φ-function vanishes exactly on $\mathcal{D}_{\boldsymbol{\Lambda}}$ of order* 1. *In particular, Φ is a nowhere vanishing holomorphic section of the Hodge line bundle on $\mathcal{M}$.*

Since the line bundle of automorphic forms on an arithmetic quotient of a symmetric bounded domain is an ample line bundle by Baily-Borel, the moduli space of Enriques surfaces is quasi-affine by Theorem 5 [8]. In fact, the quasi-affinity of the moduli space holds for wider classes of $K3$ surfaces with involution. See [36].

4 Analytic Torsion and Borcherds Φ-Function: An Analytic Counterpart

The notion of holomorphic analytic torsion was introduced by Ray-Singer [29] in their works extending the classical notion of torsion in algebraic topology to certain analytic settings; they extended the construction of torsion of finite-dimensional acyclic complex to the setting of de Rham or Dolbeault complex, in which they replaced the usual finite-dimensional determinant of the combinatorial Laplacian to the regularized determinant of the Hodge-Kodaira Laplacian. In this section, we explain the construction of the Borcherds Φ-function via analytic torsion.

4.1 Analytic Torsion

Let (M, h^{TM}) be a compact connected Kähler manifold. Let $\square_q = (\bar{\partial} + \bar{\partial}^*)^2$ be the Hodge-Kodaira Laplacian acting on $(0, q)$-forms on M. Since M is compact, the Hilbert space of square integrable $(0, q)$-forms on M splits into the direct sum $L^{0,q}_M = \bigoplus_{\lambda \in \sigma(\square_q)} E(\lambda, \square_q)$, where $\sigma(\square_q) \subset \mathbf{R}_{\geq 0}$ is the spectrum of $\square_q$ and $E(\lambda, \square_q)$ is the eigenspace of $\square_q$ with respect to the eigenvalue λ. Then $E(\lambda, \square_q)$ is of finite-dimensional. The zeta function of $\square_q$ is defined as

$$\zeta_q(s) := \sum_{\lambda \in \sigma(\square_q) \setminus \{0\}} \lambda^{-s} \dim E(\lambda, \square_q).$$

By the Weyl law of the asymptotic distribution of the eigenvalues of $\square_q$, $\zeta_q(s)$ converges absolutely for $s \in \mathbf{C}$ with $\Im s > \dim M$. From the existence of the asymptotic expansion of the trace of the heat operator $e^{-t\square_q}$ as $t \to 0$, it follows that $\zeta_q(s)$ extends to a meromorphic function on $\mathbf{C}$ and that $\zeta_q(s)$ is holomorphic at $s = 0$. After Ray-Singer [29], we make the following

Definition 4 The analytic torsion of (M, h^{TM}) is the real number defined as

$$\tau\left(M, h^{TM}\right) := \exp\left[-\sum_{q \geq 0} (-1)^q q\, \zeta_q'(0)\right].$$

When $\dim M = 1$, $\tau(M)^{-1}$ is exactly the determinant of Laplacian appearing in the formula for $\|\eta(\tau)\|$. After Theorem 1, it is natural to expect that the determinant of Laplacian or analytic torsion may produce a nice function on the moduli space. This is the main topic of this section.

One natural direction of such a generalization seems to be the study of the determinant of Laplacian for compact Riemann surfaces of higher genus $g > 1$. Among numbers of studies of the determinant of Laplacian for hyperbolic Riemann surfaces of genus $g > 1$, it is Zograf [37] and McIntyre-Takhtajan [24] who obtained a holomorphic function with infinite product expression on the Schottky space by using the determinant of Laplacian. On the other hand, Kokotov-Korotkin [17] considered the determinant of Laplacian with respect to the *flat* (but degenerate) Kähler metric $\omega \otimes \overline{\omega}$, where ω is an Abelian differential on a compact Riemann surface of genus $g > 1$. They proved that, as a function on the moduli space of pairs (C, ω), with C being a marked Riemann surfaces of genus $g > 1$ and ω being an Abelian differential on C, the determinant of Laplacian is expressed by using some classical quantities like prime forms, theta function and periods. Hence there are two different generalizations of Theorem 1 in higher genus $g > 1$.

Another direction of generalization is the study of analytic torsion for higher dimensional varieties. (For several reasons, in higher dimensions, analytic torsion seems to be more appropriate than a single determinant of Laplacian in considering a generalization of Theorem 1.) Among those varieties, we are interested in Enriques surfaces, since they can be regarded as one of the natural generalizations of elliptic curves in dimension 2. For other directions of generalization, we refer to [11, 33], where analytic torsion produces the Siegel modular form characterizing the Andreotti-Mayer locus and the section of certain line bundle on the moduli space of Calabi-Yau threefolds characterizing the discriminant locus.

