Kenji Iohara
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Symmetries, Integrable Systems and Representations

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Dedicated to Prof. Michio Jimbo for his Sixtieth Anniversary

Preface

This is the joint proceedings of the two conferences:

1. **Infinite Analysis 11—Frontier of Integrability—**
   University of Tokyo, Japan in July 25th to 29th, 2011,
2. **Symmetries, Integrable Systems and Representations**
   Université Claude Bernard Lyon 1, France in December 13th to 16th, 2011.

As both of the conferences had been organized in the occasion of 60th anniversary of Prof. Michio Jimbo, the topics covered in this proceedings are very large. Indeed, it includes combinatorics, differential equations, integrable systems, probability, representation theory, solvable lattice models, special functions etc. We hope this volume might be interesting and useful both for young researchers and experienced specialists in these domains.

We shall mention about the financial supports we had; the conference at Tokyo was supported in part by Global COE programme "The research and training center for new development in mathematics" (Graduate School of Mathematical Science, University of Tokyo), and the conference at Lyon was supported by Institut Universitaire de France, GDR 3395 'Théorie de Lie algébrique et géométrique', GDRE 571 'Representation theory', Université Lyon 1 and Université Paris 6.

Lion, France

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Contents

A Presentation of the Deformed $W_{1+\infty}$ Algebra

N. Arbesfeld and O. Schiffmann

Abstract We provide a generators and relation description of the deformed $W_{1+\infty}$-algebra introduced in previous joint work of E. Vasserot and the second author. This gives a presentation of the (spherical) cohomological Hall algebra of the one-loop quiver, or alternatively of the spherical degenerate double affine Hecke algebra of $GL(\infty)$.

1 Introduction

In the course of their work on the cohomology of the moduli space of $U(r)$-instantons on $\mathbb{P}^2$ in relation to W-algebras and the AGT conjecture (see [6]) E. Vasserot and the second author introduced a certain one-parameter deformation $\mathbf{SH^c}$ of the enveloping algebra of the Lie algebra $W_{1+\infty}$ of algebraic differential operators on $\mathbb{C}^*$. The algebra $\mathbf{SH^c}$—which is defined in terms of Cherednik's double affine Hecke algebras—acts on the above mentioned cohomology spaces (with a central character depending on the rank n of the instanton space). For the same value of the central character, $\mathbf{SH^c}$ is also strongly related to the affine W algebra of type $\mathfrak{gl}_n$, and has the same representation theory (of admissible modules) as the latter. The same algebra $\mathbf{SH^c}$ arises again as the (spherical) cohomological Hall algebra of the quiver with one vertex and one loop, and as a degeneration of the (spherical) elliptic Hall algebra (see [6, Sects. 4, 8]. It also independently appears in the work of Maulik and Okounkov on the AGT conjecture, see [5].

The definition of $\mathbf{SH^c}$ given in [6] is in terms of a stable limit of spherical degenerate double affine Hecke algebras, and does not yield a presentation by generators and relations. In this note, we provide such a presentation, which bears some resemblance with Drinfeld's new realization of quantum affine algebras and Yangians.

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Namely, we show that $\mathbf{SH^c}$ is generated by families of elements in degrees $-1, 0, 1$, modulo some simple quadratic and cubic relations (see Theorems 1, 2).

The definition of $\mathbf{SH^c}$ is recalled in Sect. 2. In the short Sect. 3 we briefly recall the links between $\mathbf{SH^c}$ and Cherednik algebras, resp. W-algebras. The presentation of $\mathbf{SH^c}$ is given in Sect. 4, and proved in Sect. 5. Although we have tried to make this note as self-contained as possible, there are multiple references to statements in [6] and the reader is advised to consult that paper (especially Sects. 1 and 8) for details.

2 Definition of $\mathbf{SH^c}$

2.1 Symmetric Functions and Sekiguchi Operators

Let κ be a formal parameter, and let us set $F = \mathbb{C}(\kappa)$. Let us denote by Λ_F the ring of symmetric polynomials in infinitely many variables with coefficients in F, i.e.

$$\Lambda_F = F[X_1, X_2, \ldots]^{\mathfrak{S}_\infty} = F[p_1, p_2, \ldots].$$

For λ a partition, we denote by J_λ the integral form of the Jack polynomial associated to λ and to the parameter $\alpha = 1/\kappa$. The integral form J_λ is characterized by the following relation:

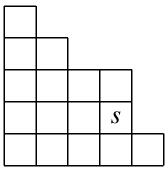
$$J_\lambda \in \bigoplus_{(1^n)<\mu\leqslant\lambda} F m_\mu + |\lambda|! m_{(1^n)}$$

where m_μ denotes the monomial symmetric function associated to a partition μ.

It is well-known that $\{J_\lambda\}$ forms a basis of Λ_F (see e.g. [7], or [6, Sects. 1.3, 1.6]). The polynomials J_λ arise as the joint spectrum of a family of commuting differential operators $\{D_{0,l}\}, l \geq 1$ called Sekiguchi operators. We will not need the expression of $D_{0,l}$ as a differential operator, but only their eigenvalues on the basis of Jack polynomials (which, of course, fully characterizes them):

$$D_{0,l}(J_\lambda) = \sum_{s\in\lambda} c(s)^{l-1} J_\lambda \tag{1}$$

where s runs through the set of boxes in the partition λ, and where $c(s) = x(s) - \kappa y(s)$ is the content of s. Here $x(s), y(s)$ denote the x and y-coordinates of the box s, when λ is drawn according to the continental convention. For example, for the box s in the partition $(5, 4^2, 2, 1)$ depicted below



we have $x(s) = 3$ and $y(s) = 1$ hence $c(s) = 3 - \kappa$.

We denote by $D_{l,0} \in \mathrm{End}(\Lambda_F)$ the operator of multiplication by the power-sum function p_l.

2.2 *The Algebras* $\mathbf{SH}^+$ *and* $\mathbf{SH}^>$

Let $\mathbf{SH}^+$ be the unital subalgebra of $\mathrm{End}(\Lambda_F)$ generated by $\{D_{0,l}, D_{l,0} \mid l \geqslant 1\}$. For $l \geq 1$ we set $D_{1,l} = [D_{0,l+1}, D_{1,0}]$. This relation is still valid when $l = 0$, and we furthermore have

$$[D_{0,l}, D_{1,k}] = D_{1,k+l-1}, \quad l \geqslant 1, k \geqslant 0. \tag{2}$$

We denote by $\mathbf{SH}^>$ the unital subalgebra of $\mathbf{SH}^+$ generated by $\{D_{1,l} \mid l \geqslant 0\}$, and by $\mathbf{SH}^0$ the unital subalgebra of $\mathbf{SH}^+$ generated by the Sekiguchi operators $\{D_{0,l} \mid l \geqslant 1\}$. It is known (and easy to check from (1)) that the $D_{0,l}$ are algebraically independent, i.e. $\mathbf{SH}^0 = F[D_{0,1}, D_{0,2}, \ldots]$.

Observe that by (2), the operators $ad(D_{0,l})$ preserve the subalgebra $\mathbf{SH}^>$. This allows us to view $\mathbf{SH}^+$ as a semi-direct product of $\mathbf{SH}^0$ and $\mathbf{SH}^>$. In fact, the multiplication map induces an isomorphism

$$\mathbf{SH}^> \otimes \mathbf{SH}^0 \simeq \mathbf{SH}^+ \tag{3}$$

(see [6, Proposition 1.18]).

2.3 *Grading and Filtration*

The algebra $\mathbf{SH}^+$ carries an $\mathbb{N}$-grading, defined by setting $D_{0,l}, D_{1,k}$ in degrees zero and one respectively. This grading, which corresponds to the degrees as operators on polynomials will be called the *rank* grading. It also carries an $\mathbb{N}$-filtration compatible with the rank grading, induced from the filtration by the order of differential operators. It may alternatively be characterized as follows, see [6, Proposition 1.2]: $\mathbf{SH}^+[\leqslant d]$ is the space of elements $u \in \mathbf{SH}^>$ satisfying

$$ad(z_1) \circ \cdots \circ ad(z_{d+1})(u) = 0$$

for all $z_1, \ldots, z_{d+1} \in F[D_{1,0}, D_{2,0}, \ldots]$. We have $\mathbf{SH}^>[\leqslant 0] = F[D_{1,0}, D_{2,0}, \ldots]$. The following is proved in [6, Lemma 1.21]. Set $D_{r,d} = [D_{0,d+1}, D_{r,0}]$ for $r \geq 1, d \geq 0$.

Proposition 1 (i) *The associated graded algebra* $gr\,\mathbf{SH}^+$ *is equal to the free commutative polynomial algebra in the generators* $D_{r,d} \in gr\,\mathbf{SH}^+[r, d]$, *for* $r \geqslant 0, d \geqslant 0$, $(r, d) \neq (0, 0)$.

(ii) *The associated graded algebra* $gr\,\mathbf{SH}^>$ *is equal to the free commutative polynomial algebra in the generators* $D_{r,d} \in gr\,\mathbf{SH}^+[r, d]$, *for* $r \geqslant 1, d \geqslant 0$.

We will need the following slight variant of the above result, which can easily be deduced from [6, Proposition 1.38]. For $r \geqslant 1$, set $D'_{r,d} = ad(D_{0,2})^d(D_{r,0})$. Then

$$D'_{r,d} \in r^{d-1} D_{r,d} \oplus \mathbf{SH}^{>}[r, \leqslant d-1]. \tag{4}$$

In particular, $gr\,\mathbf{SH}^{>}$ is also freely generated by the elements $D'_{r,d} \in gr\,\mathbf{SH}^{>}[r,d]$.

2.4 *The Algebra* $\mathbf{SH^c}$

Let $\mathbf{SH}^{<}$ be the opposite algebra of $\mathbf{SH}^{>}$. We denote the generator of $\mathbf{SH}^{>}$ corresponding to $D_{1,l}$ by $D_{-1,l}$. The algebra $\mathbf{SH^c}$ is generated by $\mathbf{SH}^{>}, \mathbf{SH}^0, \mathbf{SH}^{<}$ together with a family of central elements $\mathbf{c} = (c_0, c_1, \ldots)$ indexed by $\mathbb{N}$, modulo a certain set of relations involving the commutators $[D_{-1,k}, D_{1,l}]$ (see [6, Sect. 1. 8]). In order to write down these relations, we need a few notations. Set $\xi = 1 - \kappa$ and

$$G_0(s) = -\log(s), \qquad G_l(s) = \big(s^{-1} - 1\big)/l, \quad l \geqslant 1,$$

$$\varphi_l(s) = \sum_{q=1,-\xi,-\kappa} s^l \big(G_l(1-qs) - G_l(1+qs)\big), \quad l \geqslant 1,$$

$$\phi_l(s) = s^l G_l(1+\xi s).$$

We may now define $\mathbf{SH^c}$ as the algebra generated by $\mathbf{SH}^{>}, \mathbf{SH}^{<}, \mathbf{SH}^0$ and $F[c_0, c_1, \ldots]$ modulo the following relations:

$$[D_{0,l}, D_{1,k}] = D_{1,k+l-1}, \qquad [D_{-1,k}, D_{0,l}] = D_{-1,k+l-1}, \tag{5}$$

$$[D_{-1,k}, D_{1,l}] = E_{k+l}, \quad l,k \geqslant 0, \tag{6}$$

where the elements E_h are determined through the formulas

$$1 + \xi \sum_{l \geqslant 0} E_l s^{l+1} = \exp\Big(\sum_{l \geqslant 0} (-1)^{l+1} c_l \phi_l(s)\Big) \exp\Big(\sum_{l \geqslant 0} D_{0,l+1} \varphi_l(s)\Big). \tag{7}$$

Set $\mathbf{SH}^{0,\mathbf{c}} = \mathbf{SH}^0 \otimes F[c_0, c_1, \ldots]$. One can show that the multiplication map provides an isomorphism of F-vector spaces

$$\mathbf{SH}^{>} \otimes \mathbf{SH}^{0,\mathbf{c}} \otimes \mathbf{SH}^{<} \simeq \mathbf{SH^c}.$$

Putting the generators $D_{\pm 1,k}$ in degree ± 1 and the generators $D_{0,l}, c_i$ in degree zero induces an $\mathbb{Z}$-grading on $\mathbf{SH^c}$. One can show that the order filtration on $\mathbf{SH}^{>}, \mathbf{SH}^{<}$ can be extended to a filtration on the whole $\mathbf{SH^c}$, but we won't need this last fact.

3 Link to W-Algebras, Cherednik Algebras and Shuffle Algebras

3.1 Relation the Cherednik Algebras

Let ω be a new formal parameter and let $\mathbf{SH}^{\omega}$ be the specialization of $\mathbf{SH}$ at $c_0 = 0, c_i = -\kappa^i \omega^i$. Let $\mathbf{H}_n$ be Cherednik's degenerate (or trigonometric) double affine Hecke algebra with parameter κ (see [2]). Let $\mathbf{SH}_n \subset \mathbf{H}_n$ be its spherical subalgebra. The following result shows that $\mathbf{SH}^{\omega}$ may be thought of as the stable limit of $\mathbf{SH}_n$ as n goes to infinity (see [6, Sect. 1.7]):

Theorem *For any n there exists a surjective algebra homomorphism $\Phi_n : \mathbf{SH}^{\omega} \to \mathbf{SH}_n$ such that $\Phi_n(\omega) = n$. Moreover $\bigcap_n \operatorname{Ker} \Phi_n = \{0\}$.*

3.2 Realization as a Shuffle Algebra

Consider the rational function

$$g(z) = \frac{h(z)}{z}, \quad h(z) = (z + 1 - \kappa)(z - 1)(z + \kappa).$$

Following [3], we may associate to $g(z)$ an $\mathbb{N}$-graded associative F-algebra $A_{g(z)}$, the *symmetric shuffle algebra of* $g(z)$ as follows. As a vector space,

$$A_{g(z)} = \bigoplus_{n \geqslant 0} A_{g(z)}[n], \quad A_{g(z)}[n] = F[z_1, \ldots, z_n]^{\mathfrak{S}_n}$$

with multiplication given by

$$P(z_1, \ldots, z_r) \star Q(z_1, \ldots, z_s) = \sum_{\sigma \in Sh_{r,s}} \sigma \cdot \Bigg(\prod_{\substack{1 \leqslant i \leqslant r \\ r+1 \leqslant j \leqslant r+s}} g(z_i - z_j) \cdot P(z_1, \ldots, z_r) Q(z_{r+1}, \ldots, z_{r+s}) \Bigg)$$

where $Sh_{r,s} \subset \mathfrak{S}_{r+s}$ is the set of (r, s) shuffles inside the symmetric group $\mathfrak{S}_{r+s}$. Let $S_{g(z)} \subseteq A_{g(z)}$ denote the subalgebra generated by $A_{g(z)}[1] = F[z_1]$. The restriction of the grading on $A_{g(z)}$ yields a grading $S_{g(z)} = \bigoplus_{n \geqslant 0} S_{g(z)}[n]$. The following is proved in [6, Cor. 6.4]:

Theorem *The assignment $S_{g(z)}[1] \ni z_1^l \mapsto D_{1,l}$, $l \geqslant 0$ induces an isomorphism of F-algebras*

$$S_{g(z)} \xrightarrow{\sim} \mathbf{SH}^{>}.$$

Remark The normalization used here differs slightly from [6]. Namely, the isomorphism in [6, Cor. 6.4] is between $\mathbf{SH}^{>}$ and the shuffle algebra associated to the rational function $\frac{1}{z}(z+x+y)(z-x)(z-y)$, where x and y are formal parameters satisfying $\kappa = -y/x$. In the present note, we have applied the transformation $z \mapsto z/x$, yielding the above isomorphism.

3.3 Relation to W-Algebras

Let $W_{1+\infty}$ be the universal central extension of the Lie algebra of all differential operators on $\mathbb{C}^*$ (see e.g. [4]). This is a $\mathbb{Z}$-graded and $\mathbb{N}$-filtered Lie algebra. The following result shows that **SH** may be thought of as a deformation of the universal enveloping algebra $U(W_{1+\infty})$ of $W_{1+\infty}$ (see [6, Appendix F]):

Theorem *The specialization of* $\mathbf{SH}^{\mathbf{c}}$ *at* $\kappa = 1$ *and* $c_i = 0$ *for* $i \geqslant 1$ *is isomorphic to* $U(W_{1+\infty})$.

More interesting is the fact that, for certain good choices of the parameters $c_0, c_1, \ldots$, a suitable completion of $\mathbf{SH}^{\mathbf{c}}$ is isomorphic to the current algebra of the (affine) W-algebra $W(\mathfrak{gl}_r)$ (see e.g. [1, Sect. 3.11]). We will not need this result, so we are a bit vague here and refer to [6, Sect. 8] for the full details. Fix an integer $r \geqslant 1$, $k \in \mathbb{C}$ and let $(\varepsilon_1, \ldots, \varepsilon_r)$ be new formal parameters. Let $\mathfrak{U}(W_k(\mathfrak{gl}_r))'$ be the formal current algebra of $W(\mathfrak{gl}_r)$ at level k, defined over the field $F(\varepsilon_1, \ldots, \varepsilon_r)$ (see [6, Sect. 8.4] for details). Let $\mathbf{SH}^{(r)}$ be the specialization of $\mathbf{SH}^{\mathbf{c}}$ to $\kappa = k + r$, $c_i = \varepsilon_1^i + \cdots + \varepsilon_r^i$ for $i \geqslant 0$. The following is proved in [6, Cor. 8.24], to which we refer for details.

Theorem *There is an embedding* $\mathbf{SH}^{(r)} \to \mathfrak{U}(W_k(\mathfrak{gl}_r))'$ *with a dense image, which induces an equivalence between the category of admissible* $\mathbf{SH}^{(r)}$*-modules and the category of admissible* $\mathfrak{U}(W_k(\mathfrak{gl}_r))'$*-modules.*

4 Presentation of $\mathbf{SH}^+$ and $\mathbf{SH}^{\mathbf{c}}$

4.1 Generators and Relations for $\mathbf{SH}^+$

Consider the F-algebra $\widetilde{\mathbf{SH}}^+$ generated by elements $\{\tilde{D}_{0,l} \mid l \geqslant 1\}$ and $\{\tilde{D}_{1,k} \mid k \geqslant 0\}$ subject to the following set of relations:

$$[\tilde{D}_{0,l}, \tilde{D}_{0,k}] = 0, \quad \forall l, k \geqslant 1, \tag{8}$$

$$[\tilde{D}_{0,l}, \tilde{D}_{1,k}] = \tilde{D}_{1,l+k-1}, \quad \forall l \geqslant 1, k \geqslant 0, \tag{9}$$

$$\big(3[\tilde{D}_{1,2},\tilde{D}_{1,1}]-[\tilde{D}_{1,3},\tilde{D}_{1,0}]+[\tilde{D}_{1,1},\tilde{D}_{1,0}]\big)+\kappa(\kappa-1)\big(\tilde{D}_{1,0}^2+[\tilde{D}_{1,1},\tilde{D}_{1,0}]\big)=0, \tag{10}$$

$$\big[\tilde{D}_{1,0},[\tilde{D}_{1,0},\tilde{D}_{1,1}]\big]=0. \tag{11}$$

Let $\widetilde{\mathbf{SH}}^0=F[\tilde{D}_{0,1},\tilde{D}_{0,2},\ldots]$ denote the subalgebra of $\widetilde{\mathbf{SH}}^+$ generated by $\tilde{D}_{0,l}$, $l\geqslant 1$, and let $\widetilde{\mathbf{SH}}^>$ be the subalgebra generated by $\tilde{D}_{1,k}$, $k\geqslant 0$. The algebras $\widetilde{\mathbf{SH}}^+$, $\widetilde{\mathbf{SH}}^0$, $\widetilde{\mathbf{SH}}^>$ are all $\mathbb{N}$-graded, where $\tilde{D}_{0,l}$ and $\tilde{D}_{1,k}$ are placed in degrees zero and one respectively. According to the terminology used for $\mathbf{SH}^+$, we call this grading the *rank grading*.

Theorem 1 *The assignment $\tilde{D}_{0,l}\mapsto D_{0,l}$, $\tilde{D}_{1,k}\mapsto D_{1,k}$ for $l\geqslant 1, k\geqslant 0$ induces an isomorphism of graded F-algebras*

$$\phi:\widetilde{\mathbf{SH}}^+\xrightarrow{\sim}\mathbf{SH}^+.$$

Obviously, the map ϕ restricts to isomorphisms $\widetilde{\mathbf{SH}}^0\simeq\mathbf{SH}^0$, $\widetilde{\mathbf{SH}}^>\simeq\mathbf{SH}^>$. Note however that $\widetilde{\mathbf{SH}}^>$ is *not* generated by the elements $\tilde{D}_{1,k}$ with the sole relations (10), (11). Theorem 1 is proved in Sect. 5.

4.2 Generators and Relations for $\mathbf{SH}^{\mathbf{c}}$

For the reader's convenience, we write down the presentation of $\mathbf{SH}^{\mathbf{c}}$, an immediate corollary of Theorem 1 above. Let $\widetilde{\mathbf{SH}}^{\mathbf{c}}$ be the algebra generated by elements $\{\tilde{D}_{0,l}\mid l\geqslant 1\}$, $\{\tilde{D}_{\pm 1,k}\mid k\geqslant 0\}$ and $\{\tilde{\mathbf{c}}_i\mid i\geqslant 0\}$ subject to the following set of relations:

$$[\tilde{D}_{0,l},\tilde{D}_{0,k}]=0,\quad \forall l,k\geqslant 1, \tag{12}$$

$$[\tilde{D}_{0,l},\tilde{D}_{1,k}]=\tilde{D}_{1,l+k-1},\qquad [\tilde{D}_{-1,k},\tilde{D}_{0,l}]=\tilde{D}_{-1,l+k-1},\quad \forall l\geqslant 1,k\geqslant 0, \tag{13}$$

$$\big(3[\tilde{D}_{1,2},\tilde{D}_{1,1}]-[\tilde{D}_{1,3},\tilde{D}_{1,0}]+[\tilde{D}_{1,1},\tilde{D}_{1,0}]\big)+\kappa(\kappa-1)\big(\tilde{D}_{1,0}^2+[\tilde{D}_{1,1},\tilde{D}_{1,0}]\big)=0, \tag{14}$$

$$\begin{aligned}&\big(3[\tilde{D}_{-1,2},\tilde{D}_{-1,1}]-[\tilde{D}_{-1,3},\tilde{D}_{-1,0}]+[\tilde{D}_{-1,1},\tilde{D}_{-1,0}]\big)\\&\quad+\kappa(\kappa-1)\big(-\tilde{D}_{1,0}^2+[\tilde{D}_{-1,1},\tilde{D}_{-1,0}]\big)=0,\end{aligned} \tag{15}$$

$$\big[\tilde{D}_{1,0},[\tilde{D}_{1,0},\tilde{D}_{1,1}]\big]=0,\qquad \big[\tilde{D}_{-1,0},[\tilde{D}_{-1,0},\tilde{D}_{-1,1}]\big]=0, \tag{16}$$

$$[\tilde{D}_{-1,k},\tilde{D}_{1,l}]=\tilde{E}_{k+l},\quad l,k\geqslant 0, \tag{17}$$

where the $\tilde{E}_l$ are defined by the formula (7).

Theorem 2 *The assignment* $\tilde{D}_{0,l} \mapsto D_{0,l}, \tilde{D}_{\pm 1,k} \mapsto D_{\pm 1,k}$ *for* $l \geqslant 1, k \geqslant 0$ *and* $\tilde{\mathbf{c}}_i \mapsto \mathbf{c}_i$ *for* $i \geqslant 0$ *induces an isomorphism of* F*-algebras*

$$\phi : \widetilde{\mathbf{SH}}^{\mathbf{c}} \xrightarrow{\sim} \mathbf{SH}^{\mathbf{c}}.$$

Coupled with the Theorems in Sect. 3.3, this provides a potential'generators and relations' approach to the study of the category of admissible modules over the W-algebras $W_k(\mathfrak{gl}_r)$.

5 Proof of Theorem 1

5.1 First Reductions

Let us first observe that ϕ is a well-defined algebra map, i.e. that relations (8)–(11) hold in $\mathbf{SH}^+$. For (8), (9) this follows from the definition of $\mathbf{SH}^+$ and [6, (1.38)]. Equation (10) may be checked directly, e.g. from the Pieri rules (see [6, (1.26)]), or from the shuffle realization of $\mathbf{SH}^>$ (see Sect. 5.2 below). As for Eq. (11), we have by [6, (1.35)], $[[D_{1,1}, D_{1,0}], D_{1,0}] = [D_{2,0}, D_{1,0}] = 0$. The map ϕ is surjective by construction; in the rest of the proof, we show that it is injective as well.

Using relation (9) it is easy to see that any monomial in the generators $\tilde{D}_{0,l}, \tilde{D}_{1,k}$ may be expressed as a linear combination of similar monomials, in which all $\tilde{D}_{0,l}$ appear on the right of all $\tilde{D}_{1,k}$. Hence the multiplication map $\widetilde{\mathbf{SH}}^> \otimes \widetilde{\mathbf{SH}}^0 \to \widetilde{\mathbf{SH}}^+$ is surjective. Since ϕ clearly restricts to an isomorphism $\widetilde{\mathbf{SH}}^0 \simeq \mathbf{SH}^0$ we only have to show, by (3), that ϕ restricts to an isomorphism $\widetilde{\mathbf{SH}}^> \simeq \mathbf{SH}^>$. Our strategy will be to construct a suitable filtration on $\widetilde{\mathbf{SH}}^>$ mimicking the order filtration of $\mathbf{SH}^>$ and to pass to the associated graded algebras.

5.2 Verification in Ranks One and Two

We begin by proving directly, using the shuffle realization of $\mathbf{SH}^>$, that ϕ is an isomorphism in ranks one and two. This is obvious in rank one since ϕ is a graded map and the only relation in rank one is (9).

Suppose $\sum \alpha_i D_{1,k_i} D_{1,l_i} = 0$ is a relation in rank two. The shuffle realization then implies $\sum \alpha_i z^{k_i} \star z^{l_i} = 0$ so that

$$h(z_1 - z_2)\Big(\sum \alpha_i z_1^{k_i} z_2^{l_i}\Big) = h(z_2 - z_1)\Big(\sum \alpha_i z_1^{l_i} z_2^{k_i}\Big).$$

Therefore $\sum \alpha_i z_1^{k_i} z_2^{l_i} = h(z_2 - z_1) P(z_1, z_2)$ where $P(z_1, z_2)$ is some symmetric polynomial in z_1, z_2. Hence $\sum \alpha_i z_1^{k_i} z_2^{l_i}$ is a linear combination of polynomials of the form $h(z_2 - z_1)(z_1^k z_2^l + z_1^l z_2^l k)$ so that $\sum \alpha_i D_{1,k_i} D_{1,l_i}$ is a linear combination of expressions of the form

$$\begin{aligned}
&3[D_{1,l+2}, D_{1,k+1}] - 3[D_{1,l+1}, D_{1,k+2}] \\
&\quad - [D_{1,l+3}, D_{1,k}] + [D_{1,l}, D_{1,k+3}] + [D_{1,l+1}, D_{1,k}] - [D_{1,l}, D_{1,k+1}] \\
&\quad + \kappa(\kappa-1)\big(D_{1,k} D_{1,l} + D_{1,l} D_{1,k} + [D_{1,l+1}, D_{1,k}] - [D_{1,l}, D_{1,k+1}]\big). \quad (18)
\end{aligned}$$

If I denotes the image of (10) under the action of $F[ad\,\tilde{D}_{0,2}, ad\,\tilde{D}_{0,3}, \ldots]$ then using (9) we see that each such expression lies in $\phi(I)$ so that ϕ is indeed an isomorphism in rank two.

We remark that the relations (18) may be written in a more standard way using the generating functions $D(z) = \sum_l D_{1,l} z^{-l}$ as follows:

$$k(z-w)D(z)D(w) = -k(w-z)D(w)D(z) \quad (19)$$

where $k(u) = (u-1+\kappa)(u+1)(u-\kappa) = -h(-u)$. In particular, the defining relation (10) may be replaced by the above (19), of which it is a special case.

5.3 The Order Filtration on $\widetilde{\mathbf{SH}}^{>}$

We now turn to the definition of the analog, on $\widetilde{\mathbf{SH}}^{>}$, of the order filtration on $\mathbf{SH}^{>}$. We will proceed by induction on the rank r. For $r = 1, d \geqslant 0$, we set

$$\widetilde{\mathbf{SH}}^{>}[1, \leqslant d] = \bigoplus_{k \leqslant d} F\tilde{D}_{1,k}.$$

Assuming that $\widetilde{\mathbf{SH}}^{>}[r', \leqslant d']$ has been defined for all $r' < r$ we let $\widetilde{\mathbf{SH}}^{>}[r, \leqslant d]$ be the subspace spanned by all products

$$\widetilde{\mathbf{SH}}^{>}\big[r', \leqslant d'\big] \cdot \widetilde{\mathbf{SH}}^{>}\big[r'', \leqslant d''\big], \quad r' + r'' = r, d' + d'' = d$$

and by the spaces

$$ad(\tilde{D}_{1,l})\big(\widetilde{\mathbf{SH}}^{>}[r-1, \leqslant d-l+1]\big), \quad l = 0, \ldots, d+1.$$

From the above definition, it is clear that $\widetilde{\mathbf{SH}}^{>}$ is a $\mathbb{Z}$-filtered algebra. Note that it is not obvious at the moment that $\widetilde{\mathbf{SH}}^{>}[r, \leqslant d] = \{0\}$ for $d < 0$. Because the associated graded $gr\,\mathbf{SH}^{>}$ is commutative, it follows by induction on the rank r that $\phi : \widetilde{\mathbf{SH}}^{>} \to \mathbf{SH}^{>}$ is a morphism of filtered algebras. We denote by $gr\,\widetilde{\mathbf{SH}}^{>}$ the associated graded of $\widetilde{\mathbf{SH}}^{>}$ and we let $\overline{\phi} : gr\,\widetilde{\mathbf{SH}}^{>} \to gr\,\mathbf{SH}$ be the induced map. The map $\overline{\phi}$ is graded with respect to both rank and order. Moreover $\overline{\phi}$ is an isomorphism in ranks 1 and 2 (indeed, that the filtration as defined above coincides with the order filtration in rank 2 can be seen directly from [6, (1.84)]). The rest of the proof of

Theorem 1 consists in checking that $\overline{\phi}$ is an isomorphism. Once more, we will argue by induction. So in the remainder of the proof, *we fix an integer* $r \geqslant 3$ *and assume that* $\overline{\phi}$ *is an isomorphism in ranks* $r' < r$.

5.4 Commutativity of the Associated Graded

By our assumption above, the algebra $gr\widetilde{\mathbf{SH}}^{>}$ is commutative in ranks less than r, that is $ab = ba$ whenever $\mathrm{rank}(a) + \mathrm{rank}(b) < r$. Our first task is to extend this property to the rank r.

Lemma 1 *The algebra* $gr\widetilde{\mathbf{SH}}^{>}$ *is commutative in rank* r.

Proof We have to show that for $a \in \widetilde{\mathbf{SH}}^{>}[r_1, \leqslant d_1], b \in \widetilde{\mathbf{SH}}^{>}[r_2, \leqslant d_2]$ and $r_1 + r_2 = r$ we have

$$[a, b] \in \widetilde{\mathbf{SH}}^{>}[r, \leqslant d_1 + d_2 - 1]. \tag{20}$$

We argue by induction on r_1. If $r_1 = 1$ then (20) holds by definition of the filtration. Now let $r_1 > 1$ and let us further assume that (20) is valid for all r_1', r_2' with $r_1' + r_2' = r$ and $r_1' < r_1$. We will now prove (20) for r_1, r_2, thereby completing the induction step. According to the definition of the filtration, there are two cases to consider:

Case 1 We have $a = a_1a_2$ with $a_1 \in \widetilde{\mathbf{SH}}^{>}[s', \leqslant d'], a_2 \in \widetilde{\mathbf{SH}}^{>}[s'', \leqslant d'']$ such that $s' + s'' = r_1, d' + d'' = d_1$. Then $[a, b] = a_1[a_2, b] + [a_1, b]a_2$. By our induction hypothesis on r, $[a_2, b] \in \widetilde{\mathbf{SH}}^{>}[s'' + r_2, \leqslant d'' + d_2 - 1]$ hence $a_1[a_2, b] \in \widetilde{\mathbf{SH}}^{>}[r, \leqslant d_1 + d_2 - 1]$. The term $[a_1, b]a_2$ is dealt with in a similar fashion.

Case 2 We have $a = [\tilde{D}_{1,l}, a']$ with $a' \in \widetilde{\mathbf{SH}}^{>}[r_1 - 1, \leqslant d_1 - l + 1]$. Then $[a, b] = [[\tilde{D}_{1,l}, a'], b] = [\tilde{D}_{1,l}, [a', b]] - [a', [\tilde{D}_{1,l}, b]]$. By our induction hypothesis on r, $[a', b] \in \widetilde{\mathbf{SH}}^{>}[r_1 + r_2 - 1, \leqslant d_1 + d_2 - l]$ hence $[\tilde{D}_{1,l}, [a', b]] \in \widetilde{\mathbf{SH}}^{>}[r, \leqslant d_1 + d_2 - 1]$. Similarly, $[\tilde{D}_{1,l}, b] \in \widetilde{\mathbf{SH}}^{>}[r_2 + 1, \leqslant d_2 + l - 1]$. The inclusion $[a', [\tilde{D}_{1,l}, b]] \in \widetilde{\mathbf{SH}}^{>}[r, \leqslant d_1 + d_2 - 1]$ now follows from the induction hypothesis on r_1.

We are done. □

5.5 The Degree Zero Component

We now focus on the filtered piece of order $\leqslant 0$ of $\widetilde{\mathbf{SH}}^{>}$. We inductively define elements $\tilde{D}_{l,0}$ for $l \geqslant 2$ by

$$\tilde{D}_{l,0} = \frac{1}{l-1}[\tilde{D}_{1,1}, \tilde{D}_{l-1,0}].$$

From [6, (1.35)] we have $\phi(\tilde{D}_{l,0}) = D_{l,0}$. Since we assume are assuming that $\overline{\phi}$ is an isomorphism in ranks less than r, we have $[\tilde{D}_{l,0}, \tilde{D}_{l',0}] = 0$ whenever $l + l' < r$.

Lemma 2 *We have* $[\tilde{D}_{l,0}, \tilde{D}_{l',0}] = 0$ *for* $l + l' = r$.

Proof If $r = 3$ this reduces to the cubic relation (11). For $r = 4$ we have to consider

$$\begin{aligned}[\tilde{D}_{3,0}, \tilde{D}_{1,0}] &= \frac{1}{2}\big[[\tilde{D}_{1,1}, \tilde{D}_{2,0}], \tilde{D}_{1,0}\big] \\ &= \frac{1}{2}\big[\tilde{D}_{1,1}, [\tilde{D}_{2,0}, \tilde{D}_{1,0}]\big] - \frac{1}{2}\big[\tilde{D}_{2,0}, [\tilde{D}_{1,1}, \tilde{D}_{1,0}]\big] \\ &= -\frac{1}{2}[\tilde{D}_{2,0}, \tilde{D}_{2,0}] = 0.\end{aligned}$$

Now let us fix l, l' with $l + l' = r$. We have

$$\begin{aligned}[\tilde{D}_{l,0}, \tilde{D}_{l',0}] &= \frac{1}{l-1}\big[[\tilde{D}_{1,1}, \tilde{D}_{l-1,0}], \tilde{D}_{l',0}\big] \\ &= \frac{1}{l-1}\big[\tilde{D}_{1,1}, [\tilde{D}_{l-1,0}, \tilde{D}_{l',0}]\big] - \frac{1}{l-1}\big[\tilde{D}_{l-1,0}, [\tilde{D}_{1,1}, \tilde{D}_{l',0}]\big] \\ &= -\frac{l'}{l-1}[\tilde{D}_{l-1,0}, \tilde{D}_{l'+1,0}]. \qquad (21)\end{aligned}$$

If $r = 2k$ is even then by repeated use of (21) we get

$$[\tilde{D}_{l,0}, \tilde{D}_{l',0}] = c[\tilde{D}_k, \tilde{D}_k] = 0$$

for some constant c. Next, suppose that $r = 2k + 1$ is odd, with $k \geqslant 2$. Applying $ad(\tilde{D}_{1,1})$ to $[\tilde{D}_{k+1,0}, \tilde{D}_{k-1,0}] = 0$ yields the relation

$$(k+1)[\tilde{D}_{k+2,0}, \tilde{D}_{k-1,0}] + (k-1)[\tilde{D}_{k+1,0}, \tilde{D}_{k,0}] = 0. \qquad (22)$$

Similarly, applying $ad(\tilde{D}_{2,1})$ to $[\tilde{D}_{k,0}, \tilde{D}_{k-1,0}] = 0$ and using the relation $[D_{k,1}, D_{l,0}] = klD_{l+k,0}$ in $\mathbf{SH}^{>}$ (see [6, (1.91), (8.47)]) we obtain the relation

$$k[\tilde{D}_{k+2,0}, \tilde{D}_{k-1,0}] + (k-1)[\tilde{D}_{k,0}, \tilde{D}_{k+1,0}] = 0. \qquad (23)$$

Equations (22) and (23) imply that $[\tilde{D}_{k+2,0}, \tilde{D}_{k-1,0}] = [\tilde{D}_{k+1,0}, \tilde{D}_{k,0}] = 0$. The general case of $[\tilde{D}_{l,0}, \tilde{D}_{l',0}] = 0$ is now deduced, as in the case $r = 2k$, from repeated use of (21). □

Note that Lemma 2 above implies that $\widetilde{\mathbf{SH}}^{>}[r, \leqslant -1] = \{0\}$.

5.6 Completion of the Induction Step

Recall that $gr\,\mathbf{SH}^{>}$ is a free polynomial algebra in generators in the generators $D'_{s,d}$ for $s \geqslant 1, d \geqslant 0$. In order to prove that $\overline{\phi}$ is an isomorphism in rank r, it suffices, in virtue of Lemma 1, to show that the factor space

$$U_{r,d} = gr\widetilde{\mathbf{SH}}^{>}[r,d] \ \ / \ \left\{ \sum_{\substack{r'+r''=r \\ d'+d''=d}} gr\widetilde{\mathbf{SH}}^{>}\left[r',d'\right] \cdot gr\widetilde{\mathbf{SH}}^{>}\left[r'',d''\right] \right\}$$

is one dimensional for any $d \geqslant 0$. Let us set, for any $s \geqslant 1, d \geqslant 0$

$$\tilde{D}'_{s,d} = ad(\tilde{D}_{0,2})^d(\tilde{D}_{s,0}) \in \widetilde{\mathbf{SH}}^{>}[s, \leqslant d].$$

We will denote by the same symbol $\tilde{D}'_{s,d}$ the corresponding element of $gr\widetilde{\mathbf{SH}}^{>}[s,d]$. Note that $\tilde{D}'_{s,0} = \tilde{D}_{s,0}$ We claim that in fact $U_{r,d} = F\tilde{D}'_{r,d}$. Observe that $\phi(\tilde{D}'_{s,d}) = D'_{s,d}$ for any s, d, hence $\tilde{D}'_{s,d} \in U_{s,d}$ for any $s \leqslant r, d \geqslant 0$. Moreover, by our general induction hypothesis on r we have $U_{s,d} = F\tilde{D}'_{s,d}$ for any $s < r$ and $d \geqslant 0$.

We will prove that $U_{r,d} = F\tilde{D}'_{r,d}$ by induction on d. For $d = 0$, this comes from Lemma 2. So fix $d > 0$ and let us assume that $U_{r,l} = F\tilde{D}'_{r,l}$ for all $l < d$. By definition of the filtration on $\widetilde{\mathbf{SH}}^{>}$, $U_{r,d}$ is linearly spanned by the classes of the elements

$$\left[\tilde{D}_{1,0}, \tilde{D}'_{r-1,d+1}\right], \left[\tilde{D}_{1,1}, \tilde{D}'_{r-1,d}\right], \ldots, \left[\tilde{D}_{1,d+1}, \tilde{D}'_{r-1,0}\right].$$

By our induction hypothesis on d, the elements

$$\left[\tilde{D}_{1,0}, \tilde{D}'_{r-1,d}\right], \left[\tilde{D}_{1,1}, \tilde{D}'_{r-1,d-1}\right], \ldots, \left[\tilde{D}_{1,d}, \tilde{D}'_{r-1,0}\right]$$

all belong to $F\tilde{D}'_{r,d-1} \oplus \widetilde{\mathbf{SH}}^{>}[r, \leqslant d-2]$. Applying $ad(\tilde{D}_{0,2})$, we see that

$$\left[\tilde{D}_{1,0}, \tilde{D}'_{r-1,d+1}\right] + \left[\tilde{D}_{1,1}, \tilde{D}'_{r-1,d}\right], \ldots, \left[\tilde{D}_{1,d}, \tilde{D}'_{r-1,1}\right] + \left[\tilde{D}_{1,d+1}, \tilde{D}'_{r-1,0}\right] \quad (24)$$

all belong to $F\tilde{D}'_{r,d} \oplus \widetilde{\mathbf{SH}}^{>}[r, \leqslant d-1]$. Next, applying $ad(\tilde{D}_{0,d+2})$ to the equality $[\tilde{D}_{1,0}, \tilde{D}_{r-1,0}] = 0$ yields

$$[\tilde{D}_{1,0}, \tilde{D}_{r-1,d+1}] + [\tilde{D}_{1,d+1}, \tilde{D}_{r-1,0}] = 0$$

which implies, by (4), that

$$\begin{aligned} &\left[\tilde{D}_{1,0}, \tilde{D}'_{r-1,d+1}\right] + r^d[\tilde{D}_{1,d+1}, \tilde{D}_{r-1,0}] \\ &\quad \in \left[\tilde{D}_{1,0}, \widetilde{\mathbf{SH}}^{>}[r-1, \leqslant d]\right] \subseteq \widetilde{\mathbf{SH}}^{>}[r, \leqslant d-1]. \end{aligned} \quad (25)$$

The collection of inclusions (24), (25) may be considered as a system of linear equations in $U_{r,d}$ modulo $F\tilde{D}'_{r,d}$ in the variables $[\tilde{D}_{1,0}, \tilde{D}'_{r-1,d+1}], \ldots, [\tilde{D}_{1,d+1}, \tilde{D}'_{r-1,0}]$

whose associated matrix

$$M = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 1 & -r^d \end{pmatrix}$$

is invertible. We deduce that $[\tilde{D}_{1,0}, \tilde{D}'_{r-1,d+1}], \ldots, [\tilde{D}_{1,d+1}, \tilde{D}'_{r-1,0}]$ all belong to the space $F\tilde{D}'_{r,d} \oplus \widetilde{\mathbf{SH}}^{>}[r, \geqslant d-1]$ as wanted. This closes the induction step on d. We have therefore proved that $U_{r,d} = F\tilde{D}'_{r,d}$ for all $d \geqslant 0$, and hence that $\overline{\phi}$ and ϕ is an isomorphism in rank r. This closes the induction step on r. Theorem 1 is proved.

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Generating Series of the Poincaré Polynomials of Quasihomogeneous Hilbert Schemes

A. Buryak and B.L. Feigin

Abstract In this paper we prove that the generating series of the Poincaré polynomials of quasihomogeneous Hilbert schemes of points in the plane has a beautiful decomposition into an infinite product. We also compute the generating series of the numbers of quasihomogeneous components in a moduli space of sheaves on the projective plane. The answer is given in terms of characters of the affine Lie algebra $\widehat{sl}_m$.

1 Introduction

The Hilbert scheme $(\mathbb{C}^2)^{[n]}$ of n points in the plane $\mathbb{C}^2$ parametrizes ideals $I \subset \mathbb{C}[x,y]$ of colength n: $\dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n$. There is an open dense subset of $(\mathbb{C}^2)^{[n]}$, that parametrizes the ideals, associated with configurations of n distinct points. The Hilbert scheme of n points in the plane is a nonsingular, irreducible,

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quasiprojective algebraic variety of dimension $2n$ with a rich and much studied geometry, see [9, 22] for an introduction.

The cohomology groups of $(\mathbb{C}^2)^{[n]}$ were computed in [6] and we refer the reader to the papers [5, 15–17, 24] for the description of the ring structure in the cohomology $H^*((\mathbb{C}^2)^{[n]})$.

There is a $(\mathbb{C}^*)^2$-action on $(\mathbb{C}^2)^{[n]}$ that plays a central role in this subject. The algebraic torus $(\mathbb{C}^*)^2$ acts on $\mathbb{C}^2$ by scaling the coordinates, $(t_1,t_2)\cdot(x,y)=(t_1x,t_2y)$. This action lifts to the $(\mathbb{C}^*)^2$-action on the Hilbert scheme $(\mathbb{C}^2)^{[n]}$.

Let $T_{\alpha,\beta}=\{(t^\alpha,t^\beta)\in(\mathbb{C}^*)^2|t\in\mathbb{C}^*\}$, where $\alpha,\beta\ge 1$ and $gcd(\alpha,\beta)=1$, be a one dimensional subtorus of $(\mathbb{C}^*)^2$. The variety $((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}}$ parametrizes quasihomogeneous ideals of colength n in the ring $\mathbb{C}[x,y]$. Irreducible components of $((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}}$ were described in [7]. Poincaré polynomials of irreducible components in the case $\alpha=1$ were computed in [3]. For $\alpha=\beta=1$ it was done in [12].

For a manifold X let $H_*(X)$ denote the homology group of X with rational coefficients. Let $P_q(X)=\sum_{i\ge 0}\dim H_i(X)q^{\frac{i}{2}}$. The main result of this paper is the following theorem (it was conjectured in [3]):

Theorem 1

$$\sum_{n\ge 0}P_q\left(\left((\mathbb{C}^2)^{[n]}\right)^{T_{\alpha,\beta}}\right)t^n=\prod_{\substack{i\ge 1\\(\alpha+\beta)\nmid i}}\frac{1}{1-t^i}\prod_{i\ge 1}\frac{1}{1-qt^{(\alpha+\beta)i}}. \tag{1}$$

There is a standard method for constructing a cell decomposition of the Hilbert scheme $((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}}$ using the Bialynicki-Birula theorem. In this way the Poincaré polynomial of this Hilbert scheme can be written as a generating function for a certain statistic on Young diagrams of size n. However, it happens that this combinatorial approach doesn't help in a proof of Theorem 1. In fact, we get very nontrivial combinatorial identities as a corollary of this theorem, see Sect. 1.1.

We can describe the main geometric idea in the proof of Theorem 1 in the following way. The irreducible components of $((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}}$ can be realized as fixed point sets of a $\mathbb{C}^*$-action on cyclic quiver varieties. Theorem 4 tells us that the Betti numbers of the fixed point set are equal to the shifted Betti numbers of the quiver variety. Then known results about cohomology of quiver varieties can be used for a proof of Theorem 1.

In principle, Theorem 4 has an independent interest. However, there is another application of this theorem. In [4] we studied the generating series of the numbers of quasihomogeneous components in a moduli space of sheaves on the projective plane. Combinatorially we managed to compute it only in the simplest case. Now using Theorem 4 we can give an answer in a general case, this is Theorem 5. We show that it proves our conjecture from [4].

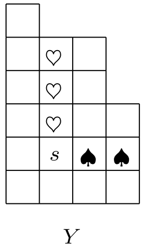

Fig. 1 Arms and legs in a Young diagram

1.1 Combinatorial Identities

Here we formulate two combinatorial identities that follow from Theorem 1. We denote by $\mathcal{Y}$ the set of all Young diagrams. For a Young diagram Y let

$$r_l(Y) = \big|\{(i,j) \in Y | j = l\}\big|,$$
$$c_l(Y) = \big|\{(i,j) \in Y | i = l\}\big|.$$

For a point $s = (i, j) \in \mathbb{Z}^2_{\geq 0}$ let

$$l_Y(s) = r_j(Y) - i - 1,$$
$$a_Y(s) = c_i(Y) - j - 1,$$

see Fig. 1. Note that $l_Y(s)$ and $a_Y(s)$ are negative, if $s \notin Y$.

The number of boxes in a Young diagram Y is denoted by $|Y|$.

Theorem 2 *Let α and β be two arbitrary positive coprime integers. Then we have*

$$\sum_{Y \in \mathcal{Y}} q^{\sharp\{s \in Y | \alpha l(s) = \beta(a(s)+1)\}} t^{|Y|} = \prod_{\substack{i \geq 1 \\ (\alpha+\beta) \nmid i}} \frac{1}{1-t^i} \prod_{i \geq 1} \frac{1}{1 - qt^{(\alpha+\beta)i}}.$$

In the case $\alpha = \beta = 1$ another identity can be derived from Theorem 1. The q-binomial coefficients are defined by

$$\begin{bmatrix} M \\ N \end{bmatrix}_q = \frac{\prod_{i=1}^{M}(1-q^i)}{\prod_{i=1}^{N}(1-q^i)\prod_{i=1}^{M-N}(1-q^i)}.$$

By $\mathcal{P}$ we denote the set of all partitions. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$, let $|\lambda| = \sum_{i=1}^{r} \lambda_i$.

Theorem 3

$$\sum_{\lambda \in \mathcal{P}} \prod_{i \geq 1} \begin{bmatrix} \lambda_i - \lambda_{i+2} + 1 \\ \lambda_{i+1} - \lambda_{i+2} \end{bmatrix}_q t^{\frac{\lambda_1(\lambda_1 - 1)}{2} + |\lambda|} = \prod_{i \geq 1} \frac{1}{(1-t^{2i-1})(1-qt^{2i})}.$$

Here for a partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r)$, $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_r$, we adopt the convention $\lambda_{>r}=0$.

1.2 Cyclic Quiver Varieties

Quiver varieties were introduced by H. Nakajima in [20]. Here we review the construction in the particular case of cyclic quiver varieties. We follow the approach from [21].

Let $m\ge 2$. We fix vector spaces $V_0,V_1,\dots,V_{m-1}$ and $W_0,W_1,\dots,W_{m-1}$ and we denote by

$$v=(\dim V_0,\dots,\dim V_{m-1}),\ w=(\dim W_0,\dots,\dim W_{m-1})\in\mathbb{Z}^m_{\ge 0}$$

the dimension vectors. We adopt the convention $V_m=V_0$. Let

$$M(v,w)=\left(\bigoplus_{k=0}^{m-1}\mathrm{Hom}(V_k,V_{k+1})\right)\oplus\left(\bigoplus_{k=0}^{m-1}\mathrm{Hom}(V_k,V_{k-1})\right)\\ \oplus\left(\bigoplus_{k=0}^{m-1}\mathrm{Hom}(W_k,V_k)\right)\oplus\left(\bigoplus_{k=0}^{m-1}\mathrm{Hom}(V_k,W_k)\right).$$

The group $G_v=\prod_{k=0}^{m-1}GL(V_k)$ acts on $M(v,w)$ by

$$g\cdot(B_1,B_2,i,j)\mapsto\left(gB_1g^{-1},gB_2g^{-1},gi,jg^{-1}\right).$$

The map $\mu\colon M(v,w)\to\bigoplus_{k=0}^{m-1}\mathrm{Hom}(V_k,V_k)$ is defined as follows

$$\mu(B_1,B_2,i,j)=[B_1,B_2]+ij.$$

Let

$$\mu^{-1}(0)^s=\left\{(B,i,j)\in\mu^{-1}(0)\,\middle|\,\begin{array}{c}\text{if a collection of subspaces } S_k\subset V_k\\ \text{is } B\text{-invariant and contains } \mathrm{Im}(i)\text{, then } S_k=V_k\end{array}\right\}.$$

The action of G_v on $\mu^{-1}(0)^s$ is free. The quiver variety $\mathfrak{M}(v,w)$ is defined as the quotient

$$\mathfrak{M}(v,w)=\mu^{-1}(0)^s/G_v,$$

see Fig. 2.

The variety $\mathfrak{M}(v,w)$ is irreducible (see e.g. [21]).

We define the $(\mathbb{C}^*)^2\times(\mathbb{C}^*)^m$-action on $\mathfrak{M}(v,w)$ as follows:

$$(t_1,t_2,e_k)\cdot(B_1,B_2,i_k,j_k)=\left(t_1B_1,t_2B_2,e_k^{-1}i_k,t_1t_2e_kj_k\right).$$

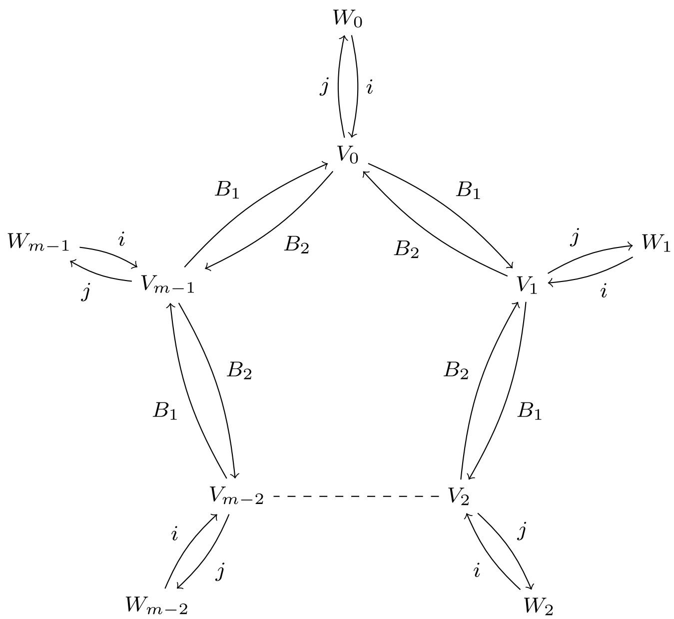

Fig. 2 Cyclic quiver variety $\mathfrak{M}(v, w)$

1.3 $\mathbb{C}^*$*-Action on* $\mathfrak{M}(v, w)$

In this section we formulate Theorem 4 that is a key step in the proofs of Theorems 1 and 5.

Let α and β be any two positive coprime integers, such that $\alpha + \beta = m$. Define the integers $\lambda_0, \lambda_1, \ldots, \lambda_{m-1} \in [-(m-1), 0]$ by the formula $\lambda_k \equiv -\alpha k \bmod m$. We define the one-dimensional subtorus $\widetilde{T}_{\alpha,\beta} \subset (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^m$ by

$$\widetilde{T}_{\alpha,\beta} = \left\{\left(t^{\alpha}, t^{\beta}, t^{\lambda_0}, t^{\lambda_1}, \ldots, t^{\lambda_{m-1}}\right) \in \left(\mathbb{C}^*\right)^2 \times \left(\mathbb{C}^*\right)^m | t \in \mathbb{C}^*\right\}.$$

For a manifold X we denote by $H_*^{BM}(X)$ the homology group of possibly infinite singular chains with locally finite support (the Borel-Moore homology) with rational coefficients. Let $P_q^{BM}(X) = \sum_{i \geq 0} \dim H_i^{BM}(X) q^{\frac{i}{2}}$.

Theorem 4 *The fixed point set* $\mathfrak{M}(v, w)^{\widetilde{T}_{\alpha,\beta}}$ *is compact and*

$$P_q^{BM}\left(\mathfrak{M}(v, w)\right) = q^{\frac{1}{2}\dim \mathfrak{M}(v,w)} P_q\left(\mathfrak{M}(v, w)^{\widetilde{T}_{\alpha,\beta}}\right).$$

1.4 Quasihomogeneous Components in the Moduli Space of Sheaves

Here we formulate our result that relates the numbers of quasihomogeneous components in a moduli space of sheaves with characters of the affine Lie algebra $\widehat{sl}_m$.

The moduli space $\mathcal{M}(r,n)$ is defined as follows (see e.g. [22]):

$$\mathcal{M}(r,n)=\left\{(B_1,B_2,i,j)\;\middle|\;\begin{array}{c}(1)\ [B_1,B_2]+ij=0\\ (2)\ \text{(stability) There is no subspace}\\ S\subsetneq\mathbb{C}^n \text{ such that } B_\alpha(S)\subset S\ (\alpha=1,2)\\ \text{and } \mathrm{Im}(i)\subset S\end{array}\right\}\Big/ GL_n(\mathbb{C}),$$

where $B_1,B_2\in\mathrm{End}(\mathbb{C}^n)$, $i\in\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^n)$ and $j\in\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^r)$ with the action of $GL_n(\mathbb{C})$ given by

$$g\cdot(B_1,B_2,i,j)=\left(gB_1g^{-1},gB_2g^{-1},gi,jg^{-1}\right),$$

for $g\in GL_n(\mathbb{C})$.

The variety $\mathcal{M}(r,n)$ has another description as the moduli space of framed torsion free sheaves on the projective plane, but for our purposes the given definition is better. We refer the reader to [22] for details. The variety $\mathcal{M}(1,n)$ is isomorphic to $(\mathbb{C}^2)^{[n]}$ (see e.g. [22]).

Define the $(\mathbb{C}^*)^2\times(\mathbb{C}^*)^r$-action on $\mathcal{M}(r,n)$ by

$$(t_1,t_2,e)\cdot\left[(B_1,B_2,i,j)\right]=\left[\left(t_1B_1,t_2B_2,ie^{-1},t_1t_2ej\right)\right].$$

Consider two positive coprime integers α and β and a vector

$$\boldsymbol{\omega}=(\omega_1,\omega_2,\dots,\omega_r)\in\mathbb{Z}^r$$

such that $0\le\omega_i<\alpha+\beta$. Let $T^{\boldsymbol{\omega}}_{\alpha,\beta}$ be the one-dimensional subtorus of $(\mathbb{C}^*)^2\times(\mathbb{C}^*)^r$ defined by

$$T^{\boldsymbol{\omega}}_{\alpha,\beta}=\left\{\left(t^\alpha,t^\beta,t^{\omega_1},t^{\omega_2},\dots,t^{\omega_r}\right)\in\left(\mathbb{C}^*\right)^2\times\left(\mathbb{C}^*\right)^r|t\in\mathbb{C}^*\right\}.$$

In [4] we studied the numbers of the irreducible components of $\mathcal{M}(r,n)^{T^{\boldsymbol{\omega}}_{\alpha,\beta}}$ and found an answer in the case $\alpha=\beta=1$. Now we can solve the general case.

We define the vector $\boldsymbol{\rho}=(\rho_0,\rho_1,\dots,\rho_{\alpha+\beta-1})\in\mathbb{Z}^{\alpha+\beta}_{\ge0}$ by $\rho_i=\sharp\{j|\omega_j=i\}$ and the vector $\boldsymbol{\mu}\in\mathbb{Z}^{\alpha+\beta}_{\ge0}$ by $\mu_i=\rho_{-i\alpha\bmod\alpha+\beta}$.

Let E_k,F_k,H_k, $k=1,2,\dots,\alpha+\beta$, be the standard generators of $\widehat{sl}_{\alpha+\beta}$. Let $\mathcal{V}$ be the irreducible highest weight representation of $\widehat{sl}_{\alpha+\beta}$ with the highest weight $\boldsymbol{\mu}$. Let $x\in\mathcal{V}$ be the highest weight vector. We denote by $\mathcal{V}_p$ the vector subspace of $\mathcal{V}$ generated by vectors $F_{i_1}F_{i_2}\dots F_{i_p}x$. The character $\chi_{\boldsymbol{\mu}}(q)$ is defined by

$$\chi_{\boldsymbol{\mu}}(q)=\sum_{p\ge0}(\dim\mathcal{V}_p)q^p.$$

We denote by $h_0(X)$ the number of connected components of a manifold X.

Theorem 5

$$\sum_{n\geq 0} h_0\big(\mathcal{M}(r,n)^{T^{\omega}_{\alpha,\beta}}\big)q^n = \chi_{\boldsymbol{\mu}}(q).$$

In [14] the authors found a combinatorial formula for characters of $\widehat{sl}_m$ in terms of Young diagrams with certain restrictions. In [8] the same combinatorics is used to give a formula for certain characters of the quantum continuous gl_∞. Comparing these two combinatorial formulas it is easy to see that Conjecture 1.2 from [4] follows from Theorem 5.

Remark 1 There is a small mistake in Conjecture 1.2 from [4]. The vector $\mathbf{a}' = (a'_0, a'_1, \dots, a'_{\alpha+\beta-1})$ should be defined by $a'_i = a_{-\alpha i \bmod \alpha+\beta}$. The rest is correct.

1.5 Organization of the Paper

We prove Theorem 4 in Sect. 2. Then using this result we prove Theorem 1 in Sect. 3. In Sect. 4 we derive the combinatorial identities as a corollary of Theorem 1. Finally, using Theorem 4 we prove Theorem 5 in Sect. 5.

2 Proof of Theorem 4

In this section we prove Theorem 4. The Grothendieck ring of quasiprojective varieties is a useful technical tool and we remind its definition and necessary properties in Sect. 2.1.

2.1 Grothendieck Ring of Quasiprojective Varieties

The Grothendieck ring $K_0(\nu_{\mathbb{C}})$ of complex quasiprojective varieties is the abelian group generated by the classes $[X]$ of all complex quasiprojective varieties X modulo the relations:

1. if varieties X and Y are isomorphic, then $[X] = [Y]$;
2. if Y is a Zariski closed subvariety of X, then $[X] = [Y] + [X \setminus Y]$.

The multiplication in $K_0(\nu_{\mathbb{C}})$ is defined by the Cartesian product of varieties: $[X_1] \cdot [X_2] = [X_1 \times X_2]$. The class $[\mathbb{A}^1_{\mathbb{C}}] \in K_0(\nu_{\mathbb{C}})$ of the complex affine line is denoted by $\mathbb{L}$.

We need the following property of the ring $K_0(\nu_{\mathbb{C}})$. There is a natural homomorphism of rings $\theta\colon \mathbb{Z}[z] \to K_0(\nu_{\mathbb{C}})$, defined by $\theta(z) = \mathbb{L}$. This homomorphism is an inclusion (see e.g. [18]).

2.2 Proof of Theorem 4

Let $r = \sum_{i=0}^{m-1} \theta_i$. For an arbitrary $\boldsymbol{v} \in \mathbb{Z}^r$ let $\Gamma^{\boldsymbol{v}}_{\alpha,\beta} \subset T^{\boldsymbol{v}}_{\alpha,\beta}$ be the subgroup of roots of 1 of degree m. Let

$$\boldsymbol{\theta} = (\underbrace{0,\dots,0}_{w_0 \text{ times}}, \underbrace{\lambda_1,\dots,\lambda_1}_{w_1 \text{ times}}, \dots, \underbrace{\lambda_{m-1},\dots,\lambda_{m-1}}_{w_{m-1} \text{ times}}) \in \mathbb{Z}^r.$$

Lemma 1 1. *We have the following decomposition into irreducible components*

$$\mathcal{M}(r,n)^{\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}} = \coprod_{\substack{v \in \mathbb{Z}^m_{\geq 0} \\ \sum v_k = n}} \mathfrak{M}(v,w). \quad (2)$$

2. *The $T^{\boldsymbol{\theta}}_{\alpha,\beta}$-action on the left-hand side of* (2) *corresponds to the $\widetilde{T}_{\alpha,\beta}$-action on the right-hand side of* (2).

Proof Let Γ_m be the group of roots of unity of degree m. By definition, a point $[(B_1, B_2, i, j)] \in \mathcal{M}(r,n)$ is fixed under the action of $\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}$ if and only if there exists a homomorphism $\lambda\colon \Gamma_m \to GL_n(\mathbb{C})$ satisfying the following conditions:

$$\begin{aligned} \zeta^\alpha B_1 &= \lambda(\zeta)^{-1} B_1 \lambda(\zeta), \\ \zeta^\beta B_2 &= \lambda(\zeta)^{-1} B_2 \lambda(\zeta), \\ i \circ \operatorname{diag}\left(\zeta^{\theta_1}, \zeta^{\theta_2}, \dots, \zeta^{\theta_r}\right)^{-1} &= \lambda(\zeta)^{-1} i, \\ \operatorname{diag}\left(\zeta^{\theta_1}, \zeta^{\theta_2}, \dots, \zeta^{\theta_r}\right) \circ j &= j \lambda(\zeta), \end{aligned} \quad (3)$$

where $\zeta = e^{\frac{2\pi\sqrt{-1}}{m}}$. Suppose that $[(B_1, B_2, i, j)]$ is a fixed point. Then we have the weight decomposition of $\mathbb{C}^n$ with respect to $\lambda(\zeta)$, i.e. $\mathbb{C}^n = \bigoplus_{k \in \mathbb{Z}/m\mathbb{Z}} V'_k$, where $V'_k = \{v \in \mathbb{C}^n | \lambda(\zeta) \cdot v = \zeta^k v\}$. We also have the weight decomposition of $\mathbb{C}^r$, i.e. $\mathbb{C}^r = \bigoplus_{k \in \mathbb{Z}/m\mathbb{Z}} W'_k$, where $W'_k = \{v \in \mathbb{C}^r | \operatorname{diag}(\zeta^{\theta_1}, \dots, \zeta^{\theta_r}) \cdot v = \zeta^k v\}$. From conditions (3) it follows that the only components of B_1, B_2, i and j that might survive are:

$$\begin{aligned} B_1 &\colon V'_k \to V'_{k-\alpha}, \\ B_2 &\colon V'_k \to V'_{k-\beta}, \\ i &\colon W'_k \to V'_k, \\ j &\colon V'_k \to W'_k. \end{aligned}$$

Let us denote $V'_{-\alpha k \bmod m}$ by V_k and $W'_{-\alpha k \bmod m}$ by W_k. Then the operators B_1, B_2, i, j act as follows: $B_{1,2}\colon V_k \to V_{k\pm 1}, i\colon W_k \to V_k, j\colon V_k \to W_k$. The first part of the lemma is proved. The second part of the lemma easily follows from the proof of the first part and from the definition of the $\widetilde{T}_{\alpha,\beta}$-action. □

In [4] it is proved that the variety $\mathcal{M}(r,n)^{T^{\boldsymbol{\theta}}_{\alpha,\beta}}$ is compact. Therefore, $\mathfrak{M}(v,w)^{\widetilde{T}_{\alpha,\beta}}$ is compact.

We denote by $\mathcal{M}(r,n)_v^{\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}}$ the irreducible component of $\mathcal{M}(r,n)^{\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}}$ corresponding to $\mathfrak{M}(v,w)$. Let $\mathcal{M}(r,n)_v^{T^{\boldsymbol{\theta}}_{\alpha,\beta}} = (\mathcal{M}(r,n)_v^{\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}})^{T^{\boldsymbol{\theta}}_{\alpha,\beta}}$. We denote by I_v the set of irreducible components of $\mathcal{M}(r,n)_v^{T^{\boldsymbol{\theta}}_{\alpha,\beta}}$ and let $\mathcal{M}(r,n)_v^{T^{\boldsymbol{\theta}}_{\alpha,\beta}} = \coprod_{i\in I_v} \mathcal{M}(r,n)_{v,i}^{T^{\boldsymbol{\theta}}_{\alpha,\beta}}$ be the decomposition into the irreducible components. We define the sets $C_{v,i}$ by

$$C_{v,i} = \left\{ z \in \mathcal{M}(r,n)_v^{\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}} \,\middle|\, \lim_{t\to 0, t\in T^{\boldsymbol{\theta}}_{\alpha,\beta}} tz \in \mathcal{M}(r,n)_{v,i}^{T^{\boldsymbol{\theta}}_{\alpha,\beta}} \right\}.$$

Lemma 2 (1) *The sets $C_{v,i}$ form a decomposition of $\mathcal{M}(r,n)_v^{\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}}$ into locally closed subvarieties.*

(2) *The subvariety $C_{v,i}$ is a locally trivial bundle over $\mathcal{M}(r,n)_{v,i}^{T^{\boldsymbol{\theta}}_{\alpha,\beta}}$ with an affine space as a fiber.*

Proof The lemma follows from the results of [1, 2]. The only thing that we need to check is that the limit $\lim_{t\to 0, t\in T^{\boldsymbol{\theta}}_{\alpha,\beta}} tz$ exists for any $z \in \mathcal{M}(r,n)_v^{\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}}$.

Consider the variety $\mathcal{M}_0(r,n)$ from [23]. It is defined as the affine algebro-geometric quotient

$$\mathcal{M}_0(r,n) = \big\{(B_1, B_2, i, j) | [B_1, B_2] + ij = 0\big\} // GL_n(\mathbb{C}).$$

It can be viewed as the set of closed orbits in $\{(B_1, B_2, i, j) | [B_1, B_2] + ij = 0\}$. There is a morphism $\pi : \mathcal{M}(r,n) \to \mathcal{M}_0(r,n)$. It maps a point $[(B_1, B_2, i, j)] \in \mathcal{M}(r,n)$ to the unique closed orbit that is contained in the closure of the orbit of (B_1, B_2, i, j) in $\{(B_1, B_2, i, j) | [B_1, B_2] + ij = 0\}$. The $(\mathbb{C}^*)^2 \times (\mathbb{C}^*)^r$-action on $\mathcal{M}_0(r,n)$ is defined in the same way as on $\mathcal{M}(r,n)$. The variety $\mathcal{M}_0(r,n)$ is affine and the morphism π is projective and equivariant (see e.g. [23]).

By [19], the coordinate ring of $\mathcal{M}_0(r,n)$ is generated by the following two types of functions:

(a) $\operatorname{tr}(B_{a_N} B_{a_{N-1}} \cdots B_{a_1} : \mathbb{C}^n \to \mathbb{C}^n)$, where $a_i = 1$ or 2.
(b) $\chi(j B_{a_N} B_{a_{N-1}} \cdots B_{a_1} i)$, where $a_i = 1$ or 2, and χ is a linear form on $\operatorname{End}(\mathbb{C}^r)$.

From the inequalities $-m < \theta_k \le 0$ it follows that both types of functions have positive weights with respect to the $T^{\boldsymbol{\theta}}_{\alpha,\beta}$-action. Therefore, for any point $z \in \mathcal{M}_0(r,n)$ we have $\lim_{t\to 0, t\in T^{\boldsymbol{\theta}}_{\alpha,\beta}} tz = 0$. The morphism π is projective, so the limit $\lim_{t\to 0, t\in T^{\boldsymbol{\theta}}_{\alpha,\beta}} tz$ exists for any $z \in \mathcal{M}(r,n)_v^{\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}}$. The lemma is proved. □

Denote by $d^+_{v,i}$ the dimension of the fiber of the locally trivial bundle $C_{v,i} \to \mathcal{M}(r,n)^{T^{\boldsymbol{\theta}}_{\alpha,\beta}}_{v,i}$.

Lemma 3 *The dimension $d^+_{v,k}$ doesn't depend on $k \in I_v$ and is equal to*

$$d^+_{v,k} = \frac{1}{2}\dim \mathfrak{M}(v,w).$$

Proof The set of fixed points of the $(\mathbb{C}^*)^2 \times (\mathbb{C}^*)^r$-action on $\mathcal{M}(r,n)$ is finite and is parametrized by the set of r-tuples $D = (D_1, D_2, \ldots, D_r)$ of Young diagrams D_i such that $\sum_{i=1}^r |D_i| = n$ (see e.g. [23]).

Let $p \in \mathcal{M}(r,n)^{(\mathbb{C}^*)^2\times(\mathbb{C}^*)^r}$ be the fixed point corresponding to an r-tuple D. Let $R((\mathbb{C}^*)^2 \times (\mathbb{C}^*)^r) = \mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, e_1^{\pm1}, e_2^{\pm1}, \ldots, e_r^{\pm1}]$ be the representation ring of $(\mathbb{C}^*)^2 \times (\mathbb{C}^*)^r$. Then the weight decomposition of the tangent space $T_p\mathcal{M}(r,n)$ of the variety $\mathcal{M}(r,n)$ at the point p is given by (see e.g. [23])

$$T_p\mathcal{M}(r,n) = \sum_{i,j=1}^r e_j e_i^{-1}\left(\sum_{s\in D_i} t_1^{-l_{D_j}(s)} t_2^{a_{D_i}(s)+1} + \sum_{s\in D_j} t_1^{l_{D_i}(s)+1} t_2^{-a_{D_j}(s)}\right). \qquad (4)$$

For a computation of $d^+_{v,k}$ we choose an arbitrary $(\mathbb{C}^*)^2 \times (\mathbb{C}^*)^r$-fixed point p in $\mathcal{M}(r,n)^{T^{\boldsymbol{\theta}}_{\alpha,\beta}}_{v,k}$. Let D be the corresponding r-tuple of Young diagrams. We have

$$\begin{aligned}
d^+_{v,k} &= \sum_{i,j} \sharp\left\{ s\in D_i \;\middle|\; \begin{matrix} \theta_j-\theta_i-\alpha l_{D_j}(s)+\beta(a_{D_i}(s)+1)\equiv 0 \bmod m \\ \theta_j-\theta_i-\alpha l_{D_j}(s)+\beta(a_{D_i}(s)+1)>0 \end{matrix} \right\} \\
&\quad + \sum_{i,j} \sharp\left\{ s\in D_j \;\middle|\; \begin{matrix} \theta_j-\theta_i+\alpha(l_{D_i}(s)+1)-\beta a_{D_j}(s)\equiv 0 \bmod m \\ \theta_j-\theta_i+\alpha(l_{D_i}(s)+1)-\beta a_{D_j}(s)>0 \end{matrix} \right\} \\
&= \sum_{i,j} \sharp\left\{ s\in D_i \;\middle|\; \begin{matrix} \theta_j-\theta_i-\alpha l_{D_j}(s)+\beta(a_{D_i}(s)+1)\equiv 0 \bmod m \\ \theta_j-\theta_i-\alpha l_{D_j}(s)+\beta(a_{D_i}(s)+1)>0 \end{matrix} \right\} \\
&\quad + \sum_{i,j} \sharp\left\{ s\in D_i \;\middle|\; \begin{matrix} \theta_j-\theta_i-\alpha l_{D_j}(s)+\beta(a_{D_i}(s)+1)\equiv 0 \bmod m \\ \theta_j-\theta_i-\alpha l_{D_j}(s)+\beta(a_{D_i}(s)+1)<m \end{matrix} \right\} \\
&= \sum_{i,j} \sharp\left\{ s\in D_i | \theta_j-\theta_i-\alpha l_{D_j}(s)+\beta(a_{D_i}(s)+1)\equiv 0 \bmod m \right\} \\
&\quad + \sum_{i,j} \sharp\left\{ s\in D_i \;\middle|\; \begin{matrix} \theta_j-\theta_i-\alpha l_{D_j}(s)+\beta(a_{D_i}(s)+1)\equiv 0 \bmod m \\ 0<\theta_j-\theta_i-\alpha l_{D_j}(s)+\beta(a_{D_i}(s)+1)<m \end{matrix} \right\}.
\end{aligned}$$

It is easy to see that the last sum is equal to zero, thus

$$d^+_{v,k} = \sum_{i,j} \sharp\left\{ s\in D_i | \theta_j-\theta_i-\alpha l_{D_j}(s)+\beta\big(a_{D_i}(s)+1\big)\equiv 0 \bmod m \right\}.$$

On the other hand

$$\begin{aligned}
&\dim \mathcal{M}(r,n)_v^{\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}} \\
&= \sum_{i,j} \sharp\big\{s \in D_i \,|\, \theta_j - \theta_i - \alpha l_{D_j}(s) + \beta\big(a_{D_i}(s)+1\big) \equiv 0 \bmod m\big\} \\
&\quad + \sum_{i,j} \sharp\big\{s \in D_j \,|\, \theta_j - \theta_i + \alpha\big(l_{D_i}(s)+1\big) - \beta a_{D_j}(s) \equiv 0 \bmod m\big\} \\
&= 2\sum_{i,j} \sharp\big\{s \in D_i \,|\, \theta_j - \theta_i - \alpha l_{D_j}(s) + \beta\big(a_{D_i}(s)+1\big) \equiv 0 \bmod m\big\}.
\end{aligned}$$

Hence $d^+_{v,k} = \frac{1}{2}\dim \mathcal{M}(r,n)_v^{\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}} = \frac{1}{2}\dim \mathfrak{M}(v,w)$. □

From Lemmas 2 and 3 it follows that

$$\big[\mathcal{M}(r,n)_v^{\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}}\big] = \mathbb{L}^{\frac{1}{2}\dim \mathfrak{M}(v,w)}\big[\mathcal{M}(r,n)_v^{T^{\boldsymbol{\theta}}_{\alpha,\beta}}\big].$$

Using the $(\mathbb{C}^*)^2 \times (\mathbb{C}^*)^r$-action it is easy to get a cell decomposition of the varieties $\mathcal{M}(r,n)_v^{\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}}$ and $\mathcal{M}(r,n)_{v,k}^{T^{\boldsymbol{\theta}}_{\alpha,\beta}}$. Therefore

$$\begin{aligned}
\big[\mathcal{M}(r,n)_v^{\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}}\big] &= P_q^{BM}\big(\mathcal{M}(r,n)_v^{\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}}\big)\big|_{q=\mathbb{L}}, \\
\big[\mathcal{M}(r,n)_{v,k}^{T^{\boldsymbol{\theta}}_{\alpha,\beta}}\big] &= P_q\big(\mathcal{M}(r,n)_{v,k}^{T^{\boldsymbol{\theta}}_{\alpha,\beta}}\big)\big|_{q=\mathbb{L}}.
\end{aligned}$$

The theorem is proved.

3 Proof of Theorem 1

In this section we prove Theorem 1. First of all, in Sect. 3.1 we remind the reader a notion of a power structure over the Grothendieck ring $K_0(\nu_{\mathbb{C}})$. This technique allows us to simplify some combinatorial computations. Then in Sect. 3.2 we review standard combinatorial constructions related to Young diagrams. In Sect. 3.3 we review a connection between Hilbert schemes and quiver varieties and do an important step in the proof of Theorem 1. Instead of considering the $T_{\alpha,\beta}$-fixed point set in the Hilbert scheme $(\mathbb{C}^2)^{[n]}$, we first look at the fixed point set of a finite subgroup of $T_{\alpha,\beta}$. Finally, in Sect. 3.4 we combine everything and prove the theorem.

3.1 Power Structure over $K_0(\nu_{\mathbb{C}})$

In [10] there was defined a notion of a power structure over a ring and there was described a natural power structure over the Grothendieck ring $K_0(\nu_{\mathbb{C}})$. This means that for a series $A(t) = 1 + a_1 t + a_2 t^2 + \cdots \in 1 + t \cdot K_0(\nu_{\mathbb{C}})[[t]]$ and for an element $m \in K_0(\nu_{\mathbb{C}})$ one defines a series $(A(t))^m \in 1 + t \cdot K_0(\nu_{\mathbb{C}})[[t]]$ so that all the usual properties of the exponential function hold. We also need the following property of this power structure. For any $i \geq 1$ and $j \geq 0$ we have (see e.g. [10])

$$\left(1 - \mathbb{L}^j t^i\right)^{\mathbb{L}} = 1 - \mathbb{L}^{j+1} t^i. \tag{5}$$

3.2 Cores and Quotients

In this section we review the well known construction of an m-core and an m-quotient of a Young diagram.

The set $Core_m$ is defined as the set of Young diagrams Y such that for any box $s \in Y$ we have $l_Y(s) + a_Y(s) + 1 \not\equiv 0 \bmod m$. For a Young diagram Y let

$$w_i(Y) = \sharp\left\{(p,q) \in Y \,|\, p + q \equiv i \bmod m\right\}.$$

We remind the reader that we consider a Young diagram as a subset of $\mathbb{Z}^2_{\geq 0}$. Let

$$\Pi^{m-1} = \left\{\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{m-1}) \in \mathbb{Z}^m \,\middle|\, \sum_{k=0}^{m-1} \lambda_k = 0\right\}.$$

Define the map $\Psi : Core_m \to \Pi^{m-1}$ by

$$Core_m \ni Y \mapsto (\lambda_0, \lambda_1, \ldots, \lambda_{m-1}), \lambda_i = w_{i+1}(Y) - w_i(Y).$$

The map Ψ is a bijection (see e.g. [13], Chap. 2.7).

There is also a bijection (see e.g. [13], Chap. 2.7)

$$\Phi : \mathcal{Y} \to Core_m \times \mathcal{Y}^m, \quad \Phi(Y) = \left(\Phi(Y)_0, \Phi(Y)_1, \ldots, \Phi(Y)_m\right).$$

We don't give a construction of this map, we will only list all necessary properties. The diagram $\Phi(Y)_0$ is called the m-core of the diagram Y and the m-tuple $(\Phi(Y)_1, \Phi(Y)_2, \ldots, \Phi(Y)_m)$ is called the m-quotient. The bijection Φ has the following properties (see e.g. [13], Chap. 2.7):

$$|Y| = \left|\Phi(Y)_0\right| + m \sum_{i=1}^{m} \left|\Phi(Y)_i\right|; \tag{6}$$

$$w_i(Y) = w_i\big(\Phi(Y)_0\big) + \sum_{i=1}^{m} \big|\Phi(Y)_i\big|; \tag{7}$$

$$\sharp\big\{s \in Y \,|\, l_Y(s) + a_Y(s) + 1 \equiv 0 \bmod m\big\} = \sum_{i=1}^{m} \big|\Phi(Y)_i\big|. \tag{8}$$

3.3 Hilbert Schemes and Quiver Varieties

For an ideal $I \subset \mathbb{C}[x, y]$ of codimension n let $V(I) = \mathbb{C}[x, y]/I$ and $B_1, B_2 \in GL(V(I))$ be the operators of the multiplications by x and y correspondingly. Let $i : \mathbb{C} \to V(I)$ be the linear map that sends $1 \in \mathbb{C}$ to the unit in $\mathbb{C}[x, y]$. Define the map $f : (\mathbb{C}^2)^{[n]} \to \mathcal{M}(1, n)$ by $I \mapsto [(B_1, B_2, i, 0)]$. This map is an isomorphism (see e.g. [22]).

For integers μ and ν let $\Gamma_{\nu,\mu}$ be the finite subgroup of $(\mathbb{C}^*)^2$ defined by $\Gamma_{\nu,\mu} = \{(\zeta^{j\nu}, \zeta^{j\mu}) \in (\mathbb{C}^*)^2 | \zeta = \exp(\frac{2\pi i}{m})\}$. It is clear that the isomorphism f transforms the $T_{\alpha,\beta}$-action on $(\mathbb{C}^2)^{[n]}$ to the $T^{\mathbf{0}}_{\alpha,\beta}$-action on $\mathcal{M}(1, n)$ and the $\Gamma_{\alpha,\beta}$-action to the $\Gamma^{\mathbf{0}}_{\alpha,\beta}$-action. Thus, by Lemma 1, we have

$$\begin{aligned}
\big((\mathbb{C}^2)^{[n]}\big)^{\Gamma_{\alpha,\beta}} &= \coprod_{\substack{v \in \mathbb{Z}^m_{\geq 0} \\ \sum v_i = n}} \mathfrak{M}(v, e_0), \\
\big((\mathbb{C}^2)^{[n]}\big)^{T_{\alpha,\beta}} &= \coprod_{\substack{v \in \mathbb{Z}^m_{\geq 0} \\ \sum v_i = n}} \mathfrak{M}(v, e_0)^{\widetilde{T}_{\alpha,\beta}},
\end{aligned} \tag{9}$$

where by e_0 we denote the vector $(1, 0, \ldots, 0) \in \mathbb{Z}^m_{\geq 0}$. Until the end of this section we consider a quiver variety $\mathfrak{M}(v, e_0)$ as a subset of $\mathcal{M}(1, \sum v_i) = (\mathbb{C}^2)^{[\sum v_i]}$.

The last factor $\mathbb{C}^*$ of the product $(\mathbb{C}^*)^2 \times \mathbb{C}^*$ acts trivially on $\mathcal{M}(1, n)$, so now we start to consider only the $(\mathbb{C}^*)^2$-action on $\mathcal{M}(1, n)$.

3.4 Proof of Theorem 1

For a vector $v \in \mathbb{Z}^m_{\geq 0}$ let $|v| = \sum_{i=0}^{m-1} v_i$. By (9) and Theorem 4, we have

$$\sum_{n \geq 0} P_q\Big(\big((\mathbb{C}^2)^{[n]}\big)^{T_{\alpha,\beta}}\Big) t^n = \sum_{v \in \mathbb{Z}^m_{\geq 0}} q^{-\frac{1}{2}\dim \mathfrak{M}(v,e_0)} P_q^{BM}\big(\mathfrak{M}(v, e_0)\big) t^{|v|}.$$

If the variety $\mathfrak{M}(v, e_0)$ is nonempty, then (see e.g. [23])

$$\dim \mathfrak{M}(v, e_0) = 2v_0 - \sum_{i=0}^{m-1} (v_i - v_{i+1})^2. \tag{10}$$

Here we follow the convention $v_m = v_0$. For $\lambda \in \Pi^{m-1}$ let

$$v_0(\lambda) = \frac{1}{2} \sum_{k=0}^{m-1} \lambda_k^2,$$

$$n(\lambda) = m v_0(\lambda) + \sum_{k=0}^{m-2} (m-1-k)\lambda_k.$$

Using these notations and formula (10) we get

$$\sum_{v \in \mathbb{Z}_{\geq 0}^m} q^{-\frac{1}{2}\dim \mathfrak{M}(v,e_0)} P_q^{BM}\big(\mathfrak{M}(v, e_0)\big) t^{|v|}$$
$$= \sum_{\lambda \in \Pi^{m-1}} t^{n(\lambda)} \sum_{\substack{v \in \mathbb{Z}_{\geq 0}^m \\ v_{i+1} - v_i = \lambda_i}} P_q^{BM}\big(\mathfrak{M}(v, e_0)\big)\big(q^{-\frac{1}{m}} t\big)^{m(v_0 - v_0(\lambda))}.$$

Lemma 4 *For any $\lambda \in \Pi^{m-1}$ we have*

$$\sum_{\substack{v \in \mathbb{Z}_{\geq 0}^m \\ v_{i+1} - v_i = \lambda_i}} P_q^{BM}\big(\mathfrak{M}(v, e_0)\big) t^{|v|} = \frac{t^{n(\lambda)}}{\prod_{i \geq 1}(1 - q^i t^{mi})^{m-1}(1 - q^{i+1} t^{mi})}.$$

Before a proof of this lemma we introduce a new notation and prove two useful lemmas.

In the proof of Lemma 2 we used the morphism $\pi : \mathcal{M}(r, n) \to \mathcal{M}_0(r, n)$. We have $\mathcal{M}_0(1, n) = S^n(\mathbb{C}^2)$ (see e.g. [22]). Slightly changing notations we denote now by π the morphism $\mathcal{M}(1, n) \to S^n(\mathbb{C}^2)$. It can be described explicitly as follows. Let $[(B_1, B_2, i, j)] \in \mathcal{M}(1, n)$. We can make B_1 and B_2 simultaneously into upper triangular matrices with numbers λ_i and μ_i on the diagonals. The morphism π is given by $\pi(B_1, B_2, i, j) = \{(\lambda_1, \mu_1), \dots, (\lambda_n, \mu_n)\}$ (see e.g. [22]).

It is useful to note that the subgroups $\Gamma_{\alpha,\beta}$ and $\Gamma_{1,-1}$ of $(\mathbb{C}^*)^2$ coincide, therefore

$$\mathcal{M}(1, n)^{\Gamma_{\alpha,\beta}} = \mathcal{M}(1, n)^{\Gamma_{1,-1}}.$$

For any $\Gamma_{1,-1}$-invariant subset $Z \subset \mathbb{C}^2$ and any vector $\lambda \in \Pi^{m-1}$ let

$$H_{Z,\lambda}(t) = \sum_{\substack{v \in \mathbb{Z}_{\geq 0}^m \\ v_{i+1} - v_i = \lambda_i}} \big[\mathfrak{M}(v, e_0) \cap \pi^{-1}\big(S^{|v|} Z\big)\big] t^{|v|}.$$

We denote by $\mathbb{C}_x$ the x-axis in the plane $\mathbb{C}^2$.

Lemma 5 *For any $\lambda \in \Pi^{m-1}$ consider the unique diagram $Y_\lambda \in Core_m$ such that $\Psi(Y_\lambda) = \lambda$. Then we have*

$$H_{\mathbb{C}_x,\lambda}(t) = \frac{t^{|Y_\lambda|}}{\prod_{i\geq 1}(1-\mathbb{L}^i t^{mi})^m}. \tag{11}$$

Proof The set of fixed points of the $(\mathbb{C}^*)^2$-action on $\mathcal{M}(1,n)$ is parametrized by the set of Young diagrams Y such that $|Y| = n$. Let p be the fixed point corresponding to a Young diagram Y, then, by (4), we have

$$T_p\big(\mathcal{M}(1,n)\big) = \sum_{s\in Y}\big(t_1^{-l_Y(s)}t_2^{a_Y(s)+1} + t_1^{l_Y(s)+1}t_2^{-a_Y(s)}\big). \tag{12}$$

We choose $\gamma \gg 1$ and for each point $p \in \mathfrak{M}(v,e_0)^{(\mathbb{C}^*)^2}$ we define the attracting set C_p as follows

$$C_p = \Big\{z \in \mathfrak{M}(v,e_0)\Big|\lim_{t\to 0, t\in T_{1,-\gamma}} tz = p\Big\}.$$

Clearly, if $z \in \mathbb{C}_x$, then $\lim_{t\to 0, t\in T_{1,-\gamma}} tz = 0$, and if $z \in \mathbb{C}^2\backslash\mathbb{C}_x$, then tz goes to infinity. By [1, 2], the sets C_p form a cell decomposition of $\mathfrak{M}(v,e_0)\cap\pi^{-1}(S^{|v|}\mathbb{C}_x)$. Using (12) we obtain

$$\big[\mathfrak{M}(v,e_0)\cap\pi^{-1}\big(S^{|v|}\mathbb{C}_x\big)\big] = \sum_{\substack{Y\in\mathcal{Y}\\ w_i(Y)=v_i}} \mathbb{L}^{\sharp\{s\in Y | l_Y(s)+a_Y(s)+1\equiv 0 \bmod m\}}. \tag{13}$$

The formula (11) follows from (13) and properties (6), (7) and (8). □

Lemma 6 *For any $Y \in Core_m$ we have $|Y| = n(\Psi(Y))$.*

Proof Consider the quiver variety $\mathfrak{M}(w(Y),e_0)$. From the properties of the bijection Φ it follows that if Y' is a Young diagram such that $|Y'| = |Y|$ and $w(Y) = w(Y')$, then $Y' = Y$. Thus, the $(\mathbb{C}^*)^2$-fixed point set in $\mathfrak{M}(w(Y),e_0)$ consists of only one point. Using the Bialynicki-Birula theorem we can construct a cell decomposition of $\mathfrak{M}(w(Y),e_0)$ and it is easy to see that the unique cell has dimension 0. Therefore, $\mathfrak{M}(w(Y),e_0)$ is just a point. By (10), $w_0(Y) = v_0(\Psi(Y))$ and clearly $|Y| = n(\Psi(Y))$. □

Proof of Lemma 4 For $\lambda = \mathbf{0}$ this lemma was proved in [11].

Since $[\mathfrak{M}(v,e_0)] = P_q^{BM}(\mathfrak{M}(v,e_0))|_{q=\mathbb{L}}$, it is sufficient to prove that

$$\sum_{\substack{v\in\mathbb{Z}^m_{\geq 0}\\ v_{i+1}-v_i=\lambda_i}} \big[\mathfrak{M}(v,e_0)\big]t^{|v|} = \frac{t^{n(\lambda)}}{\prod_{i\geq 1}(1-\mathbb{L}^i t^{mi})^{m-1}(1-\mathbb{L}^{i+1}t^{mi})}.$$

The $\Gamma_{1,-1}$-action on $\mathbb{C}^2\backslash\mathbb{C}_x$ is free. Therefore, if the intersection of $\mathfrak{M}(v,e_0)$ with $\pi^{-1}(S^{|v|}(\mathbb{C}^2\backslash\mathbb{C}_x))$ is nonempty, then $v_0=v_1=\cdots=v_{m-1}$. We get

$$H_{\mathbb{C}^2,\lambda}(t)=H_{\mathbb{C}_x,\lambda}(t)H_{\mathbb{C}^2\backslash\mathbb{C}_x,\mathbf{0}}(t). \tag{14}$$

We denote by O the origin of $\mathbb{C}^2$. Let

$$H_O(t)=\sum_{n\geq 0}\left[\mathcal{M}(1,n)\cap\pi^{-1}\left(S^n(O)\right)\right]t^n.$$

From [11] (see Theorem 1) it follows that

$$H_{\mathbb{C}^2\backslash\mathbb{C}_x,\mathbf{0}}(t)=H_O\left(t^m\right)^{[(\mathbb{C}^2\backslash\mathbb{C}_x)/\Gamma_{1,-1}]}.$$

It is easy to check that $[(\mathbb{C}^2\backslash\mathbb{C}_x)/\Gamma_{1,-1}]=\mathbb{L}^2-\mathbb{L}$. Therefore we have

$$H_{\mathbb{C}^2\backslash\mathbb{C}_x,\mathbf{0}}(t)=\left(\prod_{i\geq 1}\frac{1}{(1-\mathbb{L}^{i-1}t^{mi})}\right)^{\mathbb{L}^2-\mathbb{L}}\overset{\text{by (5)}}{=}\prod_{i\geq 1}\frac{1-\mathbb{L}^i t^{mi}}{1-\mathbb{L}^{i+1}t^{mi}}. \tag{15}$$

If we combine formulas (11), (14) and (15) and also Lemma 6, we get the proof of the lemma. □

Using Lemma 4 we get

$$\sum_{\lambda\in\Pi^{m-1}}t^{n(\lambda)}\sum_{\substack{v\in\mathbb{Z}^m_{\geq 0}\\ v_{i+1}-v_i=\lambda_i}}P_q^{BM}\left(\mathfrak{M}(v,e_0)\right)\left(q^{-\frac{1}{m}}t\right)^{m(v_0-v_0(\lambda))}$$
$$=\left(\prod_{i\geq 1}\frac{1}{(1-t^{mi})^{m-1}(1-qt^{mi})}\right)\left(\sum_{\lambda\in\Pi^{m-1}}t^{n(\lambda)}\right).$$

By Lemma 6, $\sum_{\lambda\in\Pi^{m-1}}t^{n(\lambda)}=\sum_{Y\in Core_m}t^{|Y|}$. We have (see e.g. [11])

$$\sum_{Y\in Core_m}t^{|Y|}=\prod_{i\geq 1}\frac{(1-t^{mi})^m}{(1-t^i)}.$$

This completes the proof of the theorem.

4 Proofs of Theorems 2 and 3

Here we prove two combinatorial identities from Sect. 1.1.

4.1 Proof of Theorem 2

Consider the $(\mathbb{C}^*)^2$-action on $((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}}$. Let $p \in (\mathbb{C}^2)^{[n]}$ be the fixed point corresponding to a Young diagram Y. By (4), the weight decomposition of $T_p(((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}})$ is given by

$$T_p\left(\left((\mathbb{C}^2)^{[n]}\right)^{T_{\alpha,\beta}}\right) = \sum_{\substack{s\in Y\\ \alpha(l_Y(s)+1)=\beta a_Y(s)}} t_1^{l_Y(s)+1} t_2^{-a_Y(s)} + \sum_{\substack{s\in Y\\ \alpha l_Y(s)=\beta(a_Y(s)+1)}} t_1^{-l_Y(s)} t_2^{a_Y(s)+1}. \tag{16}$$

Let γ be a big positive integer γ. By [1, 2], the variety $((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}}$ has a cellular decomposition with the cells $C_p = \{z \in ((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}} \,|\, \lim_{t\to 0, t\in T_{1,\gamma}} tz = p\}$. By (16), we have $\dim C_p = \sharp\{s \in Y | \alpha l_Y(s) = \beta(a_Y(s)+1)\}$. Thus, we have

$$\sum_{n\geq 0} P_q\left(\left((\mathbb{C}^2)^{[n]}\right)^{T_{\alpha,\beta}}\right) t^n = \sum_{Y\in\mathcal{Y}} q^{\sharp\{s\in Y | \alpha l_Y(s)=\beta(a_Y(s)+1)\}} t^{|Y|}.$$

Now Theorem 2 follows from Theorem 1.

4.2 Proof of Theorem 3

In [12] it is proved that the set of irreducible components of the variety $((\mathbb{C}^2)^{[n]})^{T_{1,1}}$ is parametrized by partitions λ such that $\frac{\lambda_1(\lambda_1-1)}{2} + |\lambda| = n$. The Poincaré polynomial of the irreducible component corresponding to a partition λ is equal to (see [12])

$$\prod_{i\geq 1} \begin{bmatrix} \lambda_i - \lambda_{i+2} + 1 \\ \lambda_{i+1} - \lambda_{i+2} \end{bmatrix}_q.$$

Combining this fact with Theorem 1 we get the proof of Theorem 3.

5 Proof of Theorem 5

Let $\alpha + \beta = m$. Similar to Lemma 1 we have the decomposition

$$\mathcal{M}(r,n)^{\Gamma^{\omega}_{\alpha,\beta}} = \coprod_{\substack{v\in\mathbb{Z}^m_{\geq 0}\\ |v|=n}} \mathfrak{M}(v, \boldsymbol{\mu}), \tag{17}$$

and the $T^{\boldsymbol{\omega}}_{\alpha,\beta}$-action on the left-hand side of (17) corresponds to the $\widetilde{T}_{\alpha,\beta}$-action on the right-hand side. Using Theorem 4 we get

$$\sum_{n\geq 0} h_0\big(\mathcal{M}(r,n)^{T^{\boldsymbol{\omega}}_{\alpha,\beta}}\big)q^n = \sum_{v\in\mathbb{Z}^m_{\geq 0}} \dim H^{BM}_{\frac{1}{2}\dim\mathfrak{M}(v,\boldsymbol{\mu})}\big(\mathfrak{M}(v,\boldsymbol{\mu})\big)q^{|v|}.$$

In [20] it is proved that the space $\bigoplus_{v\in\mathbb{Z}^m_{\geq 0}} H^{BM}_{\frac{1}{2}\dim\mathfrak{M}(v,\boldsymbol{\mu})}(\mathfrak{M}(v,\boldsymbol{\mu}))$ is an irreducible highest weight representation of $\widehat{sl}_m$ with the highest weight $\boldsymbol{\mu}$. This completes the proof of Theorem 5.

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PBW-filtration over $\mathbb{Z}$ and Compatible Bases for $V_{\mathbb{Z}}(\lambda)$ in Type $\mathtt{A}_n$ and $\mathtt{C}_n$

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Abstract We study the PBW-filtration on the highest weight representations $V(\lambda)$ of the Lie algebras of type $\mathtt{A}_n$ and $\mathtt{C}_n$. This filtration is induced by the standard degree filtration on $\mathrm{U}(\mathfrak{n}^-)$. In previous papers, the authors studied the filtration and the associated graded algebras and modules over the complex numbers. The aim of this paper is to present a proof of the results which holds over the integers and hence makes the whole construction available over any field.

1 Introduction

Let $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra, we fix a maximal torus $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Denote by R the set of roots and let P be the integral weight lattice. Corresponding to the choice of $\mathfrak{b}$, let R^+ be the set of positive roots and let P^+ be the monoid of dominant weights.

For $\lambda \in P^+$ let $V(\lambda)$ be the finite dimensional irreducible representation of highest weight λ and let v_λ be a highest weight vector. Denote by $M(\lambda)$ the Verma module corresponding to the same highest weight. For a Lie algebra $\mathfrak{a}$ denote by $\mathrm{U}(\mathfrak{a})$ its enveloping algebra. Fix a highest weight vector $m_\lambda \in M(\lambda)$. The linear map

$$\mathrm{U}(\mathfrak{n}^-) \to M(\lambda), \qquad \mathbf{n} \mapsto \mathbf{n} m_\lambda$$

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is an isomorphism of complex vector spaces. The degree filtration on $\mathrm{U}(\mathfrak{n}^-)$:

$$\mathrm{U}\big(\mathfrak{n}^-\big)_0 = \mathbb{C}1, \qquad \mathrm{U}\big(\mathfrak{n}^-\big)_s = \mathrm{span}\big\{1, x_1 \ldots x_l : x_i \in \mathfrak{n}^-, l \leq s\big\} \quad \text{for } s \geq 1,$$

induces via the isomorphism above a natural $\mathfrak{b}$-stable filtration on $M(\lambda)$:

$$M(\lambda)_s = \mathrm{U}\big(\mathfrak{n}^-\big)_s m_\lambda \quad \text{for } s \geq 0.$$

Set $\mathrm{U}(\mathfrak{n}^-)_{-1} = M(\lambda)_{-1} = 0$, then the associated q-character

$$\mathrm{char}_q M(\lambda) := \sum_{s \geq 0} \mathrm{char}\big(M(\lambda)_s / M(\lambda)_{s-1}\big) q^s$$

has a very simple form:

$$\mathrm{char}_q M(\lambda) = e^\lambda \frac{1}{\prod_{\beta \in R^+}(1 - q e^{-\beta})}.$$

This is obvious by the fact that the associated graded module $M(\lambda)^a = \bigoplus_{s \geq 0} M(\lambda)_s / M(\lambda)_{s-1}$ is a free module over the associated graded algebra $S(\mathfrak{n}^-) = \mathrm{grad}\, \mathrm{U}(\mathfrak{n}^-)$.

In contrast, the situation becomes rather complicated if one replaces $M(\lambda)$ by its finite dimensional quotient $V(\lambda)$. Again this module has an induced $\mathfrak{b}$-stable filtration $V(\lambda)_s = \mathrm{U}(\mathfrak{n}^-)_s v_\lambda$, called the *Poincaré-Brikhoff-Witt-filtration*, or, for short, just the PBW-filtration. The associated graded module $V(\lambda)^a = \bigoplus_{s \geq 0} V(\lambda)_s / V(\lambda)_{s-1}$ is a $\mathrm{U}(\mathfrak{b})$-module as well as a $S(\mathfrak{n}^-)$-module. A general closed formula for the q-character

$$\mathrm{char}_q V(\lambda) := \sum_{s \geq 0} \mathrm{char}\big(V(\lambda)_s / V(\lambda)_{s-1}\big) q^s$$

is not known, partial combinatorial answers can be found in [4, 5], more geometric interpretations can be found in [3, 6]. Another natural (and, at least in the general case, open) question is about the structure of $V(\lambda)^a$ as a cyclic $S(\mathfrak{n}^-)$-module, generated by the image of the highest weight vector.

The aim of this paper is to present a proof of the results in [4, 5] which holds over the integers and hence makes the whole construction available over any field. More precisely, for $\mathfrak{g}$ of type $\mathtt{A}_n$ or type $\mathtt{C}_n$ we want

- to describe $V^a_{\mathbb{Z}}(\lambda)$ as a cyclic $S_{\mathbb{Z}}(\mathfrak{n}^-)$-module, i.e. describe the ideal $I_{\mathbb{Z}}(\lambda) \hookrightarrow S_{\mathbb{Z}}(\mathfrak{n}^-)$ such that $V^a_{\mathbb{Z}}(\lambda) \simeq S_{\mathbb{Z}}(\mathfrak{n}^-)/I_{\mathbb{Z}}(\lambda)$;
- to find a basis of $V^a_{\mathbb{Z}}(\lambda)$, in particular, show that $V^a_{\mathbb{Z}}(\lambda)$ is torsion free;
- to get a (characteristic free) combinatorial graded character formula for $V^a_{\mathbb{Z}}(\lambda)$.

As a last remark we would like to point out that one should not confuse the PBW-filtration (discussed in this paper) neither with the Brylinski-Kostant filtration [2] (BK-filtration for short) on the weight spaces induced by a principal $\mathfrak{sl}_2$-triple (e, h, f), nor with the right Brylinski-Kostant filtration discussed in [7]. As an example, consider the case $\mathfrak{g}$ of type $\mathtt{B}_2$ and $\lambda = \omega_1 + 2\omega_2$. In Table 1 we list for

Table 1 Examples for the Poincaré polynomial of the associated graded weight spaces in $V(\lambda)$, $\lambda = \omega_1 + 2\omega_2$, $\mathfrak{g}$ of type $\mathtt{B}_2$, enumeration as in [1]

Weight	$\lambda - \alpha_1 - 3\alpha_2$	$\lambda - 2\alpha_1 - 2\alpha_2$	$\lambda - 2\alpha_1 - 3\alpha_2$	$\lambda - 2\alpha_1 - 4\alpha_2$
PBW	$q^3 + q^2$	$q^3 + 2q^2$	$2q^3 + q^2$	$q^4 + q^3 + q^2$
BK	$q^4 + q^3$	$q^4 + q^3 + q^2$	$q^5 + q^4 + q^3$	$q^6 + q^5 + q^4$
Right BK	$q^4 + q^2$	$q^4 + q^3 + q^2$	$q^5 + q^4 + q^3$	$q^6 + q^5 + q^4$

some weights the Poincaré polynomial of the associated graded weight space. For the left and right Brylinski-Kostant filtration, the polynomials have been taken from [7], for the PBW-filtration the polynomials have been calculated using Theorem 3 ($\mathtt{B}_2 = \mathtt{C}_2$).

2 The Setup over the Complex Numbers: Definitions and Notation

Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra. We fix a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Let R^+ be the set of positive roots corresponding to the choice of $\mathfrak{b}$ and let α_i, ω_i, $i = 1, \ldots, n$ be the simple roots and the fundamental weights. The height $ht(\beta)$ of a positive root is the sum of the coefficients of the expression of β as a sum of simple roots.

Let G be the simple, simply connected algebraic group such that $\operatorname{Lie} G = \mathfrak{g}$. Fix a maximal torus $T \subset G$ and a Borel subgroup $B \supset T$ such that $\operatorname{Lie} B = \mathfrak{h} \oplus \mathfrak{n}^+$ and $\operatorname{Lie} T = \mathfrak{h}$. Denote by N^- the unipotent radical of the opposite Borel subgroup.

Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be the Cartan decomposition. Consider the increasing degree filtration on the universal enveloping algebra of $\mathrm{U}(\mathfrak{n}^-)$:

$$\mathrm{U}\big(\mathfrak{n}^-\big)_s = \operatorname{span}\big\{1, x_1 \ldots x_l : \ x_i \in \mathfrak{n}^-, l \le s\big\}, \tag{1}$$

for example, $\mathrm{U}(\mathfrak{n}^-)_0 = \mathbb{C} \cdot 1$, $\mathrm{U}(\mathfrak{n}^-)_1 = \mathbb{C} \cdot 1 + \mathfrak{n}^-$, and so on. The associated graded algebra is the symmetric algebra $S(\mathfrak{n}^-)$ over $\mathfrak{n}^-$.

For a dominant integral weight λ let $\Psi : G \to \mathrm{GL}(V(\lambda))$ and $\psi : \mathfrak{g} \to \operatorname{End}(V(\lambda))$ be the corresponding irreducible representations. Fix a highest weight vector v_λ. Since $V(\lambda) = \mathrm{U}(\mathfrak{n}^-)v_\lambda$, the filtration in (1) induces an increasing filtration $V(\lambda)_s$ on $V(\lambda)$:

$$V(\lambda)_s = \mathrm{U}\big(\mathfrak{n}^-\big)_s v_\lambda.$$

Definition 1 We call this filtration the *PBW-filtration of* $V(\lambda)$ and we denote the associated graded space by $V^a(\lambda)$.

Let $\mathfrak{n}_s^- = \sum_{ht(\beta)\ge s} \mathfrak{n}_{-\beta}^- \subseteq \mathfrak{n}^-$ be the Lie subalgebra formed by the root subspaces corresponding to roots of height at least s. In fact, $\mathfrak{n}_s^- \subset \mathfrak{n}^-$ is an ideal, and the associated graded algebra $\mathfrak{n}^{-,a} = \bigoplus_{s\geq 1} \mathfrak{n}_s^-/\mathfrak{n}_{s+1}^-$ is an abelian Lie algebra. We make $\mathfrak{n}^{-,a}$ into a B- as well as a $\mathfrak{b}$-module by identifying the vector space $\mathfrak{n}^{-,a}$ with the quotient space $\mathfrak{g}/\mathfrak{b}$, which is a B- respectively $\mathfrak{b}$-module via the induced adjoint action $\overline{ad} : B \to \mathrm{GL}(\mathfrak{g}/\mathfrak{b})$.

Definition 2 Denote by $\mathfrak{g}^a$ the Lie algebra $\mathfrak{g}^a = \mathfrak{b} \oplus \mathfrak{n}^{-,a}$, where $\mathfrak{n}^{-,a}$ is an abelian ideal in $\mathfrak{g}^a$ and $\mathfrak{b}$ acts on $\mathfrak{n}^{-,a}$ via the induced adjoint action described above.

For a positive root β let $U_{-\beta} \subset G$ be the closed root subgroup corresponding to the root $-\beta$. Denote by $\mathbb{G}_a$ the additive group of the field (viewed as a one-dimensional unipotent algebraic group) and let $x_{-\beta} : \mathbb{G}_{a,\beta} \to U_{-\beta}$ be a fixed isomorphism of the root subgroup with the additive group $\mathbb{G}_a$. We add the root as an index to indicate that this copy $\mathbb{G}_{a,\beta}$ of the additive group is related to $U_{-\beta}$.

The group N^- admits a filtration by a sequence of normal subgroups: let $N_s^- = \prod_{ht(\beta)\geq s} U_{-\beta}$, then N_s^- is a normal subgroup of N^-. Denote by $N^{-,a}$ the product $N^{-,a} = \prod_{s\geq 1} N_s^-/N_{s+1}^-$, then $N^{-,a}$ is a commutative unipotent group. We can identify $N^{-,a}$ naturally with the product $\prod_{\beta\in R^+} \mathbb{G}_{a,\beta}$, viewed as a product of commuting additive groups. Here $\mathbb{G}_{a,\beta}$ gets identified with the image of $U_{-\beta}$ in $N^-_{ht(\beta)}/N^-_{ht(\beta)+1}$. The Lie algebra of $N^{-,a}$ is $\mathfrak{n}^{-,a}$.

The action $\overline{ad}$ of B on $\mathfrak{n}^{-,a}$ can be lifted to an action $\overline{Ad}$ on $N^{-,a}$ using the exponential map. To make this action more explicit, recall that for two linearly independent roots α, β we know by Chevalley's commutator formula: there exist complex numbers $c_{i,j,\alpha,\beta}$ such that

$$x_\alpha(t)x_\beta(s)x_\alpha^{-1}(t)x_\beta^{-1}(s) = \prod_{i,j>0} x_{i\alpha+j\beta}\big(c_{i,j,\alpha,\beta}t^i s^j\big)$$

for all $s, t \in \mathbb{C}$. The product is taken over all pairs $i, j \in \mathbb{Z}_{>0}$ such that $i\alpha + j\beta$ is a root and in order of increasing height of the occurring roots. We have for $m = \prod_{\beta\in R^+} x_{-\beta}(u_\beta) \in N^{-,a}$ and $x_\alpha(t) \in B$, $u_\beta, t \in \mathbb{C}$:

$$\overline{Ad}\big(x_\alpha(t)\big)(m) = \prod_{\beta\in R^+} x_{-\beta}\Bigg(u_\beta + \sum_{\substack{i,j>0,\gamma\in R^+\\ -\beta=i\alpha-j\gamma}} c_{i,j,\alpha,-\gamma}t^i u_\gamma^j\Bigg). \tag{2}$$

Definition 3 Denote by G^a the semi-direct product $G^a \simeq B \ltimes N^{-,a}$, where $N^{-,a}$ is an abelian normal subgroup in G^a and B acts on $N^{-,a}$ via the action described above.

The subspaces $V(\lambda)_s = \mathrm{U}(\mathfrak{n}^-)_s v_\lambda$ are stable with respect to the B- and the $\mathfrak{b}$-action, so we get an induced action of B as well as of $\mathfrak{b}$ on $V^a(\lambda)$. Since the application by an element $f \in \mathfrak{n}^-$ induces linear maps

$$\begin{array}{cccc} f: & V(\lambda)_s & \to & V(\lambda)_{s+1} \\ & \cup & & \cup \\ & V(\lambda)_{s-1} & \to & V(\lambda)_s \end{array}$$

we get an induced endomorphism $\psi^a(f) : V^a(\lambda) \to V^a(\lambda)$ with the property that $\psi^a(f)\psi^a(f') - \psi^a(f')\psi^a(f) : V^a(\lambda) \to V^a(\lambda)$ is the zero map for $f, f' \in \mathfrak{n}^-$. Hence we get an induced representation of the abelian Lie algebra $\mathfrak{n}^{-,a}$ and of its enveloping algebra $S(\mathfrak{n}^{-,a})$, the symmetric algebra over $\mathfrak{n}^{-,a}$. Note that $V^a(\lambda)$ is a cyclic $S(\mathfrak{n}^{-,a})$-module:

$$V^a(\lambda) = S\big(\mathfrak{n}^{-,a}\big).v_\lambda.$$

The action of $\mathfrak{n}^{-,a}$ on $V^a(\lambda)$ is compatible with the B-action on $V^a(\lambda)$ and on $\mathfrak{n}^{-,a}$: suppose $b \in B$, $f \in \mathfrak{n}^-$ and $v \in V(\lambda)_s$, then

$$b(f.v) = \big(bfb^{-1}\big)(bv) = \big(\overline{ad}(b)(f)\big)bv + m.bv \quad \text{for some } m \in \mathfrak{b},$$

and hence $bf.v = (\overline{ad}(b)(f))bv$ in $V(\lambda)_{s+1}/V(\lambda)_s$. It follows:

Proposition 1 *$V^a(\lambda)$ is a $\mathfrak{g}^a$-module, it is a cyclic $S(\mathfrak{n}^{-,a})$-module and a B-module. The B-action on $S(\mathfrak{n}^{-,a})$ is compatible with the B-action on $V^a(\lambda) = S(\mathfrak{n}^{-,a}).v_\lambda$*

The action of $U_{-\beta}$ on $V(\lambda)$ is given by:

$$\Psi\big(x_{-\beta}(t)\big)(v) = \sum_{i\geq 0} t^i \psi\left(\frac{f_\beta^i}{i!}\right)(v) \quad \text{for } v \in V(\lambda) \text{ and } t \in \mathbb{C}$$

and we get an induced action of $U_{-\beta}$ on $V^a(\lambda)$ by

$$\Psi^a\big(x_{-\beta}(t)\big)(v) = \sum_{i\geq 0} t^i \psi^a\left(\frac{f_\beta^i}{i!}\right)(v) \quad \text{for } v \in V^a(\lambda) \text{ and } t \in \mathbb{C}.$$

The action of the various $U_{-\beta}$ on $V^a(\lambda)$ commute and hence we get a representation $\Psi^a : N^{-,a} \to GL(V^a(\lambda))$. This action is compatible with the B-action on $V^a(\lambda)$ and hence:

Proposition 2 *$V^a(\lambda)$ is a representation space for G^a.*

In analogy to the classical construction we define:

Definition 4 The closure of the orbit $\overline{G^a.[v_\lambda]} \subseteq \mathbb{P}(V^a(\lambda))$ is called the degenerate flag variety $\mathcal{F}^a_\lambda$.

3 The Kostant Lattice

Let $G_{\mathbb{Z}}$ be a split and simple, simply connected algebraic $\mathbb{Z}$-group (see [8]), set $G_A = (G_{\mathbb{Z}})_A$ for any ring A. We assume without loss of generality $(G_{\mathbb{Z}})_{\mathbb{C}} = G$. We fix a split maximal torus $T_{\mathbb{Z}} \subset G_{\mathbb{Z}}$ such that $T = (T_{\mathbb{Z}})_{\mathbb{C}}$ and a Borel subgroup

$B_{\mathbb{Z}} \supset T_{\mathbb{Z}}$ such that $B = (B_{\mathbb{Z}})_{\mathbb{C}}$. Let $\mathfrak{g}_{\mathbb{Z}}$, $\mathfrak{b}_{\mathbb{Z}}$, $\mathfrak{n}^+_{\mathbb{Z}}$ etc. be the Lie algebras, then we have $\mathfrak{g} = \mathfrak{g}_{\mathbb{Z}} \otimes \mathbb{C}$, $\mathfrak{b} = \mathfrak{b}_{\mathbb{Z}} \otimes \mathbb{C}$ etc.

Fix a Chevalley basis

$$\left\{f_\beta, e_\beta : \beta \in R^+; h_1, \ldots, h_n\right\} \subset \mathfrak{g}_{\mathbb{Z}},$$

where $f_\beta \in \mathfrak{n}^-_{\mathbb{Z}}$ (respectively $e_\beta \in \mathfrak{n}^+_{\mathbb{Z}}$) is an element of the root space $\mathfrak{g}_{-\beta,\mathbb{Z}}$ (respectively $\mathfrak{g}_{\beta,\mathbb{Z}}$), and $h_i \in \mathfrak{h}_{\mathbb{Z}}$.

Let $\mathfrak{n}^-_{\mathbb{Z},s} = \sum_{ht(\beta)\geq s} \mathfrak{n}^-_{-\beta,\mathbb{Z}}$ be the Lie subalgebra formed by the root spaces corresponding to roots of height at least s. The Lie subalgebra $\mathfrak{n}^-_{\mathbb{Z},s+1} \subset \mathfrak{n}^-_{\mathbb{Z},s}$ is an ideal, and the associated graded algebra $\mathfrak{n}^{-,a}_{\mathbb{Z}} = \bigoplus_{s\geq 1} \mathfrak{n}^-_{\mathbb{Z},s}/\mathfrak{n}^-_{\mathbb{Z},s+1}$ is an abelian Lie algebra. We make $\mathfrak{n}^{-,a}_{\mathbb{Z}}$ into a $B_{\mathbb{Z}}$- as well as a $\mathfrak{b}_{\mathbb{Z}}$-module by identifying the vector space $\mathfrak{n}^{-,a}_{\mathbb{Z}}$ with the quotient module $\mathfrak{g}_{\mathbb{Z}}/\mathfrak{b}_{\mathbb{Z}}$, which is a $B_{\mathbb{Z}}$- respectively $\mathfrak{b}_{\mathbb{Z}}$-module via the adjoint action.

Definition 5 Denote by $\mathfrak{g}^a_{\mathbb{Z}}$ the Lie algebra $\mathfrak{g}^a_{\mathbb{Z}} = \mathfrak{b}_{\mathbb{Z}} \oplus \mathfrak{n}^{-,a}_{\mathbb{Z}}$, where $\mathfrak{n}^{-,a}_{\mathbb{Z}}$ is an abelian ideal in $\mathfrak{g}^a_{\mathbb{Z}}$ and $\mathfrak{b}_{\mathbb{Z}}$ acts on $\mathfrak{n}^{-,a}_{\mathbb{Z}}$ via the induced adjoint action described above.

We write $e^{(m)}_\beta$, $f^{(m)}_\beta$ for the divided powers $\frac{f^m_\beta}{m!}$ and $\frac{e^m_\beta}{m!}$ in the enveloping algebra $\mathrm{U}(\mathfrak{g})$. We denote by $\binom{h_i}{m}$ the following element in $\mathrm{U}(\mathfrak{g})$:

$$\binom{h_i}{m} = \frac{h_i(h_i-1)\cdots(h_i-m+1)}{m!}.$$

Let now $\mathrm{U}_{\mathbb{Z}}(\mathfrak{g})$ be the Kostant lattice in $\mathrm{U}(\mathfrak{g})$, i.e. the subalgebra generated by the $\binom{h_i}{m}$ and the divided powers $e^{(m)}_\beta$, $f^{(m)}_\beta$. We identify $\mathrm{U}_{\mathbb{Z}}(\mathfrak{g})$ with $\mathrm{Dist}(G_{\mathbb{Z}})$, the algebra of distributions or the hyperalgebra of $G_{\mathbb{Z}}$. We fix an enumeration of the positive roots $\{\beta_1, \ldots, \beta_N\}$. Given an N-tuple $\mathbf{m} = (m_1, \ldots, m_N)$ of non-negative integers, we set

$$f^{(\mathbf{m})} = f^{(m_1)}_{\beta_1} \cdots f^{(m_N)}_{\beta_N}, \qquad e^{(\mathbf{m})} = e^{(m_1)}_{\beta_1} \cdots e^{(m_N)}_{\beta_N},$$

and given an n-tuple $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_n)$, set

$$h^{(\boldsymbol{\ell})} = \binom{h_1}{\ell_1} \cdots \binom{h_n}{\ell_n}.$$

The ordered monomials

$$f^{(\mathbf{m})} h^{(\boldsymbol{\ell})} e^{(\mathbf{k})}, \quad \text{where } \mathbf{m}, \mathbf{k} \text{ are } N\text{-tuples}, \boldsymbol{\ell} \text{ is an } n\text{-tuple of natural numbers},$$

form a $\mathbb{Z}$-basis of $\mathrm{U}_{\mathbb{Z}}(\mathfrak{g})$ as a free $\mathbb{Z}$-module. The subalgebras $\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^-)$ and $\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^+)$ admit the ordered monomials

$$\left\{f^{(\mathbf{m})} \mid m_1, \ldots, m_N \in \mathbb{Z}_{\geq 0}\right\}$$

respectively

$$\left\{e^{(\mathbf{m})} \mid m_1, \dots, m_N \in \mathbb{Z}_{\geq 0}\right\}$$

as bases.

Let $\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^-)_s$ be the $\mathbb{Z}$-span of the monomials of degree at most s:

$$\mathrm{U}_{\mathbb{Z}}\left(\mathfrak{n}^-\right)_s = \left\langle f_{\gamma_1}^{(m_1)} \dots f_{\gamma_\ell}^{(m_\ell)} \mid m_1 + \cdots + m_\ell \leq s, \gamma_1, \dots, \gamma_\ell \in R^+ \right\rangle_{\mathbb{Z}}, \qquad (3)$$

where the degree of $f_{\gamma_1}^{(m_1)} \dots f_{\gamma_\ell}^{(m_\ell)}$ is the sum $m_1 + \cdots + m_\ell$. Since changing the ordering is commutative up to terms of smaller degree, the $\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^-)_s$ define a filtration of the algebra $\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^-)$. By abuse of notation denote by $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ the associated graded algebra. Note that $\mathfrak{n}_{\mathbb{Z}}^{-,a} \subset S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$. In fact, $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ is a divided power analogue of the symmetric algebra over $\mathfrak{n}_{\mathbb{Z}}^{-,a}$. This algebra can be described as the quotient of a polynomial algebra in infinitely many generators (the "symbols" $\mathfrak{f}_\beta^{(m)}$): $\mathbb{Z}[\mathfrak{f}_\beta^{(m)} \mid m \in \mathbb{Z}_{\geq 0}, \beta \in R^+]$ modulo the ideal $\mathfrak{J}$ generated by the following identities:

$$\mathfrak{J} = \left\langle \mathfrak{f}_\beta^{(m)} \mathfrak{f}_\beta^{(k)} - \binom{m+k}{m} \mathfrak{f}_\beta^{(m+k)} \,\middle|\, k, m \geq 1, \beta \in R^+ \right\rangle. \qquad (4)$$

So we have:

$$S_{\mathbb{Z}}\left(\mathfrak{n}^{-,a}\right) \simeq \mathbb{Z}\left[\mathfrak{f}_\beta^{(m)} \mid m \in \mathbb{Z}_{\geq 0}, \beta \in R^+\right] / \mathfrak{J}.$$

The isomorphism above sends the basis given by classes of the monomials in the symbols $\mathfrak{f}_{\beta_1}^{(m_1)} \cdots \mathfrak{f}_{\beta_N}^{(m_N)}$ to the basis of $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ given by the monomials $f_{\beta_1}^{(m_1)} \cdots f_{\beta_N}^{(m_N)}$.

Let $\mathrm{U}_{\mathbb{Z}}^+(\mathfrak{h} + \mathfrak{n}^+) \subset \mathrm{U}_{\mathbb{Z}}(\mathfrak{g})$ be the span of the monomials $h^{(\boldsymbol{\ell})} e^{(\mathbf{k})}$ such that $\sum_{i=1}^n \ell_i + \sum_{j=1}^N k_j > 0$. The natural map which sends a monomial to its class in the quotient:

$$\mathrm{U}_{\mathbb{Z}}\left(\mathfrak{n}^-\right) \to \mathrm{U}_{\mathbb{Z}}(\mathfrak{g}) / \mathrm{U}_{\mathbb{Z}}\left(\mathfrak{n}^-\right) \mathrm{U}_{\mathbb{Z}}^+\left(\mathfrak{h} + \mathfrak{n}^+\right), \qquad f^{(\mathbf{m})} \to \overline{f^{(\mathbf{m})}},$$

is an isomorphism of free $\mathbb{Z}$-modules. Recall that $\mathrm{U}_{\mathbb{Z}}(\mathfrak{g})$ is naturally a $B_{\mathbb{Z}}$-module and a $\mathrm{U}_{\mathbb{Z}}(\mathfrak{b})$-module via the adjoint action, and $\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^-)\mathrm{U}_{\mathbb{Z}}^+(\mathfrak{h} + \mathfrak{n}^+)$ is a proper submodule. Via the identification above, we get an induced structure on $\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^-)$ as a $B_{\mathbb{Z}}$-module and a $\mathrm{U}_{\mathbb{Z}}(\mathfrak{b})$-module. The filtration of $\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^-)$ by the $\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^-)_s$ is stable under this $B_{\mathbb{Z}}$- and $\mathrm{U}_{\mathbb{Z}}(\mathfrak{b})$-action and hence:

Lemma 1 *The $B_{\mathbb{Z}}$-module structure and the $\mathrm{U}_{\mathbb{Z}}(\mathfrak{b})$-module structure on $\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^-)$ induce a $B_{\mathbb{Z}}$-module structure and a $\mathrm{U}_{\mathbb{Z}}(\mathfrak{b})$-module structure on $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$.*

For a dominant integral weight $\lambda = m_1\omega_1 + \cdots + m_n\omega_n$ fix a highest weight vector v_λ and let $V_{\mathbb{Z}}(\lambda) = \mathrm{U}_{\mathbb{Z}}(\mathfrak{g})v_\lambda \subset V(\lambda)$ be the corresponding lattice in the complex representation space. Since $V_{\mathbb{Z}}(\lambda) = \mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^-)v_\lambda$, the filtration (3) induces an increasing filtration $V_{\mathbb{Z}}(\lambda)_s$ on $V_{\mathbb{Z}}(\lambda)$:

$$V_{\mathbb{Z}}(\lambda)_s = \mathrm{U}_{\mathbb{Z}}\big(\mathfrak{n}^-\big)_s v_\lambda. \tag{5}$$

We denote the associated graded space by $V^a_{\mathbb{Z}}(\lambda)$. Since $B_{\mathbb{Z}} V_{\mathbb{Z}}(\lambda)_s \subset V_{\mathbb{Z}}(\lambda)_s$, $V^a_{\mathbb{Z}}(\lambda)$ becomes naturally a $B_{\mathbb{Z}}$-module. The application by an element $f_\beta^{(m)} \in \mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^-)$ provides linear maps for all s:

$$\begin{array}{ccc} f_\beta^{(m)}: \ V_{\mathbb{Z}}(\lambda)_s & \to & V_{\mathbb{Z}}(\lambda)_{s+m} \\ \cup & & \cup \\ V_{\mathbb{Z}}(\lambda)_{s-1} & \to & V_{\mathbb{Z}}(\lambda)_{s+m-1}, \end{array}$$

and we get an induced endomorphism $\psi^a(f_\beta^{(m)}) : V^a_{\mathbb{Z}}(\lambda) \to V^a_{\mathbb{Z}}(\lambda)$ such that $\psi^a(f_\beta^{(m)})\psi^a(f_\gamma^{(\ell)}) = \psi^a(f_\gamma^{(\ell)})\psi^a(f_\beta^{(m)})$, and hence we get an induced representation of the abelian Lie algebra $\mathfrak{n}^{-,a}_{\mathbb{Z}}$ and of the algebra $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$. Note that $V^a_{\mathbb{Z}}(\lambda)$ is a cyclic $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$-module:

$$V^a_{\mathbb{Z}}(\lambda) = S_{\mathbb{Z}}\big(\mathfrak{n}^{-,a}\big) v_\lambda.$$

The action of $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ on $V^a_{\mathbb{Z}}(\lambda)$ is compatible with the $B_{\mathbb{Z}}$-action on $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ and on $V^a(\lambda)$, so summarizing we have:

Proposition 3 *$V^a_{\mathbb{Z}}(\lambda)$ is a $\mathfrak{g}^a_{\mathbb{Z}}$-module, it is a cyclic $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$-module and a $B_{\mathbb{Z}}$-module. The $B_{\mathbb{Z}}$-action on $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ is compatible with the $B_{\mathbb{Z}}$-action on $V^a_{\mathbb{Z}}(\lambda) = S_{\mathbb{Z}}(\mathfrak{n}^{-,a}).v_\lambda$.*

For a positive root β let $U_{-\beta,\mathbb{Z}} \subset G_{\mathbb{Z}}$ be the closed root subgroup corresponding to the root $-\beta$. We denote by $x_{-\beta} : \mathbb{G}_{a,\mathbb{Z},\beta} \to U_{-\beta,\mathbb{Z}}$ a fixed isomorphism of the root subgroup with the additive group $\mathbb{G}_{a,\mathbb{Z}}$. We add the root as an index to indicate that this copy $\mathbb{G}_{a,\mathbb{Z},\beta}$ of the additive group is supposed to be identified with $U_{-\beta,\mathbb{Z}}$.

As in the case before over the complex numbers, the group $N^-_{\mathbb{Z}}$ admits a filtration by a sequence of normal subgroups: set $N^-_{\mathbb{Z},s} = \prod_{ht(\beta)\geq s} U_{-\beta,\mathbb{Z}}$, the product $N^{-,a}_{\mathbb{Z}} = \prod_{s\geq 1} N^-_{\mathbb{Z},s}/N^-_{\mathbb{Z},s+1}$, is a commutative group. We can identify $N^{-,a}_{\mathbb{Z}}$ naturally with the product $\prod_{\beta\in R^+} \mathbb{G}_{a,\mathbb{Z},\beta}$, viewed as a product of commuting additive groups. Again, $\mathbb{G}_{a,\mathbb{Z},\beta}$ gets identified with the image of $U_{-\beta,\mathbb{Z}}$ in $N^-_{\mathbb{Z},ht(\beta)}/N^-_{\mathbb{Z},ht(\beta)+1}$. The Lie algebra of $N^{-,a}_{\mathbb{Z}}$ is $\mathfrak{n}^{-,a}_{\mathbb{Z}}$.

The action of $U_{-\beta,\mathbb{Z}}$ on $V_{\mathbb{Z}}(\lambda)$ is given by:

$$\Psi\big(u_{-\beta}(t)\big)(v) = \sum_{i\geq 0} t^i \psi\big(f_\beta^{(i)}\big)(v) \quad \text{for } v \in V_{\mathbb{Z}}(\lambda) \text{ and } t \in \mathbb{Z}$$

and we get an induced action of $U_{-\beta,\mathbb{Z}}$ on $V^a_{\mathbb{Z}}(\lambda)$ by

$$\Psi^a\big(u_{-\beta}(t)\big)(v) = \sum_{i\geq 0} t^i \psi^a\big(f_\beta^{(i)}\big)(v) \quad \text{for } v \in V^a_{\mathbb{Z}}(\lambda) \text{ and } t \in \mathbb{Z}.$$

The action of the various $U_{-\beta,\mathbb{Z}}$ on $V^a_{\mathbb{Z}}(\lambda)$ commute and hence we get a representation $\Psi^a : N^{-,a}_{\mathbb{Z}} \to \mathrm{GL}(V^a_{\mathbb{Z}}(\lambda))$. Since we started with a Chevalley basis, by [9], §6, or [10], §3.6, the coefficients in (2) are integral, so we get an action of $B_{\mathbb{Z}}$ on $N^{-,a}_{\mathbb{Z}}$. Denote by $G^a_{\mathbb{Z}}$ the semi-direct product $B_{\mathbb{Z}} \ltimes N^{-,a}_{\mathbb{Z}}$. The actions of $B_{\mathbb{Z}}$ and $N^{-,a}_{\mathbb{Z}}$ on $V^a_{\mathbb{Z}}(\lambda)$ are compatible and hence we get

Proposition 4 *$V^a_{\mathbb{Z}}(\lambda)$ is a $G^a_{\mathbb{Z}}$-module.*

As a consequence, given a field k, we have the group $G^a_k = (G^a_{\mathbb{Z}})_k$, the representation space $V^a_k = (V^a_{\mathbb{Z}})_k$ and the degenerate flag variety $\mathcal{F}^a_{\lambda,k} := \overline{G^a_k.[v_\lambda]} \subset \mathbb{P}(V^a_k(\lambda))$. Here are some natural questions:

(i) is the graded character of $V^a_k(\lambda)$ independent of the characteristic?
(ii) is $V^a_{\mathbb{Z}}(\lambda)$ torsion free?

An explicit monomial basis for $V^a_{\mathbb{C}}(\lambda)$ has been constructed for $G = SL_n$ in [4] and for $G = Sp_{2n}$ in [5]. Another natural question:

(iii) is this basis of $V^a(\lambda)$ compatible with the lattice construction in this section? Or, to put it differently: is $V^a_{\mathbb{Z}}(\lambda)$ a free $\mathbb{Z}$-module with basis $\{f^{(\mathbf{s})}v_\lambda \mid \mathbf{s} \in S(\lambda)\}$? (For the notation see the next sections.)

The aim of the next sections is to give an affirmative answer to these questions for $G = SL_n$ and $G = Sp_{2n}$.