4.2 Borcherds Φ-Function as the Analytic Torsion of Enriques Surface

As in the case of elliptic curves, we choose some special Kähler metric to construct an invariant of an Enriques surface. Since $c_1(Y)_{\mathbf{R}} = 0$ for an Enriques surface Y, there exists by Yau [31] a unique Ricci-flat Kähler form in each Kähler class on Y. In contrast to elliptic curves, the condition of Ricci-flatness with normalized volume 1 does not determine a unique Kähler form on Y, because the space of Kähler classes on Y has real dimension 10. Even though, we get the following:

Theorem 6 ([34]) *Let Y be an Enriques surface and let γ be a Ricci-flat Kähler metric on Y with normalized volume* 1. *Then the analytic torsion $\tau(Y, \gamma)$ is independent of the choice of such a Kähler metric γ. In particular, $\tau(Y, \gamma)$ is an invariant of Y.*

After Theorem 6, we may write $\tau(Y)$ for $\tau(Y, \gamma)$. Then the analytic torsion gives rise to the function on the moduli space of Enriques surfaces

$$\tau : \mathcal{M} \ni [Y] \to \tau(Y) \in \mathbf{R}.$$

Recall that the Petersson norm of the Borcherds Φ-function $\|\Phi\|$ is $O^+(\Lambda)$-invariant and hence it descends to a function on $\mathcal{M}$. We write $\|\Phi(Y)\|$ for $\|\Phi(\varpi(\widetilde{Y}, \alpha))\|$.

Theorem 7 ([34]) *There exists an absolute constant $C \neq 0$ such that for every Enriques surface Y, the following equality holds*

$$\tau(Y) = C \left\| \Phi(Y) \right\|^{-1/4}.$$

The proofs of Theorems 6 and 7 are based on the curvature formula for (equivariant) Quillen metrics [4–6, 19] and the immersion formula for (equivariant) Quillen metrics [2, 3]. We compare the $\partial\bar{\partial}$ of $\log \tau$ and $\log \|\Phi\|$ as currents on the Baily-Borel compactification of $\Omega_{\Lambda}^{+}/O^{+}(\Lambda)$. For this, the curvature formula and the immersion formula for (equivariant) Quillen metrics play crucial roles. We refer the reader to [34] for the details of the proofs of Theorems 6 and 7.

As in the case of elliptic curves, we get an analytic expression of the Borcherds Φ-function by using analytic torsion. In fact, we can extend this result to arbitrary $K3$ surfaces with anti-symplectic involution. Namely, for a $K3$ surface X equipped with an involution $\iota\colon X \to X$ acting non-trivially on $H^0(X, \Omega_X^2)$, we can construct an invariant $\tau_M(X, \iota)$ by using the *equivariant* analytic torsion of (X, ι), the analytic torsion of the fixed-point-set of ι and a certain Bott-Chern secondary class. Here M refers to the isometry class of the invariant sublattice of $H^2(X, \mathbf{Z})$ with respect to the ι-action, which determines the topological type of ι. When $M = \mathbb{U}(2) \oplus \mathbb{E}_8(-2)$, we get the analytic torsion of Enriques surface τ as above. It is worth remarking that we can construct the invariant $\tau_M(X, \iota)$ without assuming the existence of Ricci-flat Kähler metrics on X. After fixing M, i.e., the topological type of the involution, the invariant $\tau_M(X, \iota)$ gives rise to a function on the moduli space of $K3$ surfaces with involution, which is again a certain arithmetic quotient of a symmetric bounded domain of type IV, with the discriminant divisor removed. As before in Theorem 7, the resulting function τ_M is the Petersson norm of an automorphic form on the moduli space of $K3$ surfaces with involution. It is remarkable that the corresponding automorphic form on the moduli space of $K3$ surfaces with involution thus obtained, is very often expressed as the product of a certain Borcherds lift and Igusa's Siegel modular form. We refer the reader to [34, 36] for more details about the analytic torsion invariant τ_M of $K3$ surfaces with involution.

5 Resultants and Borcherds Φ-Function: An Algebraic Counter Part

In this section, we explain an algebro-geometric counterpart of the Borcherds Φ-function.

5.1 (2, 2, 2)-Model of an Enriques Surface

Let

$$f_1(x), g_1(x), h_1(x) \in \mathbf{C}[x_1, x_2, x_3], \qquad f_2(x), g_2(x), h_2(x) \in \mathbf{C}[x_4, x_5, x_6]$$

be homogeneous polynomials of degree 2. We define $f, g, h \in \mathbf{C}[x_1, x_2, x_3, x_4, x_5, x_6]$ by

$$f(x) := f_1(x) + f_2(x), \qquad g(x) := g_1(x) + g_2(x), \qquad h(x) := h_1(x) + h_2(x)$$

and the corresponding surface $X_{(f,g,h)}$ by

$$X_{(f,g,h)} := \left\{[x] \in \mathbf{P}^5;\ f(x) = g(x) = h(x) = 0\right\}.$$

If the quadratic forms f_1, g_1, h_1, f_2, g_2, h_2 are generic enough, then $X_{(f,g,h)}$ equipped with the line bundle $\mathcal{O}_{\mathbf{P}^5}(1)$ is a $K3$ surface of degree 8 by the adjunction formula. Let ι be the involution on $\mathbf{C}^6$ defined as

$$\iota(x_1, x_2, x_3, x_4, x_5, x_6) := (x_1, x_2, x_3, -x_4, -x_5, -x_6).$$

The involution on $\mathbf{P}^5$ induced by ι is again denoted by the same symbol ι. Since the set of fixed points of the ι-action on $\mathbf{P}^5$ is the disjoint union of two projective planes $P_1 := \{x_1 = x_2 = x_3 = 0\}$ and $P_2 := \{x_4 = x_5 = x_6 = 0\}$, we see that $X^{\iota}_{(f,g,h)}$, the set of fixed points of the ι-action on $X_{(f,g,h)}$, is given by