4 Roots and Relations in Type $\mathtt{A}$ and $\mathtt{C}$

Let R^+ be the set of positive roots of $\mathfrak{sl}_{n+1}$. Let α_i, ω_i $i = 1,\dots,n$ be the simple roots and the fundamental weights. All positive roots of $\mathfrak{sl}_{n+1}$ are of the form $\alpha_p + \alpha_{p+1} + \cdots + \alpha_q$ for some $1 \le p \le q \le n$. In the following we denote such a root by $\alpha_{p,q}$, for example $\alpha_i = \alpha_{i,i}$.

Let now R^+ be the set of positive roots of $\mathfrak{sp}_{2n}$. Let α_i, ω_i $i = 1,\dots,n$ be the simple roots and the fundamental weights. All positive roots of $\mathfrak{sp}_{2n}$ can be divided into two groups:

$$\begin{aligned} &\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \quad 1 \le i \le j \le n, \\ &\alpha_{i,\overline{j}} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_n + \alpha_{n-1} + \cdots + \alpha_j, \quad 1 \le i \le j \le n \end{aligned} \tag{6}$$

(note that $\alpha_{i,n} = \alpha_{i,\overline{n}}$). We will use the following short versions

$$\alpha_i = \alpha_{i,i}, \qquad \alpha_{\overline{i}} = \alpha_{i,\overline{i}}.$$

We recall the usual order on the alphabet $J = \{1,\dots,n,\overline{n-1},\dots,\overline{1}\}$

$$1 < 2 < \cdots < n-1 < n < \overline{n-1} < \cdots < \overline{1}. \tag{7}$$

Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be the Cartan decomposition. By Lemma 1, the $\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^+)$-module structure on $\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^-)$ induces a $\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^+)$-module structure on $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$. We want to make this action more explicit for $\mathfrak{g}$ of type $\mathtt{A}$ and $\mathtt{C}$.

If $\alpha = \beta$ or if the root vectors commute, then

$$\big(\mathrm{ad}\, e_\alpha^{(k)}\big)\big(f_\beta^{(m)}\big) = 0. \tag{8}$$

If $\alpha, \gamma, \beta = \alpha + \gamma$ are positive roots spanning a subsystem of type $\mathtt{A}_2$, then

$$\big(\mathrm{ad}\, e_\alpha^{(k)}\big)\big(f_\beta^{(m)}\big) = \begin{cases} \pm f_\gamma^{(k)} f_\beta^{(m-k)}, & \text{if } k \leq m, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

If $\alpha, \gamma, \alpha + \gamma, \alpha + 2\gamma$ span a subrootsystem of type $\mathtt{B}_2 = \mathtt{C}_2$, then

$$\big(\mathrm{ad}\, e_\alpha^{(k)}\big)\big(f_{\alpha+\gamma}^{(m)}\big) = \begin{cases} \pm f_\gamma^{(k)} f_{\alpha+\gamma}^{(m-k)}, & \text{if } k \leq m, \\ 0, & \text{otherwise,} \end{cases} \tag{10}$$

and

$$\big(\mathrm{ad}\, e_{\alpha+\gamma}^{(k)}\big)\big(f_{\alpha+2\gamma}^{(m)}\big) = \begin{cases} \pm f_\gamma^{(k)} f_{\alpha+2\gamma}^{(m-k)}, & \text{if } k \leq m, \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

and

$$\big(\mathrm{ad}\, e_\gamma^{(k)}\big)\big(f_{\alpha+\gamma}^{(m)}\big) = \begin{cases} \pm 2^k f_\alpha^{(k)} f_{\alpha+\gamma}^{(m-k)}, & \text{if } k \leq m, \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

and

$$\big(\mathrm{ad}\, e_\gamma^{(k)}\big)\big(f_{\alpha+2\gamma}^{(m)}\big) = \begin{cases} \pm f_{\alpha+\gamma}^{(k)} f_{\alpha+2\gamma}^{(m-k)} \\ \quad + \sum_{\substack{c > m-k \\ a+b+c=m}} r_{a,b,c} f_\alpha^{(a)} f_{\alpha+\gamma}^{(b)} f_{\alpha+2\gamma}^{(c)}, & \text{if } k \leq m, \\ 0, & \text{otherwise,} \end{cases} \tag{13}$$

where the coefficients $r_{a,b,c}$ are integers.

5 The Spanning Property for SL_{n+1}

We first recall the definition of a Dyck path in the SL_{n+1}-case:

Definition 6 A *Dyck path* (or simply *a path*) is a sequence

$$\mathbf{p} = \big(\beta(0), \beta(1), \dots, \beta(k)\big), \quad k \geq 0$$

of positive roots satisfying the following conditions:

(i) the first and last elements are simple roots. More precisely, $\beta(0) = \alpha_i$ and $\beta(k) = \alpha_j$ for some $1 \le i \le j \le n$;
(ii) the elements in between obey the following recursion rule: If $\beta(s) = \alpha_{p,q}$ then the next element in the sequence is of the form either $\beta(s+1) = \alpha_{p,q+1}$ or $\beta(s+1) = \alpha_{p+1,q}$.

Example 1 Here is an example for a Dyck path for $\mathfrak{sl}_6$:

$$\mathbf{p} = (\alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4, \alpha_4 + \alpha_5, \alpha_5).$$

For a multi-exponent $\mathbf{s} = \{s_\beta\}_{\beta>0}$, $s_\beta \in \mathbb{Z}_{\ge 0}$, let $f^{(\mathbf{s})}$ be the element

$$f^{(\mathbf{s})} = \prod_{\beta \in R^+} f_\beta^{(s_\beta)} \in S_{\mathbb{Z}}\big(\mathfrak{n}^{-,a}\big).$$

Definition 7 For an integral dominant $\mathfrak{sl}_{n+1}$-weight $\lambda = \sum_{i=1}^n m_i \omega_i$ let $S(\lambda)$ be the set of all multi-exponents $\mathbf{s} = (s_\beta)_{\beta \in R^+} \in \mathbb{Z}_{\ge 0}^{R^+}$ such that for all Dyck paths $\mathbf{p} = (\beta(0), \ldots, \beta(k))$

$$s_{\beta(0)} + s_{\beta(1)} + \cdots + s_{\beta(k)} \le m_i + m_{i+1} + \cdots + m_j, \tag{14}$$

where $\beta(0) = \alpha_i$ and $\beta(k) = \alpha_j$.

The space $V_{\mathbb{Z}}^a(\lambda)$ is endowed with the structure of a cyclic $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$-module, hence $V_{\mathbb{Z}}^a(\lambda) = S_{\mathbb{Z}}(\mathfrak{n}^{-,a})/I_{\mathbb{Z}}(\lambda)$ for some ideal $I_{\mathbb{Z}}(\lambda) \subseteq S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$. Our aim is to prove that the elements $f^{(\mathbf{s})} v_\lambda$, $\mathbf{s} \in S(\lambda)$, span $V_{\mathbb{Z}}^a(\lambda)$.

Let $\lambda = m_1\omega_1 + \cdots + m_n\omega_n$. The strategy is as follows: $f_\alpha^{((\lambda,\alpha)+1)} v_\lambda = 0$ in $V_{\mathbb{Z}}(\lambda)$ for all positive roots α, so for $\alpha = \alpha_i + \cdots + \alpha_j$, $i \le j$, we have the relation

$$f_{\alpha_i + \cdots + \alpha_j}^{(m_i + \cdots + m_j + 1)} \in I_{\mathbb{Z}}(\lambda).$$

In addition we have the operators $e_\alpha^{(m)}$ acting on $V_{\mathbb{Z}}^a(\lambda)$. We note that $I_{\mathbb{Z}}(\lambda)$ is stable with respect to the induced action of the $e_\alpha^{(m)}$ on $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ (Lemma 1). By applying the operators $e_\alpha^{(m)}$ to $f_{\alpha_i + \cdots + \alpha_j}^{(m_i + \cdots + m_j + 1)}$, we obtain new relations. We prove that these relations are enough to rewrite any vector $f^{(\mathbf{t})} v_\lambda$ as an integral linear combination of $f^{(\mathbf{s})} v_\lambda$ with $\mathbf{s} \in S(\lambda)$.

To simplify the notation we use the following abbreviations: we write just $f_{i,j}$ for $f_{\alpha_i + \cdots + \alpha_j}$, $i \le j$, and we write $f_{i,i}^{(s_{i,j})}$ for $f_{\alpha_i + \cdots + \alpha_j}^{(s_{\alpha_i + \cdots + \alpha_j})}$.

By the degree $\deg \mathbf{s}$ of a multi-exponent we mean the degree of the corresponding monomial $f^{(\mathbf{s})} = \prod_{1 \le i \le j \le n} f_{i,j}^{(s_{i,j})}$ in $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$, i.e. $\deg \mathbf{s} = \sum s_{i,j}$.

We are going to define an *order* on the monomials in $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$. To begin with, we define a total order on the $f_{i,j}$, $1 \le i \le j \le n$. We say that $(i, j) \succ (k, l)$ if $i > k$

or if $i = k$ and $j > l$. Correspondingly we say that $f_{i,j} \succ f_{k,l}$ if $(i, j) \succ (k, l)$, so

$$f_{n,n} \succ f_{n-1,n} \succ f_{n-1,n-1} \succ f_{n-2,n} \succ \cdots \succ f_{2,3} \succ f_{2,2} \succ f_{1,n} \succ \cdots \succ f_{1,1}.$$

We use a sort of associated *homogeneous lexicographic ordering* on the set of multi-exponents, i.e. for two multi-exponents $\mathbf{s}$ and $\mathbf{t}$ we write $\mathbf{s} \succ \mathbf{t}$:

(i) if $\deg \mathbf{s} > \deg \mathbf{t}$,
(ii) if $\deg \mathbf{s} = \deg \mathbf{t}$ and there exist $1 \le i_0 \le j_0 \le n$ such that $s_{i_0 j_0} > t_{i_0 j_0}$ and for $i > i_0$ and ($i = i_0$ and $j > j_0$) we have $s_{i,j} = t_{i,j}$.

We use the "same" total order on the set of monomials, i.e. $f^{(\mathbf{s})} \succ f^{(\mathbf{t})}$ if and only if $\mathbf{s} \succ \mathbf{t}$.

Proposition 5 *Let* $\mathbf{p} = (p(0), \dots, p(k))$ *be a Dyck path with* $p(0) = \alpha_i$ *and* $p(k) = \alpha_j$. *Let* $\mathbf{s}$ *be a multi-exponent supported on* $\mathbf{p}$, *i.e.* $s_\alpha = 0$ *for* $\alpha \notin \mathbf{p}$. *Assume further that*

$$\sum_{l=0}^{k} s_{p(l)} > m_i + \cdots + m_j.$$

Then there exist some constants $c_{\mathbf{t}} \in \mathbb{Z}$ *labeled by multi-exponents* $\mathbf{t}$ *such that*

$$f^{(\mathbf{s})} + \sum_{\mathbf{t} \prec \mathbf{s}} c_{\mathbf{t}} f^{(\mathbf{t})} \in I_{\mathbb{Z}}(\lambda) \tag{15}$$

($\mathbf{t}$ *does not have to be supported on* $\mathbf{p}$).

Remark 1 We refer to (15) as a *straightening law* because it implies

$$f^{(\mathbf{s})} = -\sum_{\mathbf{t} \prec \mathbf{s}} c_{\mathbf{t}} f^{(\mathbf{t})} \quad \text{in } S_{\mathbb{Z}}\big(\mathfrak{n}^{-,a}\big)/I_{\mathbb{Z}}(\lambda) \simeq V^a_{\mathbb{Z}}(\lambda).$$

Proof We start with the case $p(0) = \alpha_1$ and $p(k) = \alpha_n$ (so, $k = 2n - 2$). This assumption is just for convenience. In the general case $\mathbf{p}$ starts with $p(0) = \alpha_i$, $p(k) = \alpha_j$ and one would start with the relation $f_{i,j}^{(m_i + \cdots + m_j + 1)} \in I_{\mathbb{Z}}(\lambda)$ instead of the relation $f_{1,n}^{(m_1 + \cdots + m_n + 1)} \in I_{\mathbb{Z}}(\lambda)$ below.

So from now on we assume without loss of generality that $p(0) = \alpha_1$ and $p(k) = \alpha_n$. In the following we use the differential operators $\partial_\alpha^{(k)}$ defined by

$$\partial_\alpha^{(k)} f_\beta^{(m)} = \begin{cases} f_{\beta-\alpha}^{(k)} f_\beta^{(m-k)}, & \text{if } \beta - \alpha \in \triangle^+ \text{ and } k \le m, \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

The operators $\partial_\alpha^{(k)}$ satisfy the property

$$\partial_\alpha^{(k)} f_\beta^{(m)} = \pm\big(\operatorname{ad} e_\alpha^{(k)}\big)\big(f_\beta^{(m)}\big).$$

In the following we use very often the following consequence: if a monomial $f_{\beta_1}^{(m_1)} \dots f_{\beta_l}^{(m_l)} \in I_{\mathbb{Z}}(\lambda)$, then for any sequence of positive roots $\alpha_1, \dots, \alpha_s$ and any sequence of integers $k_1, \dots, k_s \in \mathbb{Z}_{>0}$ we have:

$$\partial_{\alpha_1}^{(k_1)} \dots \partial_{\alpha_s}^{(k_s)} f_{\beta_1}^{(m_1)} \dots f_{\beta_l}^{(m_l)} \in I_{\mathbb{Z}}(\lambda).$$

Since $f_{1,n}^{(m_1+\dots+m_n+1)} v_\lambda = 0$ in $V_{\mathbb{Z}}^a(\lambda)$ and $s_{p(0)} + \dots + s_{p(k)} > m_1 + \dots + m_n$ by assumption, it follows that

$$f_{1,n}^{(s_{p(0)}+\dots+s_{p(k)})} \in I_{\mathbb{Z}}(\lambda).$$

Write $\partial_{i,j}^{(m)}$ for $\partial_{\alpha_{i,j}}^{(m)}$, and for $i, j = 1, \dots, n$ set

$$s_{\bullet,j} = \sum_{i=1}^{j} s_{i,j}, \qquad s_{i,\bullet} = \sum_{j=i}^{n} s_{i,j}.$$

We first consider the vector

$$\partial_{n,n}^{(s_{\bullet,n-1})} \partial_{n-1,n}^{(s_{\bullet,n-2})} \cdots \partial_{2,n}^{(s_{\bullet,1})} f_{1,n}^{(s_{p(0)}+\dots+s_{p(k)})} \in I_{\mathbb{Z}}(\lambda). \tag{17}$$

By means of formula (16) we get:

$$\partial_{2n}^{(s_{\bullet,1})} f_{1,n}^{(s_{p(0)}+\dots+s_{p(k)})} = f_{1,n}^{(s_{p(0)}+\dots+s_{p(k)}-s_{\bullet,1})} f_{1,1}^{(s_{\bullet,1})}$$

and

$$\partial_{3n}^{(s_{\bullet,2})} \partial_{2n}^{(s_{\bullet,1})} f_{1,n}^{(s_{p(0)}+\dots+s_{p(k)})} = f_{1,n}^{(s_{p(0)}+\dots+s_{p(k)}-s_{\bullet 1}-s_{\bullet 2})} f_{1,1}^{(s_{\bullet 1})} f_{1,2}^{(s_{\bullet 2})}.$$

Summarizing, the vector (17) is equal to

$$f_{1,1}^{(s_{\bullet,1})} f_{1,2}^{(s_{\bullet,2})} \cdots f_{1,n}^{(s_{\bullet,n})} \in I_{\mathbb{Z}}(\lambda).$$

To prove the proposition, we apply more differential operators to the monomial $f_{1,1}^{(s_{\bullet,1})} f_{1,2}^{(s_{\bullet,2})} \dots f_{1,n}^{(s_{\bullet,n})}$. Consider the following element in $I_{\mathbb{Z}}(\lambda) \subset S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$:

$$A = \partial_{1,1}^{(s_{2,\bullet})} \partial_{1,2}^{(s_{3,\bullet})} \dots \partial_{1,n-1}^{(s_{n,\bullet})} f_{1,1}^{(s_{\bullet,1})} f_{1,2}^{(s_{\bullet,2})} \dots f_{1,n}^{(s_{\bullet,n})}. \tag{18}$$

Claim

$$A = \sum_{\mathbf{t} \preceq \mathbf{s}} c_{\mathbf{t}} f^{(\mathbf{t})} \quad \textit{where } c_{\mathbf{s}} = 1. \tag{19}$$

Now $A \in I_{\mathbb{Z}}(\lambda)$ by construction, so the claim proves the proposition.

Proof of the claim In order to prove the claim we need to introduce some more notation. For $j = 1, \dots, n-1$ set

$$A_j = \partial_{1,j}^{(s_{j+1,\bullet})} \partial_{1,j+1}^{(s_{j+2,\bullet})} \dots \partial_{1,n-1}^{(s_{n,\bullet})} f_{1,1}^{(s_{\bullet,1})} f_{1,2}^{(s_{\bullet,2})} \dots f_{1,n}^{(s_{\bullet,n})}, \tag{20}$$

so $A_1 = A$. To start an inductive procedure, we begin with A_{n-1}:

$$A_{n-1} = \partial_{1,n-1}^{(s_{n,\bullet})} f_{1,1}^{(s_{\bullet,1})} f_{1,2}^{(s_{\bullet,2})} \cdots f_{1,n}^{(s_{\bullet,n})}.$$

Now $s_{n,\bullet} = s_{n,n}$ and $\partial_{1,n-1}^{(x)} f_{1,j}^{(y)} = 0$ for $j \neq n$, so

$$A_{n-1} = f_{1,1}^{(s_{\bullet,1})} f_{1,2}^{(s_{\bullet,2})} \cdots f_{1,n}^{(s_{\bullet,n}-s_{n,n})} f_{n,n}^{(s_{n,n})}. \tag{21}$$

We proceed with the proof using decreasing induction. Since the induction procedure is quite involved and the initial step does not reflect the problems occurring in the procedure, we discuss for convenience the case A_{n-2} separately.

Consider A_{n-2}, we have:

$$A_{n-2} = \partial_{1,n-2}^{(s_{n-1,\bullet})} f_{1,1}^{(s_{\bullet,1})} f_{1,2}^{(s_{\bullet,2})} \cdots f_{1,n}^{(s_{\bullet,n}-s_{n,n})} f_{n,n}^{(s_{n,n})}.$$

Now $\partial_{1,n-2}^{(k)} f_{1,j}^{(m)} = 0$ for $j \neq n-1, n$, $\partial_{1,n-2}^{(k)} f_{n,n}^{(m)} = 0$, and $\partial^{(k)}(xy) = \sum_{i=0}^{k} \partial^{(k-i)}(x)\partial^{(i)}(y)$, so

$$A_{n-2} = \sum_{\ell=0}^{s_{n-1,\bullet}} f_{1,1}^{(s_{\bullet,1})} f_{1,2}^{(s_{\bullet,2})} \cdots f_{1,n-1}^{(s_{\bullet,n-1}-s_{n-1,\bullet}+\ell)} f_{1,n}^{(s_{\bullet,n}-s_{n,n}-\ell)} f_{n-1,n-1}^{(s_{n-1,\bullet}-\ell)} f_{n-1,n}^{(\ell)} f_{n,n}^{(s_{n,n})}.$$

We need to control which divided powers $f_{n-1,n}^{(\ell)}$ can occur. Recall that $\mathbf{s}$ has support in $\mathbf{p}$. If $\alpha_{n-1} \notin \mathbf{p}$, then $s_{n-1,n-1} = 0$ and $s_{n-1,\bullet} = s_{n-1,n}$, so $f_{n-1,n}^{(s_{n-1,n})}$ is the highest divided power occurring in the sum. Next suppose $\alpha_{n-1} \in \mathbf{p}$. This implies $\alpha_{j,n} \notin \mathbf{p}$ unless $j = n-1$ or n. Since $\mathbf{s}$ has support in $\mathbf{p}$, this implies

$$s_{\bullet,n} = s_{1,n} + \cdots + s_{n-1,n} + s_{n,n} = s_{n-1,n} + s_{n,n},$$

and hence again the highest divided power of $f_{n-1,n}$ which can occur is $f_{n-1,n}^{(s_{n-1,n})}$, and the coefficient is 1. So we can write

$$A_{n-2} = \sum_{\ell=0}^{s_{n-1,n}} f_{1,1}^{(s_{\bullet,1})} \cdots f_{1,n-1}^{(s_{\bullet,n-1}-s_{n-1,\bullet}+\ell)} f_{1,n}^{(s_{\bullet,n}-s_{n,n}-\ell)} f_{n-1,n-1}^{(s_{n-1,\bullet}-\ell)} f_{n-1,n}^{(\ell)} f_{n,n}^{(s_{n,n})}. \tag{22}$$

For the inductive procedure we make the following assumption:

A_j is a sum with integral coefficients of monomials of the form

$$\underbrace{f_{1,1}^{(s_{\bullet,1})} \cdots f_{1,j}^{(s_{\bullet,j})} f_{1,j+1}^{(s_{\bullet,j+1}-*)} \cdots f_{1,n}^{(s_{\bullet,n}-*)}}_{X} \underbrace{f_{j+1,j+1}^{(t_{j+1,j+1})} f_{j+1,j+2}^{(t_{j+1,j+2})} \cdots f_{n-1,n}^{(t_{n-1,n})} f_{n,n}^{(t_{n,n})}}_{Y} \tag{23}$$

having the following properties:

(i) With respect to the homogeneous lexicographic ordering, all the multi-exponents of the summands, except one, are strictly smaller than $\mathbf{s}$.

(ii) More precisely, there exists a pair (k_0, ℓ_0) such that $k_0 \geq j+1$, $s_{k_0\ell_0} > t_{k_0\ell_0}$ and $s_{k\ell} = t_{k\ell}$ for all $k > k_0$ and all pairs (k_0, ℓ) such that $\ell > \ell_0$.

(iii) The only exception is the summand such that $t_{\ell,m} = s_{\ell,m}$ for all $\ell \geq j+1$ and all m, and in this case the coefficient is equal to 1.

The calculations above show that this assumption holds for A_{n-1} and A_{n-2}.

We start now with the induction procedure and we consider $A_{j-1} = \partial_{1,j-1}^{(s_{j,\bullet})} A_j$. Note that $\partial_{1,j-1}^{(k)} f_{1,\ell}^{(m)} = 0$ for $\ell < j$, and for $\ell \geq j$ we have $\partial_{1,j-1}^{(p)} f_{1,\ell}^{(q)} = f_{j,\ell}^{(p)} f_{1,\ell}^{(q-p)}$ for $p \leq q$, and the result is 0 for $p > q$.

Furthermore, $\partial_{1,j-1}^{(p)} f_{k,\ell}^{(q)} = 0$ for $k \geq j+1$, so applying $\partial_{1,j-1}^{(p)}$ to a summand of the form (23) does not change the Y-part in (23). Summarizing, applying $\partial_{1,j-1}^{(s_{j,\bullet})}$ to a summand of the form (23) gives a sum of monomials of the form

$$\underbrace{f_{1,1}^{(s_{\bullet,1})} \cdots f_{1,j-1}^{(s_{\bullet,j-1})} f_{1,j}^{(s_{\bullet,j}-*)} \cdots f_{1,n}^{(s_{\bullet,n}-*)}}_{X'} \underbrace{f_{j,j}^{(t_{j,j})} \cdots f_{j,n}^{(t_{j,n})}}_{Z}$$
$$\underbrace{f_{j+1,j+1}^{(t_{j+1,j+1})} f_{j+1,j+2}^{(t_{j+1,j+2})} \cdots f_{n,n}^{(t_{n,n})}}_{Y}. \tag{24}$$

We have to show that these summands satisfy again the conditions (i)–(iii) above (but now for the index $(j-1)$). If we start in (23) with a summand which is not the maximal summand, but such that (i) and (ii) hold for the index j, then the same holds obviously also for the index $(j-1)$ for all summands in (24) because the Y-part remains unchanged.

So it remains to investigate the summands of the form (24) obtained by applying $\partial_{1j-1}^{(s_{j,\bullet})}$ to the only summand in (23) satisfying (iii).

To formalize the arguments used in the calculation for A_{n-2} we need the following notation. Let $1 \leq k_1 \leq k_2 \leq \cdots \leq k_n \leq n$ be numbers defined by

$$k_i = \max\{j : \alpha_{i,j} \in \mathbf{p}\}.$$

For convenience we set $k_0 = 1$.

Example 2 For $\mathbf{p} = (\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,n-1}, \alpha_{1,n}, \alpha_{2,n}, \alpha_{3,n}, \alpha_{4,n}, \ldots, \alpha_{n,n})$ we have $k_i = n$ for all $i = 1, \ldots, n$.

Since $\mathbf{s}$ is supported on $\mathbf{p}$ we have

$$s_{i,\bullet} = \sum_{\ell=k_{i-1}}^{k_i} s_{i,\ell}, \qquad s_{\bullet,\ell} = \sum_{i:\, k_{i-1} \leq \ell \leq k_i} s_{i,\ell}. \tag{25}$$

Suppose now that we have a summand of the form in (24) obtained by applying $\partial_{1j-1}^{(s_{j,\bullet})}$ to the only summand in (23) satisfying (iii). Since the Y-part remains unchanged, this implies already $t_{n,n} = s_{n,n}, \ldots, t_{j+1,j+1} = s_{j+1,j+1}$. Assume that we have already shown $t_{j,n} = s_{j,n}, \dots, t_{j,\ell_0+1} = s_{j,\ell_0+1}$, then we have to show that $t_{j,\ell_0} \le s_{j,\ell_0}$.

We consider five cases:

(i) $\ell_0 > k_j$. In this case the root α_{j,ℓ_0} is not in the support of $\mathbf{p}$ and hence $s_{j,\ell_0} = 0$. Since $\ell_0 > k_j \ge k_{j-1} \ge \dots \ge k_1$, for the same reason we have $s_{i,\ell_0} = 0$ for $i \le j$. Recall that the divided power of $f_{1,\ell_0}^{(*)}$ in A_{j-1} in (20) is equal to $s_{\bullet,\ell_0}$. Now $s_{\bullet,\ell_0} = \sum_{i>j} s_{i,\ell_0}$ by the discussion above, and hence $f_{1,\ell_0}^{(s_{\bullet,\ell_0})}$ has already been transformed completely by the operators $\partial_{1,i}^{(*)}$, $i > j$, and hence $t_{j,\ell_0} = 0 = s_{j,\ell_0}$.

(ii) $k_{j-1} < \ell_0 \le k_j$. Since $\ell_0 > k_{j-1} \ge \dots \ge k_1$, for the same reason as above we have $s_{i,\ell_0} = 0$ for $i < j$, so $s_{\bullet,\ell_0} = \sum_{i \ge j} s_{i,\ell_0}$. The same arguments as above show that for the operator $\partial_{1,j-1}^{(*)}$ only the power $f_{1,\ell_0}^{(s_{j,\ell_0})}$ is left to be transformed into a divided power of f_{j,ℓ_0}, so necessarily $t_{j,\ell_0} \le s_{j,\ell_0}$.

(iii) $k_{j-1} = \ell_0 = k_j$. In this case $s_{j,\bullet} = s_{j,\ell_0}$ and thus the operator $\partial_{1,j-1}^{s_{j,\bullet}} = \partial_{1,j-1}^{s_{j,\ell_0}}$ can transform a divided power $f_{1,\ell_0}^{(*)}$ in A_j only into a power $f_{j,\ell_0}^{(q)}$ with q at most s_{j,ℓ_0}.

(iv) $k_{j-1} = \ell_0 < k_j$. In this case $s_{j,\bullet} = s_{j,\ell_0} + s_{j,\ell_0+1} + \dots + s_{j,k_j}$. Applying $\partial_{1,j-1}^{(s_{j,\bullet})}$ to the only summand in (23) satisfying (iii), the assumption $t_{j,n} = s_{j,n}, \dots, t_{j,\ell_0+1} = s_{j,\ell_0+1}$ implies that one has to apply $\partial_{1,j-1}^{(s_{j,k_j})}$ to $f_{1,k_j}^{(*)}$ and $\partial_{1,j-1}^{(s_{j,k_j-1})}$ to $f_{1,k_j-1}^{(*)}$ etc. to get the demanded divided powers of the root vectors. So for $f_{1,\ell_0}^{(*)}$ only the operator $\partial_{1,j-1}^{(s_{j,\ell_0})}$ is left for transformations into a divided power of f_{j,ℓ_0}, and hence $t_{j,\ell_0} \le s_{j,\ell_0}$.

(v) $\ell_0 < k_{j-1}$. In this case $s_{j,\ell_0} = 0$ because the root is not in the support. Since $t_{j,\ell} = s_{j,\ell}$ for $\ell > \ell_0$ and $s_{j,\ell} = 0$ for $\ell \le \ell_0$ (same reason as above) we obtain

$$\partial_{1,j-1}^{(s_{j,\bullet})} = \partial_{1,j-1}^{(\sum_{\ell>\ell_0} s_{j,\ell})}.$$

But by assumption we know that $\partial_{1,j-1}^{(s_{j,\ell})}$ is needed to transform the power $f_{1,\ell}^{(s_{j,\ell})}$ into $f_{j,\ell}^{(s_{j,\ell})}$ for all $\ell > \ell_0$, so no divided power of $\partial_{1,j-1}$ is left and thus $t_{j,\ell_0} = 0 = s_{j,\ell_0}$.

It follows that all summands except one satisfy the conditions (i), (ii) above. The only exception is the term where the divided powers of the operator $\partial_{1,j-1}^{(s_{j,\bullet})}$ are distributed as follows:

$$f_{1,1}^{(s_{\bullet,1})} \dots f_{1,j-1}^{(s_{\bullet,j-1})} \big(\partial_{1,j-1}^{(s_{j,j})} f_{1,j}^{(s_{\bullet,j})}\big)\big(\partial_{1,j-1}^{(s_{j,j+1})} f_{1,j+1}^{(s_{\bullet,j+1}-*)}\big)$$
$$\dots \big(\partial_{1,j-1}^{(s_{j,n})} f_{1,n}^{(s_{\bullet,n}-*)}\big) f_{j+1,j+1}^{(s_{j+1,j+1})} \dots f_{n,n}^{(s_{n,n})}.$$

By construction, this term has coefficient 1 and satisfies the condition (iii), which finishes the proof of the proposition. □

Theorem 1 *The elements $f^{(\mathbf{s})}v_\lambda$ with $\mathbf{s} \in S(\lambda)$ (see Definition 7) span the module $V^a_{\mathbb{Z}}(\lambda)$.*

Proof The elements $f^{(\mathbf{s})}$, $\mathbf{s}$ arbitrary multi-exponent, span $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$, so the elements $f^{(\mathbf{s})}v_\lambda$, $\mathbf{s}$ arbitrary multi-exponent, span $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})/I_{\mathbb{Z}}(\lambda) \simeq V^a_{\mathbb{Z}}(\lambda)$. We use now the Eq. (15) in Proposition 5 as a straightening algorithm to express $f^{(\mathbf{s})}v_\lambda$, $\mathbf{s}$ arbitrary, as a linear combination of elements $f^{(\mathbf{t})}v_\lambda$ such that $\mathbf{t} \in S(\lambda)$.

Let $\lambda = \sum_{i=1}^n m_i\omega_i$ and suppose $\mathbf{s} \notin S(\lambda)$, then there exists a Dyck path $\mathbf{p} = (p(0), \dots, p(k))$ with $p(0) = \alpha_i$, $p(k) = \alpha_j$ such that

$$\sum_{l=0}^{k} s_{p(l)} > m_i + \cdots + m_j.$$

We define a new multi-exponent $\mathbf{s}'$ by setting

$$\mathbf{s}'_\alpha = \begin{cases} s_\alpha, & \text{if } \alpha \in \mathbf{p}, \\ 0, & \text{otherwise.} \end{cases}$$

For the new multi-exponent $\mathbf{s}'$ we still have

$$\sum_{l=0}^{k} s'_{p(l)} > m_i + \cdots + m_j.$$

We can now apply Proposition 5 to $\mathbf{s}'$ and conclude

$$f^{(\mathbf{s}')} = \sum_{\mathbf{s}' \succ \mathbf{t}'} c_{\mathbf{t}'} f^{(\mathbf{t}')} \quad \text{in } S_{\mathbb{Z}}\big(\mathfrak{n}^{-,a}\big)/I_{\mathbb{Z}}(\lambda),$$

where $c_{\mathbf{t}'} \in \mathbb{Z}$. We get $f^{(\mathbf{s})}$ back as $f^{(\mathbf{s})} = f^{(\mathbf{s}')} \prod_{\beta \notin \mathbf{p}} f_\beta^{(s_\beta)}$. For a multi-exponent $\mathbf{t}'$ occurring in the sum with $c_{\mathbf{t}'} \neq 0$ let the multi-exponent $\mathbf{t}$ and $c_{\mathbf{t}} \in \mathbb{Z}$ be such that $c_{\mathbf{t}'} f^{(\mathbf{t}')} \prod_{\beta \notin \mathbf{p}} f_\beta^{(s_\beta)} = c_{\mathbf{t}} f^{(\mathbf{t})}$ (recall (4)). Since we have a monomial order it follows:

$$f^{(\mathbf{s})} = f^{(\mathbf{s}')} \prod_{\beta \notin \mathbf{p}} f_\beta^{(s_\beta)} = \sum_{\mathbf{s} \succ \mathbf{t}} c_{\mathbf{t}} f^{(\mathbf{t})} \quad \text{in } S_{\mathbb{Z}}\big(\mathfrak{n}^{-,a}\big)/I_{\mathbb{Z}}(\lambda). \tag{26}$$

Equation (26) provides an algorithm to express $f^{(\mathbf{s})}$ in $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})/I_{\mathbb{Z}}(\lambda)$ as a sum of elements of the desired form: if some of the $\mathbf{t}$ are not elements of $S(\lambda)$, then we can repeat the procedure and express the $f^{(\mathbf{t})}$ in $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})/I_{\mathbb{Z}}(\lambda)$ as a sum of $f^{(\mathbf{r})}$ with $\mathbf{r} \prec \mathbf{t}$. For the chosen ordering any strictly decreasing sequence of multi-exponents (all of the same total degree) is finite, so after a finite number of steps one obtains an expression of the form $f^{(\mathbf{s})} = \sum c_{\mathbf{r}} f^{(\mathbf{r})}$ in $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})/I_{\mathbb{Z}}(\lambda)$ such that $\mathbf{r} \in S(\lambda)$ for all $\mathbf{r}$. □

6 The Main Theorem for SL_{n+1}

Theorem 2 *The elements* $\{f^{(\mathbf{s})}v_\lambda \mid \mathbf{s}\in S(\lambda)\}$ *form a basis for the module* $V^a_{\mathbb{Z}}(\lambda)$ *and the ideal* $I_{\mathbb{Z}}(\lambda)$ *is generated by the subspace*

$$\left\langle \mathrm{U}_{\mathbb{Z}}\big(\mathfrak{n}^+\big)\, f_{\alpha_{i,j}}^{(m_i+\cdots+m_j+1)} \mid 1\le i\le j\le n-1\right\rangle.$$

As an immediate consequence we see:

Corollary 1

(i) $V^a_{\mathbb{Z}}(\lambda)$ *is a free* $\mathbb{Z}$*-module.*

(ii) *For every* $\mathbf{s}\in S(\lambda)$ *fix a total order on the set of positive roots and denote by abuse of notation by* $f^{(\mathbf{s})}\in \mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^-)$ *also the corresponding product of divided powers. The* $\{f^{(\mathbf{s})}v_\lambda \mid \mathbf{s}\in S(\lambda)\}$ *form a basis for the module* $V_{\mathbb{Z}}(\lambda)$ *and for all* $s<s'$ *we have* $V_{\mathbb{Z}}(\lambda)_s$ *is a direct summand of* $V_{\mathbb{Z}}(\lambda)_{s'}$ *as a* $\mathbb{Z}$*-module.* (*See* (5) *for the filtration.*)

(iii) *With the notation as above*: *let* k *be a field and denote by* $V_k(\lambda)=V_{\mathbb{Z}}(\lambda)\otimes_{\mathbb{Z}} k$, $\mathrm{U}_k(\mathfrak{g})=\mathrm{U}_{\mathbb{Z}}(\mathfrak{g})\otimes_{\mathbb{Z}} k$, $\mathrm{U}_k(\mathfrak{n}^-)=\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^-)\otimes_{\mathbb{Z}} k$ *etc. the objects obtained by base change. The* $\{f^{(\mathbf{s})}v_\lambda \mid \mathbf{s}\in S(\lambda)\}$ *form a basis for the module* $V_k(\lambda)$.

Proof We know that the elements $f^{(\mathbf{s})}v_\lambda$, $\mathbf{s}\in S(\lambda)$, span $V^a_{\mathbb{Z}}(\lambda)$, see Theorem 1. By [6], the number $\sharp S(\lambda)$ is equal to $\dim V(\lambda)$, which implies the linear independence. By lifting the elements to $V_{\mathbb{Z}}(\lambda)$, we get a basis of $V_{\mathbb{Z}}(\lambda)$ which is (by construction) compatible with the PBW-filtration: set

$$S(\lambda)_r=\left\{\mathbf{s}\in S(\lambda)\;\middle|\; \sum_{\beta\in R^+} s_\beta\le r\right\},$$

then the elements $f^{(\mathbf{s})}v_\lambda$, $\mathbf{s}\in S(\lambda)_r$, span $V_{\mathbb{Z}}(\lambda)_r$.

Let $I\subset S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ be the ideal generated by

$$\left\langle \mathrm{U}_{\mathbb{Z}}\big(\mathfrak{n}^+\big)\circ f_{\alpha_{i,j}}^{(m_i+\cdots+m_j+1)} \mid 1\le i\le j\le n-1\right\rangle,$$

by construction we know $I\subseteq I_{\mathbb{Z}}(\lambda)$. But we also know that the relations in I are sufficient to rewrite every element in $V^a_{\mathbb{Z}}(\lambda)$ in terms of the basis elements $f^{(\mathbf{s})}v_\lambda$, $\mathbf{s}\in S(\lambda)$, which implies that the canonical surjective map $S_{\mathbb{Z}}(\mathfrak{n}^-)/I\to S_{\mathbb{Z}}(\mathfrak{n}^-)/I_{\mathbb{Z}}(\lambda)\simeq V_{\mathbb{Z}}(\lambda)$ is injective. □

7 Symplectic Dyck Paths

We recall the notion of the symplectic Dyck paths:

Definition 8 A symplectic Dyck path (or simply a path) is a sequence

$$\mathbf{p} = \big(\beta(0), \beta(1), \ldots, \beta(k)\big), \quad k \geq 0$$

of positive roots satisfying the following conditions:

(i) the first root is simple, $\beta(0) = \alpha_i$ for some $1 \leq i \leq n$;
(ii) the last root is either simple or the highest root of a symplectic subalgebra, more precisely $\beta(k) = \alpha_j$ or $\beta(k) = \alpha_{j\bar{j}}$ for some $i \leq j \leq n$;
(iii) the elements in between obey the following recursion rule: If $\beta(s) = \alpha_{p,q}$ with $p, q \in J$ (see (7)) then the next element in the sequence is of the form either $\beta(s+1) = \alpha_{p,q+1}$ or $\beta(s+1) = \alpha_{p+1,q}$, where $x+1$ denotes the smallest element in J which is bigger than x.

Denote by $\mathbb{D}$ the set of all Dyck paths. For a dominant weight $\lambda = \sum_{i=1}^{n} m_i \omega_i$ let $P(\lambda) \subset \mathbb{R}^{n^2}_{\geq 0}$ be the polytope

$$P(\lambda) := \left\{ (s_\alpha)_{\alpha>0} \mid \forall \mathbf{p} \in \mathbb{D} : \begin{array}{l} \text{If } \beta(0) = \alpha_i, \beta(k) = \alpha_j, \text{ then} \\ s_{\beta(0)} + \cdots + s_{\beta(k)} \leq m_i + \cdots + m_j, \\ \text{if } \beta(0) = \alpha_i, \beta(k) = \alpha_{\bar{j}}, \text{ then} \\ s_{\beta(0)} + \cdots + s_{\beta(k)} \leq m_i + \cdots + m_n \end{array} \right\}, \tag{27}$$

and let $S(\lambda)$ be the set of integral points in $P(\lambda)$.

For a multi-exponent $\mathbf{s} = \{s_\beta\}_{\beta>0}$, $s_\beta \in \mathbb{Z}_{\geq 0}$, let $f^{(\mathbf{s})}$ be the element

$$f^{(\mathbf{s})} = \prod_{\beta \in R^+} f_\beta^{(s_\beta)} \in S_{\mathbb{Z}}\big(\mathfrak{n}^{-,a}\big).$$

8 The Spanning Property for the Symplectic Lie Algebra

Our aim is to prove that the set $f^{(\mathbf{s})} v_\lambda$, $\mathbf{s} \in S(\lambda)$, forms a basis of $V^a_{\mathbb{Z}}(\lambda)$. As a first step we will prove that these elements span $V^a_{\mathbb{Z}}(\lambda)$.

Lemma 2 *Let $\lambda = \sum_{i=1}^{n} m_i \omega_i$ be the $\mathfrak{sp}_{2n}$-weight and let $V_{\mathbb{Z}}(\lambda) \subset V(\lambda)$ be the corresponding lattice in the highest weight module with highest weight vector v_λ. Then*

$$f_{\alpha_{i,j}}^{(m_i+\cdots+m_j+1)} v_\lambda = 0, \quad 1 \leq i \leq j \leq n-1, \tag{28}$$

$$f_{\alpha_{i,\bar{i}}}^{(m_i+\cdots+m_n+1)} v_\lambda = 0, \quad 1 \leq i \leq n. \tag{29}$$

Proof The lemma follows immediately from the $\mathfrak{sl}_2$-theory. □

In the following we use the operators $\partial_\alpha^{(k)}$ defined by $\partial_\alpha^{(k)}(f_\beta^{(m)}) = 0$ if $\alpha = \beta$ or if the root vectors commute, and if $\alpha, \gamma, \beta = \alpha + \gamma$ are positive roots spanning a subsystem of type $\mathtt{A}_2$, then

$$\partial_\alpha^{(k)}\big(f_\beta^{(m)}\big)=\begin{cases} f_\gamma^{(k)} f_\beta^{(m-k)}, & \text{if } k\le m,\\ 0, & \text{otherwise.}\end{cases} \tag{30}$$

If $\alpha,\gamma,\alpha+\gamma,\alpha+2\gamma$ span a subrootsystem of type $\mathtt{B}_2=\mathtt{C}_2$, then

$$\partial_\alpha^{(k)}\big(f_{\alpha+\gamma}^{(m)}\big)=\begin{cases} f_\gamma^{(k)} f_{\alpha+\gamma}^{(m-k)}, & \text{if } k\le m,\\ 0, & \text{otherwise,}\end{cases} \tag{31}$$

and

$$\partial_{\alpha+\gamma}^{(k)}\big(f_{\alpha+2\gamma}^{(m)}\big)=\begin{cases} f_\gamma^{(k)} f_{\alpha+2\gamma}^{(m-k)}, & \text{if } k\le m,\\ 0, & \text{otherwise,}\end{cases} \tag{32}$$

and

$$\partial_{\gamma}^{(k)}\big(f_{\alpha+\gamma}^{(m)}\big)=\begin{cases} 2^k f_\alpha^{(k)} f_{\alpha+\gamma}^{(m-k)}, & \text{if } k\le m,\\ 0, & \text{otherwise,}\end{cases} \tag{33}$$

and

$$\partial_{\gamma}^{(k)}\big(f_{\alpha+2\gamma}^{(m)}\big)=\begin{cases} f_{\alpha+\gamma}^{(k)} f_{\alpha+2\gamma}^{(m-k)} \\ \quad +\sum_{\substack{c>m-k\\ a+b+c=k}} c_{a,b,c} f_\alpha^{(a)} f_{\alpha+\gamma}^{(b)} f_{\alpha+2\gamma}^{(c)}, & \text{if } k\le m,\\ 0, & \text{otherwise,}\end{cases} \tag{34}$$

with the coefficients $c_{a,b,c}$ chosen such that $\partial_\gamma^{(k)}(f_{\alpha+2\gamma}^{(m)})=\pm(\mathrm{ad}\, e_\gamma^{(k)}(f_{\alpha+2\gamma}^{(m)}))$. Note that all the operators are such that $\partial_\gamma^{(k)}=\pm(\mathrm{ad}\, e_\gamma^{(k)})$ (see (8)–(13)).

In the following we often just write $f_{i,j}$ and $f_{i,\bar{j}}$ instead of $f_{\alpha_{i,j}}$ and $f_{\alpha_{i,\bar{j}}}$. We use the same abbreviation for the differential operators and the multi-exponents, so we write $\partial_{i,j}$ and $\partial_{i,\bar{j}}$ instead of $\partial_{\alpha_{i,j}}$ and $\partial_{\alpha_{i,\bar{j}}}$, similarly we replace $s_{\alpha_{i,j}}$ and $s_{\alpha_{i,\bar{j}}}$ by $s_{i,j}$ and $s_{i,\bar{j}}$. Recall that $\alpha_{i,\overline{n}}=\alpha_{i,n}$ (see (6)).

Lemma 3 *The only non-trivial vectors of the form $\partial_\beta f_\alpha$, $\alpha,\beta>0$ are as follows: for $\alpha=\alpha_{i,j}$, $1\le i\le j\le n$*

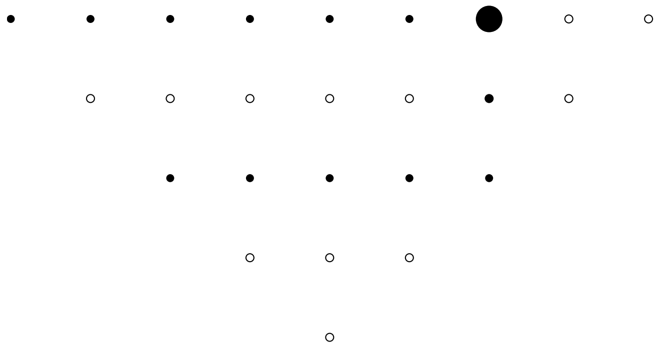
$$\partial_{i,s} f_{i,j}=f_{s+1,j},\quad i\le s<j,\qquad \partial_{s,j} f_{i,j}=f_{i,s-1},\quad i<s\le j, \tag{35}$$

and for $\alpha=\alpha_{i,\bar{j}}$, $1\le i\le j\le n$

$$\begin{aligned} &\partial_{i,s} f_{i,\bar{j}}=f_{s+1,\bar{j}},\quad i\le s<j, \qquad \partial_{i,s} f_{i,\bar{j}}=f_{j,\overline{s+1}},\quad j\le s,\\ &\partial_{i,\bar{s}} f_{i,\bar{j}}=f_{j,s-1},\quad j<s, \end{aligned} \tag{36}$$

$$\begin{aligned} &\partial_{s+1,\bar{j}} f_{i,\bar{j}}=f_{i,s},\quad i\le s<j, \qquad \partial_{j,\overline{s+1}} f_{i,\bar{j}}=f_{i,s},\quad j\le s,\\ &\partial_{j,s-1} f_{i,\bar{j}}=f_{i,\bar{s}},\quad j<s. \end{aligned} \tag{37}$$

Let us illustrate this lemma by the following picture in type $\mathtt{C}_5$.



Here all circles correspond to the positive roots of the root system of type $\mathtt{C}_5$ in the following way: in the upper row we have from left to right $\alpha_{1,1},\dots,\alpha_{1,5},\alpha_{1,\overline{4}}$, $\dots,\alpha_{1,\overline{1}}$, in the second row we have from left to right $\alpha_{2,2},\dots,\alpha_{2,5},\alpha_{2,\overline{4}},\dots,\alpha_{2,\overline{2}}$, and the last line corresponds to the root $\alpha_{5,5}$. Now let us take the root $\alpha_{1,\overline{3}}$ (which corresponds to the fat circle). Then all roots that can be obtained by applying the operators ∂_β are depicted as filled small circles.

Theorem 3

(i) *The vectors* $f^{(\mathbf{s})}v_\lambda$, $\mathbf{s}\in S(\lambda)$ *span* $V^a_{\mathbb{Z}}(\lambda)$.

(ii) *Let* $I_{\mathbb{Z}}(\lambda)=S_{\mathbb{Z}}(\mathfrak{n}^-)(\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^+)\mathfrak{R})$, *i.e.* $I_{\mathbb{Z}}(\lambda)$ *is the ideal generated by the elements obtained from* $\mathfrak{R}$ *by the* $\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^+)$*-action, where*

$$\mathfrak{R}=\operatorname{span}\left\{f_{\alpha_{i,j}}^{(m_i+\cdots+m_j+1)},\ 1\le i\le j\le n-1,\ f_{\alpha_{i,\overline{i}}}^{(m_i+\cdots+m_n+1)},\ 1\le i\le n\right\}.$$

There exists an order "$\succ_{\mathrm{mon}}$" on the ring $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ *such that for any* $\mathbf{s}\notin S(\lambda)$ *there exists a homogeneous expression* (*a straightening law*) *of the form*

$$f^{(\mathbf{s})}-\sum_{\mathbf{s}\succ_{\mathrm{mon}}\mathbf{t}}c_{\mathbf{t}}f^{(\mathbf{t})}\in I_{\mathbb{Z}}(\lambda). \tag{38}$$

Remark 2 In the following we refer to (38) as a *straightening law* for $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ with respect to the ideal $I_{\mathbb{Z}}(\lambda)$. Such a straightening law implies that in the quotient ring $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})/I_{\mathbb{Z}}(\lambda)$ we can express $f^{(\mathbf{s})}$ as a linear combination of monomials which are smaller in the order, but of the same total degree since the expression in (38) is homogeneous.

First we show that (ii) implies (i):

Proof [(ii) $\Rightarrow$ (i)] The elements in $\mathfrak{R}$ obviously annihilate $v_\lambda\in V^a_{\mathbb{Z}}(\lambda)$, and so do the elements of $\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^+)\mathfrak{R}$, and hence so do the elements of the ideal I generated by $\mathrm{U}_{\mathbb{Z}}(\mathfrak{n}^+)\mathfrak{R}$. As a consequence we get a surjective map $S(\mathfrak{n}^-)/I\to V^a_{\mathbb{Z}}(\lambda)$.

Assume $\mathbf{s} \notin S(\lambda)$. We know by (ii) that $f^{(\mathbf{s})} = \sum_{\mathbf{s} \succ_{\text{mon}} \mathbf{t}} c_{\mathbf{t}} f^{(\mathbf{t})}$ in $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})/I$. If any of the $\mathbf{t}$ with nonzero coefficient $c_{\mathbf{t}}$ is not an element in $S(\lambda)$, then we can again apply a straightening law and replace $f^{(\mathbf{t})}$ by a linear combination of smaller monomials. Since there are only a finite number of monomials of the same total degree, by repeating the procedure if necessary, after a finite number of steps we obtain an expression of $f^{(\mathbf{s})}$ in $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})/I$ as a linear combination of elements $f^{(\mathbf{t})}$, $\mathbf{t} \in S(\lambda)$. It follows that $\{f^{(\mathbf{t})} \mid \mathbf{t} \in S(\lambda)\}$ is a spanning set for $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})/I$, and hence, by the surjection above, we get a spanning set $\{f^{(\mathbf{t})} v_\lambda \mid \mathbf{t} \in S(\lambda)\}$ for $V^a_{\mathbb{Z}}(\lambda)$. □

To prove the second part we need to define the total order. We start by defining a total order on the variables:

$$
\begin{aligned}
f_{1,1} < f_{1,2} < \cdots < f_{1,n-1} < f_{1,n} < f_{1,\overline{n-1}} < \cdots < f_{1,\overline{2}} < f_{1,\overline{1}} \\
< \cdots < \cdots < \cdots < \\
< f_{n-2,n-2} < f_{n-2,n-1} < f_{n-2,n} < f_{n-2,\overline{n-1}} < f_{n-2,\overline{n-2}} \\
< f_{n-1,n-1} < f_{n-1,n} < f_{n-1,\overline{n-1}} \\
< f_{n,n},
\end{aligned}
\tag{39}
$$

so, given an element $f_{x,y}$, the elements in the rows below and the elements on the right side in the same row are larger than $f_{x,y}$.

Remark 3 If we omit in (39) above the elements $f_{i,\overline{j}}$, $i = 1, \ldots, n$, $i \le j \le n-1$, then we have the order in the case $\mathfrak{g} = \mathfrak{sl}_n$.

We use the same notation for the induced homogeneous lexicographic ordering on the monomials. Note that this monomial order $>$ is not the order $\succ_{\text{mon}}$ we define now. Let

$$
s_{\bullet,j} = \sum_{i=1}^{j} s_{i,j}, \qquad s_{\bullet,\overline{j}} = \sum_{i=1}^{j} s_{i,\overline{j}},
$$

$$
s_{i,\bullet} = \sum_{j=i}^{n} s_{i,j} + \sum_{j=i}^{n-1} s_{i,\overline{j}}.
$$

Define a map d from the set of multi-exponents $\mathbf{s}$ to $\mathbb{Z}^n_{\ge 0}$:

$$
d(\mathbf{s}) = (s_{n,\bullet}, s_{n-1,\bullet}, \ldots, s_{1,\bullet}).
$$

So, $d(\mathbf{s})_i = s_{n-i+1,\bullet}$. We say $d(\mathbf{s}) > d(\mathbf{t})$ if there exists an i such that

$$
d(\mathbf{s})_1 = d(\mathbf{t})_1, \ldots, d(\mathbf{s})_i = d(\mathbf{t})_i, d(\mathbf{s})_{i+1} > d(\mathbf{t})_{i+1}.
$$

Definition 9 For two monomials $f^{(\mathbf{s})}$ and $f^{(\mathbf{t})}$ we say $f^{(\mathbf{s})} \succ_{\text{mon}} f^{(\mathbf{t})}$ if

(a) the total degree of $f^{(\mathbf{s})}$ is bigger than the total degree of $f^{(\mathbf{t})}$;
(b) both have the same total degree but $d(\mathbf{s}) < d(\mathbf{t})$;
(c) both have the same total degree, $d(\mathbf{s}) = d(\mathbf{t})$, but $f^{(\mathbf{s})} > f^{(\mathbf{t})}$.

In other words: if both have the same total degree, this definition means that $f^{(\mathbf{s})}$ is greater than $f^{(\mathbf{t})}$ if $d(\mathbf{s})$ is smaller than $d(\mathbf{t})$, or $d(\mathbf{s}) = d(\mathbf{t})$ but $f^{(\mathbf{s})} > f^{(\mathbf{t})}$ with respect to the homogeneous lexicographic ordering on $S_{\mathbb{Z}}(\mathfrak{n}^-)$.

Remark 4 It is easy to check that "$\succ_{\mathrm{mon}}$" defines a "*monomial ordering*" in the following sense: if $f^{(\mathbf{s})} \succ_{\mathrm{mon}} f^{(\mathbf{t})}$ and $f^{(\mathbf{m})} \neq 1$, then

$$f^{(\mathbf{s}+\mathbf{m})} \succ_{\mathrm{mon}} f^{(\mathbf{t}+\mathbf{m})} \succ_{\mathrm{mon}} f^{(\mathbf{t})}.$$

By abuse of notation we use the same symbol also for the multi-exponents: we write $\mathbf{s} \succ_{\mathrm{mon}} \mathbf{t}$ if and only if $f^{(\mathbf{s})} \succ_{\mathrm{mon}} f^{(\mathbf{t})}$.

Proof of Theorem 3(ii) Let $\mathbf{s}$ be a multi-exponent violating some of the Dyck path conditions from the definition of $S(\lambda)$. As in the proof of Theorem 1, it suffices to consider the case where $\mathbf{s} \notin S(\lambda)$ and $\mathbf{s}$ is supported on a Dyck path $\mathbf{p}$ and $\mathbf{s}$ violates the Dyck path condition for $S(\lambda)$ for this path $\mathbf{p}$.

Suppose first that the Dyck path $\mathbf{p}$ is such that $p(0) = \alpha_i,\ p(k) = \alpha_j$ for some $1 \leq i \leq j < n$. In this case the Dyck path involves only roots which belong to the Lie subalgebra $\mathfrak{sl}_n \subset \mathfrak{sp}_{2n}$, and we get a straightening law by the results in Sect. 5. By (19) and Lemma 3, the application of the ∂-operators produces only summands such that $d(\mathbf{s}) = d(\mathbf{t})$ for any $\mathbf{t}$ occurring in the sum with a nonzero coefficient. Hence we can replace "$\succ$" by "$\succ_{\mathrm{mon}}$" in (15), which finishes the proof of the theorem in this case.

Now assume $p(0) = \alpha_{i,i}$ and $p(k) = \alpha_{j,\bar{j}}$ for some $j \geq i$. We include the case $j = n$ by writing $\alpha_{n,n} = \alpha_{n,\bar{n}}$. We proceed by induction on n. For $n = 1$ we have $\mathfrak{sp}_2 = \mathfrak{sl}_2$, so we can refer to Sect. 5. Now assume that we have proved the existence of a straightening law for all symplectic algebras of rank strictly smaller than n. If $i > 1$, then the Dyck path is also a Dyck path for the symplectic subalgebra $L \simeq \mathfrak{sp}_{2n-2(i-1)}$ generated by $e_{\alpha_{k,k}}, f_{\alpha_{k,k}}, h_{\alpha_{k,k}}, i \leq k \leq n$. Let $\mathfrak{n}_L^+, \mathfrak{n}_L^-$ etc. be defined by the intersection of $\mathfrak{n}^+, \mathfrak{n}^-$ etc. with L and set $\lambda_L = \sum_{k=i}^n m_k\omega_k$. It is now easy to see that the straightening law for $f^{(\mathbf{s})}$ viewed as an element in $S_{\mathbb{Z}}(\mathfrak{n}_L^{-,a})$ with respect to $I_{\mathbb{Z},L}(\lambda_L)$ defines also a straightening law for $f^{(\mathbf{s})}$ viewed as an element in $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ with respect to $I_{\mathbb{Z}}(\lambda)$.

So from now on we fix $p(0) = \alpha_1$ and $p(k) = \alpha_{i,\bar{i}}$ for some $i \in \{1, \ldots, n\}$. For a multi-exponent $\mathbf{s}$ supported on $\mathbf{p}$, set

$$\Sigma = \sum_{l=0}^{k} s_{p(l)} > m_1 + \cdots + m_n.$$

Obviously we have $f_{1,\bar{1}}^{(\Sigma)} \in I(\lambda)$. Now we consider two operators

$$\Delta_1 := \partial_{1,i-1}^{(s_{\bullet,\bar{i}}+s_{i,\bullet})} \underbrace{\partial_{i+1,\overline{i+1}}^{(s_{\bullet,i})} \dots \partial_{n,\bar{n}}^{(s_{\bullet,n-1})}}_{\delta_3} \underbrace{\partial_{1,n-1}^{(s_{\bullet,n-1}+s_{\bullet,\bar{n}})} \dots \partial_{1,i}^{(s_{\bullet,i}+s_{\bullet,\overline{i+1}})}}_{\delta_2}$$
$$\cdot \underbrace{\partial_{1,\bar{i}}^{(s_{\bullet,i-1})} \dots \partial_{1,\bar{3}}^{(s_{\bullet,2})} \partial_{1,\bar{2}}^{(s_{\bullet,1})}}_{\delta_1}$$

and

$$\Delta_2 := \partial_{1,1}^{(s_{2,\bullet})} \partial_{1,2}^{(s_{3,\bullet})} \dots \partial_{1,i-2}^{(s_{i-1,\bullet})},$$

and we will show that

$$\Delta_2 \Delta_1 f_{1,\bar{1}}^{(\Sigma)} = f^{(\mathbf{s})} + \sum_{\mathbf{s} \succ_{\mathrm{mon}} \mathbf{t}} c_{\mathbf{t}} f^{(\mathbf{t})} \tag{40}$$

with integral coefficients $c_{\mathbf{t}}$. Since $\Delta_2 \Delta_1 f_{1,\bar{1}}^{(\Sigma)} \in I_{\mathbb{Z}}(\lambda)$, the proof of (40) finishes the proof of the theorem. A first step in the proof of (40) is the following Lemma 4 below.

Recall the alphabet $J = \{1, \dots, n, \overline{n-1}, \dots, \bar{1}\}$. Let $q_1, \dots, q_i \in J$ be a sequence of increasing elements defined by

$$q_k = \max\{l \in J : \alpha_{k,l} \in \mathbf{p}\}.$$

For example, $q_i = \bar{i}$. All roots of $\mathbf{p}$ are of the form

$$\alpha_{1,1}, \dots, \alpha_{1,q_1}, \alpha_{2,q_1}, \dots, \alpha_{2,q_2}, \dots, \alpha_{i,q_{i-1}}, \dots, \alpha_{i,q_i}.$$ □

Lemma 4 *Set* $f^{(\mathbf{s}')} = f_{1,1}^{(s_{\bullet,1})} f_{1,2}^{(s_{\bullet,2})} \dots f_{1,q_{i-1}}^{(s_{\bullet,q_{i-1}}-s_{i,q_{i-1}})} f_{i,q_{i-1}}^{(s_{i,q_{i-1}})} \dots f_{i,\bar{i}}^{(s_{i,\bar{i}})}$, *then*

$$\Delta_1 f_{1,\bar{1}}^{(\Sigma)} = f^{(\mathbf{s}')} + \sum_{\mathbf{s}' \succ_{\mathrm{mon}} \mathbf{t}} c_{\mathbf{t}} f^{(\mathbf{t})}. \tag{41}$$

If $f^{(\mathbf{t})}$, $\mathbf{t} \neq \mathbf{s}'$, *is a monomial occurring in this sum, then either there exists an index* j *such that* $d(\mathbf{t})_j > 0$ *for some* $j \in \{1, 2, \dots, n-i\}$, *or* $d(\mathbf{t})_j = 0$ *for all* $j \in \{1, 2, \dots, n-i\}$ *and* $d(\mathbf{t})_{n-j+1} > s_{i,\bullet}$, *or* $d(\mathbf{t}) = d(\mathbf{s}')$ *and* $f_{i,i}^{(t_{i,i})} f_{i,i+1}^{(t_{i,i+1})} \cdots f_{i,\bar{i}}^{(t_{i,\bar{i}})} < f_{i,i}^{(s_{i,i})} f_{i,i+1}^{(s_{i,i+1})} \cdots f_{i,\bar{i}}^{(s_{i,\bar{i}})}$.

Corollary 2 *If* $f^{\mathbf{t}} \neq f^{\mathbf{s}'}$ *is a monomial occurring in* (41), *then either* $\Delta_2 f^{\mathbf{t}} = 0$, *or* $\Delta_2 f^{\mathbf{t}}$ *is a sum of monomials* $f^{\mathbf{k}}$ *such that* $f^{\mathbf{s}} \succ_{\mathrm{mon}} f^{\mathbf{k}}$.

Proof of the lemma One easily sees by induction that

$$\delta_1\big(f_{1,\bar{1}}^{(\Sigma)}\big) = f_{1,1}^{(s_{\bullet,1})} f_{1,2}^{(s_{\bullet,2})} \dots f_{1,i-1}^{(s_{\bullet,i-1})} f_{1,\bar{1}}^{(\Sigma - s_{\bullet,1} - s_{\bullet,2} - \dots - s_{\bullet,i-1})}.$$

Note that the roots used in the operator are $\alpha_{1,\bar{2}}, \ldots, \alpha_{1,\bar{i}}$, and they are applied to $f_{1,\bar{1}}$ of weight $\alpha_{1,\bar{1}}$. In terms of (10)–(13), we apply $\partial^{(*)}_{\alpha+\gamma}$ to $f^{(*)}_{\alpha+2\gamma}$, so rule (11) applies.

Since $\alpha_{1,j} - \alpha_{1,\ell}$, $1 \le j < i$, $i < \ell \le n$, and $\alpha_{1,j} - \alpha_{\ell,\bar{\ell}}$, $1 \le j < i$, $i < \ell \le n$, and $\alpha_{1,j} - \alpha_{1,i-1}$, $1 \le j < i$, are never positive roots, one has

$$\partial_{1,i-1}^{(s_{\bullet,\bar{i}}+s_{i,\bullet})} \delta_3 \delta_2 \underbrace{\left(f_{1,1}^{(s_{\bullet,1})} f_{1,2}^{(s_{\bullet,2})} \cdots f_{1,i-1}^{(s_{\bullet,i-1})}\right)}_{f^{(\mathbf{x})}} = 0,$$

so it remains to consider $f^{(\mathbf{x})} \partial_{1,i-1}^{(s_{\bullet,\bar{i}}+s_{i,\bullet})} \delta_3 \delta_2 (f_{1,\bar{1}}^{(\Sigma - s_{\bullet,1} - s_{\bullet,2} - \cdots - s_{\bullet,i-1})})$.

To better visualize the following procedure, one should think of the variables $f_{i,j}$ as being arranged in a triangle like in the picture after Lemma 3, or in the following example (type $\mathtt{C}_4$):

f_{11}	f_{12}	f_{13}	f_{14}	$f_{1\bar{3}}$	$f_{1\bar{2}}$	$f_{1\bar{1}}$
	f_{22}	f_{23}	f_{24}	$f_{2\bar{3}}$	$f_{2\bar{2}}$	
		f_{33}	f_{34}	$f_{3\bar{3}}$		
			f_{44}			

(42)

With respect to the ordering ">", the largest element is located in the bottom row and the smallest element is written in the top row on the left side. We enumerate the rows and columns like the indices of the variables, so the top row is the 1-st row, the bottom row the n-th row, the columns are enumerated from the left to the right, so we have the 1-st column on the left side and the most right one is the $\bar{1}$-st column. We refer to row k, column j as the $(k; j)$ entry. Similarly, we refer to row k, column $\bar{j}$ as the $(k; \bar{j})$ entry.

The operator $\partial_{1,q}$, $1 \le q \le n-1$, kills all $f_{1,j}$ for $1 \le j \le q$, $\partial_{1,q}(f_{1,j}) = f_{q+1,j}$ for $j = q+1, \ldots, \overline{q+1}$ (rule (9) applies), $\partial_{1,q}(f_{1,\bar{j}}) = f_{j,\overline{q+1}}$ for $j = 1, \ldots, q$ (rule (9) applies), and $\partial_{1,q}$ kills all $f_{k,\ell}$ for $k \ge 2$. Because of the set of indices of the operators occurring in δ_2, the operator applied to $f_{1,\bar{1}}^{(\Sigma - s_{\bullet,1} - s_{\bullet,2} - \cdots - s_{\bullet,i-1})}$ never increases the zero entries in positions $(1; \bar{i})$ through $(1; \bar{2})$. As a consequence, the application of δ_2 produces the sum of monomials

$$f^{(\mathbf{x})} f_{1,\overline{i+1}}^{(s_{\bullet,i}+s_{\bullet,\overline{i+1}})} \cdots f_{1,\overline{n-1}}^{(s_{\bullet,n-2}+s_{\bullet,\overline{n-1}})} f_{1,n}^{(s_{\bullet,n-1}+s_{\bullet,n})} f_{1,\bar{1}}^{(s_{\bullet,\bar{i}})} + \sum c_{\mathbf{k}} f^{(\mathbf{k})},$$

where the monomials $f^{(\mathbf{k})}$ occurring in the sum are such that the corresponding triangle (see (42)) has at least one non-zero entry in one of the positions between the $(i+1)$-th and the n-th row (counted from top to bottom). This implies $d(\mathbf{k})_j > 0$ for some $j = 1, \ldots, n-i$. The operators δ_3 and $\partial_{1,i-1}^{(s_{\bullet,\bar{i}}+s_{i,\bullet})}$ do not change this property because (in the language of the scheme (42) above) the operators $\partial_{j,\bar{j}}$ used to compose δ_3 either kill a monomial or, in the language of the scheme (42), they subtract from an $(k, \bar{j})$ entry and add to a $(k, j-1)$ entry. The operator $\partial_{1,i-1}$ subtracts from the entries in the top row and, since the entries in the positions $(1, \overline{i-1})$ up to $(1,\bar{2})$ are zero, adds to the entries in the i-th row. The only exception is $\partial_{1,i-1}$ applied to $f_{1,\bar{1}}$, the result is $f_{1,\bar{i}}$. It follows that the monomials $f^{(\mathbf{k}')}$ occurring in $\partial_{1,i-1}^{(s_{\bullet,\bar{i}}+s_{i,\bullet})}\delta_3 f^{(\mathbf{k})}$ have already the desired properties because we have just seen that $d(\mathbf{k}')_j > 0$ for some $j = 1,\dots,n-i$.

So to finish the proof of the lemma, in the following it suffices to consider

$$f^{\mathbf{x}}\partial_{1,i-1}^{(s_{\bullet,\bar{i}}+s_{i,\bullet})}\delta_3 f_{1,\overline{i+1}}^{(s_{\bullet,i}+s_{\bullet,\overline{i+1}})}\cdots f_{1,\overline{n-1}}^{(s_{\bullet,n-2}+s_{\bullet,\overline{n-1}})} f_{1,n}^{(s_{\bullet,n-1}+s_{\bullet,n})} f_{1,\bar{1}}^{(s_{\bullet,\bar{i}})}$$
$$= f^{\mathbf{x}}\partial_{1,i-1}^{(s_{\bullet,\bar{i}}+s_{i,\bullet})} f_{1,i}^{(s_{\bullet,i})} f_{1,i+1}^{(s_{\bullet,i+1})}\cdots f_{1,n}^{(s_{\bullet,n})} f_{1,\overline{n-1}}^{(s_{\bullet,\overline{n-1}})}\cdots f_{1,\overline{i+1}}^{(s_{\bullet,\overline{i+1}})} f_{1,\bar{1}}^{(s_{\bullet,\bar{i}})}. \quad (43)$$

Note that the operators in δ_3 are of the form $\partial_{j,\bar{j}}$, $j = i+1,\dots,n$, and they are applied to $f_{1,\bar{\ell}}$, $\ell = i+1,\dots,n$, so $\partial_{j,\bar{j}}^{(k)} f_{1,\bar{\ell}}^{(p)} = 0$ for $\ell \neq j$ and for $j = \ell$ we set $\alpha = \alpha_{j,\bar{j}}$, $\gamma = \alpha_{1,j-1}$, $\partial_{j,\bar{j}} = \partial_\alpha$, $f_{1,\bar{j}} = f_{\alpha+\gamma}$, so rule (10) applies and the coefficient in (43) is 1.

It remains to consider the operator $\partial_{1,i-1}^{(s_{\bullet,\bar{i}}+s_{i,\bullet})}$. There are three possibilities: applying $\partial_{1,i-1}$ to the monomial above increases the degree with respect to the variables $f_{i,*}$, or the operator is applied to a variable killed by the operator, or the operator is applied to a factor $f_{1,\bar{1}}$, in which case the result is $f_{1,\bar{i}}$ (note that in this case rule (11) applies). So the right hand side of (43) can be written as a linear combination $\sum c_{\mathbf{k}} f^{(\mathbf{k})}$ of monomials such that $d(\mathbf{k})_j = 0$ for $j = 1,\dots,n-i$ and $d(\mathbf{k})_{n-i+1} \geq s_{i,\bullet}$.

It remains to consider the case where $d(\mathbf{k})_{n-i+1} = s_{i,\bullet}$. This is only possible if $\partial_{1,i-1}$ is applied $s_{\bullet,\bar{i}}$-times to $f_{1,\bar{1}}^{s_{\bullet,\bar{i}}}$, in which case $d(\mathbf{k})$ has only two non-zero entries: $d(\mathbf{k})_1 = \Sigma - s_{i,\bullet}$ and $d(\mathbf{k})_{n-i+1} = s_{i,\bullet}$, so $d(\mathbf{k}) = d(\mathbf{s}')$. If $\mathbf{k} \neq \mathbf{s}'$, then necessarily $f_{i,i}^{(t_{i,i})} f_{i,i+1}^{(t_{i,i+1})}\cdots f_{i,\bar{i}}^{(t_{i,\bar{i}})} < f_{i,i}^{(s_{i,i})} f_{i,i+1}^{(s_{i,i+1})}\cdots f_{i,\bar{i}}^{(s_{i,\bar{i}})}$. □

Proof of the corollary The operators used to compose Δ_2 do not change anymore the entries of $d(\mathbf{t})$ for the first $n-i+1$ indices.

Suppose first $\mathbf{t}$ is such that there exists an index j such that $d(\mathbf{t})_j > 0$ for some $j \in \{1,2,\dots,n-i\}$ or $d(\mathbf{t})_{i,\bar{i}} > s_{i,\bullet}$. By the description of the operators occurring in Δ_2, every monomial $f^{(\mathbf{k})}$ occurring with a nonzero coefficient in $\Delta_2 f^{(\mathbf{t})}$ has this property too and hence $f^{(\mathbf{s})} \succ_{\text{mon}} f^{(\mathbf{k})}$.

Next assume $d(\mathbf{t}) = d(\mathbf{s}')$ and $f_{i,i}^{(t_{i,i})} f_{i,i+1}^{(t_{i,i+1})}\cdots f_{i,\bar{i}}^{(t_{i,\bar{i}})} < f_{i,i}^{(s_{i,i})} f_{i,i+1}^{(s_{i,i+1})}\cdots f_{i,\bar{i}}^{(s_{i,\bar{i}})}$. Recall that $\mathbf{t}_{1,\overline{i-1}} = \cdots = \mathbf{t}_{1,\bar{1}} = 0$. It follows that the operators occurring in Δ_2 always only subtract from one of the entries in the top row and add to the entry in the same column and a corresponding row (of index strictly smaller than i). It follows that all monomials $f^{(\mathbf{k})}$ occurring in $\Delta_2(f^{(\mathbf{t})})$ have the property: $d(\mathbf{k}) = d(\mathbf{s})$. Since $f_{i,i}^{(t_{i,i})} f_{i,i+1}^{(t_{i,i+1})}\cdots f_{i,\bar{i}}^{(t_{i,\bar{i}})} < f_{i,i}^{(s_{i,i})} f_{i,i+1}^{(s_{i,i+1})}\cdots f_{i,\bar{i}}^{(s_{i,\bar{i}})}$, it follows that $f^{(\mathbf{s})} > f^{(\mathbf{k})}$ and hence $f^{(\mathbf{s})} \succ_{\text{mon}} f^{(\mathbf{k})}$. □

Continuation of the proof of Theorem 3(ii) We have seen that, in order to prove Theorem 3(ii), it suffices to prove (40). Recall the definition of the multi-index $(\mathbf{s}')$

in Lemma 4:

$$f^{(\mathbf{s}')} = f_{1,1}^{(s_{\bullet,1})} f_{1,2}^{(s_{\bullet,2})} \dots f_{1,q_{i-1}}^{(s_{\bullet,q_{i-1}}-s_{i,q_{i-1}})} f_{i,q_{i-1}}^{(s_{i,q_{i-1}})} \dots f_{i,\bar{i}}^{(s_{i,\bar{i}})}. \tag{44}$$

To prove the theorem it remains to prove (using Lemma 4 and Corollary 2) for $f^{(\mathbf{s}')}$ that $\Delta_2 f^{(\mathbf{s}')}$ is a linear combination of $f^{(\mathbf{s})}$ with coefficient 1 and monomials strictly smaller than $f^{(\mathbf{s})}$. The following lemma proves this claim and hence finishes the proof of the theorem. □

Lemma 5 *The operator* $\Delta_2 := \partial_{1,1}^{(s_{2,\bullet})} \partial_{1,2}^{(s_{3,\bullet})} \dots \partial_{1,i-2}^{(s_{i-1,\bullet})}$ *applied to the monomial* $f^{(\mathbf{s}')}$ *(see* (44) *for the multi-index* $(\mathbf{s}')$*) is a linear combination of* $f^{(\mathbf{s})}$ *and smaller monomials*:

$$\Delta_2 f^{(\mathbf{s}')} = f^{(\mathbf{s})} + \sum_{\mathbf{s} \succ_{\text{mon}} \mathbf{t}} c_{\mathbf{t}} f^{(\mathbf{t})}. \tag{45}$$

Proof First note that all monomials $f^{(\mathbf{k})}$ occurring in $\Delta_2 f^{(\mathbf{s}')}$ have the same total degree. Recall that $\mathbf{s}'_{1,\overline{i-1}} = \dots = \mathbf{s}'_{1,\bar{1}} = 0$. It follows that the operators occurring in Δ_2 always only subtract from one of the entries in the top row and add to the entry in the same column and a corresponding row (of index strictly smaller than i and strictly greater than 1). It follows that all monomials $f^{(\mathbf{k})}$ occurring in $\Delta_2(f^{(\mathbf{s}')})$ have the same multidegree $d(\mathbf{s})$, in fact, we will see below that $f^{\mathbf{s}}$ is a summand and hence $d(\mathbf{k}) = d(\mathbf{s})$.

So in the following we can replace the ordering $\succ_{\text{mon}}$ by $>$ since, in this special case, the latter implies the first.

The elements $f_{i,j}$ and $f_{i,\bar{j}}$, $2 \le i \le j \le n$, are in the kernel of the operators $\partial_{1,k}$ for all $1 \le k \le n$, and so are the variables $f_{1,j}$, $j \le k$ in the first k columns.

The operator $\partial_{1,k}$, $1 \le k \le n$, "moves" the variables $f_{1,j}$, $k+1 \le j \le n$ from the first row to the variable $f_{k+1,j}$ in the same column, in this case rule (9) applies.

The operator $\partial_{1,k}$, $1 \le k \le n$ "moves" the variables $f_{1,\bar{j}}$, $k+1 \le j \le n$ from the first row to the variable $f_{k+1,\bar{j}}$ in the same column. Note that here rule (9) applies, except for $j = k+1$, in this case set rule (10) applies.

For $j \le k$, the operator makes the variables switch the column, it moves the variable $f_{1,\bar{j}}$ to the variable $f_{j,\overline{k+1}}$ in the j-th row and $(\overline{k+1})$-th column. In this situation rule (9) applies, except if $j = 1$. But note that $j = 1$ can be excluded in our case because $j = 1$ implies $i = 1$ for the path, and this implies that Δ_2 is the identity operator, so there is no operator $\partial_{1,k}$ in this case.

We proceed by induction on i. If $i = 1, 2$, then Δ_2 is the identity operator, $f^{(\mathbf{s})} = f^{(\mathbf{s}')}$ and hence the lemma is trivially true. Now assume $i \ge 3$ and the lemma holds for all numbers less than i. We note that the monomial

$$f_{1,1}^{(s_{1,1})} \dots f_{1,q_1}^{(s_{1,q_1})} \cdot \left(\partial_{1,1}^{(s_{2,q_1})} f_{1,q_1}^{(s_{2,q_1})} \dots \partial_{1,1}^{(s_{2,q_2})} f_{1,q_2}^{(s_{2,q_2})}\right)$$
$$\dots \cdot \left(\partial_{1,i-2}^{(s_{i-1,q_{i-2}})} f_{1,q_{i-2}}^{(s_{i-1,q_{i-2}})} \dots \partial_{1,i-2}^{(s_{i-1,q_{i-1}})} f_{1,q_{i-1}}^{(s_{i-1,q_{i-1}})}\right)\left(f_{i,q_{i-1}}^{(s_{i,q_{i-1}})} \dots f_{i,\bar{i}}^{(s_{i,\bar{i}})}\right)$$

is equal to $f^{\mathbf{s}}$ (only the rules (9) and (10) apply) and appears as a summand in $\Delta_2 f^{(\mathbf{s}')}$. Our aim is to show that all other monomials in $\Delta_2 f^{(\mathbf{s}')}$ are less than $f^{(\mathbf{s})}$.

All monomials share the common factor $(f_{i,q_{i-1}}^{(s_{i,q_{i-1}})} \dots f_{i,\bar{i}}^{(s_{i,\bar{i}})})$, the maximal variable smaller than the ones occurring in the divisor is the variable $f_{i-1,q_{i-1}}$. Note that if $j < i-1$ then for any $q \in J$ the variable $\partial_{1,j} f_{1,q}$ lies in the $(j+1)$-th row, note that $j+1 < i$. The operator $\partial_{1,i-2}$ is applied $s_{i-1,\bullet}$-times, the unique maximal monomial in the sum expression of $\partial_{1,i-2}^{(s_{i-1,\bullet})} f^{(\mathbf{s}')}$ is

$$f_{1,1}^{(s_{\bullet,1})} f_{1,2}^{(s_{\bullet,2})} \dots f_{1,q_{i-2}}^{(s_{\bullet,q_{i-2}}-s_{i-1,q_{i-2}})} \big(f_{i-1,q_{i-2}}^{(s_{i-1,q_{i-2}})} \dots f_{i-1,q_{i-1}}^{(s_{i-1,q_{i-1}})}\big)\big(f_{i,q_{i-1}}^{(s_{i,q_{i-1}})} \dots f_{i,\bar{i}}^{(s_{i,\bar{i}})}\big),$$

because applying the operator $\partial_{1,i-2}$ to any of the variables $f_{1,j}$ such that $j \neq q_{i-2}, \dots, q_{i-1}$, gives a monomial smaller in the order $>$, and the exponents $s_{i-1,j}$, $j = q_{i-2}, \dots, q_{i-1}$, are the maximal powers such that $\partial_{1,i-2}^{(*)}$ can be applied to $f_{1,j}^{(y)}$ because either $q_{i-2} < j < q_{i-1}$, and then $y = s_{\bullet,j} = s_{i-1,j}$, or $j = q_{i-1}$, then $s_{i-1,q_{i-1}}$ is the power with which the variable occurs in $f^{(\mathbf{s}')}$, or $j = q_{i-2}$, then only the power $s_{i-1,q_{i-2}}$ of the operator is left.

Repeating the arguments for the operators $\partial_{1,i-3}$ etc. finishes the proof of the lemma. □

9 The Tensor Product Property

In the following section let $\mathfrak{g} = SL_n$ or Sp_{2n}.

Proposition 6 *For two dominant weights λ and μ the $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$-module $V_{\mathbb{Z}}^a(\lambda+\mu)$ is embedded into the tensor product $V_{\mathbb{Z}}^a(\lambda) \otimes_{\mathbb{Z}} V_{\mathbb{Z}}^a(\mu)$ as the highest weight component, i.e. there exists a unique injective homomorphism of $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$-modules:*

$$V_{\mathbb{Z}}^a(\lambda+\mu) \hookrightarrow V_{\mathbb{Z}}^a(\lambda) \otimes V_{\mathbb{Z}}^a(\mu) \quad \textit{such that} \quad v_{\lambda+\mu} \mapsto v_\lambda \otimes v_\mu. \tag{46}$$

Proof Using the defining relations for $V_{\mathbb{Z}}^a(\lambda+\mu)$, it is easy to see that we have a canonical map $V_{\mathbb{Z}}^a(\lambda+\mu) \to V_{\mathbb{Z}}^a(\lambda) \otimes V_{\mathbb{Z}}^a(\mu)$ sending $v_{\lambda+\mu}$ to $v_\lambda \otimes v_\mu$. We know that $V_{\mathbb{Z}}^a(\lambda) \subset V^a(\lambda)$ and $V_{\mathbb{Z}}^a(\mu) \subset V^a(\mu)$ are lattices in the corresponding complex vector spaces, and, by [4] and [5], we know that $S(\mathfrak{n}^{-,a})(v_\lambda \otimes v_\mu) \subset V^a(\lambda) \otimes V^a(\mu)$ is isomorphic to $V^a(\lambda+\mu)$, the isomorphism being given by

$$V^a(\lambda+\mu) \ni m.v_{\lambda+\mu} \mapsto m.v_\lambda \otimes v_\mu \in V^a(\lambda) \otimes V^a(\mu) \quad \text{for } m \in S\big(\mathfrak{n}^{-,a}\big).$$

It follows that the induced map $V_{\mathbb{Z}}^a(\lambda+\mu) \to V_{\mathbb{Z}}^a(\lambda) \otimes V_{\mathbb{Z}}^a(\mu)$ between the lattices is injective and hence an isomorphism onto the image. □

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On the Subgeneric Restricted Blocks of Affine Category $\mathcal{O}$ at the Critical Level

Peter Fiebig

Abstract We determine the endomorphism algebra of a projective generator in a subgeneric restricted block of the critical level category $\mathcal{O}$ over an affine Kac–Moody algebra.

1 Introduction

This article complements the results of the articles [1] and [2]. There we studied the structure of restricted critical level representations for affine Kac–Moody algebras. The two main results we obtained are the following. The first is a multiplicity formula for restricted Verma modules with a *subgeneric* critical highest weight, and the second is a linkage principle together with a block decomposition for the restricted category $\mathcal{O}$. In this article we use these results in order to describe the categorical structure of the subgeneric restricted blocks of $\mathcal{O}$.

We would like to be able to describe the structure of all restricted blocks and to establish more general multiplicity and character formulas. Generically, a restricted critical level block contains a unique simple object which is, moreover, projective. This implies that such a block is equivalent to the category of $\mathbb{C}$-vector spaces. The next simplest situation is already much more involved. Each *subgeneric* block contains infinitely many simples. Every subgeneric restricted Verma module has a two-step Jordan–Hölder filtration, and the restricted version of BGGH-reciprocity (see [2]) tells us that a restricted subgeneric indecomposable projective object is a non-split extension of two Verma modules. In this note we describe the endomorphism algebra of a projective generator in such a subgeneric block.

2 Affine Kac–Moody Algebras

In this section we collect the main structural results on affine Kac–Moody algebras. Let $\mathfrak{g}$ be a simple complex Lie algebra and let $\widehat{\mathfrak{g}}$ be the corresponding affine Kac–

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Moody algebra. As a vector space, $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$, and the Lie bracket on $\widehat{\mathfrak{g}}$ is determined by the rules

$$\left[x \otimes t^m, y \otimes t^n\right] = [x, y] \otimes t^{m+n} + m\delta_{m,-n}(x, y)K,$$
$$[K, \widehat{\mathfrak{g}}] = \{0\},$$
$$\left[D, x \otimes t^n\right] = nx \otimes t^n,$$

where x and y are elements of $\mathfrak{g}$, m and n are integers, $\delta_{a,b}$ is the Kronecker symbol, and $(\cdot, \cdot)\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ denotes the Killing form on $\mathfrak{g}$.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and $\mathfrak{b} \subset \mathfrak{g}$ a Borel subalgebra containing $\mathfrak{h}$. Then

$$\widehat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D,$$
$$\widehat{\mathfrak{b}} := \mathfrak{g} \otimes t\mathbb{C}[t] \oplus \mathfrak{b} \oplus \mathbb{C}K \oplus \mathbb{C}D$$

denote the corresponding affine Cartan and Borel subalgebras of $\widehat{\mathfrak{g}}$, respectively.

2.1 Affine Roots

We denote by $V^\star$ the dual of a vector space V. Let $R \subset \mathfrak{h}^\star$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We consider $\mathfrak{h}^\star$ as a subspace in $\widehat{\mathfrak{h}}^\star$ by letting each $\lambda \in \mathfrak{h}^\star$ act trivially on $\mathbb{C}K \oplus \mathbb{C}D$. We define $\delta \in \widehat{\mathfrak{h}}^\star$ by

$$\delta(\mathfrak{h} \oplus \mathbb{C}K) = \{0\},$$
$$\delta(D) = 1.$$

The set $\widehat{R} \subset \widehat{\mathfrak{h}}^\star$ of roots of $\widehat{\mathfrak{g}}$ with respect to $\widehat{\mathfrak{h}}$ is

$$\widehat{R} = \{\alpha + n\delta \mid \alpha \in R, n \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{Z}, n \neq 0\}.$$

The subsets

$$\widehat{R}^{\text{re}} := \{\alpha + n\delta \mid \alpha \in R, n \in \mathbb{Z}\},$$
$$\widehat{R}^{\text{im}} := \{n\delta \mid n \in \mathbb{Z}, n \neq 0\}$$

are called the sets of *real* roots and of *imaginary* roots, resp.

We denote by $R^+ \subset R$ the positive (finite) roots, i.e. the set of roots of $\mathfrak{b}$ with respect to $\mathfrak{h}$. Then the set of positive affine roots, i.e. the set of roots of $\widehat{\mathfrak{b}}$ with respect to $\widehat{\mathfrak{h}}$, is

$$\widehat{R}^+ := \{\alpha + n\delta \mid \alpha \in R, n \geq 1\} \cup R^+ \cup \{n\delta \mid n \geq 1\}.$$

The partial order "$\leq$" on $\widehat{\mathfrak{h}}^\star$ is defined as follows. We have $\lambda \leq \mu$ if $\mu - \lambda$ is a sum of positive affine roots.

2.2 The Invariant Bilinear Form

There is an extension of the Killing form $(\cdot,\cdot)$ on $\mathfrak{g}$ to a symmetric bilinear form on $\widehat{\mathfrak{g}}$. It is determined by the following:

$$\begin{aligned}\left(x\otimes t^n, y\otimes t^m\right) &= \delta_{n,-m}(x,y),\\ \left(K, \mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}\left[t,t^{-1}\right]\oplus\mathbb{C}K\right) &= \{0\},\\ \left(D, \mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}\left[t,t^{-1}\right]\oplus\mathbb{C}D\right) &= \{0\},\\ (K,D) &= 1\end{aligned}$$

for $x, y\in\mathfrak{g}$ and $m, n\in\mathbb{Z}$. This form is again non-degenerate and *invariant*, i.e. it satisfies $([x,y],z) = (x,[y,z])$ for all $x,y,z\in\widehat{\mathfrak{g}}$. Moreover, it induces a non-degenerate bilinear form on the Cartan subalgebra $\widehat{\mathfrak{h}}$ and hence yields an isomorphism $\widehat{\mathfrak{h}}\xrightarrow{\sim}\widehat{\mathfrak{h}}^\star$. We get an induced symmetric non-degenerate bilinear form on the dual $\widehat{\mathfrak{h}}^\star$, which we again denote by the symbol $(\cdot,\cdot)$.

Remark 1 The definitions immediately imply that the isomorphism $\widehat{\mathfrak{h}}\to\widehat{\mathfrak{h}}^\star$ from above maps K to δ, i.e. for any $\lambda\in\widehat{\mathfrak{h}}^\star$ we have

$$\lambda(K) = (\delta,\lambda).$$

In particular, $(\delta,\gamma) = 0$ for any $\gamma\in\widehat{R}$.

2.3 The Weyl Group

For each real affine root $\alpha+n\delta$ we have $(\alpha+n\delta,\alpha+n\delta) = (\alpha,\alpha)\neq 0$, hence we can define the reflection

$$\begin{aligned}s_{\alpha,n}\colon \widehat{\mathfrak{h}}^\star &\to \widehat{\mathfrak{h}}^\star\\ \lambda &\mapsto \lambda - 2\frac{(\lambda,\alpha+n\delta)}{(\alpha,\alpha)}(\alpha+n\delta).\end{aligned}$$

This is a reflection as it stabilizes the hyperplane $(\cdot,\alpha+n\delta) = 0$ and maps $\alpha+n\delta$ to $-\alpha-n\delta$.

We denote by $\widehat{\mathcal{W}}\subset \mathrm{GL}(\widehat{\mathfrak{h}}^\star)$ the affine Weyl group, i.e. the subgroup generated by the reflections $s_{\alpha,n}$ for $\alpha\in R$ and $n\in\mathbb{Z}$. The subgroup $\mathcal{W}\subset\widehat{\mathcal{W}}$ generated by the reflections $s_{\alpha,0}$ with $\alpha\in R$ leaves the subset $\mathfrak{h}^\star\subset\widehat{\mathfrak{h}}^\star$ stable and can be identified with the Weyl group of $\mathfrak{g}$.

Let $\rho\in\widehat{\mathfrak{h}}^\star$ be an element that takes the value 1 on any simple affine coroot. This element is defined only up to addition of a multiple of δ. Nevertheless, nothing in what follows will depend on the value of ρ on D in an essential way.

The *dot-action* $\widehat{\mathcal{W}} \times \widehat{\mathfrak{h}}^\star \to \widehat{\mathfrak{h}}^\star$, $(w, \lambda) \mapsto w.\lambda$, of the affine Weyl group on $\widehat{\mathfrak{h}}^\star$ is obtained by shifting the linear action in such a way that $-\rho$ becomes a fixed point, i.e. it is given by

$$w.\lambda := w(\lambda + \rho) - \rho$$

for $w \in \widehat{\mathcal{W}}$ and $\lambda \in \widehat{\mathfrak{h}}^\star$. Note that since $(\delta, \alpha + n\delta) = 0$ we have $s_{\alpha,n}(\delta) = \delta$ for all $\alpha + n\delta \in \widehat{R}^{\mathrm{re}}$. Hence $w(\delta) = \delta$ for all $w \in \widehat{\mathcal{W}}$ (so the dot-action is independent of the choice of ρ).

3 The Affine Category $\mathcal{O}$

We denote by $\widehat{\mathcal{O}}$ the full subcategory of the category of representations of $\widehat{\mathfrak{g}}$ that contains an object M if and only if it has the following properties:

- M is semisimple under the action of $\widehat{\mathfrak{h}}$,
- M is locally finite under the action of $\widehat{\mathfrak{b}}$.

The first condition means that $M = \bigoplus_{\lambda \in \widehat{\mathfrak{h}}^\star} M_\lambda$, where $M_\lambda = \{m \in M \mid h.m = \lambda(h)m \text{ for all } h \in \widehat{\mathfrak{h}}\}$, and the second that each $m \in M$ is contained in a finite-dimensional sub-$\widehat{\mathfrak{b}}$-module of M.

For any $\lambda \in \widehat{\mathfrak{h}}^\star$ we denote by $\Delta(\lambda)$ the Verma module with highest weight λ, and by $L(\lambda)$ its unique irreducible quotient. The $L(\lambda)$ for $\lambda \in \widehat{\mathfrak{h}}^\star$ are a system of representatives of the simple objects in the category $\widehat{\mathcal{O}}$.

Remark 2 Most often the Verma modules are denoted by $M(\lambda)$ instead of $\Delta(\lambda)$. However, the notation $\Delta(\lambda)$ appears in categorical contexts in order to signify *standard* modules. The dual standard modules are then denoted by $\nabla(\lambda)$. This viewpoint, in particular, was taken in the article [2].

For an object M of $\widehat{\mathcal{O}}$ and a simple object L in $\widehat{\mathcal{O}}$ we denote by $[M : L] \in \mathbb{N}$ the corresponding Jordan–Hölder multiplicity, whenever this makes sense (see [3]). In general, we write $[M : L] \neq 0$ if L is isomorphic to a subquotient (i.e. a quotient of a subobject) of M.

3.1 Projective Objects in $\widehat{\mathcal{O}}$

In order to describe the categorical structure of $\widehat{\mathcal{O}}$ we want to describe the endomorphism algebra of a projective generator. Now $\widehat{\mathcal{O}}$ does not contain enough projectives. Fortunately, it is filtered by "truncated subcategories" that do contain enough projectives, which for us is good enough.

In order to define the truncated subcategories, we need the following topology on $\widehat{\mathfrak{h}}^\star$.

Definition 1 A subset $\mathcal{J}$ of $\widehat{\mathfrak{h}}^\star$ is called *open* if it is downwardly closed with respect to the partial order "$\leq$", i.e. if it satisfies the following condition: If $\lambda \in \mathcal{J}$ and $\mu < \lambda$, then $\mu \in \mathcal{J}$. An open subset $\mathcal{J}$ of $\widehat{\mathfrak{h}}^\star$ is called *bounded* (rather, *locally bounded from above*), if for any $\lambda \in \mathcal{J}$, the set $\{\nu \in \mathcal{J} \mid \nu > \lambda\}$ is finite.

Now we can define the truncated subcategories.

Definition 2 Let $\mathcal{J} \subset \widehat{\mathfrak{h}}^\star$ be open. Then $\widehat{\mathcal{O}}^{\mathcal{J}}$ is the full subcategory of $\widehat{\mathcal{O}}$ that contains all objects M with the property that $M_\lambda \neq \{0\}$ implies $\lambda \in \mathcal{J}$.

For any $\lambda \in \widehat{\mathfrak{h}}^\star$ the set $\{\mu \in \widehat{\mathfrak{h}}^\star \mid \mu \leq \lambda\}$ is open. We use the notation $\widehat{\mathcal{O}}^{\leq\lambda}$ instead of $\widehat{\mathcal{O}}^{\{\mu\in\widehat{\mathfrak{h}}^\star|\mu\leq\lambda\}}$. Note that $L(\lambda)$ is contained in $\widehat{\mathcal{O}}^{\mathcal{J}}$ if and only if $\lambda \in \mathcal{J}$. The inclusion functor $\widehat{\mathcal{O}}^{\mathcal{J}} \to \widehat{\mathcal{O}}$ has a left adjoint that we denote by $M \mapsto M^{\mathcal{J}}$. It is defined as follows: Let $\mathcal{I} = \widehat{\mathfrak{h}}^\star \setminus \mathcal{J}$ be the closed complement of $\mathcal{J}$ and let $M_{\mathcal{I}} \subset M$ be the submodule generated by all weight spaces M_λ with $\lambda \in \mathcal{I}$. Then set

$$M^{\mathcal{J}} := M/M_{\mathcal{I}}.$$

This definition clearly is functorial. We will need the following notion.

Definition 3 Let $M \in \widehat{\mathcal{O}}$. We say that M *admits a Verma flag* if there is a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

with $M_i/M_{i-1} \cong \Delta(\mu_i)$ for some $\mu_1, \ldots, \mu_n \in \widehat{\mathfrak{h}}^\star$.

In case M admits a Verma flag, the *Verma multiplicity* $(M : \Delta(\mu_i)) = \#\{i \in \{1, \ldots, n\} \mid \mu_i = \mu\}$ is independent of the chosen filtration. The following is proven in [8] (see also [4]).

Theorem 1 *Let $\mathcal{J} \subset \mathfrak{h}^\star$ be open and bounded and let $\lambda \in \mathcal{J}$.*

1. *There exists a projective cover $P^{\mathcal{J}}(\lambda) \to L(\lambda)$ in $\widehat{\mathcal{O}}^{\mathcal{J}}$ and the object $P^{\mathcal{J}}(\lambda)$ admits a Verma flag.*
2. *BGGH-reciprocity*

$$\left(P^{\mathcal{J}}(\lambda) : \Delta(\mu)\right) = \begin{cases} [\Delta(\mu) : L(\lambda)], & \text{if } \mu \in \mathcal{J}, \\ 0, & \text{if } \mu \notin \mathcal{J} \end{cases}$$

holds for the Jordan-Hölder and Verma multiplicities.

3. *If $\mathcal{J}' \subset \mathcal{J}$ is an open subset, then $P^{\mathcal{J}}(\lambda)^{\mathcal{J}'} \cong P^{\mathcal{J}'}(\lambda)$.*
4. *For any $M \in \widehat{\mathcal{O}}^{\mathcal{J}}$ such that $[M : L(\lambda)]$ is finite we have*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\widehat{\mathcal{O}}}\left(P^{\mathcal{J}}(\lambda), M\right) = \left[M : L(\lambda)\right].$$

3.2 The Block Decomposition of Category $\widehat{\mathcal{O}}$

We quickly summarize the basic facts about the block decomposition of category $\widehat{\mathcal{O}}$. Recall that the simple isomorphism classes are parametrized by $\widehat{\mathfrak{h}}^\star$ by means of their highest weight. The block decomposition in particular yields a partition of the simple isomorphism classes. In terms of their parameters, this partition is given as follows.

Definition 4 Let "$\sim$" be the equivalence relation on $\widehat{\mathfrak{h}}^\star$ that is generated by the following. We have $\lambda \sim \mu$ if there exists a positive affine root $\gamma \in \widehat{R}^+$ and $n \in \mathbb{Z}$ such that $2(\lambda + \rho, \gamma) = n(\gamma, \gamma)$ and $\mu = \lambda - n\gamma$.

For an equivalence class $\Lambda \subset \widehat{\mathfrak{h}}^\star$ with respect to "$\sim$" we let $\widehat{\mathcal{O}}_\Lambda$ be the full subcategory of $\widehat{\mathcal{O}}$ that contains all objects M with the property that $[M : L(\lambda)] \neq 0$ implies $\lambda \in \Lambda$. The linkage principle (see [5]) together with BGGH-reciprocity mentioned above now yields the following.

Theorem 2 *The functor*

$$\prod_{\Lambda \in \widehat{\mathfrak{h}}^\star/\sim} \widehat{\mathcal{O}}_\Lambda \to \widehat{\mathcal{O}},$$

$$\{M_\Lambda\} \mapsto \bigoplus_{\Lambda \in \widehat{\mathfrak{h}}^\star/\sim} M_\Lambda$$

is an equivalence of categories.

3.3 The Level

As the central line $\mathbb{C}K$ of $\widehat{\mathfrak{g}}$ is contained in $\widehat{\mathfrak{h}}$, it acts on each object M of $\widehat{\mathcal{O}}$ by semisimple endomorphisms. For each $k \in \mathbb{C}$, we denote by M_k the eigenspace of the action of K with eigenvalue k. We define $\widehat{\mathfrak{h}}^\star_k \subset \widehat{\mathfrak{h}}^\star$ as the affine hyperplane that contains all λ with $\lambda(K) = k$, so $M_k = \bigoplus_{\lambda \in \widehat{\mathfrak{h}}^\star_k} M_\lambda$. The eigenspace decomposition $M = \bigoplus_{k \in \mathbb{C}} M_k$ is a decomposition into sub-$\widehat{\mathfrak{g}}$-modules of M. When $M = M_k$ for some k we call k the *level* of M, and we let $\widehat{\mathcal{O}}_k$ be the full subcategory of $\widehat{\mathcal{O}}$ that contains all objects of level k.

If $\lambda \sim \mu$, then λ and μ differ by a sum of affine roots. As $\gamma(K) = 0$ for any $\gamma \in \widehat{R}$, for each equivalence class Λ there is a $k = k_\Lambda$ with $\Lambda \subset \widehat{\mathfrak{h}}^\star_k$, i.e. each block $\widehat{\mathcal{O}}_\Lambda$ determines a level.

There is a specific level that we denote by "crit" and that is distinguished in more than one respect. It is $\text{crit} = -\rho(K)$ (this is another instance of the above mentioned independence of the choice of ρ). In the usual normalization, this is $-h^\vee$, where $h^\vee$ denotes the dual Coxeter number of $\mathfrak{g}$. The elements in $\widehat{\mathfrak{h}}^\star_{\text{crit}}$ are called *critical weights*, and, analogously, we call an equivalence class Λ *critical* if $\Lambda \subset \widehat{\mathfrak{h}}^\star_{\text{crit}}$.

3.4 The Structure of Equivalence Classes

For any two affine roots α and β we have $2(\beta, \alpha) \in \mathbb{Z}(\alpha, \alpha)$. Since equivalent weights differ by a sum of affine roots, this implies,

$$\left\{\alpha \in \widehat{R} \mid 2(\lambda + \rho, \alpha) \in \mathbb{Z}(\alpha, \alpha)\right\} = \left\{\alpha \in \widehat{R} \mid 2(\mu + \rho, \alpha) \in \mathbb{Z}(\alpha, \alpha)\right\}$$

whenever $\lambda \sim \mu$. If Λ is a $\sim$-equivalence class, we can hence define

$$\widehat{R}(\Lambda) := \left\{\alpha \in \widehat{R} \mid 2(\lambda + \rho, \alpha) \in \mathbb{Z}(\alpha, \alpha) \text{ for some (all) } \lambda \in \Lambda\right\}.$$

Lemma 1 *Let Λ be a $\sim$-equivalence class. Then the following are equivalent*:

1. $\delta \in \widehat{R}(\Lambda)$,
2. $\mathbb{Z}\delta \subset \widehat{R}(\Lambda)$,
3. Λ *is critical*, *i.e.* $\Lambda \subset \widehat{\mathfrak{h}}^{\star}_{\text{crit}}$.

Proof Note that $(\delta, \delta) = 0$. Let $\lambda \in \Lambda$. We have $\delta \in \widehat{R}(\Lambda)$ if and only if $(\lambda + \rho, \delta) = 0$. This is the case if and only if $(\lambda + \rho, n\delta) = 0$ for all $n \in \mathbb{Z}$, i.e. if and only if $\mathbb{Z}\delta \subset \widehat{R}(\Lambda)$. Finally, $(\lambda + \rho, \delta) = (\lambda + \rho)(K)$ by Remark 1, and this equals 0 if and only if $\lambda(K) = -\rho(K)$, i.e. if and only if λ is critical. □

Lemma 2 *Suppose that λ is critical and $\alpha \in R$. Then the following are equivalent.*

1. $\alpha + n\delta \in \widehat{R}(\lambda)$ *for some* $n \in \mathbb{Z}$,
2. $\alpha + n\delta \in \widehat{R}(\lambda)$ *for all* $n \in \mathbb{Z}$.

Proof Note that $(\alpha + n\delta, \alpha + n\delta) = (\alpha, \alpha)$, as $(\delta, \gamma) = 0$ for any affine root γ by Remark 1. As λ is critical, $(\lambda + \rho, \delta) = 0$. Hence, both statements are equivalent to $2(\lambda + \rho, \alpha) \in \mathbb{Z}(\alpha, \alpha)$. □

For any $\sim$-equivalence class Λ we define

$$\widehat{\mathcal{W}}(\Lambda) = \left\langle s_{\alpha,n} \mid \alpha + n\delta \in \widehat{R}(\Lambda)\right\rangle.$$

Lemma 3

1. *Suppose that Λ is not critical. Then*

$$\Lambda = \widehat{\mathcal{W}}(\Lambda).\lambda$$

   for any $\lambda \in \Lambda$.
2. *Suppose that Λ is critical. Then*

$$\Lambda = \widehat{\mathcal{W}}(\Lambda).\lambda + \mathbb{Z}\delta$$

   for any $\lambda \in \Lambda$.

Proof Let $\lambda, \mu \in \widehat{\mathfrak{h}}^\star$, $\gamma \in \widehat{R}^+$ and $n \in \mathbb{Z}$ be as in Definition 4. Then $\lambda - \mu = n\gamma$ and $2(\lambda + \rho, \gamma) = n(\gamma, \gamma)$. Note that if γ is real, then $\lambda = s_\gamma.\mu$. If γ is imaginary, then $\gamma = m\delta$ for some $m \neq 0$ and $(\gamma, \gamma) = 0$ and $(\lambda + \rho, \delta) = 0$, which implies $\lambda + n\delta \sim \lambda$ for all $n \in \mathbb{Z}$. This, together with the fact that $\widehat{R}(\lambda) = \widehat{R}(\mu)$, implies the statements. □

4 Extensions of Neighbouring Verma Modules

In this section we collect some results about extensions of $\Delta(\lambda)$ and $\Delta(\mu)$ in $\widehat{\mathcal{O}}$, where λ and μ are "neighbouring". By this we mean the following.

Definition 5 The elements $\lambda, \mu \in \widehat{\mathfrak{h}}^\star$ are called *neighbouring* if the following conditions are satisfied:

1. There is $\alpha \in \widehat{R}^+ \cap \widehat{R}^{\mathrm{re}}$ and $n \in \mathbb{N}$ with $2(\lambda, \alpha) = n(\alpha, \alpha)$ and $\mu = \lambda + n\alpha$. In particular, $\lambda \sim \mu$ and $\lambda < \mu$.
2. There is no $\nu \in \widehat{\mathfrak{h}}^\star$ that is $\sim$-equivalent to both λ and μ with $\lambda < \nu < \mu$.

Our first result is the following:

Lemma 4 *Suppose that λ and μ are neighbouring and $\lambda < \mu$. Then $[\Delta(\mu) : L(\lambda)] = 1$.*

Proof Let

$$\Delta(\mu) = M_0 \supset M_1 \supset M_2 \supset \cdots$$

be the Jantzen filtration (for this and the sum formula below, see [5]). Then $\Delta(\mu)/M_1 \cong L(\mu)$. The Jantzen sum formula says

$$\sum_{i>0} \operatorname{ch} M_i = \sum_{\substack{\alpha \in \widehat{R}^+, n \in \mathbb{N}, \\ 2(\mu+\rho,\alpha)=n(\alpha,\alpha)}} \operatorname{ch} \Delta(\mu - n\alpha),$$

where the roots should be counted with their multiplicities (i.e. the imaginary roots should be counted $\operatorname{rk}\mathfrak{g}$-times). Now on the right hand side, $\Delta(\lambda)$ occurs exactly once, and otherwise only $\Delta(\nu)$ appear with $\nu \not\geq \lambda$. Hence $[M_1 : L(\lambda)] = 1$, so $[\Delta(\mu) : L(\lambda)] = 1$. □

Lemma 5 *Suppose that λ and μ are neighbouring and $\lambda < \mu$. Then*

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_{\widehat{\mathcal{O}}}\big(\Delta(\lambda), \Delta(\mu)\big) = 1.$$

Proof It is enough to calculate Ext^1 in the subcategory $\widehat{\mathcal{O}}^{\leq\mu}$ of $\widehat{\mathcal{O}}$. By Lemma 4 and BGGH-reciprocity we have $(P(\lambda)^{\leq\mu} : \Delta(\mu)) = (P(\lambda)^{\leq\mu} : \Delta(\lambda)) = 1$ and all other multiplicities are 0. Hence there is a short exact sequence

$$0 \to \Delta(\mu) \cong P(\mu)^{\leq \mu} \to P(\lambda)^{\leq \mu} \to \Delta(\lambda) \to 0$$

which is already a projective resolution of $\Delta(\lambda)$ in $\widehat{\mathcal{O}}^{\leq \mu}$. Applying $\mathrm{Hom}_{\widehat{\mathcal{O}}^{\leq \mu}}(\cdot, \Delta(\mu))$ to $0 \to \Delta(\mu) \to P(\lambda)^{\leq \mu} \to 0$ yields

$$0 \to \mathrm{Hom}_{\widehat{\mathcal{O}}^{\leq \mu}}\big(P(\lambda)^{\leq \mu}, \Delta(\mu)\big) \to \mathrm{Hom}_{\widehat{\mathcal{O}}^{\leq \mu}}\big(\Delta(\mu), \Delta(\mu)\big) \to 0.$$

Both Hom-spaces are one-dimensional (the first again by Lemma 4), and each non-zero homomorphism $P(\lambda)^{\leq \mu} \to \Delta(\mu)$ factors through an inclusion $\Delta(\lambda) \to \Delta(\mu)$, hence has $\Delta(\mu) \subset P(\lambda)^{\leq \mu}$ in its kernel. So the middle homomorphism in the above sequence vanishes, so the dimension of $\mathrm{Ext}^1_{\widehat{\mathcal{O}}^{\leq \mu}}(\Delta(\lambda), \Delta(\mu))$ is 1. □

We denote by $Z(\lambda, \mu) \in \widehat{\mathcal{O}}$ the (unique up to isomorphism) non-split extension of $\Delta(\mu)$ and $\Delta(\lambda)$ for neighbouring λ and μ.

Lemma 6 *Suppose that λ and μ are neighbouring and that $\lambda < \mu$. Then $P^{\leq \mu}(\lambda) \cong Z(\lambda, \mu)$.*

Proof By BGGH-reciprocity, $P^{\leq \mu}(\lambda)$ has a two-step Verma flag with subquotients isomorphic to $\Delta(\mu)$ and $\Delta(\lambda)$. This filtration is non-split, as $\Delta(\mu)$ is not a quotient of $P^{\leq \mu}(\lambda)$, since $L(\mu)$ is not. Hence the claim. □

Note that for any $\lambda, \mu \in \widehat{\mathfrak{h}}^\star$ we have $\dim_{\mathbb{C}} \mathrm{Hom}_{\widehat{\mathcal{O}}}(\Delta(\lambda), \Delta(\lambda + n\delta)) \leq [\Delta(\lambda + n\delta) : L(\lambda)]$. We now study the particular situation in which this is an equality.

Lemma 7 *Suppose that λ and μ are neighbouring and $\lambda < \mu$. Let $n > 0$ and suppose that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\widehat{\mathcal{O}}}\big(\Delta(\lambda), \Delta(\lambda + n\delta)\big) = \big[\Delta(\lambda + n\delta) : L(\lambda)\big].$$

Then every homomorphism $Z(\lambda, \mu) \to \Delta(\lambda + n\delta)$ factors through a homomorphism $\Delta(\lambda) \to \Delta(\lambda + n\delta)$.

Proof Let $\mathcal{J}$ be open and bounded and suppose it contains all relevant weights λ, μ and $\lambda + n\delta$. By the previous lemma, we have a surjection $P^{\mathcal{J}}(\lambda) \to Z(\lambda, \mu)$. So the chain of surjections

$$P^{\mathcal{J}}(\lambda) \twoheadrightarrow Z(\lambda, \mu) \twoheadrightarrow \Delta(\lambda)$$

induces a chain of injections

$$\begin{aligned} \mathrm{Hom}_{\widehat{\mathcal{O}}}\big(\Delta(\lambda), \Delta(\lambda + n\delta)\big) &\hookrightarrow \mathrm{Hom}_{\widehat{\mathcal{O}}}\big(Z(\lambda, \mu), \Delta(\lambda + n\delta)\big) \\ &\hookrightarrow \mathrm{Hom}_{\widehat{\mathcal{O}}}\big(P^{\mathcal{J}}(\lambda), \Delta(\lambda + n\delta)\big). \end{aligned}$$

Now the dimension of the space on the right is $[\Delta(\lambda+n\delta):L(\lambda)]$, as $P^{\mathcal{J}}(\lambda)$ is a projective cover of $L(\lambda)$ in $\widehat{\mathcal{O}}^{\mathcal{J}}$, so our assumptions imply that the above injections are bijections. This proves the lemma. □

4.1 The Tilting Equivalence

Let $\mathcal{M}$ be the full subcategory of $\widehat{\mathcal{O}}$ that contains all objects that admit a Verma flag.

Theorem 3 ([9, Corollary 2.3]) *There is an equivalence $t\colon \mathcal{M}\to\mathcal{M}^{opp}$ that maps short exact sequences to short exact sequences and satisfies*

$$t\big(\Delta(\lambda)\big)\cong\Delta(-2\rho-\lambda).$$

(Note that this statement does depend on the choice of ρ.)

Note that t stabilizes $\widehat{\mathcal{O}}_{\text{crit}}$, as $(-2\rho-\lambda)(K)=2\,\text{crit}-\text{crit}=\text{crit}$ for all $\lambda\in\widehat{\mathfrak{h}}^{\star}_{\text{crit}}$.

Lemma 8 *Suppose that λ and μ are neighbouring and that $\lambda<\mu$. Then $tZ(\lambda,\mu)\cong Z(-2\rho-\mu,-2\rho-\lambda)$.*

Proof Applying the tilting equivalence to the short exact sequence

$$0\to\Delta(\mu)\to Z(\lambda,\mu)\to\Delta(\lambda)\to 0$$

yields a non-split short exact sequence

$$0\to\Delta(-2\rho-\lambda)\to tZ(\lambda,\mu)\to\Delta(-2\rho-\mu)\to 0.$$

Lemma 5 now immediately implies the statement. □

Applying the tilting equivalence to the statement of Lemma 7 and using the previous lemma we obtain:

Lemma 9 *Suppose that λ, μ are neighbouring and $\lambda<\mu$. Suppose that*

$$\dim_{\mathbb{C}}\operatorname{Hom}_{\widehat{\mathcal{O}}}\big(\Delta(-2\rho-\mu),\Delta(-2\rho-\mu+n\delta)\big)=\big[\Delta(-2\rho-\mu+n\delta):L(-2\rho-\mu)\big].$$

Then every homomorphism $\Delta(\mu-n\delta)\to Z(\lambda,\mu)$ factors through a homomorphism $\Delta(\mu-n\delta)\to\Delta(\mu)$.

5 Restricted Critical Level Representations

We will now define the subcategory $\overline{\mathcal{O}}_{\text{crit}}$ of $\widehat{\mathcal{O}}_{\text{crit}}$ that contains all *restricted* representations, and we will review structural results on this subcategory that resemble the ones we discussed in Sect. 3 (references for the following are [1] and [2]).

5.1 The Feigin–Frenkel Center

Let us denote by

$$\mathcal{Z} = \bigoplus_{n \in \mathbb{Z}} \mathcal{Z}_n = \mathbb{C}\big[p_s^{(i)}, i = 1, \dots, \mathrm{rk}\,\mathfrak{g}, s \in \mathbb{Z}\big]$$

the polynomial ring (of infinite rank) constructed from the center of the critical level vertex algebra (see [1, Sect. 5]). We consider it as a $\mathbb{Z}$-graded algebra with $p_s^{(i)}$ being homogeneous of degree s.

The algebra $\mathcal{Z}$ acts on objects in $\widehat{\mathcal{O}}_{\mathrm{crit}}$ in the following way. The simple highest weight module $L(\delta)$ is invertible, i.e. it is one-dimensional and $L(\delta) \otimes_{\mathbb{C}} L(-\delta)$ is isomorphic to the trivial $\widehat{\mathfrak{g}}$-module $L(0)$. Hence, the functor

$$\begin{aligned} T \colon \widehat{\mathcal{O}} &\to \widehat{\mathcal{O}} \\ M &\mapsto M \otimes_{\mathbb{C}} L(\delta) \end{aligned}$$

is an equivalence with inverse $M \mapsto M \otimes_{\mathbb{C}} L(-\delta)$. As the level of a tensor product equals the sum of the levels of its factors, and as $L(\delta)$ is of level zero, the functor T preserves the subcategories $\widehat{\mathcal{O}}_k$ for any $k \in \mathbb{C}$. We will henceforth restrict it to $\widehat{\mathcal{O}}_{\mathrm{crit}}$.

Note that $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}K$ acts trivially on $L(\delta)$, while, as $\delta(D) = 1$, the grading element D acts as the identity.

Lemma 10 ([1]) *Let $z \in \mathcal{Z}_n$. For any $M \in \widehat{\mathcal{O}}_{\mathrm{crit}}$, z defines a homomorphism $z^M \colon T^n M \to M$.*

5.2 Restricted Representations

We define restricted representations by the following vanishing condition on the action of $\mathcal{Z}$:

Definition 6 An object $M \in \widehat{\mathcal{O}}_{\mathrm{crit}}$ is called *restricted* if for any $n \neq 0$ and any $z \in \mathcal{Z}_n$ we have that z^M is zero.

We denote by $\overline{\mathcal{O}}_{\mathrm{crit}}$ the full subcategory of $\widehat{\mathcal{O}}_{\mathrm{crit}}$ that contains all restricted objects. There is a functor $(\cdot)^{\mathrm{res}} \colon \widehat{\mathcal{O}}_{\mathrm{crit}} \to \overline{\mathcal{O}}_{\mathrm{crit}}$ that is left adjoint to the inclusion $\overline{\mathcal{O}}_{\mathrm{crit}} \subset \widehat{\mathcal{O}}_{\mathrm{crit}}$. It is defined as

$$M^{\mathrm{res}} := M/M',$$

where M' is the submodule of M that is generated by the images of all homomorphisms z^M with $z \in \mathcal{Z}_n$ and $n \neq 0$.

For any $\lambda \in \widehat{\mathfrak{h}}^\star$, the restricted Verma module with highest weight $\lambda \in \widehat{\mathfrak{h}}^\star$ is defined as

$$\overline{\Delta}(\lambda) := \Delta(\lambda)^{\text{res}}.$$

The next definition is the obvious restricted version of the earlier notion of a Verma flag.

Definition 7 Let $M \in \widehat{\mathcal{O}}_{\text{crit}}$. We say that M *admits a restricted Verma flag* if there is a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

with $M_i/M_{i-1} \cong \overline{\Delta}(\mu_i)$ for some $\mu_1, \ldots, \mu_n \in \widehat{\mathfrak{h}}^\star_{\text{crit}}$.

In case M admits a restricted Verma flag, the *restricted Verma multiplicity* $(M : \overline{\Delta}(\mu_i)) = \#\{i \in \{1, \ldots, n\} \mid \mu_i = \mu\}$ is again independent of the chosen filtration.

5.3 Restricted Projective Objects

For any open bounded subset $\mathcal{J}$ of $\widehat{\mathfrak{h}}^\star_{\text{crit}}$ set $\overline{\mathcal{O}}^{\mathcal{J}}_{\text{crit}} := \overline{\mathcal{O}}_{\text{crit}} \cap \widehat{\mathcal{O}}^{\mathcal{J}}_{\text{crit}}$. The following is an analogue of Theorem 1 in the restricted setting.

Theorem 4 ([2], see also [4, Theorems 4.3 and 5.4]) *Let $\mathcal{J} \subset \widehat{\mathfrak{h}}^\star_{\text{crit}}$ be an open bounded subset and let $\lambda \in \mathcal{J}$.*

1. *There exists a projective cover $\overline{P}^{\mathcal{J}}(\lambda) \to L(\lambda)$ of $L(\lambda)$ in $\overline{\mathcal{O}}^{\mathcal{J}}_{\text{crit}}$ and the object $\overline{P}^{\mathcal{J}}(\lambda)$ admits a restricted Verma flag.*
2. *For the multiplicities we have*

$$\left(\overline{P}^{\mathcal{J}}(\lambda) : \overline{\Delta}(\mu)\right) = \begin{cases} [\overline{\Delta}(\mu) : L(\lambda)], & \text{if } \mu \in \mathcal{J}, \\ 0, & \text{if } \mu \notin \mathcal{J}. \end{cases}$$

3. *For any open subset $\mathcal{J}' \subset \mathcal{J}$ we have $\overline{P}^{\mathcal{J}}(\lambda)^{\mathcal{J}'} \cong \overline{P}^{\mathcal{J}'}(\lambda)$.*
4. *For any $M \in \overline{\mathcal{O}}^{\mathcal{J}}_{\text{crit}}$ such that $[M : L(\lambda)]$ is finite we have*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\overline{\mathcal{O}}_{\text{crit}}}\left(\overline{P}^{\mathcal{J}}(\lambda), M\right) = \left[M : L(\lambda)\right].$$

In [2] we showed that one obtains $\overline{P}^{\mathcal{J}}(\lambda)$ from $P^{\mathcal{J}}(\lambda)$ by applying the restriction functor, i.e.

$$\overline{P}^{\mathcal{J}}(\lambda) = P^{\mathcal{J}}(\lambda)^{\text{res}}$$

for any $\lambda \in \mathcal{J}$.

5.4 The Restricted Block Decomposition

The block decomposition $\widehat{\mathcal{O}} = \prod_{\Lambda \in \widehat{\mathfrak{h}}^\star/\sim} \widehat{\mathcal{O}}_\Lambda$ of Theorem 2 clearly induces a block decomposition of $\overline{\mathcal{O}}_{\text{crit}}$. It turns out that a component $\overline{\mathcal{O}}_{\text{crit}} \cap \widehat{\mathcal{O}}_\Lambda$ is, in general, no longer indecomposable. Note that the following definition is a version of Definition 4, that only utilizes real affine roots instead of all affine roots.

Definition 8 Let "$\overline{\sim}$" be the equivalence relation on $\widehat{\mathfrak{h}}^\star$ that is generated by the following. We have $\lambda \overline{\sim} \mu$ if there exists a positive *real* root $\gamma \in \widehat{R}^{\text{re}} \cap \widehat{R}^+$ and $n \in \mathbb{Z}$ such that $2(\lambda + \rho, \gamma) = n(\gamma, \gamma)$ and $\mu = \lambda - n\gamma$.

Clearly, "$\overline{\sim}$" is a finer equivalence relation than "$\sim$" and it coincides with "$\sim$" on the affine hyperplanes $\widehat{\mathfrak{h}}^\star_k$ for all $k \neq \text{crit}$.

For a $\overline{\sim}$-equivalence class $\Gamma \subset \widehat{\mathfrak{h}}^\star_{\text{crit}}$ we let $\overline{\mathcal{O}}_\Gamma$ be the full subcategory of $\overline{\mathcal{O}}_{\text{crit}}$ that contains all objects M with the property that $[M : L(\lambda)] \neq 0$ implies $\lambda \in \Gamma$. Then we have the following analogue of Theorem 2.

Theorem 5 ([2]) *The functor*

$$\prod_{\Gamma \in \widehat{\mathfrak{h}}^\star_{\text{crit}}/\overline{\sim}} \overline{\mathcal{O}}_\Gamma \to \overline{\mathcal{O}}_{\text{crit}},$$

$$\{M_\Gamma\} \mapsto \bigoplus_{\Gamma \in \widehat{\mathfrak{h}}^\star_{\text{crit}}/\overline{\sim}} M_\Gamma$$

is an equivalence of categories.

For the restricted equivalence relation, we get the following analogue of Lemma 3, (1), which is proven using the analogous arguments.

Lemma 11 *Let Γ be a critical $\overline{\sim}$-equivalence class. Then*

$$\Gamma = \widehat{\mathcal{W}}(\Gamma).\lambda$$

for any $\lambda \in \Gamma$.

6 The Structure of Subgeneric Critical Restricted Blocks

In this section we describe the structure of $\overline{\mathcal{O}}_\Gamma$ in the case that Γ is a *subgeneric* $\overline{\sim}$-equivalence class.

6.1 Subgeneric Critical Equivalence Classes

Let $\alpha \in R^+$ be a *finite* positive root, and let $\widehat{\mathcal{W}}^\alpha \subset \widehat{\mathcal{W}}$ be the subgroup generated by the reflections $s_{\alpha,n}$ with $n \in \mathbb{Z}$. Then $\widehat{\mathcal{W}}^\alpha$ is the affine Weyl group of type A_1, and it is generated by $s_{\alpha,0}$ and $s_{\alpha,-1}$.

Definition 9 Let $\gamma \in \widehat{\mathfrak{h}}^\star_{\text{crit}}$ and let $\Gamma \subset \widehat{\mathfrak{h}}^\star_{\text{crit}}$ be its $\overline{\sim}$-equivalence class. We say that γ is α-*subgeneric* if the following holds:

1. $\alpha \in \widehat{R}(\gamma)$ (hence, as γ is critical, $\alpha + n\delta \in \widehat{R}(\gamma)$ for all $n \in \mathbb{Z}$ by Lemma 2),
2. γ is α-*regular*, i.e. $s_{\alpha,0}.\gamma \neq \gamma$,
3. $\Gamma = \widehat{\mathcal{W}}^\alpha.\gamma$.

Let $\nu \in \widehat{\mathfrak{h}}^\star_{\text{crit}}$. Then $(\nu + \rho, \delta) = 0$, hence

$$s_{\alpha,n}.\nu = \nu - \frac{2(\nu + \rho, \alpha)}{(\alpha, \alpha)}(\alpha + n\delta).$$

We call ν α-*dominant*, if $(\nu + \rho, \alpha) \in \mathbb{Z}_{\geq 0}$. If ν is α-dominant, then $s_{\alpha,0}.\nu \leq \nu$ and $s_{\alpha,-1} \geq \nu$ (as $-\alpha + \delta$ is a positive affine root). We call ν α-*antidominant*, if $(\nu + \rho, \alpha) \in \mathbb{Z}_{\leq 0}$. If ν is α-antidominant, then $s_{\alpha,0}.\nu \geq \nu$ and $s_{\alpha,-1} \leq \nu$. Moreover, ν is α-dominant if and only if $s_{\alpha,0}.\nu$ is α-antidominant, which is the case if and only if $s_{\alpha,n}.\nu$ is α-antidominant for all $n \in \mathbb{Z}$.

So suppose that γ is α-subgeneric. As $\widehat{\mathcal{W}}^\alpha$ is generated by $s_{\alpha,0}$ and $s_{\alpha,-1}$, we conclude from the above that the equivalence class Γ of γ is a totally ordered set with respect to "$\leqslant$". For any $\nu \in \Gamma$ we define

$$\begin{aligned}\alpha \uparrow \nu &:= \min\{s_{\alpha,n}.\nu \mid s_{\alpha,n}.\nu > \nu\} \\ &= \begin{cases} s_{\alpha,-1}.\nu, & \text{if } \nu \text{ is } \alpha\text{-dominant}, \\ s_{\alpha,0}.\nu. & \text{if } \nu \text{ is } \alpha\text{-antidominant}. \end{cases}\end{aligned}$$

Then $\alpha \uparrow (\cdot)\colon \Gamma \to \Gamma$ is a bijection and we denote by $\alpha \uparrow^n (\cdot)\colon \Gamma \to \Gamma$ its n-fold composition for $n \in \mathbb{Z}$, and we set $\alpha \downarrow^n (\cdot) := \alpha \uparrow^{-n} (\cdot)$. Then

$$\Gamma = \left\{\ldots, \alpha \downarrow^2 \nu, \alpha \downarrow \nu, \nu, \alpha \uparrow \nu, \alpha \uparrow^2 \nu, \ldots\right\}.$$

6.2 Multiplicities in the Subgeneric Case

The main result of [1] is the following multiplicity formula for α-subgeneric γ: for any $\mu \in \widehat{\mathfrak{h}}^\star_{\text{crit}}$ we have

$$\left[\overline{\Delta}(\gamma) : L(\mu)\right] = \begin{cases} 1, & \text{if } \mu \in \{\gamma, \alpha \downarrow \gamma\}, \\ 0, & \text{if } \mu \notin \{\gamma, \alpha \downarrow \gamma\}. \end{cases}$$

Suppose that $\mathcal{J}$ is an open and bounded subset of $\widehat{\mathfrak{h}}^{\star}_{\text{crit}}$ that contains γ and $\alpha \uparrow \gamma$. From BGGH-reciprocity we then obtain that $\overline{P}^{\mathcal{J}}(\gamma)$ has a two-step Verma flag with subquotients $\overline{\Delta}(\alpha \uparrow \gamma)$ and $\overline{\Delta}(\gamma)$. Clearly, $\overline{\Delta}(\alpha \uparrow \gamma)$ has to occur as a submodule, so we obtain a short exact sequence

$$0 \to \overline{\Delta}(\alpha \uparrow \gamma) \to \overline{P}^{\mathcal{J}}(\gamma) \to \overline{\Delta}(\gamma) \to 0.$$

Hence, in the subgeneric situations, the $\overline{P}^{\mathcal{J}}(\gamma)$ stabilize (with respect to the partially ordered set of open subsets $\mathcal{J}$ in $\widehat{\mathfrak{h}}^{\star}_{\text{crit}}$), so there is a well-defined object

$$\overline{P}(\gamma) := \varprojlim_{\mathcal{J}} \overline{P}^{\mathcal{J}}(\gamma)$$

for any α-subgeneric γ. From BGGH-reciprocity and the multiplicity statement above we obtain that

$$\left[\overline{P}(\mu), L(\gamma)\right] = \begin{cases} 1, & \text{if } \mu \in \{\alpha \uparrow \gamma, \alpha \downarrow \gamma\}, \\ 2, & \text{if } \mu = \gamma, \\ 0, & \text{if } \mu \notin \{\alpha \downarrow \gamma, \gamma, \alpha \uparrow \gamma\}, \end{cases}$$

hence

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\overline{\mathcal{O}}}\left(\overline{P}(\gamma), \overline{P}(\mu)\right) = \begin{cases} 1, & \text{if } \mu \in \{\alpha \uparrow \gamma, \alpha \downarrow \gamma\}, \\ 2, & \text{if } \mu = \gamma, \\ 0, & \text{if } \mu \notin \{\alpha \downarrow \gamma, \gamma, \alpha \uparrow \gamma\}. \end{cases}$$

6.3 The Partial Restriction Functor

We will need "partially restricted" objects. Let

$$\mathcal{Z}^{+} := \mathbb{C}\left[p_s^{(i)}, i = 1, \ldots, \operatorname{rk}\mathfrak{g}, s > 0\right].$$

We set $\mathcal{Z}^{+}_n := \mathcal{Z}^{+} \cap \mathcal{Z}_n$. Then $\mathcal{Z}^{+}$ is a positively graded subalgebra of $\mathcal{Z}$.

Definition 10 An object $M \in \widehat{\mathcal{O}}_{\text{crit}}$ is called *positively restricted* if for any $n > 0$ and any $z \in \mathcal{Z}^{+}_n$ we have that z^M is zero.

Note that, for example, each non-restricted Verma module $\Delta(\gamma)$ is positively restricted. We denote by $\widehat{\mathcal{O}}^{+}_{\text{crit}}$ the full subcategory of $\widehat{\mathcal{O}}_{\text{crit}}$ that contains all positively restricted objects. Again we have an obvious left adjoint to the inclusion functor $\widehat{\mathcal{O}}^{+}_{\text{crit}} \subset \widehat{\mathcal{O}}_{\text{crit}}$. We let $\mathcal{Z}^{+}M$ be the submodule of M generated by the images of all homomorphisms z^M with $z \in \mathcal{Z}^{+}_n$ and $n > 0$, and we set

$$M^{+} := M/\mathcal{Z}^{+}M.$$

This yields the functor from $\widehat{\mathcal{O}}_{\text{crit}}$ to $\widehat{\mathcal{O}}^{+}_{\text{crit}}$ that is left adjoint to the inclusion functor.

Analogously, we define

$$\mathcal{Z}^{-} := \mathbb{C}\big[p_s^{(i)}, i = 1, \ldots, \operatorname{rk}\mathfrak{g}, s < 0\big].$$

By replacing $\mathcal{Z}^{+}$ by $\mathcal{Z}^{-}$ in the definitions above, we obtain the analogous notion of *negatively restricted* objects, the corresponding category $\widehat{\mathcal{O}}^{-}_{\text{crit}}$ and a functor $M \mapsto M^{-}$ that is left adjoint to the inclusion $\widehat{\mathcal{O}}^{-}_{\text{crit}} \subset \widehat{\mathcal{O}}_{\text{crit}}$. As $\mathcal{Z}$ is generated by its subalgebras $\mathcal{Z}^{-}$ and $\mathcal{Z}^{+}$ we have

$$M^{\text{res}} = \big(M^{+}\big)^{-} = \big(M^{-}\big)^{+}$$

for all M in $\widehat{\mathcal{O}}_{\text{crit}}$.