$$X^{\iota}_{(f,g,h)} = (X_{(f,g,h)} \cap P_1) \amalg (X_{(f,g,h)} \cap P_2).$$

For three quadratic forms in three variables $q_1(x, y, z)$, $q_2(x, y, z)$, $q_3(x, y, z)$, let $R(q_1, q_2, q_3)$ be the resultant of q_1, q_2, q_3. Then $R(q_1, q_2, q_3)$ is the polynomial of degree 12 of the coefficients of q_1, q_2, q_3 characterizing the existence of common intersection points of the three conics of $\mathbf{P}^2$ defined by $q_1 = 0$, $q_2 = 0$ and $q_3 = 0$. Namely,

$$R(q_1, q_2, q_3) = 0 \quad \Longleftrightarrow \quad \left\{(x : y : z) \in \mathbf{P}^2;\ q_1 = q_2 = q_3 = 0\right\} \neq \emptyset.$$

If $q_i(x, y, z) = a_{i1}x^2 + a_{i2}y^2 + a_{i3}z^2 + a_{i4}xy + a_{i5}xz + a_{i6}yz$, then $R(q_1, q_2, q_3)$ is expressed as an explicit integral linear combination of the polynomials of the form

$$[j_1, j_2, j_3][k_1, k_2, k_3][l_1, l_2, l_3][m_1, m_2, m_3],$$

where

$$[j_1, j_2, j_3] := \begin{vmatrix} a_{1,j_1} & a_{1,j_2} & a_{1,j_3} \\ a_{2,j_1} & a_{2,j_2} & a_{2,j_3} \\ a_{3,j_1} & a_{3,j_2} & a_{3,j_3} \end{vmatrix}.$$

See [15, p. 215 Table 1] for an explicit formula for $R(q_1, q_2, q_3)$.

If the quadrics $f_1, g_1, h_1, f_2, g_2, h_2$ are generic enough, then we may assume that $R(f_1, g_1, h_1)R(f_2, g_2, h_2) \neq 0$, so that ι has no fixed points on $X_{(f,g,h)}$ in that case. Hence, if $R(f_1, g_1, h_1)R(f_2, g_2, h_2) \neq 0$ and $X_{(f,g,h)}$ is smooth, then

$$Y_{(f,g,h)} := X_{(f,g,h)}/\iota$$

is an Enriques surface. Let us see that a generic Enriques surface is constructed in this manner.

Assume that $R(f_1, g_1, h_1)R(f_2, g_2, h_2) \neq 0$ and that $X_{(f,g,h)}$ is smooth. For simplicity, set $X_0 := X_{(f,g,h)}$. Let $S := \mathrm{Gr}_3(\mathrm{Sym}^2\mathbf{C}^6) \cong \mathrm{Gr}_3(\mathbf{C}^{\binom{7}{2}})$ be the Grassmann variety of 3-dimensional subspaces in the vector space of quadratic forms in the variables $x_1, \ldots, x_6$. Then S is equipped with the ι-action induced from the one on $\mathbf{C}^6$ and with the $PGL(\mathbf{C}^6)$-action induced from the standard $GL(\mathbf{C}^6)$-action on $\mathbf{C}^6$. By choosing $f_1, g_1, h_1, f_2, g_2, h_2$ generic enough, we may assume that $\mathfrak{sl}(\mathbf{C}^6)$ is a subspace of the tangent space of S at the point $\mathrm{Span}\{f, g, h\} \in S$.

For $s \in S$, we define $X_s := \{[x] \in \mathbf{P}^5; q(x) = 0\,(\forall q \in s)\}$. Then we get a flat family $\pi : X \to S$ with $\pi^{-1}(s) = X_s$. Write $[X_0] \in S$ for $\mathrm{Span}\{f, g, h\} \in S$. We get a flat deformation $\pi : (X, X_0) \to (S, [X_0])$ of $K3$ surfaces of degree 8. Since ι preserves X_0 and hence $\iota([X_0]) = [X_0]$, we get a subfamily $\pi : (X|_{S^\iota}, \iota, X_0) \to (S^\iota, [X_0])$ of $K3$ surfaces with involution, where $S^\iota := \{s \in S; \iota(s) = s\}$ is the fixed-point-set of the ι-action on S. Since ι has no fixed points on X_0 by assumption and since the set of fixed points of the ι-action on X is a closed subset of X, we see that ι has no fixed points on X_s if $s \in S^\iota$ is sufficiently close to $[X_0]$. We define $Y := (X|_{S^\iota})/\iota$ and $Y_0 := X_0/\iota$. Let $p : Y \to S$ be the projection induced from $\pi : X \to S$. Since ι has no fixed points on X_s, Y_s is an Enriques surface for $s \in S$ sufficiently close to $[X_0]$. Hence $p : (Y, Y_0) \to (S^\iota, [X_0])$ is a flat deformation of Y_0.

Let $\rho_{X_0} : T_{[X_0]}S \to H^1(X_0, \Theta_{X_0})$ and $\rho_{Y_0} : T_{[X_0]}S^\iota \to H^1(Y_0, \Theta_{Y_0})$ be the Kodaira-Spencer maps of the deformations $\pi : (X, X_0) \to (S, [X_0])$ and $p : (Y, Y_0) \to (S^\iota, [X_0])$, respectively. Let $(T_{[X_0]}S)_+$ and $H^1(X_0, \Theta_{X_0})_+$ be the invariant subspaces of $T_{[X_0]}S$ and $H^1(X_0, \Theta_{X_0})$ with respect to the ι-action, respectively. Since ρ_{X_0} commutes with the ι-action, we set $(\rho_{X_0})_+ := \rho_{X_0}|_{(T_{[X_0]}S)_+} : (T_{[X_0]}S)_+ \to H^1(X_0, \Theta_{X_0})_+$. Since $(\rho_{X_0})_+$ can be identified with ρ_{Y_0} under the identifications $(T_{[X_0]}S)_+ = T_{[X_0]}S^\iota$ and $H^1(X_0, \Theta_{X_0})_+ = H^1(Y_0, \Theta_{Y_0})$, we get

$$\ker \rho_{Y_0} \cong \ker(\rho_{X_0})_+ = \mathfrak{sl}\big(\mathbf{C}^6\big) \cap \ker(\iota_* - 1) \cong \mathfrak{sl}\big(\mathbf{C}^3\big) \oplus \mathfrak{sl}\big(\mathbf{C}^3\big) \oplus \mathbf{C} \cong \mathbf{C}^{17}.$$

Here the second equality follows from the equality $\ker \rho_{X_0} = \mathfrak{sl}(\mathbf{C}^6)$, which is a consequence of the fact that $X_s \cong X_{s'}$ as polarized $K3$ surfaces of degree 8 if and only if s and s' lie on the same $PGL(\mathbf{C}^6)$-orbit. (We can also see the equality $\ker \rho_{X_0} = \mathfrak{sl}(\mathbf{C}^6)$ as follows. Set $\mathcal{L}_0 := \mathcal{O}_{\mathbf{P}^5}(1)|_{X_0}$. We consider the semiuniversal deformation $q : ((\mathfrak{X}, \mathcal{L}), (X_0, \mathcal{L}_0)) \to (\mathrm{Def}(X_0, \mathcal{L}_0), [X_0])$ of the polarized $K3$ surface (X_0, L_0) of degree 8. Since $\mathcal{L}_0$ is very ample on X_0, we may assume that $\mathcal{L}$ is very ample on $\mathfrak{X}_t$ for $t \in \mathrm{Def}(X_0, \mathcal{L}_0)$. Since $\deg \mathcal{L}|_{\mathfrak{X}_t} = 8$, the image of the projective embedding $\Phi_{|\mathcal{L}|_{\mathfrak{X}_t}|} : \mathfrak{X}_t \to \mathbf{P}^5$ must be a $(2, 2, 2)$-complete intersection. Namely, $(\mathfrak{X}_t, \mathcal{L}|_{\mathfrak{X}_t})$ is isomorphic to $(X_s, \mathcal{O}_{\mathbf{P}^5}(1))$ for some $s \in S$. Hence the deformation germ of polarized $K3$ surfaces $\pi: (X, X_0) \to (S, [X_0])$ is complete, which implies the equality $\dim\ker\rho_{X_0} = \dim S - \dim \mathrm{Def}(X_0, \mathcal{L}_0) = 35 = \dim \mathfrak{sl}(\mathbf{C}^6)$. This, together with the inclusion $\mathfrak{sl}(\mathbf{C}^6) \subset \ker\rho_{X_0}$, yields the equality $\ker\rho_{X_0} = \mathfrak{sl}(\mathbf{C}^6)$.)

Since $\dim S^{\iota} = 27$ and $\dim\ker\rho_{Y_0} = 17$, we get $\dim\mathrm{Im}\,\rho_{Y_0} = 27 - 17 = 10 = \dim H^1(Y_0, \Theta_{Y_0})$. Hence the Kodaira-Spencer map ρ_{Y_0} is surjective and the family $p\colon (Y, Y_0) \to (S^{\iota}, [X_0])$ is complete.

Set $U := \{s \in S^{\iota};\ \mathrm{Sing}\, X_s = X_s^{\iota} = \emptyset\}$. Then U is a Zariski open subset of S^{ι}. For $s \in U$, $Y_s = X_s/\iota$ is an Enriques surface. Let $\varpi: U \ni s \to \varpi(X_s/\iota) \in \mathcal{M}$ be the period mapping for the family of Enriques surfaces $p\colon Y|_U \to U$. By the Borel-Kobayashi-Ochiai extension theorem, ϖ extends to a rational map from S^{ι} to the Baily-Borel compactification of $\Omega^+_{\Lambda}/O^+(\Lambda)$. By the completeness of the deformation germ $p\colon (Y, Y_0) \to (S^{\iota}, [X_0])$, the image of ϖ contains a dense Zariski open subset of $\mathcal{M}$, say $\mathcal{U}$. If Y is an Enriques surface with $\varpi(Y) \in \mathcal{U}$, then $Y = Y_{(F,G,H)}$ for some quadratic forms F, G, H.

5.2 An Algebraic Expression of Borcherds Φ-Function

Since we have a nice projective model of Enriques surfaces of degree 4, it is natural to expect that the Borcherds Φ-function may admit an algebraic expression analogous to the one for the Dedekind η-function associated to the plane cubic model or the $(2, 2)$-complete intersection model. In fact, this is the case.