We now collect some results on the partial restriction functor that we need later on.

Proposition 1 *Let $\gamma \in \widehat{\mathfrak{h}}^{\star}_{\text{crit}}$ be α-subgeneric and let $\mathcal{J} \subset \widehat{\mathfrak{h}}^{\star}_{\text{crit}}$ be open and bounded such that $\gamma, \alpha \uparrow \gamma \in \mathcal{J}$. Then $P^{\mathcal{J}}(\gamma)^{+}$ is a non-split extension of $\Delta(\gamma)$ and $\Delta(\alpha \uparrow \gamma)$, hence isomorphic to $Z(\gamma, \alpha \uparrow \gamma)$.*

Proof Let $P := P^{\mathcal{J}}(\gamma)$. Then $P^{\leqslant \alpha\uparrow\gamma} \cong P^{\leqslant \alpha\uparrow\gamma}(\gamma)$. Then γ and $\alpha \uparrow \gamma$ are neighbouring, hence

$$\big(P^{\leqslant \alpha\uparrow\gamma} : \Delta(\gamma)\big) = \big(P^{\leqslant \alpha\uparrow\gamma} : \Delta(\alpha \uparrow \gamma)\big) = 1$$

and all other multiplicities are zero, so we have a short exact sequence

$$0 \to \Delta(\alpha \uparrow \gamma) \to P^{\leqslant \alpha\uparrow\gamma} \to \Delta(\gamma) \to 0.$$

This is a non-split short exact sequence, as $\Delta(\alpha \uparrow \gamma)$ is not a quotient of $P^{\leqslant \alpha\uparrow\gamma}$. So $P^{\leqslant \alpha\uparrow\gamma} \cong Z(\gamma, \alpha \uparrow \gamma)$.

Note that the kernel of the homomorphism $P \to P^{\leqslant \alpha\uparrow\gamma}$ is generated by all weight spaces P_μ with $\mu \nleq \alpha \uparrow \lambda$. Now P is generated by its γ-weight space, so $\mathcal{Z}^{+}P$ is generated by its weight spaces $(\mathcal{Z}^{+}P)_{\gamma+n\delta}$ for $n > 0$. As $\gamma + n\delta \nleq \alpha \uparrow \gamma$ for all $n > 0$, we obtain an induced map $P^{+} \to P^{\leqslant \alpha\uparrow\gamma}$. We claim that this map is an isomorphism, which, by the above, implies the statement of the proposition.

Clearly this map is surjective. If it is not injective, then there exists a μ with $\mu \nleq \alpha \uparrow \gamma$ and $P^{+}_{\mu} \neq 0$. Let us in this case choose a maximal such μ. Then we have $(P^{\text{res}})_\mu = ((P^{+})^{-})_\mu \neq 0$, which contradicts the fact that P^{res} is an extension of $\Delta(\gamma)$ and $\Delta(\alpha \uparrow \gamma)$, so all its weights are $\leq \alpha \uparrow \gamma$. □

For simplicity we will denote $P^{\mathcal{J}}(\gamma)^{+}$ by $P(\gamma)^{+}$ in the following.

6.4 Homomorphisms Between Projectives

We will now construct a basis of the homomorphism space $\mathrm{Hom}_{\overline{\mathcal{O}}}(\overline{P}(\lambda), \overline{P}(\mu))$ for λ and μ in a α-subgeneric equivalence class. We have already seen that this space is one-dimensional if $\mu \in \{\alpha \downarrow \lambda, \alpha \uparrow \lambda\}$, two-dimensional in case $\lambda = \mu$, and it is the trivial space otherwise.

To start with, let us fix, for any α-subgeneric ν, an inclusion

$$j_\nu \colon \Delta(\nu) \to \Delta(\alpha \uparrow \nu).$$

We denote by $\overline{j}_\nu \colon \overline{\Delta}(\nu) \to \overline{\Delta}(\alpha \uparrow \nu)$ the homomorphism j_ν^- (which coincides with j_ν^{res}, as Verma modules are already positively restricted). Note that $\nu \not\leq \alpha \uparrow \nu - n\delta$ for any $n > 0$, hence $\overline{j}_\nu$ is non-zero. Let $\mathcal{J} \subset \widehat{\mathfrak{h}}^\star$ be open and bounded and suppose that γ and $\alpha \uparrow \gamma$ are contained in $\mathcal{J}$. We also fix a surjection

$$\pi_\nu \colon P^{\mathcal{J}}(\nu) \to \Delta(\nu)$$

and an inclusion

$$i_{\alpha\uparrow\nu} \colon \Delta(\alpha \uparrow \nu) \to P(\nu)^+.$$

In particular, we have a short exact sequence

$$0 \to \Delta(\alpha \uparrow \nu) \xrightarrow{i_{\alpha\uparrow\nu}} P(\nu)^+ \xrightarrow{\pi_\nu^+} \Delta(\nu) \to 0.$$

As the action of $\mathcal{Z}^-$ on Verma modules is free (see [7] and [6, Theorem 9.5.3]), this induces, after applying the functor $(\cdot)^-$, a short exact sequence

$$0 \to \overline{\Delta}(\alpha \uparrow \nu) \xrightarrow{\overline{i}_{\alpha\uparrow\nu}} \overline{P}(\nu) \xrightarrow{\overline{\pi}_\nu} \overline{\Delta}(\nu) \to 0.$$

Now we can find, by projectivity, a homomorphism

$$a_\gamma \colon P^{\mathcal{J}}(\gamma) \to P^{\mathcal{J}}(\alpha \uparrow \gamma)$$

such that the diagram

$$\begin{array}{ccc} P^{\mathcal{J}}(\gamma) & \xrightarrow{\;a_\gamma\;} & P^{\mathcal{J}}(\alpha \uparrow \gamma) \\ {\scriptstyle \pi_\gamma}\downarrow & & \downarrow{\scriptstyle \pi_{\alpha\uparrow\gamma}} \\ \Delta(\gamma) & \xrightarrow{\;j_\gamma\;} & \Delta(\alpha \uparrow \gamma) \end{array}$$

commutes. Applying the functor $(\cdot)^+$ yields a commuting diagram

$$\begin{array}{ccc} P(\gamma)^+ & \xrightarrow{a_\gamma^+} & P(\alpha \uparrow \gamma)^+ \\ \pi_\gamma^+ \downarrow & & \downarrow \pi_{\alpha\uparrow\gamma}^+ \\ \Delta(\gamma) & \xrightarrow{j_\gamma} & \Delta(\alpha \uparrow \gamma). \end{array}$$

The following is the crucial technical result of this paper.

Lemma 12 *The composition* $\Delta(\alpha \uparrow \gamma) \xrightarrow{i_{\alpha\uparrow\gamma}} P(\gamma)^+ \xrightarrow{a_\gamma^+} P(\alpha \uparrow \gamma)^+$ *is non-zero.*

Proof Suppose the composition were zero. Then we could factor the map a_γ^+ over a homomorphism $\Delta(\gamma) \to P(\alpha \uparrow \gamma)^+$. By Proposition 1, $P(\alpha \uparrow \gamma)^+ \cong Z(\alpha \uparrow \gamma, \alpha \uparrow^2 \gamma)$.

Note that for any α-subgeneric ν, the weights ν and $\alpha \uparrow \nu$ are neighbouring. Moreover, with ν also $-2\rho - \nu$ is α-subgeneric. In [1] it is shown that for subgeneric ν we have

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\widehat{\mathcal{O}}}\big(\Delta(\nu - n\delta), \Delta(\nu)\big) = \big[\Delta(\nu) : L(\nu - n\delta)\big]$$

for all $n \in \mathbb{Z}$. Finally, $\alpha \uparrow^2 \nu = \nu + n\delta$ for some $n > 0$. The above statements now allow us to apply Lemma 9 and we conclude that the map a_γ^+ would factor over a homomorphism $\Delta(\gamma) \to \Delta(\alpha \uparrow^2 \gamma) \to P(\alpha \uparrow \gamma)^+$. But this contradicts its construction. □

Now apply the restriction functor $(\cdot)^{\mathrm{res}}$ to a_γ. We obtain a homomorphism $\overline{a}_\gamma : \overline{P}(\gamma) \to \overline{P}(\alpha \uparrow \gamma)$ such that the diagram

$$\begin{array}{ccc} \overline{P}(\gamma) & \xrightarrow{\overline{a}_\gamma} & \overline{P}(\alpha \uparrow \gamma) \\ \overline{\pi}_\gamma \downarrow & & \downarrow \overline{\pi}_{\alpha\uparrow\gamma} \\ \overline{\Delta}(\gamma) & \xrightarrow{\overline{j}_\gamma} & \overline{\Delta}(\alpha \uparrow \gamma) \end{array}$$

commutes. From Lemma 12 (and some weight considerations) we conclude:

Lemma 13 *The composition*

$$\overline{\Delta}(\alpha \uparrow \gamma) \xrightarrow{\overline{i}_{\alpha\uparrow\gamma}} \overline{P}(\gamma) \xrightarrow{\overline{a}_\gamma} \overline{P}(\alpha \uparrow \gamma)$$

is non-zero.

In particular, $\overline{a}_\gamma$ is non-zero, hence a generator of $\mathrm{Hom}_{\overline{\mathcal{O}}}(\overline{P}(\gamma), \overline{P}(\alpha \uparrow \gamma))$.

Let $b_\gamma : \overline{P}(\gamma) \to \overline{P}(\alpha \downarrow \gamma)$ be the following composition:

$$b_\gamma : \overline{P}(\gamma) \xrightarrow{\overline{\pi}_\gamma} \overline{\Delta}(\gamma) \xrightarrow{\overline{i}_\gamma} \overline{P}(\alpha \downarrow \gamma).$$

This composition is clearly non-zero, hence b_γ is a basis of $\mathrm{Hom}_{\overline{\mathcal{O}}}(\overline{P}(\gamma), \overline{P}(\alpha \downarrow \gamma))$.

Finally, let $\overline{n}_\gamma : \overline{P}(\gamma) \to \overline{P}(\gamma)$ be the composition

$$\overline{n}_\gamma : \overline{P}(\gamma) \xrightarrow{\overline{\pi}_\gamma} \overline{\Delta}(\gamma) \xrightarrow{\overline{j}_\gamma} \overline{\Delta}(\alpha \uparrow \gamma) \xrightarrow{\overline{i}_{\alpha\uparrow\gamma}} \overline{P}(\gamma).$$

Again, this is non-zero and obviously not invertible (we even have $\overline{n}_\gamma^2 = 0$), so $\{n_\gamma, \mathrm{id}\}$ is a basis of $\mathrm{End}_{\overline{\mathcal{O}}}(\overline{P}(\gamma))$.

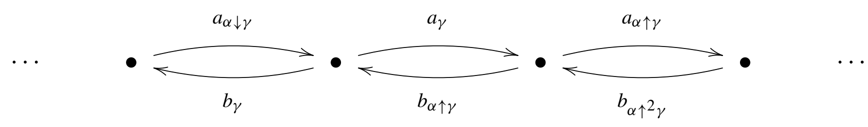
We have now exhibited a basis for any non-zero space $\mathrm{Hom}_{\overline{\mathcal{O}}}(\overline{P}(\gamma), \overline{P}(\mu))$. The following theorem describes all possible (non-trivial) compositions, hence gives a full description of the subgeneric endomorphism algebra $\mathrm{End}_{\overline{\mathcal{O}}}(\bigoplus_{\gamma\in\Gamma} \overline{P}(\gamma))$, where Γ is the $\overline{\sim}$-equivalence class of γ.

Theorem 6 *Let $\gamma \in \widehat{\mathfrak{h}}^\star_{\mathrm{crit}}$ be α-subgeneric. Then we have the following relations*:

1. *$b_{\alpha\uparrow\gamma} \circ a_\gamma$ and $a_{\alpha\downarrow\gamma} \circ b_\gamma$ are non-zero scalar multiples of n_γ.*
2. *$a_{\alpha\uparrow\gamma} \circ a_\gamma = 0$ and $b_{\alpha\downarrow\gamma} \circ b_\gamma = 0$.*
3. *$n_{\alpha\uparrow\gamma} \circ a_\gamma = 0$ and $n_{\alpha\downarrow\gamma} \circ b_\gamma = 0$.*
4. *$n_\gamma \circ n_\gamma = 0$.*

Proof Note that (2) is obvious, as the homomorphism spaces in question vanish. Then (3) and (4) follow immediately from (1) and (2). So we are left to prove (1). Note that both compositions are clearly not automorphisms of $\overline{P}(\gamma)$, so we only have to prove that they are non-zero. From the construction it immediately follows that $b_{\alpha\uparrow\gamma} \circ a_\gamma \neq 0$. That $a_{\alpha\uparrow\gamma} \circ b_\gamma$ is non-zero follows from Lemma 13. □

Hence we see that the endomorphism algebra of $\bigoplus_{\gamma\in\Gamma} \overline{P}(\gamma)$ is given by the following infinite quiver



with relations $a_{\alpha\downarrow\gamma} \circ b_\gamma = b_{\alpha\uparrow\gamma} \circ a_\gamma$ and $a_{\alpha\uparrow\gamma} \circ a_\gamma = 0$ and $b_\gamma \circ b_{\alpha\uparrow\gamma} = 0$ for all $\gamma \in \Gamma$.

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Slavnov Determinants, Yang–Mills Structure Constants, and Discrete KP

Omar Foda and Michael Wheeler

Abstract Using Slavnov's scalar product of a Bethe eigenstate and a generic state in closed XXZ spin-$\frac{1}{2}$ chains, with possibly twisted boundary conditions, we obtain determinant expressions for tree-level structure constants in 1-loop conformally-invariant sectors in various planar (super) Yang-Mills theories. When certain rapidity variables are allowed to be free rather than satisfy Bethe equations, these determinants become discrete KP τ-functions.

1 Overview

Classical integrable models, in the sense of integrable hierarchies of nonlinear partial differential equations that admit soliton solutions, and quantum integrable models, in the sense of Yang-Baxter integrability, are topics that Prof M. Jimbo continues to make profound contributions to since more than three decades.

They are also topics that, since the late 1980's, have made increasingly frequent contacts with, and have lead to definite advances in modern quantum field theory. Amongst the most important of these contacts are discoveries of integrable structures on both sides of Maldacena's conjectured AdS/CFT correspondence [1]. From 2002 onward, classical integrability was discovered in free superstrings[1] on the AdS side of AdS/CFT [2, 3], and quantum integrability in the planar limit[2] of $\mathcal{N}=4$ supersymmetric Yang-Mills on the CFT side [4–6]. Further, examples of integrability that are restricted 1-loop level were discovered in planar Yang-Mills theories with

[1]Superstrings with tree-level interactions only, and no spacetime loops.

[2]The limit in which the number of colours $N_c \to \infty$, the gauge coupling $g_{YM} \to 0$, while the 't Hooft coupling $\lambda = g_{YM}^2 N_c$ remains finite.

Dedicated to Professor M. Jimbo on his 60th birthday.

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fewer supersymmetries and in QCD [7, 8]. In the sequel, we use YM for Yang-Mills theories in general, and $\mathrm{SYM}_{\mathcal{N}}$ for $\mathcal{N}$-extended supersymmetric Yang-Mills.

1.1 Scope of This Work

In this work, we restrict our attention to quantum field theories that are (1) planar, so that the methods of integrability have a chance to work, (2) weakly-coupled, so that perturbation theory makes sense and we can focus our attention to 1-loop level, and (3) conformally-invariant at 1-loop level, so they allow an exact mapping to Heisenberg spin-chains, that is spin-chains with nearest neighbour interactions that can be solved using the algebraic Bethe Ansatz. In the sequel, we consider only Heisenberg spin-$\frac{1}{2}$ chains.

Even within the above restrictions, our subject is still very broad and we can only review the basics needed to obtain our results. For an introduction to the vast subject of integrability in AdS/CFT, we refer to [9] and references therein.[3]

1.2 Conformal Invariance and 2-Point Functions

Any 1-loop conformally-invariant quantum field theory contains (up to 1-loop order) a basis of local scalar primary conformal composite operators[4] $\{\mathcal{O}\}$ such that the 2-point functions can be written as

$$\left\langle \mathcal{O}_i(x)\overline{\mathcal{O}}_j(y)\right\rangle = \delta_{ij}\frac{\mathcal{N}_i}{|x-y|^{2\Delta_i}} \tag{1}$$

where $\overline{\mathcal{O}}_i$ is the Wick conjugate of $\mathcal{O}_i$, Δ_i is the conformal dimension of $\mathcal{O}_i$ and $\mathcal{N}_i$ is a normalization factor. Later, we choose $\mathcal{N}_i$ to be (the square root of) the Gaudin norm of the corresponding spin-chain state.

The primary goal of studies of integrability on the CFT side of AdS/CFT in the past ten years has arguably been the calculation of the spectrum of conformal

[3]Further highlights of integrability in modern quantum field theory and in string theory include (1) Classical integrable hierarchies in matrix models of non-critical strings, from the late 1980's [10], (2) Finite gap solutions in Seiberg-Witten theory of low-energy SYM_2 in the mid 1990's [11–14], (3) Integrability in QCD scattering amplitudes in the mid 1990's [8, 15–17], (4) Free fermion methods in works of Nekrasov, Okounkov, Nakatsu, Takasaki and others on Seiberg-Witten theory, in the 2000's [18, 19], (5) Integrable spin chains in works of Nekrasov, Shatashvili and others on SYM_2, in the 2000's [20], (6) Integrable structures, particularly the Yangian, that appear in recent studies of SYM_4 scattering amplitudes [21, 22]. There are many more.

[4]In this work, we restrict our attention to this class of local composite operators. In particular, we do not consider descendants or operators with non-zero spin, for which the 2-point and 3-point functions are different.

dimensions $\{\Delta_{\mathcal{O}}\}$ of local composite operators $\{\mathcal{O}\}$, and matching them with corresponding results from the strong coupling AdS side of AdS/CFT. This goal has by and large been achieved [9], and the next logical step is to study 3-point functions and their structure constants [23–29].

1.3 3-Point Functions and Structure Constants

The 3-point function of three basis local operators such as those that appear in (1) is restricted (up to 1-loop order) by conformal symmetry to be of the form

$$\begin{aligned} &\left\langle \mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\mathcal{O}_k(x_k)\right\rangle \\ &= \left(\mathcal{N}_i \quad \mathcal{N}_j \quad \mathcal{N}_k\right)^{1/2} \frac{C_{ijk}}{|x_{ij}|^{\Delta_i+\Delta_j-\Delta_k}|x_{jk}|^{\Delta_j+\Delta_k-\Delta_i}|x_{ki}|^{\Delta_k+\Delta_i-\Delta_j}} \end{aligned} \tag{2}$$

where $x_{ij} = x_i - x_j$, and C_{ijk} are structure constants. The structure constants C_{ijk} are the subject of this work. In [26–28], Escobedo, Gromov, Sever and Vieira (EGSV) obtained sum expressions for the structure constants of non-extremal single-trace operators in the scalar sector of SYM_4. In [29], the sum expressions of EGSV were evaluated, and determinant expressions for the same structure constants were obtained.[5]

1.4 Aims of This Work

We extend the results of [29] to a number of YM theories that are conformally invariant at least up to 1-loop level. We also show that the determinants that we obtain are discrete KP τ-functions.

More precisely, (1) We recall, and make explicit, a generalization of the restricted Slavnov scalar product used in [29] to twisted, closed and homogeneous XXZ spin-$\frac{1}{2}$ chains. That is, we allow for an anisotropy parameter $\Delta \neq 1$, as well as a twist parameter $\theta \neq 0$ in the boundary conditions. The result is still a determinant. We use this result to obtain determinant expressions for the YM theories listed in Subsect. 1.5.[6] (2) Allowing certain rapidity variables in the determinant expressions to be free, rather than satisfy Bethe equations, we show that these rapidities can be regarded as Miwa variables. In terms of these Miwa variables, the determinants satisfy Hirota-Miwa equations and become discrete KP τ-functions. The structure constants are recovered by requiring that the free variables are rapidities that label a gauge-invariant composite operator and satisfy Bethe equations.

[5]Three operators $\mathcal{O}_i$, of length L_i, $i \in \{1, 2, 3\}$, are non-extremal if $l_{ij} = L_i + L_j - L_k > 0$.

[6]The SYM_4 expression of [29] is a special case of the general expression obtained here.

1.5 Type-A and Type-B YM Theories

We consider six planar, weakly-coupled YM theories. 1. SYM_4 [30, 31], 2. SYM_4^M, which is an order-M Abelian orbifold of SYM_4 that is $\mathcal{N} = 2$ supersymmetric [32, 33], and 3. SYM_4^β, which is a Leigh-Strassler marginal real-β deformation of SYM_4 that is $\mathcal{N} = 1$ supersymmetric [33, 35–38]. 4. The complex scalar sector of pure SYM_2 [7, 39], 5. The gluino sector of pure SYM_1 [7], and 6. The gauge sector of QCD [7, 8].

These six theories are naturally divisible into two types. Type-**A** contains theories 1, 2 and 3, which are conformally-invariant to all orders in perturbation theory. Type-**B** contains theories 4, 5 and 6, which are conformally-invariant to 1-loop level only.[7]

Conformal invariance at 1-loop level, which is the case in all theories that we consider, is necessary and sufficient for our purposes because the mapping to spin-$\frac{1}{2}$ chains with nearest neighbour interactions breaks down at higher loops. Our results are valid only up to 1-loop level.

1.6 Non-extremal Operators

In [26–29], structure constants of three operators $\mathcal{O}_i$ of length L_i, $i \in \{1, 2, 3\}$ were considered, and the condition that the operators are non-extremal, that is $l_{ij} = L_i + L_j - L_k > 0$, for all distinct i, j and k, was emphasized. The reason is that, in these works, one wished to compute the structure constants of three non-BPS operators. Using the analysis presented in this work, one can show that this requires the condition $l_{ij} > 0$. One can of course consider the special case where one of these parameters $l_{ij} = 0$, but then at least one of the three operators has to be BPS.

In type-**A** theories, which include SYM_4, we can compute non-trivial structure constants of three non-BPS operators, so we do that, and the condition $l_{ij} > 0$ is satisfied. The case where one of these parameters vanishes, for example $l_{23} = L_2 + L_3 - L_1 = 0$, is allowed, but then either $\mathcal{O}_2$ or $\mathcal{O}_3$ has to be BPS. In type-**B** theories, we find that one of the three operators, which we choose to be $\mathcal{O}_3$, has to be BPS, hence the condition $l_{ij} > 0$ is no longer significant and we consider operators such that $l_{23} = L_2 + L_3 - L_1 = 0$.

[7] There are definitely more gauge theories that are conformally-invariant at 1-loop or more, with $SU(2)$ sectors that map to states in spin-$\frac{1}{2}$ chains. Here we consider only samples of theories with different supersymmetries and operator content.

1.7 $SU(2)$ Sectors that Map to Spin-$\frac{1}{2}$ Chains

We will not list the full set of fundamental fields in the gauge theories that we consider, but only those fundamental fields that form $SU(2)$ doublets that map to states in spin-$\frac{1}{2}$ chains. All fields are in the adjoint of $SU(N_c)$ and can be represented in terms of $N_c \times N_c$ matrices.

1. SYM_4 contains six real scalars that form three complex scalars $\{X, Y, Z\}$, and their charge conjugates $\{\bar{X}, \bar{Y}, \bar{Z}\}$. Any pair of non-charge-conjugate scalars, *e.g.* $\{Z, X\}$, or $\{Z, \bar{X}\}$, forms a doublet that maps to a state in a closed periodic XXX spin-$\frac{1}{2}$ chain[8] [4, 31].
2. SYM_4^M has the same fundamental charged scalar fields $\{X, Y, Z\}$ and their charge conjugates, as SYM_4, so the same scalars form $SU(2)$ doublets. Due to the orbifolding of the $SU(2)$ sectors by the action of the discrete group Γ_M, these doublets map to states in a closed twisted XXX spin-$\frac{1}{2}$ chain. The twist parameter is a (real) phase $\theta = \frac{2\pi}{M}$ [33].
3. SYM_4^β has the same fundamental charged scalar fields $\{X, Y, Z\}$ and their charge conjugates, as SYM_4, so the same scalars form $SU(2)$ doublets. Due to the real-β deformation, these doublets map to states in a closed twisted XXX spin-$\frac{1}{2}$ chain. The twist parameter is a (real) phase $\theta = \beta$, where β is the deformation parameter. [33, 34].
4. SYM_2 has a gluino field λ and its conjugate $\bar{\lambda}$ that form a doublet that maps to a state in a closed untwisted XXZ spin-$\frac{1}{2}$ chain with $\Delta = 3$ [7, 39].
5. SYM_1 has a complex scalar ϕ and its conjugate $\bar{\phi}$ that form a doublet that maps to a state in a closed untwisted XXZ spin-$\frac{1}{2}$ chain with $\Delta = \frac{1}{2}$ [7].
6. Pure QCD has light-cone derivatives $\{\partial_+ A, \partial_+ \bar{A}\}$, where A and $\bar{A}$ are the transverse components of the gauge field A_μ, that form a doublet that maps to a state in a closed untwisted XXZ spin–$\frac{1}{2}$ chain with $\Delta = -\frac{11}{3}$ [7].

1.8 Remark

Theories 1, 2 and 3, that are conformally invariant to all orders, contain three charged scalars and their conjugates. These combine into various $SU(2)$ doublets. Theories 4, 5 and 6, on the other hand, contain only one doublet. This fact affects the type of structure constants that we can compute in determinant form in Sects. 5 and 6.[9]

[8] XXX spin-$\frac{1}{2}$ chains are XXZ spin-$\frac{1}{2}$ chains with an anisotropy parameter $\Delta = 1$.

[9] The fact that the structure constants in these two types of theories should be handled differently was pointed out to us by C. Ahn and R. Nepomechie.

1.9 Outline of Contents

In Sect. 2, we recall basic background information related to integrability in weakly coupled YM. In Sect. 3, we review standard facts on closed XXZ spin-$\frac{1}{2}$ chains with twisted boundary conditions. In particular, following [42], we introduce restricted versions $S[L, N_1, N_2]$ of Slavnov's scalar product, that can be evaluated in determinant form.[10]

In Sect. 4, we review standard facts on the trigonometric six-vertex model, which is regarded as another way to view XXZ spin-$\frac{1}{2}$ chains in terms of diagrams that are convenient for our purposes. Following [43], we introduce the $[L, N_1, N_2]$-configurations that are central to our result. The determinant $S[L, N_1, N_2]$, obtained in Sect. 3, turns out to be the partition function of these $[L, N_1, N_2]$-configurations.

In Sect. 5, we recall the EGSV formulation of the structure constants of three non-extremal composite operators in the scalar sector of SYM_4. Since all Type-**A** theories, which include SYM_4 and two other theories that are closely related to it, share the same set of fundamental charged scalar fields, namely $\{X, Y, Z\}$ and their charge conjugates $\{\bar{X}, \bar{Y}, \bar{Z}\}$, our discussion applies to all of them in one go. Since the composite operators that we are interested in map to states in (generally twisted) XXX spin-$\frac{1}{2}$ chains, we express these structure functions in terms of rational six-vertex model configurations, and obtain determinant expressions for them.

In Sect. 6, we extend the above discussion to Type-**B** theories, which contain theories with only one $SU(2)$ doublet that we can work with. Since the composite operators that we are interested in map to states in periodic XXZ spin-$\frac{1}{2}$ chains, we express these structure functions in terms of trigonometric six-vertex model configurations. We find that our method applies only when one of the operators is BPS-like (a single-trace of a power of one type of fundamental fields). We obtain determinant expressions for these objects, and find that the result is identical to that in type-**A**, apart from the fact that one of the operators in BPS-like.

In Sect. 7, we show that the determinant expressions are solutions of Hirota-Miwa equations, and thereby τ-functions of the discrete KP hierarchy. In Sect. 8, we summarize our results.

2 Background

Let us recall basic facts on integrability on the CFT side of AdS/CFT.

2.1 Integrability in AdS/CFT

In its strongest sense, the anti-de Sitter/conformal field theory (AdS/CFT) correspondence is the postulate that all physics, including gravity, in an anti-de Sitter

[10] In [29], $S[L, N_1, N_2]$ was denoted by $S[L, \{N\}]$.

space can be reproduced in terms of a conformal field theory that lives on the boundary of that space [40]. The first and most thoroughly studied example of the correspondence is Maldacena's original proposal that type-IIB superstring theory in an $\text{AdS}^5 \times S^5$ geometry is equivalent to planar SYM_4 on the 4-dimensional boundary of AdS^5 [1].

Since its proposal in 1997, the AdS/CFT correspondence has passed every single check that it was subject to, and there was a large number of these. However, because the correspondence typically identifies one theory in a regime that is easy to study (for example, a weakly-coupled planar quantum field theory) to another theory in a regime that is hard to study (for example, a quantum free superstring theory in a strongly curved geometry), it has so far not been possible to prove it [9].

2.2 *The Dilatation Operator*

The generators of the conformal group in 4-dimensions, $SO(4, 2)$, contain a dilatation operator D [41]. Every gauge-invariant operator $\mathcal{O}$ in a YM theory, that is 1-loop conformally-invariant, is an eigenstate of D to that order in perturbation theory. The corresponding eigenvalue $\Delta_{\mathcal{O}}$, which is the conformal dimension of $\mathcal{O}$, is the analogue of mass in massive, non-conformal theories.

2.3 SYM_4 *and Spin Chains. 1-Loop Results*

An $SU(2)$ doublet of fundamental fields $\{u, d\}$, which could be any of those discussed in Subsect. 1.7 above, is analogous to the $\{\uparrow, \downarrow\}$ states of a spin variable on a single site in a spin-$\frac{1}{2}$ chain. Furthermore, the local gauge-invariant operators formed by taking single traces of a product of an arbitrary combination of u and d fields, such as $\text{Tr}[uudud duu \cdots uu]$, is analogous to a state in a closed spin-$\frac{1}{2}$ chain.

In [4], Minahan and Zarembo made the above intuitive analogies exact correspondences by showing that the action of the 1-loop dilatation operator on single-trace operators in the $SU(2)$ scalar subsector of SYM_4 is identical to the action of the nearest-neighbour Hamiltonian on the states in a closed periodic XXX spin-$\frac{1}{2}$ chain.[11] In this mapping, valid up to 1-loop level[12] single-trace operators with well-defined conformal dimensions map to eigenstates of the XXX Hamiltonian. The corresponding eigenvalues are the conformal dimensions $\Delta_{\mathcal{O}}$.

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[11] Minahan and Zarembo obtained their remarkable result in the context of the complete scalar sector of SYM_4. The relevant spin chain in that case is $SO(6)$ symmetric. Here we focus our attention on the restriction of their result to the $SU(2)$ scalar subsector.

[12] We are interested in local single-trace composite operators that consist of many fundamental fields. These fields are interacting. In a weakly-interacting quantum field theory, one can consistently choose to ignore all interactions beyond a chosen order in perturbation theory. In the planar theory under consideration, perturbation theory can be arranged according to the number of loops in Feynman diagrams computed. In a 1-loop approximation, one keeps only 1-loop diagrams.

The above brief outline is all we need for the purposes of this work. For an in-depth overview, we refer the reader to [9].

3 The XXZ Spin-$\frac{1}{2}$ Chain

In this section, we recall basic facts related to the XXZ spin-$\frac{1}{2}$ chain that are needed in later sections. The presentation closely follows that in [29, 43], but adapted to closed XXZ spin chains with twisted boundary conditions.

3.1 1-Dimensional Lattice Segments and Spin Variables

Consider a length-L 1-dimensional lattice, and label the sites with $i \in \{1, 2, \dots, L\}$. Assign site i a 2-dimensional vector space h_i with the basis

$$|\wedge\rangle_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i, \qquad |\vee\rangle_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_i \tag{3}$$

which we refer to as 'up' and 'down' states, and a spin variable s_i which can be equal to either of these states. The space of states $\mathcal{H}$ is the tensor product $\mathcal{H} = h_1 \otimes \cdots \otimes h_L$. Every state in $\mathcal{H}$ is an assignment $\{s_1, s_2, \dots, s_L\}$ of L definite-value (either up or down) spin variables to the sites of the spin chain. In computing scalar products, as we do shortly, we think of states in $\mathcal{H}$ as initial states.

3.2 Initial Spin-Up and Spin-Down Reference States

$\mathcal{H}$ contains two distinguished states,

$$|L^\wedge\rangle = \bigotimes_{i=1}^{L} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i, \qquad |L^\vee\rangle = \bigotimes_{i=1}^{L} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_i \tag{4}$$

where $L^\wedge$ indicates L spin states that are all up, and $L^\vee$ indicates L spin states that are all down. These are the initial spin-up and spin-down reference states, respectively.

3.3 Final Spin-Up and Spin-Down Reference States, and a Variation

Consider a length-L spin chain, and assign each site i a 2-dimensional vector space h_i^* with the basis

$$ {}_i\langle\wedge| = \begin{pmatrix}1 & 0\end{pmatrix}_i, \qquad {}_i\langle\vee| = \begin{pmatrix}0 & 1\end{pmatrix}_i \tag{5} $$

We construct a final space of states as the tensor product $\mathcal{H}^* = h_1^* \otimes \cdots \otimes h_L^*$. $\mathcal{H}^*$ contains two distinguished states

$$ \langle L^\wedge| = \bigotimes_{i=1}^{L} \begin{pmatrix}1 & 0\end{pmatrix}_i, \qquad \langle L^\vee| = \bigotimes_{i=1}^{L} \begin{pmatrix}0 & 1\end{pmatrix}_i \tag{6} $$

where all spins are up, and all spins are down. These are the final spin-up and spin-down reference states. respectively. Finally, we consider the variation

$$ \langle N_3{}^\vee, (L-N_3)^\wedge| = \bigotimes_{1\le i\le N_3} \begin{pmatrix}0 & 1\end{pmatrix}_i \bigotimes_{(N_3+1)\le i\le L} \begin{pmatrix}1 & 0\end{pmatrix}_i \tag{7} $$

where the first N_3 spins from the left are down, and all remaining spins are up.

3.4 Pauli Matrices

We define the Pauli matrices

$$ \sigma_m^x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}_m, \qquad \sigma_m^y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}_m, \qquad \sigma_m^z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_m \tag{8} $$

with $i = \sqrt{-1}$, and the spin raising/lowering matrices

$$ \sigma_m^+ = \frac{1}{2}(\sigma_m^x + i\sigma_m^y) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}_m, \qquad \sigma_m^- = \frac{1}{2}(\sigma_m^x - i\sigma_m^y) = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}_m \tag{9} $$

where in all cases the subscript m is used to indicate that the matrices act in the vector space h_m.

3.5 The Hamiltonian H

The Hamiltonian of the finite length XXZ spin-$\frac{1}{2}$ chain is given by the equivalent expressions

$$ H = \frac{1}{2}\sum_{m=1}^{L}\left(\sigma_m^x\sigma_{m+1}^x + \sigma_m^y\sigma_{m+1}^y + \Delta\left(\sigma_m^z\sigma_{m+1}^z - 1\right)\right) $$

$$= \sum_{m=1}^{L} \left(\sigma_m^+ \sigma_{m+1}^- + \sigma_m^- \sigma_{m+1}^+ + \frac{\Delta}{2} \left(\sigma_m^z \sigma_{m+1}^z - 1 \right) \right) \tag{10}$$

where Δ is the anisotropy parameter of the model, and where we assume the 'twisted' periodicity conditions

$$\sigma_{L+1}^{\pm} = e^{\pm i\theta} \sigma_1^{\pm}, \qquad \sigma_{L+1}^z = \sigma_1^z \tag{11}$$

3.6 The R-Matrix

From an initial reference state, we can generate all other states in $\mathcal{H}$ using operators that flip the spin variables, one spin at a time. Defining these operators requires defining a sequence of objects. (1) The R-matrix, (2) The L-matrix, and finally, (3) The monodromy or M-matrix.

The R-matrix is an element of $\mathrm{End}(h_a \otimes h_b)$, where h_a, h_b are two 2-dimensional auxiliary vector spaces. The variables u_a, u_b are the corresponding rapidity variables. The R-matrix intertwines these spaces, and it has the (4×4) structure

$$R_{ab}(u_a, u_b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b[u_a, u_b] & c[u_a, u_b] & 0 \\ 0 & c[u_a, u_b] & b[u_a, u_b] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{ab} \tag{12}$$

where we have defined the functions

$$b[u_a, u_b] = \frac{[u_a - u_b]}{[u_a - u_b + \eta]}, \qquad c[u_a, u_b] = \frac{[\eta]}{[u_a - u_b + \eta]}, \qquad [u] \equiv \sinh(u) \tag{13}$$

The R-matrix satisfies unitarity, crossing symmetry and the crucial Yang-Baxter equation that is required for integrability, given by

$$R_{ab}(u_a, u_b) R_{ac}(u_a, u_c) R_{bc}(u_b, u_c) = R_{bc}(u_b, u_c) R_{ac}(u_a, u_c) R_{ab}(u_a, u_b) \tag{14}$$

which holds in $\mathrm{End}(h_a \otimes h_b \otimes h_c)$ for all u_a, u_b, u_c.

As we will see in Sect. 4, the elements of the R-matrix (12) are the weights of the vertices of the trigonometric six-vertex model. This is the origin of the connection of the two models. One can graphically represent the elements of (12) to obtain the six vertices of the trigonometric six-vertex model in Fig. 2.

3.7 The L-Matrix

The L-matrix of the XXZ spin chain is a local operator that depends on a single rapidity u_a, and acts in the auxiliary space h_a. Its entries are operators acting at the

m-th lattice site, and identically everywhere else. It has the form

$$L_{am}(u_a) = \begin{pmatrix} [u_a + \frac{\eta}{2}\sigma_m^z] & [\eta]\sigma_m^- \\ [\eta]\sigma_m^+ & [u_a - \frac{\eta}{2}\sigma_m^z] \end{pmatrix}_a \tag{15}$$

Using the definition of the R-matrix and the L-matrix, (12) and (15) respectively, the local intertwining equation is given by

$$R_{ab}(u_a, u_b) L_{am}(u_a) L_{bm}(u_b) = L_{bm}(u_b) L_{am}(u_a) R_{ab}(u_a, u_b) \tag{16}$$

The proof of (16) is immediate, if one uses the matrix representations of $\sigma_m^z, \sigma_m^+, \sigma_m^-$ to write

$$\begin{aligned} L_{am}(u_a) &= \begin{pmatrix} [u_a + \frac{\eta}{2}] & 0 & 0 & 0 \\ 0 & [u_a - \frac{\eta}{2}] & [\eta] & 0 \\ 0 & [\eta] & [u_a - \frac{\eta}{2}] & 0 \\ 0 & 0 & 0 & [u_a + \frac{\eta}{2}] \end{pmatrix}_{am} \\ &= [u_a + \eta/2] R_{am}(u_a, \eta/2) \end{aligned} \tag{17}$$

This means that the L-matrix is equal to the R-matrix $R_{am}(u_a, z_m)$ with $z_m = \eta/2$, up to an overall multiplicative factor. Cancelling these common factors from (16), it becomes

$$\begin{aligned} & R_{ab}(u_a, u_b) R_{am}(u_a, \eta/2) R_{bm}(u_b, \eta/2) \\ & \quad = R_{bm}(u_b, \eta/2) R_{am}(u_a, \eta/2) R_{ab}(u_a, u_b) \end{aligned} \tag{18}$$

which is simply a corollary of the Yang-Baxter equation (14).

3.8 The Monodromy Matrix M

The monodromy or M-matrix is a global operator that acts on all sites in the spin chain. It is constructed as an ordered direct product of the L-matrices that act on single sites,

$$M_a(u_a) = L_{a1}(u_a) \dots L_{aL}(u_a) \Omega_a(\theta) \tag{19}$$

where $\Omega_a(\theta)$ is a twist matrix given by

$$\Omega_a(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}_a \tag{20}$$

The monodromy matrix is essential in the algebraic Bethe Ansatz approach to the diagonalization of the Hamiltonian H. It is convenient to define an inhomogeneous version, as an ordered direct product of R-matrices $R_{am}(u_a, z_m)$,

$$M_a\big(u_a, \{z\}_L\big) = R_{a1}(u_a, z_1) \dots R_{aL}(u_a, z_L) \Omega_a(\theta) \tag{21}$$

The variables $\{z_1, \dots, z_L\}$ are parameters corresponding with the sites of the spin chain and the homogeneous monodromy matrix, given by (19), is recovered by setting $z_m = \eta/2$ for all $1 \le m \le L$. The inclusion of the variables $\{z_1, \dots, z_L\}$ simplifies many later calculations, even though it is the homogeneous limit which actually interests us. We write the inhomogeneous monodromy matrix in (2×2) block form, by defining

$$M_a\big(u_a, \{z\}_L\big) = \begin{pmatrix} e^{i\theta} A(u_a) & e^{-i\theta} B(u_a) \\ e^{i\theta} C(u_a) & e^{-i\theta} D(u_a) \end{pmatrix}_a \tag{22}$$

where the matrix entries are operators that act in $\mathcal{H} = h_1 \otimes \cdots \otimes h_L$. To simplify the notation, we have omitted the dependence of the elements of the M-matrix on the quantum rapidities $\{z_1, \dots, z_L\}$. This dependence is implied from now on.

The operator entries of the monodromy matrix satisfy a set of commutation relations, which are determined by the equation

$$\begin{aligned} &R_{ab}(u_a, u_b) M_a\big(u_a, \{z\}_L\big) M_b\big(u_b, \{z\}_L\big) \\ &\quad = M_b\big(u_b, \{z\}_L\big) M_a\big(u_a, \{z\}_L\big) R_{ab}(u_a, u_b) \end{aligned} \tag{23}$$

which is a direct consequence of the Yang-Baxter equation (14) and the property

$$\big[R_{ab}(u_a, u_b), \Omega_a(\theta)\Omega_b(\theta)\big] = 0 \tag{24}$$

of the twist matrix. Typical examples of these commutation relations, which are particularly important in the algebraic Bethe Ansatz, are

$$B(u)B(v) = B(v)B(u) \tag{25}$$

$$[u - v + \eta]B(u)A(v) = [\eta]B(v)A(u) + [u - v]A(v)B(u) \tag{26}$$

$$[\eta]B(u)D(v) + [u - v]D(u)B(v) = [u - v + \eta]B(v)D(u) \tag{27}$$

In Sect. 4, we identify the operator entries of the monodromy matrix (22) with rows of vertices from the six-vertex model, see Fig. 3.

3.9 The Transfer Matrix T

The transfer matrix $T(u_a, \{z\}_L)$ is defined as the trace of the inhomogeneous monodromy matrix

$$T\big(u_a, \{z\}_L\big) = \mathrm{Tr}_a\, M_a\big(u_a, \{z\}_L\big) = e^{i\theta} A(u_a) + e^{-i\theta} D(u_a) \tag{28}$$

The Hamiltonian (10) is recovered via the quantum trace identity

$$H = [\eta] \frac{d}{du} \log T(u) \bigg|_{u = \frac{\eta}{2}}, \quad \text{where } T(u) = T\big(u, \{z\}_L\big)\big|_{z_1 = \cdots = z_L = \frac{\eta}{2}} \tag{29}$$

where the anisotropy parameter in (10) is defined as $\Delta = \cosh(\eta)$. In this equation all quantum parameters have been set equal, so for the purpose of reconstructing the Hamiltonian H we see that the homogeneous monodromy matrix is sufficient. However, in all subsequent calculations we preserve the variables $\{z_1, \dots, z_L\}$ and seek eigenvectors of $T(u, \{z\}_L)$. By (29), they are clearly also eigenvectors of H.

3.10 Generic States, Eigenstates and Bethe Equations

The initial and final spin-up reference states $|L^\wedge\rangle$ and $\langle L^\wedge|$ are eigenstates of the diagonal elements of the monodromy matrix. They satisfy the equations

$$A\big(u, \{z\}_L\big)\big|L^\wedge\big\rangle = a(u)\big|L^\wedge\big\rangle, \qquad D\big(u, \{z\}_L\big)\big|L^\wedge\big\rangle = d(u)\big|L^\wedge\big\rangle \tag{30}$$

$$\big\langle L^\wedge\big|A\big(u, \{z\}_L\big) = a(u)\big\langle L^\wedge\big|, \qquad \big\langle L^\wedge\big|D\big(u, \{z\}_L\big) = d(u)\big\langle L^\wedge\big| \tag{31}$$

where we have defined the eigenvalues

$$a(u) = 1, \qquad d(u) = \prod_{i=1}^{L} \frac{[u - z_i]}{[u - z_i + \eta]} \tag{32}$$

This makes $|L^\wedge\rangle$ and $\langle L^\wedge|$ eigenstates of the transfer matrix $T(u, \{z\}_L)$. The rest of the eigenstates $\{\mathcal{O}\}$ of $T(u, \{z\}_L)$, that is, states that satisfy

$$T\big(u, \{z\}_L\big)|\mathcal{O}\rangle_\beta = \big(e^{i\theta} A(u) + e^{-i\theta} D(u)\big)|\mathcal{O}\rangle_\beta = E_{\mathcal{O}}(u)|\mathcal{O}\rangle_\beta \tag{33}$$

where $E_{\mathcal{O}}(u)$ is the corresponding eigenvalue, are generated using the Bethe Ansatz. This is the statement that all eigenstates of $T(u, \{z\}_L)$ are created in two steps. 1. One acts on the initial reference state with the B-element of the monodromy matrix

$$|\mathcal{O}\rangle_\beta = B(u_{\beta_N}) \cdots B(u_{\beta_1})\big|L^\wedge\big\rangle \tag{34}$$

where $N \leq L$, since acting on $|L^\wedge\rangle$ with more B-operators than the number of sites in the spin chain annihilates it. This generates a 'generic Bethe state'. 2. We require that the auxiliary space rapidity variables $\{u_{\beta_1}, \dots, u_{\beta_N}\}$ satisfy Bethe equations, hence the use of the subscript β.[13] We call the resulting state a 'Bethe eigenstate'. That is, $|\mathcal{O}\rangle_\beta$ is an eigenstate of $T(u, \{z\}_L)$ if and only if

$$\frac{a(u_{\beta_i})}{d(u_{\beta_i})} = \prod_{j=1}^{L} \frac{[u_{\beta_i} - z_j + \eta]}{[u_{\beta_i} - z_j]} = e^{-2i\theta} \prod_{j \neq i}^{N} \frac{[u_{\beta_j} - u_{\beta_i} - \eta]}{[u_{\beta_j} - u_{\beta_i} + \eta]}, \tag{35}$$

[13] We use β in two different ways. 1. To indicate the deformation parameter in SYM_4^β theories, and 2. To indicate that a certain state is a Bethe eigenstate of the spin-chain Hamiltonian. There should be no confusion with 1, in which β is a parameter but never a subscript, while in 2 it is always a subscript.

for all $1 \le i \le N$. This fact can be proved using the commutation relations (26) and (27), as well as (30) and (31). As remarked earlier, by virtue of (29), eigenstates of the transfer matrix $T(u, \{z\}_L)$ are also eigenstates of the spin-chain Hamiltonian H. The latter is the spin-chain version of the 1-loop dilatation operator in SYM_4. We construct eigenstates of $T(u, \{z\}_L)$ in $\mathcal{H}^*$ using the C-element of the M-matrix

$$ {}_{\beta}\langle \mathcal{O} | = \left\langle L^{\wedge} \right| C(u_{\beta_1}) \dots C(u_{\beta_N}) \tag{36} $$

where $N \le L$ to obtain a non vanishing result, and requiring that the auxiliary space rapidity variables satisfy the Bethe equations.

3.11 Scalar Products that Are Determinants

Following [42, 43] we define the scalar product $S[L, N_1, N_2]$, $0 \le N_2 \le N_1$, that involves $(N_1 + N_2)$ operators, N_1 B-operators with auxiliary rapidities that satisfy Bethe equations, and N_2 C-operators with auxiliary rapidities that are free.[14] For $N_2 = 0$, we obtain, up to a non-dynamical factor, the domain wall partition function. For $N_2 = N_1$, we obtain Slavnov's scalar product [45]. As we will see in Sect. 4, $S[L, N_1, N_2]$ is the partition function (weighted sum over all internal configurations) of a lattice in an $[L, N_1, N_2]$-configuration, see Fig. 9.

Let $\{u_\beta\}_{N_1} = \{u_{\beta_1}, \dots, u_{\beta_{N_1}}\}$, $\{v\}_{N_2} = \{v_1, \dots, v_{N_2}\}$, $\{z\}_L = \{z_1, \dots, z_L\}$ be three sets of variables the first of which satisfies Bethe equations, $0 \le N_2 \le N_1$ and $1 \le N_1 \le L$. We define the scalar products

$$ \begin{aligned} &S[L, N_1, N_2]\big(\{u_\beta\}_{N_1}, \{v\}_{N_2}, \{z\}_L\big) \\ &\quad = \left\langle N_3^{\vee}, (L - N_3)^{\wedge} \right| \prod_{i=1}^{N_2} \mathbb{C}(v_i) \prod_{j=1}^{N_1} \mathbb{B}(u_{\beta_j}) \left| L^{\wedge} \right\rangle \end{aligned} \tag{37} $$

with $N_3 = N_1 - N_2$, and where we have defined the normalized B- and C-operators

$$ \mathbb{B}(u) = \frac{B(u)}{d(u)}, \qquad \mathbb{C}(v) = \frac{C(v)}{d(v)} \tag{38} $$

which are introduced only as a matter of convention. It is clear that for $N_2 = 0$, we obtain a domain wall partition function, while for $N_2 = N_1$, we obtain Slavnov's scalar product. In all cases, we assume that the auxiliary rapidities $\{u_\beta\}_{N_1}$ obey the Bethe equations (35), and use the subscript β to emphasize that, while the auxiliary rapidities $\{v\}_{N_2}$ are either free or also satisfy their own set of Bethe equations. When the latter is the case, this fact is not used. The quantum rapidities $\{z\}_L$ are taken to be equal to the same constant value in the homogeneous limit.

[14] To simplify the notation, we use N_1, N_2 and $N_3 = N_1 - N_2$, instead of the corresponding notation used in [42, 43]. These variables match the corresponding ones in Sect. 5.

3.12 A Determinant Expression for the Slavnov Scalar Product $S[L, N_1, N_2]$

Following [42, 43], we consider the $(N_1 \times N_1)$ matrix

$$\mathcal{S}\big(\{u_\beta\}_{N_1}, \{v\}_{N_2}, \{z\}_L\big) = \begin{pmatrix} f_1(z_1) & \cdots & f_1(z_{N_3}) & g_1(v_{N_2}) & \cdots & g_1(v_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ f_{N_1}(z_1) & \cdots & f_{N_1}(z_{N_3}) & g_{N_1}(v_{N_2}) & \cdots & g_{N_1}(v_1) \end{pmatrix} \quad (39)$$

whose entries are the functions

$$f_i(z_j) = \left(\frac{[\eta]}{[u_{\beta_i} - z_j + \eta][u_{\beta_i} - z_j]}\right) \prod_{k=1}^{N_2} \frac{1}{[v_k - z_j]} \quad (40)$$

$$g_i(v_j) = \left(\frac{[\eta]}{[u_{\beta_i} - v_j]}\right)\left(\left(\prod_{k=1}^{L} \frac{[v_j - z_k + \eta]}{[v_j - z_k]} \prod_{k\neq i}^{N_1} [u_{\beta_k} - v_j + \eta]\right) - e^{-2i\theta} \prod_{k\neq i}^{N_1} [u_{\beta_k} - v_j - \eta]\right) \quad (41)$$

and where $N_3 = N_1 - N_2$. Since the auxiliary rapidities $\{u_\beta\}_{N_1}$ satisfy Bethe equations (35), following [42, 43] it is possible to show that

$$S[L, N_1, N_2] = \frac{\prod_{i=1}^{N_1} \prod_{j=1}^{N_3} [u_{\beta_i} - z_j + \eta] \det \mathcal{S}(\{u_\beta\}_{N_1}, \{v\}_{N_2}, \{z\}_L)}{\prod_{1\le i<j\le N_1} [u_{\beta_j} - u_{\beta_i}] \prod_{1\le i<j\le N_2} [v_i - v_j] \prod_{1\le i<j\le N_3} [z_i - z_j]} \quad (42)$$

3.13 The Slavnov Scalar Product $S[L, N_1, N_1]$

Consider the special case $N_1 = N_2 = N$, which corresponds to Slavnov's scalar product itself. In this case we obtain the $(N \times N)$ matrix

$$\mathcal{S}\big(\{u_\beta\}_N, \{v\}_N, \{z\}_L\big) = \begin{pmatrix} g_1(v_N) & \cdots & g_1(v_1) \\ \vdots & & \vdots \\ g_N(v_N) & \cdots & g_N(v_1) \end{pmatrix} \quad (43)$$

whose entries are the functions

$$g_i(v_j) = \left(\frac{[\eta]}{[u_{\beta_i} - v_j]}\right)\left(\left(\prod_{k=1}^{L} \frac{[v_j - z_k + \eta]}{[v_j - z_k]} \prod_{k\neq i}^{N} [u_{\beta_k} - v_j + \eta]\right)\right.$$

$$- e^{-2i\theta} \prod_{k\neq i}^{N} [u_{\beta_k} - v_j - \eta] \Bigg) \tag{44}$$

The Slavnov scalar product $S[L,N,N]$ is then given by

$$S[L,N,N] = \frac{\det \mathcal{S}(\{u_\beta\}_N, \{v\}_N, \{z\}_L)}{\prod_{1\le i<j\le N}[u_{\beta_i} - u_{\beta_j}] \prod_{1\le i<j\le N}[v_i - v_j]} \tag{45}$$

3.14 Restrictions

There is a simple relation between the scalar products $S[L,N_1,N_1]$ and $S[L,N_1, N_2]$, which was used in [43] to provide a recursive proof of Slavnov's scalar product formula [45]. It is easy to show that by restricting the free variables $v_{N_1},\dots,v_{N_2+1}$ in (45) to the values $z_1,\dots,z_{N_3}$, one obtains the recursion relation

$$\left(\prod_{i=N_2+1}^{N_1} \prod_{j=1}^{L} [v_i - z_j] S[L,N_1,N_1] \right) \Bigg|_{\substack{v_{N_1}=z_1 \\ \vdots \\ v_{(N_2+1)}=z_{N_3}}} = \prod_{i=1}^{N_3} \prod_{j=1}^{L} [z_i - z_j + \eta] S[L,N_1,N_2] \tag{46}$$

As we show in Sect. 4, the scalar products $S[L,N_1,N_1]$ and $S[L,N_1,N_2]$ are in direct correspondence with the partition function of an $[L,N_1,N_1]$- and $[L,N_1,N_2]$-configuration, respectively. Accordingly, we expect that the recursion relation (46) has a natural interpretation at the level of six-vertex model lattice configurations, and indeed this turns out to be the case.

3.15 The Homogeneous Limit of $S[L,N_1,N_2]$

For the result in this paper, we need the homogeneous limit of $S[L,N_1,N_2]$, which we denote by $S^{hom}[L,N_1,N_2]$. Taking the limit $z_i \to z$, $i \in \{1,\dots,L\}$, the result is

$$S^{hom}[L,N_1,N_2] = \frac{\prod_{i=1}^{N_1}[u_{\beta_i} - z + \eta]^{N_3} \det \mathcal{S}^{hom}(\{u_\beta\}_{N_1}, \{v\}_{N_2}, z)}{\prod_{1\le i<j\le N_1}[u_{\beta_j} - u_{\beta_i}] \prod_{1\le i<j\le N_2}[v_i - v_j]} \tag{47}$$

$$\mathcal{S}^{hom}\big(\{u_\beta\}_{N_1}, \{v\}_{N_2}, z\big) = \begin{pmatrix} \Phi_1^{(0)}(z) & \cdots & \Phi_1^{(N_3-1)}(z) & g_1^{hom}(v_{N_2}) & \cdots & g_1^{hom}(v_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ \Phi_{N_1}^{(0)}(z) & \cdots & \Phi_{N_1}^{(N_3-1)}(z) & g_{N_1}^{hom}(v_{N_2}) & \cdots & g_{N_1}^{hom}(v_1) \end{pmatrix} \tag{48}$$

where $\Phi_i^{(j)} = \frac{1}{j!}\partial_z^{(j)} f_i(z)$, and

$$g_i^{hom}(v_j) = \frac{[\eta]}{[u_{\beta_i} - v_j]}\left(\left(\frac{[v_j - z + \eta]}{[v_j - z]}\right)^L \prod_{k\neq i}^{N_1}[u_{\beta_k} - v_j + \eta] - e^{-2i\theta}\prod_{k\neq i}^{N_1}[u_{\beta_k} - v_j - \eta]\right) \tag{49}$$

3.16 The Gaudin Norm

Let us consider the original, unrestricted Slavnov scalar product in the homogeneous limit $z_i \to z$, $S[L, N_1, N_1](\{u_\beta\}_{N_1}, \{v\}_{N_1}, z)$, and set $\{v\}_{N_1} = \{u_\beta\}_{N_1}$ to obtain the Gaudin norm $\mathcal{N}(\{u_\beta\}_{N_1})$ which is the square of the norm of the Bethe eigenstate with auxiliary rapidities $\{u_\beta\}_{N_1}$. It inherits a determinant expression that can be computed starting from that of the Slavnov scalar product that we begin with and taking the limit $\{v\}_{N_1} \to \{u_\beta\}_{N_1}$. Following [42], one obtains

$$\mathcal{N}\big(\{u_\beta\}_{N_1}\big) = \big(e^{-2i\theta}[\eta]\big)^{N_1}\left(\prod_{i\neq j}^{N_1}\frac{[u_i - u_j + \eta]}{[u_i - u_j]}\right)\det \Phi'\big(\{u_\beta\}_{N_1}\big) \tag{50}$$

where

$$\Phi'_{ij}\big(\{u_\beta\}_{N_1}\big) = -\partial_{u_j}\ln\left(\left(\frac{[u_i - z + \eta]}{[u_i - z]}\right)^L \prod_{\substack{k=1\\k\neq i}}^{N_1}\frac{[u_k - u_i + \eta]}{[u_k - u_i - \eta]}\right) \tag{51}$$

We need the Gaudin norm to normalize the Bethe eigenstates that form the 3-point functions whose structure constants we are interested in.

4 The Trigonometric Six-Vertex Model

This section follows almost *verbatim* the exposition in [29], up to straightforward adjustments to account for the fact that here we are interested in the trigonometric, rather than the rational six-vertex model. We recall the 2-dimensional trigonometric six-vertex model in the absence of external fields. From now on, 'six-vertex model' refers to that. It is equivalent to the XXZ spin-$\frac{1}{2}$ chain that appears in [26–28], but affords a diagrammatic representation that suits our purposes. We introduce quite a few terms to make this correspondence clear and the presentation precise, but the reader with basic familiarity with exactly solvable lattice models can skip all these.

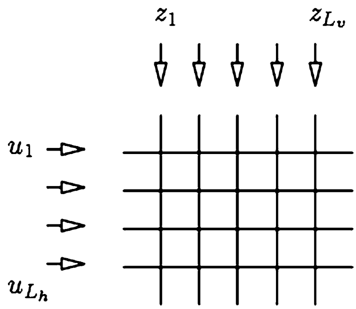

Fig. 1 A square lattice with oriented lines and rapidity variables. Lattice lines are assigned the orientations indicated by the *white arrows*

4.1 Lattice Lines, Orientations, and Rapidity Variables

Consider a square lattice with L_h horizontal lines and L_v vertical lines that intersect at $L_h \times L_v$ points. There is no restriction, at this stage, on L_h or L_v. We order the horizontal lines from top to bottom and assign the i-th line an orientation from left to right and a rapidity variable u_i. We order the vertical lines from left to right and assign the j-th line an orientation from top to bottom and a rapidity variable z_j. See Fig. 1. The orientations that we assign to the lattice lines are matters of convention and are only meant to make the vertices of the six-vertex model, that we introduce shortly, unambiguous. We orient the vertical lines from top to bottom to agree with the direction of the 'spin set evolution' that we introduce shortly.

4.2 Line Segments, Arrows, and Vertices

Each lattice line is split into segments by all other lines that are perpendicular to it. 'Bulk segments' are attached to two intersection points, and 'boundary segments' are attached to one intersection point only. Assign each segment an arrow that can point in either direction, and define the vertex v_{ij} as the union of 1. The intersection point of the i-th horizontal line and the j-th vertical line, 2. The four line segments attached to this intersection point, and 3. The arrows on these segments (regardless of their orientations). Assign v_{ij} a weight that depends on the specific orientations of its arrows, and the rapidities u_i and z_j that flow through it.

4.3 Six Vertices that Conserve 'Arrow Flow'

Since every arrow can point in either direction, there are $2^4 = 16$ possible types of vertices. In this note, we are interested in a model such that only those vertices that conserves 'arrow flow' (that is, the number of arrows that point toward the intersection point is equal to the number of arrows that point away from it) have non-zero weights. There are six such vertices. They are shown in Fig. 2. We assign

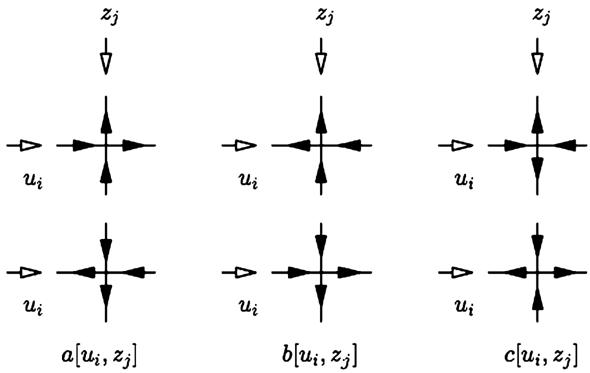

Fig. 2 The non-vanishing-weight vertices of the six-vertex model. Pairs of vertices in the same column share the weight that is shown below that column. The *white arrows* indicate the line orientations needed to specify the vertices without ambiguity

these vertices non-vanishing weights. We assign the rest of the 16 possible vertices zero weights [51].

In the trigonometric six-vertex model, and in the absence of external fields, the six vertices with non-zero weights form three equal-weight pairs of vertices, as in Fig. 2. Two vertices that form a pair are related by reversing all arrows, thus the vertex weights are invariant under reversing all arrows. In the notation of Fig. 2, the weights of the trigonometric six-vertex model, in the absence of external fields, are

$$a[u_i, z_j] = 1, \qquad b[u_i, z_j] = \frac{[u_i - z_j]}{[u_i - z_j + \eta]}, \qquad c[u_i, z_j] = \frac{[\eta]}{[u_i - z_j + \eta]} \tag{52}$$

where we use the definition $[x] = \sinh(x)$ to simplify notation.[15] The assignment of weights in (52) satisfies unitarity, crossing symmetry, and most importantly the Yang-Baxter equations [51]. It is not unique since one can multiply all weights by the same factor without changing the final physical results.

4.4 Correspondence with the XXZ R-Matrix

The connection with the R-matrix of the XXZ spin-$\frac{1}{2}$ chain is straightforward. One can think of the R-matrix (12) as assigning a weight to the transition from a pair of initial spin states (for example, the definite spin states on the right and upper segments that meet at a certain vertex) to a pair of final spin states (the definite spin states on the left and lower segments that meet at the same vertex as the initial ones). In the case of the trigonometric XXZ spin-$\frac{1}{2}$ chain, this is a transition between four possible initial spin states and four final spin states, and accordingly the R-matrix is (4×4). The six non-zero entries of (12) correspond with the vertices in Fig. 2.

[15]The weights of the six-vertex model (52) and the entries of the XXZ R-matrix (12) are identical. This is the origin of the connection between the two models. We have chosen to write down these functions twice for clarity and to emphasize this fact.

4.5 Remarks

1. The spin chains that are relevant to integrability in YM theories are typically homogeneous since all quantum rapidities are set equal to the same constant value z. In our conventions, $z = \frac{1}{2}\sqrt{-1}$. 2. The trigonometric six-vertex model that corresponds to the homogeneous XXZ spin-$\frac{1}{2}$ chain used in [26–28] has, in our conventions, all vertical rapidity variables equal to $\frac{1}{2}\sqrt{-1}$. In this note, we start with inhomogeneous vertical rapidities, then take the homogeneous limit at the end. 3. In a 2-dimensional vertex model with no external fields, the horizontal lines are on equal footing with the vertical lines. To make contact with spin chains, we treat these two sets of lines differently. 4. In all figures in this note, a line segment with an arrow on it obviously indicates a definite arrow assignment. A line segment with no arrow on it implies a sum over both arrow assignments.

4.6 Weighted Configurations and Partition Functions

By assigning every vertex v_{ij} a weight w_{ij}, a vertex model lattice configuration with a definite assignment of arrows is assigned a weight equal to the product of the weights of its vertices. The partition function of a lattice configuration is the sum of the weights of all possible configurations that the vertices can take and that respect the boundary conditions. Since the vertex weights are invariant under reversal of all arrows, the partition function is also invariant under reversal of all arrows.

4.7 Rows of Segments, Spin Systems, Spin System States and Net Spin

A 'row of segments' is a set of *vertical* line segments that start and/or end on the same horizontal line(s). An $L_h \times L_v$ six-vertex lattice configuration has $(L_v + 1)$ rows of segments. On every length-L_h row of segments, one can assign a definite spin configuration, whereby each segment carries a spin variable (an arrow) that can point either up or down. A 'spin system' on a specific row of segments is a set of all possible definite spin configurations that one can assign to that row. A 'spin system state' is one such definite configuration. Two neighbouring spin systems (or spin system states) are separated by a horizontal lattice line. The spin systems that live on the top and the bottom rows of segments are initial and final spin systems, respectively. Consider a specific spin system state. Assign each up-spin the value $+1$ and each down-spin the value -1. The sum of these values is the net spin of this spin system state. In this paper, we only consider six-vertex model configurations such that all elements in a spin system have the same net spin.

4.8 Initial and Final Spin-Up and Spin-Down Reference States, and a Variation

An initial (final) spin-up reference state $|L^{\wedge}\rangle$ ($\langle L^{\wedge}|$) is a spin system state on a top (bottom) row of segments with L arrows that are all up. An initial (final) spin-down reference state $|L^{\vee}\rangle$ ($\langle L^{\vee}|$) is a spin system state on a top (bottom) row of segments with L arrows that are all down. The state $\langle N_3^{\vee}, (L - N_3)^{\wedge}|$ is a spin system state on a bottom row of segments with L arrows such that the first N_3 arrows from the left are down, while the remaining $(L - N_3)$ arrows are up. We do not need the initial version of this state.

4.9 Correspondence with XXZ Spin Chain States

The connection to the XXZ spin-$\frac{1}{2}$ chain of Sect. 3 is clear. Every state of the periodic spin chain is analogous to a spin system state in the six-vertex model. Periodicity is not manifest in the latter representation for the same reason that it is not manifest once we choose a labeling system. The initial and final spin-up/down reference states are the six-vertex analogues of those discussed in Sect. 3.

4.10 Remarks

1. There is of course no 'time variable' in the six-vertex model, but one can think of a spin system as a dynamical system that evolves in discrete steps as one scans a lattice configuration from top to bottom. Starting from an initial spin set and scanning the configuration from top to bottom, the intermediate spin sets are consecutive states in the history of a dynamical system, ending with the final spin set. This evolution is caused by the action of the horizontal line elements. 2. In this paper, all elements in a spin system, that live on a certain row of segments, have the same net spin. The reason is that vertically adjacent spin systems are separated by horizontal lines of a fixed type that change the net spin by the same amount (± 1) or keep it unchanged. Since we consider only lattice configurations with given horizontal lines (and do not sum over different types), the net spin of all elements in a spin system changes by the same amount.

4.11 Four Types of Horizontal Lines

Each horizontal line has two boundary segments. Each boundary segment has as an arrow that can point into the configuration or away from it. Accordingly, we can

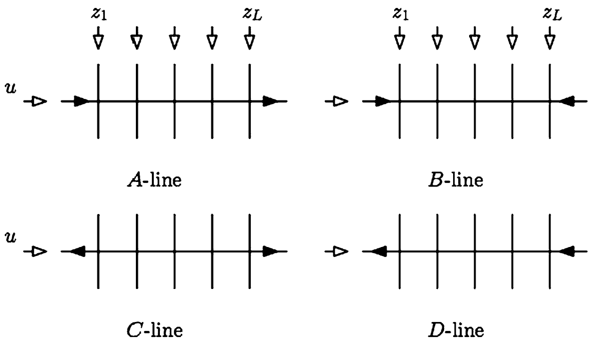

Fig. 3 There are four types of horizontal lines in a six-vertex model lattice configuration

distinguish four types of horizontal lines, as in Fig. 3. We refer to them as A-, B-, C- and D-lines.

An important property of a horizontal line is how the net spin changes as one moves across it from top to bottom. Given that all vertices conserve 'arrow flow', one can easily show that, scanning a configuration from top to bottom, B-lines change the net spin by -1, C-lines change it by $+1$, while A- and D-lines preserve the net spin. This can be easily understood by working out a few simple examples.

4.12 *Correspondence with Monodromy Matrix Entries*

The A-, B-, C- and D-lines in Fig. 3 are the six-vertex model representation of the corresponding elements of the M-matrix in Sect. 3. This graphical representation is used frequently throughout the rest of the paper.

4.13 *Four Types of Configurations*

1. A B-configuration is a lattice configuration with L vertical lines and N horizontal lines, $N \leq L$, such that (A) The initial spin system is an initial reference state $|L^\wedge\rangle$, and (B) All horizontal lines are B-lines. An example is on the left hand side of Fig. 4.

2. A C-configuration is a lattice configuration with L vertical lines and N horizontal lines, $N \leq L$, such that (A) All horizontal lines are C-lines, and (B) The final spin system is a final reference state $\langle L^\wedge|$. An example is on the right hand side of Fig. 4.

3. A BC-configuration is a lattice configuration with L vertical lines and $2N_1$ horizontal lines, $0 \leq N_1 \leq L$, such that (A) The initial spin system is an initial reference state $|L^\wedge\rangle$, (B) The first N_1 horizontal lines from top to bottom are B-

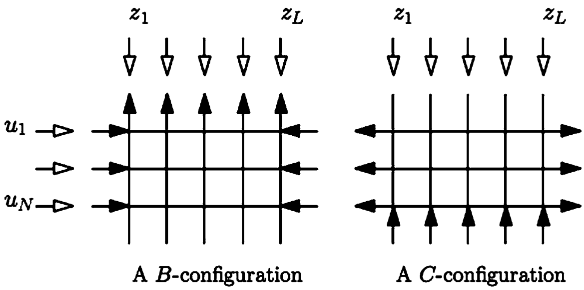

Fig. 4 *On the left*, a B-configuration, generated by the action of N B-lines on an initial length-L reference state, $N \leq L$. A weighted sum over all possible configurations of segments with no arrows is implied. *On the right*, the corresponding C-configuration

lines, (C) The following N_1 horizontal lines are C-lines, (D) The final spin system is a final reference state $\langle L^\wedge|$. See Fig. 5.[16]

4. An $[L, N_1, N_2]$-configuration, $0 \leq N_2 \leq N_1$, is identical to a BC-configuration except that it has N_1 B-lines, and N_2 C-lines. When $N_3 = N_1 - N_2 = 0$, we evidently recover a BC-configuration. The case $N_2 = 0$ is discussed below. For intermediate values of N_2, we obtain restricted BC-configurations whose partition functions turn out to be essentially the structure constants.

4.14 Correspondence with Generic Bethe States

An initial (final) generic Bethe state is represented in six-vertex model terms as a B-configuration (C-configuration), as illustrated on the left (right) hand side of Fig. 4. Note that the outcome of the action of the N B-lines (C-lines) on the initial (final) length-L spin-up reference state is an initial (final) spin system that can assume all possible spin states of net spin $(L - N)$. Each of these definite spin states is weighted by the weight of the corresponding lattice configuration.

4.15 Correspondence with $S[L, N_1, N_1]$ Scalar Products and $S[L, N_1, N_2]$ Restricted Scalar Products

In the language of the six-vertex model, the scalar product $S[L, N_1, N_1]$ corresponds with a BC-configuration with N_1 B-lines and N_1 C-lines, as illustrated in Fig. 5. The restricted scalar product $S[L, N_1, N_2]$ corresponds with an $[L, N_1, N_2]$-configuration, as illustrated in Fig. 9. Compared with the definition of $S[L, N_1, N_2]$ in (37), the partition function of an $[L, N_1, N_2]$-configuration differs only up to an overall normalization. To translate between the two, one should divide the latter by $d(u)$ for every B-line with rapidity u and by $d(v)$ for every C-line with rapidity v.

[16]For visual clarity, we have allowed for a gap between the B-lines and the C-lines in Fig. 5. There is also a gap between the N_3-th and $(N_3 + 1)$-th vertical lines, where $N_3 = 3$ in the example shown, that indicates separate portions of the lattice that will be relevant shortly. The reader should ignore this at this stage.

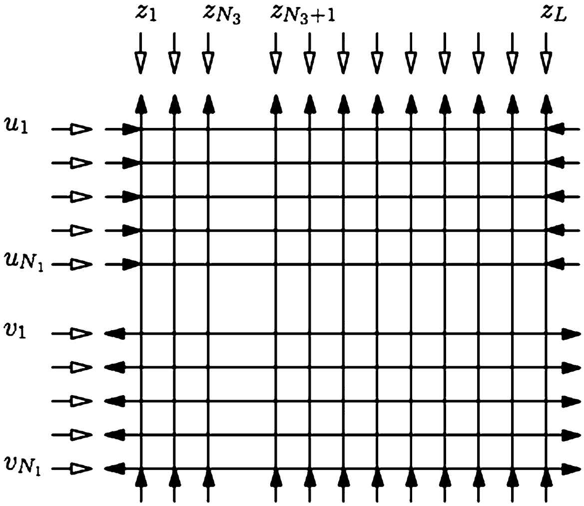

Fig. 5 A six-vertex model BC-configuration. $L = 12$, and $N_1 = 5$, or equivalently $L_h = 2 \times 5 = 10$ and $L_v = 12$. The top N_1 horizontal lines represent B-operators. The bottom N_1 horizontal lines represent C-operators. The initial (top) as well as the final (bottom) boundary spin systems are reference states

4.16 $[L, N_1, N_2]$*-Configurations as Restrictions of BC-Configurations*

Consider a BC-configuration with no restrictions. To be specific, let us consider the configuration in Fig. 5, where $N_1 = 5$ and $L = 12$. Both sets of rapidities $\{u\}$ and $\{v\}$ are labeled from top to bottom, as usual.

Consider the vertex at the bottom-left corner of Fig. 5. From Fig. 2, it is easy to see that this can be either a b- or a c-vertex. Since the $\{v\}$ variables are free, set $v_5 = z_1$, thereby setting the weight of all configurations with a b-vertex at this corner to zero, and forcing the vertex at this corner to be a c-vertex.

Referring to Fig. 2 again, one can see that not only is the corner vertex forced to be a c-vertex, but the orientations of all arrows on the horizontal lattice line with rapidity v_5, as well as all arrows on the vertical line with rapidity z_1 but below the horizontal line with rapidity u_{N_1} are also frozen to fixed values as in Fig. 6.

The above exercise in '*freezing*' vertices and arrows can be repeated and to produce a non-trivial example, we do it two more times. Setting $v_4 = z_2$ forces the vertex at the intersection of the lines carrying the rapidities v_4 and z_2 to be a c-vertex and freezes all arrows to the right as well as all arrows above that vertex and along C-lines, as in Fig. 7.

Setting $v_3 = z_3$, we end up with the lattice configuration in Fig. 8, from which we can see that (1) All arrows on the lower N_3 horizontal lines, where $N_3 = 3$ in the specific example shown, are frozen, and (2) All lines on the N_3 left most vertical lines in the lower half of the diagram, where they intersect with C-lines. Removing the lower N_3 C-lines we obtain the configuration in Fig. 9. This configuration has a subset (rectangular shape on lower left corner) that is also completely frozen. All vertices in this part are a-vertices, hence from (52), their contribution to the partition function of this configuration is trivial.

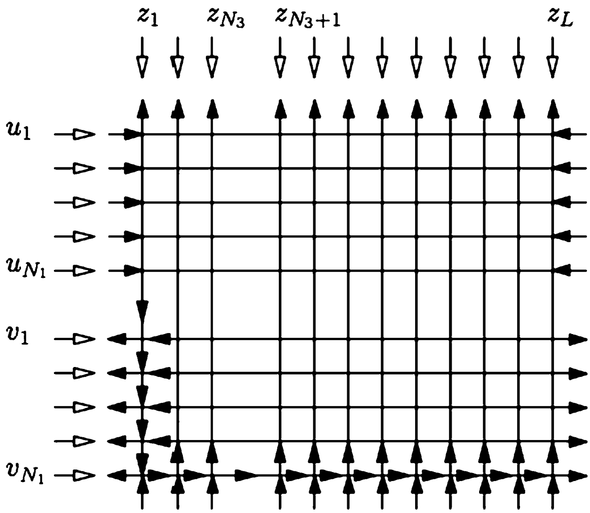

Fig. 6 Setting v_{N_1} to z_1 in Fig. 5, we freeze (1) the vertex at the lower left corner to be type-c, (2) all vertices to the right of the frozen corner to be type-a, and (3) all vertices above the frozen corner, but on the lower half of the diagram, to be type-b

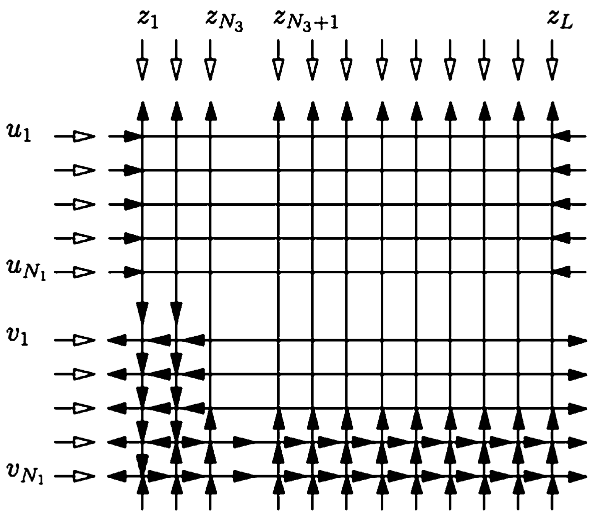

Fig. 7 Setting v_{N_1-1} (on second horizontal line from below) to z_2 (on second vertical line from left) in Fig. 6, we freeze (1) the vertex at the intersection of the lines that carry rapidities v_{N_1-1} and z_2 to be type-c, (2) all vertices to the right of the most recently-frozen corner to be type-a, and (3) all vertices above the same vertex, but on the lower half of the diagram, to be type-b

An $[L, N_1, N_2]$-configuration, as in Fig. 9, interpolates between an initial reference state $|L^{\wedge}\rangle$ and a final $\langle N_3^{\vee}, (L - N_3)^{\wedge}|$ state, using N_1 B-lines followed by N_2 C-lines.

Setting $v_{N_1-i+1} = z_i$ for $i = 1, \ldots, N_1$, we freeze all arrows that are on C-lines or on segments that end on C-lines. Discarding these we obtain the lattice configuration in Fig. 10.

Removing all frozen vertices (as well as the extra space between two sets of vertical lines, that is no longer necessary), one obtains the *domain wall configuration* in Fig. 11, which is characterized as follows. All arrows on the left and right boundaries point inwards, and all arrows on the upper and lower boundaries point

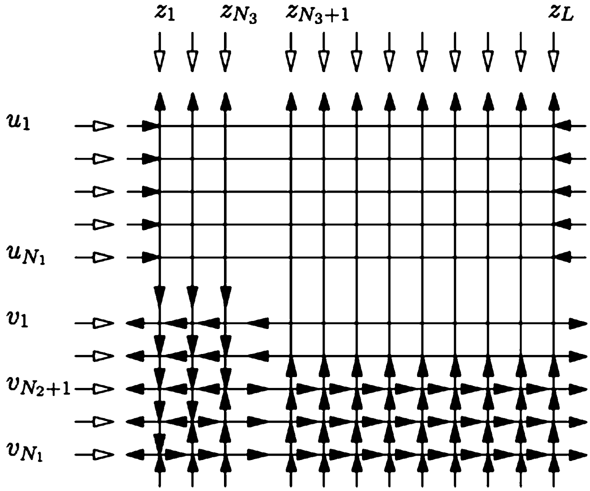

Fig. 8 The effect of forcing the three vertices at the intersection of the lines that carry the pairs of rapidities $\{v_{N_1}, z_1\}$, $\{v_{N_1-1}, z_2\}$ and $\{v_{N_1-2}, z_3\}$ to be c-vertices. We used the notation $N_3 = N_1 - N_2$

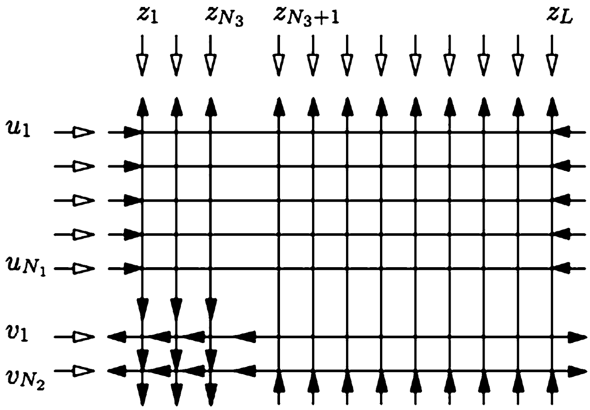

Fig. 9 An $[L, N_1, N_2]$-configuration. In this example, $N_1 = 5$, $N_2 = 2$, and as always $N_3 = N_1 - N_2$

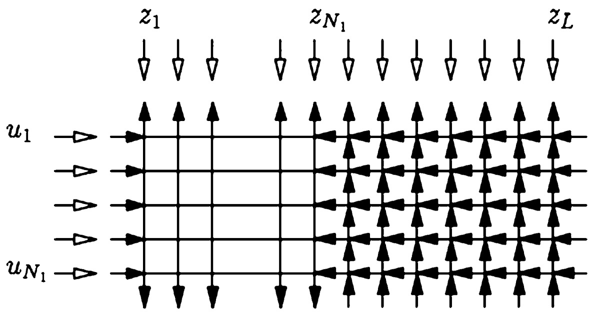

Fig. 10 Another $[L, N_1, N_2]$-configuration. In this example, $N_2 = 0$ and $N_1 = 5$. Equivalently, the left half is an $(N_1 \times N_1)$ domain wall configuration, where $N_1 = 5$, with an additional totally frozen lattice configuration to its right

outwards. The internal arrows remain free, and the configurations that are consistent with the boundary conditions are summed over. Reversing the orientation of all arrows on all boundaries is a dual a domain wall configuration.

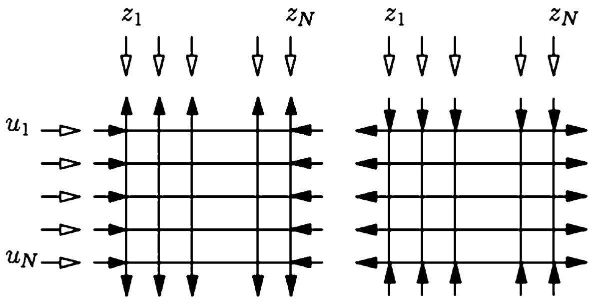

Fig. 11 *The left hand side* is an $(N \times N)$ domain wall configuration, where $N = 5$. *The right hand side* is the corresponding dual configuration

4.17 Remarks on Domain Wall Configurations

1. One can generate a domain wall configuration directly starting from a length-N initial reference state followed by N B-lines, 2. One can generate a dual domain wall configuration directly starting from a length-N dual initial reference state followed by N C-lines, 3. A BC-configuration with length-L initial and final reference states, L B-lines and L C-lines, factorizes into a product of a domain wall configuration and a dual domain wall configuration, 4. The restriction of BC-configurations to $[L, N_1, N_2]$-configurations, where $N_2 < N_1$, produces a recursion relation that was used in [43] to provide a recursive proof of Slavnov's determinant expression for the scalar product of a Bethe eigenstate and a generic state in the corresponding spin chain, 5. The partition function of a domain wall configuration has a determinant expression found by Izergin, that can be derived in six-vertex model terms (without reference to spin chains or the BA) [50].

4.18 Izergin's Domain Wall Partition Function

Let $\{u\}_N = \{u_1, \ldots, u_N\}$ and $\{z\}_N = \{z_1, \ldots, z_N\}$ be two sets of rapidity variables[17] and define $Z_N(\{u\}_N, \{z\}_N)$ to be the partition function of the domain wall lattice configuration on the left hand side of Fig. 11, after dividing by $d(u)$ for every B-line with rapidity u. Izergin's determinant expression for the domain wall partition function is

$$Z_N\big(\{u\}_N, \{z\}_N\big) = \frac{\prod_{i,j=1}^{N}[u_i - z_j + \eta]}{\prod_{1 \le i < j \le N}[u_i - u_j][z_j - z_i]} \det\left(\frac{[\eta]}{[u_i - z_j + \eta][u_i - z_j]}\right)_{1 \le i,j \le N} \tag{53}$$

Dual domain wall configurations have the same partition functions due to invariance under reversing all arrows. For the result of this note, we need the homogeneous limit of the above expression. Taking the limit $z_i \to z$, $\{i = 1, \ldots, L\}$, we

[17]The following result does not require that any set of rapidities satisfy Bethe equations.

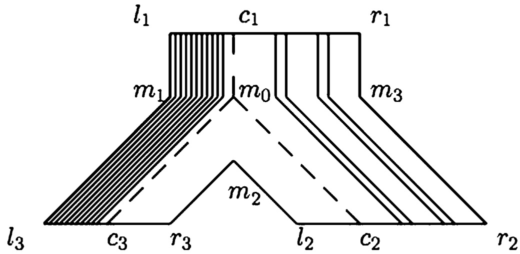

Fig. 12 A schematic representation of a 3-point function. State $\mathcal{O}_1$ is on top. $\mathcal{O}_2$ and $\mathcal{O}_3$ are below, to the right and to the left. Type-**A** 3-point functions are (initially) in this '*wide-pants*' form

obtain

$$Z_N^{hom}\big(\{u\}_N, z\big) = \frac{\prod_{i=1}^N [u_i - z + \eta]^N}{\prod_{1\le i<j\le N}[u_i - u_j]} \det\big(\phi^{(j-1)}(u_i, z)\big)_{1\le i,j\le N}$$
$$\text{where } \phi^{(j)}(u_i, z) = \frac{1}{j!}\partial_z^{(j)}\left(\frac{[\eta]}{[u_i - z + \eta][u_i - z]}\right) \tag{54}$$

5 Structure Constants in Type-A Theories

In this section, we recall the discussion of SYM_4 tree-level structure constants of [26–29] but now in the context of the Type-**A** theories in Subsect. 1.5, and construct determinant expressions for structure constants of three non-extremal $SU(2)$ single-trace operators.

Since theory 1 is SYM_4, theory 2 is an Abelian orbifolding of SYM_4, and theory 3 is a real-β-deformation of it, all three theories share the same fundamental charged scalar field content, that is $\{X, Y, Z\}$ and their charge conjugates $\{\bar{X}, \bar{Y}, \bar{Z}\}$, and all are conformally invariant up to all loops [33]. This makes our discussion a straightforward paraphrasing of that in [26–29].

5.1 Tree-Level Structure Constants

We consider tree-level 3-point functions of $SU(2)$ single-trace operators that (1) have well-defined conformal dimensions at 1-loop level, and (2) can be mapped to Bethe eigenstates in closed spin-$\frac{1}{2}$ chains.

These 3-point functions can be represented schematically as in Fig. 12. Identify the pairs of corner points $\{l_1, r_1\}$, $\{l_2, r_2\}$, $\{l_3, r_3\}$, as well as the triple $\{m_1, m_2, m_3\}$ to obtain a pants diagram. The structure constants have a perturbative expansion in the 't Hooft coupling constant λ,

$$C_{ijk} = c_{ijk}^{(0)} + \lambda c_{ijk}^{(1)} + \cdots \tag{55}$$

We restrict our attention to the leading coefficient $c_{ijk}^{(0)}$. In the limit $\lambda \to 0$, many single-trace operators have the same conformal dimension. This degeneracy is lifted at 1-loop level and certain linear combinations of single-trace operators have definite 1-loop conformal dimension. This is why although we compute tree-level structure constants, we insist on 1-loop conformal invariance: We identify operators with well defined conformal dimensions.

As explained in Sect. 2, these linear combinations correspond to eigenstates of a closed spin-$\frac{1}{2}$ chain. Their conformal dimensions are the corresponding Bethe eigenvalues. These closed spin chain states correspond to the circles at the boundaries of the pants diagram that can be constructed from Fig. 12 as discussed above.

5.2 Remark

In computing 3-point functions, the three composite operators *may or may not belong to the same $SU(2)$* doublet. In particular, in [26–28], EGSV use operators from the doublets $\{Z, X\}$, $\{\bar{Z}, \bar{X}\}$, and $\{Z, \bar{X}\}$. In [29], this procedure allowed us to construct determinant expressions for structure constants of non-extremal 3-point functions. This applies to all Type-**A** theories. Type-**B** structure constants are constructed differently. In particular, the non-extremal case $l_{23} = 0$ is considered.

5.3 Constructing 3-Point Functions

To construct three-point functions at the gauge theory operator level, the fundamental fields in the operators $\mathcal{O}_i$, $i = \{1, 2, 3\}$ are contracted by free propagators. Each propagator connects two fields, hence $L_1 + L_2 + L_3$ is an even number. The number of propagators between $\mathcal{O}_i$ and $\mathcal{O}_j$ is

$$l_{ij} = \frac{1}{2}(L_i + L_j - L_k) \tag{56}$$

where (i, j, k) take distinct values in $(1, 2, 3)$. We restrict our attention, in this section, to the non-extremal case, that is, all l_{ij}'s are strictly positive. The free propagators reproduce the factor $1/|x_i - x_j|^{\Delta_i + \Delta_j - \Delta_k}$ in (2), where $\Delta_i = \Delta_i^{(0)}$, the tree-level conformal dimension. See Fig. 12 for a schematic representation of a three point function of the type discussed in this note. The horizontal line segment between l_i and r_i represents the operator $\mathcal{O}_i$. The lines that start at $\mathcal{O}_1$ and end at either $\mathcal{O}_2$ or $\mathcal{O}_3$ represent one type of propagator.

5.4 From Single-Trace Operators to Spin-Chain States

One represents the single-trace operator $\mathcal{O}_i$ of well-defined 1-loop conformal dimension Δ_i by a closed spin-chain Bethe eigenstate $|\mathcal{O}_i\rangle_\beta$. Its eigenvalue E_i is

Table 1 Identification of operator content of $\mathcal{O}_i$, $i \in \{1,2,3\}$ with initial and final spin-chain states

Operator	$\binom{1}{0}$	$\binom{0}{1}$	$(1\quad 0)$	$(0\quad 1)$
$\mathcal{O}_1$	Z	X	$\bar{Z}$	$\bar{X}$
$\mathcal{O}_2$	$\bar{Z}$	$\bar{X}$	Z	X
$\mathcal{O}_3$	Z	$\bar{X}$	$\bar{Z}$	X

equal to Δ_i. The number of fundamental fields L_i in the trace is the length of the spin chain.

The single-trace operator $\mathcal{O}_i$ is a composite operator built from weighted sums over traces of products of two fundamental fields $\{u, d\}$. These fundamental fields are mapped to definite spin states. To perform suitable mappings that lead to non-vanishing results, we need to decide on which state(s) are in-state(s) from the viewpoint of the lattice representation, and which are out-state(s).

5.5 Type-A. Fundamental Field Content of the States

All three Type-**A** theories have the same fundamental field content, namely that of SYM$_4$, and thereby, more than one doublet. We focus on the doublets formed from the fields Z, X and their conjugates. Following [26–28], we identify the fundamental field content of $\mathcal{O}_i$, $i \in \{1,2,3\}$ with spin-chain spin states as shown in Table 1, where $\bar{Z}$ and $\bar{X}$ are the conjugates of Z and X. That is, if Z appears on one side of a propagator and $\bar{Z}$ appears on the other side, then that propagator is not identically vanishing, and Z and $\bar{Z}$ can be Wick contracted. Similarly for X and $\bar{X}$.

In our conventions

$$\langle \bar{Z} Z \rangle = \langle Z | Z \rangle = 1, \qquad \langle Z Z \rangle = \langle \bar{Z} | Z \rangle = 0 \tag{57}$$

and similarly for X and $\bar{X}$. In (57), $\langle \bar{f}\ f \rangle$ with no vertical bar between the two operators is a propagator, while $\langle f | f \rangle$ with a vertical bar between the two operators is a scalar product of an initial state $|f\rangle$ and a final state $\langle f|$.

From Table 1, one can read the fundamental-scalar operator content of each single-trace operator $\mathcal{O}_i$, $i \in \{1,2,3\}$, when it is an initial state and when it is a final state. For example, the fundamental field content of the initial state $|\mathcal{O}_1\rangle$ is $\{Z, X\}$, and that of the corresponding final state $\langle \mathcal{O}_1|$ is $\{\bar{Z}, \bar{X}\}$. The content of an initial state and the corresponding final state are related by the ‘flipping’ operation of [26–28] described below.

5.6 Structure Constants in Terms of Spin-Chains

Having mapped the single-trace operators $\mathcal{O}_i$, $i \in \{1,2,3\}$ to spin-chain eigenstates, EGSV construct the structure constants in three steps.

5.6.1 Step 1. Split the Lattice Configurations that Correspond to Closed Spin-Chain Eigenstates into Two Parts

Consider the open 1-dimensional lattice configuration that corresponds to the i-th closed spin-chain eigenstate, $i \in \{1, 2, 3\}$. This is schematically represented by a line in Fig. 12 that starts at l_i and ends at r_i. Split that, at point c_i into *left* and *right* sub-lattice configurations of lengths $L_{i,l} = \frac{1}{2}(L_i + L_j - L_k)$ and $L_{i,r} = \frac{1}{2}(L_i + L_k - L_j)$ respectively. Note that the lengths of the sub-lattices is fully determined by L_1, L_2 and L_3 which are fixed.

Following [44], we express the single lattice configuration of the original closed spin chain state as a weighted sum of tensor products of states that live in two smaller Hilbert spaces. The latter correspond to closed spin chains of lengths $L_{i,l}$ and $L_{i,r}$ respectively. That is, $|\mathcal{O}_i\rangle = \sum H_{l,r} |\mathcal{O}_i\rangle_l \otimes |\mathcal{O}_i\rangle_r$. The factors $H_{l,r}$ were computed in [44] and were needed in [26–28], where one of the scalar products is generic and had to be expressed as an explicit sum. They are not needed in this work as we use Bethe equations to evaluate this very sum as a determinant.

5.6.2 Step 2. From Initial to Final States

Map $|\mathcal{O}_i\rangle_l \otimes |\mathcal{O}_i\rangle_r \ \rightarrow |\mathcal{O}_i\rangle_l \otimes {}_r\langle\mathcal{O}_i|$, using the operator $\mathcal{F}$ that acts as follows.

$$\mathcal{F}\big(|f_1 f_2 \cdots f_{L-1} f_L\rangle\big) = \langle \bar{f}_L \bar{f}_{L-1} \cdots \bar{f}_2 \bar{f}_1| \tag{58}$$

In particular,

$$\begin{aligned} \langle ZZ \cdots Z|ZZ \cdots Z\rangle &= \langle \bar{Z}\bar{Z} \cdots \bar{Z}|\bar{Z}\bar{Z} \cdots \bar{Z}\rangle = 1, \\ \text{and} \quad \langle \bar{Z}\bar{Z} \cdots \bar{Z}|ZZ \cdots Z\rangle &= 0 \end{aligned} \tag{59}$$

More generally

$$\langle f_{i_1} f_{i_2} \cdots f_{i_L} | f_{j_1} f_{j_2} \cdots f_{j_L}\rangle \sim \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_L j_L} \tag{60}$$

The ‘flipping’ operation in (58) is the origin of the differences in assignments of fundamental fields to initial and final operator states in Table 1. For example, $|\mathcal{O}_1\rangle$ has fundamental field content $\{Z, X\}$, but $\langle\mathcal{O}_1|$ has fundamental field content $\{\bar{Z}, \bar{X}\}$. This agrees with the fact that in computing $\langle\mathcal{O}_i|\mathcal{O}_i\rangle$, free propagators can only connect conjugate fundamental fields.

5.6.3 Step 3. Compute Scalar Products

Wick contract pairs of initial states $|\mathcal{O}_i\rangle_r$ and final states $|\mathcal{O}_{i+1}\rangle_l$, where $i \in \{1, 2, 3\}$ and $i + 3 \equiv i$. The spin-chain equivalent of that is to compute the scalar products ${}_r\langle\mathcal{O}_i|\mathcal{O}_{i+1}\rangle_l$, which in six-vertex model terms are BC-configurations. The most general scalar product that we can consider is the generic scalar product between two generic Bethe states

$$S_{generic}\big(\{u\},\{v\}\big) = \langle 0| \prod_{j=1}^{N} \mathbb{C}(v_j) \prod_{j=1}^{N} \mathbb{B}(u_j)|0\rangle \tag{61}$$

A computationally tractable evaluation of $S_{generic}(\{u\},\{v\})$ using the commutation relations of BA operators is known [46]. Simpler expressions are obtained when the auxiliary rapidities of one (or both) states satisfies Bethe equations. The result in this case is a determinant. When only one set satisfies Bethe equations, one obtains a Slavnov scalar product. This was discussed in Sect. 3.

5.7 Type-A. An Unevaluated Expression

The above three steps lead to the following preliminary, unevaluated expression

$$c_{123}^{(0)} = \mathcal{N}_{123} \sum\nolimits_{r} \langle \mathcal{O}_3|\mathcal{O}_1\rangle_{lr} \langle \mathcal{O}_1|\mathcal{O}_2\rangle_{lr} \langle \mathcal{O}_2|\mathcal{O}_3\rangle_{l} \tag{62}$$

where the normalization factor $\mathcal{N}_{123}$, that turns out to be a non-trivial object that depends on the norms of the Bethe eigenstates, is

$$\mathcal{N}_{123} = \sqrt{\frac{L_1 L_2 L_3}{\mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3}} \tag{63}$$

In (63), L_i is the number of sites in the closed spin chain that represents state $\mathcal{O}_i$. $\mathcal{N}_i$ is the Gaudin norm of state $\mathcal{O}_i$ as in (50). The sum in (62) is to be understood as follows. 1. It is a sum over all possible ways to split the sites of each closed spin chain (represented as a segment in a 1-dimensional lattice) into a left part and a right part. We will see shortly that only one term in this sum survives. 2. It is a sum over all possible ways of partitioning the X or $\bar{X}$ content of a spin chain state between the two parts that spin chain was split into. We will see shortly that only one sum survives.

5.8 Type-A. Simplifying the Unevaluated Expression

Wick contracting single-trace operators, we can only contract a fundamental field with its conjugate. Given the assignments in Table 1, one can see that (1) *All* Z fields in $\mathcal{O}_3$ must contract with $\bar{Z}$ fields in $\mathcal{O}_2$. The reason is that there are $\bar{Z}$ fields only in $\mathcal{O}_2$, and none in $\mathcal{O}_1$. (2) All $\bar{X}$ fields in $\mathcal{O}_3$ contract with X fields in $\mathcal{O}_1$. The reason is that there are X fields only in $\mathcal{O}_1$, and none in $\mathcal{O}_2$. If the total number of scalar fields in $\mathcal{O}_i$ is L_i, and the number of $\{X, \bar{X}\}$-type scalar fields is N_i, then

$$l_{13} = N_3, \qquad l_{23} = L_3 - N_3, \qquad l_{12} = L_1 - N_3 \tag{64}$$

and we have the constraint

$$N_1 = N_2 + N_3 \tag{65}$$

From (64) and (65), we have the following 4 simplifications. 1. There is only one way to split each lattice configuration that represents a spin chain into a left part and a right part, 2. The scalar product ${}_r\langle\mathcal{O}_2|\mathcal{O}_3\rangle_l$ involves the fundamental field Z (and only Z) in the initial state $|\mathcal{O}_3\rangle_l$ as well as in the final state ${}_r\langle\mathcal{O}_2|$. Using Table 1, we find that these states translate to an initial and a final spin-up reference state, respectively. This is represented in Fig. 12 by the fact that no connecting lines (that stand for propagators of $\{X, \bar{X}\}$ states) connect $\mathcal{O}_2$ and $\mathcal{O}_3$. The scalar product of the two reference states is ${}_r\langle\mathcal{O}_2|\mathcal{O}_3\rangle_l = 1$, 3. The scalar product ${}_r\langle\mathcal{O}_3|\mathcal{O}_1\rangle_l$ involves the fundamental fields X (and only X) in the initial state $|\mathcal{O}_1\rangle_l$ as well as in the final state ${}_r\langle\mathcal{O}_3|$. Using Table 1, we find that these states translate to an initial spin-up and a final spin-down reference state, respectively. This is represented in Fig. 12 by the high density of connecting lines (that stand for propagators of $\{X, \bar{X}\}$ states) between $\mathcal{O}_1$ and $\mathcal{O}_3$. This scalar product is straightforward to evaluate in terms of the domain wall partition function, 4. In the remaining scalar product ${}_r\langle\mathcal{O}_1|\mathcal{O}_2\rangle_l$, both the initial state $|\mathcal{O}_2\rangle_l$ and the final state ${}_r\langle\mathcal{O}_1|$ involve $\{\bar{X}, \bar{Z}\}$. These fields translate to up and down spin states and the scalar product is generic. Using the BA commutation relations, it can be evaluated as a weighted sum [44].

5.9 Type-A. Evaluating the Expression

The idea of [29] is to identify the expression in (62), up to simple factors, with the partition function of an $[L_1, N_1, N_2]$-configuration. Since this partition function is a restricted scalar product $S[L_1, N_1, N_2]$, it can be evaluated as a determinant. This is achieved in two steps.

5.9.1 Step 1. Re-writing One of the Scalar Products

We use the facts that (1) ${}_r\langle\mathcal{O}_2|\mathcal{O}_3\rangle_l = 1$, and (2) ${}_r\langle\mathcal{O}_2|\mathcal{O}_1\rangle_l = {}_l\langle\mathcal{O}_1|\mathcal{O}_2\rangle_r$, which is true for all scalar products, to re-write (62) in the form

$$\begin{aligned} c_{123}^{(0)} &= \mathcal{N}_{123} \sum_{\alpha\cup\bar{\alpha}=\{u_\beta\}_{N_1}} {}_r\langle\mathcal{O}_3|\mathcal{O}_1\rangle_l \; {}_l\langle\mathcal{O}_2|\mathcal{O}_1\rangle_r \\ &= \mathcal{N}_{123}\big({}_r\langle\mathcal{O}_3| \otimes {}_l\langle\mathcal{O}_2|\big)|\mathcal{O}_1\rangle \end{aligned} \tag{66}$$

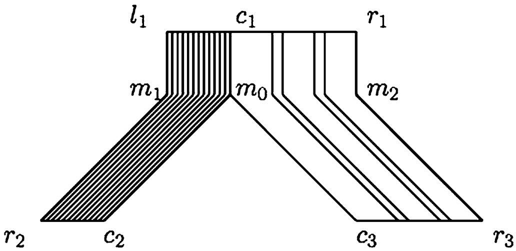

Fig. 13 A schematic representation of a 3-point function after removal of a contraction between the left part of $\mathcal{O}_2$ and the right part of $\mathcal{O}_3$, that evaluates to a factor of 1. Type-**B** 3-point functions are in this '*narrow pants*' form from the outset

where the right hand side of (66) is a scalar product of the full initials tate $|\mathcal{O}_1\rangle$ (so we no longer have a sum over partitions of the rapidities $\{u_\beta\}_{N_1}$ since we no longer split the state $\mathcal{O}_1$) and two states that are pieces of original states that were split. Deleting the scalar product corresponding to contracting the left part of state $\mathcal{O}_2$ with the right part of state $\mathcal{O}_3$, since that contraction leads to a factor of unity, the object that we are evaluating can be schematically drawn as in Fig. 13.

This right hand side is identical to an $[L_1, N_1, N_2]$-configuration, apart from the fact that it includes an $(N_3 \times N_3)$-domain wall configuration, that corresponds to the spin-down reference state contribution of ${}_r\langle N_3{}^{\vee}|$, that is not included in an $[L_1, N_1, N_2]$-configuration.

5.9.2 Step 2. The Domain Wall Partition Functions

Accounting for the domain wall partition function, and working in the homogeneous limit where all quantum rapidities are set to $z = \frac{1}{2}\sqrt{-1}$, we obtain our result for the structure constants, which up to a factor, is in determinant form.

$$\boxed{\begin{aligned} c_{123}^{(0)} &= \mathcal{N}_{123}\ Z_{N_3}^{hom}\left(\{w\}_{N_3}, \frac{1}{2}\sqrt{-1}\right) \\ &\quad \times S^{hom}[L_1, N_1, N_2]\left(\{u_\beta\}_{N_1}, \{v\}_{N_2}, \frac{1}{2}\sqrt{-1}\right) \end{aligned}} \tag{67}$$

where the normalization $\mathcal{N}_{123}$ is defined in (63), the $(N_3 \times N_3)$ domain wall partition function $Z_{N_3}^{hom}(\{w\}_{N_3}, \frac{1}{2}\sqrt{-1})$ is given in (54). The term $S^{hom}[L_1, N_1, N_2]$ $(\{u_\beta\}_{N_1}, \{v\}_{N_2}, \frac{1}{2}\sqrt{-1})$ is an $(N_1 \times N_1)$ determinant expression of the partition function of an $[L_1, N_1, N_2]$-configuration, given in (47). The auxiliary rapidities $\{u\}$, $\{v\}$ and $\{w\}$ are those of the eigenstates $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ in [26–28], respectively. Notice that $\{v\}$ and $\{w\}$ are actually $\{v\}_\beta$ and $\{w\}_\beta$, that is, they satisfy Bethe equations, but this fact is not used. In six-vertex model terms, the object that we are evaluating is drawn in Fig. 14.

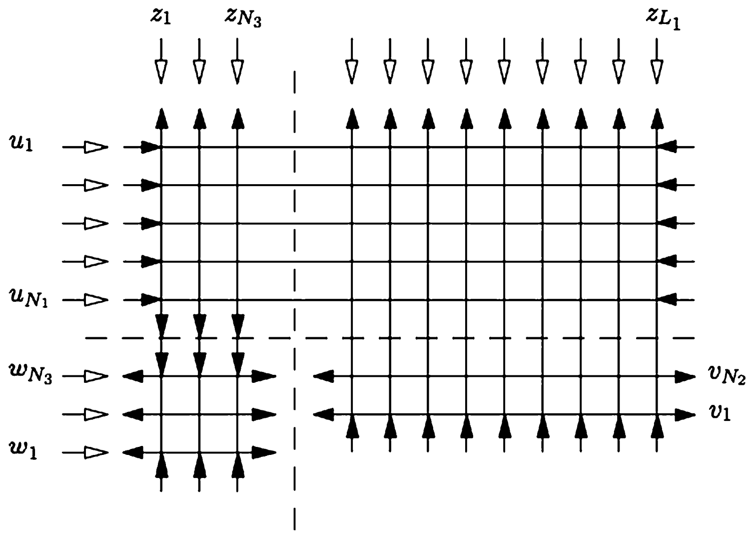

Fig. 14 The six-vertex lattice configuration that corresponds, up to a normalization factor $\mathcal{N}_{123}$, to the structure constant $c_{123}^{(0)}$

5.10 Type-A Specializations

Equation (67) is quite general. To obtain an expression specific to a certain Type-**A** theory, we need to use the values of the spin-chain parameters appropriate to that theory, as were given in Subsect. 1.7. All Type-**A** theories map to XXX spin-$\frac{1}{2}$ chains, hence the anisotropy parameter $\Delta = 1$, but with different values for the twist parameter θ. Theory 1 is SYM_4 and $\theta = 0$. Theory 2 is SYM_4^M is an Abelian orbifold version of SYM_4 and $\theta = \frac{2\pi}{M}$. Theory 3 is a real-β-deformed version of SYM_4 and $\theta = \beta$.

6 Structure Constants in Type-B Theories

In this section, we consider structure constants in Type-**B** theories. Our approach is parallel to that used in Type-**A**. The difference is that each Type-**B** theory has only one doublet, and therefore requires a slightly modified treatment.[18]

In type-**A** theories, the left part of $\mathcal{O}_2$ gets trivially contracted with the right part of $\mathcal{O}_3$, and the pants diagram is reduced to the '*narrow pants* diagram' in Fig. 13. As we will see, the starting point in the case of Type-**B** theories is a 'narrow pants' diagram.

[18]The conclusion that, in order to obtain a determinant formula, one of the single-trace operators should be BPS-like, was obtained in discussions with C. Ahn and R. Nepomechie.

Table 2 Identification of Type-**B** operator content of $\mathcal{O}_i$, $i \in \{1,2,3\}$ with initial and final spin-chain states

Operator	$\binom{1}{0}$	$\binom{0}{1}$	$(1 \quad 0)$	$(0 \quad 1)$
$\mathcal{O}_1$	ζ	$\bar{\zeta}$	$\bar{\zeta}$	ζ
$\mathcal{O}_2$	$\bar{\zeta}$	ζ	ζ	$\bar{\zeta}$
$\mathcal{O}_3$	$\bar{\zeta}$	ζ	ζ	$\bar{\zeta}$

This implies that in Type-**B** theories $\mathcal{O}_3$ must be chosen to be a BPS-like state, with *one* type of fundamental field in the composite operator $\mathcal{O}_3$. On the other hand, since the missing contraction (that between the left part of $\mathcal{O}_2$ and the right part of $\mathcal{O}_3$) was trivial for Type-**A** theories, the final result remains the same.

6.1 Type-B. Fundamental field content of the states

As in Type-**A**, we consider single-trace operators in an $SU(2)$ sector of a 1-loop conformally-invariant gauge theory, that is $\mathrm{Tr}(f_1 f_2 f_3 \cdots)$, where $f_i \in \{u, d\}$ is a fundamental field that belongs to an $SU(2)$ doublet.

The new feature in Type-**B** theories is that we have only one doublet to work with. The doublets relevant to Type-**B** theories were given in Subsect. 1.7. Theory 4 is pure gauge SYM_2, and the doublet consists of the gluino and its conjugate $\{\lambda, \bar{\lambda}\}$. Theory 5 is pure gauge SYM_1, and the doublet consists of the complex scalar and its conjugate $\{\phi, \bar{\phi}\}$. Theory 6 is pure QCD and the doublet consists of the light cone derivative of the gauge field component A and its conjugate $\bar{A}$, that is, $\{\partial_+ A, \partial_+ \bar{A}\}$. In the following, we deal with all three theories in one go, using the notation $\{\zeta, \bar{\zeta}\}$ for a generic single doublet.

Since we have only one doublet to construct composite operators from, we identify the fundamental field content of $\mathcal{O}_i$, $i \in \{1,2,3\}$ with spin-chain spin states as shown in Table 2.

Once again, in our conventions

$$\langle \bar{\zeta} \zeta \rangle = \langle \zeta | \zeta \rangle = 1, \qquad \langle \zeta \zeta \rangle = \langle \bar{\zeta} | \zeta \rangle = 0 \tag{68}$$

From Table 2, one can read the fundamental-scalar operator content of each single-trace operator $\mathcal{O}_i$, $i \in \{1,2,3\}$, when it is an initial state and when it is a final state.

6.2 Similarities Between Type-A and Type-B Theories

Steps 1, 2 and 3 from the EGSV construction of the structure constants apply unchanged to Type-**B** theories. In other words, (1) The splitting of each lattice, (2) The flipping procedure, and (3) The contraction of left and right halves to form scalar products, are replicated in the case of Type-**B** theories. Therefore we see that Eq. (62) continues to hold, and we assume that as our starting point.

6.3 Differences Between Type-A and Type-B Theories

1. In the case of Type-**A** theories, $\mathcal{O}_3$ contains Z fields that can only contract with $\bar{Z}$ fields in $\mathcal{O}_2$. This is because there are no fields that they can contract with in $\mathcal{O}_1$. This trivializes the ${}_l\langle \mathcal{O}_2 | \mathcal{O}_3 \rangle_r$ scalar product.

This is not the case in Type-**B** theories, where we have only a single doublet that must be used to populate all three states $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$. Because of that, one can see that if there is a contraction between $\mathcal{O}_2$ and $\mathcal{O}_3$, it is in general non-trivial. This is sufficient to prevent us from duplicating our Type-**A** arguments in the case of Type-**B** theories. In fact, there is yet another difference.

2. In the case of Type-**A** theories, $\mathcal{O}_3$ contains $\bar{X}$ fields that can contract only with X fields in $\mathcal{O}_1$. The reason is that there are no X fields in $\mathcal{O}_2$. This trivializes the scalar product that involves the left part of $\mathcal{O}_1$ and the right part of $\mathcal{O}_3$, leading to a domain wall partition function.

Once again, in the case of Type-**B** theories, the above trivial contraction is no longer the case, and contractions between $\mathcal{O}_1$ and $\mathcal{O}_3$ are in general non-trivial.

6.4 One of the Operators Must Be BPS-Like

Because of the above reasons, *we cannot map the most general $SU(2)$ structure constants of Type-**B** operators onto a restricted Slavnov scalar product.* However, both problems are overcome if we take $\mathcal{O}_3$ to be BPS-like, that is, a single-trace operator of the form $\mathrm{Tr}[\bar{\zeta}\ \bar{\zeta} \cdots \bar{\zeta}]$. This means that we *demand* that $N_3 = L_3$, or equivalently, that $l_{23} = L_3 - N_3 = 0$. In other words, the fields in $\mathcal{O}_3$ are all of the same type $\bar{\zeta}$ (magnons) and they contract with a subset of the fields in $\mathcal{O}_1$, while there are no contractions between $\mathcal{O}_3$ and $\mathcal{O}_2$. From this, we conclude that the starting point of the Type-**B** structure constants that we can compute in determinant form is the ‘narrow pants’ diagram in Fig. 13.

But we know that the partition function of the lattice configuration corresponding to Fig. 13 is given by a restricted Slavnov scalar product. Therefore for Type-**B** structure constants for which $\mathcal{O}_3$ is BPS-like, that is $L_3 = N_3$, we obtain

$$c_{123}^{(0)} = \mathcal{N}_{123} Z_{N_3}^{hom}\left(\{w\}_{N_3}, \frac{1}{2}\sqrt{-1}\right) \times S^{hom}[L_1, N_1, N_2]\left(\{u_\beta\}_{N_1}, \{v\}_{N_2}, \frac{1}{2}\sqrt{-1}\right) \tag{69}$$

This is the same result as the Type-**A** case, but with the caveat that we are restricting our attention to the situation $L_3 = N_3$. As a result the Gaudin norm $\mathcal{N}_3$, which occurs in the normalization factor $\mathcal{N}_{123}$, is equal to the partition function of a BC-configuration with length-N_3 initial and final reference states, and N_3 B-lines and C-lines. As we commented in Subsect. 4.17, such a configuration factorizes into a product of domain wall partition functions. Hence we are able to cancel the factor

$Z^{hom}_{N_3}(\{w\}_{N_3}, \frac{1}{2}\sqrt{-1})$ in (69) at the expense of the factor $\sqrt{\mathcal{N}_3}$ in the denominator, and obtain the final expression

$$\boxed{c^{(0)}_{123} = \sqrt{\frac{L_1 L_2 L_3}{\mathcal{N}_1 \mathcal{N}_2}}\ S^{hom}[L_1, N_1, N_2]\left(\{u_\beta\}_{N_1}, \{v\}_{N_2}, \frac{1}{2}\sqrt{-1}\right)} \tag{70}$$

6.5 Type-B Specializations

As in the previous section, (69) is quite general. To obtain an expression specific to a certain Type-**B** theory, we need to use the values of the spin-chain parameters appropriate to that theory, as were given in Subsect. 1.7. All Type-**B** theories map to periodic XXZ spin-$\frac{1}{2}$ chains, hence the twist parameter $\theta = 0$, but with different values of the anisotropy parameter Δ. Theory 4 is pure SYM_2 and $\Delta = 3$ [7, 39]. Theory 5 is pure SYM_1 and $\Delta = \frac{1}{2}$ [7]. Theory 6 is pure gauge QCD and $\Delta = -\frac{11}{3}$ [7].

7 Discrete KP τ-Functions

In this section we closely follow [47], where it was shown that Slavnov's scalar product is a τ-function of the discrete KP hierarchy. The only differences in this work are (1) A more compact expression for the τ-function itself, see (99), (2) The inclusion of the twist parameter θ in the τ-function, and (3) A discussion of restricting the Miwa variables to the values of the quantum inhomogeneities.

7.1 Notation Related to Sets of Variables

We use $\{x\}$ for the set of finitely many variables $\{x_1, x_2, \ldots, x_N\}$, and $\{\widehat{x}_m\}$ for $\{x\}$ with the element x_m omitted. In the case of sets with a repeated variable x_i, we use the superscript (m_i) to indicate the multiplicity of x_i, as in $x_i^{(m_i)}$. For example, $\{x_1^{(3)}, x_2, x_3^{(2)}, x_4, \ldots\}$ is the same as $\{x_1, x_1, x_1, x_2, x_3, x_3, x_4, \ldots\}$ and $f\{\ldots, x_i^{(m_i)}, \ldots\}$ is equivalent to saying that f depends on m_i distinct variables all of which have the same value x_i. For simplicity, we use x_i to indicate $x_i^{(1)}$.

7.2 The Complete Symmetric Function $h_i\{x\}$

Let $\{x\}$ denote a set of N variables $\{x_1, x_2, \ldots, x_N\}$. The complete symmetric function $h_i\{x\}$ is the coefficient of k^i in the power series expansion

$$\prod_{i=1}^{N} \frac{1}{1 - x_i\, k} = \sum_{i=0}^{\infty} h_i\{x\}\, k^i \tag{71}$$

For example, $h_0\{x\} = 1$, $h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$, $h_2(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$, and $h_i\{x\} = 0$ for $i < 0$.

7.3 Useful Identities for $h_i\{x\}$

From (71), it is straightforward to show that

$$h_i\{x\} = h_i\{\widehat{x}_m\} + x_m h_{i-1}\{x\} \tag{72}$$

Then from (72) one obtains

$$(x_m - x_n)h_{i-1}\{x\} = h_i\{\widehat{x}_n\} - h_i\{\widehat{x}_m\} \tag{73}$$

$$(x_m - x_n)h_i\{x\} = x_m h_i\{\widehat{x}_n\} - x_n h_i\{\widehat{x}_m\} \tag{74}$$

7.4 Discrete Derivatives

The discrete derivative $\Delta_m h_i\{x\}$ of $h_i\{x\}$ with respect to any one variable $x_m \in \{x\}$ is defined using (72) as

$$\Delta_m h_i\{x\} = \frac{h_i\{x\} - h_i\{\widehat{x}_m\}}{x_m} = h_{i-1}\{x\} \tag{75}$$

Note that the effect of applying Δ_m to $h_i\{x\}$ is a complete symmetric function $h_{i-1}\{x\}$ of degree $i - 1$ in the same set of variables $\{x\}$.

7.5 The Discrete KP Hierarchy

Discrete KP is an infinite hierarchy of integrable partial *difference* equations in an infinite set of continuous Miwa variables $\{x\}$, where time evolution is obtained by changing the multiplicities $\{m\}$ of these variables. In this work, we are interested in the situation where the total number of continuous Miwa variables is finite, which corresponds to setting to zero all continuous Miwa variables apart from

$\{x_1, \ldots, x_N\}$. In this case, the discrete KP hierarchy can be written in bilinear form as the $n \times n$ determinant equations

$$\det \begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-2} & x_1^{n-2}\tau_{+1}\{x\}\tau_{-1}\{x\} \\ 1 & x_2 & \cdots & x_2^{n-2} & x_2^{n-2}\tau_{+2}\{x\}\tau_{-2}\{x\} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} & x_n^{n-2}\tau_{+n}\{x\}\tau_{-n}\{x\} \end{pmatrix} = 0 \tag{76}$$

where $3 \leq n \leq N$, and

$$\begin{aligned} \tau_{+i}\{x\} &= \tau\left\{x_1^{(m_1)}, \ldots, x_i^{(m_i+1)}, \ldots, x_N^{(m_N)}\right\} \\ \tau_{-i}\{x\} &= \tau\left\{x_1^{(m_1+1)}, \ldots, x_i^{(m_i)}, \ldots, x_N^{(m_N+1)}\right\} \end{aligned} \tag{77}$$

In other words, if $\tau\{x\}$ has m_i copies of the variable x_i, then $\tau_{+i}\{x\}$ has $m_i + 1$ copies of x_i and the multiplicities of all other variables remain the same, while $\tau_{-i}\{x\}$ has one more copy of each variable except x_i. Equivalently, one can use the simpler notation

$$\begin{aligned} \tau_{+i}\{x\} &= \tau\left\{m_1, \ldots, (m_i + 1), \ldots, m_N\right\} \\ \tau_{-i}\{x\} &= \tau\left\{(m_1 + 1), \ldots, m_i, \ldots, (m_N + 1)\right\} \end{aligned} \tag{78}$$

The simplest discrete KP bilinear difference equation, in the notation of (78), is

$$\begin{aligned} & x_i(x_j - x_k)\tau\{m_i + 1, m_j, m_k\}\tau\{m_i, m_j + 1, m_k + 1\} \\ & \quad + x_j(x_k - x_i)\tau\{m_i, m_j + 1, m_k\}\tau\{m_i + 1, m_j, m_k + 1\} \\ & \quad + x_k(x_i - x_j)\tau\{m_i, m_j, m_k + 1\}\tau\{m_i + 1, m_j + 1, m_k\} = 0 \end{aligned} \tag{79}$$

where $\{x_i, x_j, x_k\} \in \{x\}$ and $\{m_i, m_j, m_k\} \in \{m\}$ are any two (corresponding) triples in the sets of continuous and discrete (integral valued) Miwa variables. Equation (79) is the discrete analogue of the KP equation in continuous time variables.

7.6 Casoratian Matrices and Determinants

A Casoratian matrix Ω of the type that appears in this paper is such that its matrix elements ω_{ij} satisfy

$$\omega_{i,j+1}\{x\} = \Delta_m \omega_{ij}\{x\} \tag{80}$$

where the discrete derivative Δ_m is taken with respect to any one variable $x_m \in \{x\}$ (it is redundant to specify which variable, since $\omega_{ij}\{x\}$ is symmetric in $\{x\}$). From the definition of the discrete derivative Δ_m, it is clear that the entries of Casoratian matrices satisfy

$$\begin{aligned}\omega_{ij}&\left\{x_1,\ldots,x_m^{(2)},\ldots,x_N\right\}\\&=\omega_{ij}\{x_1,\ldots,x_N\}+x_m\omega_{i,j+1}\left\{x_1,\ldots,x_m^{(2)},\ldots,x_N\right\}\end{aligned}\tag{81}$$

which, in turn, gives rise to the identity

$$\begin{aligned}(x_r-x_s)&\omega_{ij}\left\{x_1,\ldots,x_r^{(2)},\ldots,x_s^{(2)},\ldots x_N\right\}\\&=x_r\omega_{ij}\left\{x_1,\ldots,x_r^{(2)},\ldots,x_N\right\}-x_s\ \omega_{ij}\left\{x_1,\ldots,x_s^{(2)},\ldots,x_N\right\}\end{aligned}\tag{82}$$

If Ω is a Casoratian matrix, then $\det\Omega$ is a Casoratian determinant. Casoratian determinants are discrete analogues of Wronskian determinants.

7.7 Notation for Column Vectors with Elements $\boldsymbol{\omega}_{ij}$

We need the column vector

$$\boldsymbol{\omega}_j=\begin{pmatrix}\omega_{1j}\{x_1^{(m_1)},\ldots,x_N^{(m_N)}\}\\\omega_{2j}\{x_1^{(m_1)},\ldots,x_N^{(m_N)}\}\\\vdots\\\omega_{Nj}\{x_1^{(m_1)},\ldots,x_N^{(m_N)}\}\end{pmatrix}\tag{83}$$

and write

$$\boldsymbol{\omega}_j^{[k_1,\ldots,k_n]}=\begin{pmatrix}\omega_{1j}\{x_1^{(m_1)},\ldots,x_{k_1}^{(m_{k_1}+1)},\ldots,x_{k_n}^{(m_{k_n}+1)},\ldots,x_N^{(m_N)}\}\\\omega_{2j}\{x_1^{(m_1)},\ldots,x_{k_1}^{(m_{k_1}+1)},\ldots,x_{k_n}^{(m_{k_n}+1)},\ldots,x_N^{(m_N)}\}\\\vdots\\\omega_{Nj}\{x_1^{(m_1)},\ldots,x_{k_1}^{(m_{k_1}+1)},\ldots,x_{k_n}^{(m_{k_n}+1)},\ldots,x_N^{(m_N)}\}\end{pmatrix}\tag{84}$$

for the corresponding column vector where the multiplicities of the variables $x_{k_1},\ldots,x_{k_n}$ are increased by 1.

7.8 Notation for Determinants with Elements $\boldsymbol{\omega}_{ij}$

We also need the determinant

$$\tau=\det(\boldsymbol{\omega}_1\quad\boldsymbol{\omega}_2\quad\cdots\quad\boldsymbol{\omega}_N)=|\boldsymbol{\omega}_1\quad\boldsymbol{\omega}_2\quad\cdots\quad\boldsymbol{\omega}_N|\tag{85}$$

and the notation

$$\tau^{[k_1,\ldots,k_n]} = \left| \boldsymbol{\omega}_1^{[k_1,\ldots,k_n]} \quad \boldsymbol{\omega}_2^{[k_1,\ldots,k_n]} \quad \cdots \quad \boldsymbol{\omega}_N^{[k_1,\ldots,k_n]} \right| \tag{86}$$

for the determinant with shifted multiplicities.

7.9 Identities Satisfied by Casoratian Determinants

Two identities, which are needed in the sequel, are

$$x_1^{n-2}\tau^{[1]} = \left| \boldsymbol{\omega}_1 \quad \boldsymbol{\omega}_2 \quad \cdots \quad \boldsymbol{\omega}_{N-1} \quad \boldsymbol{\omega}_{N-n+2}^{[1]} \right| \tag{87}$$

$$\prod_{1\le r<s\le n} (x_r - x_s)\tau^{[1,\ldots,n]} = \left| \boldsymbol{\omega}_1 \quad \ldots \quad \boldsymbol{\omega}_{N-n} \quad \boldsymbol{\omega}_{N-n+1}^{[n]} \quad \boldsymbol{\omega}_{N-n+1}^{[n-1]} \quad \ldots \quad \boldsymbol{\omega}_{N-n+1}^{[1]} \right| \tag{88}$$

These identities may be proved by using the (81) and (82) to perform column operations in the determinant expressions for $\tau^{[1]}$ and $\tau^{[1,\ldots,n]}$. To keep the exposition concise we do not present these proofs, but full details can be found in [47].

7.10 Casoratians Are Discrete KP τ-Functions

Following [48], consider the $2N \times 2N$ determinant

$$\det \begin{pmatrix} \boldsymbol{\omega}_1 & \cdots & \boldsymbol{\omega}_{N-1} & \boldsymbol{\omega}_{N-n+2}^{[1]} & 0_1 & \cdots & 0_{N-n+1} & \boldsymbol{\omega}_{N-n+2}^{[n]} \cdots \boldsymbol{\omega}_{N-n+2}^{[2]} \\ 0_1 & \cdots & 0_{N-1} & \boldsymbol{\omega}_{N-n+2}^{[1]} & \boldsymbol{\omega}_1 & \cdots & \boldsymbol{\omega}_{N-n+1} & \boldsymbol{\omega}_{N-n+2}^{[n]} \cdots \boldsymbol{\omega}_{N-n+2}^{[2]} \end{pmatrix} = 0 \tag{89}$$

which is identically zero. For notational clarity, we have used subscripts to label the position of columns of zeros. Performing a Laplace expansion of the left hand side of (89) in $N \times N$ minors along the top $N \times 2N$ block, we obtain

$$\sum_{k=1}^{n} (-)^{k-1} \left| \boldsymbol{\omega}_1 \cdots \boldsymbol{\omega}_{N-1} \boldsymbol{\omega}_{N-n+2}^{[k]} \right| \times \left| \boldsymbol{\omega}_1 \cdots \boldsymbol{\omega}_{N-n+1} \boldsymbol{\omega}_{N-n+2}^{[n]} \cdots \boldsymbol{\omega}_{N-n+2}^{[k+1]} \boldsymbol{\omega}_{N-n+2}^{[k-1]} \cdots \boldsymbol{\omega}_{N-n+2}^{[1]} \right| = 0 \tag{90}$$

By virtue of (87) and (88), (90) becomes

$$\sum_{k=1}^{n} (-)^{k-1} x_k^{n-2}\, \tau^{[k]} \prod_{\substack{1\le r<s\le n \\ r,s\neq k}} (x_r - x_s) \tau^{[1,\ldots\hat{k}\ldots,n]} = 0 \tag{91}$$

Using the Vandermonde determinant identity

$$\det\begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-2} \\ \vdots & \vdots & & \vdots \\ \langle 1 & x_k & \cdots & x_k^{n-2}\rangle \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} \end{pmatrix} = \prod_{\substack{1\le r<s\le n \\ r,s\neq k}} (x_r - x_s) \tag{92}$$

with $\langle 1\ x_k\ \cdots\ x_k^{n-2}\rangle$ denoting the omission of the k-th row of the matrix, we recognize (91) as the cofactor expansion of the determinant in (76) along its last column. Hence we conclude that Casoratian determinants satisfy the bilinear difference equations of discrete KP.

7.11 Change of Variables

To interpret the Slavnov determinant (45) as a τ-function of discrete KP in the sense described above, it is necessary to adopt a change of variables as follows

$$\left\{e^{-2v_i}, e^{2u_{\beta_i}}, e^{2z_i}, e^{2\eta}\right\} \to \{x_i, y_i, z_i, q\} \tag{93}$$

In other words, our new variables (of which $\{x_1, \ldots, x_N\}$ end up being the continuous Miwa variables of discrete KP) are expressed as exponentials of the original variables. Furthermore, we consider a new normalization of the scalar product, given by

$$\begin{aligned} &\mathbb{S}[L, N, N] \\ &= e^{N^2\eta} \prod_{i=1}^{N} e^{(L-1)(u_{\beta_i} - v_i)} \prod_{i=1}^{L} e^{2N z_i} \prod_{j=1}^{N}\prod_{k=1}^{L} [v_j - z_k][u_{\beta_j} - z_k] S[L, N, N] \end{aligned} \tag{94}$$

Applying this normalization to (45), performing trivial rearrangements within the determinant and making the change of variables as prescribed by (93), we obtain

$$\begin{aligned} \mathbb{S}[L, N, N] = &\frac{(q-1)^N \prod_{i=1}^{N}\prod_{j=1}^{L}(y_i - z_j)}{\prod_{1\le i<j\le N}(x_i - x_j)(y_i - y_j)} \\ &\times \det\left(\frac{e^{-2i\theta} q^{N-1} \prod_{k\neq i}^{N}(1 - x_j \frac{y_k}{q}) \prod_{k=1}^{L}(1 - x_j z_k)}{1 - x_j y_i}\right. \\ &\left. - \frac{q^{\frac{L}{2}} \prod_{k\neq i}^{N}(1 - q x_j y_k) \prod_{k=1}^{L}(1 - x_j \frac{z_k}{q})}{1 - x_j y_i}\right)_{1\le i,j\le N} \end{aligned} \tag{95}$$

Our goal is to show that $\mathbb{S}[L, N, N]$ has the form of a Casoratian determinant, where the discrete derivative is taken with respect to the variables $\{x_1, \dots, x_N\}$.

7.12 Removing the Pole in the Slavnov Scalar Product

For all $1 \le i \le N$, define the function γ_i as

$$\gamma_i = e^{-2i\theta} q^{N-1} \prod_{j\neq i}^{N} \left(1 - \frac{y_j}{q y_i}\right) \prod_{j=1}^{L} \left(1 - \frac{z_j}{y_i}\right) - q^{\frac{L}{2}} \prod_{j\neq i}^{N} \left(1 - \frac{q y_j}{y_i}\right) \prod_{j=1}^{L} \left(1 - \frac{z_j}{q y_i}\right) \tag{96}$$

These functions provide a convenient way of expressing the Bethe equations (35) under the change of variables (93), namely

$$\gamma_i = 0, \quad \text{for all } 1 \le i \le N. \tag{97}$$

Recalling that these equations are assumed to apply to the variables $\{y_1, \dots, y_N\}$, we see that the pole at $x_j = 1/y_i$ in the determinant of (95) can be removed. We omit the details here as they are mechanical, and state only the result of this calculation, which reads

$$\mathbb{S}[L, N, N] = \frac{(q-1)^N \prod_{i=1}^{N} \prod_{j=1}^{L} (y_i - z_j)}{\prod_{1\le i<j\le N} (x_i - x_j)(y_i - y_j)} \det\left(\sum_{k=0}^{L+N-2} \left[y_i^k \gamma_i \right]_+ x_j^k \right)_{1\le i,j\le N} \tag{98}$$

where $[y_i^k \gamma_i]_+$ denotes all terms in the Laurent expansion of $y_i^k \gamma_i$ which have non-negative degree in y_i.

7.13 The Slavnov Scalar Product is a Discrete KP τ-Function

Using identities (72) and (73) to perform elementary column operations in the determinant of (98), it is possible to remove the Vandermonde $\prod_{1\le i<j\le N}(x_i - x_j)$ from the denominator of this equation. This procedure is directly analogous to the proof of the Jacobi-Trudi identity for Schur functions [49]. The result obtained is

$$\mathbb{S}[L, N, N] = \frac{(q-1)^N \prod_{i=1}^{N} \prod_{j=1}^{L} (y_i - z_j)}{\prod_{1\le i<j\le N} (y_j - y_i)} \det\left(\sum_{k=0}^{L+N-2} \left[y_i^k \gamma_i \right]_+ h_{k-j+1}\{x\} \right)_{1\le i,j\le N} \tag{99}$$

Up to an overall multiplicative factor which does not depend on the variables $\{x\}$, the normalized scalar product $\mathbb{S}[L, N, N]$ is a determinant of the form $\det \Omega$, where the matrix Ω has entries ω_{ij} which satisfy

$$\omega_{i,j+1} = \Delta_m \omega_{i,j}, \qquad \omega_{i,1} = \sum_{k=0}^{L+N-2} \left[y_i^k \gamma_i\right]_+ h_k\{x\} \tag{100}$$

Hence $\mathbb{S}[L, N, N]$ has the form of a Casoratian determinant, making it a discrete KP τ-function in the variables $\{x\} = \{x_1, \ldots, x_N\}$.

7.14 Restrictions of $\mathbb{S}[L, N_1, N_1]$

Similarly to (94), we define a new normalization of the restricted scalar product $S[L, N_1, N_2]$ as follows

$$\begin{aligned}\mathbb{S}[L, N_1, N_2] = e^{N_1^2 \eta} \prod_{i=1}^{N_1} e^{(L-1)u_{\beta_i}} \prod_{i=1}^{N_2} e^{-(L-1)v_i} \prod_{i=1}^{L} e^{(N_1+N_2)z_i} \\ \times \prod_{j=1}^{N_2} \prod_{k=1}^{L} [v_j - z_k] \prod_{j=1}^{N_1} \prod_{k=1}^{L} [u_{\beta_j} - z_k] S[L, N_1, N_2]\end{aligned} \tag{101}$$

Normalizing both sides of (46) using (94) and (101), and working in terms of the variables introduced by (93), we obtain the result

$$\begin{aligned}\mathbb{S}[L, N_1, N_1]\Big|_{\substack{x_{N_1} = 1/z_1 \\ \vdots \\ x_{(N_2+1)} = 1/z_{N_3}}} \\ = (z_1 \ldots z_{N_3})^{1/2} \prod_{i=1}^{N_3} \prod_{j=1}^{L} \left(q^{1/2} - q^{-1/2} z_j/z_i\right) \mathbb{S}[L, N_1, N_2]\end{aligned} \tag{102}$$

Hence the function $\mathbb{S}[L, N_1, N_2]$ is (up to an overall multiplicative factor) a restriction of $\mathbb{S}[L, N_1, N_1]$, obtained by setting the variables $x_{N_1}, \ldots, x_{N_2+1}$ to the values $1/z_1, \ldots, 1/z_{N_3}$. Since $\mathbb{S}[L, N_1, N_1]$ is a discrete KP τ-function in the variables $\{x_1, \ldots, x_{N_1}\}$, it is clear that $\mathbb{S}[L, N_1, N_2]$ is also a τ-function in the unrestricted set of variables $\{x_1, \ldots, x_{N_2}\}$.

8 Summary and Comments

Following [29], we obtained determinant expressions for two types of structure constants. 1. Structure constants of non-extremal 3-point functions of single-trace non-BPS operators in the scalar sector of SYM_4 and two close variations on it (an Abelian orbifolding of SYM_4 and a real-β-deformation of it. The operators involved map to states in closed XXX spin-$\frac{1}{2}$ chains, that are periodic in the case of SYM_4, and twisted in the other two cases. 2. Structure constants of extremal 3-point functions of two non-BPS and one BPS single-trace operators in (not necessarily scalar, but spin-zero) sectors of pure gauge SYM_2, SYM_1 and QCD. The operators involved map to states in closed periodic XXZ spin-$\frac{1}{2}$ chains, with different values of the anisotropy parameter, as identified in [7, 39]. One of the operators must be BPS-like.

Our expressions are basically special cases of Slavnov's determinant for the scalar product of a Bethe eigenstate and a generic state in a (generally twisted) closed XXZ spin chain. Finally, following [47], we showed that all these determinants are discrete KP τ-functions, in the sense that they obey the Hirota-Miwa equations.

The study of 3-point functions is a continuing activity. In [52], a systematic study, using perturbation theory, of 3-point functions in planar SYM_4 at 1-loop level, involving scalar field operators up to length 5 is reported on. In [53, 54], quantum corrections to 3-point functions of the very same type studied in this work planar SYM_4 are studied using integrability. At 1-loop level, new algebraic structures are found that govern all 2-loop corrections to the mixing of the operators as well as automatically incorporate all 1-loop corrections to the tree-level computations.

In [55], operator product expansions of local single-trace operators composed of self-dual components of the field strength tensor in planar QCD are considered. Using methods that extend those used in this work to spin-1 chains, a determinant expression for certain tree-level structure constants that appear in the operator product expansion is obtained. More recently, in [56, 57], the classical limit of the determinant form of the structure constants that appear in this work, was obtained.

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Monodromy of Partial KZ Functors for Rational Cherednik Algebras

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Abstract We study the monodromy of the Bezrukavnikov-Etingof induction and restriction functors for rational Cherednik algebras of type $G(\ell, 1, n)$. We show that these produce an $\mathfrak{sl}$-categorification on the category $\mathcal{O}$'s for these algebras, and that, through the KZ-functor, it is compatible with a corresponding categorification on cyclotomic Hecke algebra representations.

1 Introduction

Shan has proved that the categories $\mathcal{O}_c(W_n)$ for rational Cherednik algebras of type $W_n = W(G(\ell, 1, n)) = \mathfrak{S}_n \ltimes (\mu_\ell)^n$ with n varying, together with decompositions of the parabolic induction and restriction functors of Bezrukavnikov-Etingof, provide a categorification of an integrable $\tilde{\mathfrak{sl}}_e$ Fock space representation $\mathcal{F}(\mathbf{m})$, [18]. The parameters $\mathbf{m} \in \mathbb{Z}^\ell$ and $e \in \mathbb{N} \cup \{\infty\}$ arise from the choice of parameters c for the rational Cherednik algebra. This categorification gives rise to a crystal structure on the set of irreducible rational Cherednik algebra representations that belong to category $\mathcal{O}_c$; it is isomorphic to the crystal introduced by Jimbo-Misra-Miwa-Okado, [11].

Works of many authors, including Kleshchev, Brundan, Lascoux-Leclerc-Thibon, Ariki, Grojnowski-Vazirani, Grojnowski and Chuang-Rouquier, show that the categories $\mathcal{H}_q(W_n)$-mod for Hecke algebras of type W_n with n varying, together with decompositions of the parabolic induction and restriction functors, provide a categorification of an irreducible integrable $\tilde{\mathfrak{sl}}_e$-representation $L(\Lambda)$, [6]. The weight Λ arises from the choice of parameters q for the Hecke algebra. This gives rise to a crystal structure on the set of irreducible Hecke algebra representations.

To Professor Jimbo, with admiration.

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The Fock space is substantially more interesting than the representation $L(\Lambda)$. It is not irreducible, and in fact has an infinite number of non-zero isotypic components. This reducibility reveals itself through distinct canonical bases one can define on $\mathcal{F}(\mathbf{m})$, each of which produces a corresponding crystal. Nonetheless for each $n \in \mathbb{N}$ there is an exact functor $\mathrm{KZ}_n : \mathcal{O}_c(W_n) \to \mathcal{H}_q(W_n)$-mod, [10], intertwining the parabolic induction and restriction functors for Cherednik algebras and Hecke algebras and which produces a compatibility between the corresponding crystals: the component of the Cherednik crystal containing the irreducible representation of $W_0 = \{1\}$ is isomorphic to the Hecke crystal.

In this paper we give another construction of a decomposition of the parabolic induction and restriction functors for the rational Cherednik algebra of type an arbitrary complex reflection group. Our construction uses the monodromy of these functors. Together with an appropriate transitivity result for restriction, we explain how these give rise to an $\tilde{\mathfrak{sl}}_e$-categorification and crystal structure on $\mathcal{F}(\mathbf{m})$ via $\mathcal{O}_c$. The structure of the proof of these last claims is as in [18]. We also show via a homotopy calculation that the decomposition we obtain is naturally isomorphic to the decomposition introduced in [18].

The only small novelty in our approach is that we do not make use of the double centralizer property of the KZ-functor. It is a fundamental and fruitful technique of [18] to use this property to extend results systematically from Hecke algebras to Cherednik algebras, obtaining in this way definitive results. Optimistically, however, we hope that using the monodromy of the induction and restriction functors alone may be helpful towards generalizations, for instance to Cherednik algebras of varieties with a finite group action, where less is known about the corresponding KZ-functor and where one may imagine branching rules for affine type B and D appear, amongst other things.

Our results were proved in the second half of 2008, and announced by the first author at the conference "Algebraic Lie Structures with Origins in Physics" at the Isaac Newton Institute in March 2009. It is important to record that although we mentioned then that we knew biadjointness of parabolic restriction and induction, our proof for that turned out to be incomplete. This is one of the most useful results in [18]; it is also proved by Losev without use of the KZ-functor, [13]. We use this result here, although it is not needed to obtain the crystal structure. Furthermore we want to say that the presentation of [18] has helped to simplify several of our arguments significantly.

The outline of the paper is as follows. We recall the restriction and induction functors in Sect. 2, in both the algebraic and holomorphic settings. In Sect. 3 we study the monodromy actions on restrictions of modules. Finally in Sect. 4 we specialize to the W_n case to define i-restriction and i-restriction, and to explain the categorification that then arises. We also check that this does indeed match up with Shan's original results.

2 Definitions and Notation

2.1 Rational Cherednik Algebras

Let $\mathfrak{h}$ be a finite dimensional vector space over $\mathbb{C}$ and $W < GL(\mathfrak{h})$ be a finite subgroup generated by complex reflections. Let $\mathcal{S}$ be the set of complex reflections in W and $\mathcal{A}$ be the corresponding set of reflecting hyperplanes. For each $s \in \mathcal{S}$, let $H_s = \ker(\alpha_s)$ denote the reflecting hyperplane of s, and define $\alpha_s^\vee \in \mathfrak{h}$ to be an element such that $\mathfrak{h} = H_s \oplus \mathbb{C}\alpha_s^\vee$ is the s-stable decomposition of $\mathfrak{h}$, normalized by $\langle \alpha_s, \alpha_s^\vee \rangle = 2$.

Let $c : \mathcal{S} \to \mathbb{C}$ be constant on W-conjugacy classes. The rational Cherednik algebra attached to W with parameter c is the quotient $H_c(W, \mathfrak{h})$ of $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes W$, the smash product of $\mathbb{C}W$ and the tensor algebra of $\mathfrak{h} \oplus \mathfrak{h}^*$, by the relations

$$\left[x, x'\right] = 0, \qquad \left[y, y'\right] = 0, \qquad [y, x] = \langle x, y\rangle - \sum_{s\in\mathcal{S}} c_s \langle \alpha_s, y\rangle \left\langle x, \alpha_s^\vee \right\rangle s,$$

for all $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$, [8, (1.15)].

There is a faithful representation of $H_c(W, \mathfrak{h})$ on $\mathbb{C}[\mathfrak{h}]$ where $\mathfrak{h}^*$ and W act naturally, and each $y \in \mathfrak{h}$ acts via the Dunkl operator

$$D_y := \partial_y + \sum_{s\in\mathcal{S}} \frac{2c_s}{1 - \det_{\mathfrak{h}^*}(s)} \frac{\alpha_s(y)}{\alpha_s}(s - 1)$$

where ∂_y is the partial derivative in the direction of y, [8, §4]. Let $\{x_i\}$ be a basis of $\mathfrak{h}^*$ and let $\{y_i\}$ be the dual basis. Then

$$\mathbf{eu} = \sum_i x_i y_i + \frac{\dim(\mathfrak{h})}{2} - \sum_{s\in\mathcal{S}} \frac{2c_s}{1 - \det_{\mathfrak{h}^*}(s)} s$$

is an analogue of the Euler element in $H_c(W, \mathfrak{h})$. We have

$$[\mathbf{eu}, x] = x, \qquad [\mathbf{eu}, y] = -y, \qquad [\mathbf{eu}, w] = 0,$$

for all $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$ and $w \in W$.

2.2 Centralizer Algebras and the Isomorphism of Bezrukavnikov-Etingof

Let $b \in \mathfrak{h}$ and $W_b \subset W$ the stabilizer of b, and let c_b denote the restriction of the function c to $\mathcal{S} \cap W_b$. We write $\mathbb{C}[\mathfrak{h}]^{\wedge b}$ for the completion of $\mathbb{C}[\mathfrak{h}]$ at b in $\mathfrak{h}$, and $\mathbb{C}[\mathfrak{h}]^{\wedge[b]}$ for the completion of $\mathbb{C}[\mathfrak{h}]$ at the W-orbit of b in $\mathfrak{h}$. For any $\mathbb{C}[\mathfrak{h}]$-module M let $M^{\wedge b} = \mathbb{C}[\mathfrak{h}]^{\wedge b} \otimes_{\mathbb{C}[\mathfrak{h}]} M$ and $M^{\wedge[b]} = \mathbb{C}[\mathfrak{h}]^{\wedge[b]} \otimes_{\mathbb{C}[\mathfrak{h}]} M$.

The completion $H_c(W,\mathfrak{h})^{\wedge[b]}$ can be identified with the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}]^{\wedge[b]})$ generated by $\mathbb{C}[\mathfrak{h}]^{\wedge[b]}$, the Dunkl operators D_y for $y \in \mathfrak{h}$, and the group W. Let $P = \mathrm{Fun}_{W_b}(W, H_{c_b}(W_b,\mathfrak{h})^{\wedge b})$ and $Z(W, W_b, H_{c_b}(W_b,\mathfrak{h})^{\wedge b})$ be the ring of endomorphisms of the right $H_{c_b}(W_b,\mathfrak{h})^{\wedge b}$-module P.

Theorem 1 ([3] Theorem 3.2) *For any $b \in \mathfrak{h}$ there is an isomorphism of algebras*

$$\Theta_b : H_c(W,\mathfrak{h})^{\wedge[b]} \longrightarrow Z\big(W, W_b, H_{c_b}(W_b,\mathfrak{h})^{\wedge b}\big)$$

defined as follows: *for $f \in P$, $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$, $u, w \in W$,*

$$\big(\Theta_b(u) f\big)(w) = f(wu),$$

$$\big(\Theta_b(x) f\big)(w) = w(x) f(w),$$

$$\big(\Theta_b(y) f\big)(w) = w(y) f(w) + \sum_{s \in \mathcal{S}\setminus W_b} \frac{2c_s}{1-\det_{\mathfrak{h}^*}(s)} \frac{\alpha_s(wy)}{x_{\alpha_s}} \big(f(sw) - f(w)\big).$$

Thus we can identify $H_c(W,\mathfrak{h})^{\wedge[b]}$-modules with $Z(W, W_b, H_{c_b}(W_b,\mathfrak{h})^{\wedge b})$-modules. A choice of decomposition of algebras $\mathbb{C}[\mathfrak{h}]^{\wedge[b]} \cong \bigoplus_{p \in Wb} \mathbb{C}[\mathfrak{h}]^{\wedge p}$ produces a non-canonical isomorphism of algebras

$$\Phi : Z\big(W, W_b, H_{c_b}(W_b,\mathfrak{h})^{\wedge b}\big) \to \mathrm{Mat}_{|W/W_b|}\big(H_{c_b}(W_b,\mathfrak{h})^{\wedge b}\big).$$

Let $x_b \in \mathbb{C}[\mathfrak{h}]^{\wedge[b]}$ be the idempotent corresponding to 1 under the inclusion $\mathbb{C}[\mathfrak{h}]^{\wedge b} \hookrightarrow \mathbb{C}[\mathfrak{h}]^{\wedge[b]}$. Then we can identify $H_{c_b}(W_b,\mathfrak{h})^{\wedge b}$ with $x_b H_c(W,\mathfrak{h})^{\wedge[b]} x_b$. If we denote by $\mathcal{O}_c(W,\mathfrak{h})^{\wedge[b]}$ the category of $H_c(W,\mathfrak{h})^{\wedge[b]}$-modules that are finitely generated over $\mathbb{C}[\mathfrak{h}]^{\wedge[b]}$, and similarly for $\mathcal{O}_{c_b}(W_b,\mathfrak{h})^{\wedge b}$, then there are quasi-inverse equivalences

$$J : \mathcal{O}_{c_b}(W_b,\mathfrak{h})^{\wedge b} \to \mathcal{O}_c(W,\mathfrak{h})^{\wedge[b]}, \quad M \mapsto H_c(W,\mathfrak{h})^{\wedge[b]} x_b \otimes_{H_{c_b}(W_b,\mathfrak{h})^{\wedge b}} M,$$

and

$$R : \mathcal{O}_c(W,\mathfrak{h})^{\wedge[b]} \to \mathcal{O}_{c_b}(W_b,\mathfrak{h})^{\wedge b}, \quad N \mapsto x_b N.$$

2.3 Category $\mathcal{O}$ and Parabolic Restriction and Induction

The standard reference for material on category $\mathcal{O}$ is [10]. Category $\mathcal{O}_c(W,\mathfrak{h})$ is the full subcategory of the category of $H_c(W,\mathfrak{h})$-modules consisting of objects that are finitely generated as $\mathbb{C}[\mathfrak{h}]$-modules and $\mathfrak{h}$-locally nilpotent. A module $M \in \mathcal{O}$ is $\mathfrak{h}$-locally nilpotent if and only if it is **eu**-locally finite, [3]. The category $\mathcal{O}_c(W,\mathfrak{h})$ is a highest weight category. Its standard modules are parametrized by $\mathrm{Irr}(W)$, the irreducible complex representations of W: for any $\lambda \in \mathrm{Irr}(W)$ set $\Delta(\lambda) = H_c(W,\mathfrak{h}) \otimes_{\mathbb{C}[\mathfrak{h}^*] \rtimes \mathbb{C}W} \lambda$ where $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$ acts by zero on λ. We denote the irreducible head of $\Delta(\lambda)$ by $L(\lambda)$.

Parabolic restriction and induction functors were defined in [3, §2.3]. Let $b \in \mathfrak{h}$ with stabilizer W_b and set $\mathfrak{h}_b = \mathfrak{h}/\mathfrak{h}^{W_b}$. There is an adjoint pair of exact functors $((\cdot)^{\wedge[b]}, E^b)$ defined by

$$(\cdot)^{\wedge[b]} : \mathcal{O}_c(W,\mathfrak{h}) \to \mathcal{O}_c(W,\mathfrak{h})^{\wedge[b]}, \quad M \mapsto M^{\wedge[b]},$$

and

$$E^b : \mathcal{O}_c(W,\mathfrak{h})^{\wedge[b]} \to \mathcal{O}_c(W,\mathfrak{h}), \quad N \mapsto N^{\mathsf{ln}},$$

where $N^{\mathsf{ln}} \subset N$ denotes the locally nilpotent part of N under the action of $\mathfrak{h}$. Consider also

$$E : \mathcal{O}_c(W_b,\mathfrak{h})^{\wedge b} \to \mathcal{O}_c(W_b,\mathfrak{h}), \quad N \mapsto N^{\mathsf{eu}},$$

where $N^{\mathbf{eu}} \subset N$ denotes the locally finite part of N under the action of **eu**.

There is an equivalence of categories

$$\zeta : \mathcal{O}_{c_b}(W_b,\mathfrak{h}) \to \mathcal{O}_{c_b}(W_b,\mathfrak{h}_b), \quad M \mapsto \left\{v \in M : \; yv = 0 \text{ for all } y \in \mathfrak{h}^{W_b}\right\}$$

with quasi-inverse

$$\zeta^{-1} : \mathcal{O}_{c_b}(W_b,\mathfrak{h}_b) \to \mathcal{O}_{c_b}(W_b,\mathfrak{h}), \quad N \mapsto N \otimes \mathbb{C}\left[\mathfrak{h}^{W_b}\right],$$

where $\mathbb{C}[\mathfrak{h}^{W_b}]$ is the polynomial representation of the ring of polynomial differential operators $\mathcal{D}_{\mathrm{pol}}(\mathfrak{h}^{W_b})$.

The parabolic restriction and induction functors are then defined as follows, [3, §3.5]:

$$\mathrm{Res}_b : \mathcal{O}_c(W,\mathfrak{h}) \to \mathcal{O}_{c_b}(W_b,\mathfrak{h}_b), \quad M \mapsto \zeta \circ E \circ R\left(M^{\wedge[b]}\right),$$

$$\mathrm{Ind}_b : \mathcal{O}_{c_b}(W_b,\mathfrak{h}_b) \to \mathcal{O}_c(W,\mathfrak{h}), \quad N \mapsto E^b \circ J\left(\zeta^{-1}(N)^{\wedge b}\right).$$

By [3, Theorem 3.10] Ind_b is right adjoint to Res_b.

2.4 Basechange

Let S denote the formal power series ring $\mathbb{C}[[\overline{c}_s - c_s]]_{s\in\mathcal{S}}$, where the $\overline{c}_s$ denote indeterminates such that $\overline{c}_s = \overline{c}_{s'}$ if s and s' are conjugate in W. We denote by $\mathfrak{m}$ the maximal ideal of S and let K be the quotient field of S. The rational Cherednik algebras over these base rings, defined at the formal parameters $\overline{c}_s$, are denoted by $H_S(W,\mathfrak{h})$ and $H_K(W,\mathfrak{h})$ respectively.

The definition of category $\mathcal{O}$ makes sense over K, and is denoted $\mathcal{O}_K(W,\mathfrak{h})$. This is a semisimple category whose simple objects are given by the standard modules $\Delta_K(\lambda)$ for $\lambda \in \mathrm{Irr}(W)$. We can also construct standard modules over S, $\Delta_S(\lambda)$, and we define $\mathcal{O}_S^{\Delta}(W,\mathfrak{h})$ to be the full subcategory of finitely generated $H_S(W,\mathfrak{h})$-modules that are free as $S[\mathfrak{h}]$-modules. This is motivated by [10, Proposition 2.21] which states that the objects of $\mathcal{O}_c^{\Delta}(W,\mathfrak{h})$, the full subcategory of $\mathcal{O}_c(W,\mathfrak{h})$ consisting of modules with filtrations by standard modules, are precisely those for which the action of $\mathbb{C}[\mathfrak{h}]$ is free.

All of the preceding constructions are well-defined over S and K. Thus we can define restriction functors $\mathrm{Res}_{b,S}$ and $\mathrm{Res}_{b,K}$.

Lemma 1 *The following diagram commutes*:

$$\begin{array}{ccccc} \mathcal{O}_{\mathrm{c}}^{\Delta}(W,\mathfrak{h}) & \longleftarrow & \mathcal{O}_S^{\Delta}(W,\mathfrak{h}) & \longrightarrow & \mathcal{O}_K(W,\mathfrak{h}) \\ \Big\downarrow{\scriptstyle \mathrm{Res}_b} & & \Big\downarrow{\scriptstyle \mathrm{Res}_{b,S}} & & \Big\downarrow{\scriptstyle \mathrm{Res}_{b,K}} \\ \mathcal{O}_{\mathrm{c_b}}^{\Delta}(W_b,\mathfrak{h}_b) & \longleftarrow & \mathcal{O}_S^{\Delta}(W_b,\mathfrak{h}_b) & \longrightarrow & \mathcal{O}_K(W_b,\mathfrak{h}_b), \end{array} \tag{1}$$

where the arrows to the left denote $-\otimes_S S/\mathfrak{m}$ *and arrows to the right denote* $-\otimes_S K$.

Proof Let $Z_S = Z(W, W_b, H_S(W_b,\mathfrak{h})^{\wedge_b})$ and $Z_K = Z(W, W_b, H_K(W_b,\mathfrak{h})^{\wedge_b})$. There is a commutative diagram of homomorphisms:

$$\begin{array}{ccccc} H_c(W,\mathfrak{h})^{\wedge[b]} & \longleftarrow & H_S(W,\mathfrak{h})^{\wedge[b]} & \longrightarrow & H_K(W,\mathfrak{h})^{\wedge[b]} \\ \Big\downarrow{\scriptstyle \Theta} & & \Big\downarrow{\scriptstyle \Theta_S} & & \Big\downarrow{\scriptstyle \Theta_K} \\ Z & \longleftarrow & Z_S & \longrightarrow & Z_K \end{array}$$

where the horizontal arrows denote the homomorphisms corresponding to the natural maps $S \to S/\mathfrak{m}$ and $S \to K$.

Let $M \in \mathcal{O}_S^{\Delta}(W,\mathfrak{h})$. We claim that the idempotent x_b lifts uniquely from Z to Z_S. Indeed, the isomorphism Φ is derived from a choice of decomposition $\mathbb{C}[\mathfrak{h}]^{\wedge[b]} \cong \prod_{p\in W\cdot b}\mathbb{C}[\mathfrak{h}]^{\wedge p}$. Let $\mathfrak{n} \subset \mathbb{C}[\mathfrak{h}]$ denote the ideal of functions vanishing on $W\cdot b$. Then by the unique lifting property for complete rings we can lift $(x_b)_n$ uniquely to $S[\mathfrak{h}]/(\mathfrak{n})^n$ along the homomorphism $S[\mathfrak{h}]/(\mathfrak{n})^n \to \mathbb{C}[\mathfrak{h}]/(\mathfrak{n})^n$, where $(x_b)_n$ denotes the image of x_b in $\mathbb{C}[\mathfrak{h}]^{\wedge[b]}/(\mathfrak{n})^n$. Therefore, x_b lifts uniquely to $S[\mathfrak{h}]^{\wedge[b]}$. The claim now follows. As a consequence of this lifting, we have $R_S(M^{\wedge[b]})\otimes_S S/\mathfrak{m} \cong R(M^{\wedge[b]}\otimes_S S/\mathfrak{m})$.

We show that the left hand square of (1) commutes by establishing that

$$\big(E_S\circ R_S\big(M^{\wedge[b]}\big)\big)\otimes_S S/\mathfrak{m} \cong E\circ R\big(M^{\wedge[b]}\big).$$

The module $N = R_S(M^{\wedge[b]})$ is a finitely generated $S[\mathfrak{h}]^{\wedge[b]}$-module. Let m be the ideal of positive degree formal power series, and let $\widetilde{\ }$ denote the completion with respect to m. By the proof of [3, Theorem 2.3], which is valid over S, we have $\widetilde{E_S(N)} = N$. In particular, there exists a set $n_1,\dots,n_l \in N$ consisting of generalized

eu-eigenvectors that generate N as an $S[\mathfrak{h}]^{\wedge[b]}$-module. Let us denote by $\overline{n}_i$ the image of n_i in $N \otimes_S S/\mathfrak{m}$. By [3, Theorem 2.3] again, $E_S(N) = S[\mathfrak{h}]n_1 + \cdots + S[\mathfrak{h}]n_l$ and so $E_S(N) \otimes_S S/\mathfrak{m} = \mathbb{C}[\mathfrak{h}]\overline{n}_1 + \cdots + \mathbb{C}[\mathfrak{h}]\overline{n}_l$. On the other hand, by the previous paragraph, $N \otimes_S S/\mathfrak{m} = \mathbb{C}[\mathfrak{h}]^{\wedge[b]}\overline{n}_1 + \cdots + \mathbb{C}[\mathfrak{h}]^{\wedge[b]}\overline{n}_l$, and so $E(N \otimes_S S/\mathfrak{m}) = \mathbb{C}[\mathfrak{h}]\overline{n}_1 + \cdots + \mathbb{C}[\mathfrak{h}]\overline{n}_l = E_S(N) \otimes_S \mathbb{C}$.

It remains to show that the right-hand square of (1) commutes. It is clear that

$$\big(R_S\big(M^{\wedge[b]}\big)\big) \otimes_S K \cong R_K\big(M^{\wedge[b]}\big).$$

By the arguments above, $N = R_S(M^{\wedge[b]})$ is generated by a finite set of generalized **eu**-eigenvectors. Therefore $\{n \in N \otimes_S K : n \text{ is locally finite for } \mathbf{eu}\}$ is the image of the set of **eu**-finite vectors in N by the functor $- \otimes_S K$. □

2.5 Holomorphic Version

Let X be a complex manifold and define $\mathcal{K}_X$ to be the sheaf of holomorphic functions on X. For $x \in X$, we denote by $\mathcal{K}_{X,x}$ the germs of holomorphic functions around x. We also denote by $\widehat{\mathcal{K}}_{X,x}$ the algebra of formal series in local coordinates around x. There is an injective algebra homomorphism $\mathcal{K}_{X,x} \to \widehat{\mathcal{K}}_{X,x}$, sending a germ to its Maclaurin series around x. Given a sheaf $\mathcal{M}$ of $\mathcal{K}_X$-modules, we denote its stalk at x by $\mathcal{M}_x$ and define $\mathcal{M}^{\wedge x}$ to be $\widehat{\mathcal{K}}_{X,x} \otimes_{\mathcal{K}_{X,x}} \mathcal{M}_x$.

We now consider $\mathfrak{h}$ with the complex topology. Let $U \subseteq \mathfrak{h}$ be a connected W-stable open subset containing b. Define $H_c(W)|_U$ to be the sheaf of algebras on U/W whose sections at a W-invariant open subset $V \subseteq U$ are the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{K}_U(V))$ generated by W, $\mathcal{K}_U(V)$ and the Dunkl operators D_y for $y \in \mathfrak{h}$. For a W-stable open subset $V \subseteq U$, let $H_c(W, V) = H_c(W)|_U(V)$.

Let $W' \subseteq W$ be a subgroup, not necessarily parabolic. Let $U \subseteq \mathfrak{h}$ be an W'-stable open subset such that

$$w \cdot U \cap U = \emptyset, \quad \text{for } w \in W \setminus W'.$$

Thus $W \cdot U = \bigsqcup_{w \in W/W'} w \cdot U$. Let c' denote the restriction of c to $\mathcal{S} \cap W'$. For each $y \in \mathfrak{h}$ define Dunkl operators

$$D'_y := \partial_y + \sum_{s \in \mathcal{S} \cap W'} \frac{2c_s}{1 - \det_{\mathfrak{h}^*}(s)} \frac{\alpha_s(y)}{\alpha_s}(s - 1). \tag{2}$$

As above, we can define a sheaf of algebras on U/W', which we denote $H_{c'}(W')|_U$. The endomorphisms of the sheaf $\mathrm{Fun}_{W'}(W, H_{c'}(W)|_U)$ form a sheaf of algebras $Z(W, W', H_{c'}(W')|_U)$.

Theorem 2 ([3]) *Let W' and U be as above. Then there is an isomorphism of sheaves of algebras*

$$\Theta_U : H_c(W)|_{W \cdot U} \longrightarrow Z\big(W, W', H_{c'}\big(W'\big)|_U\big),$$

which is given as follows. Let f be a section of $\mathrm{Fun}_{W'}(W, H_c(W)|_{W\cdot U})$, *then*

$$\begin{aligned}
\big(\Theta_U(u)f\big)(w) &= f(wu),\\
\big(\Theta_U(\phi)f\big)(w) &= \big(\phi^w|_V\big)f(w),\\
\big(\Theta_U(y)f\big)(w) &= w(y)f(w) + \sum_{s\in\mathcal{S}, s\notin W'} \frac{2c_s}{1-\det_{\mathfrak{h}^*}(s)}\frac{\alpha_s(wy)}{x_{\alpha_s}}\big(f(sw)-f(w)\big),
\end{aligned}$$

where $u, w \in W$, ϕ is a section of $\mathcal{K}_{W\cdot U}$ and $y \in \mathfrak{h}$.

Let $\mathcal{O}_c(W, W\cdot U)$ be the category of $H_c(W)|_{W\cdot U}$-modules that are coherent as $\mathcal{K}_{W\cdot U}$-modules. Letting 1_U play the analogous role to the element x_b defined in 2.2, there are quasi-inverse equivalences

$$J_U : \mathcal{O}_{c'}\big(W', U\big) \to \mathcal{O}_c(W, W\cdot U), \quad \mathcal{M} \mapsto H_c(W)_{W\cdot U}1_U \otimes_{H_{c'}(W')_U} \mathcal{M},$$

and

$$R_U : \mathcal{O}_c(W, W\cdot U) \to \mathcal{O}_{c'}\big(W', U\big), \quad \mathcal{N} \mapsto 1_U\mathcal{N}. \tag{3}$$

Let $\mathcal{M}$ be a sheaf on $V \subseteq \mathfrak{h}$, where V is an open set containing the orbit $W\cdot b$. Then we can define $\mathcal{M}^{\wedge[b]} = \bigoplus_{p\in W\cdot b}\mathcal{M}^{\wedge p}$.

Lemma 2 *Let $M \in \mathcal{O}_c(W, \mathfrak{h})$ and set $\mathcal{M} = \mathcal{K}_{\mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{h}]} M$. Let $U \subset \mathfrak{h}$ and $W' \subseteq W$ be as above, and assume that $b \in U$ has stabilizer W_b contained in W'. Then there are natural isomorphisms of $H_{c_b}(W_b, \mathfrak{h})^{\wedge b}$-modules*:

$$R\big(M^{\wedge[b]}\big) \cong R\big((\mathcal{M}|_{W\cdot U})^{\wedge[b]}\big) \cong R\big(\big(R_U(\mathcal{M}|_{W\cdot U})\big)^{\wedge W'\cdot b}\big).$$

Proof The first isomorphism is clear since $M^{\wedge[b]} \cong (\mathcal{M}|_{W\cdot U})^{\wedge[b]}$. For the second isomorphism we can find a connected W_b-stable open subset V of U containing b such that $w\cdot V \cap V = \emptyset$ for $w \in W' \setminus W_b$. Let $\mathcal{N} = R_U(\mathcal{M}|_{W\cdot U})$. We have $R(\mathcal{N}^{\wedge W'\cdot b}) \cong R((\mathcal{N}|_{W'\cdot V})^{\wedge W'\cdot b})$. There is a decomposition

$$\mathcal{N}|_{W'\cdot V} \cong \bigoplus_{w\in W'/W_b} \mathcal{N}|_{w\cdot V},$$

such that $R_V(\mathcal{N}|_{W'\cdot V})$ is the projection onto $\mathcal{N}|_V$, and we have a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{w\in W'/W_b}\mathcal{N}^{\wedge wb} & \xrightarrow{\cdot x_b} & \bigoplus_{w\in W'/W_b}\mathcal{N}^{\wedge wb}\\
\uparrow & & \uparrow\\
\bigoplus_{w\in W'/W_b}\mathcal{N}|_{w\cdot V} & \xrightarrow[\cdot 1_V]{} & \bigoplus_{w\in W'/W_b}\mathcal{N}|_{w\cdot V}
\end{array}$$

where the vertical arrows denote the canonical morphisms into completions. Thus $R((\mathcal{N}|_{W'\cdot V})^{\wedge W'\cdot b}) \cong (R_V(\mathcal{N}|_{W'\cdot V}))^{\wedge b}$. We have $1_V 1_U = 1_V$ on $W \cdot V$, so that $R_V(\mathcal{N}|_{W'\cdot V}) \cong R_V(\mathcal{M}|_{W\cdot V})$. Hence, using for the second isomorphism below the same commutative diagram logic, we deduce

$$\begin{aligned} R\big((\mathcal{M}|_{W\cdot U})^{\wedge[b]}\big) &\cong R\big((\mathcal{M}|_{W\cdot V})^{\wedge[b]}\big) \\ &\cong \big(R_V(\mathcal{M}|_{W\cdot V})\big)^{\wedge b} \\ &\cong \big(R_V(\mathcal{N}|_{W'\cdot V})\big)^{\wedge b} \\ &\cong R\big(\mathcal{N}^{\wedge W'\cdot b}\big), \end{aligned}$$

as required. □

We define

$$\mathrm{Res}_{b,U} : \mathcal{O}_c(W, \mathfrak{h}) \to \mathcal{O}_{c_b}(W_b, \mathfrak{h}_b), \quad M \mapsto \zeta \circ E \circ R\Big(\big(R_U(\mathcal{M}|_{W\cdot U})\big)^{\wedge W'\cdot b}\Big).$$

It then follows from Lemma 2 that

Corollary 1 *The functors* $\mathrm{Res}_{b,U}$ *and* Res_b *are naturally isomorphic.*

Now note that the module $R_U(\mathcal{M}|_{W\cdot U})$ has an action of W', since U is W'-stable. Assume that W' normalizes W_b. For each $w \in W'$, there is a well-defined automorphism of $H_{c_b}(W_b, \mathfrak{h}_b)$ given by $a \mapsto waw^{-1}$ for all $a \in H_{c_b}(W_b, \mathfrak{h}_b)$. Similarly, we obtain an isomorphism $x_b H_c(W, \mathfrak{h})^{\wedge[b]} x_b \cong x_{wb} H_c(W, \mathfrak{h})^{\wedge[b]} x_{wb}$ via conjugation. There is a commutative diagram of algebra homomorphisms

$$\begin{array}{ccc} x_b H_c(W, \mathfrak{h})^{\wedge[b]} x_b & \xrightarrow{w(\cdot)w^{-1}} & x_{wb} H_c(W, \mathfrak{h})^{\wedge[b]} x_{wb} \\ \uparrow & & \uparrow \\ H_{c_b}(W_b, \mathfrak{h}_b) & \xrightarrow[w(\cdot)w^{-1}]{} & H_{c_b}(W_b, \mathfrak{h}_b) \end{array}$$

where the vertical arrow denote inclusion maps.

Lemma 3 *Let* $w \in W'$ *and let* $N \in \mathcal{O}_c(W, \mathfrak{h})^{\wedge[b]}$. *Then, as an* $H_{c_b}(W_b, \mathfrak{h}_b)$*-module,* $x_{wb}N$ *is isomorphic to the twist of* $x_b N$ *by the automorphism* $w(\cdot)w^{-1}$. *In particular, if* w *centralizes* $H_{c_b}(W_b, \mathfrak{h}_b)$, *then these modules are isomorphic.*

3 Monodromy Actions

3.1 Fundamental Groups

Let us fix throughout a W-invariant hermitian form $(-,-)$ on $\mathfrak{h}$, and let $\|\cdot\|$ be the associated hermitian product. Let $b \in \mathfrak{h}$, $W_b \subseteq W$ the stabilizer of b, and $W' = N_W(W_b)$ be the normalizer of W_b in W. Let $\mathfrak{h}^{W_b}$ the fixed point set and $\mathfrak{h}_b$ the orthogonal complement to $\mathfrak{h}^{W_b}$ in $\mathfrak{h}$. Note that the decomposition $\mathfrak{h} = \mathfrak{h}^{W_b} \oplus \mathfrak{h}_b$ is W'-stable. If we wish to consider b as an element of $\mathfrak{h}^{W_b}$ we will write it as b', so that $b = (b', 0) \in \mathfrak{h}^{W_b} \oplus \mathfrak{h}_b$.

Proposition 1 ([14]) *There is a subgroup $N \subseteq W'$ such that $N \cap W_b = 1$ and $W' = W_b \rtimes N$.*

This proposition allows us to define $C \subseteq N$ to be the pointwise stabilizer of $\mathfrak{h}_b$. In particular C centralizes W_b.

Let $\mathcal{S}_b = \mathcal{S} \cap W_b$ and $\mathcal{A}_b \subseteq \mathcal{A}$ denote the corresponding reflecting hyperplanes. We define $\mathcal{S}'$ and $\mathcal{A}'$ similarly, using W'. Let

$$\mathfrak{h}_{\mathrm{r}}^{W_b} = \{x \in \mathfrak{h} : W_x = W_b\}.$$

Let $B_C = \pi_1(\mathfrak{h}_{\mathrm{r}}^{W_b}/C, b')$ and $B_N = \pi_1(\mathfrak{h}_{\mathrm{r}}^{W_b}/N, b')$, where we abuse notation by letting b' denote the image of $b' \in \mathfrak{h}_{\mathrm{r}}^{W_b}$ in the relevant orbit space. Since $\mathfrak{h}_{\mathrm{r}}^{W_b}/C \to \mathfrak{h}_{\mathrm{r}}^{W_b}/N$ is a normal covering, we have an exact sequence

$$1 \longrightarrow B_C \longrightarrow B_N \xrightarrow{\beta} N/C \longrightarrow 1. \tag{4}$$

Note that in the special case that b is generic, we have $W_b = 1$, $N = W$ and B_W is the braid group attached to $(W, \mathfrak{h})$.

Let $\epsilon > 0$ be a real number. Let X denote the open ball of radius ϵ around 0 in $\mathfrak{h}_b$. For any $v \in \mathfrak{h}^{W_b}$ we let $\mathcal{B}_v(\epsilon)$ denote the open ball in $\mathfrak{h}^{W_b}$ with centre v and radius ϵ. Take the annulus

$$Y = \bigcup_{v \in \mathfrak{h}^{W_b};\ \|v\| = \|b'\|} \mathcal{B}_v(\epsilon),$$

and let $Y_{\mathrm{r}} = Y \cap \mathfrak{h}_{\mathrm{r}}^{W_b}$. Set $U = X \times Y_{\mathrm{r}}$. This open subset is W'-stable, and we choose ϵ small enough so that $w \cdot U \cap U = \emptyset$ for all $w \in W \setminus W'$. In particular, U intersects only the reflecting hyperplanes in $\mathcal{A}^b$. Since Y_{r} is homotopic to $\mathfrak{h}_{\mathrm{r}}^{W_b}$ and X is simply connected we have

$$\pi_1(U/N, b) \cong \pi_1\big(\mathfrak{h}_{\mathrm{r}}^{W_b}/N, b'\big) = B_N.$$

3.2 Holomorphic Differential Operators

Given a complex manifold V, let $\mathcal{D}_V$ denote the sheaf of holomorphic differential operators on V.

Lemma 4 *Keep the above notation. Let $p_1 : U \to X$ and $p_2 : U \to Y_r$ be the projections. The sheaf $H_{c'}(W', \mathfrak{h})|_U$ contains subsheaves $p_1^{-1} H_{c'}(W_b, \mathfrak{h}_b)|_X$ and $p_2^{-1}(\mathcal{D}_{Y_r} \rtimes N)$. Furthermore $p_1^{-1} H_{c'}(W_b, \mathfrak{h}_b)|_X$ and $p_2^{-1}(\mathcal{D}_{Y_r})$ commute.*

Proof Let $V \subseteq U$ be an W'-stable open subset. By our assumptions on Y_r and X, the functions $\frac{1}{\alpha_s}$ for $s \in \mathcal{S} \setminus W_b$ are well-defined on all of U. So $H_{c'}(W', V)$ contains the operators

$$D'_y = D_y - \sum_{s \in \mathcal{S}, s \notin W_b} \frac{2c_s}{1 - \det_{\mathfrak{h}^*}(s)} \frac{\alpha_s(y)}{\alpha_s}(s-1),$$

defined in (2) for all $y \in \mathfrak{h}$. The subalgebra generated by $\mathcal{K}_X(p_1(V))$, W_b and D'_y for $y \in \mathfrak{h}_b$ generate a copy of $H_{c_b}(W_b, p_1(V))$. Similarly, the algebra generated by $\mathcal{K}_{Y_r}(p_2(V))$, N and the $D'_y = \partial_y$ for $y \in \mathfrak{h}^{W_b}$ yield a copy of $\mathcal{D}_{Y_r}(p_2(V)) \rtimes N$. It is straightforward to check that the final assertion holds. □

3.3 Monodromy

Let $\lambda \in \mathrm{Irr}(W)$, and let $M = \Delta(\lambda)$ be the corresponding standard module. In the notation of (3), we set $\mathcal{N} = R_U(\mathcal{M}|_{W \cdot U})$. By Lemma 4, the action of ∂_y, $y \in \mathfrak{h}^{W_b}$, on $\mathcal{N}(U)$ defines an N-equivariant connection on Y_r with parameters in $\mathcal{K}_X(X)$, see [19, §13] for information on linear differential equations with parameters.

Proposition 2 ([3], Proposition 3.20) *The local system on Y_r attached to $\mathcal{N}(U)$ is given by the connection form*

$$\sum_{s \in \mathcal{S} \setminus W_b} \frac{2c_s}{1 - \det_{\mathfrak{h}^*}(s)} \frac{d\alpha_s}{\alpha_s}(s-1).$$

This is a connection with parameters in $\mathcal{K}_X(X)$ on the trivial bundle on Y_r taking values in $\bigoplus_{\nu \in \mathrm{Irr}(W_b)} \mathrm{Hom}_{W_b}(\nu, \lambda\downarrow_{W_b})$.

We denote this connection by ∇^λ, and for a pair $(c, p) \in \mathbb{C}^{|\mathcal{S}/W|} \times X$ we denote its specialization to this point as $\nabla^\lambda_{c,p}$.

Let $m = \dim E$. By [16], there exist functions $\phi_1, \dots, \phi_m$, holomorphic on $\mathbb{C}^{|\mathcal{S}/W|} \times X \times Y'$, where $Y' \subset Y_r$ is an open ball containing b', such that specialising to a point $(c, p) \in \mathbb{C}^{|\mathcal{S}/W|} \times X$ yields the horizontal sections of $\nabla^\lambda_{c,p}$ in Y'.

Since the connection is N-equivariant, we can associate to each $g \in B_N$ a monodromy matrix acting on the span of the ϕ_j. We call this g^λ and its specialization $g^\lambda_{c,p}$.

Corollary 2 *The stalk at b' of the local system $\mathcal{N}^{\nabla^\lambda}$ on Y_r can be evaluated by taking horizontal sections of ∇^λ in either $\mathcal{N}_{b'}$ or $\mathcal{N}^{\wedge b'}$. In particular, we can identify the completion of $\mathcal{N}_{b'}^{\nabla^\lambda}$ at $0 \in \mathfrak{h}_b$ with $\zeta(\mathrm{Res}_b(M))^{\wedge 0}$.*

Proof By the proposition, the connection on $\nabla^E_{(c,p)}$ has regular singularities at b' for all (c, p). The equality of horizontal sections in the convergent or formal setting is then well-known, see [15] for example. The second claim follows from Corollary 1. □

Lemma 5 *The elements $w \in W_b$, $f \in \widehat{\mathcal{K}}_{X,0}$ and D'_y for $y \in \mathfrak{h}_b$ act on $(\mathcal{N}_{b'}^{\nabla^\lambda})^{\wedge 0}$. Let P denote any of these operators. These P commute with g^λ via $(g^\lambda)^{-1}Pg^\lambda = \beta(g)(P)$ where β is defined in (4).*

Proof By Lemma 4, any $w \in W_b$, $f \in \widehat{\mathcal{K}}_{X,0}$ and D'_y, $y \in \mathfrak{h}_b$, act on the completion of $\mathcal{N}_{b'}$ at $0 \in \mathfrak{h}_b$, and commute with the action of $(\pi_2^{-1}\mathcal{D}_{Y_r})_b \cong (\mathcal{D}_{Y_r})_{b'}$. So the operators certainly act.

Recall that g^λ is calculated as follows. We represent g by a path p from b' to nb' for some $n \in N$. Then we let A_p denote the analytic continuation operator along p, a linear isomorphism $A_p : \mathcal{N}_{b'}^{\nabla^\lambda} \to \mathcal{N}_{nb'}^{\nabla^\lambda}$. Then $g^\lambda : \mathcal{N}_{b'}^{\nabla^\lambda} \to \mathcal{N}_{b'}^{\nabla^\lambda}$ is given by $n^{-1}A_p$.

Let $v_{b'} \in \mathcal{N}_{b'}^{\nabla^\lambda_{(c,0)}}$ and P denote one of the operators w, f or D'_y as above. Uniqueness of analytic continuation implies that $A_p(Pv_{b'}) = P(A_p v_{b'})$ and so if $\overline{A}_p$ denotes the reverse path to A_p we have

$$\begin{aligned}
\left(g^E\right)^{-1} Pg^E(v_{b'}) &= \overline{A}_p n\left(Pn^{-1}(A_p v_{b'})\right) \\
&= \left(\overline{A}_p P^n A_p\right)(v_{b'}) \\
&= P^n v_{b'} \\
&= \beta(g)(P)v_{b'},
\end{aligned}$$

as required. □

Using the short exact sequence (4) we define we define an action of B_N on $H_{c'}(W_b, \mathfrak{h}_b)$ via the quotient B_N/B_C.

Theorem 3 *Let $M \in \mathcal{O}_c(W, \mathfrak{h})$. Then there is an action of $H_{c_b}(W_b, \mathfrak{h}_b) \rtimes B_N$ on* $\mathrm{Res}_b(M)$*. The B_N-action is functorial.*

Proof Let us first suppose that $M = \Delta(\lambda)$ for some $\lambda \in \operatorname{Irr}(W)$. By Lemma 5 and Corollary 2, there is an action of $H_{c_b}(W_b, \mathfrak{h}_b)^{\wedge 0} \rtimes B_N$ on $\zeta(\operatorname{Res}_b(M))^{\wedge 0}$. Since $(\cdot)^{\wedge 0}$ and ζ are equivalences of categories, we thus obtain an action of $H_{c_b}(W_b, \mathfrak{h}_b) \rtimes B_N$ on $\operatorname{Res}_b(M)$.

We use the argument from [10, §5.3] to extend this to any $M \in \mathcal{O}_c(W, \mathfrak{h})$. By 3.3 and Lemma 1, we can extend our constructions to the base rings S and K introduced in 2.4: there is an action of $H_S(W_b, \mathfrak{h}_b) \rtimes SB_N$ on $\operatorname{Res}_{b,S}(\Delta_S(\lambda))$, and similarly for $\operatorname{Res}_{b,K}(\Delta_K(\lambda))$. These actions are compatible with the natural maps $\mathbb{C} \twoheadleftarrow S \hookrightarrow K$. Let $M \in \mathcal{O}_S^{\Delta}(W, \mathfrak{h})$. Since $M \otimes_S K$ embeds into a direct sum of standard modules, we establish the result for M by using Lemma 1. By [10, Corollary 2.7] and basechange, the result holds for projective modules in $\mathcal{O}_c(W, \mathfrak{h})$. By construction, for any morphism between projective modules $P \to Q$, the resulting map $\operatorname{Res}_b(P) \to \operatorname{Res}_b(Q)$ is a map of B_N-modules. But for any $M \in \mathcal{O}_c(W, \mathfrak{h})$ there is an exact sequence $P_1 \xrightarrow{f} P_0 \to M \to 0$, where $P_0, P_1 \in \mathcal{O}_c(W, \mathfrak{h})$ are projective. We deduce that $H_{c_b}(W_b, \mathfrak{h}_b) \rtimes B_N$ acts on $\operatorname{Res}_b(M) \cong \operatorname{Res}_b(P_0)/\operatorname{Im}\operatorname{Res}_b(f)$. The functoriality also follows, completing the proof. □

We rephrase the statement of the theorem as the existence of an exact functor

$$\operatorname{Res}_{W_b}^{W} : \mathcal{O}_c(W, \mathfrak{h}) \to \left(\mathcal{O}_{c_b}(W_b, \mathfrak{h}_b) \boxtimes \operatorname{Loc}\left(\mathfrak{h}_{\mathrm{r}}^{W_b}\right)\right)^{N_W(W_b)},$$

where the superscript $N_W(W_b)$ denotes the $N_W(W_b)$-equivariant structure. This equivariance is obtained by extending the N-action to $N_W(W_b) = W_b \rtimes N$ by using the inner W_b action on $H_{c_b}(W_b, \mathfrak{h}_b)$. The functor Res_b is the composition of $\operatorname{Res}_{W_b}^{W}$ with the functor that forgets the local system on $\mathfrak{h}_{\mathrm{r}}^{W_b}$ and the equivariant structure.

3.4 Decomposition of Induction and Restriction

This extra structure allows us to decompose both Res_b and Ind_b. Consider again equivariant local systems on $\mathfrak{h}_{\mathrm{r}}^{W_b}$ as representations of B_N and restrict them to B_C. Let $P \in \mathcal{O}(W, \mathfrak{h})$ be a projective generator, and let $A = \mathbb{C}[B_C]/\operatorname{Ann}_{B_C}(\operatorname{Res}_b(P))$, a finite dimensional algebra which is the image of $\mathbb{C}[B_C]$ in $\operatorname{End}_{\mathcal{O}_c(W,\mathfrak{h})}(P)$. Since Res_b is exact and P is a generator, the $\mathbb{C}[B_C]$-action on $\operatorname{Res}_b(M)$ factors through A for all $M \in \mathcal{O}_c(W, \mathfrak{h})$. Let I denote a labelling set for the blocks of A and for any $i \in I$ let $e_i \in Z(A)$ be the corresponding primitive central idempotent of A. By Theorem 3 we may consider $e_i \in \operatorname{End}(\operatorname{Res}_b)$ and then we have

$$\operatorname{Res}_b(M) = \bigoplus_{i \in I} \operatorname{Res}_b^i, \quad \text{where } \operatorname{Res}_b^i := e_i \circ \operatorname{Res}_b. \tag{5}$$

As $(\operatorname{Res}_b, \operatorname{Ind}_b)$ is an adjoint pair, we may apply [6, §4.1.5] to see that there exists corresponding adjoint pairs $(\operatorname{Res}_b^i, \operatorname{Ind}_b^i)$ for each $i \in I$ and such that $\operatorname{Ind}_b = \bigoplus_{i \in I} \operatorname{Ind}_b^i$.

3.5 Transitivity

Let us take two parabolic subgroups $W_1 \le W_2$. These produce decompositions $\mathfrak{h} = \mathfrak{h}^{W_1} \oplus \mathfrak{h}_1 = \mathfrak{h}^{W_2} \oplus \mathfrak{h}_2$ and $\mathfrak{h}_2 = \mathfrak{h}_2^{W_1} \oplus \mathfrak{h}_1$, where $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are the non-trivial isotypic components of $\mathfrak{h}$ with respect to the actions of W_1 and W_2 respectively. Note that $\mathfrak{h}_1 \subseteq \mathfrak{h}_2$ and $\mathfrak{h}^{W_1} \supseteq \mathfrak{h}^{W_2}$. By taking W_1-invariants we find $\mathfrak{h}^{W_1} = \mathfrak{h}^{W_2} \oplus \mathfrak{h}_2^{W_1}$. It is not true that one of $\mathfrak{h}_{\mathrm{r}}^{W_1}$ and $\mathfrak{h}_{\mathrm{r}}^{W_2} \times \mathfrak{h}_{2,\mathrm{r}}^{W_1}$ is contained in the other, but nevertheless we may pick $b_1, b_2 \in \mathfrak{h}$ such that $b_1 \in \mathfrak{h}_{\mathrm{r}}^{W_1}$ and $b_2 \in \mathfrak{h}_{\mathrm{r}}^{W_2}$ and $b_1' = (b_2', b_1'') \in \mathfrak{h}^{W_2} \times \mathfrak{h}_2^{W_1}$.

Let $N_W(W_1, W_2) = N_W(W_1) \cap N_W(W_2)$, which acts on $\mathfrak{h}_{\mathrm{r}}^{W_1}$, $\mathfrak{h}_{\mathrm{r}}^{W_2}$ and $\mathfrak{h}_{2,\mathrm{r}}^{W_1}$. There is a homomorphism

$$\iota_{W_1,W_2} : \pi_1\big(\mathfrak{h}_{\mathrm{r}}^{W_2} \times \mathfrak{h}_{2,\mathrm{r}}^{W_1}/N_W(W_1, W_2), \big(b_2', b_1''\big)\big) \longrightarrow \pi_1\big(\mathfrak{h}_{\mathrm{r}}^{W_1}/N_W(W_1, W_2), b_1'\big). \tag{6}$$

To construct this, note first that there is a free action of $N_W(W_1, W_2)/W_1$ on both $\mathfrak{h}_{\mathrm{r}}^{W_2} \times \mathfrak{h}_{2,\mathrm{r}}^{W_1}$ and $\mathfrak{h}_{\mathrm{r}}^{W_1}$. Therefore it is enough to produce an $N_W(W_1, W_2)$-equivariant homomorphism from homotopy classes of paths in $\mathfrak{h}_{\mathrm{r}}^{W_2} \times \mathfrak{h}_{2,\mathrm{r}}^{W_1}$ to homotopy classes of paths in $\mathfrak{h}_{\mathrm{r}}^{W_1}$. Given a path $\gamma = (\gamma_1, \gamma_2) : [0, 1] \to \mathfrak{h}_{\mathrm{r}}^{W_2} \times \mathfrak{h}_{2,\mathrm{r}}^{W_1}$, we may adjust γ_2 sufficiently inside $\mathfrak{h}_{2,\mathrm{r}}^{W_1}$, depending on how close γ_1 passes to the reflecting hyperplanes in $\mathfrak{h}$ that do not contain $\mathfrak{h}^{W_2}$, to ensure that the image of γ belongs to $\mathfrak{h}_{\mathrm{r}}^{W_1}$. Inclusion then provides the homomorphism.

Theorem 4 *There is a natural isomorphism of functors from $\mathcal{O}(W, \mathfrak{h})$ to $(\mathcal{O}(W_1, \mathfrak{h}_1) \boxtimes \mathrm{Loc}(\mathfrak{h}_{2,r}^{W_1}) \boxtimes \mathrm{Loc}(\mathfrak{h}_r^{W_2}))^{N_W(W_1,W_2)}$,*

$$\iota^*_{W_1,W_2} \circ \downarrow^{N_W(W_1)}_{N_W(W_1,W_2)} \circ \mathrm{Res}^W_{W_1} \cong \mathrm{Res}^{W_2}_{W_1} \boxtimes \mathrm{id} \circ \downarrow^{N_W(W_2)}_{N_W(W_1,W_2)} \circ \mathrm{Res}^W_{W_2},$$

where $\downarrow$ denotes the restriction of equivariant structure to a subgroup.

Proof It is unpleasant to deal directly with the path manipulation appearing in the construction of ι_{W_1,W_2}. To avoid this we work instead with an intermediate version of Theorems 1 and 2 and the restriction functors. For $b \in \mathfrak{h}$ with stabilizer W_b this version takes place within a formal neighbourhood of $W \cdot \mathfrak{h}_{\mathrm{r}}^{W_b}$ in $\mathfrak{h}$. The space $\mathfrak{h}_{\mathrm{r}}^{W_b}$ is an affine open algebraic subset of $\mathfrak{h}^{W_b}$ given by the non-vanishing of the polynomial $\pi := \prod_{s \in \mathcal{S} \setminus W_b} \alpha_s$. Set $\overline{\pi} = \prod_{w \in W} w \cdot \pi$, a polynomial whose non-vanishing defines $W \cdot \mathfrak{h}_{\mathrm{r}}^{W_b} = \{x \in \mathfrak{h} \mid W_x \text{ is conjugate to } W_b\}$. We denote by $H_c(W, \mathfrak{h})^{\wedge}{}_{[\mathfrak{h}_{\mathrm{r}}^{W_b}]}$ the subalgebra of $\mathbb{C}$-linear endomorphisms of $\mathbb{C}[\mathfrak{h}]^{\wedge}{}_{[\mathfrak{h}_{\mathrm{r}}^{W_b}]} := \mathbb{C}[\mathfrak{h}]^{\wedge}{}_{[\mathfrak{h}^{W_b}]}[\overline{\pi}^{-1}]$ generated by $\mathbb{C}[\mathfrak{h}]^{\wedge}{}_{[\mathfrak{h}_{\mathrm{r}}^{W_b}]}$, D_y for $y \in \mathfrak{h}$ and $w \in W$. There is then an isomorphism of algebras

$$\Theta_b : H_c(W, \mathfrak{h})^{\wedge}{}_{[\mathfrak{h}_{\mathrm{r}}^{W_b}]} \longrightarrow Z\big(W, W_b, H_{c_b}(W_b, \mathfrak{h})^{\wedge}{}_{\mathfrak{h}_{\mathrm{r}}^{W_b}}\big),$$

which is defined exactly as in Theorem 1. There is furthermore an isomorphism $H_{c_b}(W_b, \mathfrak{h})^{\wedge}{}_{\mathfrak{h}_{\mathrm{r}}^{W_b}} \cong \mathcal{D}_{\mathrm{pol}}(\mathfrak{h}_{\mathrm{r}}^{W_b}) \otimes H_{c_b}(W_b, \mathfrak{h}_b)^{\wedge 0}$. The restriction functors are then defined as usual, splitting the centralizer with an element we label by $1^W_{W_b}$, then taking locally-finite vectors with respect to $\mathbf{eu} \in H_{c_b}(W_b, \mathfrak{h})$.

Now we move onto the proof. Corresponding to the natural algebra homomorphisms

$$\begin{aligned}\mathbb{C}[\mathfrak{h}]^{\wedge \mathfrak{h}_r^{W_1}} &= \mathbb{C}\big[\mathfrak{h}_r^{W_1}\big] \otimes \mathbb{C}[[\mathfrak{h}_1]] \\ &\to \mathbb{C}\big[\mathfrak{h}_r^{W_2}\big] \otimes \mathbb{C}\big[\big[\mathfrak{h}_2^{W_1}\big]\big]\big[\alpha_s^{-1} : s \in W_2 \setminus W_1\big] \otimes \mathbb{C}[[\mathfrak{h}_1]] \\ &\to \mathbb{C}\big[\mathfrak{h}_r^{W_2}\big] \otimes \mathbb{C}\big[\big[\mathfrak{h}_2^{W_1} \times \mathfrak{h}_1\big]\big]\big[\alpha_s^{-1} : s \in W_2 \setminus W_1\big] = \mathbb{C}[\mathfrak{h}]^{\wedge \mathfrak{h}_r^{W_2}}|_{\mathfrak{h}_{2,r}^{W_1} \times \mathfrak{h}_1}\end{aligned}$$

we see that successive restriction to smaller formal neighbourhoods produces $1_{W_2}^{W}(M^{\wedge [\mathfrak{h}_r^{W_1}]}) \mapsto 1_{W_2}^{W}(M^{\wedge [\mathfrak{h}_r^{W_2}]})|_{W_2 \cdot (\mathfrak{h}_{2,r}^{W_1} \times \mathfrak{h}_1)}$. The space $1_{W_1}^{W}(M^{\wedge [\mathfrak{h}_r^{W_1}]})$ has a natural $D(\mathfrak{h}_r^{W_1})$-structure which extends to a $\mathcal{D}(\mathfrak{h}_r^{W_2}) \otimes \mathcal{D}(\mathfrak{h}_{2,r}^{W_1})^{\wedge 0}$-structure on $1_{W_1}^{W_2}(1_{W_2}^{W}(M^{\wedge [\mathfrak{h}_r^{W_2}]})|_{W_{2 \cdot (\mathfrak{h}_{2,r}^{W_1} \times \mathfrak{h}_1)}})$.

The natural inclusion

$$\big(\big(1_{W_2}^{W}\big(M^{\wedge [\mathfrak{h}_r^{W_2}]}\big)\big)^{\mathbf{eu}_2}\big)|_{W_2 \cdot (\mathfrak{h}_{2,r}^{W_1} \times \mathfrak{h}_1)} \to \big(\big(1_{W_2}^{W}\big(M^{\wedge [\mathfrak{h}_r^{W_2}]}\big)\big)|_{W_2 \cdot (\mathfrak{h}_{2,r}^{W_1} \times \mathfrak{h}_1)}\big)^{\mathbf{eu}_2}$$

is an isomorphism since the $\{\alpha_s\}_{s \in W_2}$ have positive $\mathbf{eu}_2$-weights. For $y \in \mathfrak{h}_r^{W_1} \subset \mathfrak{h}_2$ we have that

$$\partial_y = D_y - \sum_{s \in W_2 \setminus W_1} \frac{2c_s}{1 - \det_{\mathfrak{h}^*}(s)} \frac{\alpha_s(y)}{\alpha_s}(s-1) \in H_{c_2}(W_2, \mathfrak{h}_2)\big[\alpha_s^{-1} : s \in W_2 \setminus W_1\big]$$

so that there is an action of $D(\mathfrak{h}_r^{W_2}) \otimes D(\mathfrak{h}_{2,r}^{W_1})$ on

$$\big(\big(1_{W_2}^{W}\big(M^{\wedge [\mathfrak{h}\text{ tinyr}^{W_2}]}\big)\big)|_{W_2 \cdot (\mathfrak{h}_{2,r}^{W_1} \times \mathfrak{h}_1)}\big)^{\mathbf{eu}_2}.$$

Completing at $0 \in \mathfrak{h}_2^{W_1}$ produces another action of $\mathcal{D}(\mathfrak{h}_r^{W_2}) \otimes \mathcal{D}(\mathfrak{h}_{2,r}^{W_1})^{\wedge 0}$ on $(1_{W_2}^{W}(M^{\wedge [\mathfrak{h}_r^{W_2}]}))|_{W_2 \cdot (\mathfrak{h}_{2,r}^{W_1} \times \mathfrak{h}_1)}$.

A similar argument to [18, Lemma 2.2], using the version of the comparison theorem over the formal neighbourhood of a subvariety due to Kashiwara-Schapira, [12, Corollary 6.2], allows us to deduce that the monodromy representations of the two above local systems on $\mathfrak{h}_r^{W_2} \times \mathfrak{h}_{2,r}^{W_1}$ agree.

We then have a functorial morphism

$$\begin{aligned}1_{W_1}^{W}\big(M^{\wedge [\mathfrak{h}_r^{W_1}]}\big)^{\mathbf{eu}_1} &\mapsto 1_{W_1}^{W_2}\big(1_{W_2}^{W}\big(M^{[\mathfrak{h}_r^{W_2}]}\big)|_{W_2 \cdot (\mathfrak{h}_{2,r}^{W_1} \times \mathfrak{h}_1)}\big)^{\mathbf{eu}_1} \\ &\mapsto 1_{W_1}^{W_2}\big(\big(1_{W_2}^{W}\big(M^{\wedge [\mathfrak{h}_r^{W_2}]}\big)|_{W_2 \cdot (\mathfrak{h}_{2,r}^{W_1} \times \mathfrak{h}_1)}\big)^{\mathbf{eu}_2}\big)^{\mathbf{eu}_1} \\ &\mapsto 1_{W_1}^{W_2}\big(\big(1_{W_2}^{W}\big(M^{\wedge [\mathfrak{h}_r^{W_2}]}\big)^{\mathbf{eu}_2}\big)|_{W_2 \cdot (\mathfrak{h}_{2,r}^{W_1} \times \mathfrak{h}_1)}\big)^{\mathbf{eu}_1} \\ &\mapsto 1_{W_1}^{W_2}\big(\big(1_{W_2}^{W}\big(M^{\wedge [\mathfrak{h}_r^{W_2}]}\big)^{\mathbf{eu}_2}\big)^{\wedge [\mathfrak{h}_{2,r}^{W_1}]}\big)^{\mathbf{eu}_1}.\end{aligned}$$

This realizes the functor $\iota^*_{W_1,W_2}$. Since the initial term is $\mathrm{Res}^W_{W_1}(M)$ and the final term is $\mathrm{Res}^{W_2}_{W_1}(\mathrm{Res}^W_{W_2}(M))$, this completes the proof. □

4 Consequences for $G(\ell, 1, n)$

4.1 The Groups $G(\ell, 1, n)$

Fix $\ell \in \mathbb{N}$, and let $W_n = W(G(\ell, 1, n)) = \mu_\ell^n \rtimes \mathfrak{S}_n$ for any $n \in \mathbb{N}$, a complex reflection group with a Coxeter style presentation

$$\big\langle t, s_1, \dots, s_{n-1} \mid s_i^2 = t^\ell = 1, s_1ts_1t = ts_1ts_1, s_it = ts_i \text{ if } i > 1,$$
$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, s_is_j = s_js_i \text{ if } |i-j| > 1\big\rangle.$$

The reflection representation $\mathfrak{h}_n = \mathfrak{h}$ of W_n is the vector space $\mathbb{C}^n = \mathrm{span}\{y_1, \dots, y_n\}$. With respect to the standard basis the s_i generate a copy of the symmetric group acting by place permutation, and t acts by $\mathrm{diag}(\eta, 1, \dots, 1)$, where $\eta = \exp(2\pi\sqrt{-1}/\ell)$. We write $t_i = (1, \dots, 1, \eta, 1, \dots, 1)$ where η appears at the i^{th} coordinate.

When $n > 1$ there are ℓ conjugacy classes of reflections, the set of conjugates of the s_i and the set of conjugates of t^r for $1 \le r \le \ell - 1$; when $n = 1$ there are no s_i and so only $\ell - 1$ classes. The parameters we choose for the rational Cherednik algebra are

$$c_{s_i} = -k \quad \text{and} \quad c_{t^r} = -\frac{1}{2}\left(1 + \sum_{j=1}^{\ell} k(m_{j+1} - m_j)\eta^{-rj}\right) \quad \text{for } 1 \le r \le \ell - 1$$

where $k \in \mathbb{C}$, $\mathbf{m} = (m_1, \dots, m_\ell) \in \mathbb{Z}^\ell$ and we set $m_{\ell+1} = m_1$.

We identify the irreducible representations $\mathsf{Irr}(W_n)$ of W_n with the set $\mathcal{P}_\ell(n)$ of ℓ-multipartitions of n, [17, 6.1.1]. Set $\mathcal{P}_\ell = \bigcup_{n \ge 0} \mathcal{P}_\ell(n)$. We write $\lambda = (\lambda^1, \dots, \lambda^\ell)$ for the multipartition and the corresponding representation and we will often identify λ with an ℓ-tuple of Young diagrams. If a box $p \in \lambda$ is in position (i, j) of the Young diagram of λ^t we set $\beta(p) = t$, and define the residue as $\mathrm{res}(p) = j - i$ and the $\mathbf{m}$-shifted residue $\mathrm{res}^{\mathbf{m}}(p) = \mathrm{res}(p) + m_{\beta(p)}$.

4.2 Induction and Restriction

Set $b_n = b = (0, \dots, 0, n)$. Then $W_b = W_{n-1}$, $N_W(W_{n-1}) = W_{n-1} \times N$ where $N = \langle t_n \rangle \cong \mu_\ell$. Clearly $\mathfrak{h}_b = \{(a_1, a_2, \dots, a_{n-1}, 0) : a_i \in \mathbb{C}\}$ and $\mathfrak{h}_{\mathrm{r}}^{W_{n-1}} = \{(0, \dots, 0, a_n) : 0 \ne a_n \in \mathbb{C}\}$. We have

$$\mathrm{Res}^{W_n}_{W_{n-1}} : \mathcal{O}_c(W_n) \to \left(\mathcal{O}_c(W_{n-1}) \boxtimes \mathrm{Loc}\big(\mathfrak{h}_{\mathrm{r}}^{W_{n-1}}\big)\right)^{N_W(W_{n-1})}.$$

Let $\lambda \in \mathsf{Irr}(W)$. Recall from Proposition 2 that the local system on $\mathfrak{h}_{\mathrm{r}}^{W_{n-1}} \cong \mathbb{C}^\times$ attached to $\mathrm{Res}_{W_{n-1}}^{W}(\Delta(\lambda))$ arises from the N-equivariant connection

$$\sum_{s\in\mathcal{S}\setminus W_{n-1}} \frac{2c_s}{1-\det_{\mathfrak{h}^*}(s)}\frac{d\alpha_s}{\alpha_s}(s-1)$$

on $\mathrm{Hom}_{W_{n-1}}(\nu, \lambda\downarrow_{W_{n-1}})$ where we run over all $\nu \in \mathsf{Irr}(W_{n-1})$.

There are two types of $s \in \mathcal{S}\setminus W_{n-1}$: $(i\,n)t_i^r t_n^{-r}$ for $1 \le i \le n-1, 0 \le r \le \ell-1$, with $\alpha_s = -\eta^r y_i + y_n$; t_n^r for $1 \le r \le \ell-1$ with $\alpha_s = y_n$. Thus the above connection can be written explicitly as the following μ_ℓ-equivariant connection on $\bigoplus_{\nu\in\mathsf{Irr}(W_{n-1})}\mathrm{Hom}_{W_{n-1}}(\nu, \lambda\downarrow_{W_{n-1}})$ over $\mathbb{C}^\times$

$$-\sum_{i=1}^{n-1}\sum_{r=0}^{\ell-1} k\frac{dz}{z}\big((i\,n)t_i^r t_n^{-r}-1\big) - \sum_{r=1}^{\ell-1}\frac{1+\sum_{j=1}^{\ell}k(m_{j+1}-m_j)\eta^{-rj}}{1-\eta^{-r}}\frac{dz}{z}\big(t_n^r-1\big).$$

To calculate the monodromy of this connection we will apply the following result which essentially appears in [4, Theorem 4.12]

Lemma 6 *Let $\mathcal{V}$ be a trivial vector bundle over $\mathbb{C}^\times$, equipped with a μ_ℓ-equivariant structure, and take a μ_ℓ-equivariant connection on $\mathcal{V}$ of the form $\omega = \sum_{r=0}^{\ell-1}\alpha_r\epsilon_r\frac{dz}{z}$, where $\epsilon_r = \frac{1}{\ell}\sum_{j=0}^{l-1}\exp(-2\pi\sqrt{-1}r)^j t_n^j \in \mathbb{C}[\mu_\ell]$. Let Σ be the monodromy operator corresponding to the generator of $\pi_1(\mathbb{C}^\times,\cdot)$, an anticlockwise loop about the origin. Then the part of the Σ corresponding to the term $\alpha_r\epsilon_r dz/z$ is given by multiplication by* $\exp(2\pi\sqrt{-1}(\alpha_r - r)/\ell)$.

The space $M = \mathrm{Hom}_{W_{n-1}}(\nu, \lambda\downarrow_{W_{n-1}})$ is either zero or one-dimensional, with the non-zero spaces occurring precisely when p is a box of λ such that $\nu = \lambda\setminus\{p\}$. Assume we are in this case. Then t_n acts on M by multiplication by $\eta^{\beta(p)-1}$. The element $\sum_{i=1}^{n-1}\sum_{r=0}^{\ell-1}(i\,n)t_i^r t_n^{-r}$ is the nth Jucys-Murphy element of $\mathbb{C}[W_n]$: it acts on M by multiplication by $\ell\cdot\mathrm{res}(p)$. Therefore on M the coefficient of the connection is multiplication by $k\ell(n-1) - k\ell\mathrm{res}(p) + k\ell(m_1 - m_{\beta(p)}) + \beta(p) - 1$. Thus by Lemma 6 the monodromy $\Sigma_{\Delta(\lambda)}$ of the local system arising from $\mathrm{Res}_{W_{n-1}}^{W_n}(\Delta(\lambda))$ satisfies the relation

$$\prod_p\big(\Sigma_{\Delta(\lambda)} - q^{(\mathrm{res}^{\mathbf{m}}(p)+1-n)-m_1}\big) = 0,$$

where p runs over all removable boxes of λ and $q = \exp(-2\pi\sqrt{-1}k)$.

Proposition 3 *Let $M \in \mathcal{O}_c(W_n)$ and let Σ_M denote the monodromy operator on* $\mathrm{Res}_b(M)$ *arising from the local system over $\mathfrak{h}_r^{W_{n-1}}$ attached to* $\mathrm{Res}_{W_{n-1}}^{W_n}(M)$. *The eigenvalues of Σ_M lie in $\{q^i \mid i \in \mathbb{Z}\}$.*

Proof By the calculation above we know that $\Sigma_{\Delta(\lambda)}$ has eigenvalues in this set for all $\lambda \in \mathsf{Irr}(W_n)$. Since $\mathrm{Res}_b(L(\lambda))$ is a quotient of $\mathrm{Res}_b(\Delta(\lambda))$, the same is true for any $\Sigma_{L(\lambda)}$ and so for any Σ_M, as required. □

We can now decompose Res_b as in (5). Let i be an integer. We define the functor $\mathrm{Res}^i(n) := \pi_i \circ \mathrm{Res}_b$, where π_i is projection on to the $q^{i+1-n-m_1}$ generalized eigenspace of the monodromy operator Σ. We have a decomposition

$$\mathrm{Res}_b = \bigoplus_{i \in \mathbb{Z}/\sim} \mathrm{Res}^i(n)$$

where $i \sim j$ if and only $q^i = q^j$.

Let $\mathrm{Ind}^i(n)$ denote the right adjoint of $\mathrm{Res}^i(n)$.

4.3 $\tilde{\mathfrak{sl}}_e$-Categorification

We proceed to the theorem that has been proved by Shan in [18, Theorem 5.1] using the KZ-functor and its double centralizer property. We will show in Proposition 4 our approach is identical to [18]. Nonetheless, we outline the result and its proof here as it avoids using the double centralizer.

Recall the choice of parameters from 4.1: $k \in \mathbb{C}$ and $\mathbf{m} = (m_1, \ldots, m_\ell) \in \mathbb{Z}^\ell$. Let $e' \in \mathbb{N} \cup \{\infty\}$ be the multiplicative order of $q = \exp(-2\pi\sqrt{-1}k) \in \mathbb{C}^\times$ and set $e = e'$ if $e' \neq 1$ and set $e = \infty$ if $e' = 1$. Let $\mathcal{F}(\mathbf{m})$ denote the Fock space with multicharge $\mathbf{m}$, an integrable $\tilde{\mathfrak{sl}}_e$-representation, see for instance [11]. As a vector space we have $\mathcal{F}(\mathbf{m}) = \bigoplus_{\lambda \in \mathcal{P}_\ell} \mathbb{C}\lambda$; for $i \in \mathbb{Z}/e\mathbb{Z}$ the corresponding Chevalley generators act as

$$e_i(\lambda) = \sum_{|\lambda/\mu|=1, \mathrm{res}^{\mathbf{m}}(\lambda/\mu) \equiv i} \mu, \qquad f_i(\lambda) = \sum_{|\mu/\lambda|=1, \mathrm{res}^{\mathbf{m}}(\mu/\lambda) \equiv i} \mu;$$

finally $\partial(\lambda) = -\tau_0 \lambda$ where τ_0 is the number of boxes in λ with $\mathbf{m}$-shifted residue divisible by e. The weight spaces of $\mathcal{F}(m)$ are $\mathcal{F}(m)_\tau$ for $\tau = (\tau_0, \ldots, \tau_{e-1}) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}/e\mathbb{Z}$ where $\mathcal{F}(m)_\tau$ is spanned by the multipartitions having exactly τ_i boxes with $\mathbf{m}$-shifted residue equal to i for each $i \in \mathbb{Z}/e\mathbb{Z}$. Such elements have weight $\sum_{j=1}^{\ell} \Lambda_{m_j} - \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \tau_i \alpha_i$ where Λ_i is the i^{th} fundamental weight of $\tilde{\mathfrak{sl}}_e$.

Let $\mathcal{O}_c(\mathbb{N}) = \bigoplus_{n \geq 0} \mathcal{O}_c(W_n, \mathfrak{h}_n)$. Set $\mathsf{E} = \bigoplus_{n \geq 0} \mathrm{Res}_{b_n}$, $\mathsf{F} = \bigoplus_{n \geq 0} \mathrm{Ind}_{b_n} : \mathcal{O}_c(\mathbb{N}) \to \mathcal{O}_c(\mathbb{N})$. Define $X \in \mathrm{End}(\mathsf{E})$ as the direct sum over $n \geq 0$ of the operators $\Sigma \in \pi_1(\mathfrak{h}_{\mathrm{r}}^{W_{n-1}}/\mu_\ell, \cdot)$. By Theorem 4 we may identify E^2 with the direct sum of the restrictions $\mathrm{Res}_{\tilde{b}_n} : \mathcal{O}(W_n, \mathfrak{h}_n) \to \mathcal{O}(W_{n-2}, \mathfrak{h}_{n-2})$ where $\tilde{b}_n = (0, \ldots, 0, n-1, n) \in \mathfrak{h}_n$. Then $\mathfrak{h}_{\mathrm{r}}^{W_{\tilde{b}_n}} = \{(a_{n-1}, a_n) \in \mathbb{C}^2 : a_{n-1}, a_n \neq 0 \text{ and } a_{n-1} \neq \eta^j a_n \text{ for } 0 \leq j \leq \ell - 1\}$ and $N(W_{\tilde{b}_n}) = W_{\tilde{b}_n} \rtimes W_2'$ where $W_2' = \langle s_{n-1}, t_{n-1}, t_n \rangle$. Let $T \in \mathrm{End}(\mathrm{Res}_{\tilde{b}_n})$ be the operator arising from the generator of monodromy in $\pi_1(\mathfrak{h}_{\mathrm{r}}^{W_{\tilde{b}_n}}/W_2', \cdot)$ attached to the reflecting hyperplane $a_{n-1} = a_n$ in $\mathfrak{h}^{W_{\tilde{b}_n}}$, [4, Appendix 1]. We may decompose $\mathcal{O}_c(\mathbb{N}) = \bigoplus_\tau \mathcal{O}_c(\mathbb{N})_\tau$ where $\tau = (\tau_0, \ldots, \tau_{e-1})$ and $\mathcal{O}_c(\mathbb{N})_\tau$ is the full subcategory of $\mathcal{O}_c(W_{|\tau|}, \mathfrak{h}_{|\tau|})$ generated by all the $L(\lambda)$ where λ has exactly τ_i boxes with $\mathbf{m}$-shifted residue equal to i for each $i \in \mathbb{Z}/e\mathbb{Z}$. By [7, Theorem 4.1] $\mathcal{O}_c(\mathbb{N})_\tau$ is a sum of blocks of $\mathcal{O}_c(\mathbb{N})$.

Theorem 5 *Let c be the parameters for the rational Cherednik algebra given in 4.1 and keep the notation above.*

1. *The adjoint pair* (E, F), $X \in \mathrm{End}(\mathsf{E})$, $T \in \mathrm{End}(\mathsf{E}^2)$ *and the block decomposition* $\mathcal{O}_c(\mathbb{N}) = \bigoplus_{\tau \in P} \mathcal{O}_c(\mathbb{N})_\tau$ *gives an* $\tilde{\mathfrak{sl}}_e$*-categorification of* $\mathcal{F}(\mathbf{m})$ *by* $\mathcal{O}_c(\mathbb{N})$.
2. *The simple objects in* $\mathcal{O}_c(\mathbb{N})$ *give an* $\tilde{\mathfrak{sl}}_e$*-crystal basis for the Grothendieck group* $[\mathcal{O}_c(\mathbb{N})]$ *which is isomorphic to the crystal of the Fock space* $\mathcal{F}(\mathbf{m})$.

Proof (1) By construction the eigenoperators of E under the action of X are the sums of $\mathrm{Res}^i(n)$ over all n. The standard modules $\{[\Delta(\lambda)] : \lambda \in \mathcal{P}_\ell\}$ give a basis of $\mathbb{Q} \otimes K(\mathcal{O}_c(\mathbb{N}))$. Identify $\mathbb{Q} \otimes K(\mathcal{O}_c(\mathbb{N}))$ with $\mathcal{F}(\mathbf{m})$ by sending $[\Delta(\lambda)]$ to λ. We have seen in 4.2 that $\mathrm{Res}^i([\Delta(\lambda)]) = \sum_{|\lambda/\mu|=1, \mathrm{res}^{\mathbf{m}}(\lambda/\mu)\equiv i} [\Delta(\mu)]$ for any $\lambda \in \mathcal{P}_\ell$. It then follows by adjunction and the elementary fact that $\mathrm{Res}^i([\Delta(\lambda)])$ has a standard filtration, see [18, Proposition 1.9], that $\mathrm{Ind}^i([\nabla(\lambda)]) = \sum_{|\mu/\lambda|=1, \mathrm{res}^{\mathbf{m}}(\mu/\lambda)\equiv i} [\nabla(\mu)]$ where $\nabla(-)$ denotes a costandard module in $\mathcal{O}_c(\mathbb{N})$. Since $[\Delta(\lambda)] = [\nabla(\lambda)]$ for any $\lambda \in \mathcal{P}_\ell$, [10, Proposition 3.3], it follows that we have a weak $\tilde{\mathfrak{sl}}_e$-categorification of $\mathcal{F}(\mathbf{m})$.

For the full $\tilde{\mathfrak{sl}}_e$-categorification we also need that F is a left adjoint of E: this is a theorem of Shan, [18, Proposition 2.9], and Losev, [13]. Finally, the compatibilities and equalities required of T and X all follow from Theorem 4 and standard monodromy calculations in $\mathfrak{h}_{\mathrm{r}}^{W_{\tilde{b}_n}}$ and $\mathfrak{h}_{\mathrm{r}}^{W_{\hat{b}_n}}$ with the connections of Proposition 2, where $\tilde{b}_n = (0, \ldots, 0, n-1, n)$ and $\hat{b}_n = (0, \ldots, n-2, n-1, n)$.

(2) This follows formally, as explained in [18, Theorem 6.3]. □

4.4 Monodromy and the KZ Functor

We now compare $\mathrm{Res}^i(n)$ with $\mathsf{E}_i(n)$, the functor of i-restriction from [18, Definition 4.2]. We choose the basepoint $x_0 = (1, 2, \ldots, n) \in \mathfrak{h}_{\mathrm{r}}^{\{1\}} =: \mathfrak{h}_{\mathrm{r}}$ and recall that the KZ-functor $\mathrm{KZ}_n : \mathcal{O}_c(W_n, \mathfrak{h}) \to \mathcal{H}_q(W_n)$-mod is obtained from the local system over $\mathfrak{h}_{\mathrm{r}}/W_n$ attached to $\mathrm{Res}_{\{1\}}^{W_n}$. We have $\pi_1(\mathfrak{h}_{\mathrm{r}}/W_n, x_0) = B_n$, the braid group attached to W_n, which following [4] may be presented as

$$\langle \tau, \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_1\tau\sigma_1\tau = \tau\sigma_1\tau\sigma_1, \tau\sigma_i = \sigma_i\tau \text{ if } i > 1$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1\rangle,$$

and $\mathcal{H}_q(W_n)$ is the quotient of $\mathbb{C}[B_n]$ by the relations

$$(\sigma_i - 1)(\sigma_i + q^{-1}) = 0 \quad \text{for } 1 \le i \le n-1, \quad \text{and} \quad \prod_{i=1}^{\ell}(\tau - q^{m_i - m_1}) = 0. \quad (7)$$

There is an algebra isomorphism $\gamma : Z(\mathcal{O}_c(W_n, \mathfrak{h})) \xrightarrow{\sim} Z(\mathcal{H}_q(W_n))$ such that $zM = \gamma(z)\mathrm{KZ}_n(M)$ for all $z \in Z(\mathcal{O}_c(W_n, \mathfrak{h}))$. Let $\mathsf{J}_0, \ldots, \mathsf{J}_{n-1}$ denote the Jucys-Murphy elements in $\mathcal{H}_q(W_n)$, defined by

$$\mathsf{J}_0 = q^{m_1}\tau, \qquad \mathsf{J}_i = q^{m_1+i-1}\sigma_i \ldots \sigma_1 \tau \sigma_1 \ldots \sigma_i,$$

for $1 \le i \le n-1$. (See [9, Definition 5.2.3], but note that our normalization of the Hecke algebra $\mathcal{H}_q(W_n)$ differs from [loc.cit] so we have a slightly different definition.) Set $C_n(z) = \prod_{i=0}^{n-1}(z - \mathsf{J}_i)$, a polynomial in the variable z whose coefficients lie in $Z(\mathcal{H}_q(W_n))$. Let $D_n(z) = \gamma^{-1}(C_n(z))$. For $a(z) \in \mathbb{C}(z)$ let $Q_{n,a(z)}$ be the exact endo-functor of $\mathcal{O}_c(W_n)$ that maps an object M to the generalized eigenspace of $D_n(z)$ in M with the eigenvalue $a(z)$. The functor

$$\mathsf{E}_i(n) : \mathcal{O}_c(W_n) \to \mathcal{O}_c(W_{n-1})$$

is given by $\mathsf{E}_i(n) = \bigoplus_{a(z)\in\mathbb{C}(z)} Q_{n-1,a(z)/(z-q^i)} \circ \mathrm{Res}_b \circ Q_{n,a(z)}$, where b is chosen as in 4.2.

The following result shows that the categorification here of $\mathcal{F}(\mathbf{m})$ arising from the monodromy of the restriction and induction functors is the same as that of [18].

Proposition 4 *For any i and any n there is a natural isomorphism* $\mathrm{Res}^i(n) \cong \mathsf{E}_i(n)$.

Proof We will prove that $\mathrm{KZ}_{n-1} \circ \mathrm{Res}^i(n) \cong \mathrm{KZ}_{n-1} \circ \mathsf{E}_i(n)$, so the result follows from [18, Lemma 2.4]. Let $M \in \mathcal{O}_c^{\Delta}(W_n, \mathfrak{h})$. We first consider $\mathrm{KZ}_{n-1} \circ \mathsf{E}_i(n)(M)$. By [18, Theorem 2.1], this is the restriction of $\mathrm{KZ}_n(M)$ to $\mathcal{H}_q(W_{n-1})$ followed by the projection onto a block corresponding to the eigenvalue $a(z)$. The blocks of such a restriction are determined by the generalized eigenspaces of J_{n-1} in $\mathcal{H}_q(W_n)$ acting on the restriction: on removing a box p from a multipartition, J_{n-1} acts by $q^{\mathrm{res}^{\mathbf{m}}(p)}$.

On the other hand $\mathrm{KZ}_{n-1} \circ \mathrm{Res}^i(n)(M)$ equals the monodromy of the local system attached to $\mathrm{Res}_{\{1\}}^{W_{n-1}} \circ \mathrm{Res}^i(n)(M)$. By Theorem 4 this in turn equals the generalized eigenspace of the image of Σ_M in B_n acting on $\mathrm{KZ}_n(M)$. We saw in 4.2 that on removing a box p, Σ acts by $q^{\mathrm{res}^{\mathbf{m}}(p)+1-n-m_1}$. By Lemma 7 below, the element Σ is mapped to $\sigma_{n-1} \ldots \sigma_1 \tau \sigma_1 \ldots \sigma_{n-1} = q^{1-n-m_1}\mathsf{J}_{n-1}$ under the homomorphism $B_n \to \mathcal{H}_q(W_n, \mathfrak{h})$. Thus we have the required equality for objects with Δ-filtrations.

The general case follows since projective objects have Δ-filtrations and $\mathcal{O}_c(W_n, \mathfrak{h}_n)$ has finite global dimension. □

It remains only to explain the following lemma used above.

Lemma 7 *Recall the notation* ι_{W_1,W_2} *of* (6). *Let* $\psi : \langle\Sigma\rangle \longrightarrow B_n$ *be the canonical composition*

$$\begin{aligned}\pi_1\big(\mathfrak{h}_r^{W_2}/\langle t_n\rangle, b\big) &\longrightarrow \pi_1\big(\big(\mathfrak{h}_r^{W_2}/\langle t_n\rangle \times \mathfrak{h}_{2,r}/W_{n-1}, x_0\big)\\ &\stackrel{\iota_{\{1\},W_{n-1}}}{\longrightarrow} \pi_1\big(\mathfrak{h}_r/N(W_{n-1}), x_0\big)\\ &\longrightarrow \pi_1(\mathfrak{h}_r/W, x_0).\end{aligned}$$

Then $\psi(\Sigma) = \sigma_{n-1}\ldots\sigma_1\tau\sigma_1\ldots\sigma_{n-1}$.

Proof If we calculate by lifting paths to $\mathfrak{h}_r$, then a representative of $\psi(\Sigma)$ is given by the path $\gamma_{x_0}^n$ where for any $1 \le i \le n$ we set $\gamma_{x_0}^i(s) = (1,\ldots,i-1,\exp(2\pi\sqrt{-1}s)i, i+1,\ldots,n)$ for $s \in [0,1]$. For any $1 \le i \le n-1$, set $I_{[i,i+1]} = \{(a_1,a_2) : i \le |a_1|, |a_2| \le i+1\}$. One can then prove by an unenlightening (for us) calculation in the space $\{1\}\times\cdots\times\{i-1\}\times I_{[i,i+1]}\times\{i+2\}\times\{n\}\subset\mathfrak{h}_r$ that $\sigma_i\gamma_{x_0}^i\sigma_i = \gamma_{x_0}^{i+1}$. Since $\tau = \gamma_{x_0}^1$ this confirms the lemma. □

4.5 The KZ-Component of the Crystal

We end with a remark on the Cherednik crystal of irreducible representations in $\mathcal{O}_c(\mathbb{N})$. Let

$$\mathbf{B}_n = \big\{\lambda \in \mathsf{Irr}(W_n) : \mathrm{KZ}_n\big(L(\lambda)\big) \neq 0\big\}$$

and set

$$\mathbf{B} = \coprod_n \mathbf{B}_n.$$

We claim that $\mathbf{B}\cup\{0\}$ is stable under the crystal operators $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in \mathbb{Z}/e\mathbb{Z}$. Indeed $\tilde{f}_i\tilde{e}_i b = b$ which ensures that $\mathrm{KZ}(\tilde{e}_i b) = \mathrm{KZ}(\mathrm{socRes}^i(\tilde{e}_i b)) \neq 0$, so that $\mathrm{Res}^i\mathrm{KZ}(\tilde{e}_i(b)) \cong \mathrm{KZ}(\mathrm{Res}^i(\tilde{e}_i b)) \neq 0$ and so $\mathrm{KZ}(\tilde{e}_i(b)) \neq 0$. The argument for $\tilde{f}_i b$ is similar.

By [5, Corollary 5.8] the set $\mathbf{B}$ equals the subset of Uglov multipartitions, that is the subset of multipartitions that label the canonical basis of $L(\Lambda_{m_1}+\cdots+\Lambda_{m_\ell}) \subset F(\mathbf{m})$ constructed by Uglov. It follows from the argument in [2, Theorem 6.1] that this crystal *equals* the crystal defined from Uglov's canonical basis of the Fock space provided that we know that the decomposition matrix of the Hecke algebra is given by the evaluation of Uglov's canonical basis at 1. This is the well-known result of Ariki, [1]. Thus we have an explicit identification of the so-called KZ-component of the crystal with the combinatorial crystal on Uglov multipartitions.

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Category of Finite Dimensional Modules over an Orthosymplectic Lie Superalgebra: Small Rank Examples

Caroline Gruson and Vera Serganova

Abstract The aim is to give a concrete picture of simple and projective finite dimensional modules over $\mathfrak{osp}(5,4)$, together with a summary of our papers (Gruson, C., Serganova, V. in Proc. Lond. Math. Soc. 101(3):852–892, 2010 and in Mosc. Math. J., to appear).

1 Introduction

Let $\mathfrak{g}$ be a complex orthosymplectic Lie superalgebra and let G be the corresponding algebraic supergroup $SOSP(m,2n)$. Consider the category $\mathcal{F}$ of finite dimensional G-modules such that the parity of a weight space coincides with the parity of the corresponding weight. In previous work ([6, 7]), we proved results concerning the character of simple objects in $\mathcal{F}$ and projective indecomposable modules. In particular, we showed that a Bernstein-Gel'fand-Gel'fand reciprocity law holds in $\mathcal{F}$.

The aim of this presentation is to describe the algorithms introduced in [6] and [7] in low rank examples. We start with a summary of those two papers in the case $\mathfrak{osp}(2m+1,2n)$. We then give a complete description of the algorithms for the maximally atypical weights of $\mathfrak{osp}(5,4)$. Using these algorithms, we are able to give multiplicities of simple modules occurring in a projective indecomposable module: up to now, such explicit computations were available only for weights of atypicality degree less or equal to 1 (here we get atypicality degree 2). In the last section, we consider the case $\mathfrak{osp}(7,6)$, where such a complete description is rather more complicated and we draw the picture for "generic weights" (such a picture is also obtained for $\mathfrak{osp}(2n+1,2n)$). We completely describe the "exceptional moves" for

Dedicated to M. Jimbo.

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$\mathfrak{osp}(7,6)$, this is the smallest case where these moves can start from infiniteley many weights.

We encode dominant weights by weight diagrams, following the idea of Brundan and Stroppel for the $\mathfrak{gl}(m,n)$ case ([2]). The category $\mathcal{F}$ splits into blocks, which are indexed by the core of these weight diagram. We only consider maximally atypical weights since we know that, with the help of translation functors all the other cases can be reduced to that one, see Theorem 2 in [6]. We restrict ourselves to algebras of type $\mathfrak{osp}(2m+1,2n)$ in order to limit the notations...

2 Context

Let us first recall a few facts about Lie superalgebras.

It is well-known that the representation theory of simple Lie superalgebras is not a straightforward adaptation of the theory in the non graded case. In 1977, Kac in [8], classified the simple Lie superalgebras, and emphasized on the fact that the finite dimensional modules are not semi-simple. When the Lie superalgebra is basic classical, the simple modules have a highest weight, which is a dominant weight for the reductive Lie algebra which forms the even part. He asked the question of computing the characters for simple modules and introduced the Kac modules for the case of $\mathfrak{gl}(m,n)$: there is a parabolic subalgebra $\mathfrak{p}$ with a purely odd complement space. A Kac module is obtained by inflating a simple module from the Levi part $\mathfrak{gl}(m)\times\mathfrak{gl}(n)$ of $\mathfrak{p}$ to $\mathfrak{p}$, then by inducing from $\mathfrak{p}$ to $\mathfrak{gl}(m,n)$: the induced module is still finite dimensional and there is a neat character formula for them. Moreover, Kac modules play the role of standard modules in the BGG reciprocity law in the category of finite dimensional modules, as is first mentioned in [12]. This category, for $\mathfrak{gl}(m,n)$, is now quite well understood ([1–5, 10]).

It is tempting to do the same with orthosymplectic superalgebras, but they have no such parabolic subalgebras, hence in this case, Kac modules no longer exist. However, one can give a geometric interpretation Borel-Weil-Bott like for Kac modules for $\mathfrak{gl}(m,n)$, as the space of sections of a line bundle over the super flag variety. Hence, one can make the corresponding construction in the $\mathfrak{osp}$ case ([9]): now the cohomology is no longer concentrated in degree 0, and as is first mentioned in [11], we introduce the *Euler characteristic* which is a virtual module in the Grothendieck group $\mathcal{K}(\mathcal{F})$ of the category defined as the alternating sum of the cohomology groups: we will be more precise later.

Those virtual modules stand for the standard objects for $\mathcal{F}$, meaning that they have computable composition series in terms of the simple modules ([6]), and the indecomposable projective modules can be uniquely expressed as linear combinations (with not necessarily positive integral coefficients) of Euler characteristics. Moreover, a BGG reciprocity law holds ([7]). It is to be noted that there are less standard objects than projective or simple modules, since they are labelled by weights belonging to a smaller set.

We also want to emphasize that for $\mathfrak{osp}(2m+1,2n)$, the multiplicity of a simple module in any Euler characteristic is at most 1 (but not for $\mathfrak{osp}(2m,2n)$ in general).

Now let us be a little more precise. Let $\mathfrak{g} = \mathfrak{osp}(2m+1, 2n)$, we denote by $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ the decomposition into even and odd parts. We choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ together with a basis $(\varepsilon_1, \dots, \varepsilon_m, \delta_1, \dots \delta_n)$ of $\mathfrak{h}^*$, denote by W the associated Weyl group. The roots split into the roots of $\mathfrak{g}_0$ with respect to $\mathfrak{h}$, Δ_0, and the odd roots Δ_1 are the weights of $\mathfrak{g}_1$. The Killing form on $\mathfrak{g}$ restricts to a non-degenerate bilinear form on $\mathfrak{h}$ up to a scalar, it is given by $(\varepsilon_i, \varepsilon_j) = \delta_{ij} = -(\delta_i, \delta_j)$, and $(\varepsilon_i, \delta_j) = 0$. We choose the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ (and in doing so we get a choice of positive roots), such that:

- If $\mathfrak{g} = \mathfrak{osp}(2m+1, 2n)$ and $m \geq n$, the simple roots are

$$\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{m-n+1} - \delta_1, \delta_1 - \varepsilon_{m-n+2}, \dots, \varepsilon_m - \delta_n, \delta_n,$$

$$\rho = -\frac{1}{2}\sum_{i=1}^{m} \varepsilon_i + \frac{1}{2}\sum_{j+1}^{n} \delta_j + \sum_{i=1}^{m-n} (m-n-i+1)\varepsilon_i;$$

- If $\mathfrak{g} = \mathfrak{osp}(2m+1, 2n)$ and $m < n$, the simple roots are

$$\delta_1 - \delta_2, \dots, \delta_{n-m} - \varepsilon_1, \varepsilon_1 - \delta_{n-m+1}, \dots, \varepsilon_m - \delta_n, \delta_n,$$

$$\rho = -\frac{1}{2}\sum_{i=1}^{m} \varepsilon_i + \frac{1}{2}\sum_{j+1}^{n} \delta_j + \sum_{j=1}^{n-m} (n-m-j)\delta_j,$$

  here $\rho = \rho_0 - \rho_1$ is the graded version of half sum of positive roots, where $\rho_i = \frac{1}{2}\sum_{\alpha \in \Delta_i^+} \alpha$.

Recall (see [6] Corollary 3) that λ is the highest weight of a simple finite dimensional $\mathfrak{g}$-module (or λ is integral dominant) if and only if

$$\lambda + \rho = a_1\varepsilon_1 + \dots + a_m\varepsilon_m + c_1\delta_1 + \dots + c_n\delta_n,$$

where $a_i, c_j \in \frac{1}{2} + \mathbb{Z}$, and either

$$a_1 > a_2 > \dots > a_m \geq \frac{1}{2}, \qquad c_1 > c_2 > \dots > c_n \geq \frac{1}{2},$$

or there exists $\ell \in \{0, \dots, \min(m, n)\}$ such that

$$\begin{cases} a_1 > a_2 > \dots > a_{m-\ell-1} > a_{m-\ell} = \dots = a_m = -\dfrac{1}{2}, \\ c_1 > c_2 > \dots > c_{n-\ell-1} \geq c_{n-\ell} = \dots = c_n = \dfrac{1}{2}. \end{cases}$$

There is a partial ordering on the set of dominant weights, namely $\lambda \leq \mu$ iff $\mu - \lambda = \sum_{\alpha \in \Delta^+} n_\alpha \alpha$ with $n_\alpha \in \mathbb{Z}_+$.

Moreover, recall that a weight λ is *atypical* if there exist isotropic odd root(s) α such that $(\lambda+\rho,\alpha)=0$. The degree of atypicality is defined in Definition 2 in [6], we will explain in the next section how to compute it with the weight diagrams.

Let G be the algebraic supergroup $SOSP(2m+1,2n)$ and Q be a parabolic subgroup containing B, the Borel subgroup of G with Lie algebra $\mathfrak{b}$. There is a structure of algebraic supervariety on the flag manifold G/Q. Let λ be a dominant weight, one can associate to λ a vector bundle $\mathcal{L}_{G/Q}(\lambda)$ over G/Q and a structure of $\mathfrak{g}$-module on the cohomology groups $H^i(G/Q,\mathcal{L}(\lambda))$. The Euler characteristic is the following virtual module:

$$\mathcal{E}_{G/Q}(\lambda)=\sum_{0\leq i\leq \dim(G/Q)}(-1)^i\big[H^i\big(G/Q,\mathcal{L}(\lambda)\big)\big]\in\mathcal{K}(\mathcal{F}).$$

In most cases, the Euler characteristic mentioned above is $\mathcal{E}(\lambda)=\mathcal{E}_{G/B}(\lambda)$, but for certain weights, namely when λ has a tail (see [6] after Lemma 15 and next section), it turns out that $\mathcal{E}_{G/B}(\lambda)$ vanishes and then one finds a proper parabolic subgroup Q_λ associated to λ, such that $\mathcal{E}(\lambda)=\mathcal{E}_{G/Q_\lambda}(\lambda)$ is non-zero.

3 Summary of [6] and [7] in the $\mathfrak{osp}(2m+1,2n)$ Case

A dominant weight λ such that

$$\lambda+\rho=a_1\varepsilon_1+\cdots+a_m\varepsilon_m+c_1\delta_1+\cdots+c_n\delta_n$$

is encoded in the *weight diagram* denoted f_λ constructed as follows:

A weight diagram is a assignation of zero, one or several symbols $<$, $>$, or $\times$ to positions $t=\frac{2r+1}{2}$, $r\in\mathbb{Z}_{\geq 0}$, maybe endowed with a sign $(+)$ or $(-)$:

(1) put one symbol $>$ at position t for every i such that $|a_i|=t$;
(2) put one symbol $<$ at position t for every i such that $c_i=t$;
(3) for every t, replace a pair of symbols $>$ and $<$, by a single $\times$, as many times as possible;
(4) if $t=\frac{1}{2}$ and the smallest value of a_i for which $|a_i|=\frac{1}{2}$ is positive (resp. negative), put a $(+)$ (resp. $(-)$) in front of the diagram.

Remark 1

(1) There is a one-to-one correspondence between dominant weights and weight diagrams.
(2) Due to the dominance conditions, there is at most one symbol at a position $t\neq\frac{1}{2}$.
(3) The *atypicality degree* of λ is by definition the maximal number of mutually orthogonal isotropic roots which are orthogonal to $\lambda+\rho$, such roots are necessarily odd, and it turns out to be the total number of $\times$ in f_λ.

(4) The position $t = \frac{1}{2}$ can contain at most one of the symbols $>$ or $<$, and up to the maximal possible atypicality degree symbols $\times$.

Definition 1

(1) The position $t = \frac{1}{2}$ is called the tail position.
(2) The length of the tail of a diagram (and the corresponding weight) is equal to the number of $\times$ at the tail position if the diagram does not have sign or the sign is $(-)$; the number of $\times$ at the tail position minus 1 if the diagrams has sign $(+)$. The diagram is tailless if the length of the tail is 0.
(3) The core of λ is the weight diagram (for a smaller rank Lie superalgebra of the same type) obtained when removing all the $\times$ of f_λ. The core determines the block of $\mathcal{F}$ containing the modules L_λ, $\mathcal{E}(\lambda)$ and P_λ. The core symbols are all the symbols $<$ and $>$.

Theorem 1 ([6])

(1) *Two simple modules L_λ and L_μ belong to the same block of $\mathcal{F}$ if and only if weight diagrams of λ and μ have the same core, and therefore the same number of $\times$.*
(2) *Two blocks B_1 and B_2 of $\mathcal{F}$ are equivalent if and only if: let $L_\lambda \in B_1$, $L_\mu \in B_2$, f_λ and f_μ have the same number of $\times$.*

Example 1 (1) If $\lambda = (\frac{9}{2}, \frac{7}{2}, \frac{-1}{2} | \frac{7}{2}, \frac{5}{2}, \frac{1}{2})$, then f_λ is $(-) \times \circ < \times > \cdots$. The symbol $\circ$ stands for an empty position, all positions to the right of $>$ are empty. The atypicality degree is 2, and the length of the tail is 1.

(2) If $\lambda = (\frac{9}{2}, \frac{7}{2}, \frac{1}{2}, \frac{-1}{2} | \frac{7}{2}, \frac{5}{2}, \frac{1}{2}, \frac{1}{2})$ then f_λ is

$$\begin{array}{l}(-) \times \circ < \times > \cdots \\ \quad\ \ \times\end{array}$$

the atypicality degree is 3 and the length of the tail is 2.

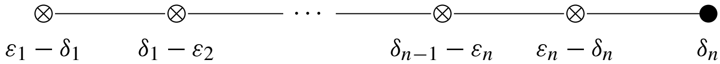
Recall that the *translation functors* are functors in $\mathcal{F}$ sending a block to another one (or possibly the same one). A translation functor is a composition of tensoring with the standard representation of $\mathfrak{osp}(2m+1, 2n)$ and projecting on the appropriate block. See for details [6] Sect. 5.

Important Remark Both papers describe algorithms giving, in the first one, the composition series of $\mathcal{E}_{G/B}(\lambda)$ or $\mathcal{E}_{G/Q_\lambda}(\lambda)$ if λ has a tail, in terms of simple modules, and in the second one an expression of a projective indecomposable as a linear combination of Euler characteristics for tailless weights, $\mathcal{E}_{G/B}(\mu)$.

3.1 Summary of [6] for $\mathfrak{osp}(2n+1,2n)$

This paper is focused on the character formula for simple modules. We restrict our attention to the maximally atypical block of $\mathfrak{osp}(2n+1,2n)$ since the translation functors lead us to understand all the other blocks, once this family of blocks is understood, see Theorem 2 and Corollary 5 in [6].

The Dynkin diagram of $\mathfrak{osp}(2n+1,2n)$ is the following:

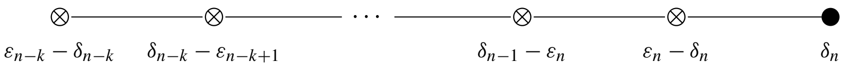


The principle of the method is as follows: the Euler characteristics have a character which is easy to compute, so the idea is to write the composition series of the Euler characteristics in terms of simple modules. Note that the highest weights of these simple modules are lower than the dominant weight of the Euler characteristic: thus one gets a triangular matrix with 1 on the diagonal. Inverting this matrix expresses a simple module in terms of Euler characteristics, and we deduce its character by applying the character formula for the Euler characteristics.

Let Q be a parabolic subgroup of G containing B and μ be an integral dominant weight which induces a one-dimensional representation of Q. Recall that

$$\mathcal{E}_{G/Q}(\mu)=\sum_{i=1}^{\dim G_0/Q_0}(-1)^i\big[H^i\big(G/Q,\mathcal{O}(-\mu)\big)^*\big].$$

If μ has a tail, then $\mathcal{E}_{G/B}(\mu)=0$. If the length of the tail of μ is $k+1$, we define $\mathfrak{q}_\mu$ as the parabolic subalgebra containing $\mathfrak{b}$ such that the semi-simple part of its Levi subalgebra has the following Dynkin diagram:



which is the Dynkin diagram of Lie superalgebra of the same type as $\mathfrak{osp}(2n+1,2n)$. Note that for a tailless μ, $\mathfrak{q}_\mu=\mathfrak{b}$.

As an element in the Grothendieck group of $\mathcal{F}$, the Euler characteristic $\mathcal{E}_{G/Q_\mu}(\mu)$ has a decomposition

$$\mathcal{E}_{G/Q_\mu}(\mu)=\sum a(\mu,\lambda)[L_\lambda].$$

Furthermore, $a(\mu,\mu)=1$ and $a(\mu,\lambda)\neq 0$ implies $\lambda\leq\mu$. The main result of [6] is a combinatorial algorithm for calculating $a(\mu,\lambda)$. Below we describe this algorithm.

Since in our case λ and μ are maximally atypical, their weight diagrams don't have any core symbols.

We say f_μ is obtained from f_λ by an elementary move if one or two $\times$ of f_λ are moved to some empty positions to the right according the following rules.

(1) Exceptional moves: can be made when λ has two $\times$ at the tail position, which are both moved simultaneously: see Definition 6, Sect. 11 of [6] for a precise definition, see the list of exceptional moves in the following sections for $\mathfrak{osp}(5,4)$ and $\mathfrak{osp}(7,6)$.
(2) Legal moves (resp. legal tail moves): take a $\times$ of f_λ at position s, $s \neq 1/2$ (resp. $s = 1/2$), move it to the right to an empty position $t > s$ of f_λ and obtain a new diagram f_μ. The $\times$ starts with 1 life (resp. 2 times the number of $\times$ at the tail position of f_μ), it looses 1 life going over an empty position, it gains one life over a $\times$ and should never have a negative number of lives. The number of lives that this moving $\times$ has at position t is called the degree (or the weight) of the corresponding legal move.

We say that f_μ is obtained from f_λ be a decreasing sequence of elementary moves $\lambda = \mu^0 \to \mu^1 \to \cdots \to \mu^k = \mu$ if f_{μ^i} is obtained from $f_{\mu^{i-1}}$ by moving a $\times$ to position t_i by a legal (or legal tail) move or two $\times$ to positions $s_i < t_i$ by an exceptional move and we have $t_1 > t_2 > \cdots > t_k$. The degree $l(\gamma)$ of a decreasing sequence γ of elementary moves is the sum of the degrees of the elementary moves included in the sequence.

Theorem 3 in [6] states that

$$a(\mu,\lambda) = \sum_{\gamma \in S(\lambda,\mu)} (-1)^{l(\gamma)},$$

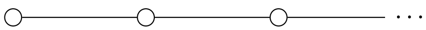
where the summation is taken over the set $S(\lambda, \mu)$ of all decreasing sequences of elementary moves from λ to μ.

Remark 2 It is proven in [6] that $a(\mu, \lambda) = \pm 1$ or 0 for all dominant integral λ, μ.

3.2 Summary of [7] for $\mathfrak{osp}(2m+1, 2n)$

This second paper contains several results. First of all, it explains in a more general context that a Bernstein-Gel'fand-Gel'fand reciprocity law holds in the category $\mathcal{F}$, in other words the multiplicity of a simple module L_λ in the Euler characteristic $\mathcal{E}_{G/B}(\mu)$ is the same as the multiplicity of $\mathcal{E}_{G/B}(\mu)$ in the projective indecomposable module P_λ, this equality holding in the Grothendieck ring of $\mathcal{F}$: it is to be noted that, in this paper, the only flag variety involved is G/B.

It also contains a categorification of the Lie algebra with Dynkin diagram



in orthosymplectic terms which allows us to interpret most of the translation functors as linear operators satisfying Serre relations.

The result we are interested in for this survey is the fact that one can express any projective indecomposable module as a linear combination with integral coefficients of Euler characteristics of tailless weights. Caution, these coefficients might be negative. We explain an algorithm on the weight diagrams which gives this combination.

Start with a tailless dominant weight λ, and consider its weight diagram. Construct the *cap diagram* as follows:

consider the rightmost $\times$ of f_λ and join it to the next free position on the right. This position is no longer free. Repeat for the next $\times$ on the left, and so on until there is no $\times$ left. Leave all the symbols corresponding to the core where they are.

Example 2 For the following weight diagram,

$$\circ \quad \times \quad > \quad < \quad \times \quad \circ \quad \circ$$

the caps are the following:

Denote by $\mathcal{P}(\lambda)$ the set $\mathcal{P}(\lambda) := \{\mu$ dominant, f_μ is obtained from f_λ by moving 0 or any number of $\times$ along the caps$\}$.

Now assume that λ has a tail: we construct a tailless weight $\bar{\lambda}$ the following way:

Ignore the sign before the diagram if it exists. In the beginning, forget about the tail position of f_λ and draw the corresponding cap diagram. Then circle the $\times$, getting $\otimes$, at the tail position, and move them according to the following rules:

- if λ has no core symbol at $\frac{1}{2}$ move all the $\otimes$ but one at the tail position to the free positions number 2, 4, 6, etc.
- if λ has a core symbol at $\frac{1}{2}$, then move all the $\otimes$ at the tail position to the free positions number 1, 3, 5, etc.

Now draw the cap diagram of this new weight $\bar{\lambda}$.

We are now ready to state the result:

Theorem 2

(1) *If λ is tailless, then one has*

$$P_\lambda = \sum_{\mu \in \mathcal{P}(\lambda)} \mathcal{E}_{G/B}(\mu).$$

(2) *If f_λ has a core symbol or a $(-)$ sign,*

$$P_\lambda = \sum_{\mu \in \mathcal{P}(\bar{\lambda})} (-1)^{c(\lambda,\mu)} \mathcal{E}_{G/B}(\mu),$$

where $c(\lambda, \mu)$ is the number of $\otimes$ in λ plus the number of $\otimes$ in $f_{\bar{\lambda}}$ moved along a cap in order to get f_μ from $f_{\bar{\lambda}}$.

(3) *If the sign before* f_λ *is* $(+)$, *use the preceding formula and change the sign of all the* $\mathcal{E}_{G/B}(\mu)$ *such that* f_μ *has a symbol at the tail position.*

The proof of this result involves a massive use of translation functors.

4 Computing Characters for a Simple Maximally Atypical Module over $\mathfrak{osp}(5,4)$

From now on, for any dominant λ we will abuse notation and set $\mathcal{E}(\lambda)$ for $\mathcal{E}_{G/Q_\lambda}(\lambda)$ if λ has a tail and $\mathcal{E}_{G/B}(\lambda)$ if λ is tailless.

In this case, a dominant weight has the form:

$$\lambda+\rho=(a_1,a_2|c_1,c_2)$$

with $a_1>a_2\geq-\frac{1}{2}$ or $a_1=a_2=-\frac{1}{2}$ and $c_1>c_2\geq\frac{1}{2}$ or $c_1=c_2=\frac{1}{2}$. It is maximally atypical iff $|a_1|=c_1$ and $|a_2|=c_2$. The weight diagram of a maximally atypical weight contains two $\times$, one at $|a_1|$ and the other at $|a_2|$, together with a sign. If there are two $\times$ at the tail position or one $\times$ and a $(-)$ sign, then the weight has a tail and the parabolic subgroup Q_λ of the previous section is obtained by adding the opposite of the roots $\varepsilon_1-\delta_1, \delta_1-\varepsilon_2$ unless the weight is trivial in which case $Q_\lambda=G$. Another difficulty occurs when one gets close to the *wall* $a_1=a_2+1$.

In [6] Sect. 11, we described a series of moves which can be made with the $\times$ of the weight diagram: if there is a (authorised) move from the weight diagram f_λ to the weight diagram f_μ of weight (or degree) i, it means that the simple module L_λ is in the cohomology group of degree i corresponding to the Euler characteristic $\mathcal{E}(\mu)$, so that it occurs with the sign $(-1)^i$ in the composition series. Nevertheless, it doesn't mean that L_λ appears in $\mathcal{E}(\mu)$ because one also has to consider paths, which are sequences of moves, and it can lead to cancellations.

There are several kinds of moves: *regular* ones, which take a $\times$ at a non-tail position and move it to the right according to specific rules, *tail moves*, which deal with one $\times$ at the tail position, and *exceptional moves* which move simultaneously two $\times$ at the tail position (see Proposition 6 in [6]), in this case there is no exceptional moves.

One can check by hand all the possibilities which occur.

In Fig. 1, we have represented a maximally atypical weight $\lambda+\rho=(a_1,a_2||a_1|, |a_2|)$ by the point (a_1,a_2) in the plane and we join two points if there exists a legal move taking the weight diagram of the first weight to the weight diagram of the second one. We have equipped all the arrows with their weights.

Now, we want to compute the multiplicity of the simple module L_λ in the Euler characteristic $\mathcal{E}_{G/Q_\mu}(\mu)$. We have to consider:

(1) Arrow going from λ to μ with weight i (there is at most one), we will say we have a *path of length one* P and weight $wt(P):=i$.

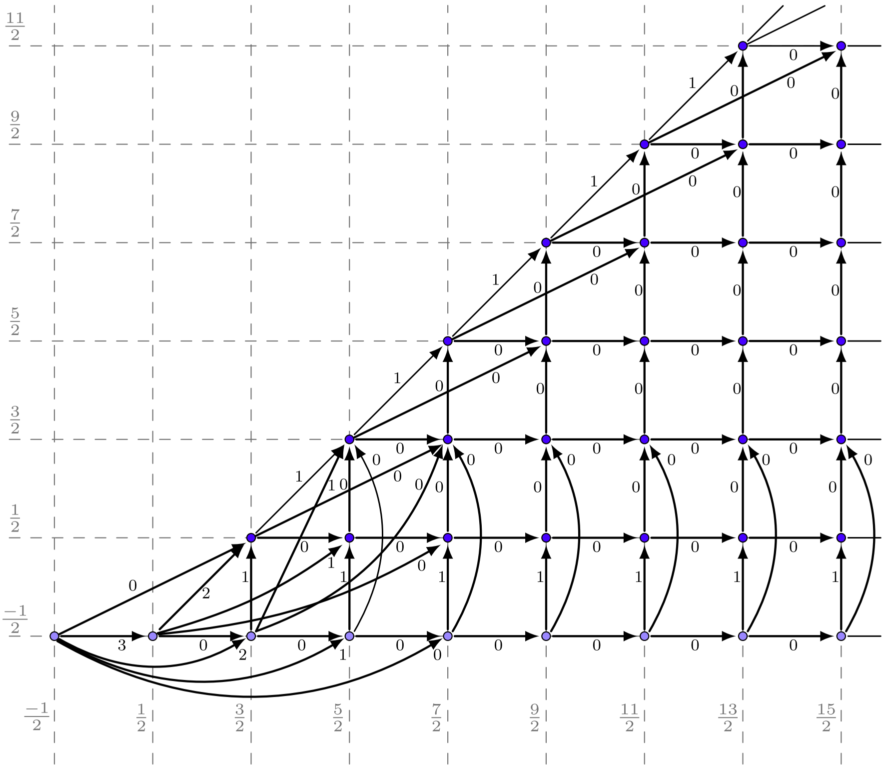


Fig. 1 $\mathfrak{osp}(5,4)$

(2) *regular paths of length two* from λ to μ: a regular path P_i is a sequence of two arrows, one, f_1 of weight i_1, from λ to a certain λ_1 and one, f_2 with weight i_2, from λ_1 to μ, such that the first one f_1 is going East or North-East in the picture (meaning that this arrow can increase the horizontal coordinate and possibly the vertical one) and the second one f_2 goes straight North (so the horizontal coordinate cannot be increased). The weight of the corresponding path is $wt(P_i) := i_1 + i_2$.

Proposition 1 *Let λ and μ be two dominant weights such that $\lambda \leq \mu$. Then the multiplicities are as follows*:

(1) $[\mathcal{E}(\mu) : L_\lambda] = 1$ *if* $\lambda = \mu$,
(2) *If $\lambda < \mu$, look at all the paths of length one and two from λ to μ, denote this set* $P(\lambda,\mu)$, $[\mathcal{E}(\mu) : L_\lambda] = \sum_{P \in P(\lambda,\mu)} (-1)^{wt(P)}$.

One can check on the picture that the module of the multiplicity of L_λ in the Euler characteristic $\mathcal{E}_{G/_\mu}(\mu)$ is at most one. This is a general phenomenon for algebras $\mathfrak{osp}(2n+1, 2m)$.

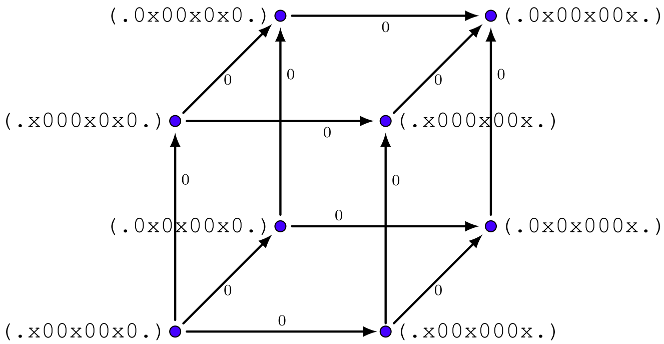


Fig. 2 $\mathfrak{osp}(7,6)$, generic case

Remark 3

(1) If μ is far from the walls, meaning that $a_1 \geq a_2 + 3 \geq 5/2$, then the partial picture explaining which L_λs appear in the Euler characteristic is just a square of size 1×1.

If one looks at the same picture for maximally atypical weights of $\mathfrak{osp}(2n + 1, 2n)$, if the weight diagram of μ has two empty positions between each couple of $\times$ and μ is far enough from the tail, the weights λ occurring in the Euler characteristic are the vertices of the hypercube with "greater vertex" μ. See Fig. 2 for the case $\mathfrak{osp}(7,6)$.

(2) As long as μ is far enough from the origin, $a_1 > 9/2$, the pattern along the walls is always the same.

We put all this information in a big (infinite) triangular unipotent matrix M labelled by all dominant maximally atypical weights, the line labelled by the weight λ encoding in which Euler characteristics L_λ occurs, and with which multiplicity. This matrix gives us the composition series of all the Euler characteristics, and since we know the character of the Euler characteristics, if we invert M and hence obtain a simple module as a linear combination of Euler characteristics, we are able to compute the character of the simple module. Well, able might be abusing language, since no one wants to explicitely compute all this...

Let us show the matrix (Table 1) for small weights, with the conventions of the Fig. 1 for the weights.

For instance, let us explain how we get the column corresponding to $(\frac{3}{2}, \frac{1}{2})$: look at the picture, and the arrows coming to this weight: one gets $(\frac{3}{2}, -\frac{1}{2})$ with weight -1, then $(\frac{1}{2}, -\frac{1}{2})$ with weight 2, but it cancels with the path $(\frac{1}{2}, -\frac{1}{2}) \to (\frac{3}{2}, -\frac{1}{2}) \to (\frac{3}{2}, \frac{1}{2})$ which is of weight 1, and then the path $(-\frac{1}{2}, -\frac{1}{2}) \to (\frac{1}{2}, -\frac{1}{2}) \to (\frac{3}{2}, \frac{1}{2})$ which has weight 5 cancels with the arrow coming from the exceptional move $(-\frac{1}{2}, -\frac{1}{2}) \to (\frac{3}{2}, \frac{1}{2})$. Finally, $(\frac{3}{2}, \frac{1}{2})$ itself appears with multiplicity 1, hence the column.

Table 1 Partial decomposition matrix

	$(\frac{-1}{2},\frac{-1}{2})$	$(\frac{1}{2},\frac{-1}{2})$	$(\frac{3}{2},\frac{-1}{2})$	$(\frac{5}{2},\frac{-1}{2})$	$(\frac{7}{2},\frac{-1}{2})$	$(\frac{9}{2},\frac{-1}{2})$	$(\frac{3}{2},\frac{1}{2})$	$(\frac{5}{2},\frac{1}{2})$	$(\frac{7}{2},\frac{1}{2})$	$(\frac{9}{2},\frac{1}{2})$	$(\frac{5}{2},\frac{3}{2})$	$(\frac{7}{2},\frac{3}{2})$	$(\frac{9}{2},\frac{3}{2})$	$(\frac{7}{2},\frac{5}{2})$	$(\frac{9}{2},\frac{5}{2})$	$(\frac{9}{2},\frac{7}{2})$
$(\frac{-1}{2},\frac{-1}{2})$	1	−1	1	−1	1	0	0	1	−1	0	−1	1	0	0	0	0
$(\frac{1}{2},\frac{-1}{2})$		1	1	0	0	0	0	−1	1	0	−1	1	0	0	0	0
$(\frac{3}{2},\frac{-1}{2})$			1	1	0	0	−1	−1	0	0	0	1	0	1	0	0
$(\frac{5}{2},\frac{-1}{2})$				1	1	0	0	−1	−1	0	1	1	0	0	0	0
$(\frac{7}{2},\frac{-1}{2})$					1	1	0	0	−1	−1	0	1	1	0	0	0
$(\frac{9}{2},\frac{-1}{2})$						1	0	0	0	−1	0	0	1	0	0	0
$(\frac{3}{2},\frac{1}{2})$							1	1	0	0	0	1	0	1	0	0
$(\frac{5}{2},\frac{1}{2})$								1	1	0	1	1	0	0	0	0
$(\frac{7}{2},\frac{1}{2})$									1	1	0	1	1	0	0	0
$(\frac{9}{2},\frac{1}{2})$										1	0	0	1	0	0	0
$(\frac{5}{2},\frac{3}{2})$											1	1	0	0	1	1
$(\frac{7}{2},\frac{3}{2})$												1	1	1	1	0
$(\frac{9}{2},\frac{3}{2})$													1	0	1	0
$(\frac{7}{2},\frac{5}{2})$														1	1	0
$(\frac{9}{2},\frac{5}{2})$															1	1

5 Projective Indecomposable Modules for $\mathfrak{osp}(5,4)$, Maximally Atypical Case

In [7], we showed that one can express any projective indecomposable module in the Grothendieck group $\mathcal{K}(\mathcal{F})$ as a linear combination with integral coefficients (possibly negative) of Euler characteristic for tailless weights, hence the underlying algebraic supermanifold is the flag variety G/B. We also showed that there is a (weak version of) Bernstein-Gel'fand-Gel'fand reciprocity law (see [7], Theorem 1):

Proposition 2 *Let λ and μ be two dominant weights such that μ is tailless, one has*:

$$\left[\mathcal{E}(\mu) : L_\lambda\right] = \left[P_\lambda : \mathcal{E}(\mu)\right].$$

Remark 4 Note that Euler characteristics for tailless weights do not form a basis in the Grothendieck group. Since our category has infinite cohomological dimension, classes of projective modules generate a proper subgroup in the Grothendieck group (see [7]). However, Euler characteristics are linearly independent, hence the presentation of the class of a projective module as a combination of Euler characteristics is unique.

Hence, actually we have already computed all the coefficients of this linear combination while computing the characters of simple modules, or, more appropriately, the multiplicity of the simple modules occurring in a given Euler characteristic for tailless weights. Note that the (partial) matrix of the previous section contains the information for Euler characteristics for weights with a tail (the lines corresponding to weights with first coordinate equal to zero), and these ones are not relevant in the computation we do now.

Thanks to the algorithm described in [7] that allows to compute the coefficients of the linear combination of Euler characteristics involved in a given projective module, we obtain the decomposition numbers of the previous section by an independent method.

Let us take the opportunity of this paper to describe the decomposition of projective indecomposable modules of maximally atypicality degree in terms of simple modules.

Let λ be a dominant weight, we write $\lambda + \rho = (a_1, a_2 || a_1|, |a_2|)$. For simplicity we encode λ by (a_1, a_2), as in the previous section. Assume that $a_1 - a_2 \geq 4$ and $a_2 \geq \frac{5}{2}$, we say that λ is *generic*, then the Euler characteristics involved are these of (a_1, a_2), $(a_1 + 1, a_2)$, $(a_1, a_2 + 1)$ and $(a_1 + 1, a_2 + 1)$ so that the simple modules involved are (see Table 2).

Let us study now the generic weights which are near the oblique wall.

Case $a_1 = a_2 + 3$, $a_2 \geq \frac{5}{2}$: The Euler characteristics involved are the same as in the generic case, but $\mathcal{E}(a_1, a_2 + 1)$ has $L_{(a_1-2,a_2)}$ as an additional composition factor. Hence Table 3.

Case $a_1 = a_2 + 2$: The Euler characteristics involved are the same as in the generic case, but $\mathcal{E}(a_1, a_2)$ has an additional composition factor which is

Table 2 Highest weights of simple modules occurring in P_λ, $\lambda = (a_1, a_2)$ generic

Coordinates of simple factor	Multiplicity
(a_1+1, a_2+1)	1
(a_1+1, a_2)	2
(a_1+1, a_2-1)	1
(a_1, a_2+1)	2
(a_1, a_2)	4
(a_1, a_2-1)	2
(a_1-1, a_2+1)	1
(a_1-1, a_2)	2
(a_1-1, a_2-1)	1

Table 3 Highest weights of simple modules occurring in P_λ, $\lambda = (a_1, a_2)$, $a_1 - a_2 = 3$, $a_2 \geq \frac{5}{2}$

Coordinates of simple factor	Multiplicity
(a_1+1, a_2+1)	1
(a_1+1, a_2)	2
(a_1+1, a_2-1)	1
(a_1, a_2+1)	2
(a_1, a_2)	4
(a_1, a_2-1)	2
(a_1-2, a_2)	1
(a_1-1, a_2+1)	1
(a_1-1, a_2)	2
(a_1-1, a_2-1)	1

Table 4 Highest weights of simple modules occurring in P_λ, $\lambda = (a_1, a_2)$, $a_1 - a_2 = 2$, $a_1 \geq \frac{5}{2}$

Coordinates of simple factor	Multiplicity
(a_1+1, a_2+1)	1
(a_1, a_2+1)	2
(a_1+1, a_2)	2
(a_1, a_2)	4
(a_1-1, a_2)	2
(a_1+1, a_2-1)	1
(a_1, a_2-1)	2
(a_1-1, a_2-1)	1
(a_1-2, a_2-1)	1

Table 5 Highest weights of simple modules occurring in P_λ, $\lambda = (a_1, a_2)$, $a_1 - a_2 = 1$, $a_2 \geq \frac{5}{2}$

Coordinates of simple factor	Multiplicity
(a_1+2, a_2+2)	1
(a_1+2, a_2+1)	2
(a_1+1, a_2+1)	1
(a_1+2, a_2)	1
(a_1+1, a_2)	2
(a_1, a_2)	4
(a_1+1, a_2-1)	1
(a_1, a_2-1)	2
(a_1-1, a_2-1)	1

Table 6 Highest weights of simple modules occurring in P_λ, $\lambda = (a_1, 3/2)$, $a_1 \geq 11/2$

Coordinates of simple factor	Multiplicity
$(a_1+1, 5/2)$	1
$(a_1, 5/2)$	2
$(a_1-1, 5/2)$	1
$(a_1+1, 3/2)$	2
$(a_1, 3/2)$	4
$(a_1-1, 3/2)$	2
$(a_1+1, 1/2)$	1
$(a_1, 1/2)$	2
$(a_1-1, 1/2)$	1
$(a_1+1, -1/2)$	1
$(a_1, -1/2)$	2
$(a_1-1, -1/2)$	1

$L_{(a_1-2,a_2-1)}$, $\mathcal{E}(a_1+1, a_2+1)$ has $L_{(a_1-1,a_2)}$ as an additional composition factor and $\mathcal{E}(a_1, a_2+1)$ is smaller than expected since it lacks $L_{(a_1-1,a_2+1)}$. Hence Table 4.

Case $a_1 = a_2 + 1$: The Euler characteristics involved are these corresponding to (a_1, a_2), (a_1+1, a_2), (a_1+2, a_2+1), (a_1+2, a_2+2). We get Table 5.

Next we study *generic weights near the tail* $a_1 \geq \frac{11}{2}$.

Case $a_2 = \frac{3}{2}$.