Theorem 8 ([16]) *Let $Y_{(f,g,h)}$ be the $(2, 2, 2)$-model of an Enriques surface defined by the quadric polynomials $f = f_1 + f_2$, $g = g_1 + g_2$, $h = h_1 + h_2 \in \mathbf{C}[x_1, x_2, x_3, x_4, x_5, x_6]$. Then the following equality holds*

$$\left\|\Phi(Y_{(f,g,h)})\right\|^2 = \left|R(f_1, g_1, h_1)R(f_2, g_2, h_2)\right| \left(\frac{2}{\pi^4}\int_{X_{(f,g,h)}} \alpha_{(f,g,h)} \wedge \overline{\alpha_{(f,g,h)}}\right)^4.$$

Here $\alpha_{(f,g,h)} \in H^0(X_{(f,g,h)}, \Omega^2_{X_{(f,g,h)}})$ is defined as the residue of f, g, h, i.e.,

$$\alpha_{(f,g,h)} := \Xi|_{X_{(f,g,h)}},$$

where Ξ is a meromorphic 2-form on $\mathbf{P}^5$ satisfying the equation

$$df \wedge dg \wedge dh \wedge \Xi = \sum_{i=1}^{6} (-1)^i x_i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_6.$$

We remark that a weaker version of this result was obtained by Maillot-Roessler [20] under a certain arithmeticity assumption on $X_{(f,g,h)}$. In their formula, the contribution from the resultants is understood as the contribution from the bad primes with respect to the reductions of $X_{(f,g,h)}$. When f, g, h are defined over the ring of integers of a number field K, Theorem 8 implies that the Borcherds Φ-function detects the degenerations of ι over $\mathrm{Spec}(\mathcal{O}_K)$, since $R(f_1, g_1, h_1)R(f_2, g_2, h_2) \in \mathfrak{p}$ for a prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K)$ if and only if ι has non-empty fixed points on the reduction $X_{(f,g,h)}(\mathcal{O}_K/\mathfrak{p})$. This picture of the Borcherds Φ-function is quite analogous to the corresponding picture of the Dedekind η-function: For an elliptic curve $E = \{y^2 = 4x^3 - g_2 x - g_3\}$ over K, $\|\eta\|^{24}$ is identified with the discriminant of E up to the L^2-norm of dx/y. Hence the algebraic part of $\|\eta\|$ detects the degenerations of E over $\mathrm{Spec}(\mathcal{O}_K)$. See [10] for more explanation of this view point.

The proof of Theorem 8 shall be given in [16]. The strategy is as follows. We compare the $\partial\bar{\partial}$ of the both hand sides as currents on S. Then it turns out that they satisfy the same $\partial\bar{\partial}$-equation of currents on S. For this, we use Theorem 7 and a formula for the asymptotic behavior of equivariant analytic torsion for degenerating family of algebraic manifolds [35]. In this way, we get the desired equality, up to an absolute constant. To fix the absolute constant, we compare the behavior of the both hand sides for certain explicit 2-parameter family of Enriques surfaces, whose universal coverings are Kummer surfaces of product type.

In fact, Theorem 8 holds even if $Y_{(f,g,h)}$ has at most rational double points by the continuity of the both hand sides at those points of S^{ι} corresponding to Enriques surfaces with rational double points. This continuity is a consequence of the existence of simultaneous resolution of 2-dimensional rational double points.

By Theorem 8, we get a Thomae type formula for the Borcherds Φ-function.

Corollary 1 ([16]) *Let* $\mathbf{v}, \mathbf{v}' \in H^2(X_{(f,g,h)}, \mathbf{Z})$ *be anti-ι-invariant, primitive, isotropic vectors with* $\langle \mathbf{v}, \mathbf{v}' \rangle = 1$ *and let* $\mathbf{v}^{\vee} \in H_2(X_{(f,g,h)}, \mathbf{Z})$ *be the Poincaré dual of* $\mathbf{v}$. *Under the identification of lattices* $(\mathbf{Z}\mathbf{v} + \mathbf{Z}\mathbf{v}')^{\perp} \cong \mathbb{U}(2) \oplus \mathbb{E}_8(-2) =: L$, *the vector*

$$z_{(f,g,h),\mathbf{v},\mathbf{v}'} := \frac{\alpha - \langle \alpha, \mathbf{v}' \rangle \mathbf{v} - \langle \alpha, \mathbf{v} \rangle \mathbf{v}'}{\langle \alpha, \mathbf{v} \rangle} \in L \otimes \mathbf{R} + i\, \mathcal{C}_L^{+}$$

is regarded as the period of $Y_{(f,g,h)}$. *Then, by a suitable choice of the* 2*-cocycles* $\{\mathbf{v}, \mathbf{v}'\}$, *one has*

$$\Phi(z_{(f,g,h),\mathbf{v},\mathbf{v}'})^2 = R(f_1, g_1, h_1)R(f_2, g_2, h_2) \left(\frac{2}{\pi^2} \int_{\mathbf{v}^{\vee}} \alpha_{(f,g,h)} \right)^8.$$

When $X_{(f,g,h)}$ is birational to a Kummer surface of product type, the 2-cycle $\mathbf{v}^{\vee}$ can be given explicitly. See [16] for the details.