The Euler characteristics involved are the usual ones and we have several additional composition factors in them, see Table 6.

Case $a_2 = \frac{1}{2}$.

The Euler characteristics involved are the usual ones and we have several additional composition factors in them. See Table 7.

Case $a_1 \geq \frac{9}{2}$, $a_2 = -\frac{1}{2}$.

See Table 8.

Table 7 Highest weights of simple modules occurring in P_λ, $\lambda = (a_1, \frac{1}{2})$, $a_1 \geq \frac{9}{2}$

Coordinates of simple factor	Multiplicity
$(a_1+1, 3/2)$	1
$(a_1, 3/2)$	2
$(a_1-1, 3/2)$	1
$(a_1+1, 1/2)$	2
$(a_1, 1/2)$	4
$(a_1-1, 1/2)$	2

Table 8 Highest weights of simple modules occurring in P_λ, $\lambda = (a_1, -\frac{1}{2})$, $a_1 \geq \frac{9}{2}$

Coordinates of simple factor	Multiplicity
$(a_1+1, 3/2)$	1
$(a_1, 3/2)$	2
$(a_1-1, 3/2)$	1
$(a_1+1, -1/2)$	2
$(a_1, -1/2)$	4
$(a_1-1, -1/2)$	2

Table 9 Highest weights of simple modules occurring in P_λ, $\lambda(9/2, 3/2)$

Coordinates of simple factor	Multiplicity
$(11/2, 5/2)$	1
$(9/2, 5/2)$	2
$(7/2, 5/2)$	1
$(11/2, 3/2)$	2
$(9/2, 3/2)$	4
$(7/2, 3/2)$	2
$(5/2, 3/2)$	1
$(11/2, 1/2)$	1
$(9/2, 1/2)$	2
$(7/2, 1/2)$	1
$(11/2, -1/2)$	1
$(9/2, -1/2)$	2
$(7/2, -1/2)$	1

The remaining weights are represented by the couples $(-\frac{1}{2}, -\frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$, $(\frac{3}{2}, -\frac{1}{2})$, $(\frac{5}{2}, -\frac{1}{2})$, $(\frac{7}{2}, -\frac{1}{2})$, $(\frac{3}{2}, \frac{1}{2})$, $(\frac{5}{2}, \frac{1}{2})$, $(\frac{7}{2}, \frac{1}{2})$, $(\frac{5}{2}, \frac{3}{2})$, $(\frac{7}{2}, \frac{3}{2})$ and $(\frac{9}{2}, \frac{3}{2})$.

We intend to use the partial matrix A we wrote in the previous section, suppressing the lines corresponding to Euler characteristics for weights with tail, and compute ${}^tA.A$. Caution, the relevant information in this matrix concerns only the weights which are labelled by (a_1, a_2) with $a_1 < 9/2$ and $a_2 < 5/2$, since we need additional information to get the other weights. We first do by hand the case $(a_1, a_2) = (9/2, 3/2)$, see Table 9.

Table 10 Partial Cartan matrix

	$(-\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2})$	$(\frac{3}{2},-\frac{1}{2})$	$(\frac{5}{2},-\frac{1}{2})$	$(\frac{7}{2},-\frac{1}{2})$	$(\frac{3}{2},\frac{1}{2})$	$(\frac{5}{2},\frac{1}{2})$	$(\frac{7}{2},\frac{1}{2})$	$(\frac{5}{2},\frac{3}{2})$	$(\frac{7}{2},\frac{3}{2})$
$(-1/2,-1/2)$	4				2	2				1
$(1/2,-1/2)$		4	2					2		1
$(3/2,-1/2)$		2	4	2	1			1	1	2
$(5/2,-1/2)$			2	4	2				2	1
$(7/2,-1/2)$	2		1	2	4	1			1	2
$(9/2,-1/2)$					2					1
$(3/2,1/2)$	2				1	4	2	1	1	2
$(5/2,1/2)$						2	4	2	2	1
$(7/2,1/2)$		2	1			1	2	4	1	2
$(9/2,1/2)$								2		1
$(5/2,3/2)$			1	2	1	1	2	1	4	2
$(7/2,3/2)$	1	1	2	1	2	2	1	2	2	4
$(9/2,3/2)$					1			1	1	2
$(7/2,5/2)$			1			1			1	2
$(9/2,5/2)$									2	1
$(9/2,7/2)$									1	

Table 10 is the result of the multiplication of matrices mentioned above, it should be read this way: the line labelled by (b_1, b_2) is the decomposition of the corresponding indecomposable projective module in terms of the simple modules labelled by the columns.

6 Generic Picture for $\mathfrak{osp}(7,6)$, Exceptional Moves for $\mathfrak{osp}(7,6)$ (and Remarks on Higher Rank Cases)

As is explained in [6], in order to get rid of the signs of the weight diagrams, it is better to look at the dominant weights of $\mathfrak{osp}(7,8)$ belonging to the same block as the trivial module. This means adding a $<$ at the tail position, move all $\times$ not at the tail one position to the right and for the $\times$ at the tail, if the sign is $(-)$ don't change anything, whether if the sign is $(+)$ move exactly one $\times$ from the tail one position to the right.

The weight diagram of a dominant maximally atypical weight has exactly three $\times$ plus a $<$ at the tail.

6.1 Generic Maximally Atypical Weights

One can draw a picture similar to Fig. 1, but it is 3-dimensional and quite intricate near the origin… Nevertheless, for a "generic" maximally atypical weight (meaning there are at least 2 empty positions between two $\times$ and it is far from the tail), the picture is easy to make, see Fig. 2. In this picture, the legal way is to go East then North then North-East.

Remark 5 For maximally atypical weights of $\mathfrak{osp}(2n+1, 2n)$ which are generic, i.e. such that the first $\times$ in the weight diagram is far from the tail position and there are at least two empty positions between two $\times$, the picture looks the same and the legal way is to move along the basis vectors corresponding first to the rightmost $\times$, then the following rightmost $\times$ and so on.

6.2 Exceptional Moves

In $\mathfrak{osp}(7, 6)$, there are infinitely many weights leading to exceptional moves, because there are more than two $\times$, see the case (5) where the rightmost $\times$ can be at any place further right. Here is a list of these moves, we indicate the parity of the weight of the corresponding move if it is not 0.

(1)

$$f_\mu = \begin{array}{l} < \\ \times \\ \times \\ \times \end{array} \longrightarrow f_\lambda = \begin{cases} < \\ \times \; \times \; \times \\ \text{or} \\ < \\ \times \; \circ \; \circ \; \times \; \times \end{cases}$$

(2)

$$f_\mu = \begin{array}{l} < \\ \times \\ \times \; \times \end{array} \longrightarrow f_\lambda = < \times \circ \times \times$$

(3)

$$f_\mu = \begin{array}{l} < \\ \times \\ \times \; \circ \; \times \end{array} \longrightarrow f_\lambda = \begin{cases} < \times \times \times \ (1) \\ \text{or} \\ < \times \times \circ \quad \times \\ \text{or} \\ < \circ \times \times \quad \times \end{cases}$$

(4)

$$f_\mu = \begin{array}{l} < \\ \times \\ \times \; \circ \; \circ \; \times \end{array} \longrightarrow f_\lambda = \begin{cases} < \times \times \times \\ \text{or} \\ < \times \circ \times \times \end{cases}$$

(5)

$$f_\mu = \begin{array}{l} < \\ \times \\ \times \circ \circ \circ \times \end{array} \longrightarrow f_\lambda = < \times \times \circ \times \text{ etc.}$$

Remark 6 This last move can be reproduced for any diagram f_μ with the same pattern at the tail and the last $\times$ at any position further on the right, with the obvious change on the diagram f_λ.

If one looks closely at the definition of admissible paths, such a move can be combined with any move concerning the $\times$ not involved in the exceptional move, so that these exceptional things are really annoying... and one has to be extremely careful in the computations. Is there still anyone wondering why we didn't draw the complete figure?

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Monoidal Categorifications of Cluster Algebras of Type *A* and *D*

David Hernandez and Bernard Leclerc

Abstract In this note, we introduce monoidal subcategories of the tensor category of finite-dimensional representations of a simply-laced quantum affine algebra, parametrized by arbitrary Dynkin quivers. For linearly oriented quivers of types A and D, we show that these categories provide monoidal categorifications of cluster algebras of the same type. The proof is purely representation-theoretical, in the spirit of Hernandez and Leclerc (Duke Math. J. 154, 265–341, 2010).

1 Introduction

The theory of cluster algebras has received a lot of attention in the recent years because of its numerous connections with many fields, in particular Lie theory and quiver representations.

One important problem is to categorify cluster algebras. In recent years, many examples of *additive* categorifications of cluster algebras have been constructed. The concept of a *monoidal* categorification of a cluster algebra, which is quite different, was introduced in [15, Definition 2.1]. If a cluster algebra has a monoidal categorification, we get informations on its structure (positivity, linear independence of cluster monomials). Conversely, if a monoidal category is a monoidal categorification of a cluster algebra of finite type, we can calculate the factorization of any simple object as a tensor product of finitely many prime objects, as well as the composition factors of a tensor product of simple objects.

To M. Jimbo on his 60th birthday.

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In [15] we have introduced a certain monoidal subcategory $\mathcal{C}_1$ of the category $\mathcal{C}$ of finite-dimensional representations of a simply-laced quantum affine algebra, and we have conjectured that $\mathcal{C}_1$ is a monoidal categorification of a cluster algebra of the same type. This conjecture was proved in [15] for types A and D_4, and in [20] for all A, D, E types. The proof in [15] relies on representation theory, and on the well-developed combinatorics of cluster algebras of finite type. Nakajima's proof is different and uses additional geometric tools: a tensor category of perverse sheaves on quiver varieties, and the Caldero-Chapoton formula for cluster variables.

The categories $\mathcal{C}_1$ of [15] are associated with bipartite Dynkin quivers. In this note, we introduce monoidal subcategories $\mathcal{C}_\xi$ of $\mathcal{C}$ associated with arbitrary Dynkin quivers. For types A and D, we show that the categories $\mathcal{C}_\xi$ corresponding to linearly oriented quivers provide new monoidal categorifications of cluster algebras of the same type. The proof is similar to [15]. However, the main calculations are much simpler because, for these choices of ξ, the irreducibility criterion for products of prime representations is more accessible than for the categories $\mathcal{C}_1$. This is why we can also treat in this note the cases D_n $(n \geq 5)$.

In his PhD thesis, Fan Qin [21] has recently generalized the geometric approach of Nakajima (partly in collaboration with Kimura), and obtained monoidal categorifications of cluster algebras associated with an arbitrary acyclic quiver (not necessarily bipartite) using perverse sheaves on quiver varieties.

2 Cluster Algebras and Their Monoidal Categorifications

We refer to [4, 17] for excellent surveys on cluster algebras.

2.1 Cluster Algebras

Let $0 \leq n < r$ be some fixed integers. If $\widetilde{B} = (b_{ij})$ is an $r \times (r-n)$-matrix with integer entries, then the *principal part* B of $\widetilde{B}$ is the square matrix obtained from $\widetilde{B}$ by deleting the last n rows. Given some $k \in [1, r-n]$ define a new $r \times (r-n)$-matrix $\mu_k(\widetilde{B}) = (b'_{ij})$ by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise,} \end{cases} \quad (1)$$

where $i \in [1, r]$ and $j \in [1, r-n]$. One calls $\mu_k(\widetilde{B})$ the *mutation* of the matrix $\widetilde{B}$ in direction k. If $\widetilde{B}$ is an integer matrix whose principal part is skew-symmetric, then it is easy to check that $\mu_k(\widetilde{B})$ is also an integer matrix with skew-symmetric principal part. We will assume from now on that $\widetilde{B}$ has skew-symmetric principal part. In this case, one can equivalently encode $\widetilde{B}$ by a quiver Γ with vertex set $\{1, \ldots, r\}$ and with b_{ij} arrows from j to i if $b_{ij} > 0$ and $-b_{ij}$ arrows from i to j if $b_{ij} < 0$.

Now Fomin and Zelevinsky define a cluster algebra $\mathcal{A}(\widetilde{B})$ as follows. Let $\mathcal{F} = \mathbb{Q}(x_1, \dots, x_r)$ be the field of rational functions in r commuting indeterminates $\mathbf{x} = (x_1, \dots, x_r)$. One calls $(\mathbf{x}, \widetilde{B})$ the *initial seed* of $\mathcal{A}(\widetilde{B})$. For $1 \leq k \leq r - n$ define

$$x_k^* = \frac{\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}}{x_k}. \tag{2}$$

The pair $(\mu_k(\mathbf{x}), \mu_k(\widetilde{B}))$, where $\mu_k(\mathbf{x})$ is obtained from $\mathbf{x}$ by replacing x_k by x_k^*, is the *mutation* of the seed $(\mathbf{x}, \widetilde{B})$ in direction k. One can iterate this procedure and obtain new seeds by mutating $(\mu_k(\mathbf{x}), \mu_k(\widetilde{B}))$ in any direction $l \in [1, r - n]$. Let $\mathcal{S}$ denote the set of all seeds obtained from $(\mathbf{x}, \widetilde{B})$ by any finite sequence of mutations. Each seed of $\mathcal{S}$ consists of an r-tuple of elements of $\mathcal{F}$ called a *cluster*, and of a matrix. The elements of a cluster are its *cluster variables*. One does not mutate the last n elements of a cluster; they are called *frozen variables* and belong to every cluster. We then define the *cluster algebra* $\mathcal{A}(\widetilde{B})$ as the subring of $\mathcal{F}$ generated by all the cluster variables of the seeds of $\mathcal{S}$. A *cluster monomial* is a monomial in the cluster variables of a *single* cluster. Two cluster variables are said to be *compatible* if they occur in the same cluster.

The first important result of the theory is that every cluster variable z of $\mathcal{A}(\widetilde{B})$ is a Laurent polynomial in $\mathbf{x}$ with coefficients in $\mathbb{Z}$. It is conjectured that the coefficients are positive.

The second main result is the classification of *cluster algebras of finite type*, i.e. with finitely many different cluster variables. Fomin and Zelevinsky proved that this happens if and only if there exists a seed $(\mathbf{z}, \widetilde{C})$ such that the quiver attached to the principal part of $\widetilde{C}$ is a Dynkin quiver (that is, an arbitrary orientation of a Dynkin diagram of type A, D, E).

In [5], Fomin and Zelevinsky have shown that the cluster variables of a cluster algebra $\mathcal{A}$ have a nice expression in terms of certain polynomials called the *F-polynomials*. In type A and D, explicit formulas for F-polynomials are known.

2.2 Monoidal Categorifications

The concept of a monoidal categorification of a cluster algebra was introduced in [15, Definition 2.1]. We say that a simple object S of a monoidal category is *real* if $S \otimes S$ is simple.

Definition 1 Let $\mathcal{A}$ be a cluster algebra and let $\mathcal{M}$ be an abelian monoidal category. We say that $\mathcal{M}$ is a monoidal categorification of $\mathcal{A}$ if there is an isomorphism between $\mathcal{A}$ and the Grothendieck ring of $\mathcal{M}$ such that the cluster monomials of $\mathcal{A}$ are the classes of all the real simple objects of $\mathcal{M}$.

A non trivial simple object S of $\mathcal{M}$ is *prime* if there exists no non trivial factorization $S \cong S_1 \otimes S_2$. By [10, Sect. 8.2], the cluster variables of $\mathcal{A}$ are the classes of all the real prime simple objects of $\mathcal{M}$. So Definition 1 coincides with the definition in [15].

As an application, we get information on the cluster algebra, as shown by the following result.

Proposition 1 ([15]) *If a cluster algebra $\mathcal{A}$ has a monoidal categorification, then*

(i) *every cluster variable of $\mathcal{A}$ has a Laurent expansion with positive coefficients with respect to any cluster;*
(ii) *the cluster monomials of $\mathcal{A}$ are linearly independent.*

Assertion (ii) can also be proved by using additive categorification, see the recent [1].

Conversely, if $\mathcal{M}$ is a monoidal categorification of a finite type cluster algebra, we can calculate the factorization of any simple object of $\mathcal{M}$ as a tensor product of finitely many prime objects, as well as the composition factors of a tensor product of simple objects of $\mathcal{M}$. Moreover, every simple object in $\mathcal{M}$ is real.

3 Categories of Finite-Dimensional Representations of $U_q(L\mathfrak{g})$

For recent surveys on the representation theory of quantum loop algebras, we invite the reader to consult [2] or [18].

3.1 Simple Lie Algebra

Let $\mathfrak{g}$ be a simple Lie algebra of type A, D, E. We denote by I the set of vertices of its Dynkin diagram, and we put $n = |I|$. The *Cartan matrix* of $\mathfrak{g}$ is the $I \times I$ matrix $C = (C_{ij})_{i,j \in I}$. We denote by α_i $(i \in I)$ and ϖ_i $(i \in I)$ the simple roots and fundamental weights of $\mathfrak{g}$, respectively.

Let $\xi : I \to \mathbb{Z}$ be a *height function*, that is $|\xi_j - \xi_i| = 1$ if $C_{ij} = -1$. It induces an orientation Q of the Dynkin diagram of $\mathfrak{g}$ such that we have an arrow $i \to j$ if $C_{ij} = -1$ and $\xi_j = \xi_i - 1$. Define

$$\widehat{I} := \big\{(i, p) \in I \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z}\big\}.$$

3.2 Quantum Loop Algebra

Let $L\mathfrak{g}$ be the loop algebra attached to $\mathfrak{g}$, and let $U_q(L\mathfrak{g})$ be the associated quantum enveloping algebra. We assume that the deformation parameter $q \in \mathbb{C}^*$ is not a root of unity.

The simple finite-dimensional irreducible $U_q(L\mathfrak{g})$-modules (of type 1) are usually labeled by Drinfeld polynomials. Here we shall use an alternative labeling by dominant monomials (see [8]). Moreover, as in [15], we shall restrict our attention to a certain tensor subcategory $\mathcal{C}_{\mathbb{Z}}$ of the category of finite-dimensional $U_q(L\mathfrak{g})$-modules. The simple modules in $\mathcal{C}_{\mathbb{Z}}$ are labeled by the dominant monomials in $\mathcal{Y} = \mathbb{Z}[Y_{i,p}^{\pm 1} \mid (i, p) \in \widehat{I}\,]$, that is monomials $m = \prod_{(i,p)\in\widehat{I}} Y_{i,p}^{u_{i,p}(m)}$ such that $u_{i,p}(m) \geq 0$ for every $(i, p) \in \widehat{I}$.

We shall denote by $L(m)$ the simple module labeled by the dominant monomial m.

By [8], every object M in $\mathcal{C}_{\mathbb{Z}}$ has a q-character $\chi_q(M) \in \mathcal{Y}$. These q-characters generate a commutative ring $\mathcal{K}$ isomorphic to the Grothendieck ring of $\mathcal{C}_{\mathbb{Z}}$.

By [7, 8], we have $\chi_q(L(m)) \in m\mathbb{Z}[A_{i,p+1}^{-1}]_{(i,p)\in\widehat{I}}$ where for $(i, p) \in \widehat{I}$ we denote

$$A_{i,p+1} = Y_{i,p} Y_{i,p+2} \prod_{j\in I, C_{ij}=-1} Y_{j,p+1}^{-1} \in \mathcal{Y}.$$

In particular, an element in $\mathcal{K}$ is characterized by the multiplicity of its dominant monomials. When m is the only dominant monomial occurring in $\chi \in \mathcal{Y}$, χ is said to be *minuscule*. We say that M is minuscule if $\chi_q(M)$ is minuscule. This implies that M is simple.

3.3 The Monoidal Category $\mathcal{C}_\xi$

Define

$$\widehat{I}_\xi := \left\{(i, \xi_i) \mid i \in I\right\} \cup \left\{(i, \xi_i + 2) \mid i \in I\right\} \subset \widehat{I},$$

and let $\mathcal{Y}_\xi$ be the subring of $\mathcal{Y}$ generated by the variables $Y_{i,p}$ $((i, p) \in \widehat{I}_\xi)$.

Definition 2 $\mathcal{C}_\xi$ is the full subcategory of $\mathcal{C}_{\mathbb{Z}}$ whose objects have all their composition factors of the form $L(m)$ where m is a dominant monomial in $\mathcal{Y}_\xi$.

When Q is a sink-source orientation, we recover the subcategories $\mathcal{C}_1$ introduced in [15]. Since $\widehat{I_\xi}$ is a "convex slice" of $\widehat{I}$, we get as in [16, Lemma 5.8]:

Lemma 1 *$\mathcal{C}_\xi$ is closed under tensor products, hence is a monoidal subcategory of $\mathcal{C}_{\mathbb{Z}}$.*

We denote by $\mathcal{K}_\xi$ the subring of $\mathcal{K}$ spanned by the q-characters $\chi_q(L(m))$ of the simple objects $L(m)$ in $\mathcal{C}_\xi$. Then $\mathcal{K}_\xi$ is isomorphic to the Grothendieck ring $\mathcal{R}_\xi$ of $\mathcal{C}_\xi$. Note that this ring is a polynomial ring over $\mathbb{Z}$ with generators the classes of the $2n$ fundamental modules

$$L(Y_{i,\xi_i}), \qquad L(Y_{i,\xi_i+2}) \quad (1 \leq i \leq n).$$

The q-character of a simple object $L(m)$ of $\mathcal{C}_\xi$ contains in general many monomials m' which do not belong to $\mathcal{Y}_\xi$. By discarding these monomials we obtain a *truncated q-character* [15]. We shall denote by $\widetilde{\chi}_q(L(m))$ the truncated q-character of $L(m)$. One checks that for a simple object $L(m)$ of $\mathcal{C}_\xi$, all the dominant monomials occurring in $\chi_q(L(m))$ belong to the truncated q-character $\widetilde{\chi}_q(L(m))$ (the proof is similar to that of [15] for the category $\mathcal{C}_1$, as for the proof of Lemma 1 above). Therefore the truncation map $\chi_q(L(m)) \mapsto \widetilde{\chi}_q(L(m))$ extends to an injective algebra homomorphism from $\mathcal{K}_\xi$ to $\mathcal{Y}_\xi$.

It is sometimes convenient to renormalize the (truncated) q-character of $L(m)$ by dividing it by m, so that its leading term becomes 1. The element of $\mathcal{Y}$ thus obtained is called a *renormalized (truncated) q-character*.

Define a partial ordering $\preceq$ on $\mathcal{Y}$ by $\chi \preceq \chi'$ if $\chi' - \chi$ is an $\mathbb{N}$-linear combination of monomials. In particular, we have $\widetilde{\chi}_q(M) \preceq \chi_q(M)$ for M in $\mathcal{C}_\xi$.

3.4 Restriction and Decomposition

Let $J \subset I$ and $\mathfrak{g}_J \subset \mathfrak{g}$ be the corresponding Lie subalgebra. Let $\widehat{I}_J = \widehat{I} \cap (J \times \mathbb{Z})$. For m a monomial, let $m_J = \prod_{(i,p)\in\widehat{I}_J} Y_{i,p}^{u_{i,p}(m)}$. If m_J is dominant, one says that m is J-dominant. In this case, let $L_J(m)$ be the sum (with multiplicities) of the monomials m' occurring in $mm_J^{-1}\chi_q(L(m_J))$ such that $m(m')^{-1}$ is a product of $A_{i,p+1}^{-1}$, $(i,p) \in \widehat{I}_J$. The image of $L_J(m)$ in $\mathbb{Z}[Y_{i,p}]_{(i,p)\in\widehat{I}_J}$, obtained by sending the $Y_{i,p}$ to 1 if $(i,p) \notin \widehat{I}_J$, is the q-character of the simple $U_q(L\mathfrak{g}_J)$ labeled by m_J [13, Lemma 5.9]. In particular we have the following:

Lemma 2 *Let m and m' be two dominant monomials such that $L(m) \otimes L(m')$ is simple. Then $L_J(m)L_J(m') = L_J(mm')$.*

For m a dominant monomial one has a decomposition [11, Proposition 3.1]

$$L(m) = \sum_{m'} \lambda_J(m') L_J(m') \tag{3}$$

where the sum runs over J-dominant monomials m'. The $\lambda_J(m') \in \mathbb{N}$ are unique. This corresponds to the decomposition of $L(m)$ in the Grothendieck ring of $U_q(L\mathfrak{g}_J)$-modules. This decomposition gives an inductive process to construct monomials occurring in $\chi_q(L(m))$. Let us start with $m_0 = m$. Then the monomials m_1 of $L_J(m_0)$ occur in $\chi_q(L(m))$. If m_1 is J_1-dominant ($J_1 \subset I$) and if $L_{J_1}(m_1)$ occurs in the decomposition (3), then the monomials m_2 of $L_{J_1}(m_1)$ occur in $\chi_q(L(m))$, and we continue. See [13, Remark 3.16] for details.

3.5 Proof of Monoidal Categorifications

In this note, we follow the proof of [15] to establish that for certain choices of ξ the category $\mathcal{C}_\xi$ is a monoidal categorification of a cluster algebra $\mathcal{A}$. Let us recall the main steps (see [15] for details):

(1) We define a family $\mathcal{P}$ of prime simple modules in $\mathcal{C}_\xi$ and we label the cluster variables of an acyclic initial seed Σ of $\mathcal{A}$ with a subset of $\mathcal{P}$.
(2) We prove that the renormalized truncated q-characters of the representations of $\mathcal{P}$ coincide with the F-polynomials with respect to Σ of all the cluster variables of $\mathcal{A}$.
(3) We prove an irreducibility criterion for tensor products of two representations in $\mathcal{P}$.
(4) By using the following general result, we factorize every simple module in $\mathcal{C}_\xi$ as a tensor product of representations in $\mathcal{P}$.

Theorem 1 ([14]) *Let $S_1, \ldots, S_N$ be simple objects in $\mathcal{C}$. Then $S_1 \otimes S_2 \otimes \cdots \otimes S_N$ is simple if and only $S_i \otimes S_j$ is simple for any $1 \leq i < j \leq N$.*

In the next sections, we follow these steps for a good choice of ξ in types A and D. We conjecture that for arbitrary choices of ξ and for every type A, D, E, $\mathcal{C}_\xi$ is the monoidal categorification of a cluster algebra of the same type. For type A, this can be proved in the same way as explained in Remark 1(b). For other types, this can be probably established by using the methods in [20].

4 Type A

4.1 A Cluster Algebra of Type A

Let $\mathcal{A}$ be a cluster algebra of type A_n in the Fomin-Zelevinsky classification. As is well-known, the combinatorics of $\mathcal{A}$ is conveniently recorded in a regular polygon **P** with $n+3$ vertices labeled from 0 to $n+2$, see [3, Sect. 12.2]. Here, each cluster variable x_{ab} $(0 \leq a < b \leq n+2)$ is labeled by the segment joining vertex a to vertex b. The cluster variables x_{ab} for which the segment $[a, b]$ is a side of the polygon are frozen. Moreover we specialize

$$x_{01} = x_{n+1,\,n+2} = x_{0,\,n+2} = 1.$$

The exchange relations (Ptolemy relations) are of the form

$$x_{ac}x_{bd} = x_{ab}x_{cd} + x_{ad}x_{bc} \quad (a < b < c < d). \tag{4}$$

The clusters of $\mathcal{A}$ correspond to the triangulations of **P**. The variables x_{0i} $(2 \leq i \leq n+1)$ together with the n frozen variables $x_{i,\,i+1}$ $(1 \leq i \leq n)$ form a cluster, whose associated quiver is

$$\begin{array}{ccccccccc} x_{02} & \to & x_{03} & \to & x_{04} & \to & \cdots & \to & x_{0,n+1} \\ \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & \nwarrow & & \nwarrow & \downarrow \\ x_{12} & & x_{23} & & x_{34} & & \cdots & & x_{n,n+1} \end{array}$$

Note that the principal part of this quiver (i.e. the subquiver with vertices the non-frozen variables) is a quiver of type A_n with linear orientation. We denote by Σ this particular seed of $\mathcal{A}$.

4.2 Cluster Structure on $\mathcal{C}_\xi$

Let $\mathfrak{g}$ be of type A_n. We will write $Y_{0,p} = Y_{n+1,p} = 1$ for $p \in \mathbb{Z}$. We choose the height function

$$\xi(i) := i \quad (i \in I),$$

corresponding to a quiver Q of type A_n with linear orientation. We define the following family of irreducible representations in $\mathcal{C}_\xi$:

$$\mathcal{P} := \big\{ L(i,j) := L(Y_{i,i}Y_{j,j+2}) \mid 0 \le i \le j \le n+1 \big\}.$$

The simple modules $L(i, j)$ are evaluation representations whose q-characters are known (see references in [2]). In particular they are prime. We have $\widetilde{\chi}_q(L(0,j)) = Y_{j,j+2}$ and if $i \neq 0$ we have

$$\widetilde{\chi}_q\big(L(i,j)\big) = Y_{i,i}Y_{j,j+2}\big(1 + A_{i,i+1}^{-1} + (A_{i,i+1}A_{i+1,i+2})^{-1} + \cdots \\ + (A_{i,i+1}A_{i+1,i+2}\cdots A_{j-1,j})^{-1}\big).$$

Dividing both sides by $Y_{i,i}Y_{j,j+2}$ and setting $t_i := A_{i,i+1}^{-1}$, we see that this formula for the renormalized truncated q-characters coincides with the formula for F-polynomials computed in [23, Example 1.14]. It is easy to deduce from this that we have the following relations in $\mathcal{R}_\xi$ (also obtained in [19]):

$$\big[L(i,k)\big]\big[L(j,l)\big] = \big[L(i,l)\big]\big[L(j,k)\big] + \big[L(i,j-1)\big]\big[L(k+1,l)\big] \\ \text{if } 0 \le i < j \le k < l \le n+1. \tag{5}$$

Therefore, comparing with (4), we see that the assignment

$$x_{ab} \mapsto \big[L(a,b-1)\big] \quad (0 \le a < b \le n+2)$$

extends to an isomorphism from the cluster algebra $\mathcal{A}$ to the Grothendieck ring $\mathcal{R}_\xi$. This isomorphism maps the seed Σ to

$$\begin{array}{ccccccc} L(0,1) & \to & L(0,2) & \to & \cdots & \to & L(0,n) \\ \downarrow & \nwarrow & \downarrow & \nwarrow & & \nwarrow & \downarrow \\ L(1,1) & & L(2,2) & & \cdots & & L(n,n) \end{array}$$

where the $L(i,i)$ $(1 \le i \le n)$ correspond to frozen variables.

We say that (i,k) and (j,l) are *crossing* if and only if $i < j \le k < l$ or $j < i \le l < k$. Otherwise, we say that (i,k) and (j,l) are *noncrossing*. The next proposition is similar to the classical irreducibility criterion for prime representations of $U_q(L\mathfrak{sl}_2)$, except that here, spectral parameters are replaced by nodes of the Dynkin diagram.

Proposition 2 *The module $L(i,j) \otimes L(k,l)$ is simple if and only if (i,j) and (k,l) are noncrossing.*

Proof The "only if" part follows from (5). We prove the "if" part. Let $M = Y_{i,i}Y_{j,j+2}Y_{k,k}Y_{l,l+2}$. We have $\widetilde{\chi}_q(L(M)) \preceq \chi = \widetilde{\chi}_q(L(i,j) \otimes L(k,l))$. We prove the other inequality. By symmetry, we are reduced to the following two cases:

(a) if $j < k$ or ($k = 0$ and $i, j \le l$) or ($1 \le k \le i$, $j = l$), then χ contains a unique dominant monomial, namely M, so $L(i,j) \otimes L(k,l)$ is simple.
(b) if $1 \le k \le i \le j < l$, then χ contains exactly two dominant monomials, namely M and

$$M' = M(A_{k,k+1}A_{k+1,k+2}\cdots A_{j,j+1})^{-1}.$$

So it suffices to prove that M' occurs in $\widetilde{\chi}(L(M))$. First, by Sect. 3.4, the monomial

$$M'' = M(A_{k,k+1}A_{k+1,k+2}\cdots A_{i-1,i})^{-1}$$

occurs in $\widetilde{\chi}(L(M))$. Hence $L_J(M'')$ occurs in the decomposition (3) for $J = \{i,\ldots,n\}$. But $L(Y_{i,-i}Y_{j,-j-2}) \otimes L(Y_{i,-i})$ is minuscule and simple. Hence, by [12, Corollary 4.11], the tensor product $L(Y_{i,i}Y_{j,j+2}) \otimes L(Y_{i,i})$ is simple, isomorphic to $L(Y_{i,i}^2Y_{j,j+2})$. So $Y_{i,i}^2Y_{j,j+2}(A_{i,i+1}\cdots A_{j,j+1})^{-1}$ occurs in $\widetilde{\chi}(L(Y_{i,i}^2Y_{j,j+2}))$ and M' occurs in $L_J(M'')$. □

Therefore, as explained in Sect. 3.5, we get the following:

Theorem 2 *$\mathcal{C}_\xi$ is a monoidal categorification of the cluster algebra $\mathcal{A}$ of type A_n.*

Remark 1 (a) It follows from Theorem 2 that when $\xi_i = i$, every simple module in $\mathcal{C}_\xi$ can be factorized as a tensor product of evaluation representations.

(b) For an arbitrary ξ, a theorem similar to Theorem 2 can be proved in an analog but slightly more complicated way. A subset $J = [i,j] \subset I$ $(1 \le i \le j \le n)$ has a natural orientation induced by ξ. Let J_+ (resp. J_-) be the set of sources (resp. sinks) of J. The prime objects in $\mathcal{C}_\xi$ are the simple modules

$$L(J) := L\left(\prod_{k\in J_-} Y_{k,\xi_k} \prod_{k\in J_+} Y_{k,\xi_k+2}\right), \qquad L(i) := L(Y_{i,\xi_i}),$$

$$L'(i) := L(Y_{i,\xi_i+2}).$$

Note that $L(J)$ is not an evaluation representation if J has several sources or several sinks.

(c) Different choices of ξ yield different subcategories $\mathcal{C}_\xi$. These subcategories seem to be quite similar, but they are not equivalent in general. For example, in type A_3, consider the categories $\mathcal{C}_\xi$ with $\xi_i = i$ and $\mathcal{C}_\phi$ with $\phi_1 = 1$, $\phi_2 = 2$, $\phi_3 = 1$. Both categories are monoidal categorifications of a cluster algebra of type A_3 with 3 coefficients. The category $\mathcal{C}_\phi$ was studied in [15]. In particular, we the have following relation in the Grothendieck ring of $\mathcal{C}_\phi$:

$$\big[L(Y_{1,1}Y_{2,4}Y_{3,1})\big]\big[L(Y_{2,2})\big] = \big[L(Y_{1,1})\big]\big[L(Y_{3,1})\big]\big[L(Y_{2,2}Y_{2,4})\big] + \big[L(Y_{1,1}Y_{1,3})\big]\big[L(Y_{3,1}Y_{3,3})\big].$$

But by (5), in the Grothendieck ring of $\mathcal{C}_\xi$, a simple constituent of the tensor product of two simple prime representations can be factorized as a tensor product of at most 2 non trivial representations. Hence, $\mathcal{C}_\xi$ and $\mathcal{C}_\phi$ are *not* equivalent.

5 Type *D*

5.1 A Cluster Algebra of Type D

Let $\mathcal{A}$ be a cluster algebra of type D_n in the Fomin-Zelevinsky classification. The clusters of $\mathcal{A}$ are now encoded by the *centrally symmetric* triangulations of a regular polygon $\mathbf{P}$ with $2n$ vertices, labeled by $a = 0,\ 1, \dots, 2n-1$ [3, Sect. 12.4] (note that a more modern way to record the combinatorics of a cluster algebra of type D_n would be by means of a once-punctured n-gon and tagged arcs [6]). A segment $[a,b]$ joining two vertices is called a *diagonal* if it meets the interior of $\mathbf{P}$, and a *side* otherwise. Let Θ be the $180°$ rotation of $\mathbf{P}$, and for a vertex a, write $\overline{a} = \Theta(a)$. Each *non frozen* cluster variable is labeled by a Θ-orbit on the set of diagonals of $\mathbf{P}$. More precisely, to each non trivial Θ-orbit ($[a,b]$, $[\overline{a},\overline{b}]$) (with $b \neq \overline{a}$) we attach a single cluster variable

$$x_{ab} = x_{\overline{a}\overline{b}}.$$

But we associate with every Θ-fixed diagonal $[a,\overline{a}]$ (or *diameter*) two *different* cluster variables

$$x_{a\overline{a}} \neq x_{\widetilde{a\overline{a}}}.$$

We may think of $[a,\overline{a}]$ and $[\widetilde{a,\overline{a}}]$ as two different Θ-orbits, supported on the same segment but with two different colors. Given two Θ-orbits, one of which at least being non trivial, we say that they are *noncrossing* if they do not meet in the interior of $\mathbf{P}$. We also declare that two Θ-fixed diagonals are *noncrossing* if and only if they have the same support or the same color. A *centrally symmetric triangulation* of $\mathbf{P}$ is then a maximal subset of pairwise noncrossing Θ-orbits of diagonals. Such a triangulation always consists of n different Θ-orbits. For instance, for $n = 4$, the following are two distinct triangulations

$$\big\{\big([1,\overline{3}],\ [\overline{1},3]\big),\ \big([2,\overline{3}],\ [\overline{2},3]\big),\ [3,\overline{3}],\ \widetilde{[3,\overline{3}]}\big\},$$
$$\big\{\big([1,\overline{3}],\ [\overline{1},3]\big),\ \big([2,\overline{3}],\ [\overline{2},3]\big),\ [3,\overline{3}], [2,\overline{2}]\big\}.$$

To the Θ-orbits of the sides $[a,b]$ of $\mathbf{P}$ we can also attach some frozen variables $x_{ab} = x_{\overline{a}\overline{b}}$. We specialize

$$x_{01} = x_{\overline{n-1},\,0} = 1.$$

Our initial seed for the cluster algebra $\mathcal{A}$ will correspond to the triangulation

$$\big\{\Theta\big([a,\overline{n-1}]\big) \mid 1 \le a \le n-2\big\} \cup \big\{[n-1,\overline{n-1}], \widetilde{[n-1,\overline{n-1}]}\big\}.$$

More precisely, it is described by the following quiver

$$\begin{array}{ccccccccc}
 & & & & & & x_{\widetilde{n-1,\overline{n-1}}} & \leftarrow & f_{n-1} \\
 & & & & & & \uparrow & \searrow & \\
x_{1,\overline{n-1}} & \to & \cdots & \to & x_{n-3,\overline{n-1}} & \to & x_{n-2,\overline{n-1}} & \leftarrow & x_{n-2,n-1} \\
\uparrow & \swarrow & & \swarrow & \uparrow & \swarrow & \downarrow & \nearrow & \\
x_{12} & & \cdots & & x_{n-3,n-2} & & x_{n-1,\overline{n-1}} & \leftarrow & f_n
\end{array}$$

where f_n and f_{n-1} are two additional frozen variables, which can not be encoded by sides of $\mathbf{P}$. The principal part of the quiver (obtained by removing the frozen vertices $x_{i,i+1}$ $(1 \le i \le n-2)$, f_{n-1}, f_n, and the arrows incident to them) is a Dynkin quiver Q of type D_n, hence $\mathcal{A}$ is indeed a cluster algebra of type D_n in the Fomin-Zelevinsky classification.

One can easily check that, because of this particular choice of frozen variables, $\mathcal{A}$ belongs to the class of cluster algebras studied in [9]. More precisely, let us label the vertices of Q by $\{1,\ldots,n\}$ so that $x_{i,\overline{n-1}}$ lies at vertex i for $i \le n-1$, and $x_{\widetilde{n-1,\overline{n-1}}}$ lies at vertex n. Then $\mathcal{A}$ is the same as the algebra attached in [9] to Q and the Weyl group element

$$w = c^2 = (s_n s_{n-1} s_{n-2} \cdots s_1)^2.$$

It follows from [9, Theorem 16.1(i)] that $\mathcal{A}$ is a polynomial ring in $2n$ generators. These generators are the initial cluster variables

$$z_i := x_{i,\overline{n-1}} \quad (1 \le i \le n-1), \qquad z_n := x_{\widetilde{n-1,\overline{n-1}}},$$

together with the new cluster variables $z_i^{\dagger}$ $(1 \le i \le n)$ produced by the sequence of mutations

$$\mu_n \circ \mu_{n-1} \circ \mu_{n-2} \circ \cdots \circ \mu_2 \circ \mu_1. \tag{6}$$

Recall from [3] that, our initial cluster being fixed, the cluster variables of $\mathcal{A}$ also have a natural labelling by almost positive roots. The correspondence is as follows. First, the Θ-orbits of the initial triangulation are labeled by negative simple roots:

$$\Theta\big([i,\overline{n-1}]\big) \mapsto -\alpha_i \quad (1 \le i \le n-2), \qquad [n-1,\overline{n-1}] \mapsto -\alpha_{n-1},$$

$$\widetilde{[n-1,\overline{n-1}]} \mapsto -\alpha_n.$$

Any other Θ-orbit x is mapped to the positive root $\sum_i c_i\alpha_i$, where the diagonals representing x cross the diagonals representing $-\alpha_i$ at c_i pairs of centrally symmetric points (counting an intersection of two diameters of different colors and support as one such pair).

In [22, 23], a different labelling for the cluster variables is used. First the choice of an *acyclic* initial seed is encoded by the choice of a Coxeter element c. For our choice of initial seed, this Coxeter element is

$$c = s_n s_{n-1} s_{n-2} \cdots s_1.$$

Next the cluster variables are labeled by weights of the form

$$c^m \varpi_i \quad \big(i \in I,\ 0 \le m \le h(i,c)\big),$$

where $h(i,c)$ is the smallest integer such that $c^{h(i,c)}\varpi_i = w_0\varpi_i$. The correspondence between the two labellings is as follows. To the fundamental weight ϖ_i corresponds $-\alpha_i$, and to the weight $c^m\varpi_i$ $(m \ge 1)$ corresponds the positive root $\beta = c^{m-1}\varpi_i - c^m\varpi_i$.

Example 1 We illustrate all these definitions in the case $n = 4$. Here $\mathbf{P}$ is a regular octogon, with vertices labeled by $0, 1, 2, 3, \overline{0}, \overline{1}, \overline{2}, \overline{3}$. Our choice of initial triangulation is

$$\big\{\big([1,\overline{3}],\ [\overline{1},3]\big),\ \big([2,\overline{3}],\ [\overline{2},3]\big),\ [3,\overline{3}],\ \widetilde{[3,\overline{3}]}\big\},$$

which corresponds to the Coxeter element $c = s_4s_3s_2s_1$. The sixteen Θ-orbits of diagonals (represented by one of their elements), and the corresponding indexings by almost positive roots, and by weights, are given in the table below:

$[1,\overline{3}]$	$-\alpha_1$	ϖ_1
$[2,\overline{3}]$	$-\alpha_2$	ϖ_2
$\widetilde{[3,\overline{3}]}$	$-\alpha_3$	ϖ_3
$[3,\overline{3}]$	$-\alpha_4$	ϖ_4
$[0,2]$	α_1	$c^3\varpi_1$
$[1,3]$	α_2	$c^2\varpi_1$
$[2,\overline{2}]$	α_3	$c\varpi_3$
$\widetilde{[2,\overline{2}]}$	α_4	$c\varpi_4$

$[0,3]$	$\alpha_1+\alpha_2$	$c^3\varpi_2$
$[1,\overline{1}]$	$\alpha_2+\alpha_3$	$c^2\varpi_4$
$\widetilde{[1,\overline{1}]}$	$\alpha_2+\alpha_4$	$c^2\varpi_3$
$[0,\overline{0}]$	$\alpha_1+\alpha_2+\alpha_3$	$c^3\varpi_3$
$\widetilde{[0,\overline{0}]}$	$\alpha_1+\alpha_2+\alpha_4$	$c^3\varpi_4$
$[2,\overline{1}]$	$\alpha_2+\alpha_3+\alpha_4$	$c\varpi_2$
$[2,\overline{0}]$	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$	$c\varpi_1$
$[1,\overline{0}]$	$\alpha_1+2\alpha_2+\alpha_3+\alpha_4$	$c^2\varpi_2$

5.2 Cluster Structure on $\mathcal{C}_\xi$

Let $\mathfrak{g}$ be of type D_n. We will write $Y_{0,p} = Y_{n+1,p} = 1$ for $p \in \mathbb{Z}$. We choose the height function $\xi_i = n - 1 - i$ if $i < n$ and $\xi_n = 0$. This induces a partial order $\preceq$ on $\{1, \dots, n\}$ defined by

$$i \prec j \iff \xi_i < \xi_j.$$

Note that $n-1$ and n are not comparable for $\preceq$. Moreover, for convenience, we extend this to $\{0, \dots, n+1\}$ by declaring that 0 is a maximal element and $n+1$ a minimal element for $\preceq$.

We define the following family $\mathcal{P}$ of representations in $\mathcal{C}_\xi$:

$$L(i, j) = L(Y_{i,\xi_i} Y_{j,\xi_j+2}) \quad (n+1 \preceq i \preceq j \preceq 0),$$
$$L(i, j)^\dagger = L(Y_{n,0} Y_{n-1,0} Y_{i,\xi_i+2} Y_{j,\xi_j+2}) \quad (n-2 \preceq j \prec i \preceq 0).$$

Since $\mathcal{A}$ and $\mathcal{R}_\xi$ are both polynomial rings over $\mathbb{Z}$ with $2n$ generators, the assignment

$$z_i \mapsto \big[L(n+1, i)\big] = \big[L(Y_{i,\xi_i+2})\big], \qquad z_i^\dagger \mapsto \big[L(i, 0)\big] = \big[L(Y_{i,\xi_i})\big] \quad (1 \le i \le n),$$

extends to a ring isomorphism $\iota \colon \mathcal{A} \xrightarrow{\sim} \mathcal{R}_\xi$. Thus $\mathcal{R}_\xi$ is endowed with the structure of a cluster algebra. Moreover, using the T-system equations for calculating the products

$$\big[L(Y_{i,\xi_i})\big]\big[L(Y_{i,\xi_i+2})\big] = z_i z_i^\dagger,$$

and comparing them with the exchange relations involved in the sequence of mutations (6), we can easily check that the frozen variables of $\mathcal{A}$ are mapped by ι to the classes $[L(i,i)] = [L(Y_{i,\xi_i} Y_{i,\xi_i+2})]$. More precisely,

$$\iota(f_{n-1}) = \big[L(n-1, n-1)\big], \qquad \iota(f_n) = \big[L(n, n)\big],$$
$$\iota(x_{i,i+1}) = \big[L(i, i)\big] \quad (1 \le i \le n-2).$$

Therefore ι maps the initial seed of $\mathcal{A}$ to

$$\begin{array}{ccccccccc}
 & & & & & L(n+1,\, n-1) & \leftarrow & L(n-1,\, n-1) \\
 & & & & & \uparrow & & \searrow \\
L(n+1,1) & \rightarrow & L(n+1,2) & \rightarrow & \cdots \rightarrow & L(n+1,\, n-2) & \leftarrow & L(n-2,\, n-2) \\
\uparrow & \swarrow & \uparrow & \swarrow & \swarrow & \downarrow & & \nearrow \\
L(1,1) & & L(2,2) & & \cdots & L(n+1,\, n) & \leftarrow & L(n,\, n)
\end{array}$$

Let us compute the truncated q-characters of the representations in $\mathcal{P}$. As in Sect. 4.2, the modules $L(i,j)$ are prime minimal affinizations. We have

$$\widetilde{\chi}_q\big(L(n+1,j)\big) = Y_{j,\xi_j+2},$$
$$\widetilde{\chi}_q\big(L(i,j)\big) = Y_{i,\xi_i}Y_{j,\xi_j+2}\big(1 + A_{i,\xi_i+1}^{-1} + \cdots + (A_{i,\xi_i+1}\cdots A_{j-1,\xi_j})^{-1}\big)$$
$$(n-1 \preceq i),$$
$$\widetilde{\chi}_q\big(L(n,j)\big) = Y_{n,0}Y_{j,\xi_j+2}\big(1 + A_{n,1}^{-1}\chi_j\big) \quad (0 \le j \le n-2),$$

where $\chi_j := 1 + A_{n-2,2}^{-1} + \cdots + (A_{n-2,2}\cdots A_{j+1,\xi_j})^{-1}$. In general the $L(i,j)^\dagger$ are not minimal affinizations. However, we have:

Lemma 3 *For $n-2 \preceq j \prec i \preceq 0$, the representation $L(i,j)^\dagger$ is prime and*

$$\widetilde{\chi}_q\big(L(i,j)^\dagger\big) = Y_{n,0}Y_{n-1,0}Y_{i,\xi_i+2}Y_{j,\xi_j+2}$$
$$\times\big(1 + \big(A_{n-1,1}^{-1} + A_{n,1}^{-1}\big)\chi_j + A_{n-1,1}^{-1}A_{n,1}^{-1}\chi_i\chi_j\big).$$

Proof As

$$\widetilde{\chi}_q\big(L(i,j)^\dagger\big) \preceq \widetilde{\chi}_q\big(L(n,j)\otimes L(n-1,i)\big)$$

and

$$\widetilde{\chi}_q\big(L(i,j)^\dagger\big) \preceq \widetilde{\chi}_q\big(L(n,i)\otimes L(n-1,j)\big),$$

there are $A, B \preceq \chi_j$ and $C \preceq \chi_i\chi_j$ such that

$$\widetilde{\chi}_q\big(L(i,j)^\dagger\big) = Y_{n,0}Y_{n-1,0}Y_{i,\xi_i+2}Y_{j,\xi_j+2}\big(1 + A_{n-1,1}^{-1}A + A_{n,1}^{-1}B + A_{n,1}^{-1}A_{n-1,1}^{-1}C\big).$$

From Proposition 2 with $J = \{1,\ldots,n-1\}$, we have

$$Y_{n,0}L_J(Y_{n-1,0}Y_{j,n-j+1})L_J(Y_{i,n-i+1}) = L_J(Y_{n,0}Y_{n-1,0}Y_{i,n-i+1}Y_{j,n-j+1}).$$

Hence, by Sect. 3.4, we have $A = \chi_j$. The proof that $B = \chi_j$ is analog. Similarly, from Proposition 2 with $J = \{1,\ldots,n-2\}$, we have

$$L_J(Y_{n-2,1}Y_{i,n-i+1})L_J(Y_{n-2,1}Y_{j,n-j+1}) = L_J\big(Y_{n-2,1}^2Y_{i,n-i+1}Y_{j,n-j+1}\big).$$

So

$$C = \big(Y_{n-2,1}^2Y_{i,n-i+1}Y_{j,n-j+1}\big)^{-1}\widetilde{\chi}_q\big(L\big(Y_{n-2,1}^2Y_{i,n-i+1}Y_{j,n-j+1}\big)\big) = \chi_i\chi_j.$$

This explicit formula shows that $\widetilde{\chi}_q(L(i,j)^\dagger)$ can not be factorized and so $L(i,j)^\dagger$ is prime. □

Let $\mathcal{P}' := \mathcal{P} \setminus \{L(i,i) \mid 1 \le i \le n\}$. We introduce the following bijection between the non frozen cluster variables of $\mathcal{A}$ and the representations in $\mathcal{P}'$.

$$\begin{array}{lll} x_{ij} & \mapsto L(j-1,i) & (0 \le i \le j-2 \le n-3), \\ x_{i\bar{j}} & \mapsto L(j,i)^{\dagger} & (0 \le j < i \le n-2), \\ x_{i\bar{i}} & \mapsto L(n-1,i) & (0 \le i \le n-2), \\ x_{\widetilde{i\bar{i}}} & \mapsto L(n,i) & (0 \le i \le n-2), \\ x_{i,\overline{n-1}} & \mapsto L(n+1,i) & (1 \le i \le n-2), \\ x_{\widetilde{n-1,\overline{n-1}}} & \mapsto L(n+1,n-1), & \\ x_{n-1,\overline{n-1}} & \mapsto L(n+1,n). & \end{array}$$

One can check that under this correspondence, the renormalized truncated q-characters for the representations in $\mathcal{P}'$ coincide with the F-polynomials of the cluster variables of $\mathcal{A}$ calculated in [22, 23]. One then deduces that this bijection is the restriction of the ring automorphism ι to the set of non frozen cluster variables.

Example 2 We continue Example 1. The table below gives the list of cluster variables of $\mathcal{A}$ together with the corresponding representations of $\mathcal{P}'$ and their truncated q-characters. Here $t_i = A^{-1}_{i,\xi_i+1}$.

x_{02}	$L(1,0)$	$Y_{1,2}(1+t_1)$
x_{03}	$L(2,0)$	$Y_{2,1}(1+t_2+t_2t_1)$
x_{13}	$L(2,1)$	$Y_{1,4}Y_{2,1}(1+t_2)$
$x_{1\bar{0}}$	$L(0,1)^{\dagger}$	$Y_{1,4}Y_{3,0}Y_{4,0}(1+t_3+t_3t_2+t_4+t_4t_2+t_3t_4+2t_3t_4t_2+t_3t_4t_2^2$ $+t_3t_4t_2t_1+t_3t_4t_2^2t_1)$
$x_{2\bar{0}}$	$L(0,2)^{\dagger}$	$Y_{2,3}Y_{3,0}Y_{4,0}(1+t_3+t_4+t_3t_4+t_3t_4t_2+t_3t_4t_2t_1)$
$x_{2,\bar{1}}$	$L(1,2)^{\dagger}$	$Y_{1,4}Y_{2,3}Y_{3,0}Y_{4,0}(1+t_3+t_4+t_3t_4+t_3t_4t_2)$
$x_{0\bar{0}}$	$L(3,0)$	$Y_{3,0}(1+t_3+t_3t_2+t_3t_2t_1)$
$x_{1\bar{1}}$	$L(3,1)$	$Y_{1,4}Y_{3,0}(1+t_3+t_3t_2)$
$x_{2\bar{2}}$	$L(3,2)$	$Y_{2,3}Y_{3,0}(1+t_3)$

$x_{\widetilde{0\bar{0}}}$	$L(4,0)$	$Y_{4,0}(1+t_4+t_4t_2+t_4t_2t_1)$
$x_{\widetilde{1\bar{1}}}$	$L(4,1)$	$Y_{1,4}Y_{4,0}(1+t_4+t_4t_2)$
$x_{\widetilde{2\bar{2}}}$	$L(4,2)$	$Y_{2,3}Y_{4,0}(1+t_4)$
$x_{1\bar{3}}$	$L(5,1)$	$Y_{1,4}$
$x_{2\bar{3}}$	$L(5,2)$	$Y_{2,3}$
$x_{\widetilde{3\bar{3}}}$	$L(5,3)$	$Y_{3,2}$
$x_{3\bar{3}}$	$L(5,4)$	$Y_{4,2}$

We now describe which tensor products of representations of $\mathcal{P}$ are simple.

Proposition 3 *We have the following*:

(a) *Suppose* $\{i,k\} \neq \{n-1,n\}$. *Then* $L(i,j) \otimes L(k,l)$ *is not simple if and only if* $i \prec k \preceq j \prec l$ *or* $k \prec i \preceq l \prec j$.
(b) *Suppose* $\{i,k\} = \{n-1,n\}$. *Then* $L(i,j) \otimes L(k,l)$ *is simple if and only if* $j = l$ *or* $i = j$ *or* $k = l$.
(c) *Suppose* $j \prec i$ *and* $l \prec k$. *Then* $L(i,j)^\dagger \otimes L(k,l)^\dagger$ *is simple if and only if* $j \preceq l \prec k \preceq i$ *or* $l \preceq j \prec i \preceq k$.
(d) *Suppose* $i \succeq n-2$ *and* $l \prec k$. *Then* $L(i,j) \otimes L(k,l)^\dagger$ *is simple if and only if* $i = j$ *or* $i \prec j \preceq l \prec k$ *or* $l \prec k \prec i \prec j$ *or* $l \prec i \prec j \preceq k$.
(e) *Suppose* $i \prec n-2$ *and* $l \prec k$. *Then* $L(i,j) \otimes L(k,l)^\dagger$ *is simple if and only if* $i = j$ *or* $((i \neq n+1)$ *and* $l \preceq j \preceq k)$ *or* $(i = n+1$ *and* $k \preceq j)$.

Proof In each case, the proof of non simplicity follows from the identification of truncated q-characters with F-polynomials in the last section. So we treat only the proof of the simplicity.

(a) The irreducibility is proved as in type A, except for the tensor product

$$L(n+1,n) \otimes L(n+1,n-1)$$

which is minuscule and so is simple.

(b) If $n-2 \preceq j = l$ or $i = j$ or $k = l$, $L(n,j) \otimes L(n-1,j)$ is minuscule and so is simple.

(c) By symmetry, we can assume $j \preceq l$. Suppose that $j \preceq l \prec k \preceq i$ and let us prove that $L(i,j)^\dagger \otimes L(k,l)^\dagger$ is simple. Let M be its highest weight monomial. It suffices to prove that any dominant monomial m occurring in $\widetilde{\chi}(L(i,j)^\dagger)\widetilde{\chi}(L(k,l)^\dagger)$ occurs in $\widetilde{\chi}_q(L(M))$. If $A_{n-1,1}^{-1}$ or $A_{n,1}^{-1}$ is not a factor of mM^{-1}, this is proved as for type A. If $A_{n-1,1}^{-2}$ is a factor of mM^{-1}, first from Sect. 3.4 $MA_{n-1,1}^{-2}$ occurs and $L_J(MA_{n-1,1}^{-2})$ occurs in the decomposition (3) for $J = \{1,\ldots,n-2,n\}$. But from type A

$$L_J\big(MA_{n-1,1}^{-2}\big) = Y_{n-1,2}^{-2} L_J(Y_{n,0}Y_{n-2,1}Y_{i,n-i+1}Y_{j,n-j+1}) \times L_J(Y_{n,0}Y_{n-2,1}Y_{k,n-k+1}Y_{l,n-l+1})$$

and we can conclude by Sect. 3.4. This is analog if $A_{n,1}^{-2}$ is a factor. So we can assume that $A_{n,1}^{-1}$ and $A_{n-1,1}^{-1}$ are factors with power 1. Then m is one of the following monomials

$$MA_{n,1}^{-1}A_{n-1,1}^{-1}A_{n-2,2}^{-1}\cdots A_{j,n-j}^{-1} \quad \text{with multiplicity 5},$$

$$MA_{n,1}^{-1}A_{n-1,1}^{-1}A_{n-2,2}^{-1}\cdots A_{l,n-l}^{-1} \quad \text{with multiplicity 2},$$

$$MA_{n,1}^{-1}A_{n-1,1}^{-1}A_{n-2,2}^{-1}\cdots A_{k,n-k}^{-1} \quad \text{with multiplicity } 1,$$

$$MA_{n,1}^{-1}A_{n-1,1}^{-1}A_{n-2,2}^{-2}\cdots A_{j,n-j}^{-2}A_{j+1,n-j-1}^{-1}\cdots A_{l,n-l}^{-1} \quad \text{with multiplicity } 1.$$

Then we conclude as above. For example for the last monomial of the list,

$$M' := MA_{n,1}^{-1}A_{n-2,2}^{-1}\cdots A_{j,n-j}^{-1}$$

occurs in $L_{\{n,n-2,\ldots,j\}}(M)$ from type A. Hence $M'A_{j,n-j}$ occurs in

$$L_{\{n-1,n-2,\ldots,j\}}\big(MA_{n,1}^{-1}\big),$$

but M' does not. So $L_{\{n-1,n-2,\ldots,j\}}(M')$ occurs in the decomposition (3). Since M is a monomial in $L_{\{n-1,n-2,\ldots,1\}}(M')$, we get the result.

(d) and (e): The proof is analog. □

Proposition 3 implies that the tensor products of representations of $\mathcal{P}$ corresponding to compatible cluster variables are simple. Indeed, two cluster variables are compatible if and only if the corresponding diagonals in **P** do not cross (with the convention that diameters of the same color do not cross each other) [3, Sect. 12.4]. This coincides with the conditions of Proposition 3.

Example 3 We continue Example 1. The following table lists the compatible pairs of non frozen variables of $\mathcal{A}$, and indicates in which case of Proposition 3 the corresponding pairs of simple modules fall.

(x_{02}, x_{03})	(a)	$(x_{02}, x_{0\bar{0}})$	(a)	$(x_{02}, x_{\widetilde{0\bar{0}}})$	(a)	$(x_{02}, x_{2\bar{2}})$	(a)
$(x_{02}, x_{\widetilde{2\bar{2}}})$	(a)	$(x_{02}, x_{2\bar{3}})$	(a)	$(x_{02}, x_{3\bar{3}})$	(a)	$(x_{02}, x_{\widetilde{3\bar{3}}})$	(a)
$(x_{03}, x_{0\bar{0}})$	(a)	$(x_{03}, x_{\widetilde{0\bar{0}}})$	(a)	(x_{03}, x_{13})	(a)	$(x_{03}, x_{3\bar{3}})$	(a)
$(x_{03}, x_{\widetilde{3\bar{3}}})$	(a)	$(x_{0\bar{0}}, x_{13})$	(a)	$(x_{0\bar{0}}, x_{1\bar{1}})$	(a)	$(x_{0\bar{0}}, x_{2\bar{2}})$	(a)
$(x_{0\bar{0}}, x_{3\bar{3}})$	(a)	$(x_{\widetilde{0\bar{0}}}, x_{13})$	(a)	$(x_{\widetilde{0\bar{0}}}, x_{\widetilde{1\bar{1}}})$	(a)	$(x_{\widetilde{0\bar{0}}}, x_{\widetilde{2\bar{2}}})$	(a)
$(x_{\widetilde{0\bar{0}}}, x_{\widetilde{3\bar{3}}})$	(a)	$(x_{13}, x_{1\bar{1}})$	(a)	$(x_{13}, x_{\widetilde{1\bar{1}}})$	(a)	$(x_{13}, x_{1\bar{3}})$	(a)
$(x_{13}, x_{3\bar{3}})$	(a)	$(x_{13}, x_{\widetilde{3\bar{3}}})$	(a)	$(x_{1\bar{1}}, x_{1\bar{3}})$	(a)	$(x_{1\bar{1}}, x_{2\bar{2}})$	(a)
$(x_{1\bar{1}}, x_{3\bar{3}})$	(a)	$(x_{\widetilde{1\bar{1}}}, x_{1\bar{3}})$	(a)	$(x_{\widetilde{1\bar{1}}}, x_{\widetilde{2\bar{2}}})$	(a)	$(x_{\widetilde{1\bar{1}}}, x_{\widetilde{3\bar{3}}})$	(a)
$(x_{1\bar{3}}, x_{2\bar{2}})$	(a)	$(x_{1\bar{3}}, x_{\widetilde{2\bar{2}}})$	(a)	$(x_{1\bar{3}}, x_{2\bar{3}})$	(a)	$(x_{1\bar{3}}, x_{3\bar{3}})$	(a)
$(x_{1\bar{3}}, x_{\widetilde{3\bar{3}}})$	(a)	$(x_{2\bar{2}}, x_{2\bar{3}})$	(a)	$(x_{\widetilde{2\bar{2}}}, x_{2\bar{3}})$	(a)	$(x_{2\bar{3}}, x_{3\bar{3}})$	(a)
$(x_{2\bar{3}}, x_{\widetilde{3\bar{3}}})$	(a)	$(x_{3\bar{3}}, x_{\widetilde{3\bar{3}}})$	(a)	$(x_{0\bar{0}}, x_{\widetilde{0\bar{0}}})$	(b)	$(x_{1\bar{1}}, x_{\widetilde{1\bar{1}}})$	(b)
$(x_{2\bar{2}}, x_{\widetilde{2\bar{2}}})$	(b)	$(x_{1\bar{0}}, x_{2\bar{0}})$	(c)	$(x_{2\bar{0}}, x_{2\bar{1}})$	(c)	$(x_{13}, x_{1\bar{0}})$	(d)
$(x_{02}, x_{2\bar{0}})$	(d)	$(x_{2\bar{2}}, x_{2\bar{1}})$	(e)	$(x_{2\bar{2}}, x_{2\bar{0}})$	(e)	$(x_{1\bar{1}}, x_{2\bar{1}})$	(e)
$(x_{1\bar{1}}, x_{2\bar{0}})$	(e)	$(x_{1\bar{1}}, x_{1\bar{0}})$	(e)	$(x_{0\bar{0}}, x_{2\bar{0}})$	(e)	$(x_{0\bar{0}}, x_{1\bar{0}})$	(e)
$(x_{\widetilde{2\bar{2}}}, x_{2\bar{1}})$	(e)	$(x_{\widetilde{2\bar{2}}}, x_{2\bar{0}})$	(e)	$(x_{\widetilde{1\bar{1}}}, x_{2\bar{1}})$	(e)	$(x_{\widetilde{1\bar{1}}}, x_{2\bar{0}})$	(e)
$(x_{\widetilde{1\bar{1}}}, x_{1\bar{0}})$	(e)	$(x_{\widetilde{0\bar{0}}}, x_{2\bar{0}})$	(e)	$(x_{\widetilde{0\bar{0}}}, x_{1\bar{0}})$	(e)	$(x_{3\bar{1}}, x_{2\bar{1}})$	(e)

Now, as explained in Sect. 3.5, we may conclude that:

Theorem 3 *$\mathcal{C}_\xi$ is a monoidal categorification of the cluster algebra $\mathcal{A}$ of type D_n.*

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A Classification of Roots of Symmetric Kac-Moody Root Systems and Its Application

Kazuki Hiroe and Toshio Oshima

Abstract We study Weyl group orbits in symmetric Kac-Moody root systems and show a finiteness of orbits of roots with a fixed index. We apply this result to the study of the Euler transform of linear ordinary differential equations on the Riemann sphere whose singular points are regular singular or unramified irregular singular points. The Euler transform induces a transformation on spectral types of the differential equations and it keeps their indices of rigidity. Then as a generalization of the result by Oshima (in Fractional calculus of Weyl algebra and Fuchsian differential equations, MSJ Memoirs 28, 2012), we show a finiteness of Euler transform orbits of spectral types with a fixed index of rigidity.

1 Introduction

Recall the definition of symmetric Kac-Moody root systems [3] (precise definition of terminology appearing below can be found in the latter section, see Sect. 2.1). For a finite index set I, define a lattice $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ with a basis $\{\alpha_i \mid i \in I\}$ and consider a symmetric bilinear form on Q which satisfies

$$\langle\alpha_i, \alpha_i\rangle = 2,$$
$$\langle\alpha_i, \alpha_j\rangle = \langle\alpha_j, \alpha_i\rangle \in \mathbb{Z}_{\leq 0} \quad (i,\ j \in I \text{ and } i \neq j).$$

The Weyl group W acting on Q is generated by simple reflections, $\sigma_i(\beta) := \beta - \langle\beta, \alpha_i\rangle\alpha_i$ for $\beta \in Q$ and $i \in I$.

Dedicated to Professor Michio Jimbo on the occasion of his 60th birthday.

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Then a certain subset of Q, called the set of roots, is defined by

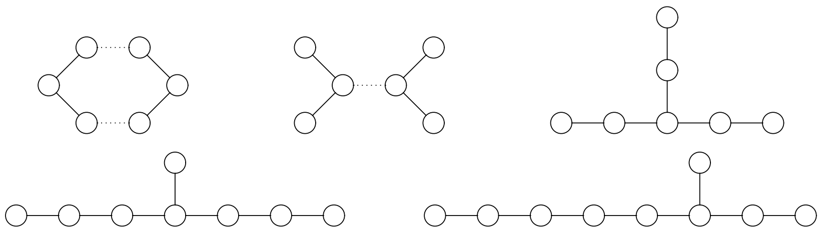
$$\Delta := \bigcup_{i \in I} W\alpha_i \sqcup WF \sqcup -WF.$$

Here $F := \{\alpha \in Q^+ \setminus \{0\} \mid \operatorname{supp}\alpha$ is connected and $\langle \alpha, \alpha_i \rangle \leq 0$ for all $i \in I\}$ with $Q^+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$. In particular we call $\Delta_{\mathrm{re}} := \bigcup_{i \in I} W\alpha_i$ the set of real roots and $\Delta_{\mathrm{im}} := WF \sqcup -WF$ the set of imaginary roots. If $\alpha \in \Delta$ is in Q^+, it is called a positive root. Moreover we call elements in F basic positive imaginary roots or shortly basic roots. Then we call the triple $(I, \langle\ ,\ \rangle, \Delta)$ or shortly $(I, \langle\ ,\ \rangle)$ the symmetric Kac-Moody root system.

A symmetric Kac-Moody root system $(\langle\ ,\ \rangle_1,\ I_1)$ is a subsystem of a symmetric Kac-Moody root system $(\langle\ ,\ \rangle_2,\ I_2)$ if there is an injective map ϕ of I_1 to I_2 such that $\langle \alpha_i, \alpha_j \rangle_1 = \langle \alpha_{\phi(i)}, \alpha_{\phi(j)} \rangle_2$ for $i, j \in I_1$ and in this case the root of $(\langle\ ,\ \rangle_1,\ I_1)$ is naturally identified with a root of $(\langle\ ,\ \rangle_2,\ I_2)$. Thus we can define the universal symmetric Kac-Moody root system by the inductive limit of symmetric Kac-Moody root systems under the injective maps defining subsystems.

One of our main aim is to classify the orbits of roots under the action of the Weyl group in the universal symmetric Kac-Moody root system. Since the real roots form a single orbit of the Weyl group, it is sufficient to classify the orbits contained in the set of positive imaginary roots, i.e., elements in $\Delta_{\mathrm{im}}^+ = \Delta_{\mathrm{im}} \cap Q^+ = WF$. Thus what we need to do is to classify elements in F, i.e., basic roots.

For an element α in a root lattice, the index of α is defined by $\operatorname{idx}\alpha := \langle \alpha, \alpha \rangle$. The classification of basic roots with index 0 is known as follows. Dynkin diagrams of supports of them are classified by the following 5 cases.



Moreover for each diagram, there exists a unique indivisible root and any basic roots are scalar multiples of one of these indivisible roots. Here $\alpha = \sum_{i \in I} m_i\alpha_i$ is indivisible if the greatest common divisor of coefficients m_i is 1.

Hence in this case, the classification of Weyl group orbits of imaginary roots is obtained by the classification of indivisible basic roots which correspond to the above finite cases.

One of the main results in this paper is to show a finiteness of basic roots with a general index. For this purpose we introduce the shape of an element in a root lattice. Fix a root lattice $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ and $\alpha = \sum_{i \in I} m_i\alpha_i \in Q$. For the Dynkin diagram of the support of α, we attach each coefficient m_i of α to the vertex corresponding to α_i, then we obtain the diagram with the coefficients, which we call the shape of α.

We say $i_1, \ldots, i_k \in \{i \in I \mid m_i \neq 0\}$ is a constant connected sequence of α if $m_{i_1} = \cdots = m_{i_k}$ and the Dynkin diagram of $\{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ is

$$\bigcirc\!\!-\!\!\bigcirc\cdots\bigcirc\!\!-\!\!\bigcirc.$$

Theorem 1 (see Corollary 1) *If a basic root* $\alpha = \sum_{i \in I} m_i \alpha_i$ *contains a constant connected sequence* $i_1, \ldots, i_k$ *of* I *satisfying* $k > 2$ *and* $\langle \alpha, \alpha_{i_\nu} \rangle = 0$ *for* $\nu = 2, \ldots, k-1$, *then the shape obtained from that of* α *by shrinking or extending the length of the sequence corresponds to a basic root with the same index. Expressing such a sequence by* $\overset{m}{\bigcirc}\cdots\overset{m}{\bigcirc}$, *we have shapes of roots which may contain such expressions. We call these shapes reduced shapes.*

Then the basic roots with a fixed nonzero index are classified by a finite number of reduced shapes. The indivisible basic roots with index 0 *are also classified by a finite number of reduced shapes.*

Moreover proceeding further from the classification of basic roots with index 0 seen above, we give the complete list of shapes of basic roots with index -2 in Sect. 2.4.

Another aim of this paper is to give a classification of orbits of linear ordinary differential equations under the action of the Euler transform as an application of our classification of basic roots.

Consider a Fuchsian system of ordinary differential equations on the Riemann sphere of the form $\frac{d}{dx} Y(x) = \sum_{i=0}^{p} \frac{A_i}{x - c_i} Y(x)$ where A_i $(i = 0, \ldots, p)$ are $n \times n$ matrices with coefficients in $\mathbb{C}$ and $Y(x)$ is a $\mathbb{C}^n$-valued function. For this system, W. Crawley-Boevey [2] constructs a representation of a quiver, more precisely, a deformed preprojective algebra, with a star-shaped quiver. His result shows that for an irreducible Fuchsian system, the dimension vector of the corresponding representation of the quiver is a positive root in the Kac-Moody root system of its quiver. Then the index of rigidity of the Fuchsian system equals the index of the root and reflection functors on representations of the quiver are obtained by algebraic transformations on Fuchsian systems, the Euler transform and the addition. Thus to study orbits of irreducible Fuchsian systems under the actions of the Euler transform and the additions, we can apply the classification of Weyl group orbits of the roots.

In [10, 11] the corresponding results for Fuchsian single differential equations together with the analysis of their global solutions, namely, integral representations of the solutions and the connection problem etc., are studied.

In [7], we consider a generalization of the result of Crawley-Boevey to ordinary differential equations whose singular points are regular singular or unramified irregular singular points. As in the case of the Fuchsian equations, there exists a Kac-Moody root system attached to a differential equation such that its spectral type corresponds to an element in the root lattice (see Theorem 6, Theorem 7 and Definition 11). Here a spectral type is a tuple of integers representing multiplicities of characteristic exponents of local formal solutions of a differential equation where we ignore integer differences of characteristic exponents (see Sect. 3.1.2 for the precise definition).

Thus for spectral types it shall be defined an analogy of basic roots, called basic pairs (see Definition 9). Then we shall consider a classification of basic pairs in Sect. 3.3 as an application of that of basic roots.

Combining this result with Theorem 1, we show the following theorem which generalizes the result of the second author [9, 11] in the Fuchsian case.

Theorem 2 (see Theorem 4) *Fix an integer $r \geq 0$ and consider linear differential equations with index of rigidity $-r$ on the Riemann sphere whose singular points are regular singular or unramified irregular singular points.*

Then we have the finiteness of orbits of spectral types of the differential equations under the actions of the Euler transform and the addition. Namely, if $r > 0$, there exist only a finite number of orbits and if $r = 0$, there exist a finite number of orbits of indivisible spectral types.

Finally in Sect. 3.3.2 and Sect. 3.3.3, we classify basic pairs with indices of rigidity 0 and -2. This gives classifications of Euler transform orbits of differential equations with these indices of rigidity. When all singular points are regular singular points, these classifications are given by V. Kostov [6] and the second author [9, 11], respectively.

2 A Classification of Basic Roots

2.1 Symmetric Kac-Moody Root Systems

Let $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ be a $\mathbb{Z}$-lattice with the basis $\{\alpha_i \mid i \in I\}$ where I is a finite set of indices. The set of positive elements in Q is written by $Q^+ := Q \cap \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$. Fix a symmetric $\mathbb{Z}$-bilinear form $\langle\ ,\ \rangle$ on Q satisfying

$$\langle \alpha_i, \alpha_i \rangle = 2 \quad (i \in I),$$
$$\langle \alpha_i, \alpha_j \rangle = \langle \alpha_j, \alpha_i \rangle \in \mathbb{Z}_{\leq 0} \quad (i,\ j \in I \text{ and } i \neq j).$$

We call this lattice Q with the bilinear form $\langle\ ,\ \rangle$ the *symmetric Kac-Moody root lattice*.

For an element $\alpha \in Q$, we define an even integer

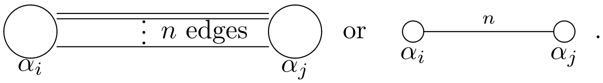
$$\operatorname{idx} \alpha := \langle \alpha, \alpha \rangle,$$

which we call the *index* of α. For each α_i $(i \in I)$, we can define a $\mathbb{Z}$-endomorphism of Q by

$$\sigma_i(\beta) = \beta - \langle \beta, \alpha_i \rangle \alpha_i \quad (\beta \in Q),$$

which is called the *simple reflection* with respect to α_i. The transformation group W on Q generated by all these σ_i $(i \in I)$ is called the *Weyl group*.

For this lattice Q, we associate a diagram which consists of edges and vertices as follows. Regard the elements in $\Pi := \{\alpha_i \mid i \in I\}$ as vertices. Connect two vertices $\alpha_i, \alpha_j \in \Pi$ by n edges if $\langle \alpha_i, \alpha_j \rangle = -n$ with a positive integer n. We express this by



We call the diagram constructed as above the *Dynkin diagram* of Q.

Let $\alpha = \sum_{i \in I} m_i \alpha_i \in Q$ with $m_i \in \mathbb{Z}$. The *support* of α is $\operatorname{supp}\alpha := \{\alpha_i \mid m_i \neq 0\}$. We say the support of α is *connected* if for any two distinct elements $\alpha_i, \alpha_j \in \operatorname{supp}\alpha$, there exists a sequence $\alpha_i = \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r} = \alpha_j$ of elements of $\operatorname{supp}\alpha$ such that $\langle \alpha_{i_k}, \alpha_{i_{k+1}} \rangle \neq 0$ for $k = 1, \ldots, r-1$. We define that α is *indivisible* if the greatest common divisor of $\{m_i \mid i \in I\}$ equals 1.

Recall the root system of Q. Each element α_i $(i \in I)$ of the basis of Q is called the *simple root*. The *real roots* are the elements of

$$\Delta_{\mathrm{re}} := \bigcup_{i \in I} W\alpha_i,$$

i.e., a real root belongs to the Weyl group orbit of a simple root α_i. Define the fundamental subset of Q,

$$F := \left\{\alpha \in Q^+ \setminus \{0\} \mid \operatorname{supp}\alpha \text{ is connected and } \langle \alpha, \alpha_i \rangle \leq 0 \text{ for all } i \in I\right\}.$$

Then the imaginary roots are the elements of

$$\Delta_{\mathrm{im}} := WF \sqcup -WF.$$

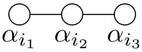
Here $WF = \{w\alpha \mid w \in W, \alpha \in F\}$ and $-WF = \{-\alpha \mid \alpha \in WF\}$. The root is the element of $\Delta := \Delta_{\mathrm{re}} \sqcup \Delta_{\mathrm{im}}$. The root in $\Delta^+ := \Delta \cap Q^+$ and that in F are called *positive* and *basic*, respectively.

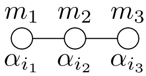
In general the symmetric Kac-Moody root system determined by the pair $\langle\ ,\ \rangle$ and I shall be denoted by $(\langle\ ,\ \rangle, I)$. A symmetric Kac-Moody root system $(\langle\ ,\ \rangle_1, I_1)$ is a subsystem of a symmetric Kac-Moody root system $(\langle\ ,\ \rangle_3, I_3)$ if there is a map ϕ of I_1 to I_3 such that $\langle \alpha_i, \alpha_j \rangle_1 = \langle \alpha_{\phi(i)}, \alpha_{\phi(j)} \rangle_3$ for $i, j \in I_1$ and in this case the root of $(\langle\ ,\ \rangle_1, I_1)$ is naturally identified with a root of $(\langle\ ,\ \rangle_3, I_3)$.

We define a root α of $(\langle\ ,\ \rangle_1, I_1)$ and a root α' of $(\langle\ ,\ \rangle_2, I_2)$ are in a same Weyl group orbit in a universal symmetric Kac-Moody root system if there exists a symmetric Kac-Moody root system $(\langle\ ,\ \rangle_3, I_3)$ such that $(\langle\ ,\ \rangle_1, I_1)$ and $(\langle\ ,\ \rangle_2, I_2)$ are subsystems of $(\langle\ ,\ \rangle_3, I_3)$ and moreover α and α' are in the same orbit under the action of the Weyl group of $(\langle\ ,\ \rangle_3, I_3)$. Namely, the universal symmetric Kac-Moody root system is defined by the inductive limit of symmetric Kac-Moody root systems under the injective maps defining subsystems.

Our purpose is to classify the Weyl group orbits in the universal symmetric Kac-Moody root system. Since the real roots form a single Weyl group orbit, it is sufficient to classify the orbits contained in the set of positive imaginary roots.

For an element $\alpha = \sum_{i \in I} m_i \alpha_i \in Q$, we consider the diagram of $\operatorname{supp}\alpha$, that is, we restrict the Dynkin diagram of Π to $\operatorname{supp}\alpha$. Then we attach each coefficient m_i of α to the vertex corresponding to α_i and obtain the diagram of the support of α with the coefficients. We call this diagram with coefficients the *shape* of α.

For example, if $\alpha = m_1\alpha_{i_1} + m_2\alpha_{i_2} + m_3\alpha_{i_3} \in Q$ with the diagram of the support 

, the diagram with coefficients is 

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Note that each Weyl group orbit contained in the set of positive imaginary roots has a unique representative in F and therefore the orbits containing positive imaginary roots are classified by the shapes of the basic roots in the orbits.

2.2 Basic Roots with a Fixed Index

First we examine some properties of the shapes of basic roots.

Fix an indivisible basic root

$$\alpha = \sum_{i \in I} m_i \alpha_i \quad (m_i \in \mathbb{Z}_{\geq 0}) \tag{1}$$

in this section and define subsets of I

$$\begin{cases} \bar{I} = \{i \in I \mid m_i > 0\}, \\ I_0 = \{i \in \bar{I} \mid \langle \alpha, \alpha_i \rangle = 0\}, \\ I_1 = \bar{I} \setminus I_0. \end{cases} \tag{2}$$

Lemma 1 *Let* $\{i_1, \ldots, i_k\} \subset J$ *for a subset* J *of* $\bar{I}$ *such that* $i_\nu \neq i_{\nu'}$ *for* $1 \leq \nu < \nu' \leq k$ *and* $\langle \alpha_{i_\nu}, \alpha_{i_{\nu+1}} \rangle \neq 0$ *for* $\nu = 1, \ldots, k-1$. *Then we call that* $i_1, \ldots, i_k$ *is a connected sequence of length* k *in* J. *Moreover if* $m_{i_1} = m_{i,2} = \cdots = m_{i,k}$, *we call* $i_1, \ldots, i_k$ *is a constant connected sequence.*

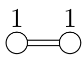
(i) *Suppose* i_1, i_2 *is a connected sequence in* $\bar{I}$ *with* $i_2 \in I_0$. *Then*

$$m_{i_1} \leq 2m_{i_2} \tag{3}$$

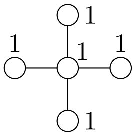
and if $m_{i_1} = 2m_{i_2}$,

$$\langle \alpha_{i_1}, \alpha_{i_2} \rangle = -1 \tag{4}$$

and $\langle \alpha_{i_2}, \alpha_\nu \rangle = 0$ *for* $\nu \in \bar{I} \setminus \{i_1, i_2\}$.

Furthermore if $i_1 \in I_0$, *then* (4) *is valid or the shape of* α *is* 

.

(ii) *Fix* $i_0 \in I_0$ *and put* $J_{i_0} = \{i \in I_0 \mid \langle \alpha_i, \alpha_{i_0} \rangle < 0\}$. *Then* $\# J_{i_0} \le 4$ *and the equality holds if and only if the shape of* α *is* 

. *If* $\# J_{i_0} = 3$, *then* $m_i < m_{i_0}$ *for* $i \in J_{i_0}$ *or* $\{m_i \mid i \in J_{i_0}\} = \{m_{i_0}, \frac{1}{2} m_{i_0}, \frac{1}{2} m_{i_0}\}$.

(iii) *Let* $i_1, \ldots, i_k$ *be a connected sequence in* I *with* $k \ge 3$. *Suppose* $i_\nu \in I_0$ *for* $\nu = 2, \ldots, k-1$ *and* $m_{i_1} \ge m_{i_2}$. *Then*

$$m_{i_1} - m_{i_k} \ge (k-1)(m_{i_1} - m_{i_2}).$$

If $m_{i_1} - m_{i_k} = (k-1)(m_{i_1} - m_{i_2})$, *then*

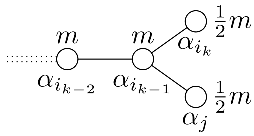
$$\langle \alpha_i, \alpha_{i_\nu} \rangle = \begin{cases} -1 & (i = i_{\nu-1} \text{ or } i_{\nu+1} \text{ and } 1 < \nu < k), \\ 0 & (i \in \bar{I} \setminus \{i_{\nu-1},\ i_\nu,\ i_{\nu+1}\} \text{ and } 1 < \nu < k). \end{cases}$$

If $m_{i_1} - m_{i_{k-1}} = (k-2)(m_{i_1} - m_{i_2})$ *and* $m_{i_1} - m_{i_k} > (k-1)(m_{i_1} - m_{i_2})$, *then there exists* $j \in \bar{I}$ *such that*

$$\langle \alpha_{i_{k-1}}, \alpha_j \rangle < 0 \quad \text{and} \quad j \in I_1 \tag{5}$$

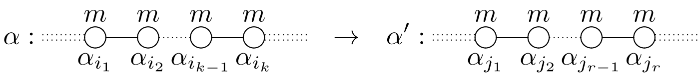
or

$$\begin{cases} m_{i_1} = m_{i_2}, i_k \ne j, m_{i_{k-1}} = 2m_{i_k} = 2m_j \\ \text{and } \langle \alpha_{i_{k-1}}, \alpha_j \rangle = \langle \alpha_{i_{k-1}}, \alpha_{i_k} \rangle = -1. \end{cases}$$



(6)

Suppose $m_{i_1} = m_{i_2} = \cdots = m_{i_k}$. *Then* $\{j \in \bar{I} \mid \langle \alpha_{i_\nu}, \alpha_j \rangle < 0\} = \{i_{\nu-1}, i_{\nu+1}\}$ *for* $\nu = 2, \ldots, k-1$. *Moreover suppose* $\langle \alpha_{i_1}, \alpha_{i_k} \rangle = 0$. *Fix* $r \in \mathbb{Z}_{>0}$, *put* $m = m_{i_1}$ *and introduce new simple roots* $\alpha_{j_1}, \ldots, \alpha_{j_r}$ *and put* $I' = (\bar{I} \cup \{j_1, \ldots, j_r\}) \setminus \{i_1, \ldots, i_k\}$. *Then the element* $\alpha' = \sum_{i \in I'} m_i \alpha_i$ *with* $m_{j_\nu} = m$ $(1 \le \nu \le r)$ *is also a basic root such that* $r = 1$ *or* $j_1, \ldots, j_r$ *is a constant connected sequence satisfying* $\langle \alpha, \alpha_{j_\nu} \rangle = 0$ *for* $\nu = 2, \ldots, r-1$ *and* $\operatorname{idx} \alpha = \operatorname{idx} \alpha'$. *Here* $\langle \alpha_j, \alpha_{j_1} \rangle = \langle \alpha_j, \alpha_{i_1} \rangle + \delta_{r,1} \langle \alpha_j, \alpha_{i_k} \rangle$ *for* $j \in \bar{I} \setminus \{i_1, \ldots, i_k\}$ *etc.*



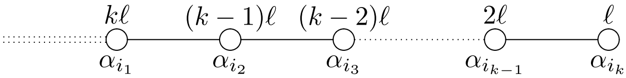
$(r = 1, 2, \ldots)$

(iv) *Suppose that* i_1, i_2 *is a connected sequence in* $\bar{I}$ *with* $i_2 \in I_0$ *and* $\ell := m_{i_1} - m_{i_2} \ge 0$. *Then there exists a connected sequence* $i_1, i_2, \ldots, i_k$ *in* $\bar{I}$ *such that*

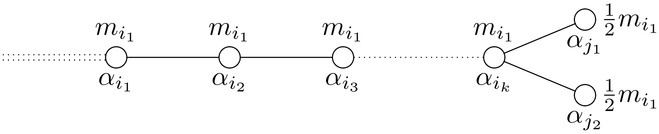
$$\langle \alpha_{i_\nu}, \alpha_i \rangle = 0 \quad (i \in \bar{I} \setminus \{i_{\nu-1}, i_\nu, i_{\nu+1}\},\ \nu = 2, \ldots, k-1)$$

and one of the following is valid.

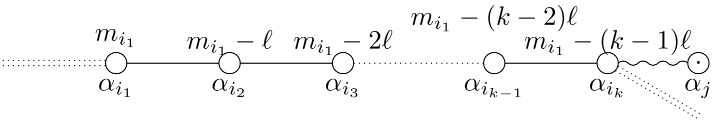
(a) $\ell > 0,\ k\ell = m_{i_1},\ m_{i_\nu} = m_{i_1} - (\nu - 1)\ell\ (0 \le \nu \le k)$,
and $\langle \alpha_{i_k}, \alpha_i \rangle = 0\ (i \in \bar{I} \setminus \{i_{k-1},\ i_k\})$: (7)



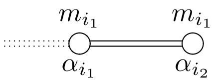
(b) $\ell = 0,\ m_{i_1} = \cdots = m_{i_{k-1}} = m_{i_k}$ *and there exist* $j_\nu \in I_0$ *for* $\nu = 1, 2$
such that $2m_{j_\nu} = m_{i_1},\ \langle \alpha_{j_\nu}, \alpha_i \rangle = 0\ (i \in \bar{I} \setminus \{i_k,\ j_\nu\})$, (8)
$\langle \alpha_{j_\nu}, \alpha_{i_k} \rangle = -1,\ \langle \alpha_{i_k}, \alpha_i \rangle = 0\ (i \in \bar{I} \setminus \{i_{k-1},\ i_k,\ j_1,\ j_2\})$:



(c) $k\ell < m_{i_1},\ m_{i_\nu} = m_{i_1} - (\nu - 1)\ell$ *for* $\nu = 1, \ldots, k$
and there exists $j \in I_1$ *with* $\langle \alpha_{i_k}, \alpha_j \rangle < 0$: (9)



(d) $k = 2,\ l = 0,\ m_{i_1} = m_{i_2},\ \langle \alpha_{i_1}, \alpha_{i_2} \rangle = -2,\ \langle \alpha_{i_2}, \alpha_i \rangle = 0\ (i \in \bar{I} \setminus \{i_1,\ i_2\})$



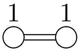
(10)

Proof (i) Since $\langle \alpha_{i_1}, \alpha_{i_2} \rangle \le -1$ and $\langle \alpha_\nu, \alpha_{i_2} \rangle \in \mathbb{Z}_{\le 0}$ for $\nu \in \bar{I} \setminus \{i_1, i_2\}$ and

$$2m_{i_2} + m_{i_1} \langle \alpha_{i_1}, \alpha_{i_2} \rangle + \sum_{\nu \in I \setminus \{i_1, i_2\}} m_\nu \langle \alpha_\nu, \alpha_{i_2} \rangle = \langle \alpha, \alpha_{i_2} \rangle = 0,$$

we have $m_{i_1} \le 2m_{i_2}$ and the condition $m_{i_1} = 2m_{i_2}$ implies $\langle \alpha_{i_1}, \alpha_{i_2} \rangle = -1$ and $\langle \alpha_\nu, \alpha_{i_2} \rangle = 0$ for $\nu \in \bar{I} \setminus \{i_1, i_2\}$.

Suppose $i_1 \in I_0$ and $\langle \alpha_{i_1}, \alpha_{i_2} \rangle < -1$. Then we have $m_{i_2} \le m_{i_1}$. In the same way we have $m_{i_1} \le m_{i_2}$ and hence $m_{i_1} = m_{i_2}$ and $\langle \alpha_{i_1}, \alpha_{i_2} \rangle = -2$ and the shape of α is



(ii) We may assume $\#J_{i_0} > 2$. Since the claim (i) shows $\langle \alpha_{i_0}, \alpha_\nu \rangle = -1$ and $m_{i_0} \le 2m_\nu$ for $\nu \in J_{i_0}$ and the condition $i_0 \in I_0$ implies $2m_{i_0} - \sum_{\nu \in J_{i_0}} m_\nu \ge 0$, we have $\#J_{i_0} \le 4$. If $\#J_{i_0} = 4$, $2m_\nu = m_{i_0}$ for $\nu \in J_{i_0}$ and the shape of α is given in the claim.

Suppose $\#J_{i_0} = 3$. Put $J_{i_0} = \{i_1, i_2, i_3\}$ with $m_{i_1} \geq m_{i_2} \geq m_{i_3}$. Then $m_{i_1} \leq 2m_{i_0} - m_{i_2} - m_{i_3} \leq 2m_{i_0} - \frac{1}{2}m_{i_0} - \frac{1}{2}m_{i_0} = m_{i_0}$. If $m_{i_1} = m_{i_0}$, $m_{i_2} = m_{i_3} = \frac{1}{2}m_{i_0}$.

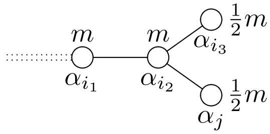
(iii) We may assume $k = 3$. Since $m_{i_1} \geq m_{i_2}$ and

$$0 = \langle \alpha, \alpha_{i_2} \rangle = 2m_{i_2} + m_{i_1} \langle \alpha_{i_1}, \alpha_{i_2} \rangle + m_{i_3} \langle \alpha_{i_3}, \alpha_{i_2} \rangle + \sum_{\nu \in I \setminus \{i_1, i_2, i_3\}} m_\nu \langle \alpha_\nu, \alpha_{i_2} \rangle,$$

we have $\langle \alpha_{i_1}, \alpha_{i_2} \rangle = -1$ and $2m_{i_2} \geq m_{i_1} + m_{i_3}$, which means $m_{i_1} - m_{i_3} \geq 2(m_{i_1} - m_{i_2})$. Moreover the condition $m_{i_1} - m_{i_3} = 2(m_{i_1} - m_{i_2})$ implies $\langle \alpha_{i_2}, \alpha_{i_3} \rangle = -1$ and $\langle \alpha_\nu, \alpha_{i_2} \rangle = 0$ for $\nu \in \bar{I} \setminus \{i_1, i_2, i_3\}$.

Suppose $2m_{i_2} > m_{i_1} + m_{i_3}$ and $i_3 \in I_0$. Then the claim (i) shows $\langle \alpha_{i_2}, \alpha_{i_3} \rangle = -1$ and there exists $j \in \bar{I} \setminus \{i_1, i_2\}$ satisfying $\langle \alpha_{i_2}, \alpha_j \rangle < 0$. Suppose $j \in I_0$. Then $2m_j \geq m_{i_2}$, $2m_{i_3} \geq m_{i_2}$ and $m_{i_2} - m_{i_3} - m_j \geq 2m_{i_2} - m_{i_1} - m_{i_3} - m_j \geq 0$ and therefore $2m_j = 2m_{i_3} = m_{i_1} = m_{i_2}$ and

$$\langle \alpha_{i_2}, \alpha_\nu \rangle = \begin{cases} -1 & (\nu = i_1,\ i_3,\ j), \\ 0 & (\nu \in \bar{I} \setminus \{i_1,\ i_2,\ i_3,\ j\}), \end{cases}$$
$$\langle \alpha_j, \alpha_\nu \rangle = 0 \quad (\nu \in \bar{I} \setminus \{i_2,\ j\}),$$
$$\langle \alpha_{i_3}, \alpha_\nu \rangle = 0 \quad (\nu \in \bar{I}\{i_2,\ i_3\}).$$



Thus we have (iii) since the last claim in (iii) is clear.

(iv) The claims easily follow from (iii). □

Now we give one of our main results in this paper.

Theorem 3 *Fix integers $N \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}_{>0}$. Let α be the basic root satisfying the following conditions*:

1. $\operatorname{idx} \alpha = -N$.
2. *$N \neq 0$ or α is indivisible.*
3. *α has no constant connected sequence in I_0 whose length is larger than M.*

Then there are only finite shapes which can be the shape of α.

Proof Since the basic roots with index 0 are well-known as are given in the next section, we may assume $N > 0$. We shall use the notation in the previous lemma. Since

$$N = -\sum_{i \in I_1} m_i \langle \alpha, \alpha_i \rangle \geq \sum_{i \in I_1} m_i, \tag{11}$$

we have

$$m_i \leq N \quad \text{for } i \in I_1 \text{ and } \#I_1 \leq N. \tag{12}$$

Let $i \in I_1$ and $j \in I_0$ and suppose $\langle \alpha_i, \alpha_j \rangle < 0$. Then

$$N = -\sum_{\nu \in I_1} m_\nu \langle \alpha, \alpha_\nu \rangle \geq -m_i \langle \alpha, \alpha_i \rangle \geq m_i m_j \left| \langle \alpha_j, \alpha_i \rangle \right| - 2m_i^2$$

and therefore

$$m_j \leq m_i^{-1} N + 2m_i \leq 3N \quad \text{and} \left| \langle \alpha_i, \alpha_j \rangle \right| \leq 3N. \tag{13}$$

Since $N = -\sum_{\nu \in I_1} m_\nu \langle \alpha, \alpha_\nu \rangle = -\sum_{i \in I} \sum_{\nu \in I_1} m_i m_\nu \langle \alpha_i, \alpha_\nu \rangle - 2 \sum_{\nu \in I_1} m_\nu^2$, we have

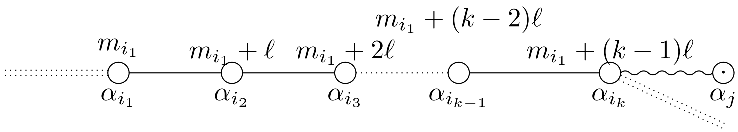
$$\sum_{i \in I_0} \sum_{\nu \in I_1} \left| \langle \alpha_i, \alpha_\nu \rangle \right| \leq N + 2 \sum_{\nu \in I_1} m_\nu^2 \leq N + 2N^2 \leq 3N^2 \tag{14}$$

and therefore $\#\partial I_0 \leq 3N^2$ by denoting $\partial I_0 := \{i \in I_0 \mid \sum_{\nu \in I_1} \langle \alpha_i, \alpha_\nu \rangle \neq 0\}$.

Fix $i_1 \in \partial I_0$. Suppose $J_{i_1} := \{j \in I_0 \mid \langle \alpha_j, \alpha_{i_1} \rangle < 0\} \neq \emptyset$. Note that $\#J_{i_1} \leq 3$. Fix $i_2 \in J_{i_1}$ and put

$$J(i_1, i_2) = \big\{ i \in I_0 \mid \exists \text{ connected sequence } i_1, i_2, \ldots, i_k = i \text{ in } I_0 \\ \text{with } i_\nu \notin \partial I_0 \ (1 < \nu < k) \big\}.$$

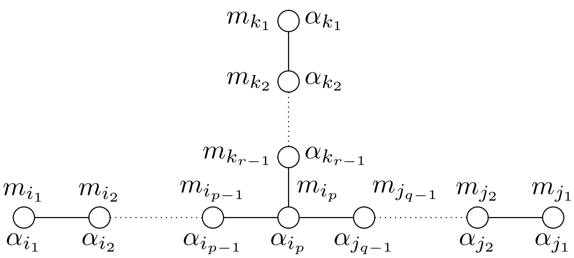
Then the Dynkin diagram of $J(i_1, i_2)$ equals that in (7) or (8) or (9) or



(15)

$$\ell \in \mathbb{Z}_{>0},\ \alpha_j \in I_1,\ J(i_1, i_2) = \{i_1, \ldots, i_k\},\ i_2, \ldots, i_{k-1} \in I_0 \setminus \partial I_0$$

or



(16)

$$m_{i_\nu} = m_{i_p} - (p - \nu)\ell_i,\ m_{j_\nu} = m_{i_p} - (q - \nu)\ell_j,\ m_{k_\nu} = m_{i_p} - (q - \nu)\ell_k,$$

$$\ell_i,\ \ell_j,\ \ell_k \in \mathbb{Z}_{>0},\ i_2, \ldots, i_p,\ j_2, \ldots, j_{q-1}, k_2, \ldots, k_{r-1} \in I_0 \setminus \partial I_0,$$

$$\{i_1, j_1, k_1\} \cap \partial I_0 \neq \emptyset,\ p \geq 2,\ q \geq 2,\ r \geq 2.$$

If the Dynkin diagram is not of the form (16), we have

$$\#J(i_1, i_2) \leq 3N + M \quad \text{and} \quad m_i \leq 3N \quad \bigl(i \in J(j_1, j_2)\bigr) \tag{17}$$

by the estimate (13).

Hence we assume the Dynkin diagram is of the form (16). We may assume $\ell_i \leq \ell_j \leq \ell_k$ without loss of generality. Since $i_p \in I_0$, we have $2m_{i_p}^2 = m_{i_p}(m_{i_p} - \ell_i) + m_{i_p}(m_{i_p} - \ell_j) + m_{i_p}(m_{i_p} - \ell_k)$ and therefore

$$\ell_i + \ell_j + \ell_k = m_{i_p}.$$

Hence $3\ell_k \geq m_{i_p}$ and $r \leq 3$.

If $k_1 \in \partial I_0$, $r = 2$ and $m_{k_1} \leq 3N$ and we have

$$m_{i_p} < 6N, \qquad \#J(i_1, i_2) < 12N \quad \text{and} \quad m_i < 6N \quad \text{for } i \in J(i_1, i_2) \tag{18}$$

because $p < 6N$ and $q < 6N$.

Suppose $k_1 \in I_0 \setminus \partial I_0$. Then $r = 2$ or $r = 3$.

If $r = 3$, $\ell_i = \ell_j = \ell_k = \frac{1}{3}m_{i_p}$ and we may assume $i_1 \in \partial I_0$ and we have the same claim (18).

Suppose $r = 2$. Then $\ell_k = \frac{1}{2}m_{i_p}$ and $\frac{1}{2}m_{i_p} > \ell_j \geq \frac{1}{4}m_{i_p}$. If $j_1 \in I_0 \setminus \partial I_0$, $4\ell_j = 4\ell_i = m_{i_p}$ or $3\ell_j = 6\ell_i = m_{i_p}$ and therefore $p \leq 5$ and

$$\#J(i_1, i_2) < 9 \quad \text{and} \quad m_i \leq 15N \quad \bigl(i \in J(i_1, i_2)\bigr). \tag{19}$$

If $j_1 \in \partial I_0$, $q \leq 3$ and $m_p \leq 9N$ and therefore

$$\#J(i_1, i_2) \leq 9N + 2 + 1 = 9N + 3 \quad \text{and} \quad m_i \leq 9N \quad \bigl(i \in J(i_1, i_2)\bigr). \tag{20}$$

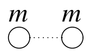
Since $\#\{(i_1, i_2) \mid i_1 \in \partial I_0,\ i_2 \in I_0,\ \langle \alpha_{i_1}, \alpha_{i_2} \rangle < 0\} \leq 3 \cdot \#\partial I_0 \leq 9N^2$, we have

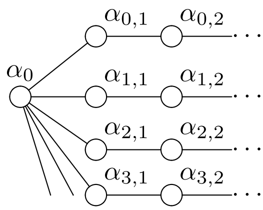
$$\#I \leq 9N^2 \cdot (12N + M) + \#\partial I_0 + \#I_1 \leq 108N^3 + 9MN^2 + 3N^2 + N,$$

$$m_i \leq 15N \quad (i \in I) \quad \text{and} \quad \bigl|\langle \alpha_i, \alpha_j \rangle\bigr| \leq 3N \quad (i,\ j \in I).$$

These estimates imply the theorem. □

The proof of Theorem 3 assures the following finiteness of the shapes.

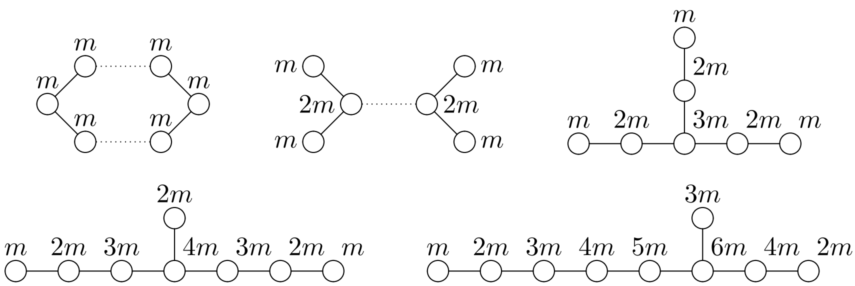
Corollary 1 *If a basic root $\alpha = \sum_{i \in I} m_i \alpha_i$ contains a constant connected sequence $i_1, \ldots, i_k$ of I such that $k \geq 2$ and $\langle \alpha, \alpha_{i_\nu} \rangle = 0$ for $\nu = 2, \ldots, k-1$, then the shape obtained from that of α by shrinking or extending the length of the sequence corresponds to a basic root with the same index. Expressing such a sequence by*  *, we have shapes of roots which may contain such expressions. We call these shapes reduced shapes. Then the basic roots with a fixed nonzero index are classified by a finite number of reduced shapes. Also the indivisible basic roots with index* 0 *are classified by a finite number of reduced shapes.*

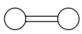
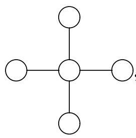
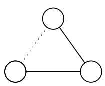
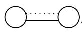
Remark 1 The Dynkin diagram of the form 

is called star-shaped. The basic roots whose shapes have star-shaped Dynkin diagrams are studied and the finiteness of such basic roots with a fixed index is proved in [9]. The number of such shapes with index 0, −2, −4, −6, ... equals 4, 13, 36, 67, ..., respectively, and the list of them is given in [11].

2.3 Basic Roots with Index 0

Theorem 3 assures that in the universal symmetric Kac-Moody root system there are only a finite number of Weyl group orbits with a fixed index. The basic roots with index 0 are well-known and we list their shapes as follows.



These are diagrams obtained by attaching coefficients to the Dynkin diagrams called *Euclidean diagrams*, which are denoted by $A_n^{(1)}$ $(n \geq 1)$, $D_n^{(1)}$ $(n \geq 4)$, $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$, respectively. Here m are positive integers and $A_n^{(1)}$ and $D_n^{(1)}$ have $n+1$ vertices. Moreover $A_1^{(1)}$ and $D_4^{(1)}$ mean  and , respectively. Hence $A_n^{(1)}$ for $n \geq 1$ shall be written by  or .

2.4 Basic Roots with Index −2

In this section, we shall give a classification of the basic roots whose indices are −2. Suppose that $\alpha = \sum_{i \in I} m_i \alpha_i \in Q$ is basic and $\operatorname{idx} \alpha = -2$. Retain the notation in

Sect. 2.2 and put $N_i = -\langle \alpha, \alpha_i \rangle \geq 0$. Then $\langle \alpha, \alpha \rangle = -\sum_{i \in I} m_i N_i$ and $I_1 = \{i \in I \mid N_i > 0,\ m_i > 0\}$.

Lemma 2 *Let $\alpha \in Q$ be as above. Put*

$$\mathrm{Ed}\,(\alpha_i) := -\sum_{j \in \bar{I} \setminus \{i\}} \langle \alpha_j, \alpha_i \rangle, \tag{21}$$

which equals the number of edges spread out from α_i. Then we have the following.

(i) *The cardinality* $\#I_1$ *is* 1 *or* 2.
(ii) *If $I_1 = \{i\}$, there are two cases.*
Case 1: $m_i = 2$, $N_i = 1$ *and* $\mathrm{Ed}\,(\alpha_i) \leq 5$.
Case 2: $m_i = 1$, $N_i = 2$ *and* $\mathrm{Ed}\,(\alpha_i) \leq 4$.
(iii) *If $I_1 = \{i, i'\}$, then $m_i = m_{i'} = N_i = N_{i'} = 1$,* $\mathrm{Ed}\,(\alpha_i) \leq 3$ *and* $\mathrm{Ed}\,(\alpha_{i'}) \leq 3$.

Proof Since $2 = \sum_{i \in I} m_i N_i = \sum_{j \in I_1} m_j N_j$, we have $\#I_1 = 1$ or 2. Then $(m_i, N_i) = (1, 2)$ or $(2, 1)$ if $I_1 = \{i\}$ and $m_i = m_{i'} = N_i = N_{i'} = 1$ if $I_1 = \{i, i'\}$. The remaining assertions follow from $N_i = \sum_{j \in \bar{I} \setminus \{i\}} m_j \langle \alpha_j, \alpha_i \rangle - 2m_i \geq \mathrm{Ed}\,(\alpha_i) - 2m_i$. □

From this lemma and Lemma 1, the basic roots with index −2 are classified by the following cases.

Case 1: $I_1 = \{i\}$, $m_i = 2$ and $\mathrm{Ed}\,(\alpha_i) \leq 5$

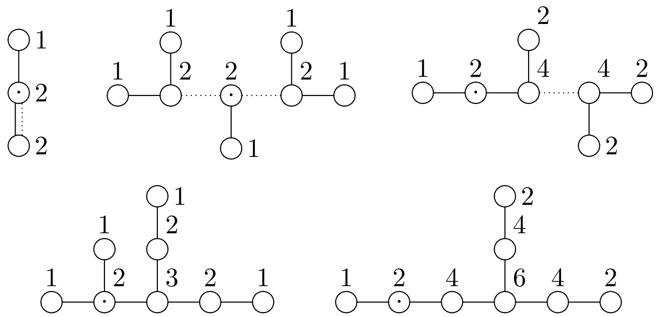
Since

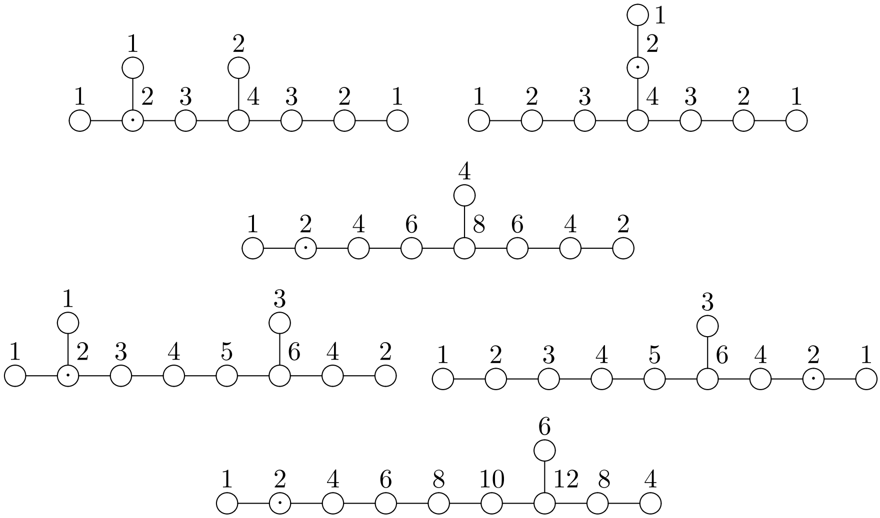
$$\sum_{\nu \in \bar{I} \setminus \{i\}} m_\nu \left| \langle \alpha_i, \alpha_\nu \rangle \right| = 5,$$

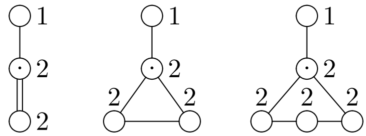
one of the following Case 1.1, Case 1.2 or Case 1.3 is valid.

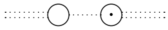
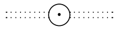
Case 1.1: There exists α_k such that $m_k = 1$ and $\langle \alpha_i, \alpha_k \rangle < 0$.

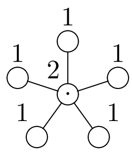
It follows from Lemma 1(i) that $\langle \alpha_j, \alpha_k \rangle = 0$ for $j \in \bar{I} \setminus \{i,\ k\}$ and $\langle \alpha_i, \alpha_k \rangle = -1$. Then the element $\alpha' = \alpha - \alpha_k \in Q^+$ satisfies $\langle \alpha', \alpha_i \rangle = 0$ for $i \in \bar{I} \setminus \{k\}$ and $\mathrm{supp}\,\alpha'$ is connected. Hence the diagram of $\{\alpha_i \mid i \in \bar{I} \setminus \{k\}\}$ is one of the Euclidean diagrams $A_n^{(1)}$, $D_n^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ given in the previous section and we have the list of the shapes with indicating α_i by dotted circles.





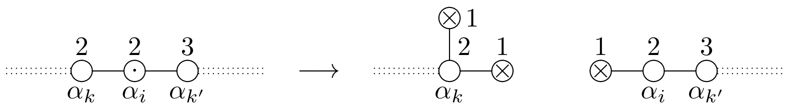
Note that the first shape represents  ⋯, etc.,

and the part  in the diagram above can be  as a special

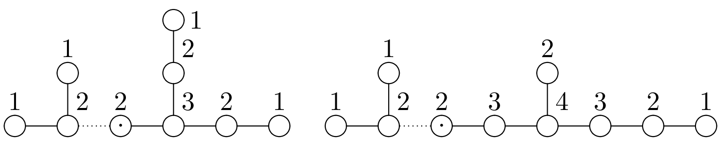
case. Hence  is a special case of the second shape.

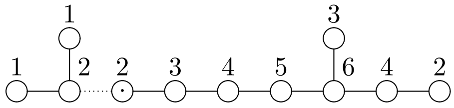
Case 1.2: There exist α_k and $\alpha_{k'}$ such that $(m_k, m_{k'}) = (2,3)$ and $\langle\alpha_i, \alpha_k\rangle = \langle\alpha_i, \alpha_{k'}\rangle = -1$.

Then cutting the shape of the basic root between α_k and α_i and adding three vertices, we have one or two Euclidean diagrams with coefficients corresponding to some basic roots of index 0:



Here each $\otimes$ represents a new vertex. It follows from the shapes given in the previous section that the corresponding diagrams are $D_n^{(1)}$ and $E_k^{(1)}$ $(k = 6,\ 7,\ 8)$ and we have the following list of shapes:





Replacing $\overset{2}{\bigcirc}\cdots\overset{2}{\odot}$ by $\overset{2}{\odot}$ in the above shapes, we may regard three shapes in Case 1.1 as special cases in Case 1.2.

Case 1.3: There uniquely exists α_k with $k \in \bar{I}$ such that $\langle \alpha_i, \alpha_k \rangle < 0$. Put

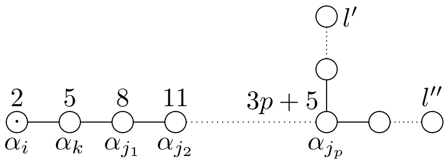
$$\left\{ j \in \bar{I} \mid \langle \alpha_k, \alpha_j \rangle < 0 \right\} = \{i, l_1, \ldots, l_r\}$$

with suitable r. Note that $m_i = 2$, $m_k = 5$, $k \in I_0$, $\ell_\nu \in I_0$ and $\langle \alpha_{l_\nu}, \alpha_k \rangle = -1$ for $\nu = 1, \ldots, r$. Since $\sum_{j \in \bar{I}} m_j \langle \alpha_k, \alpha_j \rangle = 0$, we have

$$m_{l_1} + \cdots + m_{l_r} = 2m_k - m_i = 2 \times 5 - 2 = 8$$

and Lemma 1(iv) shows $m_{l_\nu} \geq 4$ and $m_{l_\nu} \neq 5$ for $\nu = 1, \ldots, r$. Hence $\{m_{l_1}, \ldots, m_{l_r}\}$ $= \{4, 4\}$ or $\{8\}$.

Suppose $\{m_{l_1}, \ldots, m_{l_r}\} = \{8\}$. Then the shape of α is



$(p \geq 1,\ j_1 = l_1)$

and there exist positive integers p' and p'' such that $p'l' = p''l'' = 3p + 5$. The condition $j_p \in I_0$ shows

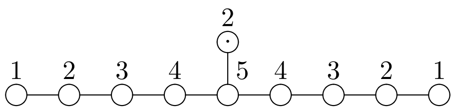
$$2(3p+5) = (3p+2) + (3p+5)\frac{p'-1}{p'} + (3p+5)\frac{p''-1}{p''},$$

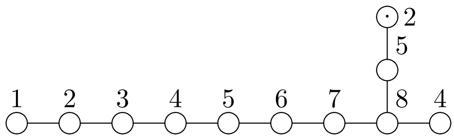
$$2 = \frac{3p+2}{3p+5} + \frac{p'-1}{p'} + \frac{p''-1}{p''},$$

$$1 = \frac{3}{3p+5} + \frac{1}{p'} + \frac{1}{p''}.$$

Since $\{1 - \frac{1}{p'} - \frac{1}{p''} \mid p',\ p'' \in \mathbb{Z}_{>0}\} \cap (0, 1) \subset [\frac{1}{6}, 1)$, it follows that $3p + 5 =$ 8, 11, 14, 17 and $1 - \frac{3}{3p+5} = \frac{5}{8}, \frac{8}{11}, \frac{11}{14}, \frac{14}{17}$. Then we can conclude $p = 1$ and $\{p', p''\} = \{2, 8\}$, which corresponds to $\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$.

Hence α is one of the following.





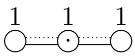
Case 2: $I_1 = \{i\}$, $m_i = 1$ and $\mathrm{Ed}\,(\alpha_i) \le 4$.
Since

$$\sum_{\nu \in \bar{I} \setminus \{i\}} m_\nu \big|\langle \alpha_i, \alpha_\nu \rangle\big| = 4,$$

one of Case 2.1, …, Case 2.4 is valid.

Case 2.1: The condition $\langle \alpha_\nu, \alpha_i \rangle \neq 0$ implies $m_\nu \le 1$.

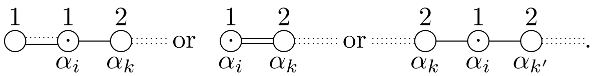
Then it follows from Lemma 1 that the shape of α is the following:



Hence the diagrams is obtained by connecting Euclidean diagrams $A_n^{(1)}$ $(n \ge 1)$ and $A_{n'}^{(1)}$ $(n' \ge 1)$ at the common vertex α_i.

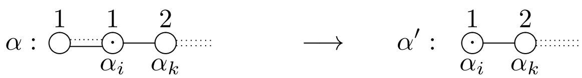
Case 2.2: There exists α_k such that $\langle \alpha_k, \alpha_i \rangle \neq 0$ and $m_k = 2$.

Then the shape of α is

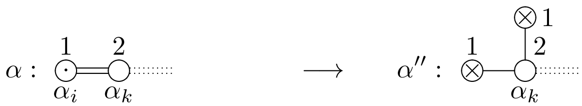


or

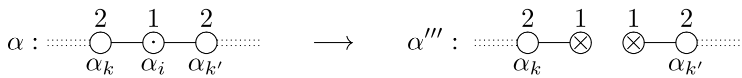
Modify these diagrams with coefficients as follows.



(22)



(23)



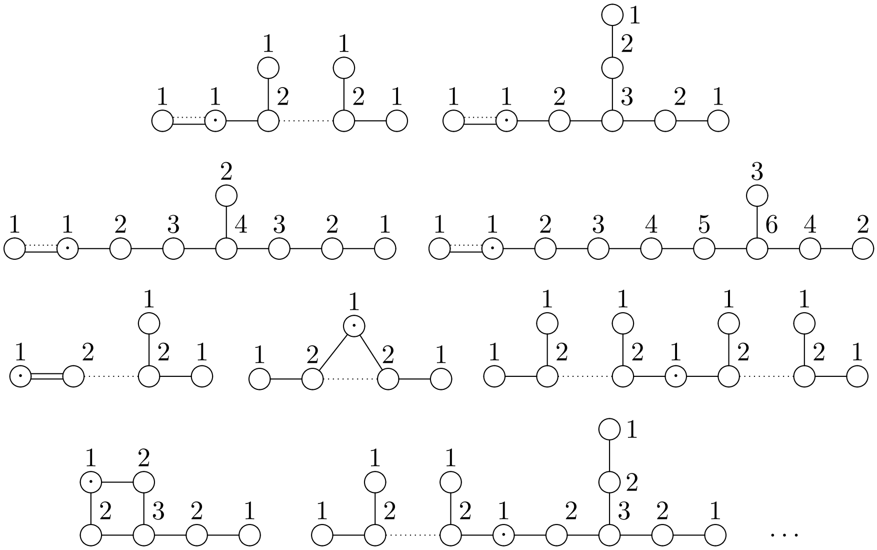
(24)

Here we do not modify the parts ⋯⋯ in the above. Then α', α'' and α''' are elements of F with the given shapes and their indices are 0.

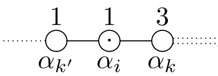
The element $\alpha' \in F$ is a basic root with the diagram $D_n^{(1)}$ or $E_7^{(1)}$ or $E_8^{(1)}$ and in this case α corresponds to the first four shapes in the list below.

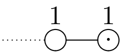
The element $\alpha'' \in F$ is an indivisible basic root with the diagram $D_n^{(1)}$ and we have the fifth shape in the list below.

The shape of $\alpha''' \in F$ is that of an indivisible basic root with the diagram $D_n^{(1)}$ or $E_6^{(1)}$ or $E_7^{(1)}$ or a disjoint union of the shapes $\mathcal{D}_1$ and $\mathcal{D}_2$ of indivisible basic roots with the diagrams $\mathcal{D}_\nu \in \{D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}\}$ for $\nu = 1$ and 2. In each case it is easy to write the shape of α and therefore we only give some examples in the list below.



Case 2.3: There exists α_k such that $\langle \alpha_k, \alpha_i \rangle \neq 0$ and $m_k = 3$.

Then α corresponds to 

. However applying Lemma 1(iv) to the part 

of this shape, we can conclude that such α does not exist.

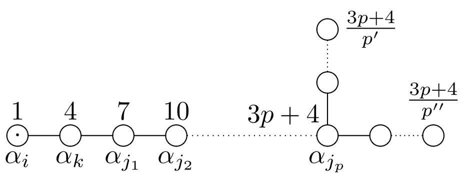
Case 2.4: There exists α_k such that $\langle \alpha_k, \alpha_i \rangle \neq 0$ and $m_k = 4$.

In the same way as in Case 1.3 we have

$$\left\{ j \in \bar{I} \mid \langle \alpha_k, \alpha_j \rangle < 0 \right\} = \{i, l_1, \dots, l_r\},$$
$$m_{l_1} + \cdots + m_{l_r} = 2m_k - m_i = 7,$$
$$m_{l_\nu} \geq 2, \qquad m_{l_\nu} \neq 4 \quad (1 \leq \nu \leq r).$$

Hence $\{m_{\ell_1}, \dots, m_{\ell_r}\} = \{3, 2, 2\}$ or $\{5, 2\}$ or $\{4, 3\}$ or $\{7\}$.

Suppose $\{m_{\ell_1}, \dots, m_{\ell_r}\} = \{7\}$. Then α corresponds to



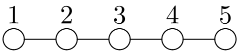
$(p \geq 1,\ j_1 = l_1)$

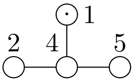
and we have

$$2(3p+4) = 3p+1+(3p+4)\frac{p'-1}{p'} + (3p+4)\frac{p''-1}{p''},$$

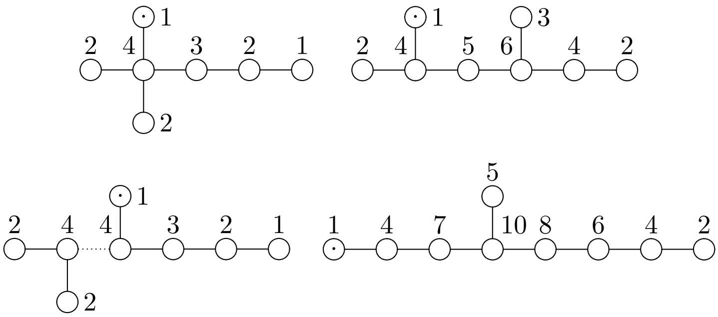
$$1 = \frac{3}{3p+4} + \frac{1}{p'} + \frac{1}{p''}.$$

Here $\frac{3}{3p+4}$ should be $\frac{3}{7}$, $\frac{3}{10}$, $\frac{3}{13}$ or $\frac{3}{16}$. It is easy to see that $p = 2$ together with $\{p', p''\} = \{2, 5\}$ is the unique solution of the above equation.

If $\{m_{\ell_1}, \ldots, m_{\ell_r}\} = \{5, 2\}$, the shape of α is obtained by replacing a part 

of the shape of a basic root with index 0 by 

. Hence α corresponds to one of the following:



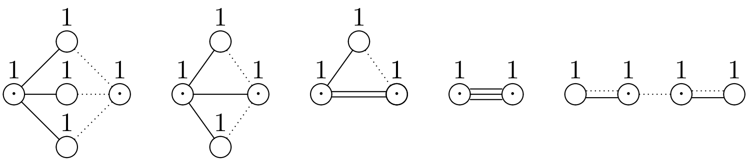
Case 3: $I_1 = \{i, i'\}$, $m_i = m_{i'} = 1$, Ed $(\alpha_i) \le 3$ and Ed $(\alpha_{i'}) \le 3$.
Since

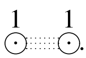
$$\begin{cases} \sum_{\nu\in\bar{I}\setminus\{i\}} m_\nu |\langle\alpha_i, \alpha_\nu\rangle| + \sum_{\nu\in\bar{I}\setminus\{i,\,i'\}} m_\nu |\langle\alpha_{i'}, \alpha_\nu\rangle| = 6, \\ \sum_{\nu\in\bar{I}\setminus\{i\}} m_\nu |\langle\alpha_i, \alpha_\nu\rangle| \ge 3,\ \sum_{\nu\in\bar{I}\setminus\{i'\}} m_\nu |\langle\alpha_{i'}, \alpha_\nu\rangle| \ge 3, \end{cases}$$

one of Case 3.1, …, Case 3.4 is valid.

Case 3.1: $\langle\alpha_k, \alpha_i\rangle = \langle\alpha_k, \alpha_{i'}\rangle = 0$ if $m_k > 1$.

It is easy to see that the shape of α is one of the following:

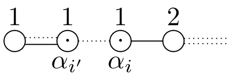
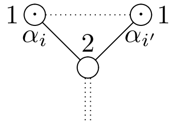


The first four shapes may be expressed by 

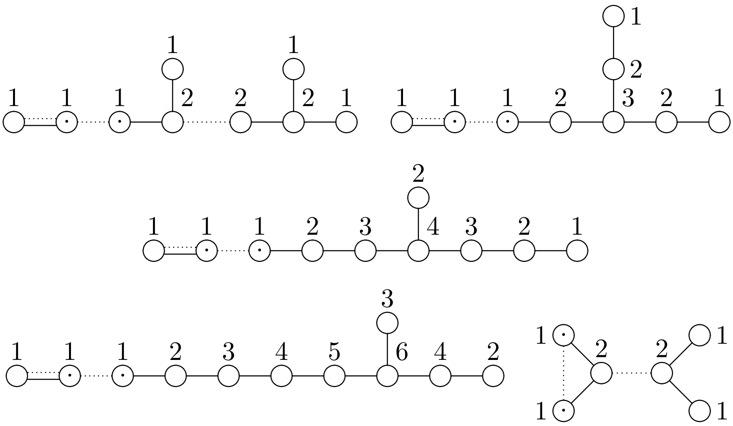
.

Case 3.2: There uniquely exists α_k such that $\langle\alpha_k, \alpha_i\rangle \neq 0$, $\langle\alpha_k, \alpha_{i'}\rangle \neq 0$ and $m_k = 2$.

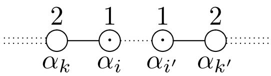
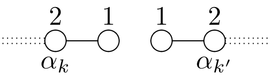
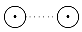
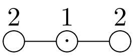
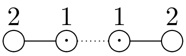
Then α is

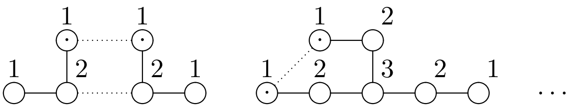
 or  .

Hence α has one of the following shapes:



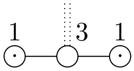
Case 3.3: There are different elements α_k and $\alpha_{k'}$ such that $\langle\alpha_k, \alpha_i\rangle \neq 0$ and $\langle\alpha_{k'}, \alpha_{i'}\rangle \neq 0$ and $m_k = m_{k'} = 2$.

Then the shape of α is  and therefore the shape  obtained by replacing  by  are one or two of the shapes of basic roots of index 0. Hence the list of the shapes of α is obtained by replacing  by  in the shapes classified in Case 2.2. For example we have



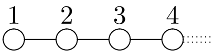
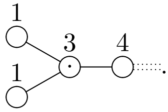
Consequently the shapes in Case 2.2 may be regarded as special cases of those in Case 3.3 except for the first five shapes listed there.

Case 3.4: There uniquely exists a pair α_k and $\alpha_{k'}$ such that $\langle\alpha_k, \alpha_i\rangle \neq 0$, $\langle\alpha_{k'}, \alpha_{i'}\rangle \neq 0$ with $m_k = m_{k'} = 3$.

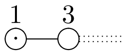
Suppose $k = k'$. Then the shape of α is . Then as in Case 1.3 we have

$$\left\{j \in \bar{I} \mid \langle \alpha_k, \alpha_j \rangle < 0\right\} = \left\{i, i', l_1, \ldots, l_r\right\},$$
$$m_{l_1} + \cdots + m_{l_r} = 2m_k - m_i - m_{i'} = 4.$$

Since $m_{l_\nu} \geq 2$, we have $\{m_{l_1}, \ldots, m_{l_r}\} = \{2, 2\}$ or $\{4\}$. If $\{m_{l_1}, \ldots, m_{l_r}\} = \{2, 2\}$, it corresponds to the first shape in the list below .

If $\{m_{l_1}, \ldots, m_{l_r}\} = \{4\}$, the shape of α is obtained by replacing a part  of the shape of a basic root with index 0 by .

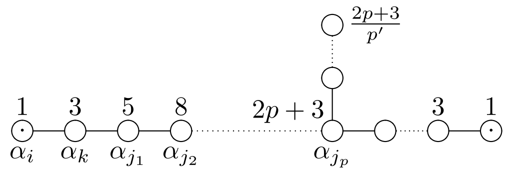
It corresponds to the second and the third shape in the below.

Suppose $k \neq k'$. Then the shape of α contains  twice and we have

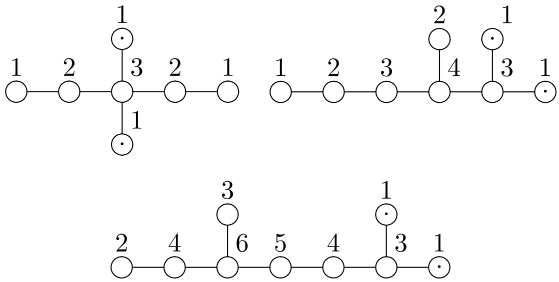
$$\left\{j \in \bar{I} \mid \langle \alpha_k, \alpha_j \rangle < 0\right\} = \{i, l_1, \ldots, l_r\},$$
$$m_{l_1} + \cdots + m_{l_r} = 2m_k - m_i = 5$$

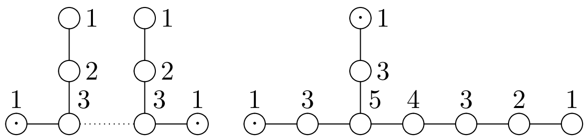
and $m_{l_\nu} \geq 2$ for $\nu = 1, \ldots, r$. Hence $\{m_{l_1}, \ldots, m_{l_r}\} = \{3, 2\}$ or $\{5\}$. If $\{m_{l_1}, \ldots, m_{l_r}\}$ $= \{3, 2\}$, Lemma 1(iv) assures that the shape of α is the forth shape in the below.

Suppose $\{m_{l_1}, \ldots, m_{l_r}\} = \{5\}$. The shape of α is



with $p \geq 1$ and $j_1 = l_1$. Then $2(2p + 3) = 2 + 2 + (2p + 3)\frac{p'-1}{p'}$, which means $1 = \frac{4}{2p+3} + \frac{1}{p'}$ and we have $(p, p') = (1, 5)$. The shape of α is the last one in below. Thus we see the following list:





3 Spectral Types of Differential Equations

In this section, we consider linear differential equations on the Riemann sphere whose singular points are regular singular or unramified irregular singular points. For these differential equations, we define spectral types as tuples of integers representing multiplicities of characteristic exponents of local formal solutions where we ignore integer differences of characteristic exponents. We shall classify orbits of spectral types under algebraic transformations on differential equations, called the Euler transform and the addition and show the finiteness of orbits with a fixed index of rigidity, where we note that the index does not change under the transformations.

First we explain that spectral types can be seen as elements of a certain $\mathbb{Z}$-lattice L which has a group action defined by these transformations. Moreover we shall see that there exists a Kac-Moody root lattice Q_L and the lattice L can be seen as a quotient lattice of Q_L. Then the group action on L coincides with the Weyl group action on Q_L and an analogy of the root system for L shall be defined. As in the previous section, we study the classification of basic roots of L, in particular we show the finiteness of basic roots with a fixed index and give lists of basic roots with index 0 and -2.

3.1 Differential Equations and Spectral Types

The detail of this section can be found in [7]. Let K be an algebraically closed field of characteristic zero. Let $W[x] = K[x][\partial]$ be the ring of differential operators with polynomial coefficients and $W(x) = K(x)[\partial]$ the ring of differential operators with coefficients in $K(x)$. Moreover $W((x))$ denotes the ring of differential operators with coefficients in $K((x))$, the quotient field of the ring of formal power series $K[[x]]$.

3.1.1 Local Structures

In this section we review the local structure of elements in $W(x)$. We fix an element in $W(x)$, $P = \sum_{i=0}^{n} a_i(x)\partial^i$ $(a_n(x) \neq 0)$. Here the non-negative integer n is called the *rank* and written by rank P. For $c \in K$ and a monomial $(x-c)^a\partial^b$, we introduce the *weight*

$$\mathrm{wt}_c\big((x-c)^a\partial^b\big) := a - b.$$

The weight of $P \in W(x) \subset W((x-c))$ is defined by

$$\mathrm{wt}_c(P) := \min\Bigl\{\mathrm{wt}_c\bigl((x-c)^i\partial^j\bigr) \mid P = \sum_{i,j} a_{i,j}(x-c)^i\partial^j,\ a_{i,j} \neq 0\Bigr\}.$$

For $f(x) \in K((x-c))$, weight $\mathrm{wt}_c(f(x))$ is defined by regarding $f(x)$ as an element in $W((x-c))$.

For an integer k, the k-homogeneous part of $P \in W((x-c))$ is

$$P_{(k)} := \sum_{i-j=k} a_{i,j}(x-c)^i\partial^j$$

if $P = \sum_{i,j} a_{i,j}(x-c)^i\partial^j$ with $a_{i,j} \in K$.

Similarly we can define wt_∞ by

$$\mathrm{wt}_\infty\bigl(x^a\partial^b\bigr) = b - a.$$

The *singular points* of P are poles of $\frac{a_i(x)}{a_n(x)}$ $(i = 1, \ldots, n)$. We also say that ∞ is a singular point of P if

$$P^{(\infty)} := \sum_{i=0}^{n} a_i\left(\frac{1}{x}\right)\bigl(-x^2\partial\bigr)^i$$

has a singular point at 0. Suppose that c $(\neq \infty)$ is a singular point of P. The $\mathrm{wt}_c(P)$-homogeneous part of P equals

$$\sum_{i-j=\mathrm{wt}_c(P)} a_{i,j}(x-c)^i\partial^j$$

and then the *characteristic polynomial* of P at c is defined by

$$\mathrm{C}_c(P)(t) := \sum_{i-j=\mathrm{wt}_c(P)} a_{i,j}t(t-1)\cdots(t-j+1).$$

If $\deg_{K[t]} \mathrm{C}_c(P)(t) = \operatorname{rank} P$, we say that c is a *regular singular point* of P. Otherwise, c is an *irregular singular point* of P. For the point ∞, we can define characteristic polynomials, regular and irregular singular points as well as the above replacing $x - c$ by $\frac{1}{x}$.

Suppose that c is an irregular singular point of P. For simplicity of notation, we put $c = 0$. There exists an algebraic extension $K((x^{\frac{1}{q}}))$ of $K((x))$ for a positive integer q and we denote the ring of differential operators with coefficients in $K((x^{\frac{1}{q}}))$ by $W_q((x))$. Then we can decompose the left-$W_q((x))$-module $W_q((x))/W_q((x))P$ as follows.

Definition 1 (Local decomposition (see [8] for example)) For $P \in W(x)$ with an irregular singular point c, there exists the algebraic extension $K(((x-c)^{\frac{1}{q}}))$ of

$K((x-c))$, distinct polynomials w_i of $(x-c)^{-\frac{1}{q}}$ with no constant terms and $P_i(t) \in K(((x-c)^{\frac{1}{q}}))[t]$ for $1 \le i \le r$ such that we have the following.

(i) Each $P_i(\vartheta_c)$ has a regular singular point at c.

(ii) We can write P as the least left common multiple of

$$\big\{P_1(\vartheta_c - w_1), \ldots, P_r(\vartheta_c - w_r)\big\}.$$

Namely there exist $R_i \in W_q((x-c))$ such that

$$P = R_i P_i(\vartheta_c - w_i) \quad \text{for } i = 1, \ldots, r.$$

Here $\vartheta_c = (x-c)\partial$ and for $Q(t) = \sum_{\nu \ge 0} q_\nu(x) t^\nu \in K(((x-c)^{\frac{1}{q}}))[t]$ and $w \in K(((x-c)^{\frac{1}{q}}))$, we put

$$Q(\vartheta_c - w) = \sum_{\nu \ge 0} q_\nu(x)(\vartheta_c - w)^\nu.$$

(iii) We have the decomposition

$$W_q\big((x-c)\big)/W_q\big((x-c)\big)P \simeq \bigoplus_{i=1}^{r} W_q\big((x-c)\big)/W_q\big((x-c)\big)P_i(\vartheta_c - w_i)$$

as $W_q((x-c))$-modules.

We call the decomposition in (iii) the *local decomposition* of P at c. Moreover we call $P_i(\vartheta_c - w_i) \in W_q((x-c))$ *local factors* and w_i the *exponential factors* of $P_i(\vartheta_c - w_i)$ for $1 \le i \le r$.

If the local decomposition at c is obtained in $W_1((x-c)) = W((x-c))$, we say that c is an *unramified* irregular singular point. Otherwise, c is called a *ramified* irregular singular point.

We introduce the notion of spectral data. Let $P \in W((x))$. We regard the left $W((x))$-module $M_P = W((x))/W((x))P$ as the $K((x))$-vector space of $\dim M_P = \operatorname{rank} P$. For a basis $\{u_1, \ldots, u_n\}$ of M_P as $K((x))$-vector space, we can represent the action of $\vartheta = x\partial$ by the matrix as follows. For $u \in M_P$, there exists $A = (a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}} \in M(n, K((x)))$ such that

$$\vartheta u_i = \sum_{j=1}^{n} a_{ij} u_j.$$

Moreover if 0 is a regular singular point of P, there exists a basis such that we can take $A \in M(n, K)$. We call this matrix $A \in M(n, K)$ a *local matrix* of P at 0. For any other regular singular point $c \in K$ and ∞, we can define a local matrix in the same way.

Definition 2 (Spectral data) Fix $m_1, \ldots, m_s \in \mathbb{Z}_{>0}$ and $\lambda_1, \ldots, \lambda_s \in K$ which satisfy

$$\lambda_i - \lambda_j \notin \mathbb{Z} \quad (i \neq j).$$

We say $P \in W(x)$ has the *spectral data*

$$\big\{(\lambda_1, \ldots, \lambda_s); (m_1, \ldots, m_s)\big\}$$

at c if P has a regular singular point at c and satisfies the following.

(i) The characteristic polynomial is

$$\mathrm{C}_c(P)(t) = C \prod_{i=1}^{s} \prod_{j=0}^{m_i-1} \big(t - (\lambda_i + j)\big)$$

for a constant C.

(ii) A local matrix of P is a semisimple matrix.

Here we note that condition (ii) does not depend on the choice of local matrices.

3.1.2 Spectral Types and the Euler Transform

Fix $P \in W(x)$ satisfying the assumption below.

Assumption 1 *We assume that $P \in W(x)$ satisfies the following.*

(i) *All singular points of P, written by $c_0 = \infty, c_1, \ldots, c_p \in K$, are regular singular or unramified irregular singular points.*

(ii) *Denote the set of local factors of P at c_i by*

$$\big\{P_{i,1}(\vartheta_{c_i} - w_{i,1}), \ldots, P_{i,k_i}(\vartheta_{c_i} - w_{i,k_i})\big\}.$$

Then there exist positive integers $m_{i,j,s}$ and $\lambda_{i,j,s} \in K$ for $i = 0, \ldots, p$, $j = 1, \ldots, k_i$, $s = 1, \ldots, l_{i,j}$ such that $\lambda_{i,j,s} - \lambda_{i,j,s'} \notin \mathbb{Z}$ if $s \neq s'$ and $P_{i,j}(\vartheta)$ have spectral data

$$\big\{(\lambda_{i,j,1}, \ldots, \lambda_{i,j,l_{i,j}}); (m_{i,j,1}, \ldots, m_{i,j,l_{i,j}})\big\},$$

respectively. Here $w_{i,j}$ are the exponential factors of the corresponding local factors.

Put

$$\lambda(P) = \big((\lambda_{i,j,1}, \ldots, \lambda_{i,j,l_{i,j}})\big)_{\substack{0 \le i \le p,\\ 1 \le j \le k_i}},$$

$$\mathbf{m}(P) = \big((m_{i,j,1}, \ldots, m_{i,j,l_{i,j}})\big)_{\substack{0 \le i \le p\\ 1 \le j \le k_i}}.$$

The *index of rigidity* is defined by

$$\begin{aligned}\operatorname{idx} P := &-\sum_{i=0}^{p}\sum_{1\le j\ne j'\le k_i} d_i\bigl(j,j'\bigr)\Biggl(\sum_{s=1}^{l_{i,j}} m_{i,j,s}\Biggr)\Biggl(\sum_{s'=1}^{l_{i,j'}} m_{i,j',s'}\Biggr)\\ &+\sum_{i=0}^{p}\sum_{j=1}^{k_i}\sum_{s=1}^{l_{i,j}} m_{i,j,s}^2-(p-1)(\operatorname{rank} P)^2 \end{aligned}\tag{25}$$

where $d_i(j,j')=-\mathrm{wt}_{c_i}(w_{i,j}-w_{i,j'})$ for $i=0,\dots,p$ and $j,\ j'=1,\dots,k_i$. Here we notice that these $d_i(j,j')$ satisfy

$$\begin{aligned} &d_i\bigl(j,j'\bigr)=0 \quad \text{if and only if} \quad j=j'\\ &d_i\bigl(j,j'\bigr)=d_i\bigl(j,j'\bigr),\\ &d_i(j_1,j_2)\le \max\bigl\{d_i(j_1,j_3),\,d_i(j_2,j_3)\bigr\}\end{aligned}\tag{26}$$

for all $i=0,\dots,p$ and $j,\ j',\ j_1,\ j_2,\ j_3\in\{1,\dots,k_i\}$.

Remark 2 The index of rigidity is defined by N. Katz in [4] and can be computed from local structures of differential equations (see Proposition 3.1 in [1] for example). One can check that our definition of the index of rigidity coincides with the original one.

Remark 3 Suppose $P\in W(x)$ satisfies Assumption 1 and put $Z_i:=\bigoplus_{j=1}^{k_i}\mathbb{Z}^{l_{i,j}}$. If $p>0$ and there exists $i_0\in\{0,\dots,p\}$ such that $k_{i_0}=1$ and $l_{i_0,1}=1$, then c_{i_0} is not a singular point of $\mathrm{Ad}\,(e^{-w_{i_0,1}})\mathrm{Ad}\,((x-c_{i_0})^{-\lambda_{i_0,1,1}})P$. Here the operator $\mathrm{Ad}\,(f(x))$ is defined in Definition 4. Hence in this case, we identify $\mathbf{m}(P)$ and $\mathrm{pr}_{\{0,\dots,p\}\setminus\{i_0\}}\mathbf{m}(P)$. Here $\mathrm{pr}_{\{0,\dots,p\}\setminus\{i_0\}}\colon \bigoplus_{i=0}^{p} Z_i\to\bigoplus_{i\in\{0,\dots,p\}\setminus\{i_0\}} Z_i$ is the natural projection.

Thus for $\mathbf{m}(P)$, we assume $k_i\cdot l_{i,k_i}>1$ for all $i=0,\dots,p$ if $p>0$.

Definition 3 (Spectral type) Choose arbitrary integers $p\in\mathbb{Z}_{\ge 0}$, $k_i\in\mathbb{Z}_{>0}$ ($i=0,\dots,p$) and $l_{i,j}\in\mathbb{Z}_{>0}$ ($i=0,\dots,p,\ j=1,\dots,k_i$). Fix integers $d_i(j,j')\in\mathbb{Z}_{\ge 0}$ satisfying the relation (26) and take a tuple of positive integers

$$\mathbf{m}=\bigl((m_{i,j,1},\dots,m_{i,j,l_{i,j}})\bigr)_{\substack{0\le i\le p\\ 1\le j\le k_i}}\in\bigoplus_{i=0}^{p}\bigoplus_{j=1}^{k_i}\mathbb{Z}_{\ge 0}^{l_{i,j}}.$$

Then we call $\mathbf{m}$ with the integers $(d_i(j,j'))_{\substack{0\le i\le p\\ 1\le j,j'\le k_i}}$ a *spectral type*.

The spectral type of $P\in W(x)$ satisfying Assumption 1 is defined by $\mathbf{m}=\mathbf{m}(P)$ and $d_i(j,j')=-\mathrm{wt}_{c_i}(w_{i,j}-w_{i,j'})$. A spectral type is called *irreducible* if there exists an irreducible operator $P\in W(x)$ with the spectral type which satisfies Assumption 1.

In the remaining of this paper, we investigate orbits of spectral types under the action of the twisted Euler transform which is defined below. The following is one of our main theorem which tells us that the finiteness of Euler transform orbits of spectral types with a fixed index of rigidity.

Theorem 4 *Fix an integer* $r \geq 0$*. If* $r > 0$*, there exist only a finite number of orbits of irreducible spectral types with index of rigidity* $-r$ *under the action of twisted Euler transforms.*

Moreover there exist a finite number of orbits of indivisible irreducible spectral types with index of rigidity 0 *under the action of twisted Euler transforms.*

Here we say that a spectral type $\mathbf{m} = ((m_{i,j,1}, \ldots, m_{i,j,l_{i,j}}))_{\substack{0\leq i\leq p\\ 1\leq j\leq k_i}}$ *with integers is indivisible if the greatest common divisor of* $\{m_{i,j,s} \mid i = 0, \ldots, p,\ \ j = 1, \ldots, k_i,\ \ s = 1, \ldots, l_{i,j}\}$ *is* 1.

This theorem follows from Theorem 7 and Theorem 8 which appear in the latter sections.

We give a brief review of algebraic transformations on $W[x]$ and $W(x)$.

Definition 4 (Addition) For $f(x) \in K(x)$, define

$$\begin{array}{rcl} \mathrm{Ad}\,(e^{\int f(x)\,dx})\colon W(x) & \longrightarrow & W(x) \\ x & \longmapsto & x \\ \partial & \longmapsto & \partial - f(x) \end{array}.$$

In particular,

$$\begin{array}{rcl} \mathrm{Ad}\,((x-c)^{\lambda})\colon W(x) & \longrightarrow & W(x) \\ x & \longmapsto & x \\ \partial & \longmapsto & \partial - \frac{\lambda}{x-c} \end{array}$$

for $c,\ \lambda \in K$ is called the *addition* at c with the parameter λ.

Definition 5 (Fourier-Laplace transform) The *Fourier-Laplace transform* is the K-algebra automorphism of $W[x]$,

$$\begin{array}{rcl} \mathcal{L}\colon W[x] & \longrightarrow & W[x] \\ x & \longmapsto & -\partial \\ \partial & \longmapsto & x \end{array}.$$

Definition 6 (Primitive component) We say that $P = \sum_{i=0}^{n} a_i(x)\partial^i \in W[x]$ is *primitive* if

(i) $\gcd_{K[x]}\{a_i(x) \mid i = 0, \ldots, n\} = 1$,
(ii) the highest term $a_n(x)$ is monic.

For $P \in W(x)$, there exist $f(x) \in K(x)$ and the primitive element $\tilde{P} \in W[x]$, and then we can decompose P by

$$P = f(x)\tilde{P},$$

uniquely.

We denote the primitive element by $\mathrm{Prim}(P)$ and call this the *primitive component* of P.

Definition 7 (Euler transform) The *Euler transform* of $P \in W(x)$ with the parameter λ is

$$E(\lambda)P := \mathcal{L} \circ \mathrm{Prim} \circ \mathrm{Ad}\left(x^{\lambda}\right) \circ \mathcal{L}^{-1} \circ \mathrm{Prim}(P) \in W[x].$$

For $P \in W(x)$ satisfying Assumption 1, we consider following special Euler transforms.

Definition 8 (Twisted Euler transform) Let $P \in W(x)$ satisfying Assumption 1. Define $\mathcal{J} := \bigoplus_{i=0}^{p}\{1, \ldots, k_i\}$. Then for $\hat{j} = (j_0, \ldots, j_p) \in \mathcal{J}$, the *twisted Euler transform* $E(\hat{j})P$ is

$$E(\hat{j})P := \prod_{i=0}^{p} \mathrm{Ad}\left(e^{w_{i,j_i}}\right) \prod_{i=1}^{p} \mathrm{Ad}\left((x-c_i)^{\lambda_{i,j_i,1}}\right)$$
$$\circ\, E\big(1-\lambda(P;\hat{j})\big) \prod_{i=1}^{p} \mathrm{Ad}\left((x-c_i)^{-\lambda_{i,j_i,1}}\right) \prod_{i=0}^{p} \mathrm{Ad}\left(e^{-w_{i,j_i}}\right) P$$

where

$$\lambda(P;\hat{j}) := \sum_{i=0}^{p} \lambda_{i,j_i,1}.$$

The following theorem gives explicit changes of spectral types induced by the twisted Euler transform.

Theorem 5 (Theorem 3.2 in [7]) *Let $P \in W(x)$ satisfying Assumption 1. Choose $\hat{j} = (j_0, \ldots, j_p) \in \mathcal{J}$ and suppose $\lambda(P)$ is generic (see [7, Theorem 2.18]).*

Then $E(\hat{j})P \in W(x)$ also satisfies conditions in Assumption 1. If the spectral type of $P_{\hat{j}} = E(\hat{j})P$ is $\mathbf{m}(P_{\hat{j}}) = ((\tilde{m}_{i,j,1}, \ldots, \tilde{m}_{i,j,l_{i,j}}))_{\substack{0 \le i \le p \\ 1 \le j \le k_i}}$ with $(\tilde{d}_i(j,j'))_{\substack{0 \le i \le p \\ 1 \le j,\, j' \le k_i}}$, then we have

$$\begin{aligned}
\tilde{m}_{i,j,1} &= m_{i,j,1} + d(\hat{j}) && \text{if } j = j_i, \\
\tilde{m}_{i,j,s} &= m_{i,j,s} && \text{otherwise}, \\
\tilde{d}_i\big(j,j'\big) &= d_i\big(j,j'\big)
\end{aligned}$$

where

$$d(\hat{j}) = \sum_{i=1}^{p}\sum_{j=1}^{k_i}\bigl(-\mathrm{wt}_{c_i}(w_{i,j} - w_{i,j_i}) + 1\bigr)\sum_{s=1}^{l_{i,j}} m_{i,j,s} + \sum_{j=1}^{k_0}\bigl(-\mathrm{wt}_{c_0}(w_{0,j} - w_{0,j_0}) - 1\bigr)\sum_{s=1}^{l_{0,j}} m_{0,j,s} - \sum_{i=0}^{p} m_{i,j_i,1}.$$

3.2 The Lattice of Spectral Types and the Root System

Theorem 5 shows that twisted Euler transforms $E(\hat{j})$ $(\hat{j} \in \mathcal{J})$ induce transformations of the spectral type $\mathbf{m}(P)$ of $P \in W(x)$ satisfying Assumption 1. From these transformations we shall construct a transformation group on a certain lattice where $\mathbf{m}(P)$ can be seen as an element in this lattice. Moreover we shall see this lattice with the transformation group is a quotient lattice of a Kac-Moody root lattice.

3.2.1 The Lattice of Spectral Types

Choose arbitrary integers $p \in \mathbb{Z}_{\geq 0}$, $k_i \in \mathbb{Z}_{>0}$ $(i = 0, \dots, p)$ and $l_{i,j} \in \mathbb{Z}_{>0}$ $(i = 0, \dots, p,\ j = 1, \dots, k_i)$. Fix integers $d_i(j, j') \in \mathbb{Z}_{\geq 0}$ satisfying the relation (26).

Then we consider the following $\mathbb{Z}$-lattice

$$L := \left\{ \bigl((m_{i,j,1}, \dots, m_{i,j,l_{i,j}})\bigr)_{\substack{0\leq i\leq p\\ 1\leq j\leq k_i}} \in \bigoplus_{i=0}^{p}\bigoplus_{j=1}^{k_i}\mathbb{Z}^{l_{i,j}} \;\middle|\; \sum_{j=1}^{k_0}\sum_{s=1}^{l_{0,j}} m_{0,j,s} = \cdots = \sum_{j=1}^{k_p}\sum_{s=1}^{l_{p,j}} m_{p,j,s} \right\}.$$

We denote the set of positive elements in L by

$$L^{+} := L \cap \bigoplus_{i=0}^{p}\bigoplus_{j=1}^{k_i}\mathbb{Z}_{\geq 0}^{l_{i,j}}$$

and define the *rank* of $\mathbf{m} = ((m_{i,j,1}, \dots, m_{i,j,l_{i,j}}))_{\substack{0\leq i\leq p\\ 1\leq j\leq k_i}} \in L$ by

$$\operatorname{rank}\mathbf{m} := \sum_{j=1}^{k_i}\sum_{s=1}^{l_{i,j}} m_{i,j,s}$$

for any $i = 0, \ldots, p$. Note that the definition of rank $\mathbf{m}$ is independent of the choice of $i = 0, \ldots, p$.

Then we define transformations on L as an analogy of the transformation of spectral types given in Theorem 5. Namely, for each $\hat{j} = (j_0, j_1, \ldots, j_p) \in \mathcal{J} := \bigoplus_{i=0}^{p} \{1, \ldots, k_i\}$, we define the lattice transformation on L,

$$\begin{array}{rccc} \sigma(\hat{j}): & L & \longrightarrow & L \\ & \mathbf{m} = \big((m_{i,j,1}, \ldots, a_{i,j,l_{i,j}})\big)_{\substack{0 \le i \le p \\ 1 \le j \le k_i}} & \longmapsto & \big((\tilde{m}_{i,j,1}, \ldots, \tilde{m}_{i,j,l_{i,j}})\big)_{\substack{0 \le i \le p \\ 1 \le j \le k_i}}, \end{array}$$

where

$$\begin{aligned} \tilde{m}_{i,j,1} &:= m_{i,j,1} + d(\mathbf{m}; \hat{j}) \quad && \text{if } (i, j) = (i, j_i), \\ \tilde{m}_{i,j,s} &:= m_{i,j,s} && \text{otherwise} \end{aligned}$$

and

$$\begin{aligned} d(\mathbf{m}; \hat{j}) := &\sum_{i'=1}^{p} \sum_{j'=1}^{k_{i'}} \big(d_{i'}\big(j', j_{i'}\big) + 1\big) \sum_{s=1}^{l_{i',j'}} m_{i',j',s} \\ &+ \sum_{j'=1}^{k_0} \big(d_0\big(j', j_0\big) - 1\big) \sum_{s=1}^{l_{0,j'}} m_{0,j',s} - \sum_{i'=0}^{p} m_{i',j_{i'},1}. \end{aligned}$$

In addition, for $i_0 = 0, \ldots, p,\ j_0 = 1, \ldots, k_{i_0},\ s_0 = 1, \ldots, l_{i_0,j_0} - 1$, we also define permutations on L,

$$\begin{array}{llll} \sigma(i_0, j_0, s_0): L(P) & \longrightarrow & L(P) & \\ m_{i_0,j_0,s_0} & \longmapsto & m_{i_0,j_0,s_0+1} & \\ m_{i_0,j_0,s_0+1} & \longmapsto & m_{i_0,j_0,s_0} & \\ m_{i,j,s} & \longmapsto & m_{i,j,s} & (i, j, s) \neq (i_0, j_0, s_0),\ (i_0, j_0, s_0 + 1). \end{array}$$

Then L has the action of the group $\overline{W}$ generated by these $\sigma(\hat{j}), \sigma(i, j, s)$, i.e.,

$$\overline{W} := \big\langle \sigma(\hat{j}), \sigma(i, j, s) \mid \hat{j} \in \mathcal{J},\ i = 0, \ldots, p,\ j = 1, \ldots, k_i,\ s = 1, \ldots, l_{i,j} - 1 \big\rangle.$$

We call L with $\overline{W}$ action the *lattice of spectral types* and denote it by $(L, \overline{W})$ or shortly by L.

3.2.2 The Lattice of Spectral Types as a Quotient Lattice

We shall explain that the lattice of spectral types $(L, \overline{W})$ can be seen as a quotient lattice of the Kac-Moody root lattice Q_L with the index set

$$I := \mathcal{J} \sqcup \{(i, j, s) \mid i = 0, \dots, p,\ j = 1, \dots, k_i,\ s = 1, \dots, l_{i,j} - 1\}$$

and the basis $\{\alpha_t \mid t \in I\}$. Namely, $Q_L := \bigoplus_{t \in I} \mathbb{Z}\alpha_t$. We define the symmetric bilinear form $\langle\ ,\ \rangle$ on Q_L,

$$\langle \alpha_{\hat{j}}, \alpha_{\hat{j}'} \rangle := 2 - \sum_{\substack{0 \le i \le p \\ j_i \ne j_i'}} \bigl(d_i\bigl(j_i, j_i'\bigr) + 1\bigr),$$

$$\langle \alpha_{\hat{j}}, \alpha_{(i,j,s)} \rangle := \begin{cases} -1 & \text{if } j_i = j \text{ and } s = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle \alpha_{(i,j,s)}, \alpha_{(i',j',s')} \rangle := \begin{cases} 2 & \text{if } (i, j, s) = (i', j', s'), \\ -1 & \text{if } (i, j) = (i', j') \text{ and } |s - s'| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\hat{j} = (j_0, \dots, j_p)$, $\hat{j}' = (j_0', \dots, j_p') \in \mathcal{J}$. Let $W_L := \langle \sigma_t \mid t \in I \rangle$ be the Weyl group of Q_L. Then we have the surjection $\Phi\colon Q_L \to L$ by which W_L action on Q_L coincides with the $\overline{W}$ action on L.

Theorem 6 (Theorem 3.3 in [7]) *Define the $\mathbb{Z}$-module homomorphism*

$$\Phi\colon Q_L \longrightarrow L$$

as follows. For

$$\alpha = \sum_{\hat{j} \in \mathcal{J}} m_{\hat{j}} \alpha_{\hat{j}} + \sum_{i=0}^{p} \sum_{j=1}^{k_i} \sum_{s=1}^{l_{i,j}-1} m_{(i,j,s)} \alpha_{(i,j,s)} \in Q_L,$$

the image $\Phi(\alpha) = ((\bar{m}_{i,j,1}, \dots, \bar{m}_{i,j,l_{i,j}}))_{\substack{0 \le i \le p \\ 1 \le j \le k_i}}$ is given by

$$\bar{m}_{i,j,1} = \sum_{\{\hat{j} \in \mathcal{J} \mid j_i = j\}} m_{\hat{j}} - m_{(i,j,1)},$$

$$\bar{m}_{i,j,s} = m_{(i,j,s-1)} - m_{(i,j,s)} \quad \textit{for } 2 \le s \le l_{i,j}.$$

Here we put $m_{(i,j,l_{i,j})} = 0$. Then we have the following.

(i) *The map Φ is surjective.*
(ii) *Φ is injective if and only if $\#\{i \in \{0, 1, \dots, p\} \mid k_i > 1\} \le 1$.*

(iii) *The Weyl group action on Q_L corresponds to the action of $\overline{W}$ on L. Namely, we have*

$$\Phi(\sigma_{\hat{j}}\alpha) = \sigma(\hat{j})\Phi(\alpha) \qquad (\alpha \in Q_L),$$

$$\Phi(\sigma_{(i,j,s)}\alpha) = \sigma(i,j,s)\Phi(\alpha) \quad (\alpha \in Q_L).$$

(iv) *If $\alpha \in \operatorname{Ker}\Phi$, then $\langle \alpha, \beta \rangle = 0$ for any $\beta \in Q_L$.*

(v) *Let $\mathbf{m} \in L$. Then we have*

$$\langle \alpha, \alpha_{\hat{j}} \rangle = -d(\mathbf{m}; \hat{j}) \quad \big(\alpha \in \Phi^{-1}(\mathbf{m}),\ \hat{j} \in \mathcal{J}\big)$$

(vi) *For $\alpha \in \Phi^{-1}(\mathbf{m})$, we have*

$$\langle \alpha, \alpha \rangle = -\sum_{i=0}^{p} \sum_{1 \le j \ne j' \le k_i} d_i\big(j, j'\big) \left(\sum_{s=1}^{l_{i,j}} m_{i,j,s}\right) \left(\sum_{s'=1}^{l_{i,j'}} m_{i,j',s'}\right) + \sum_{i=0}^{p} \sum_{j=1}^{k_i} \sum_{s=1}^{l_{i,j}} m_{i,j,s}^2 - (p-1)(\operatorname{rank}\mathbf{m})^2.$$

Form (vi) in this theorem, we define the *index of rigidity* of $\mathbf{m} \in L$ by

$$\operatorname{idx}\mathbf{m} := \operatorname{idx}\alpha = \langle \alpha, \alpha \rangle$$

for $\alpha \in \Phi^{-1}(\mathbf{m})$.

3.2.3 Φ-Root System

We shall define the Φ-root system of $(L, \overline{W})$ which is an analogue of the root system of Q_L.

First consider the following subset of L,

$$\Delta_{\mathrm{re}}^{\Phi} := \bigcup_{\hat{j} \in \mathcal{J}} \overline{W}\Phi(\alpha_{\hat{j}}),$$

i.e., the union of $\overline{W}$-orbits of $\Phi(\alpha_{\hat{j}})$, which is called the set of *Φ-real roots*. We also consider the subset

$$F^{\Phi} := \left\{ \mathbf{m} \in L^+ \setminus \{0\} \;\middle|\; \begin{array}{c} m_{i,j,1} \ge m_{i,j,2} \ge \cdots \ge m_{i,j,l_{i,j}},\ d(\mathbf{m}; \hat{j}) \ge 0 \\ \text{for all } i=0,\dots,p,\ j=1,\dots,k_i,\ \hat{j} \in \mathcal{J}, \\ \overline{W}\mathbf{m} \subset L^+ \end{array} \right\}.$$

Then the set of *Φ-imaginary roots* is

$$\Delta_{\mathrm{im}}^{\Phi} := \overline{W}F^{\Phi} \cup -\overline{W}F^{\Phi}.$$

We call

$$\Delta^{\Phi} := \Delta^{\Phi}_{\mathrm{re}} \cup \Delta^{\Phi}_{\mathrm{im}}$$

the set of *Φ-roots*.

3.2.4 Spectral Types of Differential Equations and Root Systems

We explain that for $P \in W(x)$ satisfying Assumption 1, the spectral type of P can be seen as an element in the lattice of spectral types $(L, \overline{W})$.

Suppose $P \in W(x)$ satisfies Assumption 1. If we put

$$d_i\big(j, j'\big) = -\mathrm{wt}_{c_i}(w_{i,j} - w_{i,j'})$$

for $i = 0, \ldots, p$ and $j,\ j' = 1, \ldots, k_i$, then $d_i(j, j')$ satisfy the relations (26).

Thus we can define the lattice of spectral types $(L, \overline{W})$ and see $\mathbf{m}(P) \in L$. Then the index of rigidity of P equals that of $\mathbf{m}(P) \in L$, namely, $\mathrm{idx}\, P = \mathrm{idx}\, \mathbf{m}(P)$. Also $\mathrm{rank}\, P = \mathrm{rank}\, \mathbf{m}(P)$ as well.

Theorem 5 shows that the spectral types of $P_{\hat{j}} = E(\hat{j})P$ $(\hat{j} \in \mathcal{J})$ are obtained by the transformation $\sigma(\hat{j})$ on L, i.e.,

$$\mathbf{m}(P_{\hat{j}}) = \sigma(\hat{j})\mathbf{m}(P).$$

Hence we can associate an element in Δ^{Φ} to P as follows.

Theorem 7 (Theorem 3.11 in [7]) *Suppose $\lambda(P)$ is generic* (*see* [7, *Definition* 3.8]). *If P is irreducible in $W(x)$, then we have the following.*

(i) $\mathbf{m}(P) \in \Delta^{\Phi}$.
(ii) *If* $\mathrm{idx}\, \mathbf{m}(P) > 0$, *then* $\mathrm{idx}\, \mathbf{m}(P) = 2$.
(iii) *We have*

$$\mathbf{m}(P) \in \begin{cases} \Delta^{\Phi}_{re} & \text{if } \mathrm{idx}\, \mathbf{m}(P) = 2, \\ \Delta^{\Phi}_{im} & \text{if } \mathrm{idx}\, \mathbf{m}(P) \leq 0. \end{cases}$$

3.3 A Classification of Basic Pairs

At the end of Sect. 3.2, we see that the spectral type of the irreducible operator $P \in W(x)$ satisfying Assumption 1 corresponds to an element in $\Delta^{\Phi+} = \Delta^{\Phi} \cap L^{+}$. By the definition of Δ^{Φ}, any element in $\Delta^{\Phi+}$ can be reduced to an element in $\{\Phi(\alpha_{\hat{j}}) \mid \hat{j} \in \mathcal{J}\} \sqcup F^{\Phi}$ by $\overline{W}$ action. This means that $\mathbf{m}(P)$ can be reduced to an element in $\{\Phi(\alpha_{\hat{j}}) \mid \hat{j} \in \mathcal{J}\} \sqcup F^{\Phi}$ by the Euler transform.

Thus to see Euler transform orbits of spectral types, it suffices to see elements in $F^{\Phi} \sqcup \{\Phi(\alpha_{\hat{j}}) \mid \hat{j} \in \mathcal{J}\}$.

The differential operator corresponding to an element in $\{\Phi(\alpha_{\hat{j}}) \mid \hat{j} \in \mathcal{J}\}$ is an obvious operator of the first order. Hence we study F^{Φ}.

Definition 9 (Basic pair) Let $(L, \overline{W})$ be a lattice of spectral types with $\overline{W}$-action. Denote the corresponding Kac-Moody root lattice by Q_L and the surjection by $\Phi\colon Q_L \to L$ defined as in Sect. 3.2.2. We also define the subset $F^{\Phi} \subset L$ as in Sect. 3.2.3.

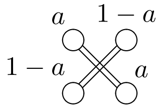
Choose an element $\mathbf{m} = ((m_{i,j,1}, \ldots, m_{i,j,l_{i,j}}))_{\substack{0\le i\le p\\ 1\le j\le k_i}} \in F^{\Phi}$ and suppose $m_{i,j,s} \neq 0$ for all $i = 0, \ldots, p,\ j = 1, \ldots, k_i$ and $s = 1, \ldots, l_{i,j}$.

Then we call $(\mathbf{m}, L, \overline{W})$ the *basic triple*. We usually omit $\overline{W}$ and call $(\mathbf{m}, L)$ the *basic pair*.

We define the shape of a basic pair $(\mathbf{m}, L)$.

Definition 10 (Shape of a basic pair) Let $(\mathbf{m}, L)$ be a basic pair. The *shape* of $(\mathbf{m}, L)$ is the set of shapes of elements in $\Phi^{-1}(\mathbf{m}) \subset Q_L$.

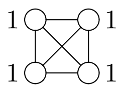
Example 1 For example, suppose $p = 1$, $k_0 = k_1 = 2$, $l_{i,j} = 1$ ($i = 0, 1$ and $j = 1, 2$), $d_0(1,2) = d_1(1,2) = 1$. Consider $\mathbf{m} = ((m_{i,j,1}))_{\substack{0\le i\le 1\\ 1\le j\le 2}}$ such that $m_{i,j,1} = 1$ for all $i,\ j$. Then $(\mathbf{m}, L)$ is a basic pair and its shape is



$$(a \in \mathbb{Z}), \tag{27}$$

where we simply denote $\{x_a \mid a \in \mathbb{Z}\}$ by x_a $(a \in \mathbb{Z})$.

Suppose $p = 0$, $k_0 = 4$, $d_0(i, j) = 2$ for $1 \le i < j \le 4$ and $l_{0,\nu} = 2$ for $1 \le \nu \le 4$. If $m_{0,j,1} = 1$ for $1 \le j \le 4$, the shape of $(\mathbf{m}, L)$ equals



(28)

Now we prepare the following lemma to have an element in $\Phi^{-1}(\mathbf{m}) \cap Q_L^{+}$.

Lemma 3 *Let* $(m_{i,j})_{\substack{0\le i\le p\\ 1\le j\le k_i}}$ *be a tuple of* $p + 1$ *partitions of a positive integer* n, *namely,* $n,\ p,\ m_{i,j}$ *and* k_i *are positive integers satisfying*

$$m_{i,1} + \cdots + m_{i,k_i} = n \quad (j = 0, \ldots, p).$$

Then there exist non-negative integers $\tilde{m}_{\nu_0,\dots,\nu_p}$ *for* $1 \le \nu_i \le k$ *and* $0 \le i \le p$ *such that*

$$\sum_{\substack{0\le j\le p\\ j\ne i}} \sum_{\nu_j=1}^{k_j} \tilde{m}_{\nu_0,\nu_1,\dots,\nu_p} = m_{i,\nu_i} \quad (0 \le i \le p,\ 1 \le \nu_i \le k_i),$$

$$\tilde{m}_{j_0,\dots,j_p} \cdot \tilde{m}_{j'_0,\dots,j'_p} \ne 0 \Rightarrow \begin{cases} j_\nu \le j'_\nu & (0 \le \nu \le p) \\ or & \\ j_\nu \ge j'_\nu & (0 \le \nu \le p), \end{cases}$$

$$\tilde{m}_{1,\dots,1} \cdot \tilde{m}_{k_0,\dots,k_p} \ne 0.$$

Proof Put

$$\tilde{m}_{\nu_0,\dots,\nu_p} = \#\big\{k \in \{1,2,\dots,n\} \mid m_{j,1} + \cdots + m_{j,\nu_j-1} < k \le m_{j,1} + \cdots + m_{j,\nu_j} \text{ for } j = 0,\dots,p\big\}.$$

Then the lemma is clear. Here we note that $\tilde{m}_{1,\dots,1} = \min\{m_{0,1},\dots,m_{p,1}\}$ and $\tilde{m}_{k_0,\dots,k_p} = \min\{m_{0,k_0},\dots,m_{p,k_p}\}$. □

Definition 11 Fix $\mathbf{m} = ((m_{i,j,1},\dots,m_{i,j,l_{i,j}}))_{\substack{0\le i\le p\\ 1\le j\le k_i}} \in L^+$. Put $n = \operatorname{rank}\mathbf{m}$ and $m_{i,j} = \sum_{j=1}^{l_{i,j}} m_{i,j,s}$. Applying Lemma 3 to $\mathbf{m}$ and putting $m_{\hat{j}} = \tilde{m}_{\nu_0,\dots,\nu_p}$ ($\hat{j} = (\nu_0,\dots,\nu_p)$) and $m_{(i,j,s)} = \sum_{t=s+1}^{l_{i,j}} m_{i,j,t}$, we define

$$\alpha(\mathbf{m}) := \sum_{\hat{j}\in\mathcal{J}} m_{\hat{j}}\alpha_{\hat{j}} + \sum_{i=0}^{p}\sum_{j=1}^{k_i}\sum_{s=1}^{l_{i,j}-1} m_{(i,j,s)}\alpha_{(i,j,s)} \in \Phi^{-1}(\mathbf{m}) \cap Q_L^+.$$

The following lemma gives some properties of $\alpha(\mathbf{m})$.

Lemma 4 *Retain the notation in Definition 11.*

Let I *be the index set of the basis of* Q_L:

$$I = \mathcal{J} \sqcup \big\{(i,j,s) \mid i = 0,\dots,p,\ j = 1,\dots,k_i,\ s = 1,\dots,l_{i,j}-1\big\}.$$

Put $\mathcal{C}_{\mathbf{m}} = \operatorname{supp}\alpha(\mathbf{m})$ *and define*

$$\bar{I} := \{i \in I \mid \alpha_i \in \mathcal{C}_{\mathbf{m}}\}, \qquad I_0 := \big\{t \in \bar{I} \mid \big\langle\alpha(\mathbf{m}),\alpha_t\big\rangle = 0\big\}, \qquad I_1 := \bar{I} \setminus I_0.$$

Assume $k_0 \ge k_1 \ge \cdots \ge k_{N-1} > k_N = \cdots = k_p = 1$. *Here* N *is a non-negative integer. Put* $\hat{j}_0 = (1,\dots,1) \in \mathcal{J}$, $\hat{j}_1 = (k_0,\dots,k_p) \in \mathcal{J}$.

(i) *The element* $\alpha(\mathbf{m})$ *is indivisible if* $\mathbf{m}$ *is indivisible.*

(ii) *We have* $m_{\hat{j}_0} > 0$, $m_{\hat{j}_1} > 0$ *and*

$$\langle \alpha_{\hat{j}_0}, \alpha_{\hat{j}_1} \rangle \le 2 - 2N,$$

$$\max\{k_0, \dots, k_p\} \le \#(\bar{I} \cap \mathcal{J}) \le 1 + \sum_{i=0}^{p} (k_i - 1),$$

$$\sum_{\hat{j} \in \bar{I} \cap \mathcal{J}} m_{\hat{j}} = \operatorname{rank} \mathbf{m}.$$

(iii) *The Dynkin diagram of a subset of* $\mathcal{C}_{\mathbf{m}}$ *is never equal to* $D_n^{(1)}$ *with* $n > 4$.

Preceding to the proof of Lemma 4, we remark the following.

Lemma 5 *Let* $\hat{j}_\nu = (j_{\nu,0}, \dots, j_{\nu,p}) \in \mathcal{J}$ *for* $\nu = 1, 2, \dots$. *Then we have*

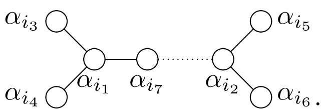
$$\langle \alpha_{\hat{j}_1}, \alpha_{\hat{j}_2} \rangle \le 2 - 2\#\big\{i \in \{0, 1, \dots, p\} \mid j_{1,i} \ne j_{2,i}\big\}, \tag{29}$$

$$\langle \alpha_{\hat{j}_1}, \alpha_{\hat{j}_2} \rangle = \langle \alpha_{\hat{j}_1}, \alpha_{\hat{j}_3} \rangle = 0 \;\Rightarrow\; \langle \alpha_{\hat{j}_2}, \alpha_{\hat{j}_3} \rangle \ne -1. \tag{30}$$

Proof Definition 11 directly shows (29). Suppose $-1 \le \langle \alpha_{\hat{j}_\nu}, \alpha_{\hat{j}_{\nu'}} \rangle \le 0$ for $1 \le \nu < \nu' \le 3$. Then there exists $l \in \mathbb{Z}_{\ge 0}$ such that $j_{1,i} = j_{\nu,i}$ for $i \in \{0, \dots, p\} \setminus \{l\}$ and therefore (30) follows from the relation (26). □

Proof of Lemma 4 The claims (i) and (ii) follow from Definition 11, Lemma 3 and (29).

(iii) Suppose the Dynkin diagram of a subset of $\mathcal{C}_{\mathbf{m}}$ is $D_n^{(1)}$ with $n > 4$:



Define $c_{\mu,\nu} = \langle \alpha_{i_\mu}, \alpha_{i_\nu} \rangle$. For $i_\nu \in I$ put $i_\nu = (j_{\nu,0}, \dots, j_{\nu,p})$ if $i_\nu \in \mathcal{J}$ and put $i_\nu = (k_\nu, j_\nu, s_\nu)$ otherwise. The proof of Lemma 5 shows that there exists l with $0 \le l \le N$ such that $j_{\nu,i} = j_{\nu',i}$ if $i \ne l$ and $i_\nu, i_{\nu'} \in \mathcal{J}$.

Suppose $i_1 \in \mathcal{J}$ and $i_2 \in \mathcal{J}$. Then (30) shows $\#(\{i_3, i_4, i_5, i_6\} \cap \mathcal{J}) \le 1$ and there exists $i_\nu \notin \mathcal{J}$ such that $i_\nu = (k_\nu, j_\nu, 1)$ with $k_\nu \ne l$ and $j_\nu = j_{1,k_\nu}$. Then $c_{\nu,1} = c_{\nu,2} = -1$, which contradicts to the Dynkin diagram.

Suppose $i_1 \notin \mathcal{J}$ and $i_2 \in \mathcal{J}$. Then $\{i_3, i_4\} \cap \mathcal{J} \ne \emptyset$. We may assume $i_3 \in \mathcal{J}$ and then the claim (i) shows $i_5 \notin \mathcal{J}$ and $i_6 \notin \mathcal{J}$, The same argument as above shows $c_{3,5} = -1$ or $c_{3,6} = -1$, which leads a contradiction.

Lastly suppose $i_1 = (k_1, j_1, 1) \notin \mathcal{J}$ and $i_2 \notin \mathcal{J}$. We may assume $i_3 \in \mathcal{J}$ and $i_5 \in \mathcal{J}$. Then there exists $\alpha_{i_7} \in \mathcal{J}$ such that $i_7 \ne i_3$, $i_7 \ne i_4$ and $c_{1,7} = -1$. Since $c_{1,7} = c_{1,3} = -1$, we have $k_1 \ne l$, $j_1 = j_{3,k_1}$ and $c_{1,5} = -1$, which leads a contradiction. □

We shall show some properties of $\alpha(\mathbf{m})$ when $(\mathbf{m}, L)$ is basic.

Lemma 6 *Retain the notation and the assumption in Lemma 4. Suppose* $(\mathbf{m}, L)$ *is basic.*

(i) $\mathcal{C}_{\mathbf{m}}$ *is connected.*

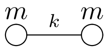
(ii) *Put* $\alpha' = \sum_{i\in\bar{I}'} m_i\alpha_i$ *for a proper subset* $\bar{I}' \subsetneqq \bar{I}$*. Then*

$$\operatorname{idx}\alpha' > \operatorname{idx}\alpha(\mathbf{m}).$$

(iii) *We have*

$$\langle\alpha_{i_1}, \alpha_{i_2}\rangle \geq \frac{1}{2}\operatorname{idx}\mathbf{m} - 2 \quad \textit{for } i_1,\ i_2 \in \bar{I} \tag{31}$$

and the equality holds if and only if the shape of $\alpha(\mathbf{m})$ *is*



$$(m = 1 \textit{ if } k \neq 2) \tag{32}$$

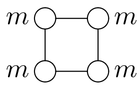
with $k = \frac{1}{2} - \operatorname{idx}\mathbf{m}$.

(iv) *We have*

$$N \leq 2 + \frac{1}{4}|\operatorname{idx}\mathbf{m}| \tag{33}$$

and the equality holds if and only if the shape of $\alpha(\mathbf{m})$ *is the one in* (32) *with* $k = 2N - 2$.

(v) *Suppose* $(\mathbf{m}, L)$ *is basic. Let* $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_K}$ *be a constant connected sequence in* I_0 $(k > 1)$*. Then* $K \leq 4$ *and* $1 \leq N \leq 2$.

If $K = 4$, *then* $N = 1$ *and the shape of* $\alpha(\mathbf{m})$ *is*  .

Suppose $N = 2$ *and* $K = 3$. *Then* $i_2 = \hat{j}_2$ *or* $i_2 = \hat{j}_3$ *by denoting*

$$\hat{j}_2 := (1, k_1, 1, \ldots, 1) \in \mathcal{J} \quad \textit{and} \quad \hat{j}_3 := (k_0, 1, 1, \ldots, 1) \in \mathcal{J}.$$

Moreover $i_1 \notin \mathcal{J}$ *and* $i_3 \notin \mathcal{J}$.

Proof (i) We say that two elements α and α' in $\mathcal{C}_{\mathbf{m}}$ are connected in $\mathcal{C}_{\mathbf{m}}$ if they belong to a connected component of the Dynkin diagram of $\mathcal{C}_{\mathbf{m}}$. Note that $\alpha_{\hat{j}_0}$ and $\alpha_{\hat{j}_1}$ are connected in $\mathcal{C}_{\mathbf{m}}$.

Fix $\alpha_{(i,j,s)}$ with $1 \leq s \leq l_{i,j} - 1$. Then there exists $\hat{j} = (j_0, \ldots, j_p)$ such that $\alpha_{\hat{j}} \in \mathcal{C}_{\mathbf{m}}$ and $j_i = j$. Then $\alpha_{(i,j,s)}$ and $\alpha_{\hat{j}}$ are connected in $\mathcal{C}_{\mathbf{m}}$.

Let $\hat{j} \in \mathcal{J}$ with $\alpha_{\hat{j}} \in \mathcal{C}_{\mathbf{m}}$. If $N \geq 3$, then $\langle\alpha_{\hat{j}_0}, \alpha_{\hat{j}}\rangle \neq 0$ or $\langle\alpha_{\hat{j}_1}, \alpha_{\hat{j}}\rangle \neq 0$, which means $\alpha_{\hat{j}}$ and $\alpha_{\hat{j}_0}$ are connected in $\mathcal{C}_{\mathbf{m}}$ and therefore $\mathcal{C}_{\mathbf{m}}$ is connected.

Hence we assume $N = 2$ and $\langle\alpha_{\hat{j}_0}, \alpha_{\hat{j}}\rangle = \langle\alpha_{\hat{j}_1}, \alpha_{\hat{j}}\rangle = 0$. Then $\hat{j} = \hat{j}_2$ or $\hat{j} = \hat{j}_3$.

Suppose $\hat{j} = \hat{j}_2$. Then $\alpha_{\hat{j}_3} \notin \mathcal{C}_{\mathbf{m}}$, which follows from Lemma 3. Since $\alpha(\mathbf{m})$ is basic, there exists $\alpha \in \mathcal{C}_{\mathbf{m}}$ satisfying $\langle\alpha, \alpha_{\hat{j}_2}\rangle < 0$.

Suppose $\alpha = \alpha_{\hat{j}'}$ with $\hat{j}' \in \mathcal{J}$, $\hat{j}' = (1, j_1, \dots, 1)$ or $\hat{j}' = (j_0, k_1, 1, \dots, 1)$. Here $1 < j_1 < k_1$ and $1 < j_0 < k_0$, respectively. If $\hat{j}' = (1, j_1, \dots, 1)$, α and $\alpha_{\hat{j}_1}$ are connected in $\mathcal{C}_{\mathbf{m}}$. We have the same conclusion when $\hat{j}' = (j_0, k_1, 1, \dots, 1)$.

Suppose $\alpha = \alpha_{(i,j,s)}$. Then $s = 1$. If $i \geq 2$ or $i = 0$, $j = 1$ and $\langle \alpha_{(i,1,1)}, \alpha_{\hat{j}_0} \rangle < 0$. If $i = 1$, $j = k_1$ and $\langle \alpha_{(i,1,1)}, \alpha_{\hat{j}_1} \rangle < 0$. Hence α and $\alpha_{\hat{j}_1}$ are connected in $\mathcal{C}_{\mathbf{m}}$ and so are $\alpha_{\hat{j}}$ and $\alpha_{\hat{j}_1}$.

In the same way as above we have the same conclusion when $\hat{j} = \hat{j}_3$. Thus we have the claim.

If $N = 0$, $\#\mathcal{J} = 1$ and the Dynkin diagram of $\mathcal{C}_{\mathbf{m}}$ is star-shaped and hence connected.

Lastly assume $N = 1$. If there exists $i \in \{2, \dots, p\}$ such that $l_{i,1} > 1$, then $\langle \alpha_{\hat{j}}, \alpha_{(i,1,1)} \rangle = -1$ for any $\hat{j} \in \bar{I} \cap \mathcal{J}$ and therefore the Dynkin diagram of $\mathcal{C}_{\mathbf{m}}$ is connected. Hence we assume $l_{i,1} = 1$ for all $i \in \{2, \dots, p\}$. If $\langle \alpha_{\hat{j}_0}, \alpha_{j} \rangle = 0$ for any $\hat{j} \in (\bar{I} \cap \mathcal{J}) \setminus \{\hat{j}_0\}$, then $\langle \alpha(\mathbf{m}), \alpha_{\hat{j}_0} \rangle = 2m_{\hat{j}} - m_{(0,1,1)} > 0$, which contradicts to the fact that $\alpha(\mathbf{m})$ is basic. Hence there exists $\hat{j} \in \bar{I} \cap \mathcal{J}$ satisfying $\langle \alpha_{\hat{j}_0}, \alpha_{\hat{j}} \rangle < 0$. Then the relation (26) assures $\langle \alpha_{\hat{j}_0}, \alpha_{\hat{j}'} \rangle < 0$ or $\langle \alpha_{\hat{j}}, \alpha_{\hat{j}'} \rangle < 0$ for any $\hat{j}' \in (\bar{I} \cap \mathcal{J}) \setminus \{\hat{j}_0, \hat{j}\}$, which proves that $\alpha_{\hat{j}_0}$ and $\alpha_{\hat{j}'}$ are connected in $\mathcal{C}_{\mathbf{m}}$ and moreover $\mathcal{C}_{\mathbf{m}}$ is connected.

(ii) The claim easily follows from the definition of the index and the connectedness of the Dynkin diagram of $\mathcal{C}_{\mathbf{m}}$.

(iii) We may assume $\langle \alpha_{i_1}, \alpha_{i_2} \rangle \leq -2$. If $m_{i_1} < m_{i_2}$, we have

$$\begin{aligned}
\operatorname{idx} \mathbf{m} &\leq \big\langle \alpha(\mathbf{m}), m_{i_1} \alpha_{i_1} \big\rangle \\
&\leq m_{i_1}^2 + m_{i_1} m_{i_2} \langle \alpha_{i_1}, \alpha_{i_2} \rangle \\
&\leq m_{i_1}^2 \big(1 + \langle \alpha_{i_1}, \alpha_{i_2} \rangle\big) + m_{i_1} \langle \alpha_{i_1}, \alpha_{i_2} \rangle \\
&\leq m_{i_1}^2 \big(1 + \langle \alpha_{i_1}, \alpha_{i_2} \rangle\big) + m_{i_1}^2 \langle \alpha_{i_1}, \alpha_{i_2} \rangle, \\
\langle \alpha_{i_1}, \alpha_{i_2} \rangle &\geq \frac{\operatorname{idx} \mathbf{m}}{2 m_{i_1}^2} - \frac{1}{2}.
\end{aligned}$$

If $m_{i_1} = m_{i_2}$, we have

$$\begin{aligned}
\operatorname{idx} \mathbf{m} &\leq \big\langle \alpha(\mathbf{m}), m_{i_1} \alpha_{i_1} \big\rangle + \big\langle \alpha(\mathbf{m}), m_{i_2} \alpha_{i_2} \big\rangle \\
&\leq 2 m_{i_1}^2 + 2 m_{i_1} m_{i_2} \langle \alpha_{i_1}, \alpha_{i_2} \rangle + 2 m_{i_2}^2, \\
\langle \alpha_{i_1}, \alpha_{i_2} \rangle &\geq \frac{\operatorname{idx} \mathbf{m}}{2 m_{i_1} m_{i_2}} - 2.
\end{aligned}$$

Hence we have $\langle \alpha_{i_1}, \alpha_{i_2} \rangle \geq \frac{1}{2} \operatorname{idx} \mathbf{m} - 2$ and the equality implies $m_{i_1} = m_{i_2}$ and moreover $m_{i_1} = 1$ if $\operatorname{idx} \mathbf{m} \neq 0$. It follows from the claim (ii) that the equality implies that the shape of $\alpha(\mathbf{m})$ is the one in (32) with $k = 2 - \frac{1}{2} \operatorname{idx} \mathbf{m}$.

(iv) Since $\langle \alpha_{\hat{j}_0}, \alpha_{\hat{j}_1} \rangle \leq 2 - 2N$, the claim in (iii) implies that in (iv).

(v) Put $A = \bar{I} \cap \mathcal{J}$ and $B = \bar{I} \setminus A$.

Let $(i, j, s) \in B$ with $s \geq 2$. Then $m_{(i,j,s-1)} > m_{(i,j,s)} > m_{i,j,s+1} \geq 0$ and $\{i \in I \mid \langle \alpha_{(i,j,s)}, \alpha_i \rangle < 0\} \subset \{(i, j, s-1), (i, j, s+1)\}$. Hence $i_\nu \neq (i, j, s)$ for $\nu = 1, \ldots, K$.

Note that $\langle \alpha_{(i,j,1)}, \alpha_{(i',j',1)} \rangle = 0$ for two different elements $(i, j, 1)$ and $(i', j', 1)$ of B. Hence there exists no constant connected sequence in B.

If $N = 0$, then $\#\mathcal{J} = 1$ and it is clear that there is no constant connected sequence.

Suppose $N \geq 3$. Then $\alpha_{\hat{j}_\nu} \notin I_0$ for $\nu = 0$ and 1 and moreover $\langle \alpha_{\hat{j}}, \alpha_{\hat{j}_0} \rangle \leq -2$ or $\langle \alpha_{\hat{j}}, \alpha_{\hat{j}_1} \rangle \leq -2$ for any $\hat{j} \in A$. Hence there exists no constant connected sequence in I_0.

Suppose $N = 2$ and $K = 3$. If $\hat{j} \in A \setminus \{\hat{j}_0, \hat{j}_1, \hat{j}_2, \hat{j}_3\}$, then $\langle \alpha_{\hat{j}}, \alpha_{\hat{j}_0} \rangle \leq -2$ or $\langle \alpha_{\hat{j}}, \alpha_{\hat{j}_1} \rangle \leq -2$. Hence $i_2 = \hat{j}_2$ or $i_2 = \hat{j}_3$. Since $\{\hat{j}_2, \hat{j}_3\} \not\subset A$, the length of the constant connected sequence in I_0 is not larger than 3 and the corresponding claim in (iii) is valid.

Lastly suppose $N = 1$. Suppose $K = 2$ and $i_2 \in B$. Then $i_1 = (j_1, 1, \ldots, 1) \in A$ and $i_2 = (0, j_1, 1)$ or $i_2 = (i, 1, 1)$ with $i > 0$. If $i_2 = (0, j_1, 1)$, then $m_{i_1} > m_{i_2}$, which implies $i_2 = (i, 1, 1)$ and $\langle \alpha_{\hat{j}}, \alpha_{i_2} \rangle < 0$ for any $\hat{j} \in A$.

Fix a constant connected sequence in I_0. The number M of the elements α_i in the sequence with $i \in B$ is not larger than 2. If $M > 0$, the number of the elements α_j with $j \in A$ in the sequence is not larger than 2 and therefore $K \leq 4$. If $M > 0$ and $K = 4$, then $M = 2$ and the shape of $\alpha(\mathbf{m})$ is the one given in (v).

Suppose $M = 0$ and $K \geq 4$. Put $i_\nu = (j_\nu, 1, \ldots, 1) \in A$ for $\nu = 1, \ldots, K$. Since $\langle \alpha_{i_1}, \alpha_{i_3} \rangle = 0$ we have $\langle \alpha_{i_3}, \alpha_{i_4} \rangle = 0$ if $\langle \alpha_{i_1}, \alpha_{i_4} \rangle = 0$. Hence $K = 4$ and $\langle \alpha_{i_1}, \alpha_{i_4} \rangle < 0$ and the condition $i_1 \in I_0$ shows the claim (v). $\square$

3.3.1 The Finiteness of Basic Pairs

We show the finiteness theorem which is an analogue of Theorem 3.

We say that a basic pair $(\mathbf{m}, L)$ is *indivisible* if the greatest common divisor of $\{m_{i,j,s} \mid i = 0, \ldots, p,\ j = 1, \ldots, k_i,\ s = 1, \ldots, l_{i,j}\}$ is 1 for $\mathbf{m} = ((m_{i,j,1}, \ldots, m_{i,j,l_{i,j}}))_{\substack{0 \leq i \leq p \\ 1 \leq j \leq k_i}}$.

We also say that a basic pair $(\mathbf{m}, L)$ is *reduced* when we have $l_{i,1} > 1$ for all $i = 0, \ldots, p$ satisfying $k_i = 1$ (cf. Remark 3).

Theorem 8 (Corollary of Theorem 3) *Fix an integer $r \geq 0$. If $r > 0$, then there exist only a finite number of reduced basic pairs $(\mathbf{m}, L)$ with $\operatorname{idx} \mathbf{m} = -r$. Moreover there exist only a finite number of reduced indivisible basic pairs $(\mathbf{m}, L)$ with $\operatorname{idx} \mathbf{m} = 0$.*

Proof Theorem 8 and Lemma 6 assure that there are only finite possibilities of shapes of $\alpha(\mathbf{m})$. Hence there exists a positive integer n_r such that $\operatorname{rank} \mathbf{m} \leq n_r$. Hence the theorem is reduced to the following lemma. $\square$

Lemma 7 *Fix integers $n > 0$ and r. Then there exist a finite number of reduced basic pairs $(\mathbf{m}, L)$ satisfying $\operatorname{rank} \mathbf{m} \leq n$ and $\operatorname{idx} \mathbf{m} \geq -r$.*

Proof Let $(\mathbf{m}, L)$ be a reduced basic pair satisfying the assumption. Since $\sum_{u=1}^{v} e_u^2 - \left(\sum_{u=1}^{v} e_u\right)^2 \le -2$ if $v \ge 2$ and $e_u \in \mathbb{Z}_{>0}$ for $u = 1, \ldots, v$, we have

$$\begin{aligned}
&\operatorname{idx}\mathbf{m} + \sum_{i=0}^{p} \sum_{1 \le j \ne j' \le k_i} d_i\big(j, j'\big)\left(\sum_{s=1}^{l_{i,j}} m_{i,j,s}\right)\left(\sum_{s'=1}^{l_{i,j'}} m_{i,j',s'}\right)\\
&= \sum_{i=0}^{p}\left(\sum_{j=1}^{k_i}\sum_{s=1}^{l_{i,j}} m_{i,j,s}^2 - (\operatorname{rank}\mathbf{m})^2\right) + 2(\operatorname{rank}\mathbf{m})^2\\
&\le -2(p+1) + 2(\operatorname{rank}\mathbf{m})^2
\end{aligned}$$

by putting $\mathbf{m} = ((m_{i,j,1}, \ldots, m_{i,j,l_{i,j}}))_{\substack{0 \le i \le p\\ 1 \le j \le k_i}}$, which implies

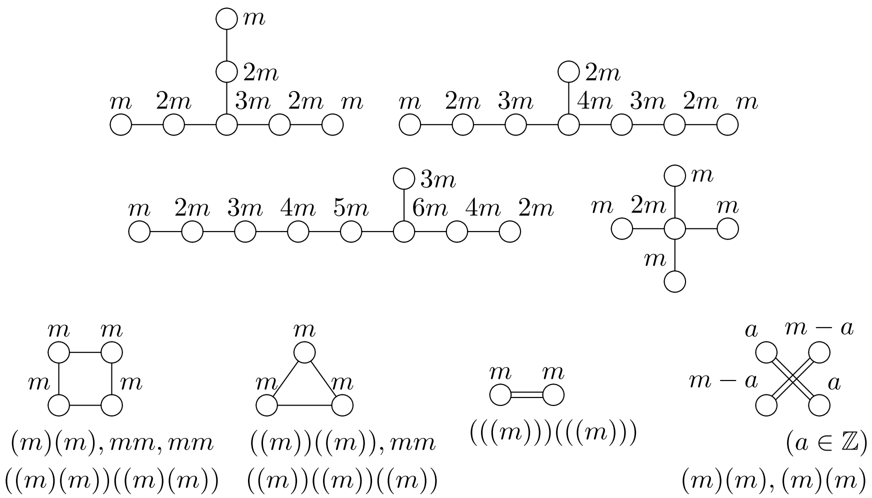
$$\begin{aligned}
2(p+1) &\le r + 2n^2,\\
d_i\big(j, j'\big) &\le 2n^2 - 2(p+1) + r\\
&\le 2n^2 + r - 2 \quad \big(0 \le i \le p,\ 1 \le j < j' \le k_i\big).
\end{aligned}$$

This shows the lemma. □

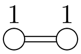
3.3.2 The Classification of Basic Pairs with idx 0

We shall give lists of shapes of basic pairs of index 0 and -2. First we consider basic pairs of index 0.

Theorem 9 *If a basic pair* $(\mathbf{m}, L)$ *satisfies* $\operatorname{idx}\mathbf{m} = 0$, *then its shape is one of the following*.



Here m are arbitrary elements in $\mathbb{Z}_{>0}$. We simply write sets $\{x_a \mid a \in \mathbb{Z}\}$ and $\{x\}$ by x_a $(a \in \mathbb{Z})$ and x, respectively. The sequences of integers written under the shapes except for star-shaped ones stand for the corresponding basic pairs $(\mathbf{m}, L)$.

Proof Retain the notation in the proof of Lemma 6. We may assume $\mathbf{m}$ is indivisible. If $N = 2$, the shape of $\alpha(\mathbf{m})$ is 

. Then rank $\mathbf{m} = 2$ and the shape of $(\mathbf{m}, L)$ is the last shape in the above list with $m = 1$.

Then we may assume $N \le 1$ and the shape of $(\mathbf{m}, L)$ corresponds to the shape of $\alpha(\mathbf{m})$. Hence the claim in Sect. 2.3, Lemma 4 and Lemma 6 show the theorem. □

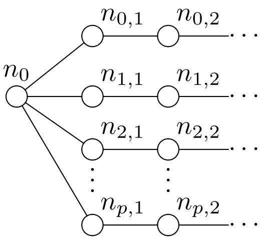
Remark 4 We shall explain the notation expressing $(\mathbf{m}, L)$ in Theorem 9. The number of parentheses ( ) represents the number $d_i(j, j')$. For instance, if $(\mathbf{m}, L)$ is written by

$$\cdots m_{i,j,1} m_{i,j,2} \ldots m_{i,j,l_{i,j}}))((m_{i,j',1} m_{i,j',2} \cdots ,$$

then we can see the double parenthesis (( )) between $m_{i,j,1} \ldots$, and $m_{i,j',1} \ldots$. This means $d_i(j, j') = 2$. Let us see an example. Consider a basic pair $(\mathbf{m}, L)$ where $p = 1$, $(k_0, k_1) = (2, 3)$, $(l_{0,1}, l_{0,2}, l_{1,1}, l_{1,2}, l_{1,3}) = (1, 2, 1, 1, 2)$ and $(d_0(1, 2), d_1(1, 2), d_1(2, 3), d_1(1, 3)) = (1, 1, 2, 2)$.

Then $\mathbf{m} = ((m_{i,j,1}, \ldots, m_{i,j,l_{i,j}}))_{\substack{0 \le i \le p \\ 1 \le j \le k_i}}$ is written by

$$(m_{0,1,1})(m_{0,2,1} m_{0,2,2}),\ \big((m_{1,1,1})(m_{1,2,1})\big)\big((m_{1,3,1} m_{1,3,2})\big).$$

Remark 5 In the above list of shapes, we omit the corresponding $(\mathbf{m}, L)$ for star-shaped diagrams. For these cases $(\mathbf{m}, L)$ are obtained as follows. Consider a shape 

and put $m_{(i,1)} = n_0 - n_{i,1}$, $m_{(i,j+1)} = n_{i,j} - n_{i,j+1}$, $m_{(i,0)} = \sum_{\substack{0 \le k \le p \\ k \ne i}} n_{k,1} - n_0$ and $m_{(0)} = \sum_{i=0}^{p} n_{i,1} - n_0$. Then the shape corresponds to the following 5 types of $(\mathbf{m}, L)$ with $0 \le i \le p$.

$$m_{(0,1)} m_{(0,2)} \ldots,\ m_{(1,1)} m_{(1,2)} \ldots,\ \ldots,\ m_{(p,1)} m_{(p,2)} \ldots,$$

$$m_{(0)} n_0,\ (m_{(0,2)} m_{(0,3)} \ldots) \ldots (m_{(p,2)} m_{(p,3)} \ldots),$$

$$m_{(i,0)} m_{(i,1)} \ldots,\ (m_{(0,2)} m_{(0,3)} \ldots) \ldots (m_{(i-1,2)} \ldots)(m_{(i+1,2)} \ldots) \ldots,$$

$$\big((m_{(i,1)} m_{(i,2)} \ldots)\big)\big((m_{(0,2)} m_{(0,3)} \ldots) \ldots (m_{(i-2,2)} \ldots)(m_{(i+1,2)} \ldots) \ldots\big),$$

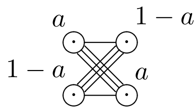
$$\big((n_0)\big)\big((m_{(0,2)} m_{(0,3)} \ldots) \ldots (m_{(p,2)} m_{(p,3)} \ldots)\big).$$

In [12], K. Takemura obtains a part of the classification in Theorem 9 under some conditions (see Proposition 4.3 in [12]).

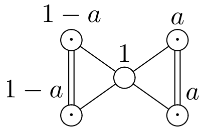
3.3.3 The Classification of Basic Pairs with idx −2

We shall give a classification of basic pairs of idx -2.

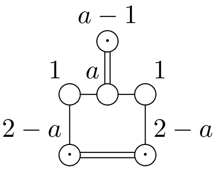
Theorem 10 *Let* $(\mathbf{m}, L)$ *be a basic pair with* $\operatorname{idx}\mathbf{m} = -2$. *Then its shape is one of the following.*
Case 1:



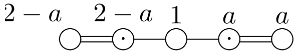
$(a \in \mathbb{Z})$
$((1))((1)), (1)(1)$



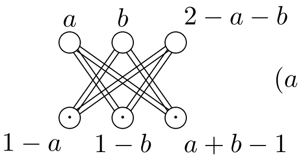
$(a \in \mathbb{Z})$
$(1)(1), (1)(1), 11$



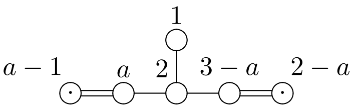
$(a \in \mathbb{Z})$
$(1)(11), (1)(11)$



$(a \in \mathbb{Z})$
$(2)(2), (2)(11)$



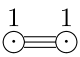
$(a, b \in \mathbb{Z})$
$(1)(1)(1), (2)(1)$



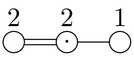
$(a \in \mathbb{Z})$
$(2)(2), (1)(111)$

Case 2:

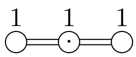
<u>*2:*</u>



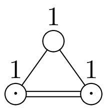
$((((1))))((((1))))$



$(((2)))(((11)))$

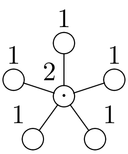


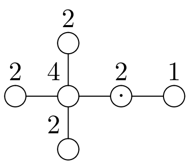
$(((1)(1)))(((1)))$

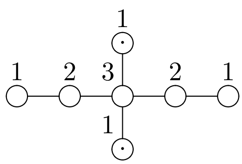


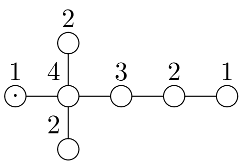
$(((1)))(((1))), 11$

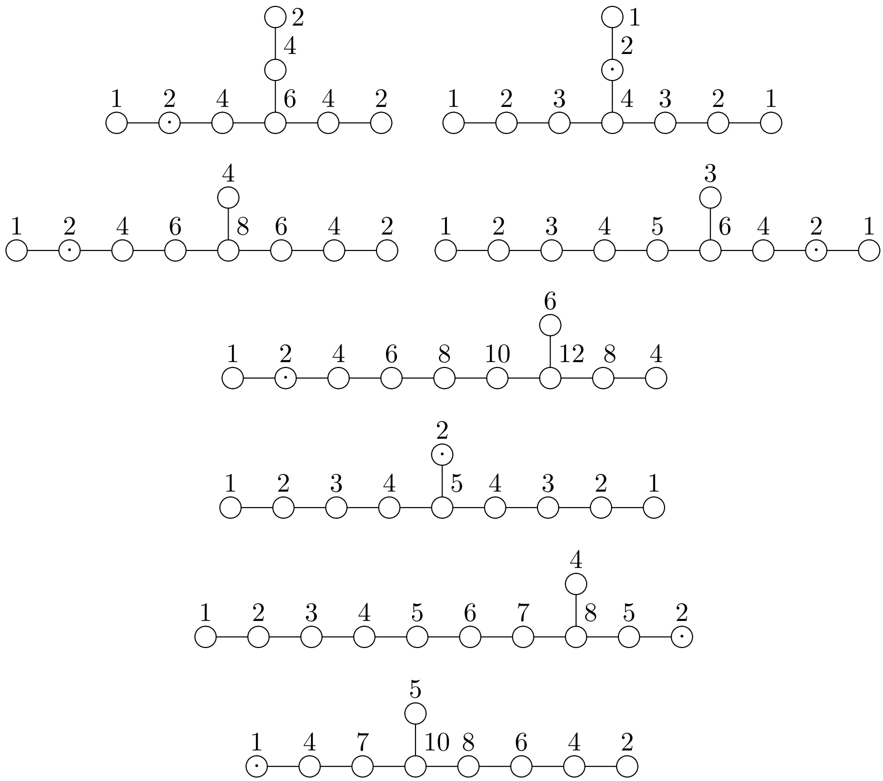
Case 3:

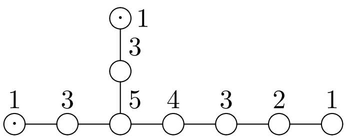


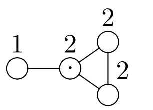




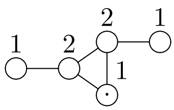




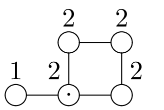




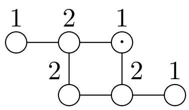
$((2))((2))((11))$
$((2))((11)), 22$
$((2))((2)), 211$



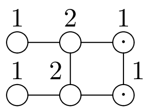
$((11))((11))((1))$
$((11))((1)), 111$
$((11))((11)), 31$



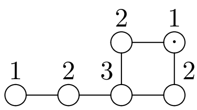
$((2)(2))((2)(11))$
$(2)(2), 22, 211$
$(2)(11), 22, 22$



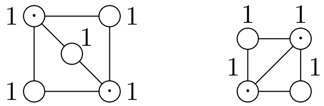
$((11)(11))((2)(1))$
$(11)(11), 22, 31$
$(2)(1), 111, 111$



$((11)(1))((11)(1))$
$((11)(1)), 21, 111$



$((1)(111))((2)(2))$
$(1)(111), 22, 22$
$(2)(2), 31, 1111$



$((1)(1))((1)(1)(1))$ $((1))((1))((1)(1))$

$(1)(1), 11, 11, 11$ $((1))((1)), 11, 11$

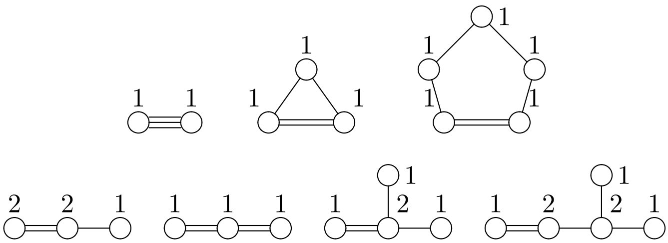
$(1)(1)(1), 21, 21$

Here we simply denote the sets $\{x_a \mid a \in \mathbb{Z}\}$ *and* $\{y\}$ *by* x_a $(a \in \mathbb{Z})$ *and* y, *respectively. The sequences of integers written under the shapes except for star-shaped ones stand for the corresponding basic pairs* $(\mathbf{m}, L)$.

Retain the notation in the previous section. To prove the theorem we may assume $k_0 \geq k_1 \geq \cdots \geq k_{N-1} > k_N = \cdots = k_p = 1$ and $l_{N,1} \geq l_{N+1,1} \geq \cdots \geq l_{p,1} > 1$. Note that Lemma 6(iv) assures $N \leq 2$.

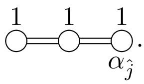
Then the proof of the theorem deduced to the following three lemmas.

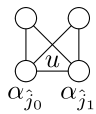
Lemma 8 *Suppose* $N = 2$. *Then the shape of* $\alpha(\mathbf{m})$ *is one of the following*.



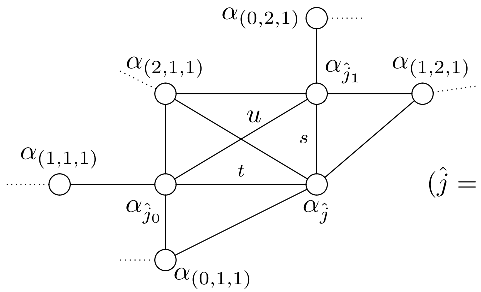
Moreover the shape of $(\mathbf{m}, L)$ *is one of the shapes in Case* 1 *in Theorem* 10.

Proof Use the notation in Lemma 6.

First suppose $k_0 \geq 3$. Then there exists $\hat{j} = (2, l, 1 \ldots, 1) \in \mathcal{J} \cap \bar{I}$. If $l \neq 1$, $\langle \alpha_{\hat{j}}, \alpha_{\hat{j}_1} \rangle \leq -2$. If $l \neq k_1$, $\langle \alpha_{\hat{j}}, \alpha_{\hat{j}_0} \rangle \leq -2$. Since $\langle \alpha_{\hat{j}_0}, \alpha_{\hat{j}_1} \rangle \leq -2$, the lists in Sect. 2.4 show that the shape of $\alpha(\mathbf{m})$ equals (E) .

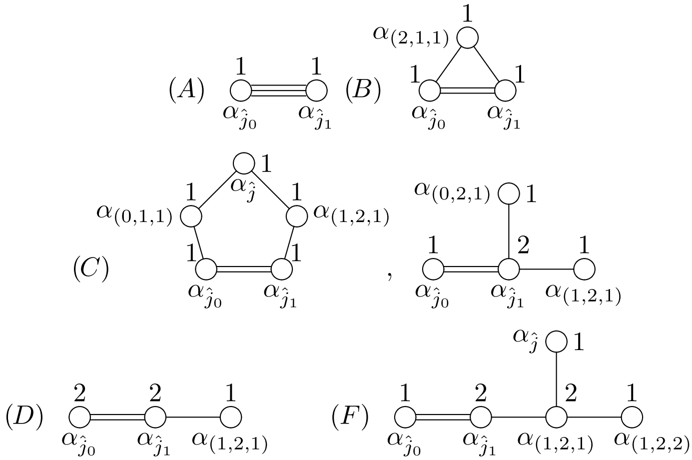
If $p \geq 3$, the Dynkin diagram of $\{\alpha_{\hat{j}_0}, \alpha_{\hat{j}_1}, \alpha_{(2,1,1)}, \alpha_{(3,1,1)}\}$ equals  with $u = -\langle \alpha_{\hat{j}_0}, \alpha_{\hat{j}_1} \rangle \geq 2$, which contradicts to the lists in Sect. 2.4.

Next suppose $k_0 = k_1 = 2$ and $p \leq 2$. Then $\#(\bar{I} \cap \mathcal{J}) \leq 3$ and the support of $\alpha(\mathbf{m})$ is a subset of the set of simple roots whose Dynkin diagram is



$$(\hat{j} = (1, 2, 1, \ldots, 1)),$$

where $s, t \geq 0$, $u = s + t + 2 \geq 2$ and $\hat{j} = (1, 2, 1 \ldots, 1)$ or $\hat{j} = (2, 1, 1, \ldots, 1)$. Here the Dynkin diagram in the case $\hat{j} = (2, 1, 1 \ldots, 1)$ is similar as above and hence we assume $\hat{j} = (1, 2, 1 \ldots, 1)$. Then the lists in Sect. 2.4 tell us that the shape of $\alpha(\mathbf{m})$ is one of the following.



Here $s = t = 0$ when $m_{\hat{j}} > 0$ and the simple roots indicated in the shape are examples corresponding to the shapes.

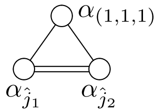
Since $\alpha \in Q_L$ and $\Phi(\alpha) = \mathbf{m}$, $\mathbf{m}$ is uniquely determined from $\alpha(\mathbf{m})$ for fixed L. Then if we write the shapes of $(\mathbf{m}, L)$ from the shapes (A), (B), (C), (D), (E) and (F), then we have the shapes in Case 1 in Theorem 10, respectively. Here we note that the shapes of $\alpha(\mathbf{m})$ labeled by (C) correspond to a single shape of $(\mathbf{m}, L)$, which is the third one in Case 1. $\square$

Next consider the case $N = 1$. We notice that Φ is injective in this case. Hence the shape of $(\mathbf{m}, L)$ consists only of the shape of $\alpha(\mathbf{m})$. Put $\mathcal{J}_0 := \{1, \ldots, k_0\}$ for simplicity.

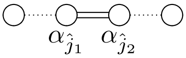
Lemma 9 *Retain the notation above. If* $\max\{d_0(j, j') \mid j,\ j' \in \mathcal{J}_0\} \geq 3$, *the shape of* $\alpha(\mathbf{m})$ *is one of the shapes in Case* 2 *in Theorem* 10.

Proof Lemma 6 proves $\max\{d_0(j, j') \mid j,\ j' \in \mathcal{J}_0\} \leq 4$ and the equality means that the shape of $\alpha(\mathbf{m})$ is the first one in Case 2.

Suppose $\max\{d_0(j, j') \mid j,\ j' \in \mathcal{J}_0\} = 3$. We may assume $d_0(1, 2) = 3$. Put $\hat{j}_\nu = (\nu, 1, \ldots, 1)$.

If $p \geq 1$, the Dynkin diagram of $\{\alpha_{\hat{j}}, \alpha_{\hat{j}'}, \alpha_{(1,1,1)}\}$ equals 

and the lists in Sect. 2.4 show that the shape of $\alpha(\mathbf{m})$ is the last one in Case 2.

If $k_0 = 2$, the shape of $\alpha(\mathbf{m})$ is 

and the lists in Sect. 2.4 show that the shape of $\alpha(\mathbf{m})$ is the second one in Case 2.

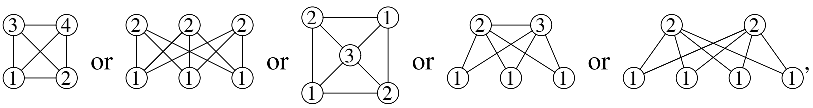
Suppose $k_0 \geq 3$, Then $d_0(1, 3) = 3$ or $d_0(2, 3) = 3$ by the relation (26). Hence the lists in Sect. 2.4 show that $k_0 \leq 3$ and moreover that if $k_0 = 3$, the shape of $\alpha(\mathbf{m})$ is the third one in Case 2. □

Lemma 10 *If* $\max\{d_0(j, j') \mid j,\ j' \in \mathcal{J}_0\} \leq 2$, *the shape of* $\alpha(\mathbf{m})$ *is one of the shapes in Case* 3 *in Theorem* 10.

Proof Define the coset decomposition of $\mathcal{J}_0$ by the following relation: for distinct $j,\ j' \in \mathcal{J}_0$, j and j' are in the same coset if and only if $d_0(j, j') = 1$.

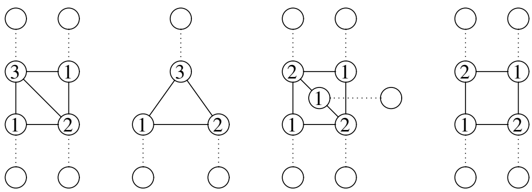
Put $\tilde{\mathcal{J}}_0 = \mathcal{J}_0 \cup \{(j, 1, 1) \mid j = 1, \ldots, p\}$ and define the coset decomposition $\tilde{\mathcal{J}}_0 = \coprod_{j=1}^{q} J(q)$ so that the coset is one of the cosets of $\mathcal{J}_0$ or $\{(j, 1, 1) \mid j = 1, \ldots, p\}$. We may assume $\#J(1) \geq \#J(2) \geq \cdots \geq \#J(q) \geq 1$.

Then we have $q \leq 3$, $\#J(2) \leq 2$ and if $q = 3$, then $\#J(2) = 1$ and $\#J(1) \leq 2$. Moreover if $\#J(2) = 2$, then $\#J(1) \leq 3$. In fact, if this is not valid, $\operatorname{supp}\alpha(\mathbf{m})$ contains a set of simple roots with the Dynkin diagram



which contradicts to the classification in Sect. 2.4. Here ⓘ corresponds to a simple root in $J(i)$.

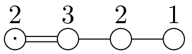
If $q = 1$ or $q = 2$ and $\#J(2) = 1$, the Dynkin diagram of the support of $\alpha(\mathbf{m})$ is star-shaped. Otherwise it is one of the following:

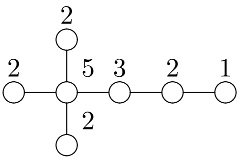


Hence we have the lemma from the classification in Sect. 2.4. □

Remark 6 We mention about a related work by H. Kawakami, A. Nakamura and H. Sakai in [5]. They consider systems of first order differential equations with index of rigidity −2 whose singular points are regular singular or unramified irregular singular points. These equations are obtained by the confluence of singular points from Fuchsian systems of first order differential equations with index of rigidity −2 whose spectral types are basic in the sense of Definition 9. We notice that spectral types can be defined for systems of first order differential equations (see [9] for instance).

We regard these spectral types as elements in lattices of spectral types and write their shapes as in Sect. 3. Then the list of shapes of these spectral types in [5] and our list of shapes of basic pairs with index −2 coincide with each other.

This coincidence is no more valid in the case when the index of rigidity is −4. Let P be a differential operator with the shape of the spectral type 

, which represents a basic root with index −4. Then P is of order 5 and has an unramified irregular singular point. The operator P is obtained by a confluence of four regular singular points of a Fuchsian differential equation with the shape of the spectral type 

, which does not corresponds to a basic root. Note that any Fuchsian differential equation of order 5 with a basic spectral type and index −4 has only three singular points (see [9, 11]).

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Fermions Acting on Quasi-local Operators in the XXZ Model

Michio Jimbo, Tetsuji Miwa, and Feodor Smirnov

Abstract This is a survey about the construction of fermions which act on the space of quasi-local operators in the XXZ model. We also include a proof of the anti-commutativity of fermionic creation operators.

1 Introduction

In this article, we give an exposition of the 'fermionic basis' found in [1, 2] for the space of operators in the XXZ spin chain. In order to explain the problem, let us begin with some historical background.

Quite generally, in integrable models one is given a large family of commuting operators which act on the space of states. The first issue is then to describe their spectra. In the case of the XXZ chain, the space of states is simply a tensor product $V^{\otimes N}$, where $V = \mathbb{C}^2$. The generating function of the commuting operators is the transfer matrix of the underlying six vertex model, and the standard machinary of the Bethe ansatz enables one to study its spectra in great detail.

The second issue is to describe expectation values of local operators

$$\mathcal{O} \in \mathrm{End}\, V^{\otimes m} \subset \mathrm{End}\, V^{\otimes N}.$$

This is a problem far more involved than the first. It has been known for some time that, for the XXZ chain in the thermodynamic limit $N \to \infty$, the expectation

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values of the standard basis elements (products of matrix units) are given by certain multiple integrals [3–6]. Subsequently it has been recognised, on many examples, that actually these integrals can be reduced to sums of products of one-dimensional integrals, with complicated rational functions as coefficients [7–10]. These findings suggested that, if one passes from the standard basis of local operators to a suitable new basis, then the corresponding expectation values simplify drastically.

It turns out to be convenient to introduce a parameter α and consider in place of $\mathrm{End}\, V^{\otimes m}$ the space of expressions of the form

$$q^{2\alpha S(0)}\mathcal{O}, \qquad S(0)=\frac{1}{2}\sum_{j=-\infty}^{0}\sigma_j^3,$$

where σ_j^3 is a Pauli matrix at site j and $\mathcal{O}$ is a local operator in the usual sense. We shall call such operators 'quasi-local' (see Sect. 3 below). The parameter α plays a role of regularisation which helps removing degeneracies from the formulas.

In [1], we have defined certain fermions $\mathbf{b}_p, \mathbf{c}_p, \mathbf{b}^*_p, \mathbf{c}^*_p$, $p \geq 1$, which act on the space of all quasi-local operators. Together with the adjoint action of the integrals of motion $\mathbf{t}^*_p$, these operators act on $q^{2\alpha S(0)}$ and create a basis which we call 'fermionic'. We have shown in [2] that for these basis elements the expectation values are given by determinants involving only two basic functions $\rho(\zeta)$ and $\omega(\zeta,\xi)$ (see Sect. 6, Theorem 2). This clarifies the reason for the simplification of the integrals mentioned above.

The aim of the present paper is to outline the construction of the fermions, leaving the proofs to the original papers. The construction is purely algebraic. It can be viewed as a sophisticated version of the algebraic Bethe ansatz, but there are new features. In particular it is applied to the spaces $\mathrm{End}\, V^{\otimes m}$ rather than $V^{\otimes m}$. Also, essential use is made of representations of the Borel subalgebra $U_q\mathfrak{b}$ of $U_q\widehat{\mathfrak{sl}}_2$. Taking this opportunity, we supply a proof of the anti-commutativity of fermionic creation operators which has not been published in the previous papers.

The text is organised as follows. In Sect. 2 we collect preliminary materials about the transfer matrix and Baxter's Q-matrices, thereby introducing our notation. In Sect. 3 we consider the action of integrals of motion on the space of quasi-local operators. In Sect. 4 we define the fermionic annihilation and creation operators, and in Sect. 5 explain their properties. In Sect. 6 we consider the expectation values. The main statement is that the expectation values of operators created by fermions from 'the primary operator' can be computed as a determinant. We give an explicit formula for the function $\omega(\zeta,\xi)$ in Appendix A. Appendix B is devoted to the proof of anti-commutativity of creation operators.

Throughout the text we shall assume that q is not a root of unity.

2 Transfer Matrix and Q-Matrices

In this section, we fix our notation and review the standard construction of the transfer matrix of the six vertex model and Baxter's Q-matrices.

Let $V=\mathbb{C}^2$, and let v_+, v_- be the standard basis. Let V_j ($j\in\mathbb{Z}$) be copies of V, and set $V_{[K,L]}=V_K\otimes\cdots\otimes V_L$ for an interval $[K,L]\subset\mathbb{Z}$. The transfer matrix is an element of End $V_{[K,L]}$ defined by

$$\begin{aligned} T_{[K,L]}(\zeta,\alpha) &= \mathrm{Tr}_a\big(\mathcal{T}_{a,[K,L]}(\zeta)\cdot q^{\alpha\sigma_a^3}\big),\\ \mathcal{T}_{a,[K,L]}(\zeta) &= \mathcal{L}_{a,L}(\zeta)\cdots\mathcal{L}_{a,K}(\zeta). \end{aligned} \tag{1}$$

Here the operator $\mathcal{L}$ is the image of the universal R matrix of $U_q\widehat{\mathfrak{sl}}_2$ in the two-dimensional evaluation representation $\pi_{a,\zeta}: U_q\widehat{\mathfrak{sl}}_2\to \mathrm{End}\, V_a$,

$$\mathcal{L}_{a,j}(\zeta/\xi)=(\pi_{a,\zeta}\otimes\pi_{j,\xi})\mathcal{R}. \tag{2}$$

It has the weight preserving property

$$\big[x\otimes x, \mathcal{L}_{a,j}(\zeta/\xi)\big]=0 \quad \text{for any diagonal } x\in \mathrm{End}\, V. \tag{3}$$

We have also introduced an arbitrary parameter α, which will play a key role later on.

Due to the Yang-Baxter relation, for each fixed α the transfer matrices (1) mutually commute,

$$\big[T_{[K,L]}(\zeta,\alpha), T_{[K,L]}\big(\zeta',\alpha\big)\big]=0 \quad \big(\forall\zeta,\zeta'\big).$$

We note also

$$\big[S_{[K,L]}, T_{[K,L]}(\zeta,\alpha)\big]=0,$$

where

$$S_{[K,L]}=\frac{1}{2}\sum_{j=K}^{L}\sigma_j^3.$$

In addition to the transfer matrices, there are also Baxter's Q-matrices among the commuting family. As we shall see below, the latter are more fundamental objects than the former.

For the construction of Q-matrices one uses representations of the Borel subalgebra $U_q\mathfrak{b}$ of $U_q\widehat{\mathfrak{sl}}_2$ [12] in place of the two-dimensional 'auxiliary space' V_a. More specifically, consider the following operators $\mathbf{a}, \mathbf{a}^*, q^{\pm D}$ on the vector space $W=\bigoplus_{k\in\mathbb{Z}}\mathbb{C}|k\rangle$:

$$q^D|k\rangle=q^k|k\rangle,\qquad \mathbf{a}|k\rangle=\big(1-q^{2k}\big)|k-1\rangle,\qquad \mathbf{a}^*|k\rangle=|k+1\rangle.$$

They satisfy the so-called q-oscillator algebra relations

$$q^D\mathbf{a}q^{-D}=q^{-1}\mathbf{a},\qquad q^D\mathbf{a}^*q^{-D}=q\mathbf{a}^*,\qquad q^{-1}\mathbf{a}\mathbf{a}^*-q\mathbf{a}^*\mathbf{a}=q^{-1}-q.$$

Then the formulas

$$\varpi^+_{A,\zeta}(e_0) = \frac{\zeta}{q-q^{-1}}\mathbf{a}, \qquad \varpi^+_{A,\zeta}(e_1) = \frac{\zeta}{q-q^{-1}}\mathbf{a}^*,$$
$$\varpi^+_{A,\zeta}\bigl(q^{h_0}\bigr)^{-1} = \varpi^+_{A,\zeta}\bigl(q^{h_1}\bigr) = q^{2D}$$

give a representation $\varpi^+_{A,\zeta} : U_q\mathfrak{b} \to \operatorname{End} W^+$, where $W^+ = \bigoplus_{k\geq 0}\mathbb{C}|k\rangle$. Interchanging the indices 0 and 1, one defines another representation $\varpi^-_{A,\zeta}$ on the quotient space $W^- = W/W^+$. (We use the letter A for 'auxiliary'. The representations $\varpi^\pm_{A,\zeta}$ are the two types of fundamental representations of $U_q\mathfrak{b}$, see [13].) Accordingly we define

$$Q^\pm_{[K,L]}(\zeta,\alpha) = \zeta^{\pm(\alpha - S_{[K,L]})}\mathrm{Tr}_A\bigl(\mathcal{T}^\pm_{A,[K,L]}(\zeta)\cdot q^{\pm 2\alpha D_A}\bigr),$$
$$\mathcal{T}^\pm_{A,[K,L]}(\zeta) = \mathcal{L}^\pm_{A,L}(\zeta)\cdots\mathcal{L}^\pm_{A,K}(\zeta),$$

where $\pm 2D_A = \varpi^\pm_A(h_1)$ and

$$\mathcal{L}^\pm_{A,j}(\zeta/\xi) = \bigl(\varpi^\pm_{A,\zeta}\otimes\pi_{j,\xi}\bigr)\mathcal{R}. \tag{4}$$

In the above, Tr is understood as analytic continuation from $|q^{\pm 2\alpha}| < 1$, e.g., $\mathrm{Tr}_{W^\pm}(q^{2\alpha D}) = \pm 1/(1-q^{2\alpha})$.

The Q-matrices commute among themselves as well as with $T_{[K,L]}(\zeta,\alpha)$ and $S_{[K,L]}$,

$$\bigl[Q^\epsilon_{[K,L]}(\zeta,\alpha), Q^{\epsilon'}_{[K,L]}\bigl(\zeta',\alpha\bigr)\bigr] = 0 \quad \bigl(\forall\zeta,\zeta'\bigr),$$
$$\bigl[Q^\epsilon_{[K,L]}(\zeta,\alpha), T_{[K,L]}\bigl(\zeta',\alpha\bigr)\bigr] = 0, \qquad \bigl[Q^\epsilon_{[K,L]}(\zeta,\alpha), S_{[K,L]}\bigr] = 0.$$

In fact, the transfer matrix and its 'higher' analogs are all expressible as quadratic combinations of the Q-matrices. For instance, dropping the suffix $[K,L]$ we have

$$\bigl(q^{\alpha-S} - q^{-\alpha+S}\bigr)\begin{vmatrix} Q^+(q^{-1/2}\zeta,\alpha) & Q^-(q^{-1/2}\zeta,\alpha) \\ Q^+(q^{1/2}\zeta,\alpha) & Q^-(q^{1/2}\zeta,\alpha)\end{vmatrix} = \mathrm{id.},$$
$$\bigl(q^{\alpha-S} - q^{-\alpha+S}\bigr)\begin{vmatrix} Q^+(q^{-1}\zeta,\alpha) & Q^-(q^{-1}\zeta,\alpha) \\ Q^+(q\zeta,\alpha) & Q^-(q\zeta,\alpha)\end{vmatrix} = T(\zeta,\alpha).$$

These 'Wronskian' like relations follow from the analysis of the composition factors of $W^+_{A,\zeta_1}\otimes W^-_{A,\zeta_2}$ [12]. They entail in particular Baxter's TQ relation

$$T(\zeta,\alpha)Q^\pm(\zeta,\alpha) = Q^\pm\bigl(q^{-1}\zeta,\alpha\bigr) + Q^\pm(q\zeta,\alpha), \tag{5}$$

which corresponds to the exact sequence of $U_q\mathfrak{b}$-modules

$$0 \longrightarrow W^\pm_{A,q^{-1}\zeta}[\mp 1] \longrightarrow V_{a,\zeta}\otimes W^\pm_{A,\zeta} \longrightarrow W^\pm_{A,q\zeta}[\pm 1] \longrightarrow 0. \tag{6}$$

Here, for a $U_q\mathfrak{b}$-module W, $W[m]$ means the $U_q\mathfrak{b}$-module structure on W where q^{h_1} acts by $q^m \times q^{h_1}$.

3 Quasi-local Operators

Our main concern in this note is *not* the space of states $V_{[K,L]}$, but rather the space of operators $\mathrm{End}\, V_{[K,L]}$. We wish to consider them all at once by letting $-K, L \to \infty$ while keeping only local operators, i.e., only those elements $\mathcal{O}$ which have finite support. Here, by *support* $\mathrm{supp}\,\mathcal{O}$ of $\mathcal{O}$, we mean the minimal interval $[k_0, l_0] \subset \mathbb{Z}$ such that $\mathcal{O}$ acts as identity on V_j for all $j \notin [k_0, l_0]$. When $\mathrm{supp}\,\mathcal{O} \subset [k, l]$, we indicate this fact by putting a suffix and writing $\mathcal{O}_{[k,l]}$. We shall also say that $\mathcal{O}$ has spin $s \in \mathbb{Z}$ if $\mathbb{S}(\mathcal{O}) = s\mathcal{O}$, where $\mathbb{S}(\cdot) = [S_{(-\infty,\infty)}, \cdot]$.

Let us look at the action of the transfer matrix on an element $\mathcal{O} \in \mathrm{End}\, V_{[K,L]}$,

$$\mathbf{t}^*_{[K,L]}(\zeta,\alpha)(\mathcal{O}) = \mathrm{Tr}_a\left\{\mathcal{T}_{a,[K,L]}(\zeta) q^{\alpha\sigma^3_a} \cdot \mathcal{O} \cdot \mathcal{T}_{a,[K,L]}(\zeta)^{-1}\right\}.$$

It is a simple consequence of the weight-preserving property (3) that, if $\mathrm{supp}\,\mathcal{O} \subset [k,l]$, then

$$\mathbf{t}^*_{[K,L]}(\zeta,\alpha)\left(q^{\alpha(\sigma^3_K+\cdots+\sigma^3_{k-1})}\mathcal{O}_{[k,l]}\right) = q^{\alpha(\sigma^3_K+\cdots+\sigma^3_{k-1})}\mathbf{t}^*_{[k,L]}(\zeta,\alpha)(\mathcal{O}_{[k,l]}). \quad (7)$$

Namely, apart from the 'tail' $q^{\alpha(\sigma^3_K+\cdots+\sigma^3_{k-1})}$, there is a reduction of the action of the operator $\mathbf{t}^*$ to the left of the support $[k,l]$ of the operand $\mathcal{O}_{[k,l]}$. Although there is no such simple reduction to the right, the following stability takes place. Consider the Taylor expansion at $\zeta^2 = 1$,

$$\mathbf{t}^*_{[k,L]}(\zeta,\alpha)(\mathcal{O}_{[k,l]}) = \sum_{p\geq 1} \mathbf{t}^*_{[k,L],p}(\mathcal{O}_{[k,l]}) \cdot \left(\zeta^2 - 1\right)^{p-1} \quad \left(\zeta^2 \to 1\right).$$

Then for each fixed p the coefficient $\mathbf{t}^*_{[k,L],p}(\mathcal{O}_{[k,l]})$ becomes independent of L if L is chosen large enough.

These properties suggest that, instead of naïvely taking $-K, L \to \infty$, it is more natural to introduce a formal element

$$q^{2\alpha S(0)} = \cdots q^{\alpha\sigma^3_{-2}} q^{\alpha\sigma^3_{-1}} q^{\alpha\sigma^3_0}, \qquad S(0) = S_{(-\infty,0]},$$

and to consider expressions of the form

$$q^{2(\alpha-s)S(0)}\mathcal{O}, \quad \mathcal{O} \text{ is local and has spin } s. \quad (8)$$

The shift of α depending on the spin s is introduced for convenience. Let $\mathcal{W}_{\alpha-s,s}$ be the set of all elements (8), and set

$$\mathcal{W}^{(\alpha)} = \bigoplus_{s\in\mathbb{Z}} \mathcal{W}_{\alpha-s,s}.$$

We shall say that an element of $\mathcal{W}^{(\alpha)}$ is a quasi-local operator. We call $q^{2\alpha S(0)} \in \mathcal{W}_{\alpha,0}$ the primary operator. Abusing the language we define the support of (8) to be $\operatorname{supp}\mathcal{O}$.

From the foregoing discussions it is clear that for each $p \geq 1$ the limit

$$\mathbf{t}^*_p\bigl(q^{2(\alpha-s)S(0)}\mathcal{O}_{[k,l]}\bigr) := \lim_{-K,L\to\infty} \mathbf{t}^*_{[K,L],p}\bigl(q^{2(\alpha-s)S_{[K,0]}}\mathcal{O}_{[k,l]}\bigr)$$

has a well-defined meaning as an operator acting on $\mathcal{W}^{(\alpha)}$. We shall use the generating series

$$\mathbf{t}^*(\zeta) = \sum_{p=1}^{\infty} \mathbf{t}^*_p\bigl(\zeta^2-1\bigr)^{p-1}. \tag{9}$$

From the definition it is equally clear that the operators $\{\mathbf{t}^*_p\}_{p\geq 1}$ mutually commute. However we are *not* interested in their diagonalisation. Indeed, the question does not even make sense because their action on $\mathcal{W}^{(\alpha)}$ turns out to be free. They generate one half of the Heisenberg algebra, and we shall use them as a part of operators which create a basis of $\mathcal{W}^{(\alpha)}$ from the primary operator $q^{2\alpha S(0)}$, see Sect. 5 below.

4 Introducing Fermions

In this section we shall introduce fermions which act on the space $\mathcal{W}^{(\alpha)}$.

Going back to the setting of a finite interval $[K, L]$, let us re-examine the derivation of the TQ relation (5). For definiteness we consider only $W^+_{A,\zeta}$ and omit the superfix $+$. The exact sequence (6) tells that, with an appropriate matrix $F_{a,A}$ of base change in $V_{a,\zeta} \otimes W_{A,\zeta}$, the product of the two $\mathcal{L}$ operators (2), (4) can be brought to a block triangular form

$$\begin{aligned}\mathcal{L}_{\{a,A\},j}(\zeta) &= F^{-1}_{a,A}\mathcal{L}_{a,j}(\zeta)\mathcal{L}_{A,j}(\zeta)F_{a,A}\\ &= \begin{pmatrix}\mathcal{L}_{A,j}(q\zeta)q^{-\sigma^3_j/2} & 0\\ * & \mathcal{L}_{A,j}(q^{-1}\zeta)q^{\sigma^3_j/2}\end{pmatrix}_a,\end{aligned}$$

where the suffix a refers to the block structure in V_a. Introducing

$$\mathcal{T}_{\{a,A\},[K,L]}(\zeta) = \mathcal{L}_{\{a,A\},L}(\zeta)\cdots\mathcal{L}_{\{a,A\},K}(\zeta),$$

we consider its action on an element $X \in \mathrm{End}(V_{[K,L]})$ by

$$\begin{aligned}&\mathcal{T}_{\{a,A\},[K,L]}(\zeta)q^{\alpha(\sigma^3_a+2D_A)}\cdot X\cdot\mathcal{T}_{\{a,A\},[K,L]}(\zeta)^{-1}\\ &= \begin{pmatrix}\mathbb{A}_{A,[K,L]}(\zeta,\alpha)(X) & 0\\ \mathbb{C}_{A,[K,L]}(\zeta,\alpha)(X) & \mathbb{D}_{A,[K,L]}(\zeta,\alpha)(X)\end{pmatrix}_a.\end{aligned}$$

If we take the trace of both sides on $V_{a,\zeta} \otimes W_{A,\zeta}$, then we obtain an End $V_{[K,L]}$ version of the TQ relation (5). Here we proceed differently and define a new operator by looking at the lower left block,

$$\mathbf{k}_{[K,L]}(\zeta,\alpha)(X) = \mathrm{Tr}_A\big(\mathbb{C}_{A,[K,L]}(\zeta,\alpha)\cdot \zeta^{\alpha-\mathbb{S}}\big(q^{-2S_{[K,L]}}X\big)\big). \tag{10}$$

For each X, the operator (10) is a rational function in ζ^2 apart from an overall power ζ^α. It has poles at $\zeta^2 = 1, q^{\pm 2}$ in the ζ^2-plane. Hence one can write the partial fraction decomposition

$$\begin{aligned}\mathbf{k}_{[K,L]}(\zeta,\alpha)(X) = \bar{\mathbf{c}}_{[K,L]}(\zeta,\alpha)(X) + \mathbf{c}_{[K,L]}(q\zeta,\alpha)(X) + \mathbf{c}_{[K,L]}\big(q^{-1}\zeta,\alpha\big)(X)\\ + \mathbf{f}_{[K,L]}(q\zeta,\alpha)(X) - \mathbf{f}_{[K,L]}\big(q^{-1}\zeta,\alpha\big)(X),\end{aligned} \tag{11}$$

demanding that $\zeta^2 = 1$ is the only pole of

$$\bar{\mathbf{c}}_{[K,L]}(\zeta,\alpha)(X), \qquad \mathbf{c}_{[K,L]}(\zeta,\alpha)(X), \qquad \mathbf{f}_{[K,L]}(\zeta,\alpha)(X).$$

(There is an ambiguity about how to share the possible polynomial part among them. The prescription is given in [1], Sect. 2.7.) We define further

$$\begin{aligned}\mathbf{b}^*_{[K,L]}(\zeta,\alpha)(X) := \mathbf{f}_{[K,L]}(q\zeta,\alpha)(X) + \mathbf{f}_{[K,L]}\big(q^{-1}\zeta,\alpha\big)(X)\\ - \mathbf{t}^*_{[K,L]}(\zeta,\alpha)\mathbf{f}_{[K,L]}(q\zeta,\alpha)(X).\end{aligned} \tag{12}$$

Notice that the right hand side is the combination which appears in the TQ relation (5). Although we are not able to give a logical explanation to the formula (12), it turns out that this operator enjoys various nice properties.

We supplement (11), (12) by giving two more definitions,

$$\begin{aligned}\mathbf{b}_{[K,L]}(\zeta,\alpha) &= N\mathbb{J}_{[K,L]}\circ \mathbf{c}_{[K,L]}(\zeta,-\alpha)\circ \mathbb{J}_{[K,L]},\\ \mathbf{c}^*_{[K,L]}(\zeta,\alpha) &= -N\mathbb{J}_{[K,L]}\circ \mathbf{b}^*_{[K,L]}(\zeta,-\alpha)\circ \mathbb{J}_{[K,L]},\end{aligned}$$

where $N = q^{-1}(q^{-\alpha+\mathbb{S}+1} - q^{\alpha-\mathbb{S}-1})$ is a normalisation and

$$\mathbb{J}_{[K,L]}(X) = J_{[K,L]}\cdot X\cdot J^{-1}_{[K,L]}, \qquad J_{[K,L]} = \prod_{j=K}^{L}\sigma^1_j$$

is an operator which flips the spin.

It can be shown that the operators $\mathbf{b}_{[K,L]}$, $\mathbf{c}_{[K,L]}$, $\mathbf{b}^*_{[K,L]}$, $\mathbf{c}^*_{[K,L]}$ introduced above have reduction properties similar to (7). The left reduction takes the form

$$\mathbf{x}_{[K,L]}(\zeta,\alpha)\big(q^{(\alpha\pm1)(\sigma^3_K+\cdots+\sigma^3_{k-1})}\mathcal{O}_{[k,l]}\big) = q^{\alpha(\sigma^3_K+\cdots+\sigma^3_{k-1})}\mathbf{x}_{[k,L]}(\zeta,\alpha)(\mathcal{O}_{[k,l]}), \tag{13}$$

where the $+$ sign is chosen for $\mathbf{x} = \mathbf{c}, \mathbf{b}^*$ and $-$ for $\mathbf{x} = \mathbf{b}, \mathbf{c}^*$. (The change of α in (13) is the reason why we introduced the shift in the definition (8) of quasi-local operators.) In addition, $\mathbf{b}^*$, $\mathbf{c}^*$ share with $\mathbf{t}^*$ the same stability properties to the right. For $\mathbf{x} = \mathbf{b}, \mathbf{c}$ the situation is even simpler, since

$$\mathbf{x}_{[K,L]}(\zeta,\alpha)(\mathcal{O}_{[k,l]}) = \mathbf{x}_{[K,l]}(\zeta,\alpha)(\mathcal{O}_{[k,l]}).$$

As it was explained for $\mathbf{t}^*_p$, these properties allow us to consider the limit $-K, L \to \infty$ of $\mathbf{b}_{[K,L]}(\zeta,\alpha)$ and so forth. We end up with the formal series

$$\mathbf{b}(\zeta) = \zeta^{-\alpha} \sum_{p=0}^{\infty} \mathbf{b}_p \bigl(\zeta^2 - 1\bigr)^{-p}, \qquad \mathbf{c}(\zeta) = \zeta^{\alpha} \sum_{p=0}^{\infty} \mathbf{c}_p \bigl(\zeta^2 - 1\bigr)^{-p}, \tag{14}$$

$$\mathbf{b}^*(\zeta) = \zeta^{\alpha-2} \sum_{p=1}^{\infty} \mathbf{b}^*_p \bigl(\zeta^2 - 1\bigr)^{p-1}, \qquad \mathbf{c}^*(\zeta) = \zeta^{-\alpha+2} \sum_{p=1}^{\infty} \mathbf{c}^*_p \bigl(\zeta^2 - 1\bigr)^{p-1}, \tag{15}$$

whose coefficients $\mathbf{b}_p, \mathbf{c}_p, \mathbf{b}^*_p, \mathbf{c}^*_p$ are well-defined operators on $\mathcal{W}^{(\alpha)}$. We shall not use the zeroth coefficients $\mathbf{b}_0, \mathbf{c}_0$ because they are not independent from $\mathbf{b}_p, \mathbf{c}_p$, $p \geq 1$.

5 Properties of Fermions

So far we have introduced the operators

$$\mathbf{t}^*_p, \quad \mathbf{b}_p, \quad \mathbf{c}_p, \quad \mathbf{b}^*_p, \quad \mathbf{c}^*_p \quad (p \geq 1), \tag{16}$$

which act on $\mathcal{W}^{(\alpha)}$ in the following manner:

$$\begin{aligned}
&\mathbf{t}^*_p \;:\; \mathcal{W}_{\alpha-s,s} \longrightarrow \mathcal{W}_{\alpha-s,s}, \\
&\mathbf{c}_p,\ \mathbf{b}^*_p \;:\; \mathcal{W}_{\alpha-s+1,s-1} \longrightarrow \mathcal{W}_{\alpha-s,s}, \\
&\mathbf{b}_p,\ \mathbf{c}^*_p \;:\; \mathcal{W}_{\alpha-s-1,s+1} \longrightarrow \mathcal{W}_{\alpha-s,s}.
\end{aligned}$$

In this section we summarize their basic properties.

Commutation Relations Among the operators in the list (16), $\mathbf{t}^*_p$ are central:

$$\bigl[\mathbf{t}^*_p, \mathbf{x}_{p'}\bigr] = 0 \quad \bigl(p, p' \geq 1,\ \mathbf{x} = \mathbf{t}^*, \mathbf{b}, \mathbf{c}, \mathbf{b}^*, \mathbf{c}^*\bigr). \tag{17}$$

The rest of the operators obey the canonical anti-commutation relations

$$[\mathbf{b}_p, \mathbf{b}_{p'}]_+ = [\mathbf{c}_p, \mathbf{c}_{p'}]_+ = [\mathbf{c}_p, \mathbf{b}_{p'}]_+ = 0, \tag{18}$$

$$\bigl[\mathbf{b}_p, \mathbf{b}^*_{p'}\bigr]_+ = \bigl[\mathbf{c}_p, \mathbf{c}^*_{p'}\bigr]_+ = \delta_{p,p'}, \qquad \bigl[\mathbf{b}_p, \mathbf{c}^*_{p'}\bigr]_+ = \bigl[\mathbf{c}_p, \mathbf{b}^*_{p'}\bigr]_+ = 0, \tag{19}$$

$$\bigl[\mathbf{b}^*_p, \mathbf{b}^*_{p'}\bigr]_+ = \bigl[\mathbf{b}^*_p, \mathbf{c}^*_{p'}\bigr]_+ = \bigl[\mathbf{c}^*_p, \mathbf{c}^*_{p'}\bigr]_+ = 0. \tag{20}$$

The proof of (18), (19) requires quite a heavy computation which occupies a large part of [1]. We give a proof of (20) when the target space is $\mathcal{W}_{\alpha,0}$ in Appendix B.

Later on we shall use (19) in the form of generating series,

$$\left[\mathbf{b}(\zeta), \mathbf{b}^*(\xi)\right]_+ = \psi(\zeta/\xi, -\alpha), \qquad \left[\mathbf{c}(\zeta), \mathbf{c}^*(\xi)\right]_+ = \psi(\zeta/\xi, \alpha),$$

where $\psi(\zeta,\alpha)$ is a Cauchy kernel defined by

$$\psi(\zeta,\alpha) = \frac{1}{2}\zeta^\alpha \frac{\zeta^2+1}{\zeta^2-1}.$$

Support Property By acting with $\mathbf{b}_p, \mathbf{c}_p$ the support of an operator does not enlarge. Namely if $X \in \mathcal{W}^{(\alpha)}$ satisfies $\operatorname{supp} X \subset [k,l]$, then

$$\operatorname{supp} \mathbf{x}_p(X) \subset [k,l] \quad (\mathbf{x} = \mathbf{b}, \mathbf{c}), \tag{21}$$

$$\mathbf{x}_p(X) = 0 \quad \text{if } p > l-k+1 \ (\mathbf{x} = \mathbf{b}, \mathbf{c}). \tag{22}$$

In particular, we have

$$\mathbf{b}_p\left(q^{2\alpha S(0)}\right) = 0, \qquad \mathbf{c}_p\left(q^{2\alpha S(0)}\right) = 0. \tag{23}$$

These properties justify calling $\mathbf{b}_p, \mathbf{c}_p$ annihilation operators.

In contrast, the support is enlarged by $\mathbf{t}^*_p, \mathbf{b}^*_p, \mathbf{c}^*_p$ according to the rule

$$\operatorname{supp} \mathbf{x}^*_p(X) \subset [k, l+p] \quad \left(\mathbf{x}^* = \mathbf{t}^*, \mathbf{b}^*, \mathbf{c}^*\right). \tag{24}$$

We call $\mathbf{t}^*_p, \mathbf{b}^*_p, \mathbf{c}^*_p$ creation operators.

Fermionic Basis The following set is a basis of $\mathcal{W}^{(\alpha)}$ [11]:

$$\left(\mathbf{t}^*_1\right)^p \mathbf{t}^*_{i_1} \cdots \mathbf{t}^*_{i_r} \mathbf{b}^*_{j_1} \cdots \mathbf{b}^*_{j_s} \mathbf{c}^*_{k_1} \cdots \mathbf{c}^*_{k_t} \left(q^{2\alpha S(0)}\right)$$
$$(i_1 \geq \cdots \geq i_r \geq 2,\ j_1 > \cdots > j_s \geq 1,\ k_1 > \cdots > k_t \geq 1,\ p \in \mathbb{Z},\ r,s,t \geq 0). \tag{25}$$

Hence $\mathcal{W}^{(\alpha)}$ may be regarded as a tensor product of Fock spaces of one boson and two kinds of fermions. (However we do not know how to construct the annihilation partner to $\mathbf{t}^*_p$.)

As we shall explain in the next section, it is in this fermionic basis that the calculation of expectation values simplify drastically.

6 Expectation Values

We now move on to the discussion of expectation values in the six vertex model. Dealing with the infinite lattice limit one has to be specific about the range of the parameters. From now on we assume that $q = e^{\pi i \nu}$, $1/2 < \nu < 1$, $\nu \notin \mathbb{Q}$.

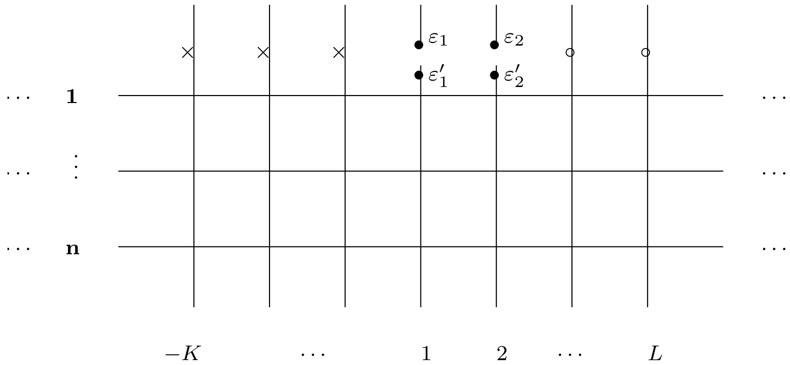

Fig. 1 Six vertex model with fields $\times = q^{(\kappa+\alpha)\sigma_j^3}$, $\circ = q^{\kappa\sigma_j^3}$. Insertion of a local field $E_1^{\varepsilon_1',\varepsilon_1} E_2^{\varepsilon_2',\varepsilon_2}$ corresponds to introducing defects (*filled circles*)

Let us consider an infinite cylinder extending to the horizontal direction. We take finitely many rows, numbered say from 1 to n, and denote them collectively by M (the letter M stands for 'Matsubara'). To each row $m = 1, \dots, n$ attach a parameter τ_m and set

$$\mathcal{T}_{[K,L],M}(\zeta) = \mathcal{T}_{K,M}(\zeta) \cdots \mathcal{T}_{L,M}(\zeta),$$
$$\mathcal{T}_{j,M}(\zeta) = \mathcal{L}_{j,n}(\zeta/\tau_n) \cdots \mathcal{L}_{j,1}(\zeta/\tau_1).$$

Further, on each vertical edge j between the n-th and the $(n+1)$-st row, i.e., the first row in the cyclic boundary condition, we assign a 'field' $q^{(\kappa+\alpha)\sigma_j^3}$, $j \le 0$ or $q^{\kappa\sigma_j^3}$, $j > 0$ (see Fig. 1).

We introduce the expectation value of a quasi-local operator $q^{2\alpha S(0)}\mathcal{O}$ as the limit of the ratio

$$Z^\kappa\left\{q^{2\alpha S(0)}\mathcal{O}\right\} = \lim_{-K,L\to\infty} \frac{\mathrm{Tr}_{[K,L],M}\{\mathcal{T}_{[K,L],M}(1) q^{2\alpha S_{[K,0]}+2\kappa S_{[K,L]}}\mathcal{O}\}}{\mathrm{Tr}_{[K,L],M}\{\mathcal{T}_{[K,L],M}(1) q^{2\alpha S_{[K,0]}+2\kappa S_{[K,L]}}\}}. \tag{26}$$

It is a linear functional $Z^\kappa : \mathcal{W}^{(\alpha)} \to \mathbb{C}$ so normalised that $Z^\kappa\{q^{2\alpha S(0)}\} = 1$. The numerator which appears in the right hand side is the partition function corresponding to a lattice with 'defects' specified by $\mathcal{O}$.

It is convenient to introduce a slightly more general object than Z^κ. Consider the transfer matrix corresponding to a column

$$T_M(\zeta,\kappa) = \mathrm{Tr}_j\left\{\mathcal{T}_{j,M}(\zeta) q^{\kappa\sigma_j^3}\right\}.$$

We call it the 'Matsubara' transfer matrix. Fix an eigencovector $\langle\Phi|$ of $T_M(\zeta,\kappa+\alpha)$ (resp. eigenvector $|\Psi\rangle$ of $T_M(\zeta,\kappa)$) with eigenvalue $T(\zeta,\kappa+\alpha)$ (resp. $T(\zeta,\kappa)$),

$$\langle\Phi|T_M(\zeta,\kappa+\alpha) = \langle\Phi|T(\zeta,\kappa+\alpha), \qquad T_M(\zeta,\kappa)|\Psi\rangle = T(\zeta,\kappa)|\Psi\rangle.$$

We assume that

$$\langle \Phi | \Psi \rangle \neq 0, \tag{27}$$

and in particular that they have the same spin. For an element $q^{2\alpha S(0)}\mathcal{O} \in \mathcal{W}_{\alpha,0}$, choose $-K, L > 0$ so that $\mathrm{supp}\,\mathcal{O} \subset [K, L]$, and set

$$Z^{\kappa}_{\Phi,\Psi}\left\{q^{2\alpha S(0)}\mathcal{O}\right\} = \frac{\langle \Phi | \mathrm{Tr}_{[K,L]}\{\mathcal{T}_{[K,L],M}(1) q^{2\kappa S_{[K,L]}+2\alpha S_{[K,0]}}\mathcal{O}\}|\Psi\rangle}{\langle \Phi|\Psi\rangle\ T(1,\kappa+\alpha)^{-K+1}T(1,\kappa)^{L}}. \tag{28}$$

Then the right hand side is independent of the choice of K, L. If $\langle \Phi| = \langle \kappa+\alpha|$, $|\Psi\rangle = |\kappa\rangle$ are the maximal eigenvectors, then by the Perron-Frobenius theorem the expectation value (26) considered above reduces to (28):

$$Z^{\kappa}\left\{q^{2\alpha S(0)}\mathcal{O}\right\} = Z^{\kappa}_{\langle\kappa+\alpha|,|\kappa\rangle}\left\{q^{2\alpha S(0)}\mathcal{O}\right\}.$$

It is not difficult to see that

$$Z^{\kappa}_{\Phi,\Psi}\left\{\mathbf{t}^*(\zeta)\big(q^{2\alpha S(0)}\big)\right\} = 2\rho(\zeta),$$

where

$$\rho(\zeta) = \frac{T(\zeta,\kappa+\alpha)}{T(\zeta,\kappa)}.$$

A more interesting example of (28) is

$$Z^{\kappa}_{\Phi,\Psi}\left\{\mathbf{b}^*(\zeta)\mathbf{c}^*(\xi)\big(q^{2\alpha S(0)}\big)\right\} = \omega(\zeta,\xi).$$

The function $\omega(\zeta,\xi)$ is determined from the data about the eigenvectors $\langle\Phi|$, $|\Psi\rangle$. An explicit formula for $\omega(\zeta,\xi)$ is given in Appendix A.

We are now in a position to state the 'Ward identities' regarding the expectation values.

Theorem 1 *For any $X \in \mathcal{W}^{(\alpha)}$ the following relations hold*:

$$Z^{\kappa}_{\Phi,\Psi}\left\{\mathbf{t}^*(\zeta)(X)\right\} = 2\rho(\zeta)Z^{\kappa}_{\Phi,\Psi}\{X\},$$

$$Z^{\kappa}_{\Phi,\Psi}\left\{\mathbf{b}^*(\zeta)(X)\right\} = \mathrm{res}_{\xi^2=1}\,\omega(\zeta,\xi)Z^{\kappa}_{\Phi,\Psi}\left\{\mathbf{c}(\xi)(X)\right\}\frac{d\xi^2}{\xi^2},$$

$$Z^{\kappa}_{\Phi,\Psi}\left\{\mathbf{c}^*(\zeta)(X)\right\} = -\mathrm{res}_{\xi^2=1}\,\omega(\xi,\zeta)Z^{\kappa}_{\Phi,\Psi}\left\{\mathbf{b}(\xi)(X)\right\}\frac{d\xi^2}{\xi^2}.$$

In the left hand side, we have the action of creation operators. In the right hand side, it becomes reduced to that of the annihilation operators. The proof given in [2] makes use of a q-difference analog of Abelian integrals on hyperelliptic Riemann surfaces.

The formulas in Theorem 1, combined with the annihilation property (23), allow one to calculate the expectation values of operators inductively. We thus arrive at the following main result of [2], which says that the calculation of $Z^{\kappa}_{\Phi,\Psi}$ on the fermionic base vectors in (25) can be performed by using the ordinary Wick theorem for fermions.

Theorem 2 *Notation being as above, we have*

$$Z^{\kappa}_{\Phi,\Psi}\big\{\mathbf{t}^*(\zeta^0_1)\cdots\mathbf{t}^*(\zeta^0_r)\mathbf{b}^*(\zeta^+_1)\cdots\mathbf{b}^*(\zeta^+_s)\mathbf{c}^*(\zeta^-_t)\cdots\mathbf{c}^*(\zeta^-_1)\big(q^{2\alpha S(0)}\big)\big\}$$
$$=\prod_{j=1}^{r}2\rho(\zeta^0_j)\times\delta_{s,t}\det\big(\omega(\zeta^+_k,\zeta^-_l)\big)_{1\le k,l\le s}.$$

7 Concluding Remarks

In this article we have outlined the construction of fermions acting on the space $\mathcal{W}^{(\alpha)}$ of quasi-local operators. In this basis the expectation values take a very simple form (Theorem 2).

As long as the number of sites n in the Matsubara direction is kept finite, $\rho(\zeta)$ and $(\zeta/\xi)^{-\alpha}\omega(\zeta,\xi)$ are rational functions (see Appendix A below). The main virtue of such a formula is that, in passing to various limits, it is enough to do that for these two functions alone. For example, for the ground state average in the XXZ spin chain, the limit $n\to\infty$ can be taken in a straightforward manner. Moreover, on the infinite lattice any operator of the form $\mathbf{t}^*_p(X)$ $(p\ge 2)$ has vanishing expectation value, so $\rho(\zeta)$ does not appear in the result. This explains the fact that the original multiple integral formula can be simplified using only one transcendental function (the limit of $\omega(\zeta,\xi)$).

In a sense, the main formula is only an existence theorem, since the transition matrix between the standard basis and the fermionic basis remains unknown in general. Nevertheless, it has non-trivial implications in the continuous limit to conformal field theory and the sine-Gordon theory. For these topics the reader is referred to [15–18].

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Appendix A: Formula for $\omega(\zeta,\xi)$

We quote an explicit formula for the function $\omega(\zeta,\xi)$ from [2], Sect. 7. For that purpose we need to prepare some notation.

As in the text, we fix an eigencovector $\langle\Phi|$ of $T_M(\zeta,\kappa+\alpha)$ and an eigenvector $|\Psi\rangle$ of $T_M(\zeta,\kappa)$ satisfying $\langle\Phi|\Psi\rangle\neq 0$. Denote their eigenvalues and those for the Q-matrices as follows:

$$\langle\Phi|T_M(\zeta,\kappa+\alpha)=\langle\Phi|T(\zeta,\kappa+\alpha),\qquad \langle\Phi|Q_M^{\pm}(\zeta,\kappa+\alpha)=\langle\Phi|Q^{\pm}(\zeta,\kappa+\alpha),$$
$$T_M(\zeta,\kappa)|\Psi\rangle=T(\zeta,\kappa)|\Psi\rangle,\qquad Q_M^{\pm}(\zeta,\kappa)|\Psi\rangle=Q^{\pm}(\zeta,\kappa)|\Psi\rangle.$$

Introduce q-difference operators Δ_ζ, $\overline{D}_\zeta$ by

$$\Delta_\zeta F(\zeta)=F(q\zeta)-F\bigl(q^{-1}\zeta\bigr),$$
$$\overline{D}_\zeta F(\zeta)=F(q\zeta)+F\bigl(q^{-1}\zeta\bigr)-2\rho(\zeta)F(\zeta).$$

Hereafter we shall use the shorthand

$$T(\zeta)=T(\zeta,\kappa),\qquad \tilde{T}(\zeta)=T(\zeta,\kappa+\alpha),$$
$$Q^{\pm}(\zeta)=Q^{\pm}(\zeta,\kappa),\qquad \tilde{Q}^{\pm}(\zeta)=Q^{\pm}(\zeta,\kappa+\alpha),$$
$$\psi(\zeta)=\psi(\zeta,\alpha).$$

Set

$$a(\zeta)=\prod_{m=1}^{n}\bigl(1-q^2\zeta^2/\tau_m^2\bigr),\qquad d(\zeta)=\prod_{m=1}^{n}\bigl(1-\zeta^2/\tau_m^2\bigr),$$
$$\varphi(\zeta)=\bigl(a(\zeta)d(\zeta)\bigr)^{-1}, \tag{29}$$

and define $\omega_{\mathrm{sym}}(\zeta,\xi)$ by

$$\begin{aligned}T(\zeta)T(\xi)\omega_{\mathrm{sym}}(\zeta,\xi)&=\bigl(4a(\xi)d(\zeta)-T(\zeta)T(\xi)\bigr)\psi(q\zeta/\xi)\\&\quad-\bigl(4a(\zeta)d(\xi)-T(\zeta)T(\xi)\bigr)\psi\bigl(q^{-1}\zeta/\xi\bigr)\\&\quad-2\bigl(T(\zeta)\tilde{T}(\xi)-T(\xi)\tilde{T}(\zeta)\bigr)\psi(\zeta/\xi).\end{aligned}$$

As a function of ζ, $\omega(\zeta,\xi)$ is characterised by the following two conditions.

1. $\zeta^{-\alpha}T(\zeta)(\omega(\zeta,\xi)-\omega_{\mathrm{sym}}(\zeta,\xi))$ is a polynomial in ζ^2 of degree n,
2. It satisfies the normalisation conditions for $m=0,1,\dots,n$:

$$\int_{\Gamma_m}T(\zeta)\bigl(\omega(\zeta,\xi)+\overline{D}_\zeta\overline{D}_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi)\bigr)\tilde{Q}^{-}(\zeta)Q^{+}(\zeta)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}=0.$$

Here Γ_0 is a contour around $\zeta^2=0$, and for $m=1,\dots,n$, Γ_m is a contour encircling $\zeta^2=\tau_m^2, q^{-2}\tau_m^2$.

As it is explained in [2], Sect. 5, the integral in (ii) does not depend on a particular choice of the 'q-primitive' $\Delta_\zeta^{-1}\psi(\zeta/\xi)$.

To be more explicit, consider the function

$$\begin{aligned}r^+(\zeta,\xi) = & T(\zeta)\Delta_\zeta^{-1}\big(\big(T(\zeta)-T(\xi)\big)\psi(\zeta/\xi)\big)\\ & +\tilde T(\zeta)\Delta_\zeta^{-1}\big(\big(\tilde T(\zeta)-\tilde T(\xi)\big)\psi(\zeta/\xi)\big)\\ & -T(\zeta)\Delta_\zeta^{-1}\big(\big(\tilde T(q\zeta)-\tilde T(\xi)\big)\psi(q\zeta/\xi)\big)\\ & -\tilde T(\zeta)\Delta_\zeta^{-1}\big(\big(T\big(q^{-1}\zeta\big)-T(\xi)\big)\psi\big(q^{-1}\zeta/\xi\big)\big)\\ & +\big(a(q\zeta)-a(\xi)\big)d(\zeta)\psi(q\zeta/\xi)\\ & -a(\zeta)\big(d\big(q^{-1}\zeta\big)-d(\xi)\big)\psi\big(q^{-1}\zeta/\xi\big).\end{aligned}$$

Then it has the form

$$(\zeta/\xi)^{-\alpha}r^+(\zeta,\xi)=\sum_{m=0}^{n}p_m^+\big(\zeta^2\big)\xi^{2m},$$

where $p_m^+(\zeta^2)$ is a polynomial in ζ^2 of degree $2n$. Using them we introduce $(n+1)\times(n+1)$ matrices $\mathcal{A}$, $\mathcal{B}$ by

$$\mathcal{A}_{i,j}=\int_{\Gamma_i}\zeta^{\alpha+2j}\,\tilde Q^-(\zeta)Q^+(\zeta)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2},$$

$$\mathcal{B}_{i,j}=\int_{\Gamma_i}\zeta^{\alpha}p_j^+\big(\zeta^2\big)\tilde Q^-(\zeta)Q^+(\zeta)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}.$$

The formula for $\omega(\zeta,\xi)$ reads

$$\omega(\zeta,\xi)=\frac{4}{T(\zeta)T(\xi)}{}^t\mathbf{v}^+(\zeta)\cdot\mathcal{A}^{-1}\mathcal{B}\cdot\mathbf{v}^-(\xi)+\omega_{\mathrm{sym}}(\zeta,\xi),$$

where $\mathbf{v}^\pm(\zeta)$ denote column vectors with entries $\mathbf{v}^\pm(\zeta)_j=\zeta^{\pm\alpha+2j}$.

For the purpose of studying various limits, it is more convenient to use an alternative expression in terms of solutions to integral equations [19]. The relevant formula can be found in [15], (3.11) (the function $\omega(\zeta,\xi)$ in the present paper is denoted $\omega_{\mathrm{rat}}(\zeta,\xi)$ there, see [15], (2.11)). In this connection one should mention the recent paper [20] where a Riemann-Hilbert problem has been formulated.

Appendix B: Anti-commutativity of Fermionic Creation Operators

In this appendix we prove the following anti-commutation relations between the creation operators.

Theorem 3 *For all* $p, p' \geq 1$, *we have*

$$\big[\mathbf{b}^*_p, \mathbf{b}^*_{p'}\big]_+ = 0 \quad on\ \mathcal{W}_{\alpha+2,-2}, \qquad \big[\mathbf{c}^*_p, \mathbf{c}^*_{p'}\big]_+ = 0 \quad on\ \mathcal{W}_{\alpha-2,2}, \tag{30}$$

$$\big[\mathbf{b}^*_p, \mathbf{c}^*_{p'}\big]_+ = 0 \quad on\ \mathcal{W}_{\alpha,0}. \tag{31}$$

Since the proofs are similar, we shall concentrate on the case (31).

The next Proposition says that the anti-commutation relation (31) holds in the sense of expectation values.