5.3 A 4-Parameter Family of Enriques Surfaces Associated to $M_{3,6}(\mathbf{C})$

For a non-zero 3×6-complex matrix $A \in M_{3,6}(\mathbf{C})$, we define

$$X_A := \left\{ [x] \in \mathbf{P}^5;\ \begin{array}{l} f(x) = a_{11}x_1^2 + a_{12}x_2^2 + a_{13}x_3^2 + a_{14}x_4^2 + a_{15}x_5^2 + a_{16}x_6^2 = 0 \\ g(x) = a_{21}x_1^2 + a_{22}x_2^2 + a_{23}x_3^2 + a_{24}x_4^2 + a_{25}x_5^2 + a_{26}x_6^2 = 0 \\ h(x) = a_{31}x_1^2 + a_{32}x_2^2 + a_{33}x_3^2 + a_{34}x_4^2 + a_{35}x_5^2 + a_{36}x_6^2 = 0 \end{array} \right\}.$$

For $A = (\mathbf{a}_1, \ldots, \mathbf{a}_6) \in M(3, 6; \mathbf{C})$ and $i < j < k$, we define

$$\Delta_{ijk}(A) = \det(\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k).$$

A matrix $A \in M(3, 6; \mathbf{C})$ is said to be *non-degenerate* if $\prod_{i<j<k} \Delta_{ijk}(A) \neq 0$. Then, for a non-degenerate $A \in M_{3,6}(\mathbf{C})$, X_A is a $K3$ surface. We write α_A for $\alpha_{(f,g,h)}$. As an immediate consequence of Theorem 8, we get the following:

Corollary 2 ([16]) *Let $A \in M_{3,6}(\mathbf{C})$ be non-degenerate. For a partition of* 6 *letters* $\{1, 2, 3, 4, 5, 6\}$

$$\binom{ijk}{lmn} := \{i, j, k\} \cup \{l, m, n\} = \{1, 2, 3, 4, 5, 6\},$$

define an involution $\iota_{\binom{ijk}{lmn}}$ on $\mathbf{P}^5$ by

$$\iota_{\binom{ijk}{lmn}}(x_i, x_j, x_k, x_l, x_m, x_n) = (x_i, x_j, x_k, -x_l, -x_m, -x_n).$$

Then $\iota_{\binom{ijk}{lmn}}$ is a free involution on X_A called a switch *such that*

$$\left\| \Phi(X_A/\iota_{\binom{ijk}{lmn}}) \right\|^2 = \left|\Delta_{ijk}(A)\right|^4 \left|\Delta_{lmn}(A)\right|^4 \left(\frac{2}{\pi^4} \int_{X_A} \alpha_A \wedge \overline{\alpha}_A \right)^4.$$

By Corollary 2 , if $A \in M_{3,6}(K)$ with $K \subset \mathbf{C}$, then for any partitions $\binom{ijk}{lmn}$ and $\binom{i'j'k'}{l'm'n'}$, one has

$$\frac{\|\Phi(X_A/\iota_{\binom{ijk}{lmn}})\|^2}{\|\Phi(X_A/\iota_{\binom{i'j'k'}{l'm'n'}})\|^2} = \frac{|\Delta_{ijk}(A)|^4 |\Delta_{lmn}(A)|^4}{|\Delta_{i'j'k'}(A)|^4 |\Delta_{l'm'n'}(A)|^4} \in K.$$

Since $|\Delta_{ijk}(A)|^4 |\Delta_{lmn}(A)|^4 / |\Delta_{i'j'k'}(A)|^4 |\Delta_{l'm'n'}(A)|^4 \neq 1$ for all pairs of partitions $\binom{ijk}{lmn}$, $\binom{i'j'k'}{l'm'n'}$ for generic non-degenerate A, we conclude that all of the 10 Enriques surfaces $X_A/\iota_{\binom{ijk}{lmn}}$ are mutually distinct for a generic choice of A.

6 Theta Function and Borcherds Φ-Function

In this section, we explain a relation between the Borcherds Φ-function and Freitag's theta function.

6.1 The Matsumoto-Sasaki-Yoshida Model

Recall that, for $A \in M_{2,4}(\mathbf{C})$, we could associate two distinct models E_A and C_A of an elliptic curve. By a similar construction, we can associate another $K3$ surface to $A \in M_{3,6}(\mathbf{C})$ as follows. For $A \in M_{3,6}(\mathbf{C})$, define a $K3$ surface

$$Z_A := \left\{ \big((x_1 : x_2 : x_3), y\big) \in \mathcal{O}_{\mathbf{P}^2}(3);\ y^2 = \prod_{i=1}^{6} (a_{1i}x_1 + a_{2i}x_2 + a_{3i}x_3) \right\},$$

which is identified with its minimal resolution. Then Z_A is (the minimal resolution of) the double covering of $\mathbf{P}^2$, whose branch divisor is the union of 6 lines in general position $a_{1i}x_1 + a_{2i}x_2 + a_{3i}x_3 = 0$ $(i = 1, \ldots, 6)$. The period mapping and its inverse for the family of $K3$ surfaces Z_A over a certain open subset of $M_{3,6}(\mathbf{C})$ were worked out by Matsumoto-Sasaki-Yoshida [23] and Matsumoto [21].

We define a holomorphic 2-form η_A on Z_A by

$$\eta_A := \frac{x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2}{y}.$$

By Matsumoto-Sasaki-Yoshida [23], there are 6 independent transcendental 2-cycles $\{\gamma_{ij}\}_{1 \le i < j \le 4}$ on Z_A and 16 independent algebraic 2-cycles on Z_A, which form a basis of $H_2(Z_A, \mathbf{Q})$.