Proposition 1 *Assume* (27). *Then for any* $X \in \mathcal{W}_{\alpha,0}$ *we have*

$$Z^{\kappa}_{\Phi,\Psi}\big\{\big[\mathbf{b}^*_p, \mathbf{c}^*_{p'}\big]_+(X)\big\} = 0 \quad \big(\forall p, p' \geq 1\big).$$

Proof Abbreviating $Z^{\kappa}_{\Phi,\Psi}$ to Z, we apply the Ward identities for the expectation values in Theorem 1,

$$\begin{aligned}
&Z\big\{\mathbf{b}^*(\zeta_1)\mathbf{c}^*(\zeta_2)(X)\big\} \\
&\quad = \mathrm{res}_{\xi_1^2=1}\omega(\zeta_1,\xi_1)Z\big\{\mathbf{c}(\xi_1)\mathbf{c}^*(\zeta_2)(X)\big\}\frac{d\xi_1^2}{\xi_1^2} \\
&\quad = \mathrm{res}_{\xi_1^2=1}\omega(\zeta_1,\xi_1)\big(-Z\big\{\mathbf{c}^*(\zeta_2)\mathbf{c}(\xi_1)(X)\big\} + \psi(\xi_1/\zeta_2,\alpha)\big)\frac{d\xi_1^2}{\xi_1^2} \\
&\quad = \mathrm{res}_{\xi_1^2,\xi_2^2=1}\omega(\zeta_1,\xi_1)\omega(\zeta_2,\xi_2)Z\big\{\mathbf{b}(\xi_2)\mathbf{c}(\xi_1)(X)\big\}\frac{d\xi_1^2}{\xi_1^2}\frac{d\xi_2^2}{\xi_2^2} + \omega(\zeta_1,\zeta_2).
\end{aligned}$$

In the second line we used the known anti-commutation relations between the creation and annihilation operators.

Similarly one calculates

$$\begin{aligned}
&Z\big\{\mathbf{c}^*(\zeta_2)\mathbf{b}^*(\zeta_1)(X)\big\} \\
&\quad = \mathrm{res}_{\xi_1^2,\xi_2^2=1}\omega(\zeta_2,\xi_2)\omega(\zeta_1,\xi_1)Z\big\{\mathbf{c}(\xi_1)\mathbf{b}(\xi_2)(X)\big\}\frac{d\xi_1^2}{\xi_1^2}\frac{d\xi_2^2}{\xi_2^2} - \omega(\zeta_1,\zeta_2).
\end{aligned}$$

Using the known anti-commutativity of $\mathbf{b}(\xi_2)$ and $\mathbf{c}(\xi_1)$, we arrive at

$$Z\big\{\mathbf{b}^*(\zeta_1)\mathbf{c}^*(\zeta_2)(X)\big\} = -Z\big\{\mathbf{c}^*(\zeta_2)\mathbf{b}^*(\zeta_1)(X)\big\},$$

which is equivalent to the assertion of Proposition. □

Before proceeding, we recall a few facts from the algebraic Bethe ansatz. Normalising the $\mathcal{L}$ operator as

$$\mathcal{L}(\zeta)=\begin{pmatrix}1-q^2\zeta^2 &&&\\ & (1-\zeta^2)q & (1-q^2)\zeta & \\ & (1-q^2)\zeta & (1-\zeta^2)q & \\ &&& 1-q^2\zeta^2\end{pmatrix},$$

we set

$$\mathcal{L}_{a,n}(\zeta/\tau_n)\cdots\mathcal{L}_{a,1}(\zeta/\tau_1)=\begin{pmatrix}A(\zeta) & B(\zeta)\\ C(\zeta) & D(\zeta)\end{pmatrix}_a .$$

Let $|0\rangle=v_+^{\otimes n}$, $\langle 0|=(v_+^*)^{\otimes n}$ be the reference vector and covector respectively, where v_+, v_- is the standard basis of $\mathbb{C}^2$ and v_+^*, v_-^* is the dual basis. Let further $l\in\{0,1,\dots,n\}$ and set for $j=1,\dots,l$

$$F_j(\xi_1,\dots,\xi_l)=a(\xi_j)\prod_{i=1}^{l}\bigl(\xi_i^2-q^{-2}\xi_j^2\bigr)+q^{-2\kappa+n-2l}d(\xi_j)\prod_{i=1}^{l}\bigl(\xi_i^2-q^2\xi_j^2\bigr),$$

where $a(\zeta)$, $d(\zeta)$ are defined in (29).

The following formula is well known [14].

Proposition 2 *Assume that* $(\xi_1,\dots,\xi_l)\in(\mathbb{C}^\times)^l$ *is a solution of the Bethe equation*

$$F_j(\xi_1,\dots,\xi_l)=0\quad (j=1,\dots,l), \tag{32}$$

and let $(\zeta_1,\dots,\zeta_l)\in(\mathbb{C}^\times)^l$ *be arbitrary. Then*

$$\langle 0|\prod_{j=1}^{l}C(\zeta_j)\prod_{j=1}^{l}B(\xi_j)|0\rangle=q^{-l(l-1-n)}\bigl(q-q^{-1}\bigr)^l \times\frac{\prod_{j=1}^{l}\zeta_j\xi_j d(\xi_j)}{\prod_{1\le i<j\le l}(\xi_i^2-\xi_j^2)(\zeta_j^2-\zeta_i^2)}\det(\Omega_{j,k})_{1\le j,k\le l},$$

$$\Omega_{j,k}=\frac{a(\zeta_k)\prod_{i=1}^{l}(q^2\xi_i^2-\zeta_k^2)}{(\xi_j^2-\zeta_k^2)(q^2\xi_j^2-\zeta_k^2)}-q^{-2\kappa+n}\frac{d(\zeta_k)\prod_{i=1}^{l}(\xi_i^2-q^2\zeta_k^2)}{(\xi_j^2-\zeta_k^2)(\xi_j^2-q^2\zeta_k^2)}.$$

We shall consider the specialisation of parameters q, $\boldsymbol{\tau}=(\tau_1,\dots,\tau_n)$ to

$$q_0=e^{\pi i/2},\qquad \boldsymbol{\tau}_0=(1,\dots,1).$$

Lemma 1 *Define* $x_j(\kappa)$ *by*

$$\frac{1-x_j(\kappa)}{1+x_j(\kappa)}=-e^{-\frac{\pi i}{n}(\kappa-\frac{n}{2}+l+2j)}\quad (j=1,\dots,n).$$

Then, for any subset $I = \{i_1, \ldots, i_l\} \subset \{1, \ldots, n\}$, $i_1 < \cdots < i_l$,

$$\left(\xi_1^2, \ldots, \xi_l^2\right) = \left(x_{i_1}(\kappa), \ldots, x_{i_l}(\kappa)\right) \tag{33}$$

is a solution of (32) *for* $(q, \boldsymbol{\tau}) = (q_0, \boldsymbol{\tau}_0)$. *If further* κ *is generic, then we have* $\xi_j^2 \neq \pm 1$, $\xi_j^2 \neq \pm \xi_k^2$ $(j \neq k)$.

Proof is straightforward.

Hereafter we choose and fix a generic κ_0. Denote by $\boldsymbol{\xi}_0^{(I)}$ the solution (33) at $(\kappa, q, \boldsymbol{\tau}) = (\kappa_0, q_0, \boldsymbol{\tau}_0)$.

Lemma 2 *We have*

$$\det\left(\frac{\partial F_j}{\partial \xi_k^2}\left(\boldsymbol{\xi}_0^{(I)}\right)\right) \neq 0.$$

Proof This follows from the calculation

$$\frac{\partial F_j}{\partial \xi_k^2}\left(\boldsymbol{\xi}_0^{(I)}\right) = \delta_{j,k} \cdot \frac{2n}{1 - \xi_j^4} a(\xi_j) \prod_{i=1}^{l} \left(\xi_i^2 + \xi_j^2\right),$$

where (32) is used. □

By Lemma 2 and the implicit function theorem, in a neighborhood of $(\kappa, q, \boldsymbol{\tau}) = (\kappa_0, q_0, \boldsymbol{\tau}_0)$ there exists a unique branch $\boldsymbol{\xi}^{(I)}(\kappa, q, \boldsymbol{\tau}) = \{\xi_1^2, \ldots, \xi_l^2\}$ of solutions to (32) such that $\boldsymbol{\xi}^{(I)}(\kappa_0, q_0, \boldsymbol{\tau}_0) = \boldsymbol{\xi}_0^{(I)}$. Denote by

$${}_I\langle \kappa, q, \boldsymbol{\tau}| = \langle 0| \prod_{j=1}^{l} C(\xi_j), \qquad |\kappa, q, \boldsymbol{\tau}\rangle_I = \prod_{j=1}^{l} B(\xi_j)|0\rangle$$

the corresponding Bethe (co)vectors.

Lemma 3 *In a neighborhood of* $(\kappa_0, q_0, \boldsymbol{\tau}_0)$, *we have*

$${}_I\left\langle \kappa', q, \boldsymbol{\tau} | \kappa, q, \boldsymbol{\tau}\right\rangle_J \neq 0 \quad \left(\kappa' \neq \kappa\right)$$

for all $I, J \subset \{1, \ldots, n\}$ *with* $\sharp I = \sharp J = l$.

Proof We apply Proposition 2 at $(q, \boldsymbol{\tau}) = (q_0, \boldsymbol{\tau}_0)$. Setting $\boldsymbol{\xi}^{(I)}(\kappa, q_0, \boldsymbol{\tau}_0) = (\xi_1, \ldots, \xi_l)$ and $\boldsymbol{\xi}^{(J)}(\kappa', q_0, \boldsymbol{\tau}_0) = (\zeta_1, \ldots, \zeta_l)$ we find

$$\begin{aligned} &{}_I\left\langle \kappa', q_0, \boldsymbol{\tau}_0 | \kappa, q_0, \boldsymbol{\tau}_0\right\rangle_J \\ &\quad = 2^l i^{-l(l-2-n)} \left(1 - e^{\pi i(\kappa' - \kappa)}\right)^l \prod_{p=1}^{l} \left(\xi_p \zeta_p d(\xi_p) a(\zeta_p)\right) \frac{\prod_{j<k} (\xi_j^2 + \xi_k^2)(\zeta_k^2 + \zeta_j^2)}{\prod_{j,k=1}^{l} (\xi_j^2 + \zeta_k^2)} \end{aligned}$$

which is non-zero. Hence the scalar product does not vanish in some neighborhood of $(\kappa_0, q_0, \boldsymbol{\tau}_0)$ and $\kappa' \neq \kappa$. □

We finish the proof with the

Proposition 3 *For any $p, p' \geq 1$ and $X \in \mathcal{W}_{\alpha,0}$ we have*

$$\left[\mathbf{b}^*_p, \mathbf{c}^*_{p'}\right]_+(X) = 0. \tag{34}$$

Proof Denote the left hand side of (34) by Y. Take $(\kappa, q, \boldsymbol{\tau})$ in a neighborhood of $(\kappa_0, q_0, \boldsymbol{\tau}_0)$ and $\alpha \neq 0$ small enough. Choose $\langle \Phi| = {}_I\langle \kappa + \alpha, q, \boldsymbol{\tau}|$ and $|\Psi\rangle = |\kappa, q, \boldsymbol{\tau}\rangle_J$, where $\sharp I = \sharp J = l$ and $0 \leq l \leq n$. Under the assumption above, we have $\langle \Phi|\Psi\rangle \neq 0$ by Lemma 3. Hence Proposition 1 is applicable, and we obtain that

$$\langle \Phi|\mathrm{Tr}_{[K,L]}\left\{\mathcal{T}_{[K,L],M}(1) q^{2\kappa S_{[K,L]}} Y\right\}|\Psi\rangle = 0.$$

Since the vectors $\{{}_I\langle \kappa + \alpha, q, \boldsymbol{\tau}|\}$, $\{|\kappa, q, \boldsymbol{\tau}\rangle_I\}$ are bases of the spin $n/2 - l$ subspace, we find

$$\mathrm{Tr}_{[K,L]}\left\{\mathcal{T}_{[K,L],M}(1) q^{2\kappa S_{[K,L]}} Y\right\} = 0.$$

If we choose $n = L - K + 1$ and $\boldsymbol{\tau} = \boldsymbol{\tau}_0$, then $\mathcal{T}_{[K,L],M}(1)$ becomes a permutation operator and the trace becomes simply $q^{2\kappa S_{[K,L]}} Y$. We conclude that $Y = 0$ provided (q, α) is close enough to $(q_0, 0)$ and $\alpha \neq 0$. But Y is rational in q, q^α, so we must have that $Y = 0$ identically. This completes the proof. □

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The Romance of the Ising Model

Barry M. McCoy

Abstract The essence of romance is mystery. In this talk, given in honor of the 60th birthday of Michio Jimbo, I will explore the meaning of this for the Ising model beginning in 1946 with Bruria Kaufman and Willis Lamb, continuing with the wedding by Jimbo and Miwa in 1980 of the Ising model with the Painlevé VI equation which had been first discovered by Picard in 1889. I will conclude with the current fascination of the magnetic susceptibility and explore some of the mysteries still outstanding.

1 Introduction

A search of Google books reveals that the observation

The essence of romance is mystery

has been made by many authors in many different ways and in many different contexts ranging from the literary to the scientific. But in all contexts romance betokens fascination and the Ising model has fascinated many people, including myself, for many decades and in spite of many breakthroughs and moments of understanding the mystery continues to this day. In this talk I will present some of the milestones of this romance.

2 Kaufman and Lamb

In his talk "The Ising model in two dimensions" [1] presented at the fifth Battelle Colloquium on Materials Science, held in Geneva and Gstaad, Switzerland, September 7–12, 1970, Lars Onsager wrote, following a discussion of his famous 1944 computation of the free energy [2] and a sketch of his 1945 proof of his conjectured spectrum of the transfer matrix,

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> Before long, however, Bruria Kaufman had developed a much better strategy.
>
> At Columbia University she first asked Willis E. Lamb to direct her work on order-disorder problems; but he was much too heavily engaged in an experimental effort, and I was asked to assume the responsibility. Unable to talk her out of the idea I suggested that she explore ... By the summer of 1946 she had a beautifully compact computation of the partition function, bypassing all tedious detail.
>
> By itself that was only a more elegant derivation of an old result but the approach looked powerful enough to produce a few more new ones. Very well, how about correlations?

The history of the Ising model from that time forth has been the study of these correlations.

But the deeper meaning of this passage from Onsager's paper completely escaped me until many years later Rodney Baxter wrote to me concerning a typescript [3] that had been given to him which is certainly a draft of Onsager and Kaufman's calculation of the spontaneous magnetization of the Ising model. Why in the world would Kaufman, who was creating pioneering mathematics, ask Lamb, an experimental physicist, to supervise her research? This question was brought into sharp focus when Baxter told me that he was going to contact her about the authorship of the typescript. She was then living in Tucson, Arizona with her husband, Willis Lamb.

So this is the first romance concerned with the Ising model. Both Bruria and Willis were married to other people in 1946 when Bruria asked Willis to be her research supervisor and he turned her down. But decades later, when Kaufman's husband died in 1992, Lamb invited her to Tucson as a Visiting Scholar at the University of Arizona where he was a professor. In 1996, after his wife died, Willis and Bruria were married.

3 Correlations and Form Factors

The great understanding of Kaufman was that the Ising partition function could be written by use of fermionic methods as the sum of four Pfaffians [4] and that this fermionic method is powerful enough to write all correlation functions of the Ising model as determinants [5].

The Ising model is a system of "spins" $\sigma_{j,k}$ at row j and column k of a square lattice which take on the values $\sigma_{j,k} = \pm 1$ and interact with their nearest neighbors with the interaction energy

$$\mathcal{E} = -\sum_{j=-L^v}^{L^v} \sum_{k=-L^h}^{L^h} \left\{ E^h \sigma_{j,k}\sigma_{j,k+1} + E^v \sigma_{j,k}\sigma_{j+1,k} \right\}. \tag{1}$$

The correlation functions studied by Kaufman and Onsager are defined as

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle = \lim_{L^v,L^h\to\infty} Z^{-1}_{L^v,L^h} \sum_{\sigma=\pm1} \sigma_{0,0}\sigma_{M,N}e^{-\mathcal{E}/k_BT} \tag{2}$$

where T is the temperature, k_B is Boltzmann's constant,

$$Z_{L^v,L^h} = \sum_{\sigma=\pm1} e^{-\mathcal{E}/k_BT} \tag{3}$$

is the partition function and the sum $\sum_{\sigma=\pm1}$ is over all values of the variables $\sigma_{j,k}$.

The discovery of Kaufman and Onsager [5] is that the row and diagonal correlations can be written as a sum of two determinants. These are further simplified by Montroll, Potts and Ward [6] to a single determinant. The diagonal $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ and the row correlations $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ can both be written as $N\times N$ Toeplitz determinants

$$D_N = \begin{vmatrix} a_0 & a_{-1} & \cdots & a_{-N+1} \\ a_1 & a_0 & \cdots & a_{-N+2} \\ \vdots & \vdots & & \vdots \\ a_{N-1} & a_{N-2} & \cdots & a_0 \end{vmatrix} \tag{4}$$

where

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-in\theta}\phi(\theta) \tag{5}$$

with

$$\phi(\theta) = \left[\frac{(1-\alpha_1 e^{i\theta})(1-\alpha_2 e^{-i\theta})}{(1-\alpha_1 e^{-i\theta})(1-\alpha_2 e^{i\theta})}\right]^{1/2}. \tag{6}$$

For $\langle\sigma_{0,0}\sigma_{N,N}\rangle$

$$\alpha_1 = 0, \alpha_2 = \left(\sinh 2E^v/k_BT \sinh 2E^h/k_BT\right)^{-1} \tag{7}$$

and for $\langle\sigma_{0,0}\sigma_{0,N}\rangle$

$$\alpha_1 = e^{-2E^v/k_BT}\tanh E^h/k_BT, \qquad \alpha_2 = e^{-2E^v/k_BT}\coth E^h/k_BT \tag{8}$$

and the square roots are defined to be positive at $\theta=\pi$. These determinants are very efficient for the calculation of the correlations when N is small.

However, when N is large the determinental representation (4) is not an efficient method of calculation and a different representation must be found.

The first step in finding this new representation is the computation of the limiting value as $N\to\infty$

$$\lim_{N\to\infty}\langle\sigma_{0,0}\sigma_{0,N}\rangle = \lim_{N\to\infty}\langle\sigma_{0,0}\sigma_{N,N}\rangle = (1-t)^{1/4} \tag{9}$$

with

$$t = \left(\sinh 2E^v/k_BT \sinh 2E^h/k_BT\right)^{-2}, \tag{10}$$

which is valid for $0 \le t \le 1$. For $t > 1$ the limit vanishes. The value of T for which $t = 1$ is called the critical temperature T_c. It is the evaluation of this limit for $\langle \sigma_{0,0}\sigma_{N,N} \rangle$ which is accomplished by Kaufman and Onsager in the manuscript recently published by Baxter [3].

The next step in the evaluation of the long distance behavior of the correlations was made in 1966 by Wu [7] who computed the first correction $f^{(2)}_{0,N}$ to (9) as $N \to \infty$ for $\langle \sigma_{0,0}\sigma_{0,N} \rangle$ for $T < T_c$ as a two-dimensional integral and the leading behavior $f^{(1)}_{0,N}$ as $N \to \infty$ of $\langle \sigma_{0,0}\sigma_{0,N} \rangle$ for $T > T_c$ as a one-dimensional integral. These are the first terms in what is now called the form factor expansion of the correlation functions, which for general M, N is written for $T < T_c$ as

$$\langle \sigma_{0,0}\sigma_{M,N} \rangle = (1-t)^{1/4} \left\{ 1 + \sum_{n=1}^{\infty} f^{(2n)}_{M,N} \right\} \tag{11}$$

and for $T > T_c$ as

$$\langle \sigma_{0,0}\sigma_{M,N} \rangle = (1-t)^{1/4} \sum_{n=0}^{\infty} f^{(2n+1)}_{M,N}, \tag{12}$$

where for $T > T_c$ we use the definition

$$t = \left(\sinh 2E^v/k_BT \sinh 2E^h/k_BT\right)^{2}. \tag{13}$$

The derivation of the complete expansions (11) and (12) has its own interesting story. In 1976 Wu, McCoy, Tracy and Barouch [8] derived an expansion valid for all N of the correlations in the form for $T < T_c$ of

$$\langle \sigma_{0,0}\sigma_{M,N} \rangle = (1-t)^{1/4} \exp \sum_{n=0}^{\infty} F^{(2n)}_{M,N} \tag{14}$$

and for $T > T_c$

$$\langle \sigma_{0,0}\sigma_{M,N} \rangle = (1-t)^{1/4} \sum_{n=0}^{\infty} G^{(2n+1)}_{M,N} \exp \sum_{n=0}^{\infty} \tilde{F}^{(2n)}_{M,N} \tag{15}$$

where $F^{(2n)}_{M,N}$ and $\tilde{F}^{(2n)}_{M,N}$ are $4n$ dimensional integrals and $G^{(2n+1)}_{M,N}$ are $4n+2$ dimensional integrals. For all three functions half of the integrals may be executed by closing a contour integral on a pole. The forms (14) and (15) of the correlation functions are called the exponential forms.

The form factor expansions (11) and (12) are obtained from the exponential forms (14) and (15) by expanding the exponentials. For a few low values of n this was done in [8] in connection with the study of the magnetic susceptibility but the general results for the $f^{(n)}_{M,N}$ were not given by Nickel [9] and [10] until 1999 and 2000.

A curious feature of the derivation given in [8] of (14) and (15) is that the method of [7] developed for the row correlation $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ is not used; instead the method used by Cheng and Wu [11] in the study of the leading terms of large separation behavior of the general correlation $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ is used. The original method [7] of Wu as applied to the correlations $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ and $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ was extended to all orders in 2007 by Lyberg and McCoy [12]. The results in [12] for the diagonal form factors $f^{(n)}_{N,N}(t)$ are for $T < T_c$

$$f^{(2n)}_{N,N}(t) = \frac{t^{n(N+n)}}{(n!)^2\pi^{2n}}\int_0^1\prod_{k=1}^{2n} dx_k x_k^N \prod_{j=1}^{n}\left(\frac{(1-tx_{2j})(x_{2j}^{-1}-1)}{(1-tx_{2j-1})(x_{2j-1}^{-1}-1)}\right)^{1/2}$$
$$\times\prod_{1\le j\le n}\prod_{1\le k\le n}\left(\frac{1}{1-tx_{2k-1}x_{2j}}\right)^2$$
$$\times\prod_{1\le j<k\le n}(x_{2j-1}-x_{2k-1})^2(x_{2j}-x_{2k})^2, \tag{16}$$

and for $T > T_c$

$$f^{(2n+1)}_{N,N}(t)$$
$$= \frac{t^{(n+1/2)N+n(n+1)}}{n!(n+1)!\pi^{2n+1}}\int_0^1\prod_{k=1}^{2n+1} dx_k\, x_k^N\prod_{j=1}^{n+1} x_{2j-1}^{-1}\left[(1-tx_{2j-1})\left(x_{2j-1}^{-1}-1\right)\right]^{-1/2}$$
$$\times\prod_{j=1}^{n} x_{2j}\left[(1-tx_{2j})\left(x_{2j}^{-1}-1\right)\right]^{1/2}\prod_{1\le j\le n+1}\prod_{1\le k\le n}\left(\frac{1}{1-tx_{2j-1}x_{2k}}\right)^2$$
$$\times\prod_{1\le j<k\le n+1}(x_{2j-1}-x_{2k-1})^2\prod_{1\le j<k\le n}(x_{2j}-x_{2k})^2. \tag{17}$$

A closely related form for the row form factor $f^{(n)}_{0,N}$ is also obtained in [12]. The results (16) and (17) have the startling feature that in the diagonal case the $f^{(n)}_{N,N}$ do not manifestly reduce term by term to the corresponding functions obtained from [8]. The reconciliation of these two forms is one of the present mysteries of the Ising model.

These diagonal form factor integrals, which on the surface may appear to be indigestible, have proven to have many very special properties.

(1) All the integrals in (16) and (17) reduce at $t = 0$ to a product of two special cases of the celebrated Selberg integral [13]

$$\int_0^1\cdots\int_0^1\prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1}\prod_{1\le i<j\le n}|t_i-t_j|^{2\gamma}dt_1\cdots dt_n. \tag{18}$$

(2) In [14] it was discovered by Maple calculations the $f^{(n)}_{N,N}$ satisfy Fuchsian differential equation with a factorized "Russian doll" structure

$$F_{2n} f^{(2n)}_{N,N} = 0 \quad \text{with } F_{2n} = L_{2n+1}(N) \cdots L_3(N) \cdot L_1(N), \tag{19}$$

$$F_{2n+1} f^{(2n+1)} = 0 \quad \text{with } F_{2n+1} = L_{2n+2}(N) \cdots L_4(N) \cdot L_2(N) \tag{20}$$

where $L_j(N)$ are linear differential operators of order j.

(3) It was also discovered in [14] by Maple calculations that the operators F_n have in addition a direct sum decomposition

$$F_{2n} = M_{2n+1}(N) \oplus \cdots \oplus M_3(N) \oplus M_1(N), \tag{21}$$

$$F_{2n+1} = M_{2n+2}(N) \oplus \cdots \oplus M_4(N) \oplus M_2(N). \tag{22}$$

(4) Furthermore, the $f^{(n)}_{N,N}(t)$ have a factorization property first found in [14] by computer computations and proven for $n = 1, 2, 3$ in [15] that

$$f^{(2n)}_{N,N}(t) = \sum_{m=0}^{n-1} K^{(2n)}_m(N) \cdot f^{(2m)}_{N,N}(t) + \sum_{m=0}^{2n} C^{(2n)}_m(N;t) \cdot F^{2n-m}_N \cdot F^m_{N+1}, \tag{23}$$

$$\frac{f^{(2n+1)}_{N,N}(t)}{t^{N/2}} = \sum_{m=0}^{n-1} K^{(2n+1)}_m(N) \cdot \frac{f^{(2m+1)}_{N,N}(t)}{t^{N/2}} + \sum_{m=0}^{2n+1} C^{(2n+1)}_m(N;t) \cdot F^{2n+1-m}_N \cdot F^m_{N+1}, \tag{24}$$

where F_N is the hypergeometric function

$$F_N = {}_2F_1(1/2, N + 1/2; N + 1; t), \tag{25}$$

and $f^{(0)}_{N,N} = 1$. The $K^{(n)}_m(N)$ depend only on N and we note in particular that

$$K^{(3)}_0(0) = \frac{1}{6}, \tag{26}$$

$$K^{(4)}_0(0) = 0, \qquad K^{(4)}_1(0) = \frac{1}{3}, \tag{27}$$

$$K^{(5)}_0(0) = -\frac{1}{120}, \qquad K^{(5)}_1(0) = \frac{1}{2}, \tag{28}$$

$$K^{(6)}_0(0) = 0, \qquad K^{(6)}_1(0) = -\frac{2}{45}, \qquad K^{(6)}_2(0) = \frac{2}{3}. \tag{29}$$

The $C^{(j)}_m(N;t)$ are polynomials in t of degree for $N \geq 1$

$$\deg C^{(2n)}_m(N;t) = \deg C^{(2n+1)}_m(N;t) = n \cdot (2N + 1), \tag{30}$$

with $C_m^{(n)}(N;t) \sim t^m$ as $t \sim 0$. which have the palindromic property

$$C_m^{(2n)}(N;t) = t^{n(2N+1)+m} \cdot C_m^{(2n)}(N;1/t), \tag{31}$$

$$C_m^{(2n+1)}(N;t) = t^{n(2N+1)+m} \cdot C_m^{(2n+1)}(N;1/t). \tag{32}$$

Explicit formulas for the polynomials $C_m^{(n)}(N,t)$ have been obtained in [15] for $n = 1, 2, 3$ and conjectured for $n = 4$. For example $K_0^{(2)} = N/2$ and

$$C_m^{(2)}(N;t) = (-1)^{m+1}\frac{N}{2}\binom{m}{2}\left[\frac{(2N+1)^2}{4N(N+1)}\right]^m t^m \sum_{n=0}^{2N+1-m} c_{m;n}^{(2)}(N)t^n, \tag{33}$$

where for $0 \le n \le N-1$

$$c_{2;n}^{(2)}(N) = c_{2;\,2N-1-n}^{(2)}(N) = \sum_{k=0}^{n} a_k(N)a_{n-k}(N), \tag{34}$$

$$c_{1;n}^{(2)}(N) = c_{1;\,2N-n}^{(2)}(N) = \sum_{k=0}^{n} a_k(N)a_{n-k}(N+1), \tag{35}$$

and for $0 \le n \le N$

$$c_{0;\,n}^{(2)}(N) = c_{0;\,2N+1-n}^{(2)}(N) = c_{2;\,n}^{(2)}(N+1), \tag{36}$$

and

$$c_{1;N}^{(2)}(N) = \left(\frac{(1/2)_N}{N!}\right)^2 \left\{1 + 2NH_N(1/2)\right\} \tag{37}$$

where

$$a_n(N) = \frac{(1/2)_N(1/2-N)_n}{(1-N)_n n!} \tag{38}$$

and

$$H_N(1/2) = \sum_{k=0}^{N-1} \frac{1}{k+1/2}. \tag{39}$$

It is certainly true (but not yet proven) that the factorizations (23) and (24) hold for all $f_{N,N}^{(n)}$. The computations in [15] are based on Fuchsian differential equations for the $f_{N,N}^{(n)}(t)$. For $n = 4$ the order of these equations is 20. These equations have a direct sum decomposition into operators which are homomorphic to symmetric powers and products of the operator which annihilates the hypergeometric function F_N.

It is furthermore very suggestive that this factorization property has been previously seen in the correlation functions of the XXZ model [16–22].

The final property of the form factors to be discussed can best be illustrated by making a “lambda extension”, first introduced in [23], of the expansions (11) and (12) by defining

$$C_{-}(M,N;\lambda) = (1-t)^{1/4}\left\{1+\sum_{n=1}^{\infty}\lambda^{2n} f_{M,N}^{(2n)}\right\} \tag{40}$$

and

$$C_{+}(M,N;\lambda) = (1-t)^{1/4}\sum_{n=0}^{\infty}\lambda^{2n} f_{M,N}^{(2n+1)}, \tag{41}$$

which reduce to the Ising correlations below and above T_c when $\lambda = 1$. By use of a remarkable set of relations presented by Orrick, Nickel, Guttmann and Perk [24] in 2001 for small values of M and N, these lambda extensions can be written in terms of theta functions [25]

$$\theta_3(u;q) = 1+2\sum_{n=1}^{\infty} q^{n^2}\cos 2nu, \tag{42}$$

$$\theta_2(u;q) = 2q^{1/4}\sum_{n=0}^{\infty} q^{n(n+1)}\cos\big[(2n+1)u\big] = q^{1/4}e^{iu}\theta_3(u+\pi\tau/2;q) \tag{43}$$

and their derivatives

$$\frac{d}{du}\theta_n(u;q) \equiv \theta_n'(u;q) \tag{44}$$

where $t^{1/2} = k$ is the modulus of elliptic functions which is related to the nome q by

$$q = e^{-\pi K'(t^{1/2})/K(t^{1/2})} \tag{45}$$

and

$$K\left(t^{1/2}\right) = \frac{\pi}{2}\,{}_2F_1(1/2,1/2;t) \tag{46}$$

is the complete elliptic integral of the first kind with $K'(t^{1/2}) = K((1-t)^{1/2})$.

The simplest example given in [14] is for the low temperature case with $M = N = 0$

$$C_{-}(0,0;\lambda) = \frac{\theta_3(u;q)}{\theta_3(0;q)} \quad \text{where } \lambda = \cos u. \tag{47}$$

For the special values $\lambda = \cos(\pi m/n)$ we find that $C_{-}(0,0;\lambda)$ and t satisfy an algebraic equation. Calling $C_{-}(0,0;\lambda) = \tau$, it is seen in [14] that for $\lambda = \cos\pi/3$

$$16\tau^{12} - 16\tau^{8} - 8(t-1)\tau^{3} + t(1-t) = 0, \tag{48}$$

which is a curve of genus one. For $\lambda = \cos(\pi/4)$,

$$16\tau^{16} + 16(t-1)\tau^8 + t^2(t-1) = 0 \tag{49}$$

is a curve of genus three which has the simple algebraic expression

$$C_-\bigl(0,0;\cos(\pi/4)\bigr) = 2^{-1/4}(1-t)^{1/16}\bigl[1+(1-t)^{1/2}\bigr]^{1/4}. \tag{50}$$

Further results in this direction are [14]

$$C_-\bigl(1,1;\cos(\pi/4)\bigr) = 2^{-3/4}(1-t)^{1/16}\bigl[1+(1-t)^{1/2}\bigr]^{3/4}, \tag{51}$$

$$C_-\bigl(2,2;\cos(\pi/4)\bigr) = 2^{-5/4}(1-t)^{1/16}\bigl[1+(1-t)^{1/2}\bigr]^{5/4}\bigl[5-(1-t)^{1/2}\bigr]/4. \tag{52}$$

Further results which follow from [24] are given in [26]

$$C_+(0,0;\lambda) = \frac{\theta_2(u;q)}{\theta_2(0;q)}, \tag{53}$$

$$C_-(1,1;\lambda) = -\frac{\theta_2'(u;q)}{\sin u\,\theta_2(0;q)\theta_3^2(0;q)}, \tag{54}$$

$$C_+(1,1;\lambda) = -\frac{\theta_3'(u;q)}{\sin u\,\theta_3(0;q)\theta_2^2(0;q)} \tag{55}$$

where is to be noted (for $N=0,1$) that $C_+(N,N;\lambda)$ is obtained from $C_-(N,N;\lambda)$ by the interchange $\theta_2 \leftrightarrow \theta_3$.

Many further results for various low values of M, N remain (in the tradition of Kaufman and Onsager) to be published by the authors of [14].

4 Jimbo, Miwa and Painlevé

The immediate object of the computation of the leading term in the form factor expansion by Wu [7] for the row correlation $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ and by Cheng and Wu [11] for the general case $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ was to compute the leading behavior of the correlations functions for large separations $R = (M^2+N^2)^{1/2}$. They found that for $T < T_c$ the correlation decays to the limiting value (9) as

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle \sim (1-t)^{1/4}\left\{1 - \frac{C_-(T)}{R^2}e^{-R/\xi_-(T)}\right\} \tag{56}$$

and vanishes for $T > T_c$ as

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle \sim (1-t)^{1/4}\frac{C_+(T)}{R^{1/2}}e^{-R/\xi_+(T)}, \tag{57}$$

where, in addition to depending on the temperature T, the R independent quantities $C_{\pm}(T)$ and $\xi_{\pm}(t)$ depend on the ratio M/N. It is found in [7] and [11] as $t \to 1$ that

$$\xi_{\pm}(T) \sim \frac{A_{\xi,\pm}}{1-t}, \tag{58}$$

$$C_{-}(T) \sim \frac{A_{-}}{(1-t)^2} \tag{59}$$

and

$$C_{+}(T) \sim \frac{A_{+}}{(1-t)^{1/2}}, \tag{60}$$

where again the amplitudes $A_{\xi.\pm}$ and $A_{\pm}$ depend on the ratio M/N. Neither of these asymptotic leading terms reduces to the result valid for $T = T_c$ (i.e. $t = 1$) where in [7] Wu found that the diagonal correlation has the leading behavior for large N

$$\langle \sigma_{0,0}\sigma_{N,N} \rangle \sim \frac{A_{T_c}}{N^{1/4}} \tag{61}$$

and

$$A_{T_c} = 2^{1/12} e^{3\zeta'(-1)} \tag{62}$$

with $\zeta'(-1)$ the derivative of Riemann's zeta function at -1.

The history of the result (61) is romantic in its own way. In the original 1949 paper of [5] there is a remark that the diagonal correlation vanishes "slowly". In 1959 Fisher [27] derived the exponent $1/4$ and remarked in footnote 8 that

> Onsager, private communication, has derived exact expressions for the correlations along the main diagonal ...

This computation was never published and perhaps there is another typescript out there waiting to be discovered.

Wu [7] also found the large N behavior of the row correlation $\langle \sigma_{0,0}\sigma_{0,N} \rangle$, which has the same dependence on N as (61) but with an amplitude

$$A_{\text{row}} = A_{T_c} \left(\cosh 2E^h/k_B T_c\right)^{1/4}. \tag{63}$$

The first purpose of the paper [8] was to connect the three different asymptotic behaviors (56), (57) and (61) by defining an interpolating function, traditionally called a scaling function,

$$G_{\pm}(r) = \lim_{M,N\to\infty, t\to 1} (1-t)^{-1/4} \langle \sigma_{0,0}\sigma_{M,N} \rangle \tag{64}$$

with

$$\left[\left(\frac{\sinh 2E^h/k_B T_c}{\sinh 2E^v/k_B T_c} \right)^{1/2} M^2 + \left(\frac{\sinh 2E^v/k_B T_c}{\sinh 2E^h/k_B T_c} \right)^{1/2} N^2 \right]^{1/2} (1-t) = r \text{ fixed.} \tag{65}$$

For this purpose the exponential representation of the correlation functions was derived. When the scaling function was computed it was discovered that $G_{\pm}(r)$ is expressed in terms of a Painlevé equation of the third kind

$$\frac{d^2\eta}{d\theta^2} = \frac{1}{\eta}\left(\frac{d\eta}{d\theta}\right)^2 - \frac{1}{\theta}\frac{d\eta}{d\theta} + \eta^3 - \eta^{-1} \tag{66}$$

as

$$G_{\pm}(r) = \frac{1 \mp \eta(r/2)}{2\eta(r/2)^{1/2}} \exp \frac{1}{4}\int_{r/2}^{\infty} d\theta\theta \frac{(1-\eta^2)^2 - (\eta')^2}{\eta}, \tag{67}$$

with the boundary condition

$$\eta(\theta) \sim 1 - \frac{2}{\pi}\lambda K_0(2\theta) \quad \text{as } \theta \to \infty, \tag{68}$$

where $K_0(2\theta)$ is the modified Bessel function and $\lambda = 1$.

This result was first announced in [28] and [29].Two different proofs were given. The first, in [8], is based on Myers' work [30] on the scattering of electromagnetic radiation from a strip and the second [23] is based on a direct manipulation of the exponential representation in the scaling limit.

It is at this point that I first learned of the existence of Sato, Miwa and Jimbo when in 1977 I received in the mail (how long ago it was that papers were sent by mail) a letter by the three of them with title "Studies on holonomic quantum fields II" [31] which generalized several of the results of [8] and made clear the relation of the Painlevé III equation with the massive Dirac equation. This letter was followed by many more where the only change in the title was that the Roman numeral was different and by a series of 5 papers with the title "Holonomic quantum field theory" [32–36]. These papers culminated in the groundbreaking paper "Studies on holonomic quantum fields XVII" [37, 38] where it is derived that the diagonal Ising correlation function for a general temperature on the lattice and not in the scaling limit satisfies the sigma form of the Painlevé VI equation

$$\begin{aligned}&\left(t\,(t-1)\frac{d^2\sigma}{dt^2}\right)^2\\ &\quad = N^2\left((t-1)\frac{d\sigma}{dt} - \sigma\right)^2 - 4\frac{d\sigma}{dt}\left((t-1)\frac{d\sigma}{dt} - \sigma - 1/4\right)\left(t\frac{d\sigma}{dt} - \sigma\right).\end{aligned} \tag{69}$$

The diagonal correlation is related to σ for $T > T_c$ by

$$\sigma(t) = t(t-1)\cdot\frac{d}{dt}\log\langle\sigma_{0,0}\sigma_{N,N}\rangle - 1/4, \tag{70}$$

with the boundary condition at $t = 0$ of

$$\langle\sigma_{0,0}\sigma_{N,N}\rangle = t^{N/2}\frac{(1/2)_N}{N!} + O\left(t^{1+N/2}\right), \tag{71}$$

and for $T < T_c$ by

$$\sigma(t) = t(t-1)\cdot\frac{d}{dt}\log\langle\sigma_{0,0}\sigma_{N,N}\rangle - t/4 \tag{72}$$

with the boundary condition

$$\langle\sigma_{0,0}\sigma_{N,N}\rangle = (1-t)^{1/4}\left\{1 - \frac{t^{N+1}}{2N+1}\left(\frac{(1/2)_{N+1}}{(N+1)!}\right)^2 + O\left(t^{N+2}\right)\right\} \tag{73}$$

where $(a)_N = a(a+1)\cdots(a+N-1)$ for $1 \leq N$ and $(a)_0 = 1$ is Pochammer's symbol. These boundary conditions are obtained from the leading terms of (16) and (17) as $t \to 0$. Furthermore the lambda extensions (40) and (41) satisfy the same Painlevé VI equation (69) where the λ appears as a boundary condition.

The six Painlevé equations have a long history [39, 40]. They are defined as those second-order nonlinear equations the location of whose branch points and essential singularities (but not poles) are independent of the boundary conditions and which cannot be reduced to simpler functions. Painlevé obtained three of these equations [41] and Gambier [42] obtained the remaining three including the PVI equation which in the general case has four parameters. However, the specific case of Painlevé VI needed for the Ising model (69) had already been obtained by Picard [43] in 1889. Subsequent to the discovery that this PVI equation characterizes the diagonal Ising model, this equation has appeared in many contexts [44–46] ranging from Poncelet polygons to mirror symmetry. The sigma form of the Painlevé equations was first obtained by Okamoto [47, 48].

5 The Susceptibility

The second purpose of the paper [8] was to begin the study of the magnetic susceptibility at zero magnetic field $\chi(T)$, which is computed in terms of the correlation functions as

$$k_B T\chi(T) = \sum_{M=-\infty}^{\infty}\sum_{N=-\infty}^{\infty}\left\{\langle\sigma_{0,0}\sigma_{M,N}\rangle - \mathcal{M}^2\right\}, \tag{74}$$

where $\mathcal{M}^2$ is the square of the spontaneous magnetization which was given in (9). In order to evaluate the sums in (74) the exponential forms (14) and (15) which were the basis of computing the Painlevé III equation cannot be used and instead the exponentials must be expanded into the form factor representations (11) and (12). Using these forms the sums over M and N are easily evaluated as geometric series and the susceptibility is written as the infinite sum of n "particle" contributions

$$k_B T\chi_+(T) = (1-t)^{1/4}t^{-1/4}\sum_{j=0}^{\infty}\chi^{(2j+1)}(T) \quad \text{for } T > T_c \tag{75}$$

$$k_BT\chi_-(T) = (1-t)^{1/4}\sum_{j=1}^{\infty}\chi^{(2j)}(T) \quad \text{for } T < T_c. \tag{76}$$

In [8] the terms $\chi^{(n)}(T)$ for $n = 1, 2, 3, 4$ were studied. In the scaling limit the scaled $\chi^{(n)}(T)$ for general n were given by Nappi [49] in 1978. For arbitrary temperature the results in the isotropic case were obtained by Nickel [9] and [10] and for $E^v \neq E^h$ in [24]

$$\chi^{(j)}(T) = \frac{\cot^j \alpha}{j!}\int_{-\pi}^{\pi}\frac{d\omega_1}{2\pi}\cdots\int_{-\pi}^{\pi}\frac{d\omega_{j-1}}{2\pi}\left(\prod_{n=1}^{j}\frac{1}{\sinh\gamma_n}\right)H^{(j)}\frac{1+\prod_{n=1}^{j}x_n}{1-\prod_{n=1}^{j}x_n}, \tag{77}$$

with

$$x_n = \cot^2\alpha\left[\xi - \cos\omega_n - \sqrt{(\xi-\cos\omega_n)^2-(\cot\alpha)^{-4}}\right], \tag{78}$$

$$\sinh\gamma_n = \cot^2\alpha\sqrt{(\xi-\cos\omega_n)^2-(\cot\alpha)^{-4}}, \tag{79}$$

where

$$\cot\alpha = \sqrt{s_h/s_v}, \tag{80}$$

$$\xi = \left(1+s_h^{-2}\right)^{1/2}\left(1+s_v^2\right)^{1/2}, \tag{81}$$

$$s_v = \sinh 2E^v/k_BT \; s_h = \sinh 2E^h/k_BT, \tag{82}$$

$$H^{(j)} = \left(\prod_{1\le i<k\le j} h_{ik}\right)^2 \tag{83}$$

with

$$h_{ik} = \cot\alpha\frac{\sin\frac{1}{2}(\omega_i-\omega_k)}{\sinh\frac{1}{2}(\gamma_i-\gamma_k)} = \frac{1}{\cot\alpha}\frac{\sinh\frac{1}{2}(\gamma_i-\gamma_k)}{\sin\frac{1}{2}(\omega_i+\omega_k)}, \tag{84}$$

and ω_j is defined in terms of the remaining ω_i from $\omega_1+\cdots\omega_j = 0 \bmod 2\pi$. We note in particular that for $E^v = E^h$

$$\chi^{(1)}(t) = \frac{t^{1/4}}{(1-t^{1/4})^2} \tag{85}$$

with t given by (13) and

$$\chi^{(2)}(t) = \frac{(1+t)E(t^{1/2})-(1-t)K(t^{1/2})}{3\pi(1-t^{1/2})(1-t)} \tag{86}$$

with t given by (10).

5.1 The Amplitude of the Susceptibility Divergence

The study of the susceptibility from the form factor expansions was initiated in 1973 in [28] where it was demonstrated that as $T \to T_c\pm$ the susceptibility diverges as

$$k_B T\chi(T)_{\pm} \sim C_{\pm}\left|\frac{s^{-1}-s}{2}\right|^{-7/4}\sqrt{2} \tag{87}$$

where in the isotropic case

$$s = \sinh 2E/k_B T. \tag{88}$$

The constants C_- and C_+ are different and are given as infinite series

$$C_- = \sum_{n=1}^{\infty} C^{(2n)}, \qquad C_+ = \sum_{n=0}^{\infty} C^{(2n+1)} \tag{89}$$

where the $C^{(n)}$ are n-fold integrals coming from the form factor expansion and have been studied both numerically for $n = 1, \ldots, 5$ [8, 28]. The first term in each of (89) has been analytically evaluated in [8, 28]

$$C^{(1)} = 1, \qquad C^{(2)} = \frac{1}{12\pi} \tag{90}$$

and the next leading term was evaluated by Tracy [50] as

$$C^{(3)} = \frac{1}{2\pi^2}\left(\frac{\pi^2}{3} + 2 - 3\sqrt{3}\mathrm{Cl}_2(\pi/3)\right) \tag{91}$$

where

$$\mathrm{Cl}_2(\theta) = \sum_{n=1}^{\infty}\frac{\sin n\theta}{n^2} \tag{92}$$

is Clausen's function and

$$C^{(4)} = \frac{1}{16\pi^3}\left(\frac{4\pi^2}{9} - \frac{1}{6} - \frac{7}{2}\zeta(3)\right). \tag{93}$$

In the tradition of Onsager and Kaufman [3] the details are only in an unpublished typescript. A curious feature of these results is that the ratio C_+/C_- is found to be closely approximated by 12π and the second terms are approximately three orders of magnitude less than the leading term. The study of the constants C_- and C_+ has been continued by high precision numerical computations [24] and the most recent evaluation [51] in 2011 is to an incredible 104 places. This is one of the most precisely determined constants in all of mathematical physics.

However, the $\chi^{(n)}(w)$ have singularities at other points besides $\sinh E/k_B T = \pm 1$ and the determination of the analytic properties of the magnetic susceptibility as a function of temperature has become the most challenging problem in the field.

Table 1 The Nickel singularities of $\chi^{(n)}$ for $n = 3, 4, 5, 6$

n	w
3	$-1/2, 1$
4	$\pm 1/2$
5	$-1, \frac{-1\pm\sqrt{5}}{4}, \frac{3\pm\sqrt{5}}{2}$
6	$\pm 1, \pm 1/3$

5.2 Nickel Singularities and the Natural Boundary Conjecture

The first studies of analytic properties after the initial computations of [8] were made in 1999 [9] and 2000 [10] when Nickel demonstrated for the isotropic case $E^v = E^h = E$ that the integrals (77) $\chi^{(n)}$ have singularities in the complex T plane on the curve

$$|\sinh 2E/k_B T| = 1, \tag{94}$$

which is the same curve on which the four Pfaffians of Kaufman's original evaluation [4] of the Ising partition function vanish. This was extended to the general case $E^v \neq E^h$ in [24] where the singularities of $\chi^{(n)}(T)$ are at

$$\begin{aligned} &\cosh 2E^v/k_B T \cosh 2E^h/k_B T \\ &\quad - \sinh 2E^h/k_B T \cos(2\pi j/n) - \sinh 2E^v/k_B T \cos(2\pi k/n) = 0 \end{aligned} \tag{95}$$

with

$$0 \leq j, k \leq [n/2], \quad j = k = 0 \text{ excluded} \tag{96}$$

where $[x]$ is the integer part of x and for n even $j + k = n/2$ is also excluded. In terms of the variable used in [52–61] for the isotropic lattice with s given by (88)

$$w^{-1} = 2\left(s + s^{-1}\right) \tag{97}$$

these singularities for $n = 3, 4, 5, 6$ are given in Table 1, where we note that $\sinh 2E/k_B T$ is real for $-1/4 \leq w \leq 1/4$ and is complex with $|\sinh 2E/k_B T| = 1$ for $1/4 < |w|$. If we call ϵ the deviation from the singular temperatures $T^{(j)}_{m,m'}$ determined by (95), then for $T > T_c$ the singularity in $\chi^{(2j+1)}(T)$ is

$$\epsilon^{2j(j+1)-1} \ln \epsilon \tag{98}$$

and for $T < T_c$ the singularity in $\chi^{(2j)}(T)$ is

$$\epsilon^{2j^2-3/2}. \tag{99}$$

It is striking that the number of singularities increases with n and becomes dense in the limit $n \to \infty$. This feature led Nickel to the conclusion that unless cancellations occur there will be a natural boundary in the susceptibility $\chi(T)$ in the complex T plane at the location (95). The existence of a natural boundary in the complex temperature plane is not contemplated in the scaling theory of critical phenomena.

5.3 Fuchsian Equations

The next step in the study of the susceptibility was begun in 2005 [52] and has continued in the series of papers [53–61]. In these papers exact Fuchsian differential equations for the $\chi^{(n)}(T)$ in the isotropic case $E^v = E^h$ are determined by use of Maple by first expanding the integrals in an appropriate variable such as w or w^2 and then using Maple programs which obtain ODE's from these series. The resulting differential equations have very special properties such as being globally nilpotent [58] which allow for extensive analysis to be carried out. These studies have uncovered several new and important features of the susceptibility; namely that the $\chi^{(n)}(w)$ have a direct sum decomposition and that they have further singularities beyond those of (95).

5.3.1 Direct Sum Decompositions

In [55] and [60] it is shown for $1 \leq n \leq 6$ that $\chi^{(n)}(w)$ have the same direct sum decomposition seen already in the diagonal form factors

$$\chi^{(2n)}(w) = \sum_{m=1}^{n-1} K_m^{(2n)} \chi^{(2m)}(w) + \Omega^{(2n)}(w), \tag{100}$$

$$\chi^{(2n+1)}(w) = \sum_{m=1}^{n-1} K_m^{(2n+1)} \chi^{(2m+1)}(w) + \Omega^{(2n+1)}(w) \tag{101}$$

where the $\Omega^{(n)}(w)$ satisfy Fuchsian equations of order m

$$L_m^{(n)} \cdot \Omega^{(n)} = 0 \tag{102}$$

with

$$\begin{array}{ccccc} n & 3 & 4 & 5 & 6 \\ m & 6 & 8 & 29 & 46 \end{array} \tag{103}$$

The $K_j^{(n)}$ are constants which for $n = 3, 4, 5, 6$ coincide with the values of $K_m^{(n)}(0)$ given in (26)–(29).

The operators in (102) factorize further. For $L_6^{(3)}$ and $L_8^{(4)}$ we have we have

$$L_6^{(3)} = L_3^{(3)} \cdot L_2^{(3)} \cdot L_1^{(3)} \tag{104}$$

and

$$L_8^{(4)} = L_4^{(4)} \cdot L_1^{(4)} \cdot \left(L_{1;a}^{(4)} \oplus L_{1;b}^{(4)} \oplus L_{1;c}^{(4)}\right) \tag{105}$$

where the numeral in the subscript indicates the order of the operators which are given in [54] and [55]. The operator $L_{29}^{(5)}$ has been found in [57, 59] and [61] to have the factorization

$$L_{29}^{(5)} = L_5^{(5)} \cdot L_{12}^{(5)} \cdot L_1^{(5)} \cdot L_{11}^{(5)} \tag{106}$$

where $L_{11}^{(5)}$ has the further direct sum decomposition (A.1) of [61]

$$L_{11}^{(5)} = (Z_2 \cdot N_1) \oplus V_2 \oplus \left(F_3 \cdot F_2 \cdot L_1^s\right). \tag{107}$$

Similarly in (56) and (57) of [60] the operator $L_{46}^{(6)}$ is shown to have the decomposition

$$L_{46}^{(6)} = L_6^{(6)} \cdot L_{23}^{(6)} \cdot L_{17}^{(6)} \tag{108}$$

where $L_{17}^{(6)}$ has a direct sum decomposition into the sum of four operators but the possible reducibility of $L_{23}^{(6)}$ has not yet been determined due to computational complexity.

5.3.2 Singularities

The location of the singularities of the operators $L_m^{(n)}$ are obtained by examining the roots of the polynomial multiplying the highest derivative d^m/dw^m and this analysis shows that there are further singularities beyond the singularities at $w = \pm 1/4, \infty$ and the Nickel singularities (95).

In [53] that the differential equation for $\chi^{(3)}(w)$ admits additional singularities at

$$w = \frac{-3 \pm i\sqrt{7}}{8} \tag{109}$$

which correspond to

$$s = \frac{-1 \pm i\sqrt{7}}{4}, \qquad |s| = \frac{1}{\sqrt{2}}, \tag{110}$$

$$s = \frac{-1 \pm i\sqrt{7}}{2}, \qquad |s| = \sqrt{2} \tag{111}$$

where we note the singularity at (110) is inside the unit circle $|s| = 1$ and thus cannot appear in the principle sheet of the integral for $\chi^{(3)}$ which is analytic for $|s| < 1$.

There are no additional singularities in $\chi^{(4)}(w)$ and the singularities of $\chi^{(5)}(w)$ are shown in (34) of [57] to be at the roots of following polynomial

$$\begin{aligned} &w^{33}(1-4w)^{22}(1+4w)^{16}(1-w)^4(1+2w)^4\left(1+3w+4w^2\right)\\ &\quad\times(1+w)\left(1-3w+w^2\right)\left(1+2w-4w^2\right)\\ &\quad\times\left(1-w-3w^2+4w^3\right)\left(1+8w+20w^2+15w^3+4w^4\right)\\ &\quad\times\left(1-7w+5w^2-4w^3\right)\\ &\quad\times\left(1+4w+8w^2\right)(1-2w). \end{aligned} \tag{112}$$

The singularities located by the roots of the first line in (112) are identical with the location of singularities of $\chi^{(3)}$ and the roots of the second line are the Nickel singularities of $\chi^{(5)}$. Most of the remaining singularities correspond to complex values of s not on $|s| = 1$.

6 Diagonal Susceptibility

The integrals (77) for the n particle contribution to the susceptibility $\chi^{(n)}(T)$ are quite complex and the Maple-based studies cannot be extended much beyond their present limits. Therefore it would be of great utility if a simpler set of integrals could be found which would still incorporate all significant analytic features of the $\chi^{(n)}$. Several such simplified modifications of the integrals have been studied [56] but by far the most natural case is to restrict the two dimensional sum over the lattice positions M, N in (74) to the lattice diagonal $M = N$ and thus to consider the susceptibility that will result if a magnetic field is applied only to the diagonal

$$k_B T \chi_d(t) = \sum_{N=-\infty}^{\infty} \left\{\langle\sigma_{0,0}\sigma_{N,N}\rangle - \mathcal{M}^2\right\}, \tag{113}$$

where the dependence on T is now for all E^v and E^h in terms of the single variable t defined by (10) for $T < T_c$ and by (13) for $T > T_c$.

This diagonal susceptibility has been studied in [62] and [63] and has been found to have the remarkable simplification over the bulk susceptibility that all singularities of the differential equations are at $s = 0, \infty$ and $|s| = 1$. There are no other complex singularities for $|s| \neq 1$ such as appear in $\chi^{(n)}(t)$. Furthermore $\chi_d^{(3)}(t)$ and $\chi_d^{(4)}(t)$ have been found to be explicitly expressed in terms of generalized hypergeometric functions ${}_{p+1}F_p$.

6.1 Integral Representations

From the integral expressions for $f^{(n)}_{N,N}(t)$ of (16) and (17) given in [12] and [14], we find in [62] the expansion for $T < T_c$

$$k_B T \chi_{d,-}(t) = (1-t)^{1/4} \sum_{n=1}^{\infty} \chi_d^{(2n)}(t) \tag{114}$$

and for $T > T_c$

$$k_B T \chi_{d,+}(t) = (1-t)^{1/4} \sum_{n=0}^{\infty} \chi_d^{(2n+1)}(t), \tag{115}$$

where

$$\begin{aligned} \chi_d^{(2n)}(t) = \frac{t^{n^2}}{(n!)^2} \frac{1}{\pi^{2n}} \cdot \int_0^1 \cdots \int_0^1 \prod_{k=1}^{2n} dx_k \cdot \frac{1 + t^n x_1 \cdots x_{2n}}{1 - t^n x_1 \cdots x_{2n}} \\ \times \prod_{j=1}^{n} \left(\frac{x_{2j-1}(1-x_{2j})(1-tx_{2j})}{x_{2j}(1-x_{2j-1})(1-t\,x_{2j-1})} \right)^{1/2} \\ \times \prod_{1 \le j \le n} \prod_{1 \le k \le n} (1 - t\, x_{2j-1}\, x_{2k})^{-2} \\ \times \prod_{1 \le j < k \le n} (x_{2j-1} - x_{2k-1})^2\, (x_{2j} - x_{2k})^2 \end{aligned} \tag{116}$$

and for $T > T_c$

$$\begin{aligned} \chi_d^{(2n+1)}(t) = \frac{t^{n(n+1))}}{\pi^{2n+1} n!(n+1)!} \cdot \int_0^1 \cdots \int_0^1 \prod_{k=1}^{2n+1} dx_k \\ \times \frac{1 + t^{n+1/2} x_1 \cdots x_{2n+1}}{1 - t^{n+1/2} x_1 \cdots x_{2n+1}} \cdot \prod_{j=1}^{n} \big((1-x_{2j})(1-t\,x_{2j}) \cdot x_{2j}\big)^{1/2} \\ \times \prod_{j=1}^{n+1} \big((1-x_{2j-1})(1-t\,x_{2j-1}) \cdot x_{2j-1}\big)^{-1/2} \\ \times \prod_{1 \le j \le n+1} \prod_{1 \le k \le n} (1 - t\, x_{2j-1}\, x_{2k})^{-2} \\ \times \prod_{1 \le j < k \le n+1} (x_{2j-1} - x_{2k-1})^2 \prod_{1 \le j < k \le n} (x_{2j} - x_{2k})^2. \end{aligned} \tag{117}$$

The expressions (116) and (117) are, indeed, much simpler than the corresponding expressions for $\chi^{(n)}$ given in (77). In particular

$$\chi_d^{(1)}(t) = \frac{1}{1 - t^{1/2}} \tag{118}$$

and

$$\chi_d^{(2)}(t) = \frac{t}{4(1-t)} \tag{119}$$

which are simpler than (85) and (86) respectively. Most noticeable is that $\chi^{(2)}(w)$ in (86) has a logarithmic singularity at $t = 1$ $(w = 1/4)$ while $\chi_d^{(2)}(t)$ in (119) does not.

6.2 Root of Unity Singularities

In addition to the singularity at $t = 1$ it is straightforward to see from the integral expressions (116) and (117) that $\chi_d^{(2n)}(t)$ has singularities at

$$t_0^n = 1 \tag{120}$$

of the form

$$\epsilon^{2n^2-1} \ln \epsilon \tag{121}$$

and $\chi_d^{(2n+1)}(t)$ has singularities

$$t_0^{n+1/2} = 1 \tag{122}$$

of the form

$$\epsilon^{(n+1)^2-1/2} \tag{123}$$

where ϵ is the deviation from t_0. These are the analogues for the diagonal susceptibility of the Nickel singularities of the bulk susceptibility $\chi^{(n)}$ of (95).

6.3 Direct Sum Decomposition

The $\chi_d^{(n)}(t)$ have the same direct sum decomposition seen already in the diagonal form factors and $\chi^{(n)}(w)$

$$\chi_d^{(2n)}(t) = \sum_{j=1}^{n-1} K_{d;j}^{(2n)} \chi_d^{(2j)}(t) + \Omega_d^{(2n)}(t), \tag{124}$$

$$\chi_d^{(2n+1)}(t) = \sum_{j=1}^{n-1} K_{d;j}^{(2n+1)} \chi_d^{(2j+1)}(t) + \Omega_d^{(2n+1)}(t) \tag{125}$$

where $K_{d;j}^{(n)}$ are constants. However, unlike $\chi^{(n)}(w)$, the operators $L_d^{(n)}$ which annihilate $\Omega_d^{(n)}(t)$ have a further direct sum decomposition

$$L_{d;5}^{(3)} = L_{d;2}^{(3)} + L_{d;3}^{(3)} \quad \text{and} \quad L_{d;7}^{(4)} = L_{d;3}^{(4)} + L_{d;4}^{(4)} \tag{126}$$

6.4 Results for $\chi_d^{(3)}(t)$

For $\chi_d^{(3)}(t)$ we explicitly find by combining [58] and [62] and setting $x = t^{1/2}$ that

$$\chi_d^{(3)}(x) = \frac{1}{3}\chi_{d;1}^{(3)}(x) + \frac{1}{2}\chi_{d;2}^{(3)}(x) - \frac{1}{6}\chi_{d;3}^{(3)}(x) \tag{127}$$

where

$$\chi_{d;1}^{(3)}(x) = \frac{1}{1-x} = \chi_d^{(1)}(x), \tag{128}$$

$$\chi_{d;2}^{(3)}(x) = \frac{1}{(1-x)^2}\,{}_2F_1\big(1/2, -1/2; 1; x^2\big) - \frac{1}{1-x}\,{}_2F_1\big(1/2, 1/2; 1; x^2\big) \tag{129}$$

and

$$\begin{aligned}\chi_{d;3}^{(3)}(x) = \frac{(1+2x)(x+2)}{(1-x)(x^2+x+1)}\Big[&F(1/6, 1/3; 1; Q)^2 \\ &+ \frac{2Q}{9}F(1/6, 1/3; 1; Q)F(7/6, 4/3; 2; Q)\Big]\end{aligned} \tag{130}$$

with

$$Q = \frac{27}{4}\frac{(1+x)^2x^2}{(x^2+x+1)^3} \tag{131}$$

where we note that

$$1 - Q = \frac{(1-x)^2(1+2x)^2(2+x)^2}{4(1+x+x^2)^3}. \tag{132}$$

From (127) and (128) we see that in (125) we have $K_1^{(1)} = 1/3$.

We see from (117) that $\chi_d^{(3)}(x)$ vanishes when $x \to 0$ as x^4. However, $\chi_{d;1}^{(3)}(x)$ and $\chi_{d;3}^{(3)}(x)$ are constant as $x \to 0$ and $\chi_{d;2}^{(3)}(x)$ vanishes linearly in x. The three constants in (127) are determined by matching with the x^4 behavior of $\chi_d^{(3)}$ and this requires that the three constants will solve a set of five (overdetermined) linear equations.

As $x \to 1$ we find that $\chi_d^{(3)}$ diverges as

$$\chi_d^{(3)}(x) = \frac{1}{1-x}\left(\frac{1}{3} - \frac{5\pi}{18\Gamma^2(5/6)\Gamma^2(2/3)} + \frac{4\pi}{\Gamma^2(1/6)\Gamma^2(1/3)}\right) = \frac{0.016329\cdots}{1-x} \tag{133}$$

Furthermore $\chi_{d;3}^{(3)}(x)$ has an additional singularity at $x \to e^{\pm 2\pi i/3}$ which, to leading order is

$$\chi_{d;3\text{sing}}^{(3)} \to \frac{3^{4/5}16}{35\pi} e^{\pm 5\pi i/12} \epsilon^{7/2}. \tag{134}$$

6.5 *Results for* $\chi_d^{(4)}(t)$

These results have been extended in [62] and [63] to $\chi_d^{(4)}(t)$ where is shown that

$$\chi_d^{(4)}(t) = \frac{1}{2^3}\chi_{d;1}^{(4)}(t) + \frac{1}{3\cdot 2^3}\chi_{d;2}^{(4)}(t) - \frac{1}{2^3}\chi_{d;3}^{(4)}(t) \tag{135}$$

where

$$\chi_{d;1}^{(4)}(t) = \chi_d^{(2)}(t), \tag{136}$$

$$\begin{aligned}\chi_{d;2}^{(4)}(t) = &\frac{1+t}{(1-t)^2}\,{}_2F_1(1/2,-1/2;1;t)^2 - {}_2F_1(1/2,1/2;1,t)^2\\ &- \frac{2t}{1-t}F(1/2,1/2;1;t)\,{}_2F_1(1/2,-1/2;1;t)\end{aligned} \tag{137}$$

and

$$\chi_{d;4}^{(4)}(t) = A_3 \cdot {}_4F_3\big([1/2,1/2,1/2,1/2];[1,1,1]t^2\big) \tag{138}$$

with

$$\begin{aligned}A_3 = &2(1+t)t^3 D_t^3 + \frac{2}{3}\frac{16t^2-t-11}{t-1}t^2 d_t^2\\ &+ \frac{1}{3}\frac{31t^2-4t-11}{t-1}tD_t + t.\end{aligned} \tag{139}$$

The singular behavior as $t \to 1$ of $\chi_{d;1}^{(4)}(t)$ and $\chi_{d;3}^{(4)}$ is easily obtained and the singularity of $\chi_{d;4}^{(4)}$ at $t = 1$ is obtained by use of the analytic continuation formula of Bühring [67]. The final result [63] is that as $t \to 1$

$$\chi^{(4)}(t) \to \frac{1}{8(1-t)}\left(1 - \frac{1}{3\pi^2}\left[64 + 16(3I_1 - 4I_2)\right]\right) + \frac{7}{16\pi^2}\ln\frac{16}{1-t} - \frac{1}{16\pi^2}\ln^2\frac{16}{1-t} \tag{140}$$

where

$$3I_1 - 4I_2 = -2.2128121\cdots \tag{141}$$

has been given to 100 digits.

At the root of unit singularity $t = -1$ the leading singular behavior is

$$\chi_d^{(4)} \to \frac{1}{26880}(1+t)^7 \ln(1+t). \tag{142}$$

6.6 $\chi_d^{(5)}(t)$

The ODE satisfied by $\chi_d^{(5)}(x)$ has been studied in [63] modulo a large prime. It is found that the minimal order ODE is of order 19 and that the operator $L_{d;19}^{(5)}$ has the decomposition

$$L_{d;19}^{(5)} = L_{d;2}^{(3)} \oplus L_{d;17}^{(5)} \tag{143}$$

where $L_{d;2}^{(3)}$ is the second order operator which annihilates $\chi_{d;2}^{(3)}(x)$ and $L_{d;17}^{(5)}$ has singularities at $x = 0, \infty,\ 1,\ -1,\ x_3 = e^{\pm 2\pi i/3},\ x_5 = e^{\pm 2\pi i/5},\ e^{\pm 4\pi i/5}$ where the non-integer exponents at x_3 are $5/2, 7/2, 7/2$ and at x_5 are $23/2$. It has been further found that

$$L_{d;17}^{(5)} = L_{d;6}^{(5)} \cdot L_{d;11}^{(5)} \tag{144}$$

with

$$L_{d;11}^{(5)} = L_{d;1}^{(3)} \oplus L_{d;3}^{(3)} \oplus \left(W_{d;1}^{(5)} \cdot U_{d;1}^{(5)}\right) \oplus \left(L_{d;4}^{(5)} \cdot V_{d;1}^{(5)} \cdot U_{d;1}^{(5)}\right) \tag{145}$$

where $L_{d;m}^{(3)}$ annihilates $\chi_{d;m}^{(3)}$ and the remaining operators in this decomposition are all given in [63].

6.7 Singularities and Cancellations

By examining the integral representations for $\chi^{(n)}(w)$ (77) and $\chi_d^{(n)}(t)$ (116), (117) it is clear that these integrals have no singularities for $|\sinh 2E/k_BT| < 1$ or $t < 1$. The singularities at $|\sinh 2E/k_BT| < 1$ of the differential equations for $\chi^{(n)}(w)$

will only appear in analytic continuations of the integral in the complex plane of the variable w. The corresponding differential equations for $\chi_d^{(n)}(t)$ are significantly simpler because they have singularities only at $t = 0, \infty$ and $|t| = 1$.

It remains to discuss the singularities in the differential equations which do lie on $|\sinh 2E/k_BT| = 1$ and to give an explanation for the observation that the singularities of the ODEs for $\chi^{(n-2m)}(w)$ and $\chi_d^{(n-2m)}(t)$ are also singularities of $\chi^{(n)}(w)$ and $\chi_d^{(n)}(t)$ respectively even though the integrands are singular only at the points given by (95) for $\chi^{(n)}(w)$ and by (120) and (122) for $\chi_d^{(n)}(t)$.

The resolution of this is easily seen for $\chi_d^{(n)}(t)$. By an examination of the integrals (116) and (117) we see that there are paths of analytic continuation possible in the complex t plane where the contour of integration must be deformed past the pole at

$$1 - t x_{2j} x_{2k+1} = 0 \tag{146}$$

and the residue at that pole will reduce the denominators in $\chi_d^{2n}(t)$ and $\chi_d^{(2n+1)}(t)$ from

$$1 - t^n x_1 \cdots x_{2n} \tag{147}$$

and

$$1 - t^{n+1/2} x_1 \cdots x_{2n+1} \tag{148}$$

to the denominators in $\chi_d^{(2n-2)}(t)$ and $\chi_d^{(2n-1)}(t)$ respectively with $n \to n-1$ and two less integration variables. Therefore, the singularities of $\chi_d^{(n-2m)}(t)$ will not appear on the principle sheet of the integral which is analytic at $t = 0$ but only on analytic continuations to non-physical branches. The similar phenomenon occurs for $\chi^{(n)}(w)$.

It remains to reconcile this non appearance of the singularities of $\chi_d^{(n-2m)}(t)$ in the physical sheet of $\chi_d^{(n)}(t)$ with the direct sum decompositions (124) and (125). This will be accomplished by showing that the term $\Omega_d^{(n)}(t)$ has singularities which exactly cancel the singularities on $\chi_d^{(n)}(t)$. This requires the solution of a global connection problem which has not yet been explicitly done even though from the examination of the original integral the resulting exact cancellation must hold.

7 Conclusion

Now that we have summarized the known features of the Ising correlations, form factors and susceptibility we can proceed to discuss what is not known. This is the fascinating, mysterious and thus romantic part of the subject.

7.1 Conformal and Quantum Field Theory

One of the most important features of the Ising model is that the scaling limit satisfies all the axioms for a massive Euclidean quantum field theory and that at $T = T_c$ the long range correlations are those of a conformal field theory with central charge $c = 1/2$. This is in fact the earliest conformal field theory known and from this beginning a vast new field of mathematics and physics has been developed in the last 30 years. However, the Ising model is much more than a conformal field theory because we have a vast number of results for $T \neq T_c$ which are the simplest examples of properties of massive Euclidean quantum field theories. Part of the romance is the exploration of how these Ising results can be used to extend massless conformal field theories into the massive region.

7.2 Form Factors, Exponential Forms and Amplitudes

The derivation [12] of the exponential and form factor expansion for the diagonal Ising correlation is much more general than this special case. Indeed in [12] it is proven that every Toeplitz determinant (4) with a generating function $\phi(\xi)$ such that $\ln\phi(\xi)$ is continuous and periodic on $|\xi| = 1$ has both an exponential and a form factor expansion. Furthermore these Toeplitz determinants are also expressible as Fredholm determinants [64] (at times in several different ways [65]). Consequently the Ising computations have subsequently been extended to several very important problems including the seminal work on the one dimensional impenetrable Bose gas and on random matrices by Jimbo, Miwa, Mori and Sato [66].

To illustrate the differences between the form factor and the exponential representation of the correlation functions, we consider the computation by Tracy [68] of the constant A_{T_c} of (62). In the scaling limit the scaled correlations in the general case where $E^v \neq E^h$ depend only on the single variable r (65). Therefore we can restrict attention to the scaled form of the diagonal correlation $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ and consider the lambda extension of the scaling form of the exponential form (14) which we write as

$$G_-(r;\lambda) = \exp\sum_{n=1}^{\infty}\lambda^{2n}g^{(2n)}(r), \tag{149}$$

where

$$g^{(2n)}(r) = \lim_{\text{scaling}} F^{(2n)}_{N,N}(t), \tag{150}$$

which depends on the single variable r instead of the two independent variables N and t. Tracy finds that, as $r \to 0$,

$$g^{(2n)}(r) = -\alpha_n \ln r + \beta_n + o(1). \tag{151}$$

Therefore, defining the lambda dependent sums

$$\alpha(\lambda)=\sum_{n=1}^{\infty}\lambda^{2n}\alpha_n,\qquad \beta(\lambda)=\sum_{n=1}^{\infty}\lambda^{2n}\beta_n,\tag{152}$$

we find

$$G_-(r;\lambda)\sim\exp\{-\alpha(\lambda)\ln r+\beta(\lambda)\}=\frac{e^{\beta(\lambda)}}{r^{\alpha(\lambda)}}.\tag{153}$$

In the Ising case where $\lambda=1$ the functions specialize to

$$\alpha(1)=1/4,\qquad \beta(1)=\ln A\tag{154}$$

where A is the constant in (61).

If, however, instead of the scaled exponential form we define the scaled limit of the form factors $f_{N,N}^{(2n)}(t)$ as

$$\tilde{f}^{(2n)}(r)=\lim_{\text{scaling}} f_{N,N}^{(2n)}(t),\tag{155}$$

then as $r\to 0$

$$\tilde{f}^{(2n)}(r)=\sum_{k=0}^{n}a_k^{(2n)}\ln^k r+o(1).\tag{156}$$

Thus, in order for (153) to agree with the $r\to 0$ behavior of the form factor expansion, we need

$$\begin{aligned}\frac{e^{\beta(\lambda)}}{r^{\alpha(\lambda)}}&=\left[\sum_{k=0}^{\infty}\frac{1}{k!}\left(\ln r\sum_{n=1}^{\infty}\lambda^{2n}\alpha_n\right)^k\right]\left[\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{n=1}^{\infty}\lambda^{2n}\beta_n\right)^k\right]\\&=1+\sum_{n=1}^{\infty}\lambda^{2n}\sum_{k=0}^{n}a_k^{(2n)}\ln^k r\end{aligned}\tag{157}$$

to hold term by term for each power λ^{2n}. The requires an infinite number of identities between the $a_k^{(2n)}$.

As an additional remark we note that if we rewrite the integral (16) for $f_{N,N}^{(2n)}$ as a contour integral, rescale the variables x_k by $x_k=t^{-1/2}y_k$ and then send $y_{2k}\to 1/y_{2k}$ we see that as $t\to 1$ the integral has logarithmic divergences as in (156). The amplitudes a_n are closely related to the special case with $\rho=1$ of the integral found by Dotsenko and Fateev [69] in their study of four point correlations in conformal field theories with central charge $c \leq 1$

$$I_{n,m}(\alpha, \beta; \rho) = \frac{1}{n!m!} \prod_{i=1}^{n} \int_0^1 dt_i t_i^{\alpha'} (1-t_i)^{\beta'} \prod_{i=1}^{m} \int_0^1 d\tau_i^{\alpha} (1-\tau_i)^{\beta} \times \left| \prod_{i<j} (t_i - t_j) \right|^{2\rho'} \left| \prod_{i<j} (\tau_i - \tau_j) \right|^{2\rho} \prod_{i,j}^{n,m} \frac{P}{(t_i - \tau_j)^2} \tag{158}$$

where P indicates the principal value and

$$\alpha' = -\rho'\alpha, \qquad \beta' = -\rho'\beta, \qquad \rho' = \rho^{-1}. \tag{159}$$

7.3 Exponentiation

Form factor expansions exist for many massive models of quantum field theory including sine-Gordon and the non-linear sigma model [70] and similar form factor expansions exist [71, 72] for the XXZ model on a chain of finite length

$$H_{XXZ} = -\sum_{j=1}^{L} \{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z + H \sigma_j^z \}. \tag{160}$$

Moreover the Feynman expansion of amplitudes in quantum field theory is also what we have called here a form factor expansion. In all of the models there are limiting cases where series of multiple dimensional integrals expand to series in powers of logarithms which need to be summed. However, unlike the Ising correlation functions these form factor expansions do not come from either Toeplitz or Fredholm determinants and thus the exponentiation methods of Ising correlations are not applicable.

Over the years an immense effort has been made to sum the form factor series of logarithms. In quantum field theory this starts with the classic 1939 paper of Bloch and Nordsiek [73] on resummation of infrared divergences in quantum electrodynamics. A second example is the Regge theory of the 60's and 70's where the $2n$th order Feynman diagram in the expansion of a four point scattering amplitude is shown to diverge as the energy $s \to \infty$ with a fixed momentum transfer t as

$$g^{2n} \alpha(t)^n \frac{\ln^n s}{n!}. \tag{161}$$

This is a "leading" log approximation and is analagous to the Ising case if only the first term in the series (152) for α_n is retained. More recently there has been a great deal of work on quantum chromodynamics[1] where many non leading terms summed by the use of an ingenious decomposition of the multidimensional integrals.

[1] This literature is also vast. For example see [74–76].

In the theory of integrable systems a great deal of effort has been devoted to compute the long range asymptotic behavior of the correlations of the XXZ model in the massless region $-1 < \Delta < 1$ from multiple integral representations. One method is presented in [77] which shows how to modify the Fredholm determinant form which holds for $\Delta = 0$ by suitably picking out the important pieces of the multiple integrals. This has led to the computation of both the exponents and the amplitude of the long range behavior of the correlations when the $H \neq 0$. The study of the correlations from the form factors is begun in [71, 72] with more results announced to be forth coming. A full exploration of the relation of these subjects is beyond the scope of this article.

7.4 Short Distance Versus Scaling Terms

In the n-particle expansions of the full (76), (75) and the diagonal (114), (115) susceptibility the $\chi^{(n)}(t)$ and the $\chi_d^{(n)}(t)$ will (for $n \geq 3$) have terms which contain powers of $\ln t$. From this it might be inferred that the susceptibility will contain terms of the form $(1-t)^{1/4+p} \ln^q (1-t)$. However, from the extensive calculations on long low and high temperature series expansions made in [24] and [51] such terms do not appear to exist. Instead the susceptibility is conjectured to have the form for $t \to 1$ of

$$k_B T \chi(t)_\pm = (1-t)^{-7/4} \sum_{j=0}^{\infty} C_\pm^{(j)} (1-t)^j + \sum_{q=0}^{\infty} \sum_{p=0}^{[\sqrt{q}]} b_\pm^{(p,q)} (1-t)^q \ln^p (1-t). \tag{162}$$

The first term is called the "scaling function". The second term is called the "background" or "short distance" term and is numerically obtained by summing correlation functions instead of form factors. In [51] it is stated that the "scaling function" is determined by conformal field theory while for the "short distance" term here is "no explicit prediction". In [24] the belief is stated that the separation into "scaling" and "short distance" parts is "tantamount to the scaling argument that in the critical region there is a single length scale proportional to $(1-t)^{-\nu}$ with $\nu = 1$". It would be highly desirable if this distinction between "scaling" and "short distance" terms could be made precise and if both terms could be obtained by use of the form factor expansion alone.

7.5 Natural Boundaries and λ Extensions

Perhaps the most perplexing question concerning the relation of the Ising model on a lattice with the scaling field theory limit is the existence of the natural boundary in the susceptibility implied by the singularities (98), (99) found by Nickel [9, 10].

The magnetic susceptibility is the second derivative of the free energy with respect to an external magnetic field H interacting with the spins as $-H\sum_{j,k}\sigma_{j,k}$. In the scaling limit the Ising model in a magnetic field is also a field theory and the analyticity properties of this field theory have been extensively studied by Fonseca and Zamolodchikov [78] with the conclusion that there is no natural boundary. How can this be reconciled with the computations of [9] and [10]?

The existence of the natural boundary suggested by Nickel in [9] and [10] rests on the accumulation of the singularities (98) and (99) and the assumption that there is no cancellation. However, for this argument to hold we need to be able to show that the limit of t approaching the location of the supposed natural boundary (95) will commute with the infinite sum over the n particle contributions $\chi^{(n)}(T)$ in (75) and (76). Since the natural boundary does not exist if only a finite number of the $\chi^{(n)}(T)$ are included this interchange need to be investigated. It is also possible that the existence of a natural boundary could depend on the value of λ in the lambda extensions of (75) and (76)

$$k_B T\chi_+(T;\lambda) = (1-t)^{1/4}t^{-1/4}\sum_{j=0}^{\infty}\lambda^{2j}\chi^{(2j+1)}(T) \quad \text{for } T > T_c, \tag{163}$$

$$k_B T\chi_-(T;\lambda) = (1-t)^{1/4}\sum_{j=1}^{\infty}\lambda^{2j}\chi^{(2j)}(T) \quad \text{for } T < T_c. \tag{164}$$

These possibilities remain to be investigated.

7.6 Row Correlations

All of the results obtained for the diagonal correlation, which depend on the single variable t, can be extended to the row correlation, which depends on the two variables α_1 and α_2 in a symmetric fashion (5). In particular it has been pointed out to me by Jean-Marie Maillard and Nicholas Witte in private conversations that the Painlevé VI results of Jimbo and Miwa [37, 38] can be extended to a two variable Garnier system.[2] However, this system must possess some most interesting properties because one of the most important properties of the Ising model is the fact that, when these two variables are rewritten as

$$k = \sinh 2E^v/k_BT \sinh 2E^h/k_BT \quad \text{and} \quad r = \frac{\sinh 2E^v/k_BT}{\sinh 2E^h/k_BT}, \tag{165}$$

the dependence on k (the modulus of the elliptic functions) and the anisotropy ratio r which is related to the spectral variable of the star triangle equation [79] is dramatically different. These results for Garnier systems have also yet to be obtained.

[2] For a modern exposition of Garnier systems see [40].

8 Romance Versus Understanding

In a lecture given in Melbourne in January 2006 [80], I gave the following definition of "understanding"

> No one can be said to understand a paper unless he is able to generalize the paper.

This definition is open to criticism on at least two grounds. Firstly the use of the word "he" has a sexist implication which is neither appropriate nor intended. Secondly, there are surely subjects which are fully understood where further generalization is pointless. An illustration of this are the laws of thermodynamics which have been fully understood by physicists for many decades (even if they are not accepted by the overwhelming majority of voters and politicians).

However, precisely because thermodynamics is fully understood, it has lost the mystery it had at the time of Gibbs, Boltzmann and Ehrenfest. This illustrates the great truth that understanding is the enemy of romance because once the mysteries are understood the romance dies.

Fortunately for romance, there are many mysteries of the Ising model which are far from being understood. The romantic in me says that, even when these mysteries have been understood, the understanding of the mysteries will generate new mysteries and the romance of the Ising model will be everlasting.

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$A_n^{(1)}$-Geometric Crystal Corresponding to Dynkin Index $i = 2$ and Its Ultra-Discretization

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Abstract Let $\mathfrak{g}$ be an affine Lie algebra with index set $I = \{0, 1, 2, \dots, n\}$ and $\mathfrak{g}^L$ be its Langlands dual. It is conjectured in Kashiwara et al. (Trans. Am. Math. Soc. 360(7):3645–3686, 2008) that for each $i \in I \setminus \{0\}$ the affine Lie algebra $\mathfrak{g}$ has a positive geometric crystal whose ultra-discretization is isomorphic to the limit of certain coherent family of perfect crystals for $\mathfrak{g}^L$. We prove this conjecture for $i = 2$ and $\mathfrak{g} = A_n^{(1)}$.

1 Introduction

Let $A = (a_{ij})_{i,j \in I}, I = \{0, 1, \dots, n\}$ be an affine Cartan matrix and $(A, \{\alpha_i\}_{i \in I}, \{\alpha_i^\vee\}_{i \in I})$ be a given Cartan datum. Let $\mathfrak{g} = \mathfrak{g}(A)$ denote the associated affine Lie algebra [8] and $U_q(\mathfrak{g})$ denote the corresponding quantum affine algebra. Let $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{Z}\delta$ and $P^\vee = \mathbb{Z}\alpha_0^\vee \oplus \mathbb{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbb{Z}\alpha_n^\vee \oplus \mathbb{Z}d$ denote the affine weight lattice and the dual affine weight lattice respectively. For a dominant weight $\lambda \in P^+ = \{\mu \in P \mid \mu(h_i) \geq 0 \text{ for all } i \in I\}$ of level $l = \lambda(c)$ ($c =$ canonical central element), Kashiwara defined the crystal base $(L(\lambda), B(\lambda))$ [13] for the integrable highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$. The crystal $B(\lambda)$ is the $q = 0$ limit of the canonical basis [21] or the global crystal basis [14]. It has many interesting combinatorial properties. To give explicit realization of the crystal $B(\lambda)$, the notion of affine crystal and perfect crystal has been introduced in [10]. In particular, it is shown in [10] that the affine crystal $B(\lambda)$ for the level $l \in \mathbb{Z}_{>0}$ integrable highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ can be realized as the semi-infinite tensor product $\cdots \otimes B_l \otimes B_l \otimes B_l$, where B_l is a perfect crystal of level l. This is known as the path realization. Subsequently it is noticed in [12] that one needs

Dedicated to Professor Michio Jimbo on the occasion of his 60th birthday.

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a coherent family of perfect crystals $\{B_l\}_{l\geq 1}$ in order to give a path realization of the Verma module $M(\lambda)$ (or $U_q^-(\mathfrak{g})$). In particular, the crystal $B(\infty)$ of $U_q^-(\mathfrak{g})$ can be realized as the semi-infinite tensor product $\cdots \otimes B_\infty \otimes B_\infty \otimes B_\infty$ where B_∞ is the limit of the coherent family of perfect crystals $\{B_l\}_{l\geq 1}$ (see [12]). At least one coherent family $\{B_l\}_{l\geq 1}$ of perfect crystals and its limit is known for $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}, D_4^{(3)}, G_2^{(1)}$ (see [11, 12, 17, 22, 30]).

A perfect crystal is indeed a crystal for certain finite dimensional module called Kirillov-Reshetikhin module (KR-module for short) of the quantum affine algebra $U_q(\mathfrak{g})$ ([4, 5, 19]). The KR-modules are parametrized by two integers (i, l), where $i \in I \setminus \{0\}$ and l any positive integer. Let $\{\varpi_i\}_{i\in I\setminus\{0\}}$ be the set of level 0 fundamental weights [15]. Hatayama et al. ([4, 5]) conjectured that any KR-module $W(l\varpi_i)$ admit a crystal base $B^{i,l}$ in the sense of Kashiwara and furthermore $B^{i,l}$ is perfect if l is a multiple of $c_i^\vee := \max(1, \frac{2}{(\alpha_i,\alpha_i)})$. This conjecture has been proved for quantum affine algebras $U_q(\mathfrak{g})$ of classical types ([2, 3, 27]). When $\{B^{i,l}\}_{l\geq 1}$ is a coherent family of perfect crystals we denote its limit by $B_\infty(\varpi_i)$ (or just B_∞ if there is no confusion).

On the other hand the notion of geometric crystal is introduced in [1] as a geometric analog to Kashiwara's crystal (or algebraic crystal) [13]. In fact, geometric crystal is defined in [1] for reductive algebraic groups and is extended to general Kac-Moody groups in [23]. For a given Cartan datum $(A, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$, the geometric crystal is defined as a quadruple $\mathcal{V}(\mathfrak{g}) = (X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$, where X is an algebraic variety, $e_i : \mathbb{C}^\times \times X \longrightarrow X$ are rational $\mathbb{C}^\times$-actions and $\gamma_i, \varepsilon_i : X \longrightarrow \mathbb{C}$ $(i \in I)$ are rational functions satisfying certain conditions (see Definition 1). A geometric crystal is said to be a positive geometric crystal if it admits a positive structure (see Definition 3). A remarkable relation between positive geometric crystals and algebraic crystals is the ultra-discretization functor $\mathcal{UD}$ between them (see Sect. 2.4). Applying this functor, positive rational functions are transfered to piecewise linear functions by the simple correspondence:

$$x \times y \longmapsto x + y, \qquad \frac{x}{y} \longmapsto x - y, \qquad x + y \longmapsto \max\{x, y\}.$$

It was conjectured in [18] that for each affine Lie algebra $\mathfrak{g}$ and each Dynkin index $i \in I \setminus 0$, there exists a positive geometric crystal $\mathcal{V}(\mathfrak{g}) = (X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ whose ultra-discretization $\mathcal{UD}(\mathcal{V})$ is isomorphic to the limit B_∞ of a coherent family of perfect crystals for the Langlands dual $\mathfrak{g}^L$. In [18], it has been shown that this conjecture is true for $i = 1$ and $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$. In [25] (resp. [6]) a positive geometric crystal for $\mathfrak{g} = G_2^{(1)}$ (resp. $\mathfrak{g} = D_4^{(3)}$) and $i = 1$ has been constructed and it is shown in [26] (resp. [7]) that the ultra-discretization of this positive geometric crystal is isomorphic to the limit of a coherent family of perfect crystals for $\mathfrak{g}^L = D_4^{(3)}$ (resp. $\mathfrak{g}^L = G_2^{(1)}$) given in [17] (resp. [22]).

In this paper we have constructed a positive geometric crystal associated with the Dynkin index $i = 2$ for the affine Lie algebra $A_n^{(1)}$ and have proved that its ultra-discretization is isomorphic to the limit $B^{2,\infty}$ of the coherent family of perfect crystals $\{B^{2,l}\}_{l\geq 1}$ for the affine Lie algebra $A_n^{(1)}$ given in [11, 28].

This paper is organized as follows. In Sect. 2, we recall necessary definitions and facts about geometric crystals. In Sect. 3, we recall from [28] (see also [11]) the coherent family of perfect crystals $\{B^{2,l}\}_{l\geq 1}$for $\mathfrak{g} = A_n^{(1)}$ and its limit $B^{2,\infty}$. In Sects. 4, we construct a positive affine geometric crystal $\mathcal{V} = \mathcal{V}(A_n^{(1)})$ explicitly. In Sect. 5, we prove that the ultra-discretization $\mathcal{X} = \mathcal{UD}(\mathcal{V})$ is isomorphic to the limit $B^{2,\infty}$ which proves the conjecture in [18, Conjecture 1.2] for $i = 2$ and $\mathfrak{g} = A_n^{(1)}$.

2 Geometric Crystals

In this section, we review Kac-Moody groups and geometric crystals following [1, 20, 23, 29].

2.1 *Kac-Moody Algebras and Kac-Moody Groups*

Fix a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ with a finite index set I. Let $(\mathfrak{t}, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$ be the associated root data, where $\mathfrak{t}$ is a vector space over $\mathbb{C}$ and $\{\alpha_i\}_{i\in I} \subset \mathfrak{t}^*$ and $\{\alpha_i^\vee\}_{i\in I} \subset \mathfrak{t}$ are linearly independent satisfying $\alpha_j(\alpha_i^\vee) = a_{ij}$.

The Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated with A is the Lie algebra over $\mathbb{C}$ generated by $\mathfrak{t}$, the Chevalley generators e_i and f_i $(i \in I)$ with the usual defining relations [9, 29]. There is the root space decomposition $\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{t}^*} \mathfrak{g}_\alpha$. Denote the set of roots by $\Delta := \{\alpha \in \mathfrak{t}^* | \alpha \neq 0,\ \mathfrak{g}_\alpha \neq (0)\}$. Set $Q = \sum_i \mathbb{Z}\alpha_i$, $Q_+ = \sum_i \mathbb{Z}_{\geq 0}\alpha_i$, $Q^\vee := \sum_i \mathbb{Z}\alpha_i^\vee$ and $\Delta_+ := \Delta \cap Q_+$. An element of Δ_+ is called a *positive root*. Let $P \subset \mathfrak{t}^*$ be a weight lattice such that $\mathbb{C} \otimes P = \mathfrak{t}^*$, whose element is called a weight.

Define simple reflections $s_i \in \mathrm{Aut}(\mathfrak{t})$ $(i \in I)$ by $s_i(h) := h - \alpha_i(h)\alpha_i^\vee$, which generate the Weyl group W. It induces the action of W on $\mathfrak{t}^*$ by $s_i(\lambda) := \lambda - \lambda(\alpha_i^\vee)\alpha_i$. Set $\Delta^{\mathrm{re}} := \{w(\alpha_i) | w \in W,\ i \in I\}$, whose element is called a real root.

Let $\mathfrak{g}'$ be the derived Lie algebra of $\mathfrak{g}$ and let G be the Kac-Moody group associated with $\mathfrak{g}'$ [29]. Let $U_\alpha := \exp \mathfrak{g}_\alpha$ $(\alpha \in \Delta^{\mathrm{re}})$ be the one-parameter subgroup of G. The group G is generated by U_α $(\alpha \in \Delta^{\mathrm{re}})$. Let $U^\pm$ be the subgroup generated by $U_{\pm\alpha}$ $(\alpha \in \Delta_+^{\mathrm{re}} = \Delta^{\mathrm{re}} \cap Q_+)$, *i.e.*, $U^\pm := \langle U_{\pm\alpha} | \alpha \in \Delta_+^{\mathrm{re}}\rangle$.

For any $i \in I$, there exists a unique homomorphism; $\phi_i : SL_2(\mathbb{C}) \to G$ such that

$$\phi_i\left(\begin{pmatrix} c & 0 \\ 0 & c^{-1}\end{pmatrix}\right) = c^{\alpha_i^\vee}, \qquad \phi_i\left(\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}\right) = \exp(te_i),$$

$$\phi_i\left(\begin{pmatrix} 1 & 0 \\ t & 1\end{pmatrix}\right) = \exp(tf_i).$$

where $c \in \mathbb{C}^\times$ and $t \in \mathbb{C}$. Set $\alpha_i^\vee(c) := c^{\alpha_i^\vee}$, $x_i(t) := \exp(te_i)$, $y_i(t) := \exp(tf_i)$, $G_i := \phi_i(SL_2(\mathbb{C}))$, $T_i := \phi_i(\{\mathrm{diag}(c, c^{-1}) | c \in \mathbb{C}^\vee\})$ and $N_i := N_{G_i}(T_i)$. Let T (resp. N) be the subgroup of G with the Lie algebra $\mathfrak{t}$ (resp. generated by the N_i's), which is called a *maximal torus* in G, and let $B^\pm = U^\pm T$ be the Borel subgroup of G. We have the isomorphism $\phi : W \xrightarrow{\sim} N/T$ defined by $\phi(s_i) = N_i T/T$. An element $\overline{s}_i := x_i(-1)y_i(1)x_i(-1) = \phi_i\begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}$ is in $N_G(T)$, which is a representative of $s_i \in W = N_G(T)/T$.

2.2 Geometric Crystals

Let X be an ind-variety, $\gamma_i : X \to \mathbb{C}$ and $\varepsilon_i : X \longrightarrow \mathbb{C}$ $(i \in I)$ rational functions on X, and $e_i : \mathbb{C}^\times \times X \longrightarrow X$ $((c, x) \mapsto e_i^c(x))$ a rational $\mathbb{C}^\times$-action.

Definition 1 A quadruple $(X, \{e_i\}_{i \in I}, \{\gamma_i,\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ is a G (or $\mathfrak{g}$)-*geometric crystal* if

1. $\{1\} \times X \subset \mathrm{dom}(e_i)$ for any $i \in I$.
2. $\gamma_j(e_i^c(x)) = c^{a_{ij}} \gamma_j(x)$.
3. e_i's satisfy the following relations.

$$
\begin{array}{ll}
e_i^{c_1} e_j^{c_2} = e_j^{c_2} e_i^{c_1} & \text{if } a_{ij} = a_{ji} = 0, \\
e_i^{c_1} e_j^{c_1 c_2} e_i^{c_2} = e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1} & \text{if } a_{ij} = a_{ji} = -1, \\
e_i^{c_1} e_j^{c_1^2 c_2} e_i^{c_1 c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1^2 c_2} e_i^{c_1} & \text{if } a_{ij} = -2,\ a_{ji} = -1, \\
e_i^{c_1} e_j^{c_1^3 c_2} e_i^{c_1^2 c_2} e_j^{c_1^3 c_2^2} e_i^{c_1 c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1^3 c_2^2} e_i^{c_1^2 c_2} e_j^{c_1^3 c_2} e_i^{c_1} & \text{if } a_{ij} = -3,\ a_{ji} = -1,
\end{array}
$$

4. $\varepsilon_i(e_i^c(x)) = c^{-1}\varepsilon_i(x)$ and $\varepsilon_i(e_j^c(x)) = \varepsilon_i(x)$ if $a_{i,j} = a_{j,i} = 0$.

The condition 4 is slightly modified from the one in [6, 25, 26].

Let W be the Weyl group associated with $\mathfrak{g}$. For $w \in W$ define $R(w)$ by

$$
R(w) := \left\{(i_1, i_2, \ldots, i_l) \in I^l | w = s_{i_1} s_{i_2} \cdots s_{i_l}\right\},
$$

where l is the length of w. Then $R(w)$ is the set of reduced words of w. For a word $\mathbf{i} = (i_1, \ldots, i_l) \in R(w)$ $(w \in W)$, set $\alpha^{(j)} := s_{i_l} \cdots s_{i_{j+1}}(\alpha_{i_j})$ $(1 \le j \le l)$ and

$$
\begin{aligned}
e_{\mathbf{i}} &: T \times X \to X \\
&(t, x) \mapsto e_{\mathbf{i}}^t(x) := e_{i_1}^{\alpha^{(1)}(t)} e_{i_2}^{\alpha^{(2)}(t)} \cdots e_{i_l}^{\alpha^{(l)}(t)}(x).
\end{aligned}
$$

Note that the condition 3 above is equivalent to the following: $e_{\mathbf{i}} = e_{\mathbf{i}'}$ for any $w \in W$, $\mathbf{i}, \mathbf{i}' \in R(w)$.

2.3 Geometric Crystal on Schubert Cell

Let $w \in W$ be a Weyl group element and take a reduced expression $w = s_{i_1} \cdots s_{i_k}$. Let $X := G/B$ be the flag variety, which is an ind-variety and $X_w \subset X$ the Schubert cell associated with w, which has a natural geometric crystal structure ([1, 23]). For $\mathbf{i} := (i_1, \dots, i_k)$, set

$$B_{\mathbf{i}}^- := \left\{ Y_{\mathbf{i}}(c_1, \dots, c_k) := Y_{i_1}(c_1) \cdots Y_{i_k}(c_k) \,|\, c_1, \dots, c_k \in \mathbb{C}^\times \right\} \subset B^-, \tag{2.1}$$

where $Y_i(c) := y_i(\frac{1}{c})\alpha_i^\vee(c)$. If $I = \{i_1, \dots, i_k\}$, this has a geometric crystal structure ([23]) isomorphic to X_w. The explicit forms of the action e_i^c, the rational function ε_i and γ_i on $B_{\mathbf{i}}^-$ are given by

$$e_i^c\big(Y_{\mathbf{i}}(c_1, \dots, c_k)\big) = Y_{\mathbf{i}}(\mathcal{C}_1, \dots, \mathcal{C}_k)),$$

where

$$\mathcal{C}_j := c_j \frac{\sum_{1 \le m \le j, i_m = i} \frac{c}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m} + \sum_{j < m \le k, i_m = i} \frac{1}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m}}{\sum_{1 \le m < j, i_m = i} \frac{c}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m} + \sum_{j \le m \le k, i_m = i} \frac{1}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m}}, \tag{2.2}$$

$$\varepsilon_i\big(Y_{\mathbf{i}}(c_1, \dots, c_k)\big) = \sum_{1 \le m \le k, i_m = i} \frac{1}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m}, \tag{2.3}$$

$$\gamma_i\big(Y_{\mathbf{i}}(c_1, \dots, c_k)\big) = c_1^{a_{i_1,i}} \cdots c_k^{a_{i_k,i}}. \tag{2.4}$$

Remark As in [23], the above setting requires the condition $I = \{i_1, \dots, i_k\}$. Otherwise, set $J := \{i_1, \dots, i_k\} \subsetneq I$ and let $\mathfrak{g}_J \subsetneq \mathfrak{g}$ be the corresponding subalgebra. Then, by arguing similarly to [23, 4.3], we can define the $\mathfrak{g}_J$-geometric crystal structure on $B_{\mathbf{i}}^-$.

2.4 Positive Structure, Ultra-Discretizations and Tropicalizations

Let us recall the notions of positive structure, ultra-discretization and tropicalization.

The setting below is same as in [18]. Let $T = (\mathbb{C}^\times)^l$ be an algebraic torus over $\mathbb{C}$ and $X^*(T) := \mathrm{Hom}(T, \mathbb{C}^\times) \cong \mathbb{Z}^l$ (resp. $X_*(T) := \mathrm{Hom}(\mathbb{C}^\times, T) \cong \mathbb{Z}^l$) be the lattice of characters (resp. co-characters) of T. Set $R := \mathbb{C}(c)$ and define

$$\begin{aligned} v : R \setminus \{0\} &\longrightarrow \mathbb{Z} \\ f(c) &\mapsto \deg\big(f(c)\big), \end{aligned}$$

where deg is the degree of poles at $c = \infty$. Here note that for $f_1, f_2 \in R \setminus \{0\}$, we have

$$v(f_1 f_2) = v(f_1) + v(f_2), \qquad v\left(\frac{f_1}{f_2}\right) = v(f_1) - v(f_2). \tag{2.5}$$

A non-zero rational function on an algebraic torus T is called *positive* if it can be written as g/h where g and h are positive linear combinations of characters of T.

Definition 2 Let $f\colon T \to T'$ be a rational morphism between two algebraic tori T and T'. We say that f is *positive*, if $\eta \circ f$ is positive for any character $\eta\colon T' \to \mathbb{C}$.

Denote by $\mathrm{Mor}^+(T, T')$ the set of positive rational morphisms from T to T'.

Lemma 1 ([1]) *For any* $f \in \mathrm{Mor}^+(T_1, T_2)$ *and* $g \in \mathrm{Mor}^+(T_2, T_3)$, *the composition* $g \circ f$ *is well-defined and belongs to* $\mathrm{Mor}^+(T_1, T_3)$.

By Lemma 1, we can define a category $\mathcal{T}_+$ whose objects are algebraic tori over $\mathbb{C}$ and arrows are positive rational morphisms.

Let $f\colon T \to T'$ be a positive rational morphism of algebraic tori T and T'. We define a map $\widehat{f}\colon X_*(T) \to X_*(T')$ by

$$\langle \eta, \widehat{f}(\xi)\rangle = v(\eta \circ f \circ \xi),$$

where $\eta \in X^*(T')$ and $\xi \in X_*(T)$.

Lemma 2 ([1]) *For any algebraic tori* T_1, T_2, T_3, *and positive rational morphisms* $f \in \mathrm{Mor}^+(T_1, T_2)$, $g \in \mathrm{Mor}^+(T_2, T_3)$, *we have* $\widehat{g \circ f} = \widehat{g} \circ \widehat{f}$.

Let $\mathfrak{Set}$ denote the category of sets with the morphisms being set maps. By the above lemma, we obtain a functor:

$$\begin{array}{rccc} \mathcal{UD}\colon & \mathcal{T}_+ & \longrightarrow & \mathfrak{Set} \\ & T & \mapsto & X_*(T) \\ & (f : T \to T') & \mapsto & (\widehat{f} : X_*(T) \to X_*(T')) \end{array}$$

Definition 3 ([1]) Let $\chi = (X, \{e_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ be a geometric crystal, T' an algebraic torus and $\theta : T' \to X$ a birational isomorphism. The isomorphism θ is called *positive structure* on χ if it satisfies

1. For any $i \in I$ the rational functions $\gamma_i \circ \theta : T' \to \mathbb{C}$ and $\varepsilon_i \circ \theta : T' \to \mathbb{C}$ are positive.
2. For any $i \in I$, the rational morphism $e_{i,\theta} : \mathbb{C}^\times \times T' \to T'$ defined by $e_{i,\theta}(c, t) := \theta^{-1} \circ e_i^c \circ \theta(t)$ is positive.

Let $\theta : T \to X$ be a positive structure on a geometric crystal $\chi = (X, \{e_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$. Applying the functor $\mathcal{UD}$ to positive rational morphisms $e_{i,\theta} : \mathbb{C}^\times \times T \to T$ and $\gamma_i \circ \theta, \varepsilon_i \circ \theta : T \to \mathbb{C}$ (the notations are as above), we obtain

$$\tilde{e}_i := \mathcal{UD}(e_{i,\theta}) : \mathbb{Z} \times X_*(T) \to X_*(T),$$
$$\mathrm{wt}_i := \mathcal{UD}(\gamma_i \circ \theta) : X_*(T') \to \mathbb{Z},$$

$$\varepsilon_i := \mathcal{UD}(\varepsilon_i \circ \theta) : X_*\big(T'\big) \to \mathbb{Z}.$$

Now, for given positive structure $\theta : T' \to X$ on a geometric crystal $\chi = (X, \{e_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$, we associate the quadruple $(X_*(T'), \{\tilde{e}_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ with a free pre-crystal structure (see [1, Sect. 7]) and denote it by $\mathcal{UD}_{\theta,T'}(\chi)$. We have the following theorem:

Theorem 1 ([1, 23]) *For any geometric crystal $\chi = (X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ and positive structure $\theta : T' \to X$, the associated pre-crystal $\mathcal{UD}_{\theta,T'}(\chi) = (X_*(T'), \{\tilde{e}_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ is a crystal (see [1, Sect. 7])*

Now, let $\mathcal{GC}^+$ be a category whose object is a triplet (χ, T', θ) where $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ is a geometric crystal and $\theta : T' \to X$ is a positive structure on χ, and morphism $f : (\chi_1, T_1', \theta_1) \longrightarrow (\chi_2, T_2', \theta_2)$ is given by a rational map $\varphi : X_1 \longrightarrow X_2$ ($\chi_i = (X_i, \ldots)$) such that

$$\varphi \circ e_i^{X_1} = e_i^{X_2} \circ \varphi, \qquad \gamma_i^{X_2} \circ \varphi = \gamma_i^{X_1}, \qquad \varepsilon_i^{X_2} \circ \varphi = \varepsilon_i^{X_1},$$
$$\text{and} \quad f := \theta_2^{-1} \circ \varphi \circ \theta_1 : T_1' \longrightarrow T_2',$$

is a positive rational morphism. Let $\mathcal{CR}$ be the category of crystals. Then by the theorem above, we have

Corollary 1 *The map $\mathcal{UD} = \mathcal{UD}_{\theta,T'}$ defined above is a functor*

$$\begin{aligned}
&\mathcal{UD} : \mathcal{GC}^+ \longrightarrow \mathcal{CR},\\
&\quad \big(\chi, T', \theta\big) \mapsto X_*\big(T'\big),\\
&\quad \big(f : \big(\chi_1, T_1', \theta_1\big) \to \big(\chi_2, T_2', \theta_2\big)\big) \mapsto \big(\widehat{f} : X_*\big(T_1'\big) \to X_*\big(T_2'\big)\big).
\end{aligned}$$

We call the functor $\mathcal{UD}$ "*ultra-discretization*" as in [23, 24] instead of "tropicalization" as in [1]. And for a crystal B, if there exists a geometric crystal χ and a positive structure $\theta : T' \to X$ on χ such that $\mathcal{UD}(\chi, T', \theta) \cong B$ as crystals, we call an object (χ, T', θ) in $\mathcal{GC}^+$ a *tropicalization* of B, which is not standard but we use such a terminology as before.

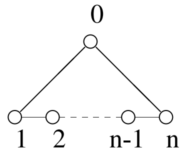
3 Perfect Crystals of Type $A_n^{(1)}$

From now on we assume $\mathfrak{g}$ to be the affine Lie algebra $A_n^{(1)}$, $n \geq 2$. In this section, we recall the coherent family of perfect crystals of type $A_n^{(1)}$, $n \geq 2$ and its limit given in [11, 28]. For basic notions of crystals, coherent family of perfect crystals and its limit we refer the reader to [12] (see also [10, 11]).

For the affine Lie algebra $A_n^{(1)}$, let $\{\alpha_0, \alpha_1, \dots \alpha_n\}$, $\{\alpha_0^\vee, \alpha_1^\vee, \dots \alpha_n^\vee\}$ and $\{\Lambda_0, \Lambda_1, \dots \Lambda_n\}$ be the set of simple roots, simple coroots and fundamental weights, respectively. The Cartan matrix $A = (a_{ij})_{i,j \in I}$, $I = \{0, 1, \dots, n\}$ is given by:

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i \equiv (j \pm 1) \bmod (n+1), \\ 0 & \text{otherwise} \end{cases}$$

and its Dynkin diagram is as follows.



The standard null root δ and the canonical central element c are given by

$$\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad c = \alpha_0^\vee + \alpha_1^\vee + \cdots + \alpha_n^\vee,$$

where $\alpha_0 = 2\Lambda_0 - \Lambda_1 - \Lambda_n + \delta$, $\alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}$, $1 \le i \le n-1$, $\alpha_n = -\Lambda_0 - \Lambda_{n-1} + 2\Lambda_n$.

For a positive integer l we introduce $A_n^{(1)}$-crystals $B^{2,l}$ and $B^{2,\infty}$ as

$$B^{2,l} = \left\{ b = (b_{ji})_{1 \le j \le 2, j \le i \le j+n-1} \;\middle|\; \begin{array}{c} b_{ji} \in \mathbb{Z}_{\ge 0}, \sum_{i=j}^{j+n-1} b_{ji} = l, 1 \le j \le 2 \\ \sum_{i=1}^{t} b_{1i} \ge \sum_{i=2}^{t+1} b_{2i}, 1 \le t \le n \end{array} \right\},$$

$$B^{2,\infty} = \left\{ b = (b_{ji})_{1 \le j \le 2, j \le i \le j+n-1} \;\middle|\; b_{ji} \in \mathbb{Z}, \sum_{i=j}^{j+n-1} b_{ji} = 0, 1 \le j \le 2 \right\}.$$

Now we describe the explicit crystal structures of $B^{2,l}$ and $B^{2,\infty}$. Indeed, most of them coincide with each other except for ε_0 and φ_0. In the rest of this section, we use the following convention: $(x)_+ = \max(x, 0)$. For $b = (b_{ji})$ we denote

$$z_i = b_{1i} - b_{2,i+1}, \quad 2 \le i \le n-1. \tag{3.6}$$

Now we define conditions (E_m) and (F_m) for $2 \le m \le n$ as follows.

$$(F_m): \quad \begin{cases} z_k + z_{k+1} + \cdots + z_{m-1} \le 0, & 2 \le k \le m-1, \\ z_m + z_{m+1} + \cdots + z_k > 0, & m \le k \le n-1, \end{cases} \tag{3.7}$$

$$(E_m): \quad \begin{cases} z_k + z_{k+1} + \cdots + z_{m-1} < 0, & 2 \le k \le m-1, \\ z_m + z_{m+1} + \cdots + z_k \ge 0, & m \le k \le n-1. \end{cases} \tag{3.8}$$

We also define

$$\Delta(m) = (b_{12} + b_{13} + \cdots + b_{1,m-1}) + (b_{2,m+1} + b_{2,m+2} + \cdots + b_{2n}), \quad 2 \le m \le n. \tag{3.9}$$

Let $\Delta = \min\{\Delta(m) \mid 2 \le m \le n\}$. Note that for $2 \le m \le n$, $\Delta = \Delta(m)$ if the condition (F_m) (or (E_m)) hold. Then for $b = (b_{ji}) \in B^{2,l}$ or $B^{2,\infty}$, $\tilde{e}_k(b), \tilde{f}_k(b), \varepsilon_k(b), \varphi_k(b), k = 0, 1, \dots, n$ are given as follows.

For $0 \le k \le n$, $\tilde{e}_k(b) = (b'_{ji})$, where

$$\begin{cases} k=0: & b'_{11}=b_{11}-1, \quad b'_{1m}=b_{1m}+1, \quad b'_{2m}=b_{2m}-1, \\ & b'_{2,n+1}=b_{2,n+1}+1 \quad \text{if } (E_m), 2\le m\le n, \\ k=1: & b'_{11}=b_{11}+1, \quad b'_{12}=b_{12}-1, \\ 2\le k\le n-1: & \begin{cases} b'_{1k}=b_{1k}+1, \quad b'_{1,k+1}=b_{1,k+1}-1 & \text{if } b_{1k}\ge b_{2,k+1}, \\ b'_{2k}=b_{2k}+1, \quad b'_{2,k+1}=b_{2,k+1}-1 & \text{if } b_{1k}< b_{2,k+1}, \end{cases} \\ k=n: & b'_{2n}=b_{2n}+1, \quad b'_{2,n+1}=b_{2,n+1}-1 \end{cases}$$

and $b'_{ji} = b_{ji}$ otherwise.

For $0 \le k \le n$, $\tilde{f}_k(b) = (b'_{ji})$, where

$$\begin{cases} k=0: & b'_{11}=b_{11}+1, \quad b'_{1m}=b_{1m}-1, \quad b'_{2m}=b_{2m}+1, \\ & b'_{2,n+1}=b_{2,n+1}-1 \quad \text{if } (F_m), 2\le m\le n, \\ k=1: & b'_{11}=b_{11}-1, \quad b'_{12}=b_{12}+1, \\ 2\le k\le n-1: & \begin{cases} b'_{1k}=b_{1k}-1, \quad b'_{1,k+1}=b_{1,k+1}+1 & \text{if } b_{1k}> b_{2,k+1}, \\ b'_{2k}=b_{2k}-1, \quad b'_{2,k+1}=b_{2,k+1}+1 & \text{if } b_{1k}\le b_{2,k+1}, \end{cases} \\ k=n: & b'_{2n}=b_{2n}-1, \quad b'_{2,n+1}=b_{2,n+1}+1 \end{cases}$$

and $b'_{ji} = b_{ji}$ otherwise. For $b \in B^{2,l}$ if $\tilde{e}_k b$ or $\tilde{f}_k b$ does not belong to $B^{2,l}$ then we understand it to be 0.

$$\varepsilon_1(b) = b_{12}, \qquad \varphi_1(b) = b_{11} - b_{22},$$

$$\varepsilon_k(b) = b_{1,k+1} + (b_{2,k+1} - b_{1,k})_+, \qquad \varphi_k(b) = b_{2k} + (b_{1k} - b_{2,k+1})_+,$$

$$\text{for } 2 \le k \le n-1,$$

$$\varepsilon_n(b) = b_{2,n+1} - b_{1n}, \qquad \varphi_n(b) = b_{2n},$$

$$\varepsilon_0(b) = \begin{cases} l - b_{2,n+1} - \Delta, & b \in B^{2,l}, \\ -b_{2,n+1} - \Delta, & b \in B^{2,\infty}, \end{cases}$$

$$\varphi_0(b) = \begin{cases} l - b_{11} - \Delta, & b \in B^{2,l}, \\ -b_{11} - \Delta, & b \in B^{2,\infty}. \end{cases}$$

Hence the weights $wt_i(b) = \varphi_i(b) - \varepsilon_i(b), 0 \le i \le n$ are:

$$\begin{cases} wt_0(b) = b_{2,n+1} - b_{11}, \\ wt_1(b) = b_{11} - b_{12} - b_{22}, \\ wt_k(b) = (b_{1k} - b_{1,k+1}) + (b_{2k} - b_{2,k+1}) \quad (1 < k < n), \\ wt_n(b) = b_{1n} + b_{2n} - b_{2,n+1}. \end{cases}$$

The following results have been proved in [11, 28]:

Theorem 2 ([11, 28])

1. *The $A_n^{(1)}$-crystal $B^{2,l}$ is a perfect crystal of level l.*
2. *The family of the perfect crystals $\{B^{2,l}\}_{l\ge1}$ forms a coherent family and the crystal $B^{2,\infty}$ is its limit with the vector $b_\infty = (0)_{2\times n}$.*

4 Affine Geometric Crystal $\mathcal{V}(A_n^{(1)})$

Let $c = \sum_{i=0}^{n} \alpha_i^\vee$ be the canonical central element in the affine Lie algebra $\mathfrak{g} = A_n^{(1)}$ and $\{\Lambda_i | i \in I\}$ be the set of fundamental weights as in the previous section. Let σ denote the Dynkin diagram automorphism. In particular, $\sigma(\Lambda_i) = \Lambda_{\overline{i+1}}$, where $\overline{i+1} = (i+1) \bmod(n+1)$. Consider the level 0 fundamental weight $\varpi_2 := \Lambda_2 - \Lambda_0$. Let $I_0 = I \setminus 0,\ I_n = I \setminus n$, and $\mathfrak{g}_i$ denote the subalgebra of $\mathfrak{g}$ associated with the index sets $I_i, i = 0, n$. Then $\mathfrak{g}_0$ as well as $\mathfrak{g}_n$ is isomorphic to A_n.

Let $W(\varpi_2)$ be the fundamental representation of $U_q'(\mathfrak{g})$ associated with ϖ_2 ([15]). By [15, Theorem 5.17], $W(\varpi_2)$ is a finite-dimensional irreducible integrable $U_q'(\mathfrak{g})$-module and has a global basis with a simple crystal. Thus, we can consider the specialization $q = 1$ and obtain the finite-dimensional $A_n^{(1)}$-module $W(\varpi_2)$, which we call a fundamental representation of $A_n^{(1)}$ and use the same notation as above. We shall present the explicit form of $W(\varpi_2)$ below.

4.1 *Fundamental Representation $W(\varpi_2)$ for $A_n^{(1)}$*

The $A_n^{(1)}$-module $W(\varpi_2)$ is an $\frac{1}{2}n(n+1)$-dimensional module with the basis,

$$\big\{(i, j) \mid 1 \le i < j \le n+1\big\},$$

where (i, j) denotes the tableaux:

$$\begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array}$$

The actions of e_i and f_i on these basis vectors are given as follows.

For $1 \le k \le n$, we have

$$f_k(i,j) = \begin{cases} (i+1,j), & i = k < j-1, \\ (i,j+1), & j = k, \\ 0, & \text{otherwise}, \end{cases}$$

$$e_k(i,j) = \begin{cases} (i-1,j), & i = k+1, \\ (i,j-1), & i < j-1 = k, \\ 0, & \text{otherwise}, \end{cases}$$

$$f_0(i,j) = \begin{cases} (1,i), & i \neq 1,\ j = n+1, \\ 0, & \text{otherwise}, \end{cases}$$

$$e_0(1,j) = \begin{cases} (j,n+1), & i \neq 1, \\ 0, & \text{otherwise}. \end{cases}$$

Furthermore the weights of the basis vectors are given by:

$$wt(i,j) = (\Lambda_i - \Lambda_{i-1} + \Lambda_j - \Lambda_{j-1}), \quad 1 \le i < j \le n+1,$$

where we understand that $\Lambda_{n+1} = \Lambda_0$. Note that in $W(\varpi_2)$, we have $(1,2)$ (resp. $(1,n+1)$) is a $\mathfrak{g}_0$ (resp. $\mathfrak{g}_n$) highest weight vector with weight $\varpi_2 = \Lambda_2 - \Lambda_0$ (resp. $\sigma^{-1}\varpi_2 = \Lambda_1 - \Lambda_n$).

4.2 Affine Geometric Crystal $\mathcal{V}(A_n^{(1)})$ in $W(\varpi_2)$

Now we will construct the affine geometric crystal $\mathcal{V}(A_n^{(1)})$ in $W(\varpi_2)$ explicitly. For $\xi \in (\mathfrak{t}_{\mathrm{cl}}^*)_0$, let $t(\xi)$ be the translation as in [15, Sect. 4] and $\widetilde{\varpi}_i$ as in [16]. Indeed, $\widetilde{\varpi}_i := \max(1, \frac{2}{(\alpha_i,\alpha_i)})\varpi_i = \varpi_i$ in our case. Then we have

$$t(\widetilde{\varpi}_2) = \sigma^2(s_{n-1}s_{n-2}\cdots s_1)(s_n s_{n-1}\cdots s_2) =: \sigma^2 w_1,$$
$$t\big(\mathrm{wt}(1,n+1)\big) = \sigma^2(s_{n-2}s_{n-3}\cdots s_0)(s_{n-1}s_{n-2}\cdots s_1) =: \sigma^2 w_2.$$

Associated with these Weyl group elements $w_1, w_2 \in W$, we define algebraic varieties $\mathcal{V}_1,\ \mathcal{V}_2 \subset W(\varpi_2)$ as follows.

$$\mathcal{V}_1 := \left\{ V_1(x) := Y_{n-1}(x_{2n-1})\cdots Y_1(x_{n+1})Y_n(x_n)\cdots Y_2(x_2)(1,2) \mid x_i \in \mathbb{C}^\times \right\},$$

$$\mathcal{V}_2 := \left\{ V_2(y) := Y_{n-2}(y_{2n-2})\cdots Y_0(y_n)Y_{n-1}(y_{n-1})\cdots Y_1(y_1)(1,n+1) \mid y_i \in \mathbb{C}^\times \right\}.$$

Using the explicit actions of f_i's on $W(\varpi_2)$ as above, we have $f_i^2 = 0$, for all $i \in I$. Therefore, we have

$$Y_i(c) = \left(1 + \frac{f_i}{c}\right)\alpha_i^\vee(c) \quad \text{for all } i \in I.$$

Thus we can get explicit forms of $V_1(x) \in \mathcal{V}_1$ and $V_2(y) \in \mathcal{V}_2$. Set

$$V_1(x) = V_1(x_2, x_3, \dots, x_{2n-1}) = \sum_{1 \le i < j \le n+1} X_{ij}(i, j),$$

$$V_2(y) = V_2(y_1, y_2, \dots, y_{2n-2}) = \sum_{1 \le i < j \le n+1} Y_{ij}(i, j),$$

where the coefficients X_{ij}'s and Y_{ij}'s can be computed explicitly. These coefficients are positive rational functions in the variables $(x_2, \dots, x_{2n-1})$ and $(y_1, \dots, y_{2n-2})$ respectively and they are given as follows:

$$X_{ij} = \begin{cases} x_{i+1} + \frac{x_{i+2}x_{n+i}}{x_{n+i+1}} + \frac{x_{i+3}x_{n+i}}{x_{n+i+2}} + \cdots + \frac{x_n x_{n+i}}{x_{2n-1}}, & j = n, \\ x_{n+j}(x_{i+1} + \frac{x_{i+2}x_{n+i}}{x_{n+i+1}} + \frac{x_{i+3}x_{n+i}}{x_{n+i+2}} + \cdots + \frac{x_j x_{n+i}}{x_{n+j-1}}), & j \le n-1, \end{cases}$$

$$X_{i,n+1} = \begin{cases} x_{n+i}, & i \ne n, \\ 1, & i = n, \end{cases}$$

$$Y_{ij} = \begin{cases} y_{n+j}(y_{i+1} + \frac{y_{i+2}y_{n+i}}{y_{n+i+1}} + \frac{y_{i+3}y_{n+i}}{y_{n+i+2}} + \cdots + \frac{y_j y_{n+i}}{y_{n+j-1}}), & j \le n-2, \\ y_{i+1} + \frac{y_{i+2}y_{n+i}}{y_{n+i+1}} + \frac{y_{i+3}y_{n+i}}{y_{n+i+2}} + \cdots + \frac{y_{n-1}y_{n+i}}{y_{2n-2}}, & j = n-1, \end{cases}$$

$$Y_{i,n} = \begin{cases} y_{n+i}, & 1 \le i \le n-2, \\ 1, & i = n-1, \end{cases}$$

$$Y_{i,n+1} = \begin{cases} y_{n+i}(y_1 + \frac{y_2 y_n}{y_{n+1}} + \frac{y_3 y_n}{y_{n+2}} + \cdots + \frac{y_i y_n}{y_{n+i-1}}), & 1 \le i \le n-2 \\ y_1 + \frac{y_2 y_n}{y_{n+1}} + \frac{y_3 y_n}{y_{n+2}} + \cdots + \frac{y_{n-1} y_n}{y_{2n-2}}, & i = n-1, \\ y_n, & i = n. \end{cases}$$

Now for a given $x = (x_2, x_3, , \dots, x_{2n-1})$ we solve the equation

$$V_2(y) = a(x)V_1(x), \tag{4.10}$$

where $a(x)$ is a rational function in $x = (x_2, x_3, \dots, x_{2n-1})$. Though this equation is over-determined, it can be solved uniquely by direct calculation and the explicit form of solution is given below.

Lemma 3 *We have the rational function $a(x)$ and the unique solution of* (4.10):

$$a(x) = \frac{1}{x_n}, \qquad y_1 = \left(\frac{x_2}{x_{n+1}} + \frac{x_3}{x_{n+2}} + \cdots + \frac{x_n}{x_{2n-1}}\right)^{-1},$$

$$y_k = x_k\left(\frac{x_{k+1}}{x_{n+k}} + \frac{x_{k+2}}{x_{n+k+1}} + \cdots + \frac{x_n}{x_{2n-1}}\right)^{-1}, \quad 2 \le k \le n-1,$$

$$y_n = \frac{1}{x_n}, \qquad y_{n+l} = \frac{x_{n+l}}{x_n}\left(\frac{x_{l+1}}{x_{n+l}} + \frac{x_{l+2}}{x_{n+l+1}} + \cdots + \frac{x_n}{x_{2n-1}}\right), \quad 1 \le l \le n-2.$$

Now using Lemma 3 we define the map

$$\overline{\sigma} : \mathcal{V}_1 \longrightarrow \mathcal{V}_2,$$
$$V_1(x_2, \ldots, x_{2n-1}) \mapsto V_2(y_1, \ldots, y_{2n-2}).$$

Then we have the following result.

Proposition 1 *The map $\overline{\sigma} : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ is a bi-positive birational isomorphism with the inverse positive rational map*

$$\overline{\sigma}^{-1} : \mathcal{V}_2 \longrightarrow \mathcal{V}_1,$$
$$V_2(y_1, \ldots, y_{2n-2}) \mapsto V_1(x_2, \ldots, x_{2n-1}).$$

given by:

$$x_k = \frac{y_k}{y_n}\left(\frac{y_1}{y_n} + \frac{y_2}{y_{n+1}} + \cdots + \frac{y_k}{y_{n+k-1}}\right)^{-1}, \quad 2 \le k \le n-1,$$
$$x_{n+l} = y_{n+l}\left(\frac{y_1}{y_n} + \frac{y_2}{y_{n+1}} + \cdots + \frac{y_l}{y_{n+l-1}}\right), \quad 1 \le k \le n-2,$$
$$x_n = \frac{1}{y_n}, \quad x_{2n-1} = \left(\frac{y_1}{y_n} + \frac{y_2}{x_{n+1}} + \cdots + \frac{y_{n-1}}{y_{2n-2}}\right).$$

Proof The fact that $\overline{\sigma}$ is a bi-positive birational map follows from the explicit formulas. The rest follows by direct calculation. □

It is known (see [18] and 2.3) that $\mathcal{V}_1$ (resp. $\mathcal{V}_2$) is a geometric crystal for $\mathfrak{g}_0$ (resp. $\mathfrak{g}_n$). Indeed, we have the $\mathfrak{g}_0$-geometric crystal structure on $\mathcal{V}_1$ by setting $Y(x) = Y(x_{2n-1}, \ldots, x_2) := Y_{n-1}(x_{2n-1}) \cdots Y_2(x_2)$, $V_1(x) = V_1(x_{2n-1}, \ldots, x_2) := Y(x)(1, 2)$ and

$$e_i^c\big(V_1(x)\big) := e_i^c\big(Y(x)\big)(1, 2), \qquad \gamma_i\big(V_1(x)\big) = \gamma_i\big(Y(x)\big),$$
$$\varepsilon_i\big(V_1(x)\big) := \varepsilon_i\big(Y(x)\big),$$

since the vector $(1, 2)$ is the highest weight vector with respect to $\mathfrak{g}_0$. Similarly, we obtain the $\mathfrak{g}_n$-geometric crystal structure on $\mathcal{V}_2$. Hence the actions of $e_i^c, \gamma_i, \varepsilon_i$ (resp. $\overline{e}_i^c, \overline{\gamma}_i, \overline{\varepsilon}_i$) on $V_1(x)$ (resp. $V_2(y)$) are described explicitly for $i \in I_0$ (resp. $i \in I_n$) by the formula in 2.3. In particular, the actions of $\overline{e}_0^c, \overline{\gamma}_0$ and $\overline{\varepsilon}_0$ on $V_2(y)$ are given by:

$$\overline{e}_0^c\big(V_2(y)\big) = V_2(y_1, \ldots, cy_n, \ldots, y_{2n-2}),$$
$$\overline{\gamma}_0\big(V_2(y)\big) = \frac{y_n^2}{y_1 y_{n+1}}, \qquad \overline{\varepsilon}_0\big(V_2(y)\big) = \frac{y_{n+1}}{y_n}.$$

In order to make $\mathcal{V}_1$ a $A_n^{(1)}$-geometric crystal we need to define the actions of e_0^c, γ_0 and ε_0 on $V_1(x)$. We define the action of e_0^c on $V_1(x)$ by

$$e_0^c V_1(x) = \overline{\sigma}^{-1} \circ \overline{e}_0^c \circ \overline{\sigma}\bigl(V_1(x)\bigr) \tag{4.11}$$

and the actions of γ_0 and ε_0 on $V_1(x)$ by

$$\gamma_0\bigl(V_1(x)\bigr) = \overline{\gamma}_0\bigl(\overline{\sigma}\bigl(V_1(x)\bigr)\bigr), \qquad \varepsilon_0\bigl(V_1(x)\bigr) := \overline{\varepsilon}_0\bigl(\overline{\sigma}\bigl(V_1(x)\bigr)\bigr). \tag{4.12}$$

Theorem 3 *Together with the actions of e_0^c, γ_0 and ε_0 on $V_1(x)$ given in* (4.11), (4.12), *we obtain a positive affine geometric crystal $\mathcal{V}(A_n^{(1)}) := (\mathcal{V}_1, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ $(I = \{0, 1, \ldots, n\})$, whose explicit form is as follows: first we have $e_i^c(V_1(x))$, $\gamma_i(V_1(x))$ and $\varepsilon_i(V_1(x))$ for $i = 1, 2, \ldots, n$ from the formula* (2.2), (2.3) *and* (2.4).

$$e_i^c\bigl(V_1(x)\bigr) = \begin{cases} V_1(x_2, \ldots, cx_{n+1}, \ldots, x_{2n-1}), & i = 1, \\ V_1(x_2, \ldots, c_i x_i, \ldots, \frac{c}{c_i} x_{n+i}, \ldots, x_{2n-1}), & 2 \le i \le n-1, \\ V_1(x_2, \ldots, cx_n, \ldots, x_{2n-1}), & i = n \end{cases}$$

where

$$c_i = \frac{c(x_i x_{n+i} + x_{i+1} x_{n+i-1})}{c x_i x_{n+i} + x_{i+1} x_{n+i-1}},$$

$$\gamma_i\bigl(V_1(x)\bigr) = \begin{cases} \frac{x_{n+1}^2}{x_2 x_{n+2}}, & i = 1, \\ \frac{x_i^2 x_{n+i}^2}{x_{i-1} x_{i+1} x_{n+i-1} x_{n+i+1}}, & 2 \le i \le n-1, \\ \frac{x_n^2}{x_{n-1} x_{2n-1}}, & i = n. \end{cases}$$

$$\varepsilon_i\bigl(V_1(x)\bigr) = \begin{cases} \frac{x_{n+2}}{x_{n+1}}, & i = 1, \\ \frac{x_{n+i+1}}{x_{n+i}} + \frac{x_{i+1} x_{n+i-1} x_{n+i+1}}{x_i x_{n+i}^2}, & 2 \le i \le n-2, \\ \frac{1}{x_{2n-1}} + \frac{x_n x_{2n-2}}{x_{n-1} x_{2n-1}^2}, & i = n-1, \\ \frac{x_{2n-1}}{x_n}, & i = n. \end{cases}$$

Using (4.11) *and* (4.12), *the explicit actions of e_0^c, ε_0 and γ_0 on $V_1(x)$ are given by*:

$$\gamma_0\bigl(V_1(x)\bigr) = \frac{1}{x_n x_{n+1}}, \qquad \varepsilon_0\bigl(V_1(x)\bigr) = x_{n+1}\left(\frac{x_2}{x_{n+1}} + \frac{x_3}{x_{n+2}} + \cdots + \frac{x_n}{x_{2n-1}}\right),$$

$$e_0^c\bigl(V_1(x)\bigr) = V_1\bigl(x'\bigr) = V_1\bigl(x_2', x_3', \ldots, x_{2n-1}'\bigr),$$

where

$$\begin{cases} x'_k = x_k \cdot \frac{\frac{x_2}{x_{n+1}}+\frac{x_3}{x_{n+2}}+\cdots+\frac{x_n}{x_{2n-1}}}{c(\frac{x_2}{x_{n+1}}+\frac{x_3}{x_{n+2}}+\cdots+\frac{x_k}{x_{n+k-1}})+(\frac{x_{k+1}}{x_{n+k}}+\cdots+\frac{x_n}{x_{2n-1}})}, & 2 \le k < n, \\ x'_n = \frac{x_n}{c}, \qquad x'_{n+1} = \frac{x_{n+1}}{c}, & \\ x'_{n+l} = x_{n+l} \cdot \frac{c(\frac{x_2}{x_{n+1}}+\frac{x_3}{x_{n+2}}+\cdots+\frac{x_l}{x_{n+l-1}})+(\frac{x_{l+1}}{x_{n+l}}+\cdots+\frac{x_n}{x_{2n-1}})}{c(\frac{x_2}{x_{n+1}}+\frac{x_3}{x_{n+2}}+\cdots+\frac{x_n}{x_{2n-1}})}, & 2 \le l < n. \end{cases}$$

Proof Since the positivity is clear from the explicit formulas, it suffices to show that $\mathcal{V}(A_n^{(1)}) := (V_1(x), \{e_i^c\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ satisfies the relations in Definition (1). Indeed, since $\mathcal{V}_1$ is a $\mathfrak{g}_0$ geometric crystal we need to check the relations involving the 0-index:

(1) $\gamma_0(e_i^c(V_1(x))) = c^{a_{i0}}\gamma_0(V_1(x)), 1 \le i \le n$,
(2) $\gamma_i(e_0^c(V_1(x))) = c^{a_{0i}}\gamma_i(V_1(x)),\ 1 \le i \le n$,
(3) $\varepsilon_0(e_0^c(V_1(x))) = c^{-1}\varepsilon_0(V_1(x))$,
(4) $e_0^c e_1^{cd} e_0^d = e_1^d e_0^{cd} e_1^c$,
(5) $e_0^c e_n^{cd} e_0^d = e_n^d e_0^{cd} e_n^c$,
(6) $e_0^c e_i^d = e_i^d e_0^c,\ 2 \le i \le n-1$.