Following Matsumoto-Sasaki-Yoshida [23], define the period of Z_A as the matrix

$$\Omega_A := \frac{1}{\eta_{34}(A)} \begin{pmatrix} \eta_{14}(A) & -\frac{\eta_{13}(A) - \sqrt{-1}\eta_{24}(A)}{1+\sqrt{-1}} \\ -\frac{\eta_{13}(A) + \sqrt{-1}\eta_{24}(A)}{1-\sqrt{-1}} & -\eta_{23}(A) \end{pmatrix},$$

where

$$\eta_{ij}(A) := \int_{\gamma_{ij}} \eta_A.$$

By a suitable choice of the cycles $\{\gamma_{ij}\}_{1 \le i < j \le 4}$, one has

$$\Omega_A \in \mathbb{D} := \left\{ T \in M_{2,2}(\mathbf{C});\ \big(T - {}^t\overline{T}\big)/2i > 0 \right\},$$

where $\mathbb{D}$ is isomorphic to a symmetric bounded domain of type IV of dimension 4.

6.2 Theta Function on $\mathbb{D}$

Write $\mathbf{e}(x) := \exp(2\pi i x)$.

Definition 5 For $\Omega \in \mathbb{D}$ and $a, b \in \mathbf{Z}[i]^2$, define the Freitag theta function as

$$\begin{aligned}&\Theta_{\frac{a}{1+i},\frac{b}{1+i}}(\Omega)\\ &\quad := \sum_{n\in\mathbb{Z}[i]^2} \mathbf{e}\left[\frac{1}{2}\left(n+\frac{a}{1+i}\right)\Omega\,{}^t\overline{\left(n+\frac{a}{1+i}\right)} + \Re\left(n+\frac{a}{1+i}\right){}^t\overline{\left(\frac{b}{1+i}\right)}\right].\end{aligned}$$

Following [32], we identify the characteristic $\binom{a}{b}$ with the partition $\binom{ijk}{lmn}$ by the rule:

$$\begin{array}{cccccccccccc}
\binom{a}{b} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} i & i \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} i & i \\ i & i \end{pmatrix} & \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ i & i \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
\binom{ijk}{lmn} & \binom{123}{456} & \binom{124}{356} & \binom{125}{346} & \binom{126}{345} & \binom{134}{256} & \binom{135}{246} & \binom{136}{245} & \binom{145}{236} & \binom{146}{235} & \binom{156}{234}
\end{array}$$

Under this identification, we define

$$\Theta_{\binom{ijk}{lmn}}(\Omega) := \Theta_{\frac{a}{1+i},\frac{b}{1+i}}(\Omega)$$

and its Petersson norm by

$$\left\|\Theta_{\binom{ijk}{lmn}}(\Omega)\right\|^2 := \det\left(\frac{\Omega - {}^t\overline{\Omega}}{2\sqrt{-1}}\right)\left|\Theta_{\binom{ijk}{lmn}}(\Omega)\right|^2.$$

Theorem 9 ([16]) *For a non-degenerate* $A = (A_1, A_2) \in M_{3,6}(\mathbf{C})$ *with* $A_1, A_2 \in M_3(\mathbf{C})$, *define*

$$A^\vee := \left({}^tA_1^{-1}, {}^tA_2^{-1}\right).$$

Then

$$\left\|\Phi(X_A/\iota_{\binom{ijk}{lmn}})\right\| = \left\|\Theta_{\binom{ijk}{lmn}}(Z_{A^\vee})\right\|^4.$$

The proof of Theorem 9 shall be given in [16]. We use Matsumoto-Terasoma's Thomae type formula [22] to rewrite the right hand side of Theorem 9. Comparing this with Theorem 8, we get the result. See [16] for the details. We remark that, after Freitag-Salvati-Manni [12, Th. 5.6], Theorem 9 is not very surprising, because they proved that the Borcherds Φ-function itself is expressed as a linear combination of certain additive Borcherds lifts.

6.3 The Case of Jacobian Kummer Surfaces

For $\lambda = (\lambda_1, \dots, \lambda_6) \in \mathbf{C}^6$ with $\lambda_i \neq \lambda_j$ $(i \neq j)$, define a genus 2 curve C_λ by the affine equation

$$C_\lambda := \left\{ (x, y) \in \mathbf{C}^2;\ y^2 = \prod_{i=1}^{6} (x - \lambda_i) \right\}.$$

Define holomorphic differentials ω_1 and ω_2 on C_λ by

$$\omega_1 := \frac{dx}{y}, \qquad \omega_2 := \frac{x\,dx}{y}.$$

Let $\{A_1, A_2, B_1, B_2\}$ be a certain symplectic basis of $H_1(C_\lambda, \mathbf{Z})$ and set

$$T_\lambda := \begin{pmatrix} \int_{B_1} \omega_1 & \int_{B_2} \omega_1 \\ \int_{B_1} \omega_2 & \int_{B_2} \omega_2 \end{pmatrix}^{-1} \begin{pmatrix} \int_{A_1} \omega_1 & \int_{A_2} \omega_1 \\ \int_{A_1} \omega_2 & \int_{A_2} \omega_2 \end{pmatrix} \in \mathfrak{S}_2.$$

Then the Kummer surface $K(C_\lambda)$ of the Jacobian variety $\mathrm{Jac}(C_\lambda)$ is expressed as follows:

$$K(C_\lambda) \cong X_A, \qquad A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \end{pmatrix} \in M_{3,6}(\mathbf{C}).$$

By Theorem 9, we get the following.

Corollary 3 ([16]) *If the partition* $\binom{pqr}{stu}$ *corresponds to the characteristic* (a, b)*, then*

$$\left\| \Phi\big(K(C_\lambda)/\iota_{\binom{pqr}{stu}}\big) \right\| = (\det \Im T_\lambda)^2 \left| \theta_{\Re(\frac{a}{1+i}), \Re(\frac{b}{1+i})}(T_\lambda)\, \theta_{\Im(\frac{a}{1+i}), \Im(\frac{b}{1+i})}(T_\lambda) \right|^4.$$

Here $\theta_{\alpha,\beta}(T)$, $\alpha, \beta \in \{0, 1/2\}^2$, *is the Riemann theta constant*

$$\theta_{\alpha,\beta}(T) := \sum_{n \in \mathbf{Z}^2} \mathbf{e}\left[\frac{1}{2}(n+\alpha) T\, {}^t\overline{(n+\alpha)} + (n+\alpha)\, {}^t\overline{\beta} \right], \quad T \in \mathfrak{S}_2.$$

Recall that Igusa's Siegel modular form Δ_5 is defined as the product of all even theta constants

$$\Delta_5(T) := \prod_{(\alpha,\beta)\ \text{even}} \theta_{\alpha,\beta}(T), \quad T \in \mathfrak{S}_2.$$

For a genus 2 curve C with period $T \in \mathfrak{S}_2$, its Petersson norm

$$\left\| \Delta_5(C) \right\|^2 := (\det \Im T)^5 \left| \Delta_5(T) \right|^2$$

is independent of the choice of a symplectic basis of $H_1(C, \mathbf{Z})$. Hence $\|\Delta_5(C)\|$ is an invariant of C. Form Corollary 3, it follows the following:

Corollary 4 ([16]) *The Igusa cusp form Δ_5 is the average of Φ with respect to the* 10 *switches, i.e.,*

$$\prod_{\binom{ijk}{lmn}} \left\| \Phi\big(K(C)\big)/\iota_{\binom{ijk}{lmn}} \right\| = \left\| \Delta_5(C) \right\|^8.$$

7 Some Problems

Problem 1 For elliptic curves, two distinct models E_A and C_A yield distinct algebro-geometric expressions of $\|\eta\|$. For projective models of Enriques surfaces distinct from the $(2, 2, 2)$-complete intersection of $\mathbf{P}^5$, find the corresponding algebro-geometric expressions of $\|\Phi\|$.

Problem 2 On a generic Jacobian Kummer surface, there exists 31 conjugacy classes of free involutions ([25, 28]), which split into three families:

- 10 switches,
- 15 Hutchinson-Göpel involutions,
- 6 Hutchinson-Weber involutions.

Recall that, as the average of the Borcherds Φ-function by 10 switches, we get Igusa's Siegel modular form Δ_5. Determine the Siegel modular form constructed as the average of the Borcherds Φ-function by the 15 Hutchinson-Göpel involutions (resp. 6 Hutchinson-Weber involutions).

Problem 3 As mentioned in Sect. 4.2, there exists an analytic torsion invariant τ_M for $K3$ surfaces with involution [34], which is often expressed as the Petersson norm of the tensor product of an explicit Borcherds lift and Igusa's Siegel modular form [36]. After Theorem 8, it is an interesting problem to find an algebro-geometric expression of τ_M for general M.

Problem 4 (The inverse of the period mapping for Enriques surfaces) For elliptic curves, the inverse of the period mapping was constructed by Jacobi by using theta constants. We ask the same problem for the $(2, 2, 2)$-model of Enriques surfaces: For $1 \leq i < j \leq 3$ and $4 \leq k < l \leq 6$, find a system of automorphic forms

$$\alpha_{ij}^{(1)}(Z),\ \alpha_{kl}^{(2)}(Z),\ \beta_{ij}^{(1)}(Z),\ \beta_{kl}^{(2)}(Z),\ \gamma_{ij}^{(1)}(Z),\ \gamma_{kl}^{(2)}(Z)$$

on Ω_{Λ}^+ for (a finite index subgroup of) $O^+(\Lambda)$ such that

$$Y_Z := X_Z/\iota, \quad \iota(x) = (x_1, x_2, x_3, -x_4, -x_5, -x_6)$$

is the Enriques surface whose period is the given by $Z \in \Omega^{+}_{\Lambda}$. Here

$$X_Z = \left\{ [x] \in \mathbf{P}^5;\ \begin{array}{l} \sum_{1\le i<j\le 3} \alpha^{(1)}_{ij}(Z) x_i x_j + \sum_{4\le k<l\le 6} \alpha^{(2)}_{kl}(Z) x_k x_l = 0 \\ \sum_{1\le i<j\le 3} \beta^{(1)}_{ij}(Z) x_i x_j + \sum_{4\le k<l\le 6} \beta^{(2)}_{kl}(Z) x_k x_l = 0 \\ \sum_{1\le i<j\le 3} \gamma^{(1)}_{ij}(Z) x_i x_j + \sum_{4\le k<l\le 6} \gamma^{(2)}_{kl}(Z) x_k x_l = 0 \end{array} \right\}.$$

Kondō [18] and Freitag-Salvati-Manni [12] constructed certain (birational) projective embeddings of the moduli space of Enriques surfaces with some level structure. Are the system of automorphic forms appearing in their embeddings regarded as the set of coefficients of the defining equations of appropriately polarized Enriques surfaces?

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