Since

$$\gamma_0\big(e_i^c\big(V_1(x)\big)\big) = \begin{cases} \frac{c^2}{x_n x_{n+1}}, & i = 0, \\ \frac{1}{c x_n x_{n+1}}, & i = 1, n, \\ \frac{1}{x_n x_{n+1}}, & 2 \le i \le n-1, \end{cases}$$

and

$$\gamma_i\big(e_0^c\big(V_1(x)\big)\big) = \begin{cases} \frac{x_{n+1}^2}{c x_n x_{n+2}}, & i = 1, \\ \frac{x_n^2}{c x_{n-1} x_{2n-1}}, & i = n, \\ \frac{x_i^2 x_{n+i}^2}{x_{i-1} x{i+1} x_{n+i-1} x_{n+i+1}}, & 2 \le i \le n-1, \end{cases}$$

we have (1) and (2) hold. We also have (3) hold since $\mathcal{V}_2$ is a $\mathfrak{g}_n$-geometric crystal and hence

$$\begin{aligned} \varepsilon_0\big(e_0^c\big(V_1(x)\big)\big) &= \overline{\varepsilon}_0\overline{\sigma}\overline{\sigma}^{-1}\overline{e}_0^c\overline{\sigma}\big(V_1(x)\big) = \overline{\varepsilon}_0\overline{e}_0^c\big(V_2(y)\big) \\ &= \overline{\varepsilon}_0\big(V_2\big(y'\big)\big) = \frac{y'_{n+1}}{y'_n} = \frac{y_{n+1}}{c y_n} = c^{-1}\varepsilon_0\big(V_1(x)\big). \end{aligned}$$

By direct calculations we see that on $V_1(x)$ we have

$$\overline{\sigma} \circ e_i^c = \overline{e}_i^c \circ \overline{\sigma}, \quad \text{for } 1 \le i \le n-1.$$

Hence for $2 \le i \le n-1$, we have

$$e_0^c e_i^d = \left(\overline{\sigma}^{-1}\overline{e}_0^c\overline{\sigma}\right)\left(\overline{\sigma}^{-1}\overline{e}_i^d\overline{\sigma}\right) = \overline{\sigma}^{-1}\overline{e}_0^c\overline{e}_i^d\overline{\sigma}$$
$$= \overline{\sigma}^{-1}\overline{e}_i^d\overline{e}_0^c\overline{\sigma} = e_i^d e_0^c,$$

and

$$e_0^c e_1^{cd} e_0^d = \left(\overline{\sigma}^{-1}\overline{e}_0^c\overline{\sigma}\right)\left(\overline{\sigma}^{-1}\overline{e}_1^{cd}\overline{\sigma}\right)\left(\overline{\sigma}^{-1}\overline{e}_0^d\overline{\sigma}\right)$$
$$= \overline{\sigma}^{-1}\overline{e}_0^c\overline{e}_i^{cd}\overline{e}_0^d\overline{\sigma} = \overline{\sigma}^{-1}\overline{e}_1^d\overline{e}_0^{cd}\overline{e}_1^c\overline{\sigma} = e_1^d e_0^{cd} e_1^c,$$

since $\mathcal{V}_2$ is a $\mathfrak{g}_n$-geometric crystal. Therefore, (4) and (6) hold.

Now for $k = 2, \ldots, n-1$ we set $X = X_k + \widetilde{X}_k$ where

$$X_k = \frac{x_2}{x_{n+1}} + \frac{x_3}{x_{n+2}} + \cdots + \frac{x_k}{x_{k+n-1}}, \qquad \widetilde{X}_k = \frac{x_{k+1}}{x_{k+n}} + \frac{x_{k+2}}{x_{k+n+1}} + \cdots + \frac{x_n}{x_{2n-1}}.$$

Observe that for any $k, l = 2, \ldots, n-1$ we have $X = X_k + \widetilde{X}_k = X_l + \widetilde{X}_l$. Recall that $e_0^c(V_1(x)) = V_1(x') = V_1(x'_2, \ldots, x'_{2n-1})$. Now we have

$$\frac{x'_k}{x'_{k+n-1}} = \frac{cX^2}{c-1}\left(\frac{1}{cX_{k-1} + \widetilde{X}_{k-1}} - \frac{1}{cX_k + \widetilde{X}_k}\right) \quad (3 \le k \le n-1,\ c \ne 1). \tag{4.13}$$

Using Eq. (4.13) we can easily see that (5) holds which completes the proof. □

5 Ultra-Discretization of $\mathcal{V}(A_n^{(1)})$

We denote the positive structure on $\mathcal{V} = \mathcal{V}(A_n^{(1)})$ as in the previous section by $\theta : T' := (\mathbb{C}^\times)^{2n-2} \longrightarrow \mathcal{V}$ $(x \mapsto V_1(x))$. Then by Corollary 1 we obtain the ultra-discretization $\mathcal{X} = \mathcal{UD}(\mathcal{V}, T', \theta)$ which is a Kashiwara's crystal. Now we show that the conjecture in [18] holds for $\mathfrak{g} = A_n^{(1)},\ i = 2$ by giving an explicit isomorphism of crystals between $\mathcal{X}$ and $B^{2,\infty}$. In order to show this isomorphism, we need the explicit crystal structure on $\mathcal{X} := \mathcal{UD}(\chi, T', \theta)$. Note that $\mathcal{X} = \mathbb{Z}^{2n-2}$ as a set. In $\mathcal{X}$, we use the same notations $c, x_0, x_2, \ldots, x_{2n-1}$ for variables as in $\mathcal{V}$.

For $x = (x_2, x_1, \ldots, x_{2n-1}) \in \mathcal{X}$, by applying the ultra-discretization functor $\mathcal{UD}$ it follows from the results in the previous section that the functions $\mathrm{wt}_i = \mathcal{UD}(\gamma_i)$, $\varepsilon_i = \mathcal{UD}(\varepsilon_i)$ and $\mathcal{UD}(e_i^c)$ for $i = 0, 1, \ldots, n$ are given by:

$$\mathrm{wt}_i(x) = \begin{cases} -x_n - x_{n+1}, & i = 0, \\ -x_2 + 2x_{n+1} - x_{n+2}, & i = 1, \\ 2x_2 - x_3 - x_{n+1} + 2x_{n+2} - x_{n+3}, & i = 2, \\ -x_{i-1} + 2x_i - x_{i+1} - x_{n+i-1} + 2x_{n+i} - x_{n+i+1}, & 3 \le i < n, \\ -x_{n-1} + 2x_n - x_{2n-1}, & i = n. \end{cases}$$

$$\varepsilon_i(x) = \begin{cases} x_{n+1} + \max_{2\le k\le n}(\beta_k), & i=0,\\ -x_{n+1}+x_{n+2}, & i=1,\\ \max(x_{n+i+1}-x_{n+i}, -x_i+x_{i+1} \\ \quad +x_{n+i-1}-2x_{n+i}+x_{n+i+1}), & 2\le i\le n-2,\\ \max(-x_{2n-1}, -x_{n-1}+x_n+x_{2n-2}-2x_{2n-1}), & i=n-1,\\ -x_n+x_{2n-1}, & i=n, \end{cases}$$

where $\beta_k := x_k - x_{n+k-1}$ for $2 \le k \le n$.

$$\mathcal{UD}\big(e_i^c\big)(x) = \begin{cases} (x_2+C_2,\dots,x_{n-1}+C_{n-1}, x_n-c, x_{n+1}-c, \\ x_{n+2}-c-C_2,\dots,x_{2n-1}-c-C_{n-1}), & i=0,\\ (x_2,\dots,x_n,x_{n+1}+c,x_{n+2},\dots,x_{2n-1}), & i=1,\\ (x_2,\dots,x_i+\overline{c}_i,\dots,x_{n+i}+c-\overline{c}_i,\dots,x_{2n-1}), & 2\le i<n,\\ (x_2,\dots,x_{n-1},x_n+c,x_{n+1},\dots,x_{2n-1}), & i=n, \end{cases}$$

where

$$C_k = \max_{2\le j\le n}(\beta_j) - \max\Big(\max_{2\le j\le k}(c+\beta_j), \max_{k<j\le n}(\beta_j)\Big), \quad 2\le k<n,$$

$$\overline{c}_i = c + \max(x_i + x_{n+i}, x_{i+1}+x_{n+i-1}) - \max(c+x_i+x_{n+i}, x_{i+1}+x_{n+i-1}),$$
$$2 \le i < n.$$

Note that the Kashiwara operators are $\tilde{e}_i(x) = \mathcal{UD}e_i^c(x)|_{c=1}$ and $\tilde{f}_i(x) = \mathcal{UD}e_i^c(x)|_{c=-1}$ on $\mathcal{X}$. In particular, for $x \in \mathcal{X}$, we have

$$\begin{cases} \tilde{f}_1(x) = (x_2,\dots,x_{n+1}-1,\dots,x_{2n-1}),\\ \tilde{f}_n(x) = (x_2,\dots,x_n-1,\dots,x_{2n-1}), \end{cases} \tag{5.14}$$

and for $2 \le i \le n-1$,

$$\tilde{f}_i(x) = \begin{cases} (x_2,\dots,x_{n+i}-1,\dots,x_{2n-1}), & \text{if } \beta_i > \beta_{i+1},\\ (x_2,\dots,x_i-1,\dots,x_{2n-1}), & \text{if } \beta_i \le \beta_{i+1}. \end{cases} \tag{5.15}$$

To determine the explicit action of $\tilde{f}_0$ we define conditions:

$$(\phi_j): \quad \beta_2,\dots,\beta_{j-1} \le \beta_j > \beta_{j+1},\dots,\beta_n \tag{5.16}$$

for each $2 \le j \le n$ where we assume $\beta_1 = 0 = \beta_{n+1}$. Note that under condition (ϕ_j) we have:

$$C_2 = \cdots = C_{j-1} = 0, \quad \text{and} \quad C_j = \cdots = C_{n-1} = 1.$$

Hence for $x \in \mathcal{X}$ and $2 \le j \le n$ we have

$$\tilde{f}_0(x) = (x_2,\dots,x_{j-1},x_j+1,x_{j+1}+1,\dots,x_{n+j-1}+1,x_{n+j},\dots,x_{2n-1}),$$

if condition (ϕ_j) hold.

Theorem 4 *The map*

$$\begin{array}{cccc} \Omega: & \mathcal{X} & \longrightarrow & B^{2,\infty}, \\ & (x_2,\dots,x_{2n-1}) & \mapsto & b=(b_{ji})_{1\le j\le 2, j\le i\le j+n-1}, \end{array}$$

defined by

$$b_{11}=x_{n+1}, \qquad b_{1i}=x_{n+i}-x_{n+i-1}, \quad 2\le i\le n-1, \qquad b_{1n}=-x_{2n-1},$$
$$b_{22}=x_2, \qquad b_{2i}=x_i-x_{i-1}, \quad 3\le i\le n, \qquad b_{2,n+1}=-x_n,$$

is an isomorphism of crystals.

Proof First we observe that the map $\Omega^{-1}: B^{2,\infty}\longrightarrow \mathcal{X}$ is given by $\Omega^{-1}(b)=x=(x_2,\dots,x_{2n-1})$ where

$$x_i=\sum_{k=2}^{i} b_{2k}, \quad 2\le i\le n,$$

$$x_{n+i}=\sum_{k=1}^{i} b_{1k}, \quad 1\le i\le n-1.$$

Hence the map Ω is bijective. To prove that Ω is an isomorphism of crystals we need to show that it commutes with the actions of $\tilde{f}_i$ and preserves the actions of the functions wt_i and ε_i. In particular we need to show that for $x\in\mathcal{X}$ and $0\le i\le n$ we have:

$$\begin{aligned} &\Omega\big(\tilde{f}_i(x)\big)=\tilde{f}_i\big(\Omega(x)\big),\\ &\mathrm{wt}_i\big(\Omega(x)\big)=\mathrm{wt}_i(x),\\ &\varepsilon_i\big(\Omega(x)\big)=\varepsilon_i(x). \end{aligned}$$

Indeed commutativity of Ω and $\tilde{e}_i$ follows similarly. For $x\in\mathcal{X}$, set $\Omega(x)=b=(b_{ji})\in B^{2,\infty}$. First let us check wt_i.

$$\begin{aligned} \mathrm{wt}_0\big(\Omega(x)\big)&=\mathrm{wt}_0(b)=b_{2,n+1}-b_{11}=-x_n-x_{n+1}=\mathrm{wt}_0(x),\\ \mathrm{wt}_1\big(\Omega(x)\big)&=\mathrm{wt}_1(b)=b_{11}-b_{12}-b_{22}=x_{n+1}-(x_{n+2}-x_{n+1})-x_2\\ &=-x_2+2x_{n+1}-x_{n+2}=\mathrm{wt}_1(x),\\ \mathrm{wt}_2\big(\Omega(x)\big)&=\mathrm{wt}_2(b)=(b_{12}-b_{13})-(b_{22}-b_{23})\\ &=x_{n+2}-x_{n+1}-x_{n+3}+x_{n+2}+x_2-x_3+x_2\\ &=2x_2-x_3-x_{n+1}+2x_{n+2}-x_{n+3}=\mathrm{wt}_2(x), \end{aligned}$$

$$\begin{aligned}
\mathrm{wt}_i\bigl(\Omega(x)\bigr) &= \mathrm{wt}_i(b) = (b_{1i} - b_{1,i+1}) + (b_{2i} - b_{2,i+1}) \\
&= x_{n+i} - x_{n+i-1} - x_{n+i+1} + x_{n+i} + x_i - x_{i-1} - x_{i+1} + x_i \\
&= -x_{i-1} + 2x_i - x_{i+1} - x_{n+i-1} + 2x_{n+i} - x_{n+i+1} = \mathrm{wt}_i(x), \\
&\qquad 3 \le i \le n-1, \\
\mathrm{wt}_n\bigl(\Omega(x)\bigr) = \mathrm{wt}_n(b) &= b_{1n} + (b_{2n} - b_{2,n+1}) \\
&= -x_{2n-1} + x_n - x_{n-1} + x_n = -x_{n_1} + 2x_n - x_{2n-1} \\
&= \mathrm{wt}_n(x).
\end{aligned}$$

Next, we shall check ε_i:

$$\begin{aligned}
\varepsilon_0\bigl(\Omega(x)\bigr) &= \varepsilon_0(b) = -b_{2,n+1} - \Delta \\
&= -b_{2,n+1} - \min_{2\le k\le n}(b_{12} + \cdots + b_{1,k-1} + b_{2,k+1} + \cdots + b_{2n}) \\
&= x_n - \min_{2\le k\le n}(x_{n+k-1} - x_{n+1} + x_n - x_k) \\
&= x_n + \max_{2\le k\le n}(-x_{n+k-1} + x_{n+1} - x_n + x_k) \\
&= x_{n+1} + \max(x_k - x_{n+k-1}) = \varepsilon_0(x). \\
\varepsilon_1\bigl(\Omega(x)\bigr) &= \mathrm{wt}_1(b) = b_{12} = x_{n+2} - x_{n+1} = \varepsilon_1(x), \\
\varepsilon_i\bigl(\Omega(x)\bigr) &= \varepsilon_i(b) = b_{1,i+1} + (b_{2,i+1} - b_{1i})_+ \\
&= \max(b_{1,i+1}, b_{1,i+1} + b_{2,i+1} - b_{1i}) \\
&= -\max(x_{n+i+1} - x_{n+i}, -x_i + x_{i+1} + x_{n+i-1} - 2x_{n+i} + x_{n+i+1}) \\
&= \varepsilon_i(x), \quad \text{for } 2 \le i \le n-2, \\
\varepsilon_{n-1}\bigl(\Omega(x)\bigr) &= \varepsilon_{n-1}(b) = \max(b_{1n}, b_{1n} + b_{2n} - b_{1,n-1}) \\
&= \max(-x_{2n-1}, -x_{n-1} + x_n + x2n - 2 - 2x_{2n-1}) = \varepsilon_{n-1}(x), \\
\varepsilon_n\bigl(\Omega(x)\bigr) &= \varepsilon_n(b) = b_{2,n+1} - b_{1n} = -x_n + x_{2n-1} = \varepsilon_n(x).
\end{aligned}$$

Now we shall check that $\Omega(\tilde{f}_i(x)) = \tilde{f}_i(\Omega(x))$ for $i = 0, 1, \ldots, n$.

$$\tilde{f}_1\bigl(\Omega(x)\bigr) = \tilde{f}_1(b) = b' = \bigl(b'_{ji}\bigr),$$

where

$$\begin{aligned}
&b'_{11} = b_{11} - 1 = x_{n+1} - 1, \qquad b'_{12} = b_{12} + 1 = x_{n+2} - x_{n+1} + 1, \\
&b'_{ji} = b_{ji}, \quad \text{otherwise.}
\end{aligned}$$

Hence $\Omega(\tilde{f}_1(x)) = \Omega(x_2, \ldots, x_{n+1} - 1, \ldots, x_{2n-1}) = \tilde{f}_1(\Omega(x))$.

$$\tilde{f}_n\bigl(\Omega(x)\bigr) = \tilde{f}_n(b) = b' = \bigl(b'_{ji}\bigr),$$

where

$$b'_{2n} = b_{2n} - 1 = x_n - x_{n-1} - 1, \qquad b'_{2,n+1} = b_{2,n+1} + 1 = -x_n + 1,$$
$$b'_{ji} = b_{ji}, \quad \text{otherwise.}$$

Hence $\Omega(\tilde{f}_n(x)) = \Omega(x_2, \ldots, x_n - 1, \ldots, x_{2n-1}) = \tilde{f}_n(\Omega(x))$. Now we check that $\Omega(\tilde{f}_i(x)) = \tilde{f}_i(\Omega(x))$ for $2 \le i \le n-1$. Let $\tilde{f}_i(\Omega(x)) = \tilde{f}_i(b) = b' = (b'_{ji})$. Note that $b_{1i} = x_{n+i} - x_{n+i-1}$ and $b_{1,i+1} = x_{i+1} - x_i$. Hence $b_{1i} > b_{2,i+1}$ (resp. $b_{1i} \le b_{2,i+1}$) if and only if $\beta_i > \beta_{i+1}$ (resp. $\beta_i \le \beta_{i+1}$).

If $x_{n+i} - x_{n+i-1} > x_{i+1} - x_i$, then $\tilde{f}_i(\Omega(x)) = \tilde{f}_i(b) = b' = (b'_{ji})$, where

$$b'_{1i} = b_{1i} - 1 = x_{n+i} - x_{n+i-1} - 1, \qquad b'_{1,i+1} = b_{1,i+1} + 1 = x_{n+i+1} - x_{n+i} + 1,$$
$$b'_{ji} = b_{ji}, \quad \text{otherwise.}$$

Hence $\Omega(\tilde{f}_i(x)) = \Omega(x_2, \ldots, x_{n+i} - 1, \ldots, x_{2n-1}) = \tilde{f}_i(\Omega(x))$ in this case.

If $x_{n+i} - x_{n+i-1} \le x_{i+1} - x_i$, then $\tilde{f}_i(\Omega(x)) = \tilde{f}_i(b) = b' = (b'_{ji})$, where

$$b'_{2i} = b_{2i} - 1 = x_i - x_{i-1} - 1, \qquad b'_{2,i+1} = b_{2,i+1} + 1 = x_{i+1} - x_i + 1,$$
$$b'_{ji} = b_{ji}, \quad \text{otherwise.}$$

Hence $\Omega(\tilde{f}_i(x)) = \Omega(x_2, \ldots, x_i - 1, \ldots, x_{2n-1}) = \tilde{f}_i(\Omega(x))$ in this case.

Finally we want to verify that $\Omega(\tilde{f}_0(x)) = \tilde{f}_0(\Omega(x))$. For $2 \le m \le n$, we have $\tilde{f}_0(\Omega(x)) = \tilde{f}_0(b) = b' = (b'_{ji})$ where

$$b'_{11} = b_{11} + 1 = x_{n+1} + 1,$$
$$b'_{1m} = b_{1m} - 1 = \begin{cases} x_{n+m} - x_{n+m-1} - 1, & \text{if } m \ne n, \\ -x_{2n-1} - 1, & \text{if } m = n, \end{cases}$$
$$b'_{2m} = b_{2m} + 1 = \begin{cases} x_2 + 1, & \text{if } m = 2, \\ x_m - x_{m-1} + 1 & \text{if } m \ne 2, \end{cases}$$
$$b'_{2,n+1} = b_{2,n+1} - 1 = -x_n - 1, \qquad b'_{ji} = b_{ji}, \quad \text{otherwise,}$$

if the condition (F_m) in (3.7) holds. Since $z_i = b_{1i} - b_{2,i+1} = (x_{n+i} - x_{n+i-1}) - (x_{i+1} - x_i) = \beta_i - \beta_{i+1}$ for $2 \le i \le n-1$, we observe that for $2 \le m \le n$, the condition (F_m) in (3.7) holds if and only if the condition (ϕ_m) in (5.16) holds. Therefore, for $2 \le m \le n$, we have

$$\Omega\big(\tilde{f}_0(x)\big) = \Omega(x_2, \ldots, x_{m-1}, x_m + 1, \ldots, x_{n+m-1} + 1, x_{n+m}, \ldots, x_{2n-1})$$
$$= \tilde{f}_0\big(\Omega(x)\big),$$

which completes the proof. □

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A $\mathbb{Z}_3$-Orbifold Theory of Lattice Vertex Operator Algebra and $\mathbb{Z}_3$-Orbifold Constructions

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Abstract For an even positive definite lattice L and its automorphism σ of order 3, we prove that a fixed point subVOA V_L^σ of a lattice VOA V_L is C_2-cofinite. Using this result and the results in arXiv:0909.3665, we present $\mathbb{Z}_3$-orbifold constructions of holomorphic VOAs from lattice VOAs V_Λ, where Λ are even unimodular positive definite lattices. One of them has the same character with the moonshine VOA $V^\natural$ and another is a new VOA corresponding to No. 32 in Schellekens' list (Theor. Mat. Fiz. 95(2), 348–360, 1993).

1 Introduction

This is a half part of the preprint [13] and we will publish the other half separately.

A concept of a vertex operator algebra (shortly VOA) $V = (V, Y, \mathbf{1}, \omega)$ was introduced by Borcherds [1] with a purpose to explain the moonshine phenomenon [3] and then as a stage for studying the phenomenon, Frenkel, Lepowsky and Meurman [7] constructed the moonshine VOA $V^\natural$ by a $\mathbb{Z}_2$-orbifold construction from the Leech lattice VOA V_Λ and an automorphism -1 on Λ.

When we consider such an orbifold construction from a lattice VOA V_L with a finite automorphism σ, the rationality of the fixed point subVOA $(V_L)^\sigma$ (i.e. all $\mathbb{N}$-gradable $(V_L)^\sigma$-modules are completely reducible) and C_2-cofiniteness of $(V_L)^\sigma$ (i.e. $\mathrm{Span}_{\mathbb{C}}\{v_{-2}u \mid v, u \in (V_L)^\sigma\}$ has a finite codimension in $(V_L)^\sigma$) are very useful conditions because there are several significant known theorems under these conditions. For an automorphism -1 of L, these properties are already known by [19] and [20]. Our target is an automorphism of order three.

Theorem A *Let L be a positive definite even lattice and V_L a lattice VOA associated with L. Let $\sigma \in \mathrm{Aut}(L)$ of order three. We use the same notation for an automorphism of V_L lifted from σ. Then a fixed point subVOA V_L^σ is C_2-cofinite.*

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We note that some special case is already studied in [17]. Generally, proving of C_2-cofiniteness looks easier than that of rationality. Moreover, we quote the following result.

Theorem ([12]) *Let V be a rational VOA of CFT-type satisfying $V' \cong V$ and σ a finite automorphism of V. If V^σ is C_2-cofinite, then V^σ is rational and every simple V^σ-module is a submodule of twisted σ^j-module of V for some j.*

Keeping the above theorem in mind, we will show the following theorem.

Theorem B *Let Λ be a positive definite even unimodular lattice with an automorphism σ of order three. We assume that the conclusions in the above theorem hold for $V = V_\Lambda$ and σ. If $\mathrm{rank}(\Lambda) - \mathrm{rank}(\Lambda^\sigma)$ is divisible by 3, then we are able to construct a holomorphic VOA $\widetilde{V}$ by a $\mathbb{Z}_3$-orbifold construction from a lattice VOA V_Λ and σ.*

At the end of paper, we will study two examples. One of them has the same character with the moonshine VOA $V^\natural$ and another is a new VOA corresponding to No. 32 in Schellekens' list [16].

2 Proof of Theorem A

Before we start the proofs, we present several properties of C_2-cofiniteness. There are several different definitions of modules for vertex operator algebras. We will consider the widest one. Namely, a V-module is just a vector space W on which all v_n $(v \in V, n \in \mathbb{Z})$ act such that vertex operators

$$Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-1-n} \in \mathrm{End}(W)[[z,z^{-1}]]$$

satisfies a truncation property on each $w \in W$, $L(-1)$-derivative property, and Borcherds' identity. One of main properties of C_2-cofiniteness is that all V-modules are $\mathbb{Z}_+$-gradable as we will see. Namely, there are no weak modules.

Proposition 1 *Let V be a VOA. We have the following*:

(i) *If V is C_2-cofinite then the number of inequivalent simple modules is finite*, [8].
(ii) *Set $V = B + C_2(V)$ for B spanned by homogeneous elements. Then any V-module W generated from one element w has a spanning set $\{v^1_{n_1} \dots v^k_{n_k} w \mid v^i \in B, n_1 < \cdots < n_k\}$. In particular, if V is C_2-cofinite, then finitely generated V-modules are C_2-cofinite*, [2, 11].
(iii) *All modules are $\mathbb{Z}_+$-gradable if and only if V is C_2-cofinite*, [11] *and* [2].
(iv) *If U, W, T are C_2-cofinite V-modules, then a fusion product $U \boxtimes W$ is a well-defined V-module and C_2-cofinite and $(U \boxtimes W) \boxtimes T \cong U \boxtimes (W \boxtimes T)$*, [9, 12, 14].

Let L be an even positive lattice and σ an automorphism of order 3. Abusing the notation, we use the same notation to denote an automorphism of a lattice VOA V_L lifted from σ. We note that all lattice VOA are C_2-cofinite [5].

One of the advantages of proving C_2-cofiniteness is that if a full subVOA is C_2-cofinite, then the larger VOA is also C_2-cofinite, see Proposition 1. Here a subVOA U of V is called full subVOA if it contains a Virasoro element of V. Hence it is sufficient to prove C_2-cofiniteness of $(V_H)^\sigma$ for any σ-invariant full sublattice H of L, where a sublattice H of L is called full if $\mathbb{Q}\otimes_{\mathbb{Z}} H=\mathbb{Q}\otimes_{\mathbb{Z}} L$. This is very useful. For example, we are able to consider H doubly even, that is, $\langle h,h\rangle\in 4\mathbb{Z}$ for $h\in H$. Furthermore, for VOAs V^1 and V^2, $V^1\otimes V^2$ is C_2-cofinite if and only if the both V^i are C_2-cofinite by the Proposition 1(iii). Therefore, it is enough to prove Theorem A for $L=\mathbb{Z}x+\mathbb{Z}y$ with $y=\sigma(x)$, $-x-y=\sigma^2(x)$ and $\langle x,x\rangle=-2\langle x,y\rangle=18M>72$ and M is even.

2.1 A Lattice VOA

Recalling the definition of lattice VOA V_L from [7], we will explain the notation of this paper. Viewing $\mathbb{C}L=\mathbb{C}x+\mathbb{C}y$ as a commutative Lie algebra with a symmetric bilinear form $\langle\cdot,\cdot\rangle$, $\mathcal{H}:=\mathbb{C}L[t,t^{-1}]\oplus\mathbb{C}$ becomes an affine Lie algebra with a product

$$\left[v\otimes t^n,u\otimes t^m\right]=\delta_{n+m,0}n\langle v,u\rangle,$$

for $v,u\in\mathbb{C}L$. Hereafter we use $v(n)$ to denote $v\otimes t^n$. We then consider its universal enveloping algebra $U(\mathcal{H})$ and its subalgebras $\mathcal{H}^+=\mathbb{C}L[t]$ and $\mathcal{H}^-=\mathbb{C}L[t^{-1}]$. Using an $U(\mathcal{H}^+)$-module $\mathbb{C}e^\gamma$ with the actions

$$\mu(0)e^\gamma=\langle\mu,\gamma\rangle e^\gamma,\quad\text{and}\quad\mu(n)e^\gamma=0\quad\text{for } n>0$$

for $\mu\in\mathbb{C}L$, we define an $U(\mathcal{H})$-module:

$$M_2(1)e^\gamma:=U(\mathcal{H})\otimes_{U(\mathcal{H}^+)}\mathbb{C}e^\gamma.$$

We note $M_2(1)e^\gamma\cong S(\mathcal{H}^-)$ as vector spaces, where $S(\mathcal{H}^-)$ denotes a symmetric tensor algebra of $\mathcal{H}^-$. We also note that $M_2(1)e^\gamma$ is spanned by

$$\left\{x(-i_1)\cdots x(-i_k)y(-j_1)\cdots y(-j_h)e^\gamma\mid i_1\geq\cdots\geq i_k>0,\ j_1\geq\cdots\geq j_h>0\right\}$$

and their weights are defined by $\sum_{s=1}^k i_s+\sum_{t=1}^h j_t+\frac{\langle\gamma,\gamma\rangle}{2}$.

We set the vector spaces for a lattice VOA V_L and a subVOA $M_2(1)$ by

$$V_L:=\bigoplus_{\gamma\in L}M_2(1)e^\gamma\quad\text{and}\quad M_2(1):=M_2(1)e^0.$$

We also define vertex operators $Y(e^\gamma, z)$ as follows:

$$\begin{cases} Y(e^0, z) := \mathrm{Id}_{V_L} \quad \text{(i.e. } e^0 \text{ is the Vacuum } \mathbf{1}\text{)}, \\ Y(e^\gamma, z) := \displaystyle\sum_{p=0}^{\infty} \frac{1}{p!}\Bigl(\sum_{n\in\mathbb{Z}_+} \frac{\gamma(-n)}{-n} z^{-n}\Bigr)^p \sum_{m=0}^{\infty} \frac{1}{m!}\Bigl(\sum_{n\in\mathbb{Z}_+} \frac{\gamma(n)}{n} z^{n}\Bigr)^m e^\gamma z^\gamma, \end{cases}$$

where $e^\gamma e^\mu = e^{\gamma+\mu}$ for $e^\mu, e^{\gamma+\mu} \in V_L$ and $z^\gamma e^\mu = z^{\langle\gamma,\mu\rangle} e^\mu$ for $e^\mu \in V_L$. We note that we usually need cocycle $c(\gamma,\mu)$ to define a product $e^\gamma e^\mu = c(\gamma,\mu)e^{\gamma+\mu}$. However, since we have chosen L enough small so that $\langle\gamma,\mu\rangle \in 4\mathbb{Z}$ for any $\gamma,\mu \in L$, we may assume $e^\gamma e^\mu = e^{\gamma+\mu}$ for any $\gamma,\mu \in L$ by choosing suitable basis $\{e^\gamma \mid \gamma \in L\}$.

For general elements u, we define $Y(u,z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}$ inductively by using normal products

$$\bigl(v(-m)\alpha\bigr)_n = \sum_{i=0}^{\infty} (-1)^i \binom{-m}{i} \bigl\{v(-m-i)\alpha_{n+i} - (-1)^m \alpha_{-m+n-i} v(i)\bigr\} \tag{2.1}$$

for $v \in \mathbb{C}L$ and $\alpha \in M_n(1)e^\gamma$ and $u = v(-m)\alpha$, where $\binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{i!}$. We will frequently use this normal product expansion (2.1).

2.2 *In the Free Bosonic Fock Space* $M_2(1)$

We will treat two VOAs $M_2(1)$ and $M_2(1)^\sigma$ at the same time. In order to avoid the confusion, $C_2(W)$ denotes only $\mathrm{Span}_{\mathbb{C}}\{v_{-2}w \mid w \in W, v \in M_2(1)^\sigma\}$ by viewing W as an $M_2(1)^\sigma$-module and $\equiv_2$ denotes a congruence relation modulo $C_2(M_2(1)^\sigma)$. We will also use

$$C_1(W) := \mathrm{Span}_{\mathbb{C}}\bigl\{v_{-1}w \mid w \in W, v \in \bigl(M_2(1)\bigr)^\sigma, \mathrm{wt}(v) > 0\bigr\}.$$

Set $\xi = e^{2\pi\sqrt{-1}/3}$. Viewing $\mathbb{C}L$ as a $\mathbb{C}[\sigma]$-module, there are $a, a' \in \mathbb{C}L$ such that $\sigma(a) = \xi a$, $\sigma(a') = \xi^{-1}a'$ and $\langle a, a'\rangle = 1$. Hence $M_2(1)$ is spanned by

$$\bigl\{a(-i_1)\cdots a(-i_h)a'(-j_1)\cdots a'(-j_k)\mathbf{1} \mid i_1 \geq \cdots \geq i_h > 0, j_1 \geq \cdots \geq j_k > 0\bigr\}.$$

Clearly, $u = a(-i_1)\cdots a(-i_h)a'(-j_1)\cdots a'(-j_k)\mathbf{1}$ is σ-invariant if and only if $h - k \equiv 0 \pmod 3$. We note that $\omega = a(-1)a' = a'(-1)a$ is the Virasoro element of $M_2(1)$. Since we will use an induction often, we introduce left ideals

$$\mathcal{P}_k = \mathrm{Span}_{\mathbb{C}}\bigl\{\alpha^1(i_1)\cdots\alpha^h(i_h) \mid i_h \geq k, \alpha^j \in L\bigr\} \subseteq \mathrm{End}(V_L)$$

of $U(\mathcal{H})$ for $k = 0, 1$. We note $\mathcal{P}_0 e^0 = 0$ and $\mathcal{P}_1 e^\gamma = 0$.

Lemma 1 *For $k = 0, 1$ and $m, n, l > 0$ and $v \in M_2(1)$, there are $\lambda_{s,t}, \lambda_{s,t,u} \in \mathbb{Q}$ such that*

$$a(-l)a'(-m)v - \sum_{s,t>0} \lambda_{s,t}\big(a(-s)a'(-t)\mathbf{1}\big)_{k-1} v \in \mathcal{P}_k v,$$

$$a(-l)a(-m)a(-n)v - \sum \lambda_{s,t,u}\big(a(-s)a(-t)a(-u)\mathbf{1}\big)_{k-1} v \in \mathcal{P}_k v.$$

Proof These comes from the normal product (2.1). We prove only the second case for $k = 0$. Since

$$\big(a(-i)\gamma\big)_{-1} v = \sum_{j=0}^{\infty} \binom{-i}{j} (-1)^j \big\{a(-i-j)\gamma_{-1+i} - (-1)^i \gamma_{-i-1-j} a(j)\big\} v,$$

there are $\lambda_i, \lambda_{ij} \in \mathbb{Q}$ and $\lambda_0 = 1$ such that

$$\begin{aligned}
\big(a(-s)a(-t)a(-u)\mathbf{1}\big)_{-1} v &\in \sum_{i=0}^{\infty} \lambda_i a(-s-i)\big\{a(-t)a(-u)\mathbf{1}\big\}_{-1+i} v + \mathcal{P}_0 v \\
&= \sum_{i=0}^{\infty} \lambda_{i,j} a(-s-i) \sum_{j=0}^{\infty} a(-t-j)\big\{a(-u)\mathbf{1}\big\}_{-1+i+j} v + \mathcal{P}_0 v \\
&= \sum_{i=0}^{\infty} \lambda_{i,j} a(-s-i) \sum_{j=0}^{\infty} a(-t-j)a(-u+i+j)v + \mathcal{P}_0 v \\
&= a(-s)a(-t)a(-u)v + \sum_{p=1}^{\infty}\sum_{i=0}^{p} \lambda_{i,p-i} a(-s-i)a(-t-p+i)a(-u+p)v \\
&\quad + \mathcal{P}_0 v.
\end{aligned}$$

Hence $a(-l)a(-m)a(-n)v - (a(-l)a(-m)a(-n)\mathbf{1})_{-1} v$ is equivalent to a $\mathbb{Q}$-linear combination of $\{a(-s-i)a(-t-p+i)a(-u+p)v \mid p > 0\}$ modulo $\mathcal{P}_0 v$. Iterating these steps, we can reduce them to zero modulo $\mathcal{P}_0 v$. □

2.3 *Modulo $C_2(M_2(1)^\sigma)$*

For a while, we consider the following element:

$$u = a(-i_1)\cdots a(-i_h)a'(-j_1)\cdots a'(-j_k)\mathbf{1} \quad \text{with } i_s, j_t > 0. \tag{2.2}$$

We will call u a "bloated element" if $\mathrm{wt}(u) > h + k$. Set

$$\mathcal{S}_2 = \left\{ \begin{matrix} a(-1)^i a'(-1)^j \mathbf{1}, a(-i)a(-j)a, a'(-i)a'(-j)a' \\ a(-i)a'(-1)^2 a, a(-i)a', \mathbf{1} \end{matrix} \;\middle|\; i, j \in \mathbb{N} \right\}.$$

Proposition 2 *Let* $u = a(-i_1)\cdots a(-i_h)a'(-j_1)\cdots a'(-j_k)\mathbf{1}$ *be a bloated element.*
If $|h-k| \geq 4$, *then* $u \in C_2(M_2(1)^\sigma)$.
If $h-k=3$, *then* $u \in \mathrm{Span}_{\mathbb{C}}\{a(-i_1)a(-i_2)a \mid i_1, i_2 \in \mathbb{N}\} + C_2(M_2(1)^\sigma)$.
If $h=k$, *then*

$$u \in \mathrm{Span}_{\mathbb{C}}\left\{a\left(-\mathrm{wt}(u)+3\right)a'(-1)^2a, a\left(-\mathrm{wt}(u)+1\right)a'\right\} + C_2\left(M_2(1)^\sigma\right).$$

In particular, we have $M_2(1)^\sigma = C_2(M_2(1)^\sigma) + \mathrm{Span}_{\mathbb{C}}\{\mathcal{S}_2\}$.

Proof We will prove the last statement. The others come from the same arguments. We note $\omega_0\beta = \beta_{-2}\mathbf{1} \in C_2(M_2(1)^\sigma)$ for $\beta \in M_2(1)^\sigma$ and Virasoro element ω. Furthermore $a(-h)a(-k)a(-m)\mathbf{1}$ is congruent to a linear sum of elements of type $a(-i)a(-j)a$ since

$$\omega_0\left(a(-r_1)\cdots a(-r_k)\mathbf{1}\right) = \sum_{i=1}^{k} r_i a(-r_1)\cdots a(-r_i-1)\cdots a(-r_k)\mathbf{1}.$$

Suppose $h-k \geq 4$ and $u \notin C_2(M_2(1)^\sigma) + \mathrm{Span}_{\mathbb{C}}\{\mathcal{S}_2\}$. We take u such that the total number $h+k$ is minimal. At least one of i_s, j_t in (2.2) is not 1. Since $h \geq 4$, by using a suitable triple term of a, we may assume $i_1 = 1$ by Lemma 1. Then by choosing another suitable triple a-term, we may also assume $i_2 = 1$ and then $i_3 = 2$. Then

$$2u - \left(a(-1)a(-1)a\right)_{-2}a(-i_4)\cdots a(-i_h)a'(-i_1)\cdots a'(-i_k)\mathbf{1}$$

is congruent to a linear sum of elements whose the total number of terms are less than $h+k$, which contradicts the choice of u. We next treat the case $h-k=3$. By applying the same arguments to $a(-n)a'(-m)$, we can reduce to the case $h=3$ and $k=0$ in (2.2) as we desired. If $h=k$ and $h \geq 3$, then using the same argument as above, we can reduce to $u = a(-n)a'(-m)a(-1)a'(-1)$ and $n \geq 2$. If $m \geq 2$, then u is congruent to a linear sum of $a(-n-m+1)a'(-1)^2a$ and $a(-n-m-1)a'$ by Lemma 1. Therefore we obtain $M_2(1)^\sigma = C_2(M_2(1)^\sigma) + \mathrm{Span}_{\mathbb{C}}\{\mathcal{S}_2\}$. □

2.4 A Subring

We first note that $M_2(1)^\sigma/C_2(M_2(1)^\sigma)$ is a commutative associative algebra by -1-normal product. Let $\mathcal{O}$ be the subspace of $M_2(1)^\sigma/C_2(M_2(1)^\sigma)$ spanned by elements with the same number of a-terms and b-terms and $\mathcal{O}^{even}$ the subspace of $\mathcal{O}$ spanned by elements with even weights. By abusing the notation, we may view $\mathcal{O}$ as a subset of $M_2(1)^\sigma$ modulo $C_2(M_2(1)^\sigma)$. Clearly, $\mathcal{O}$ and $\mathcal{O}^{even}$ are subrings of $M_2(1)^\sigma/C_2(M_2(1)^\sigma)$. We will study an algebraic structure of $\mathcal{O}^{even}$ modulo $C_2(M_2(1)^\sigma)$.

Set $\gamma(n) = a(-n+1)a'$ and we sometimes omit subscript -1 denoting -1-normal product, for example, $\gamma(n)\gamma(m)$ denotes $\gamma(n)_{-1}\gamma(m)$. From $0 \equiv \omega_0(a(-n)a'(-m)\mathbf{1}) = na(-n-1)a'(-m)\mathbf{1} + ma(-n)a'(-m-1)\mathbf{1}$ mod $\omega_0(M_2(1)^\sigma) \subseteq C_2(M_2(1)^\sigma)$, we have:

Lemma 2

$$a(-n)a'(-m-1)\mathbf{1} \equiv \binom{-n}{m}\gamma(n+m+1) \quad \bigl(\mathrm{mod}\ \omega_0 M_2(1)\bigr).$$

Proposition 3

$$a(-r)a(-m)a'(-n)a' \equiv_2 \binom{-m}{n-1}\gamma(r+1)\gamma(m+n) - \frac{(-1)^{n-1}(r+m+n-1)!(m+n+r+1)}{(r-1)!(m-1)!(n-1)!(m+1)(r+n)}\gamma(t),$$

where $t = r+m+n+1$. In particular, by replacing r with m, we have

$$\gamma(n+3) \equiv_2 \frac{6}{(n-1)(n-2)(n+3)}\bigl\{\gamma(3)\gamma(n) - (n-1)\gamma(2)\gamma(n+1)\bigr\}$$

for $n \geq 3$ and $\gamma(n) \in C_1(M_2(1)^\sigma)$ for $n \geq 6$.

Proof The assertion comes from the direct calculation:

$$\begin{aligned}
&\binom{-m}{n-1}\gamma(r+1)\gamma(m+n)\\
&\equiv_2 \bigl(a(-r)a'\bigr)_{-1}a(-m)a'(-n)\mathbf{1}\\
&\equiv_2 \sum_i \binom{-r}{i}(-1)^i\bigl\{a(-r-i)a'(-1+i) - (-1)^{-r}a'(-r-1-i)a(i)\bigr\}\\
&\quad \times a(-m)a'(-n)\mathbf{1}\\
&\equiv_2 a(-r)a'(-1)a(-m)a'(-n)\mathbf{1} + \binom{r+m}{m+1}ma(-r-m-1)a'(-n)\mathbf{1}\\
&\quad - (-1)^r\binom{r+n-1}{n}a'(-r-1-n)na(-m)\mathbf{1}\\
&\equiv_2 a(-r)a(-m)a'(-n)a'\\
&\quad + \Biggl\{\binom{r+m}{m+1}m\binom{-r-m-1}{n-1}\\
&\quad - (-1)^n\binom{r+n-1}{n}n\binom{m+r+n-1}{r+n}\Biggr\}\gamma(t)\\
&\equiv_2 a(-r)a(-m)a'(-n)a' + \frac{(-1)^{n-1}(r+m+n-1)!(m+r+n+1)}{(r-1)!(m-1)!(n-1)!(m+1)(r+n)}\gamma(t).
\end{aligned}$$

□

For example, we will use the following:

$$2\gamma(6) \equiv_2 \gamma(3)\gamma(3) - 2\gamma(2)\gamma(4), \qquad 7\gamma(7) \equiv_2 \gamma(3)\gamma(4) - 3\gamma(2)\gamma(5),$$
$$16\gamma(8) \equiv_2 \gamma(3)\gamma(5) - 4\gamma(2)\gamma(6), \qquad 30\gamma(8) \equiv_2 \gamma(4)\gamma(4) - 6\gamma(2)\gamma(6). \tag{2.3}$$

Lemma 3 $\mathcal{O} = \mathrm{Span}_{\mathbb{C}}\{\gamma(2)^n, \gamma(n+1), \gamma(2)\gamma(m)\mathbf{1} \mid n, m = 2, \ldots\} + C_2(M_2(1)^\sigma)$.

Proof By Proposition 2, $\mathcal{O}$ is spanned by $\{a(-1)^n a'(-1)^n\mathbf{1}, a(-n)a'(-1)^2 a, \gamma(m)\}$ modulo $C_2(M_2(1)^\sigma)$. By Proposition 3, we get $a(-n)a'(-1)^2a - \gamma(2)\gamma(n+1) \in \mathbb{Q}\gamma(n+3)$. We also have that $a(-1)^n a'(-1)^n\mathbf{1} - \gamma(2)^n$ is congruence to a linear sum of $a(-2n+3)a'(-1)^2a$ and $\gamma(2n)$ modulo $C_2(M_2(1)^\sigma)$, which proves the desired result. □

Set $\mathcal{S}_1 = \{a(-i_1)a(-i_2)a, a'(-i_1)a'(-i_2)a', a(-i_3)a', \mathbf{1} \mid i_1, i_2 \leq 5, i_3 \leq 4\}$.

Proposition 4 *$M_2(1)^\sigma = C_1(M_2(1)^\sigma) + \mathrm{Span}_{\mathbb{C}}\mathcal{S}_1$. In particular, $M_2(1)^\sigma$ is C_1-cofinite.*

Proof To simplify the notation, set $C_1 = C_1(M_2(1))^\sigma$ in this proof. Suppose that the proposition is false and let

$$u = a(-i_1)\cdots a(-i_h)a'(-j_1)\cdots a'(-j_k)\mathbf{1} \notin C_1 + \mathrm{Span}_{\mathbb{C}}\mathcal{S}_1.$$

We take u such that the number of terms is minimal. By Lemma 1 and 2, we may assume $u = a(-i_1)a(-i_2)a$ or $u = a(-m)a'$. By Lemma 2 and Proposition 3, we obtain $a(-m)a' \in C_1$ for $m \geq 5$. Furthermore, since C_1 is closed by the 0-normal product, we have:

(1) $C_1 \ni \big(a(-k+1)a'\big)_0\big(a(-1)a(-1)a\big) = 3(k-1)a(-k)a(-1)^2\mathbf{1}$ and so

(2) $C_1 \ni \big(a(-n)a'\big)_0 a(-1)^2 a(-k)\mathbf{1}$
$= 2a(-n-1)a(-k)a + ka(-n-k)a(-1)a$

for $k \geq 6$ and any n, which contradicts the choice of u. □

We next express $\mathcal{O}$ as a $\mathbb{C}[\gamma(2)]$-module. We need the following lemma.

Lemma 4

$$120\gamma(7)\mathbf{1} \equiv_2 8\gamma(2)\gamma(5)\mathbf{1} + \gamma(2)^2\gamma(3)\mathbf{1} \quad \textit{and}$$
$$60\gamma(8)\mathbf{1} \equiv_2 6\gamma(2)\gamma(3)^2\mathbf{1} - 13\gamma(2)^2\gamma(4)\mathbf{1}.$$

Proof Since $0 \equiv_2 (a(-1)a(-1)a)_{-2}a'(-1)a'(-1)a' \equiv_2 3a(-1)^2a(-2)a'(-1)^2a' + 18a(-4)a(-1)a'(-1)a' + 18a(-3)a(-2)a'(-1)a' + 18\gamma(7)$, we have $a(-1)^2 \times a(-2)a'(-1)^2a' \equiv_2 -6a(-4)a(-1)a'(-1)a' - 6a(-3)a(-2)a'(-1)a' - 6\gamma(7)$.

Using Proposition 3 and the above lemma, we obtain the first congruence expression:

$$\begin{aligned}
\gamma(2)^2\gamma(3) \equiv_2 & \big(a(-1)a'\big)_{-1}\{a(-1)a'(-1)a(-2)a'\} \\
& + \gamma(2)\{2a(-4)a' + a'(-3)a(-2)\mathbf{1}\} \\
\equiv_2 & \, a(-1)a'(-1)a(-1)a'(-1)a(-2)a' + a(-3)a'(-1)a(-2)a' \\
& + 2a(-4)a(-1)a'(-1)a' + 2a'(-3)a(-1)a(-2)a' + 5\gamma(2)\gamma(5) \\
\equiv_2 & -6a(-4)a(-1)a'(-1)a' - 6a(-3)a(-2)a'(-1)a' - 6\gamma(7) \\
& + a(-3)a'(-1)a(-2)a' + 2a(-4)a'(-1)^2 a \\
& + 2a'(-3)a(-1)a(-2)a' + 5\gamma(2)\gamma(5) \\
\equiv_2 & -4a(-4)a(-1)a'(-1)a' - 5a(-3)a(-2)a'(-1)a' - 6\gamma(7) \\
& + 2a'(-3)a(-1)a(-2)a' + 5\gamma(2)\gamma(5) \\
\equiv_2 & -4\left\{\binom{-4}{0}\gamma(2)\gamma(5) - \left[4 + \binom{-4}{2}\right]\gamma(7)\right\} \\
& - 5\{3\gamma(2)\gamma(5) - 28\gamma(7)\} - 6\gamma(7) \\
& + 2\left\{\binom{-2}{2}\gamma(2)\gamma(5) - \left[2\binom{-4}{2} + 3\binom{-2}{4}\right]\gamma(7)\right\} + 5\gamma(2)\gamma(5) \\
\equiv_2 & \, 120\gamma(7) - 8\gamma(2)\gamma(5),
\end{aligned}$$

which proves the first equation.

By expanding $0 \equiv_2 (a(-1)a(-1)a)_{-2}a'(-2)a'(-1)a'$, we have,

$$\begin{aligned}
& -\big(a(-2)a(-1)a(-1)\big)a'(-2)a'(-1)a' \\
& \equiv_2 4a(-4)a(-1)a'(-2)a' + 4a(-3)a(-2)a'(-2)a' + 4a(-5)a(-1)a'(-1)a' \\
& \quad + 4a(-4)a(-2)a'(-1)a' + 2a(-3)a(-3)a'(-1)a' + 8\gamma(8)
\end{aligned}$$

and then we obtain:

$$\begin{aligned}
& 2\gamma(2)\gamma(2)\gamma(4) \\
& \equiv_2 -\big(a(-1)a'\big)_{-1}\{a(-1)a(-2)a'(-2)a' - 16\gamma(6)\} \\
& \equiv_2 -a(-1)^2a'(-1)^2a(-2)a'(-2) - a(-3)a(-2)a'(-2)a' \\
& \quad - 2a(-4)a(-1)a'(-1)a'(-2) - a'(-3)a(-1)a(-2)a'(-2) \\
& \quad - 2a'(-4)a(-1)a'(-1)a(-2) + 16\gamma(2)\gamma(6) \\
& \equiv_2 4a(-4)a(-1)a'(-2)a' + 4a(-3)a(-2)a'(-2)a' + 4a(-5)a(-1)a'(-1)a' \\
& \quad + 4a(-4)a(-2)a'(-1)a' + 2a(-3)a(-3)a'(-1)a' + 8\gamma(8)
\end{aligned}$$

$$
\begin{aligned}
&\quad - a(-3)a'(-1)a(-2)a'(-2) - 2a(-4)a(-1)a'(-1)a'(-2) \\
&\quad - a'(-3)a(-1)a(-2)a'(-2) - 2a'(-4)a(-1)a'(-1)a(-2) + 16\gamma(2)\gamma(6) \\
&\equiv_2 2a(-3)a(-2)a'(-2)a' + 4a(-5)a(-1)a'(-1)a' + 4a(-4)a(-2)a'(-1)a' \\
&\quad + 2a(-3)a(-3)a'(-1)a' + 16\gamma(2)\gamma(6) + 8\gamma(8) \\
&\equiv_2 2\big(180\gamma(8) - 3\gamma(3)\gamma(5)\big) + 4\big(\gamma(2)\gamma(6) - 20\gamma(8)\big) \\
&\quad + 4\big(\gamma(3)\gamma(5) - 64\gamma(8)\big) + 2\big(-90\gamma(8) + \gamma(4)\gamma(4)\big) + 16\gamma(2)\gamma(6) + 8\gamma(8) \\
&\equiv_2 -120\gamma(8) + 12\gamma(2)\gamma(3)^2 - 24\gamma(2)^2\gamma(4),
\end{aligned}
$$

which proves the second equation. □

By the above lemma, the direct calculation shows:

$$
\begin{aligned}
2\gamma(4)\gamma(4) &\equiv_2 12\gamma(2)\gamma(6) + 60\gamma(8) \equiv_2 12\gamma(2)\gamma(3)^2 - 25\gamma(2)^2\gamma(4), \\
15\gamma(3)\gamma(5) &\equiv_2 60\gamma(2)\gamma(6) + 240\gamma(8) \equiv_2 54\gamma(2)\gamma(3)^2 - 112\gamma(2)^2\gamma(4), \\
120\gamma(3)\gamma(4) &\equiv_2 120\big(7\gamma(7) + 3\gamma(2)\gamma(5)\big) \equiv_2 7\gamma(2)^2\gamma(3) + 416\gamma(2)\gamma(5).
\end{aligned}
$$

Therefore, $\mathcal{O}^{even}$ has a subring

$$
\mathcal{O}^{even}_{\mathbb{Q}} = \mathbb{Q}\big[\gamma(2)\big]\gamma(2) + \mathbb{Q}\big[\gamma(2)\big]\gamma(3)\gamma(3) + \mathbb{Q}\big[\gamma(2)\big]\gamma(4).
$$

2.5 *Elements $a(-1)a(-1)a$*

We denote $a(-1)a(-1)a$ and $a'(-1)a'(-1)a'$ by α and β, respectively.

Lemma 5 $\gamma(2)_{-1}\gamma(2)_{-1}\gamma(2) \equiv_2 \alpha_{-1}\beta - 264\gamma(2)_{-1}\gamma(4)\mathbf{1} + 117\gamma(3)_{-1}\gamma(3)$.

Proof From the direct calculation, we have:

$$
\begin{aligned}
\alpha_{-1}\beta &\equiv_2 \big(a(-1)a(-1)a\big)_{-1}a'(-1)a'(-1)a' \\
&\equiv_2 a(-1)^3a'(-1)^3\mathbf{1} + 18a(-3)a(-1)a'(-1)a' + 9a(-2)a(-2)a'(-1)a' \\
&\qquad + 18a(-5)a'.
\end{aligned}
$$

Therefore, by Proposition 3, we obtain:

$$
\begin{aligned}
\gamma(2)^3 &\equiv_2 \big(a(-1)a'\big)_{-1}\big\{a(-1)a'(-1)a(-1)a' + 2\gamma(4)\big\} \\
&\equiv_2 a(-1)^3a'(-1)^3\mathbf{1} + 2a(-3)a(-1)a'(-1)a' + 2a'(-3)a(-1)a'(-1)a \\
&\qquad + 2\gamma(2)\gamma(4)
\end{aligned}
$$

$$
\begin{aligned}
&\equiv_2 \alpha_{-1}\beta - 14\{\gamma(2)\gamma(4) - 9\gamma(6)\} - 9\{\gamma(3)^2 - 16\gamma(6)\} - 18\gamma(6) \\
&\quad + 2\gamma(2)\gamma(4) \\
&\equiv_2 \alpha_{-1}\beta - 264\gamma(2)\gamma(4) + 117\gamma(3)^2. \qquad \square
\end{aligned}
$$

2.6 The Action of $\gamma(4)$

By (2.3), we have shown that $\mathcal{O}_{\mathbb{Q}}^{even}$ is closed by the -1-normal product and

$$\mathcal{A}_{\mathbb{Q}}^{even} = \mathbb{Q}[\gamma(2)]\gamma(4) + \mathbb{Q}[\gamma(2)]\gamma(3)\gamma(3)$$

is an ideal modulo $C_2(M_2(1)^\sigma)$. Let $\mathcal{Q}$ be an ideal generated by $\alpha_{-1}\beta$. We note

$$\alpha_{-1}\beta \equiv_2 \gamma(2)^3 + 264\gamma(2)\gamma(4) - 117\gamma(3)^2.$$

We will see the action of $\gamma(4)$ on $\mathcal{A}_{\mathbb{Q}}^{even}$.

Lemma 6 $\mathcal{Q} = \mathcal{O}_{\mathbb{Q}}^{even}$.

Proof We already know $2\gamma(4)^2 \equiv_2 12\gamma(2)\gamma(3)^2 - 25\gamma(2)^2\gamma(4)$.
Since $\gamma(3)\gamma(5) \equiv_2 54\gamma(2)\gamma(3)^2 - 112\gamma(2)^2\gamma(4)$, we have:

$$
\begin{aligned}
1800\gamma(4)\gamma(3)^2 &\equiv_2 15\gamma(3)\{7\gamma(2)^2\gamma(3) + 416\gamma(2)\gamma(5)\} \\
&\equiv_2 105\gamma(2)^2\gamma(3)^2 + 416\gamma(2)\{54\gamma(2)\gamma(3)^2 - 112\gamma(2)^2\gamma(4)\} \\
&\equiv_2 22569\gamma(2)^2\gamma(3)^2 - 46592\gamma(2)^3\gamma(4).
\end{aligned}
$$

Therefore the action of $1800\gamma(4)$ on $\mathcal{A}_{\mathbb{Q}}^{even}$ is expressed by

$$\gamma(2)^2 \begin{pmatrix} 22569 & -46592 \\ 10800 & -22500 \end{pmatrix}.$$

The eigenpolynomial of $1800\gamma(4)$ is $X^2 - 69X - 4608900$ and its discriminant is $3\sqrt{2048929}$, which is not a rational number. Therefore, the action of $\gamma(4)/\gamma(2)^2$ on $\mathbb{Q}\gamma(2)\gamma(4) + \mathbb{Q}\gamma(3)^2$ is irreducible over $\mathbb{Q}$. Furthermore, since

$$
\begin{aligned}
(\alpha_{-1}\beta)_{-1}\gamma(4) &\equiv_2 \left(\gamma(2)^3 - 264\gamma(2)\gamma(4) - 117\gamma(3)^2\right)\gamma(4) \\
&\equiv_2 \gamma(2)^3\gamma(4) - 264\gamma(2)\left\{6\gamma(2)\gamma(3)^2 - \frac{25}{2}\gamma(2)^2\gamma(4)\right\} \\
&\quad - \frac{117}{120}\gamma(3)\{7\gamma(2)^2\gamma(3) + 416\gamma(2)\gamma(5)\} \\
&\equiv_2 3301\gamma(2)^3\gamma(4) - \left\{1584 + \frac{273}{40}\right\}\gamma(2)\gamma(3)^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{39\times 52}{5}\gamma(2)^2\left\{\frac{54}{15}\gamma(3)^2-\frac{112}{15}\gamma(2)\gamma(4)\right\}\\
\equiv_2 &\left(3301+\frac{13\times 52\times 112}{25}\right)\gamma(2)^2\gamma(4)\\
&-\left\{1584+\frac{273}{40}+\frac{39\times 52\times 18}{25}\right\}\gamma(2)^2\gamma(3)^2,
\end{aligned}
$$

we have $\mathcal{Q}_{\mathbb{Q}}^{even}\cap\mathcal{A}_{\mathbb{Q}}^{even}\neq 0$ and so

$$
\left(\mathrm{Span}_{\mathbb{Q}}\left\{\alpha_{-1}\beta,\gamma(4)\alpha_{-1}\beta,\gamma(4)^2\alpha_{-1}\beta\right\}\right)_n=\left(\mathcal{O}_{\mathbb{Q}}^{even}\right)_n \quad \text{for } n\geq 14,
$$

where $(*)_n$ denotes the homogeneous subspace of weight n. □

2.7 Nilpotency of α Modulo $C_2(V_L^\sigma)$

From now on, $C_2(W)$ denotes $\mathrm{Span}_{\mathbb{C}}\{v_{-2}w \mid w\in W, v\in V_L^\sigma\}$ and we use $\equiv$ to denote the congruence modulo $C_2(V_L^\sigma)$. A standard expression for an element μ in V_L^σ is

$$
\mu=\sum_{t=0}^{2}\sigma^t\left(a(-i_1)\cdots a(-i_h)a'(-j_1)\cdots a'(-j_k)e^\gamma\right) \quad \text{with } i_s, j_t>0.
$$

However, we will use $a(0)$ and $a'(0)$ so that we express it by

$$
\mu'=\frac{1}{\langle a,\gamma\rangle^s\langle b,\gamma\rangle^t}a(-i_1)\cdots a(-i_h)a'(-j_1)\cdots a'(-j_k)a(0)^s a'(0)^t\left(\sum_{i=0}^{2}e^{\sigma^i(\gamma)}\right),
$$

where $h+s-k-t\equiv 0 \pmod 3$ and $a(0)^s=\underbrace{a(0)\cdots a(0)}_{s}$ and $a'(0)^t=\underbrace{a'(0)\cdots a'(0)}_{t}$. From now on, E^γ denotes $\sum_{i=0}^{2}e^{\sigma^i(\gamma)}$ and we will call h and k the numbers of a-terms and a'-terms, respectively. We next show that

Lemma 7 *$(a(-i_1)a(-i_2)a)_{-1}$ and $(a'(-j_1)a'(-j_2)a')_{-1}$ are all nilpotent in $M_2(1)^\sigma/(C_2(V_L^\sigma)\cap M_2(1))$ for any i_1,i_2,j_1,j_2.*

Proof Except α and β, the square of the remainings are zero by Proposition 2. We will prove that α_{-1} is nilpotent. Since $\mathrm{wt}(e^x)=9M$, $\mathrm{wt}(e^{x-y})=27M$ and $\mathrm{wt}(e^{2x+y})=27M$ for $y=\sigma(x)$, we have $e^y_{-1-k}e^{-x}=e^{-x-y}_{-1-k}e^x=0$ for $k<9M$

and so

$$E^x_{-1-k}E^{-x}=\sum_{i=0}^{2}\sigma^i\big(E^x_{-1-k}e^{-x}\big)=\sum_{i=0}^{2}\sigma^i\big(e^x_{-1-k}e^{-x}\big)\in M_2(1)^\sigma\cap C_2\big(V_L^\sigma\big)$$

for $1<k<9M$, where E^x denotes $e^x+e^y+e^{-x-y}$. Multiplying $(\alpha_{-1})^{6M+9}$ to $E^x_{-4}e^{-x}$, the number of a-terms in $(\alpha_{-1})^{6M+9}E^x_{-4}e^{-x}$ is 6 more than that of a'-terms and so all bloated elements vanished modulo $C_2(V_L^\sigma)$. Hence

$$(\alpha_{-1})^{6M+9}E^x_{-4}e^{-x}\equiv\frac{1}{(18M+3)!}(\alpha_{-1})^{6M+9}\big(x(-1)\big)^{18M+3}\mathbf{1}\in C_2\big(V_L^\sigma\big).$$

Set $x=ra+sa'$. Since we multiply many $a(-1)$, $(\alpha_{-1})^{6M+9+k}$ annihilates all elements except for $a(-1)$ and $a'(-1)$ by Proposition 2 and so we have:

$$\begin{aligned}
&(\alpha_{-1})^{6M+9}\big(x(-1)\big)^{18M+3}\mathbf{1}\\
&\quad\equiv a(-1)^{18M+27}\big(ra(-1)+sa'(-1)\big)^{18M+3}\mathbf{1}\\
&\quad\equiv\sum_{i=0}^{18M+3}\binom{18M+3}{i}r^{18M+3-i}s^i a(-1)^{36M+30-i}\gamma(2)^i\\
&\quad\equiv\sum_{i=0}^{6M+1}\binom{18M+3}{3i}r^{18M+3-3i}s^{3i}(\alpha_{-1})^{12M+10-i}\gamma(2)^{3i}\\
&\qquad+\sum_{i=0}^{6M}\binom{18M+3}{3i+1}r^{18M+2-3i}s^{3i+1}(\alpha_{-1})^{12M+9-i}a(-1)a(-1)\gamma(2)^{3i+1}\\
&\qquad+\sum_{i=0}^{6M}\binom{18M+3}{3i+2}r^{18M+1-3i}s^{3i+2}(\alpha_{-1})^{12M+9-i}a(-1)\gamma(2)^{3i+2}.
\end{aligned}$$

Similarly, since we obtain

$$\begin{aligned}
&\alpha_{-1}^{6M+9}E^x_{-4}a(-1)e^{-x}\\
&\quad=\alpha_{-1}^{6M+9}\big(x(-1)\big)^{18M+3}a(-1)\mathbf{1}+\alpha_{-1}^{6M+9}\langle a,x\rangle\big(x(-1)\big)^{18M+4}\mathbf{1}\\
&\quad=\alpha_{-1}^{6M+9}\big(x(-1)\big)^{18M+3}a(-1)\mathbf{1}+\alpha_{-1}^{6M+9}\langle a,x\rangle E^x_{-5}e^{-x}\\
&\quad\equiv\alpha_{-1}^{6M+9}\big(x(-1)\big)^{18M+3}a(-1)\mathbf{1}\\
&\quad\equiv\sum_{i=0}^{6M+1}\binom{18M+3}{3i}r^{18M+3-3i}s^{3i}\alpha_{-1}^{12M+10-i}a(-1)\gamma(2)^{3i}
\end{aligned}$$

$$+\sum_{i=0}^{6M}\binom{18M+3}{3i+1}r^{18M+2-3i}s^{3i+1}\alpha_{-1}^{12M+9-i+1}\gamma(2)^{3i+1}$$
$$+\sum_{i=0}^{6M}\binom{18M+3}{3i+2}r^{18M+1-3i}s^{3i+2}\alpha_{-1}^{12M+9-i}a(-1)^2\gamma(2)^{3i+2}$$

and

$$\begin{aligned}
&\alpha_{-1}^{6M+9}E_{-4}^{x}a(-1)^2e^{-x}\\
&\quad=\alpha_{-1}^{6M+9}\big(x(-1)\big)^{18M+3}a(-1)^2\mathbf{1}+2\langle a,x\rangle\alpha_{-1}^{6M+9}\big(x(-1)\big)^{18M+4}a\\
&\qquad+2\langle a,x\rangle^2\alpha_{-1}^{6M+9}\big(x(-1)\big)^{18M+5}\mathbf{1},\\
&\quad\equiv\alpha_{-1}^{6M+9}\big(x(-1)\big)^{18M+3}a(-1)^2\mathbf{1}\\
&\quad\equiv\sum_{i=0}^{6M+1}\binom{18M+3}{3i}r^{18M+3-3i}s^{3i}\alpha_{-1}^{12M+10-i}a(-1)^2\gamma(2)^{3i}\\
&\qquad+\sum_{i=0}^{6M}\binom{18M+3}{3i+1}r^{18M+2-3i}s^{3i+1}\alpha_{-1}^{12M+9-i+1}a(-1)\gamma(2)^{3i+1}\\
&\qquad+\sum_{i=0}^{6M}\binom{18M+3}{3i+2}r^{18M+1-3i}s^{3i+2}\alpha_{-1}^{12M+9-i+1}\gamma(2)^{3i+2},
\end{aligned}$$

we have

$$\begin{aligned}
&a(-1)\alpha_{-1}^{6M+9}\big(x(-1)\big)^{18M+3}\mathbf{1}\in C_2(V_L),\\
&a'(-1)\alpha_{-1}^{6M+9}\big(x(-1)\big)^{18M+3}\mathbf{1}\in C_2\big(V_L^{\sigma}\big)V_L\quad\text{and}\\
&a(-1)a(-1)\alpha_{-1}^{6M+9}\big(x(-1)\big)^{18M+3}\mathbf{1}\in C_2(V_L).
\end{aligned}$$

Hereafter V_L is a $(V_L)^{\sigma}$-module. Hence

$$\alpha_{-1}^{6M+9+k}a(-1)^e a'(-1)^k\big(x(-1)\big)^{18M+3}\mathbf{1}$$

is a linear sum of elements of the form

$$\alpha_{-1}^{6M+9+k}v_{-1}\big(u\cdot\big(x(-1)\big)^{18M+3}\mathbf{1}\big),$$

where v is a σ-invariant element and $u\in\{\mathbf{1}_{-1},a(-1),a(-1)a(-1)\}$ by Lemma 1. Therefore we obtain

$$\alpha_{-1}^{6M+9+k}a(-1)^e a'(-1)^k\big(x(-1)\big)^{18M+3}\mathbf{1}\in C_2(V_L)$$

for any $e, k \geq 0$. We also get a similar result for $y = \sigma(x)$ as for x. Therefore we have:

$$\alpha_{-1}^{12M+18}\big(\lambda x(-1) + \mu y(-1)\big)^{36M+6}\mathbf{1} \in C_2(V_L)$$

for any $\lambda, \mu \in \mathbb{C}$. By choosing suitable λ and μ so that $\lambda x(-1) + \mu y(-1) = a(-1)$, we have

$$\alpha_{-1}^{12M+18}a(-1)^{36M+6}\mathbf{1} = \alpha_{-1}^{48M+24}\mathbf{1} \in C_2\big(V_L^\sigma\big),$$

which implies that α_{-1} is nilpotent modulo $C_2(V_L^\sigma)$. Similarly, β_{-1} is nilpotent.

This completes the proof of Lemma 7. □

Since α, β are nilpotent and $\mathcal{O}^{even} = \mathcal{O}^{even}\alpha_{-1}\beta$, we have the following:

Proposition 5 $\dim(M_2(1)^\sigma/(M_2(1)^\sigma \cap C_2(V_L))) < \infty$.

2.8 C_2-Cofiniteness of V_L^σ

By the previous proposition, there is an integer N_0 such that $v_{-1}^1 \cdots v_{-1}^k \gamma \in C_2(V_L^\sigma)$ for any $v^i \in \mathcal{S}_1$ and $\gamma \in V_L^\sigma$ if $\mathrm{wt}(v_{-1}^1 \cdots v_{-1}^k \mathbf{1}) \geq N_0$. Set $N = N_0 + 9M + 30$.

Our final step is to prove

$$V_L^\sigma = C_2\big(V_L^\sigma\big) + \bigoplus_{n \leq N}\big(M_2(1)\big)_n^\sigma + \bigoplus_{n \leq N}\big(M_2(1)E^x\big)_n^\sigma + \bigoplus_{n \leq N}\big(M_2(1)E^{-x}\big)_n^\sigma,$$

which implies the C_2-cofiniteness of V_L^σ. For $\mu \neq 0$, set

$$\mathcal{R}_\mu = \left\{ d_{i_k}^k \cdots d_{i_1}^1 d_{i_0}^0 a(r)a'(0)E^\mu \;\middle|\; \begin{array}{ll} \text{(i)} & \mathrm{wt}(d_{i_0}^0 a(r)a'(0)E^\mu) - \mathrm{wt}(E^\mu) \leq 30 \\ \text{(ii)} & i_k \leq \cdots \leq i_1 \leq -1,\ i_0, r \leq 0 \text{ and} \\ \text{(iii)} & d^i \in \mathcal{S}_1 \end{array} \right\}$$

Proposition 6 $(M_2(1)E^\mu)^\sigma = \mathrm{Span}_{\mathbb{C}}\{\mathcal{R}\} + C_2(V_L^\sigma)$. *In particular, if* $v \in (M_2(1)E^\mu)^\sigma$ *has a weight greater than* $\mathrm{wt}(E^\mu) + N_0 + 30$, *then* $v \in C_2(V_L^\sigma)$.

Proof Suppose false and we take $u \notin \mathrm{Span}_{\mathbb{C}}\{\mathcal{R}\} + C_2(V_L^\sigma)$ with minimal $\mathrm{wt}(u)$. Since $M_2(1)E^\mu$ is an irreducible $M_2(1)^\sigma$-module, we may assume $u = c_{i_k}^k \cdots c_{i_1}^1 E^\mu$ with $c^i \in M_2(1)^\sigma$. We choose u with the above expression such that $\sum_{i=1}^k \mathrm{wt}(c^i)$ is minimal. Moreover, among elements with the same $\sum_{i=1}^k \mathrm{wt}(c^i)$, we choose u with maximal k. Since $(e_{-1}f)_m = \sum_{i=0}^\infty (e_{-1-i}f_{m+i} + f_{m-1-i}e_i)$ and $\mathrm{wt}(e_{-1}f) = \mathrm{wt}(e) + \mathrm{wt}(f)$, we may assume $c^i \in \mathcal{S}_1$. Also, since $e_s f_t - f_t e_s = \sum_{i=0}^\infty \binom{s}{i} \times (e_i f)_{s+t-i}$ and $\mathrm{wt}(e_i f) < \mathrm{wt}(e) + \mathrm{wt}(f)$ for $i \geq 0$, we may assume $i_k \leq \cdots \leq i_1$.

By the minimality of $\mathrm{wt}(u)$, we have $0 \le i_k$ and

$$\sum_{i=1}^{k} \mathrm{wt}\big(c^i\big) - \sum_{i=1}^{k}(1+i_j) = \big(\mathrm{wt}(u) - \mathrm{wt}\big(E^{\mu}\big)\big).$$

To simplify the notation, we will call $-\sum_{i=1}^{k}(1+i_j)$ a bloated weight. Since $\mathrm{wt}(E^{\mu})$ and $\mathrm{wt}(u)$ are fixed, we have chosen $u = c^k_{i_k} \cdots c^1_{i_1} E^{\mu}$ such that a bloated weight is maximal. We note that $\mathrm{wt}(c^i) \le 11$ for $c^i \in \mathcal{S}_1$. On the other hand, by Lemma 1, u is also a linear sum of elements of the form

$$e^r_{-1} \cdots e^1_{-1} F,$$

where $e^i \in M_2(1)^{\sigma}$ and F is one of

$$\mathcal{D} = \big\{a(m+n)a'(0)E^{\mu}, a(m)a(n)a(0)E^{\mu}, a'(m)a'(n)a'(0)E^{\mu} \mid m+n<0\big\}.$$

By the minimality of $\mathrm{wt}(u)$, u is a linear sum of elements in $\mathcal{D}$ and $-m-n+\mathrm{wt}(E^{\mu}) = \mathrm{wt}(u)$. We assert that a bloated weight of u is greater than or equal to three. For elements $a(m+n)a'(0)E^{\mu}$, we get $a(m+n)a'(0)E^{\mu} = (a'(m+n-1)a)_1 E^{\mu}$, which has only bloated weight two. Before we start the proof for the remaining case, we note

$$\begin{aligned}
&\big(a'(m-1)a\big)_1\big(a'(n-1)a\big)_1 E^{\mu}\\
&\quad = \big(a'(m-1)a\big)_1 a(n)a'(0)E^{\mu}\\
&\quad = \sum_{i=0}^{\infty}\binom{m-1}{i}(-1)^i(-1)^m a(m-i)a'(i)a(n)a'(0)E^{\mu}\\
&\quad = \binom{m-1}{-n}(-1)^{m-n} n a(m+n)a'(0)E^{\mu} + (-1)^m a(m)a'(0)a(n)a'(0)E^{\mu}.
\end{aligned}$$

Suppose $a(m)a(n)a(0)E^{\mu}$ has a bloated weight greater than three. By ignoring elements with bloated weights less than three, we have

$$\begin{aligned}
&\frac{\langle b,\mu\rangle^2}{\langle a,\mu\rangle} a(m)a(n)a(0)E^{\mu}\\
&\quad = a(m)a'(0)a(n)a'(0)E^{\mu}\\
&\quad \equiv \big(a'(m-1)a\big)_1\big(a'(n-1)a\big)_1 E^{\mu}\\
&\quad = \big(a'(m-1)a\big)_1\big(a(n-1)a'\big)_1 E^{\mu} + \big(a'(m-1)a\big)_1\big(\omega_0\gamma(-n+1)+\cdots\big)_1 E^{\mu}\\
&\quad \equiv \big(a'(m-1)a\big)_1\big(a(n-1)a'\big)_1 E^{\mu} + \big(\omega_0\gamma(-n+1)\big)_1\big(a'(m-1)a\big)_1 E^{\mu}\\
&\quad \equiv \big(a'(m-1)a\big)_1 a'(n)a(0)E^{\mu} - \gamma(-n+1)_0\big(a'(m-1)a\big)_1 E^{\mu}
\end{aligned}$$

$$\equiv \sum_{i=0}^{s} \binom{m-1}{i} (-1)^i \left\{ a'(m-1-i)a(1+i) - (-1)^{m-1} a(m-i)a'(i) \right\}$$
$$\times a'(n)a(0)E^{\mu}$$
$$\equiv \binom{m-1}{-n-1} (-1)^{-n} n a'(m+n)a(0)E^{\mu} - (-1)^{m-1} a(m)a'(0)a'(n)a(0)E^{\mu}$$
$$\equiv \lambda_1 a(m+n)a'(0)E^{\mu} + \mu_1 a(m)a'(n)E^{\mu}$$
$$\equiv \lambda_2 a(m+n)a'(0)E^{\mu} + \mu_2 a'(m+n)a(0)E^{\mu} \equiv 0$$

for some λ_i and μ_j, which is a contradiction. Therefore, the bloated weight of u is less than or equal to three. In particular, $k \le 3$ and $\mathrm{wt}(u) - E^{\mu} \le 30$. Therefore, the elements $a(m+n)a'(0)E^{\mu}$ and $\gamma(-n+1)_0(a'(m-1)a)_1 E^{\mu}$ are all in $\mathrm{Span}_{\mathbb{C}}\{\mathcal{R}\} + C_2(V_L^{\sigma})$. In order to show $a(m)a(n)a(0)E^{\mu} \in \mathrm{Span}_{\mathbb{C}}\{\mathcal{R}\} + C_2(V_L^{\sigma})$, we are able to have exactly the same congruence expressions as above modulo $\mathrm{Span}_{\mathbb{C}}\{\mathcal{R}\} + C_2(V_L^{\sigma})$.

This completes the proof of Proposition 6. □

Set $K = M_2(1)^{\sigma} + (M_2(1)E^{x})^{\sigma} + (M_2(1)E^{-x})^{\sigma}$. Since we have already shown that if $v \in K$ and $\mathrm{wt}(v) > N$, then $v \in C_2(V_L^{\sigma})$. The remaining is to show

$$V_L^{\sigma} = K + C_2\big(V_L^{\sigma}\big).$$

By Proposition 6, it is enough to show that

$$a(-n)a'(0)E^{\mu} \in K + C_2\big(V_L^{\sigma}\big)$$

for $1 \le n \le 30$ and $\mu \notin \{0, \pm x, \pm y, \pm(x+y)\}$. We first treat the following case:

Lemma 8 *For any* $n + m \equiv 0 \pmod 3$, *we have* $E^{mx+ny} \in C_2(V_L^{\sigma}) + K$.

Proof We note that if $n + m \equiv 0 \pmod 3$, then there is $\gamma \in L$ such that $E^{mx+ny} = E^{\pm(\sigma(\gamma)-\gamma)}$. Set $2k = \langle \gamma, \gamma \rangle$. Then since $\langle \gamma - \sigma(\gamma), \gamma - \sigma(\gamma) \rangle = 6k$, we have

$$E^{\gamma}_{-1-k} E^{-\gamma} \in M_2(1) + E^{\sigma(\gamma)-\gamma} + E^{-\sigma(\gamma)+\gamma},$$
$$E^{\gamma}_{-k} a(-1)e^{-\gamma} \in M_2(1) + \big\langle \sigma(\gamma), a \big\rangle e^{\sigma(\gamma)-\gamma} + \big\langle \sigma^2(\gamma), a \big\rangle e^{-\sigma(\gamma)+\gamma}, \quad \text{and}$$
$$E^{\gamma}_{-k} \sum_{i=0}^{2} \sigma^i \big(a(-1)e^{-\gamma}\big) \in M_2(1) + \big\langle \sigma(\gamma), a \big\rangle E^{\sigma(\gamma)-\gamma} + \big\langle \sigma^2(\gamma), a \big\rangle E^{-\sigma(\gamma)+\gamma}.$$

Therefore, we obtain $E^{\sigma(\gamma)-\gamma}, E^{-\sigma(\gamma)+\gamma} \in C_2(V_L^{\sigma}) + M_2(1)$. □

For E^{μ} with $\mu = mx + ny$ and $m + n \equiv \pm 1 \pmod 3$, we need the following lemma.

Lemma 9 (1) *For* $m,n\in\mathbb{Z}$ *with* $m+n\equiv 1 \pmod 3$, *there are* $\gamma\in L$ *satisfying* $\gamma-\sigma^i(\gamma-\mu)=mx+ny$ *for some* $i=1,2$ *and* $\mu\in\{x,y,-x-y\}$ *such that* $\langle\gamma,-\sigma^1(\gamma-\mu)\rangle$ *and* $\langle\gamma,-\sigma^2(\gamma-\mu)\rangle$ *are both positive.*

(2) *For* $m,n\in\mathbb{Z}$ *with* $m+n\equiv 2 \pmod 3$, *there are* $\gamma\in L$, $i=1,2$, $\mu\in\{-x,-y,+x+y\}$ *such that* $\gamma-\sigma^i(\gamma-\mu)=mx+ny$, $\langle\gamma,-\sigma^1(\gamma-\mu)\rangle>0$ *and* $\langle\gamma,-\sigma^2(\gamma-\mu)\rangle>0$.

Proof We first note that for $\gamma=px+qy$ and $-\gamma-x-y$, we have

$$\begin{aligned}\big\langle\sigma(\gamma),-\gamma-x-y\big\rangle&=p^2+q^2-pq+2p-q\\&=\big(q-(p+1)/2\big)^2+\frac{3}{4}(p+1)^2-1,\\\big\langle\sigma^2(\gamma),-\gamma-x-y\big\rangle&=p^2+q^2-pq+2q-p\\&=\big(p-(q+1)/2\big)^2+\frac{3}{4}(q+1)^2-1,\end{aligned}$$

and so the both are positive except $-2\le p,q\le 1$. For $\mu=mx+ny$ with $m+n\equiv 1 \pmod 3$, we may assume $m,n\le 0$ by taking a conjugate by $\langle\sigma\rangle$. If $\mu=mx+ny\notin\{x,y,-x-y,-2y\}$, then by setting $\gamma=px+qy$ with $q=(-m-n+1)/3$ and $p=(n-2m+2)/3$, we obtain $\sigma(\gamma)-\gamma-x-y=\mu$ and $\langle\sigma(\gamma),-\gamma-x-y\rangle$ and $\langle\sigma^2(\gamma),-\gamma-x-y\rangle$ are positive. In the case $\mu=-2y$, we choose $q=(-m-n+1)/3$ and $p=(-2n+m+2)/3$, then we have $\sigma^2(\gamma)-\gamma-x-y=\mu$ and $\langle\sigma^1(\gamma),-\gamma-x-y\rangle$ and $\langle\sigma^2(\gamma),-\gamma-x-y\rangle$ are positive. (2) comes from (1) by replacing x and y by $-x$ and $-y$, respectively. □

We come back to the proof of Theorem A. By the above lemmas, for any μ, there are γ,γ' and k such that

$$E^{\gamma}_{-2-k}e^{-\gamma'}\in e^{\mu}+e^{\mu'}+M_2(1)e^{\pm x}$$

and so

$$E^{\gamma}_{-2-k}E^{-\gamma'}\in E^{\mu}+E^{\mu'}+M_2(1)E^{\pm x}.$$

We also have

$$E^{\gamma}_{-2-k+1}\sum_{i=0}^{2}\sigma^i\big(a(-1)e^{-\gamma'}\big)\in\langle a,\gamma\rangle E^{\mu}+\big\langle a,\sigma(\gamma)\big\rangle E^{\mu'}+M_2(1)E^{\pm x},$$

which implies $E^{\mu},E^{\mu'}\in M_2(1)E^{\pm x}+C_2(V_L^{\sigma})$ for any μ. The remaining is to show $a(-n)a'(0)E^{\mu}\in M_2(1)E^{\pm x}+C_2(V_L^{\sigma})$ for $n\le 30$. Actually, we obtain

$$\begin{aligned}&E^{\gamma}_{-2-k+1+n}a(-n)a(-n)e^{-\gamma'}\\&\quad\in 2n\langle a,\gamma\rangle a(-n)e^{\mu}+2n\big\langle a,\sigma(\gamma)\big\rangle a(-n)e^{\mu'}+E^{\gamma}_{-2-k+1}e^{-\gamma'}+M_2(1)e^{\pm x}\end{aligned}$$

and

$$E^{\gamma}_{-2-k+1+2n}a(-n)a(-n)a(-n)e^{-\gamma'}$$
$$\in 6n^2\langle a,\gamma\rangle a(-n)e^{\mu}+6n^2\big\langle a,\sigma(\gamma)\big\rangle a(-n)e^{\mu'}+E^{\gamma}_{-2-k+1}e^{-\gamma'}+M_2(1)e^{\pm x}.$$

Therefore, for $n\le 30$, we have

$$a(-n)e^{\mu},a(-n)e^{\mu'}\in C_2\big(V_L^{\sigma}\big)V_L+M_2(1)e^{\pm x}\quad\text{and so}$$
$$a(-n)a'(0)E^{\mu},a(-n)a'(0)E^{\mu'}\in C_2\big(V_L^{\sigma}\big)+M_2(1)E^{\pm x}.$$

This completes the proof of Theorem A.

3 $\mathbb{Z}_3$-Orbifold Construction

The purpose in this section is to show $\mathbb{Z}_3$-orbifold constructions from a lattice VOA. Let Λ be a positive definite even unimodular lattice of rank N with a triality automorphism σ and set $H=\Lambda^{\sigma}$. We note $8|N$. By the assumption, $(V_\Lambda)^{\sigma}$ is C_2-cofinite and rational and so we can apply many useful known theorems like Zhu's theory [21], orbifold theory [6] and Verlinde formula [10, 15, 18].

Since Λ is unimodular, a lattice VOA V_Λ has exactly one simple module V_Λ and all modules are completely reducible ([5]). As Dong, Li and Mason has shown in [6], V_Λ has one σ-twisted module $V_\Lambda(\sigma)$ and one σ^2-twisted module $V_\Lambda(\sigma^2)$. Decompose them into direct sums of simple V_Λ^{σ}-modules:

$$V_\Lambda=W^0\oplus W^1\oplus W^2,\qquad V_\Lambda(\sigma)=W^3\oplus W^4\oplus W^5,$$
$$V_\Lambda\big(\sigma^2\big)=W^6\oplus W^7\oplus W^8.$$

We note that V_Λ^{σ} has exactly nine simple modules $\{W^i\mid 0\le i\le 8\}$ by the assumption of Theorem B. By Zhu's theory [21], there is a 9×9-matrix $S=(S_{ij})$ such that

$$\left(\frac{1}{z}\right)^{\mathrm{wt}(v)}T_{W^i}(v;-1/\tau)=\sum_{j=1}^{9}s_{ij}T_{W^j}(v;\tau)$$

for $v\in V_{\mathrm{wt}(v)}$ with $L(1)v=0$, where

$$T_W(v;\tau):=\mathrm{Tr}_W\, v_{\mathrm{wt}(v)-1}e^{2\pi i\tau(L(0)-N/24)}.$$

Lemma 10 *Under the assumption of Theorem B, the weights of elements in V_Λ, $V(\sigma)$ and $V(\sigma^2)$ are in $\mathbb{Z}/3$.*

Proof Set $H'=\{u\in\mathbb{Q}H\mid\langle u,h\rangle\in\mathbb{Z}\text{ for }h\in H\}$ the dual of H. Set $s=\mathrm{rank}(H)$, then from the assumption $t=(N-s)/2$ is divisible by three. As it is well known, the character $T_{V_H}(\mathbf{1};\tau)$ of V_H is $\frac{\theta_H(\tau)}{\eta(\tau)^s}$, where $\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta-function. Since Λ is unimodular, $3H'\subseteq H$ and the restriction of Λ into $\mathbb{Q}H$ covers H'/H and so the weights of elements in V_H-modules are in $\mathbb{Z}/3$. Hence the powers of q in the character of simple V_H-modules are all in $-s/24+\mathbb{Z}/3$ and so are those of q in $T_{V_H}(\mathbf{1};-1/\tau)$ by Zhu's theory [21]. Since

$$\begin{aligned}T_{V_\Lambda}(\sigma,\mathbf{1};\tau)&=q^{-N/24}\frac{\theta_H(\tau)}{\prod_n(1-q^n)^s}\times\frac{1}{\prod_n(1-\xi q^n)^t(1-\xi^{-1}q^n)^t}\\&=T_{V_H}(1,\mathbf{1};\tau)\frac{\eta(\tau)^t}{\eta(3\tau)^t}\end{aligned}$$

and $T_{V(\sigma)}(1,\mathbf{1};\tau)$ is a scalar times of

$$\begin{aligned}T_{V_\Lambda}(\sigma,\mathbf{1};-1/\tau)&=T_{V_H}(1,\mathbf{1};-1/\tau)\frac{\eta(-1/\tau)^t}{\eta(-3/\tau)^t}\\&=T_{V_H}(1,\mathbf{1};-1/\tau)\frac{(-\sqrt{-1}\tau)^{t/2}\eta(\tau)^t}{(-\sqrt{-1}\tau/3)^{t/2}\eta(\tau/3)^t}\\&=3^{t/2}T_{V_H}(1,\mathbf{1};-1/\tau)\frac{\eta(\tau)^t}{\eta(\tau/3)^t}\\&=3^{t/2}q^{-2t/24}T_{V_H}(1,\mathbf{1};-1/\tau)q^{t/9}\frac{\prod_n(1-q^n)^t}{\prod_n(1-q^{n/3})^t},\end{aligned}\tag{3.1}$$

we have that the powers of q in $T_{V(\sigma)}(1,\mathbf{1};\tau)$ are in $-N/24+\mathbb{Z}/3$. □

By reordering, we may assume that the weights of elements in W^i are in $i/3+\mathbb{Z}$ for $i\geq 3$. In particular, all elements in

$$\widetilde{V}=W^0\oplus W^3\oplus W^6$$

have integer weights. We next show that $\widetilde{V}$ has a VOA-structure.

Lemma 11 *W^i are all simple currents, that is, $W^i\boxtimes W^j$ are simple modules for any i,j. Moreover, $\widetilde{V}$ is closed by the fusion products.*

Proof We will determine the entries of the matrix $S=(S_{ij})$. Decompose S into $S=(A_{ij})_{i,j=1,2,3}$ with 3×3-matrices A_{ij}. Since S is symmetric by [10], $A_{ij}={}^tA_{ji}$. Simplify the notation, we denote $T_{W^i}(v;\tau)$ by $W^i(\tau)$. As they showed in [6], there are $\lambda_i\in\mathbb{C}$ $(i=0,1,2)$ such that $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$-transformation shifts

$$W^0(\tau)+\xi^iW^1(\tau)+\xi^{2i}W^2(\tau)\to\lambda_i\big(W^{3i}(\tau)+W^{3i+1}(\tau)+W^{3i+2}(\tau)\big).$$

Namely, the first three rows of S are

$$(A_{11}A_{12}A_{13}) = \frac{1}{3}\begin{pmatrix} \lambda_0 & \lambda_0 & \lambda_0 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_2 & \lambda_2 & \lambda_2 \\ \lambda_0 & \lambda_0 & \lambda_0 & \xi^2\lambda_1 & \xi^2\lambda_1 & \xi^2\lambda_1 & \xi\lambda_2 & \xi\lambda_2 & \xi\lambda_2 \\ \lambda_0 & \lambda_0 & \lambda_0 & \xi\lambda_1 & \xi\lambda_1 & \xi\lambda_1 & \xi^2\lambda_2 & \xi^2\lambda_2 & \xi^2\lambda_2 \end{pmatrix}.$$

Since a squared S^2 of S is a permutation matrix which shifts W to its restricted dual W', we get $\lambda_i^2 = 1$. We next consider the characters $\mathrm{ch}(W) = T_W(1, \mathbf{1}; \tau)$. In this case, since $\mathrm{ch}(W') = \mathrm{ch}(W)$, we have $\mathrm{ch}(W^1) = \mathrm{ch}(W^2)$, $\mathrm{ch}(W^{3+i}) = \mathrm{ch}(W^{6+i})$ for $i = 0, 1, 2$. Clearly,

$$\left\{\mathrm{ch}\left(W^0\right), \mathrm{ch}\left(W^1\right), \mathrm{ch}\left(W^3\right), \mathrm{ch}\left(W^4\right), \mathrm{ch}\left(W^5\right)\right\}$$

is a linearly independent set. Since (3.1) has $q^{1/3+\mathbb{Z}}$-parts, $A_{12} + A_{13} \neq 0$ and so $\lambda_1 = \lambda_2$. Similarly, since $\mathrm{ch}(W^{3+i}) = \mathrm{ch}(W^{6+i})$, we have $A_{22} + A_{23} = A_{32} + A_{33}$. Furthermore, since $A_{33} = A_{22} + A_{23} - {}^tA_{23}$ is symmetric, A_{23} is symmetric and $A_{22} = A_{33}$. Again, by [6], there are $\mu_i \in \mathbb{C}$ $(i = 1, 2)$ such that $\tau \to -1/\tau$-transformation shifts

$$W^3(\tau) + \xi^i W^4(\tau) + \xi^{2i} W^5(\tau) \to \mu_i\left(W^{3i}(\tau) + \xi^2 W^{3i+1}(\tau) + \xi W^{3i+2}(\tau)\right)$$

for $i = 1, 2$. Therefore, from these information, we know the entries of S:

$$(S_{ij}) = \frac{1}{3}\begin{pmatrix} \lambda_0 & \lambda_0 & \lambda_0 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\ \lambda_0 & \lambda_0 & \lambda_0 & \xi^2\lambda_1 & \xi^2\lambda_1 & \xi^2\lambda_1 & \xi\lambda_1 & \xi\lambda_1 & \xi\lambda_1 \\ \lambda_0 & \lambda_0 & \lambda_0 & \xi\lambda_1 & \xi\lambda_1 & \xi\lambda_1 & \xi^2\lambda_1 & \xi^2\lambda_1 & \xi^2\lambda_1 \\ \lambda_1 & \xi^2\lambda_1 & \xi\lambda_1 & \mu_1 & \xi\mu_1 & \xi^2\mu_1 & \mu_2 & \xi^2\mu_2 & \xi\mu_2 \\ \lambda_1 & \xi^2\lambda_1 & \xi\lambda_1 & \xi\mu_1 & \xi^2\mu_1 & \mu_1 & \xi^2\mu_2 & \xi\mu_2 & \mu_2 \\ \lambda_1 & \xi^2\lambda_1 & \xi\lambda_1 & \xi^2\mu_1 & \mu_1 & \xi\mu_1 & \xi\mu_2 & \mu_2 & \xi^2\mu_2 \\ \lambda_1 & \xi\lambda_1 & \xi^2\lambda_1 & \mu_2 & \xi^2\mu_2 & \xi\mu_2 & \mu_1 & \xi\mu_1 & \xi^2\mu_1 \\ \lambda_1 & \xi\lambda_1 & \xi^2\lambda_1 & \xi^2\mu_2 & \xi\mu_2 & \mu_2 & \xi\mu_1 & \xi^2\mu_1 & \mu_1 \\ \lambda_1 & \xi\lambda_1 & \xi^2\lambda_1 & \xi\mu_2 & \mu_2 & \xi^2\mu_2 & \xi^2\mu_1 & \mu_1 & \xi\mu_1 \end{pmatrix},$$

with $\lambda_i^2 = \mu_1\mu_2 = 1$. This implies $\overline{S_{ih}}S_{i'h} = 1$ and so

$$N_{i,i'}^k = \sum_{h=1}^{9} \frac{S_{ih}S_{i'h}S_{hk'}}{S_{0h}} = \sum_{h=1}^{9} \frac{S_{hk'}}{S_{0h}}.$$

Therefore, $N_{i,i'}^k \neq 0$ if and only if $k = 0$ and $N_{i,i'}^0 = 1$. Namely, $W^i \boxtimes (W^i)' = V_\Lambda^\sigma$ for every i. If $R \boxtimes W^i$ is not simple for a V_L^σ-module R, then $(R \boxtimes W^i) \boxtimes (W^i)' \cong R \boxtimes (W^i \boxtimes (W^i)') \cong R$ is not simple by Proposition 1, which implies W^i are simple current. □

By considering the characters, we have:

$$T_{V_\Lambda}(\sigma, \mathbf{1}; \tau) = T_{V_H}(1, \mathbf{1}; \tau)\frac{\eta(\tau)^s}{\eta(3\tau)^s} = \mathrm{ch}\left(W^0\right) + \xi\mathrm{ch}\left(W^1\right) + \xi^2\mathrm{ch}\left(W^2\right),$$

$$T_{V_\Lambda}(\sigma, \mathbf{1}, -1/\tau) = \lambda_1\{\mathrm{ch}(W^3) + \mathrm{ch}(W^4) + \mathrm{ch}(W^5)\},$$
$$T_{V_\Lambda}\big(\sigma, \mathbf{1}, -1/(\tau+1)\big) = e^{2\pi\sqrt{-1}N/24}\lambda_1\{\mathrm{ch}(W^3) + \xi\mathrm{ch}(W^4) + \xi^2\mathrm{ch}(W^5)\},$$
$$T_{V_\Lambda}\big(\sigma, \mathbf{1}, -1/((-1/\tau)+1)\big)$$
$$= e^{2\pi\sqrt{-1}N/24}\lambda_1\mu_1\{\mathrm{ch}(W^3) + \xi^2\mathrm{ch}(W^4) + \xi\mathrm{ch}(W^5)\}$$

from the above a matrix S. On the other hand, since

$$\begin{aligned} &T_{V_\Lambda}\big(\sigma, \mathbf{1}, -1/((-1/\tau)+1)\big) \\ &= T_{V_\Lambda}\left(\sigma, \mathbf{1}, -1 - \frac{1}{\tau - 1}\right) \\ &= e^{-2\pi\sqrt{-1}N/24}T_{V_\Lambda}\big(\sigma, \mathbf{1}, -1/(\tau-1)\big) \\ &= e^{-4\pi\sqrt{-1}N/24}\lambda_1\{\mathrm{ch}(W^3) + \xi^2\mathrm{ch}(W^4) + \xi\mathrm{ch}(W^5)\} \end{aligned}$$

we have $\mu_1 = e^{-6\pi\sqrt{-1}N/24} = 1$ since $8|N$. Then the matrix S implies $\lambda_1 = \lambda_0$ and $W^3 \boxtimes W^3 = W^6$ and $W^3 \boxtimes W^6 = W^0$. Therefore,

$$\widetilde{V} = W^0 \oplus W^3 \oplus W^6$$

is a direct sum of simple current V-modules W^{3i} with integer weights and $W^{3i} \boxtimes W^{3j} = W^{3k}$ with $i + j \equiv k \pmod 3$. In order to prove that $\widetilde{V}$ has a VOA-structure, we will prove a more general statement.

Proposition 7 *Let V be a C_2-cofinite VOA of CFT-type and all V-modules are completely reducible. Let $W = \bigoplus_{i=0}^{n} W^i$ be a direct sum of simple V-module W^i with integer weights and we assume $W^i \boxtimes W^j = W^k$ for $i + j \equiv k \pmod n$, $W^0 = V$ and n is odd. Then W has a VOA structure.*

Proof Let $\mathcal{Y}^{i,j}$ be a nonzero intertwining operator of type $\binom{W^{i+j}}{W^i\ \ W^j}$, where W^{s+n} coincides W^s. We choose $\mathcal{Y}^{2,i}$ so that

$$E\big(\langle d', \mathcal{Y}^{1,i+1}(w,x)\mathcal{Y}^{1,i}(u,y)a\rangle\big) = E\big(\langle d', \mathcal{Y}^{2,i}\big(\mathcal{Y}^{1,1}(w,x-y)u,y\big)a\rangle\big)$$

for any $a \in W^i$, $w, u \in W^1$ and $d' \in (W^{i+2})'$. Set an intertwining operator $\mathcal{Y}$ of type $\binom{W}{W^1\ \ W}$ by

$$\mathcal{Y}(w,z) = \begin{pmatrix} 0 & \cdots & 0 & \mathcal{Y}^{1,n-1}(w,z) \\ \mathcal{Y}^{1,0}(w,z) & 0\cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots 0 & \mathcal{Y}^{1,n-2}(w,z) & 0 \end{pmatrix},$$

where $w \in W^1$. We note that a vertex operator Y^W of V on W is given by

$$Y(v,z) = \begin{pmatrix} Y^{0,0}(v,z) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Y^{0,n-1}(v,z) \end{pmatrix}.$$

By the definition, we have that $\mathcal{Y}(w,z)$ satisfies Commutativity with $Y^W(v,z)$ for any $w \in W$ and $v \in V$. We also have $\mathcal{Y}(L(-1)w,z) = \frac{d}{dz}\mathcal{Y}(w,z)$.

Our next aim is to prove that $\mathcal{Y}(w,z)$ satisfies Commutativity with itself. We note that $\mathcal{Y}^{i,j}$ are all integer power series. Therefore, it is sufficient to show

$$E\big(\langle d', \mathcal{Y}^{1,i+1}(w,x)\mathcal{Y}^{1,i}(u,y)a\rangle\big) = E\big(\langle d', \mathcal{Y}^{1,i+1}(u,y)\mathcal{Y}^{1,i}(w,x)a\rangle\big)$$

for any $i = 0, \ldots, n-1$, $d' \in (W^{i+2})'$, $a \in W^i$, and $w, u \in W^1$.

Recall a skew-symmetric intertwining operator

$$\sigma_{12}\big(\mathcal{Y}^{1,1}\big)(w,z)u = e^{L(-1)z}\mathcal{Y}^{1,1}(u,-z)w.$$

Since $\dim \mathcal{I}_{W^1,W^1}^{W^2} = 1$ and all W^i have integer weights, we have $\sigma_{12}^2 = 1$ on $\mathcal{I}_{W^1,W^1}^{W^2}$ and so there is $\lambda \in \{\pm 1\}$ such that $\sigma_{12}(\mathcal{Y}^{1,1}) = \lambda\mathcal{Y}^{1,1}$. Therefore we have:

$$\begin{aligned}
&E\big(\langle d', \mathcal{Y}^{1,i+1}(w,x)\mathcal{Y}^{1,i}(u,y)a\rangle\big)\\
&= E\big(\langle d', \mathcal{Y}^{2,i}\big(\mathcal{Y}^{1,1}(w,x-y)u,y\big)a\rangle\big)\\
&= E\big(\langle d', \mathcal{Y}^{2,i}\big(e^{L(-1)(x-y)}\sigma_{12}\big(\mathcal{Y}^{1,1}\big)(u,y-x)w,y\big)a\rangle\big)\\
&= E\big(\langle d', \mathcal{Y}^{2,i}\big(\sigma_{12}\big(\mathcal{Y}^{1,1}\big)(u,y-x)w,x\big)a\rangle\big)\\
&= E\big(\langle d', \lambda\mathcal{Y}^{2,i}\big(\mathcal{Y}^{1,1}(u,y-x)w,x\big)a\rangle\big)\\
&= E\big(\langle d', \lambda\mathcal{Y}^{1,i+1}(u,y)\mathcal{Y}^{1,i}(w,x)a\rangle\big).
\end{aligned}$$

Irritating it n times, we obtain

$$\begin{aligned}
&E\big(\langle e', \mathcal{Y}^{1,n-1}\big(d^1,x_1\big)\cdots\mathcal{Y}^{1,0}\big(d^n,x_n\big)\mathcal{Y}^{1,n-1}(a,z_1)\cdots\mathcal{Y}^{1,0}(c,z_{n-1})v\rangle\big)\\
&= \lambda^{n^2}E\big(\langle e', \mathcal{Y}^{1,0}(a,z_1)\cdots\mathcal{Y}^{1,1}(c,z_{n-1})\mathcal{Y}^{1,n-1}\big(d^1,x_1\big)\cdots\mathcal{Y}^{1,0}\big(d^n,x_n\big)v\rangle\big)
\end{aligned}$$

for $e' \in (W^1)'$, $\underbrace{a, \ldots, c}_{n}, d^i \in W^1$ and $v \in V$. On the other hand, from the associativity and $\overbrace{W^1 \boxtimes \cdots \boxtimes W^1}^{n} = V$, there is $0 \neq \mu \in \mathbb{C}$ such that

$$\begin{aligned}
&E\big(\langle e', \mathcal{Y}^{1,n-1}\big(d^1,x_1\big)\cdots\mathcal{Y}^{1,0}\big(d^n,x_n\big)\mathcal{Y}^{1,n-1}(a,z_1)\cdots\mathcal{Y}^{1,0}(c,z_{n-1})v\rangle\big)\\
&= E\big(\langle e', \mathcal{Y}^{1,n-1}\big(d^1,x_1\big)\cdots\mathcal{Y}^{1,0}\big(d^n,x_n\big)\\
&\quad \times \mu Y^V\big(\mathcal{Y}^{n-1,1}\cdots\mathcal{Y}^{1,1}(a,r_1)b,\cdots(r_{n-2})c,z_{n-1}\big)v\rangle\big)
\end{aligned}$$

for any $v \in V$, where we set $r_i = z_i - z_{i+1}$ to make notation short. Since $\mathcal{Y}^{i,j}$ satisfies Commutativity with Y^W, we have

$$\begin{aligned}
&E\big(\langle e', \mathcal{Y}^{1,n-1}\big(d^1, x_1\big)\cdots\mathcal{Y}^{1,0}\big(d^n, x_n\big)\mathcal{Y}^{1,n-1}(a, z_1)\cdots\mathcal{Y}^{1,0}(c, z_{n-1})v\rangle\big)\\
&\quad= E\big(\langle e', \mathcal{Y}^{1,n-1}\big(d^1, x_1\big)\cdots\mathcal{Y}^{1,0}\big(d^n, x_n\big)\\
&\qquad\times \mu Y^V\big(\mathcal{Y}^{n-1,1}\cdots\mathcal{Y}^{1,1}(a, r_1)b, \cdots(r_{n-2})c, z_{n-1}\big)v\rangle\big)\\
&\quad= E\big(\langle e', \mu Y^V\big(\mathcal{Y}^{n-1,1}\cdots\mathcal{Y}^{1,1}(a, r_1)b, \cdots(r_{n-2})c, z_{n-1}\big)\\
&\qquad\times \mathcal{Y}^{1,n-1}\big(d^1, x_1\big)\cdots\mathcal{Y}^{1,0}\big(d^n, x_n\big)v\rangle\big)\\
&\quad= E\big(\langle e', \mathcal{Y}^{1,0}(a, z_1)\cdots\mathcal{Y}^{1,1}(c, z_{n-1})\mathcal{Y}^{1,n-1}\big(d^1, x_1\big)\cdots\mathcal{Y}^{1,0}\big(d^n, x_n\big)v\rangle\big),
\end{aligned}$$

which implies $\lambda^{n^2} = 1$. Since n is odd, we have $\lambda = 1$. Thus, $\mathcal{Y}$ satisfies Commutativity with itself. By using the normal products, $\mathcal{Y}$ and Y^W generate a local system $\mathcal{O}$ with a Virasoro element $Y^W(\omega, z)$. Since V is a subVOA of $\mathcal{O}$ and its modules are completely reducible, $\mathcal{O}$ is a direct sum $\mathcal{O} = \bigoplus T^j$ of simple V-modules T^j. Clearly, the action of $\mathcal{O}$ on W induces intertwining operators of type $\binom{W}{\mathcal{O}\ W}$. Since $V \boxtimes V = V$, the multiplicity of a V-module V in $\mathcal{O}$ is one and so we have $\mathcal{O} \cong W$ as V-modules. Therefore, W has a VOA structure, which proves Proposition 7. □

This completes the proof of Theorem B.

3.1 The Character of the Moonshine VOA

The first example is the Leech lattice Λ and a fixed point free automorphism σ of Λ of order three. Then a trace function $T_{V_\Lambda}(\sigma, \mathbf{1}; \tau)$ of σ on V_Λ is

$$\begin{aligned}
&q^{-1}\left(\frac{1}{\prod_{n=1}^{\infty}(1-\xi q^n)}\right)^{12}\left(\frac{1}{\prod_{n=1}^{\infty}(1-\xi^2 q^n)}\right)^{12}\\
&\quad= q^{-1}\left(\frac{1}{\prod_{n=1}^{\infty}(1+q^n+q^{2n})}\right)^{12} = \frac{\eta(\tau)^{12}}{\eta(3\tau)^{12}}.
\end{aligned}$$

Hence, a character function of the twisted module $V_\Lambda(\sigma)$ is

$$\begin{aligned}
\mathrm{ch}\big(V_\Lambda(\sigma)\big) &= \mathrm{ch}\big(W^3\big) + \mathrm{ch}\big(W^4\big) + \mathrm{ch}\big(W^5\big) = T_{V_\Lambda}(\sigma, \mathbf{1}, -1/\tau)\\
&= 3^6 q^{-1} q^{4/3}\frac{\prod_{n=1}^{\infty}(1-q^n)^{12}}{\prod_{n=1}^{\infty}(1-q^{n/3})^{12}},
\end{aligned}$$

which implies that W^3 (also W^6) has no elements of weight 1 and $\mathrm{ch}(\widetilde{V}_\Lambda) = J(\tau)$.

By a calculation, $\dim W_2^3 = 3^6(12 + 12 + \binom{12}{2}) = 65610$, where W_2^3 denotes the weight 2 subspace of W^3 and so a triality automorphism of $\widetilde{V}$ defined by ξ^i on W^{3i} for $i = 0, 1, 2$ is corresponding to $3\mathrm{B} \in \mathbb{M}$ if $\widetilde{V} \cong V^\natural$.

3.2 A New VOA No. 32 in Schellekens' List

The second example is a Niemeier lattice N of type E_6^4 and a triality automorphism σ which acts on the first entry E_6 as fixed point free and permutes the last three E_6, where we choose $\langle(0,1,1,1),(1,1,2,0)\rangle$ as a set of glue vectors of N for E_6^4, see [4]. We note that since E_6 contains a full sublattice A_2^3, E_6 has a fixed point free automorphism of order three. Since $t=9$, in order to determine a dimension of the weight one space $(\tilde{V}_N)_1$, it is enough to see the constant term of $q^{6/24}\frac{\Theta_H(-1/\tau)}{\eta(-1/\tau)^6}$. Since the fixed point sublattice H is isomorphic to $\sqrt{3}E_6^*$, where E_6^* denotes the dual of E_6, we have

$$\Theta_H(\tau)=\frac{1}{3}\left[\phi_0(\tau)^3+\frac{1}{4}\left\{3\phi_0(3\tau)-\phi_0(\tau)\right\}^3\right],$$

where $\phi_0(\tau)=\theta_2(2\tau)\theta_2(6\tau)+\theta_3(2\tau)\theta_3(6\tau)$ and $\theta_2(\tau)=\sum_{m\in\mathbb{Z}}q^{(m+1/2)^2}$ and $\theta_3(\tau)=\sum_{m\in\mathbb{Z}}q^{m^2}$, see [4]. Applying $\phi_0(-1/\tau)=\frac{\tau}{i\sqrt{3}}\phi_0(\tau/3)$, we have

$$\frac{\Theta_H(-1/\tau)}{\eta(-1/\tau)^6}=\frac{1}{9\sqrt{3}}q^{-1/4}+\cdots$$

and so

$$\dim(\tilde{V}_N)_1=(6\times 12)/3+6+6\times 12+2\times\left\{3^{9/2}3^{-5/2}\right\}=120.$$

Clearly, from the construction we know that $(\tilde{V}_N)_1$ contains $(V_N^\sigma)_1\cong A_2^3E_{6,3}$ as a subring. Therefore, $\tilde{V}_N$ is a new VOA No 32 in the list of Schellekens [16].

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Words, Automata and Lie Theory for Tilings

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Abstract We will give a new relationship between several simple automata and formal power series as word invariants. Such an invariant is derived from certain combinatorics and algebraic structures. We review them, and especially we deal with a connection to Lie theory via through tilings.

1 Introduction

For each word w, as an abstract invariant, we obtain a formal power series $f_w(t)$ in t with real coefficients (cf. [17, 20]). On the other hand, automata sometimes create words. Therefore, we have the following picture.

$$\boxed{\text{automata}} \longrightarrow \boxed{\text{words}} \longrightarrow \boxed{\text{formal power series}}$$

Hence, it seems to be nice that there is a characterization of certain automata using formal power series. This paper consists of

(1) a new approach to characterize the following simple automata, $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ (Figs. 1, 2, 3, 4), using formal power series and a hierarchy of numbers, as well as
(2) a review of several recent results.

We note that $f_w(t)$ can be obtained as a unique solution of some quadratic equation for each word w. Such an equation is related to the decomposition rule for tensor products of standard modules. A rough story is as follows. Let $\mathcal{T}$ be a one dimensional tiling (or a bi-infinite word), and $\mathfrak{M}$ the associated monoid. Put $\mathfrak{A} = \mathbb{C}[\mathfrak{M}]$ as a monoid algebra with a bialgebra structure. Then, standard modules of $\mathfrak{A}$ are defined, and irreducible standard modules are parametrized by subwords of $\mathcal{T}$. An irreducible standard module corresponding to a subword w of $\mathcal{T}$ is denoted by V_w.

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Fig. 1 Automaton $\mathcal{A}_1$

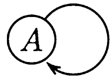


Fig. 2 Automaton $\mathcal{A}_2$

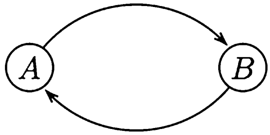


Fig. 3 Automaton $\mathcal{A}_3$

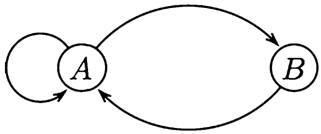
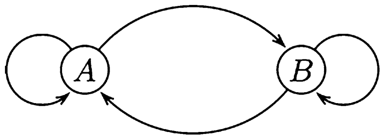


Fig. 4 Automaton $\mathcal{A}_4$



Any standard module is completely reducible. Then, we have:

$$V_w \otimes V_w = V_w \oplus \left(\bigoplus_{s \neq w} V_s^{\oplus \mu_s(w)} \right),$$

which produces the equation

$$f_w(t)^2 = f_w(t) + \sum_{s \neq w} \mu_s(w) f_s(t).$$

Here, $\mu_s(w)$ means the multiplicity, and $V^{\oplus m}$ means

$$\underbrace{V \oplus \cdots \oplus V}_{m}.$$

By our convention here, we use "lowercase" for a word, and "uppercase" for a letter, respectively. We will discuss words in Sect. 1, and formal power series as algebraic invariants in Sect. 2 (cf. [15, 17, 20]). We will characterize simple automata using formal power series in Sect. 3. We refer [13] and [14] for automata (or shifts), symbolic dynamics and combinatorics on words. In Sect. 4, we review several recent results. Our motivation is coming from the study of tilings. Using the so-called Kellendonk product (cf. [9–11]), we will construct monoids (cf. [12]) and monoid algebras (cf. [1, 15, 17, 20]). These algebras were introduced and developed deeply in [9–11], where topology, C^*-algebras and K-theory are used. We also discuss a certain representation theory. Especially we introduce group structures and

Lie algebra structures (cf. [6–8, 16]), and obtain fundamental algebraic properties (cf. [17, 20]), including Gauss decompositions and additive Gauss decompositions (cf. [4, 5, 18, 19]).

We denote by $\mathbb{C}$ (resp. $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$) the set of complex numbers (resp. real numbers, rational numbers, integers, natural numbers).

2 Words

Let Ω be a set of letters. For convenience, we usually assume that Ω is finite. (We assume $\sharp\Omega \leq 2$ in Theorem 1. But, in general, we need not assume this finiteness.) A word is a finite sequence of letters in Ω. That is,

$$w = X_1 X_2 \cdots X_n \quad (X_i \in \Omega)$$

is a word. The transpose of w means ${}^t w = X_n X_{n-1} \cdots X_1$. The length of w is denoted by $\ell(w)$, and in this case $\ell(w) = n$. A subword of w is a successive subsequence of w, namely

$$X_k X_{k+1} X_{k+2} \cdots X_l \quad (1 \leq k \leq l \leq n).$$

We say $s \prec w$ if s is a subword of w, and we set $S(w) = \{s \mid s \prec w\}$. We choose and fix a special symbol, called ϕ, which should be totally different from letters in Ω. One sometimes understand that ϕ is a special symbol meaning an empty word. To control partial similarity of w, we define a certain $n \times n$ matrix $M(w) = (m_{ij})$, whose entries are in $\Omega \cup \{\phi\}$ as follows.

$$m_{ij} = \begin{cases} X_i & \text{if } X_i = X_j, \\ \phi & \text{if } X_i \neq X_j. \end{cases}$$

Using the matrix $M(w)$, we introduce the associated graph, called $\Gamma(w)$, with vertices v_{ij} $(1 \leq i, j \leq n)$ and edges defined by saying that v_{ij} and $v_{i+1,j+1}$ are joined if $m_{ij} \neq \phi$ and $m_{i+1,j+1} \neq \phi$. Let $C(w)$ be the set of all connected components of $\Gamma(w)$. For $\gamma \in C(w)$, we define

$$w_\gamma = X_i X_{i+1} X_{i+2} \cdots X_{i+k}$$

if $\gamma = \{v_{ij}, v_{i+1,j+1}, v_{i+2,j+2}, \ldots, v_{i+k,j+k}\}$ with $k \geq 1$, or if $\gamma = \{v_{ij}\}$ with $m_{ij} \neq \phi$. Then, we define the multiplicity $\mu_s(w)$ for a subword $s \prec w$ by

$$\mu_s(w) = \sharp \left\{\gamma \in C(w) \mid w_\gamma = s\right\}.$$

Also we define

$$\mu_\phi(w) = \sharp \{v_{ij} \mid m_{ij} = \phi,\ 1 \leq i, j \leq n\}.$$

Then, we obtain

$$\mu_\phi(w) + \sum_{s \in S(w)} \mu_s(w) \cdot \ell(s) = n^2.$$

3 Algebraic Invariants

Here we abstractly consider the following quadratic equation in $x = g(t)$:

$$x^2 = x + \mu_\phi(w) f_\phi(t) + \sum_{s \in S(w), s \neq w} \mu_s(w) \cdot f_s(t), \qquad (*)$$

where $g(t)$, $f_\phi(t)$ and $f_s(t)$ are formal power series in t with real coefficients. We proceed in the following process.

(F1) Put $f_\phi(t) = t$ first, where t is a formal variable.
(F2) Suppose that the $f_s(t)$ for $s \in S(w)$ with $s \neq w$ are already defined as formal power series in t with positive constant terms.
(F3) If the equation $(*)$ has a solution $x = g(t)$ as a formal power series in t with a positive constant term, then we put $f_w(t) = g(t)$.

This recursive process works well (cf. [17, 20]). We compute several typical examples. If $w = A$, then $M(w) = (A)$ and the equation $(*)$ is

$$x^2 = x.$$

This has an expected solution $f_A(t) = 1$, which is just a constant. Next suppose $w = AB$. Then,

$$M(w) = \begin{pmatrix} A & \phi \\ \phi & B \end{pmatrix}$$

and our equation $(*)$ is

$$x^2 = x + 2t.$$

We need to solve

$$g(t)^2 = g(t) + 2t$$

satisfying $g(t) = \sum_{i=0}^{\infty} a_i t^i$ with $a_0 > 0$. Hence, we have the following.

$$\begin{aligned} a_0^2 &= a_0 \\ 2a_0 a_1 &= a_1 + 2 \\ 2a_0 a_2 + a_1^2 &= a_2 \\ 2a_0 a_3 + 2a_1 a_2 &= a_3 \\ &\vdots \end{aligned}$$

This implies $a_0 = 1$ since $a_0 > 0$. Then we inductively obtain $a_1 = 2/(2a_0 - 1) = 2$, $a_2 = -a_1^2/(2a_0 - 1) = -4$, $a_3 = -2a_1a_2/(2a_0 - 1) = 16, \ldots$. That is,

$$f_{AB}(t) = g(t) = 1 + 2t - 4t^2 + 16t^3 - \cdots,$$

where all coefficients are integers since $2a_0 - 1 = 1$. Hence, we have $f_{AB}(t) \in \mathbb{Z}[[t]]$. More precisely we easily see that a_i is an even integer for all $i > 0$. Furthermore, we examine one more example:

$$w = \underbrace{AA \cdots A}_{n}$$

with $\ell(w) = n$. In this case, we claim $f_w(t) = n$, which can be obtained by induction on $\ell(w)$. As above, we already knew that our claim is true for $\ell(w) = 1$. We suppose that our claim is true for $\ell(w) < n$. In case of $\ell(w) = n$, we have

$$M(w) = \begin{pmatrix} A & \cdots & A \\ \vdots & \ddots & \vdots \\ A & \cdots & A \end{pmatrix}$$

as an $n \times n$ matrix, and our equation ($*$) is

$$g(t)^2 = g(t) + n(n-1).$$

Here we note that $\mu_s(w) = 2$ and $f_s(t) = \ell(s)$ for $s \in S(w)$ with $s \neq w$ and $\mu_\phi(w) = 0$, and that $2(1 + 2 + 3 + \cdots + (n-1)) = n(n-1)$. Thus, we obtain $f_w(t) = g(t) = n$ if

$$w = \underbrace{AA \cdots A}_{n}.$$

We may call $f_w(t)$ is an algebraic invariant, since our equation ($*$) is coming from certain algebraic structures (standard modules and tensor product decompositions etc.) behind.

4 Automata

To create words from a given set Ω of letters, there may be a rule. Such a rule is sometimes controlled by an automaton (or a shift). Here we consider the automata called $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ (as in Introduction) with $\Omega = \{A\}$ or $\Omega = \{A, B\}$, which are very simple and fundamental. Let $W_i(X)$ be the set of all words according to

$\mathcal{A}_i$ $(i = 1, 2, 3, 4)$ and an initial letter $X \in \Omega$. Namely,

$$W_1(A) = \{A, AA, AAA, AAAA, AAAAA, AAAAAA, \ldots\},$$
$$W_2(A) = \{A, AB, ABA, ABAB, ABABA, ABABAB, \ldots\},$$
$$W_2(B) = \{B, BA, BAB, BABA, BABAB, BABABA, \ldots\},$$
$$W_3(A) = \{A, AA, AB, AAA, AAB, ABA,$$
$$AAAA, AAAB, AABA, ABAA, ABAB, \ldots\},$$
$$W_3(B) = \{B, BA, BAA, BAB, BAAA, BAAB, BABA, \ldots\},$$
$$W_4(A) = \{A, AA, AB, AAA, AAB, ABA, ABB, \ldots\},$$
$$W_4(B) = \{B, BA, BB, BAA, BAB, BBA, BBB, \ldots\}.$$

For two words w, w' in $W_i(X)$ with $\ell(w) = n$ and $\ell(w') = n + 1$, we call w' is an extension in $W_i(X)$ of w if w is a subword of w' satisfying

$$w = X_1 X_2 \cdots X_n \quad \text{and} \quad w' = wX_{n+1} = X_1 X_2 \cdots X_n X_{n+1},$$

and we call w' a unique extension in $W_i(X)$ of w if w' is unique as an extension of w. For example, ABA is a unique extension in $W_3(A)$ of AB, but ABA is not a unique extension in $W_4(A)$ of AB since ABB exists in $W_4(A)$ as another extension of AB. A word w in $W_i(X)$ is called of infinite type if there is an infinite sequence, $\{u_k\}_{k=0}^{\infty}$ with $u_0 = w$, of words in $W_i(X)$ such that u_{k+1} is a unique extension of u_k for all $k \geq 0$. A word w in $W_i(X)$ is called maximal if there is no unique extension in $W_i(X)$ of w. Let $V_i(X)$ be the set of $w \in W_i(X)$ satisfying (P1) and (P2):

(P1) $\ell(w)$ is even,
(P2) w is maximal or of infinite type.

We call $V_i(X)$ the principal part of $W_i(X)$. Then, we put

$$V_i = \bigcup_{X \in \Omega} V_i(X).$$

In fact, we have

$$V_1 = \{AA, AAAA, AAAAAA, AAAAAAAA, \ldots\},$$
$$V_2 = \{AB, BA, ABAB, BABA, ABABAB, BABABA, \ldots\},$$
$$V_3 = \{AA, BA, AAAA, AABA, ABAA, BAAA, BABA, \ldots\},$$
$$V_4 = \{AA, AB, BA, BB, AAAA, AAAB, AABA, AABB,$$
$$ABAA, ABAB, ABBA, ABBB, BAAA, BAAB,$$
$$BABA, BABB, BBAA, BBAB, BBBA, BBBB, \ldots\}.$$

Theorem 1 *Notation is as above.*

(1) $f_w(t) = \ell(w) \in \mathbb{N}$ *if* $w \in V_1$.
(2) $f_w(t) \in \mathbb{Z}[[t]]$ *if* $w \in V_2$. *There exists* $w \in V_2$ *such that* $f_w(t) \notin \mathbb{N}$.
(3) $f_w(t) \in \mathbb{Q}[[t]]$ *if* $w \in V_3$. *There exists* $w \in V_3$ *such that* $f_w(t) \notin \mathbb{Z}[[t]]$.
(4) $f_w(t) \in \mathbb{R}[[t]]$ *if* $w \in V_4$. *There exists* $w \in V_4$ *such that* $f_w(t) \notin \mathbb{Q}[[t]]$.

Proof of Theorem 1 (1) is already discussed in the previous section. For (2), we observe that $f_w(t) = m \times f_{AB}(t)$ for all $w \in V_2$ with $\ell(w) = n = 2m$ as well as that the constant term a_0 of $f_w(t) = \sum_{k=0}^{\infty} a_k t^k$ is m. We will show it by induction. We can suppose $w \in V_2(A)$, and

$$w = \underbrace{\underline{AB}\ \underline{AB}\ \cdots\ \underline{AB}}_{m}$$

with $\ell(w) = 2m$. Then, we obtain

$$M(w) = \begin{pmatrix} A & \phi & A & \phi & \cdots & \cdots & A & \phi \\ \phi & B & \phi & B & \cdots & \cdots & \phi & B \\ A & \phi & A & \phi & \ddots & \ddots & \vdots & \vdots \\ \phi & B & \phi & B & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & A & \phi \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \phi & B \\ A & \phi & \cdots & \cdots & A & \phi & A & \phi \\ \phi & B & \cdots & \cdots & \phi & B & \phi & B \end{pmatrix}$$

as a $2m \times 2m$ matrix. Our equation (∗) implies

$$\begin{aligned} g(t)^2 &= g(t) + 2\big((2m-1) + (2m-3) + \cdots + 1\big)t \\ &\qquad + 2\big((m-1) + (m-2) + \cdots + 1\big) f_{AB}(t) \\ &= g(t) + 2m^2 t + m(m-1) f_{AB}(t). \end{aligned}$$

Then, we observe that $g(t) = m \times f_{AB}(t)$ is a solution of this equation, that is,

$$\big(m \times f_{AB}(t)\big)^2 = m \times f_{AB}(t) + 2m^2 t + m(m-1) f_{AB}(t),$$

since $f_{AB}(t)$ satisfies

$$f_{AB}(t)^2 = f_{AB}(t) + 2t.$$

In particular, we obtain, as a unique solution under our condition, $f_w(t) = m \times f_{AB}(t) \in \mathbb{Z}[[t]]$. For (3), we should refer [17], where the proof of (3) is given. However, it might be better to review it here for the reader. Take $w = X_1 \cdots X_n \in W_3(A)$. Under this assumption, w satisfies that there is no pattern like $X_i X_{i+1} = BB$ with

$2 \le i \le n-1$. Then, we claim that if $w \in W_3(A)$ then $f_w(t) \in \mathbb{Q}[[t]]$ and the constant term $f_w(0)$ of $f_w(t)$ is the number, called $\ell_A(w)$, of the letter A in w. We proceed by induction. Let $D(w) = \{w_\gamma \mid \gamma \in C(w)\} \setminus \{w\}$, and put $w' = X_1 \cdots X_{n-1}$. Then, we write $f_w(t) = \sum_{i=0}^{\infty} p_i t^i$ and $f_{w'}(t) = \sum_{i=0}^{\infty} q_i t^i$ with $p_i, q_i \in \mathbb{R}$. By induction, we can assume $q_0 = \ell_A(w')$. We also note that $w'' = Y_1 \cdots Y_m \in D(w)$ implies $Y_1 = A$, which shows by induction that if $w'' \in D(w)$ then $f_{w''}(t) \in \mathbb{Q}[[t]]$ and $f_{w''}(0) = \ell_A(w'')$. First suppose $X_n = B$. If $m_{in} = B$ in $M(w)$ with $1 \le i \le n-1$, then the corresponding v_{in} is always attached to $\gamma' = \{\ldots, v_{i-1,n-1}\} \in C(w')$ in $C(w)$ to get $w_{\gamma'} B \in D(w)$, since $m_{i-1,n-1} = A$. In this case, there is no change of the number of the letter A. Hence, $p_0^2 = p_0 + (q_0 - 1)q_0 = p_0 + (\ell_A(w') - 1)\ell_A(w')$. Therefore, $p_0 = \ell_A(w') = \ell_A(w)$. Next, suppose that $X_n = A$. If $m_{in} = A \in M(w)$ with $1 \le i \le n-1$, then the corresponding v_{in} is attached to $\gamma' = \{\ldots, v_{i-1,n-1}\} \in C(w')$ in $C(w)$, or v_{in} is itself a connected component in $C(w)$. In each case, such A can give an effect "+1" to compute the corresponding constant term. Note that the number of the letter A added in the process from $C(w')$ to $C(w)$ is $2\ell_A(w') = 2q_0$ as follows. We need to count the letter A at the last column and at the last row. Since the letter A appears $\ell_A(w')$ times in w', we have

$$p_0^2 = p_0 + (q_0 - 1)q_0 + 2\ell_A\big(w'\big) = p_0 + \ell_A\big(w'\big)\big(\ell_A\big(w'\big) + 1\big).$$

Therefore, $p_0 = \ell_A(w') + 1 = \ell_A(w)$. In any case, that is, in either case $X_n = B$ or $X_n = A$, we see $p_0 = \ell_A(w)$. Now we should solve

$$g(t)^2 = g(t) + \mu_\phi(w)t + \sum_{s \in D(w)} \mu_s(w) f_s(t),$$

which can be written as

$$\sum_{k=0}^{\infty}(p_0 p_k + p_1 p_{k-1} + \cdots + p_k p_0)t^k = \left(\sum_{k=0}^{\infty} p_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^k\right).$$

By induction all the b_k are rational, and by our local recursive setting we can assume that $p_1, \ldots, p_{k-1}$ are also rational. Then, we have

$$p_0 p_k + p_1 p_{k-1} + \cdots + p_k p_0 = p_k + b_k.$$

Since $p_0 = \ell_A(w)$ is a positive integer, we see that p_k is rational, that is,

$$p_k = \frac{b_k - (p_1 p_{k-1} + p_2 p_{k-2} + \cdots + p_{k-1} p_1)}{2\ell_A(w) - 1}$$

for $k > 0$. Hence, we can recursively obtain $f_w(t) \in \mathbb{Q}[[t]]$. Now we take $w \in V_3$. Then, we see $w \in W_3(A)$ or ${}^t w \in W_3(A)$. Since $f_w(t) = f_{{}^t w}(t)$, we always obtain $f_w(t) \in \mathbb{Q}[[t]]$. To confirm (4), we choose, for example, $w = AABB \in V_4$. Then,

$$f_{AABB}(t) = \frac{1+\sqrt{17}}{2} + \frac{8}{\sqrt{17}}t - \frac{64}{17\sqrt{17}}t^2 + \cdots \notin \mathbb{Q}[[t]].$$

One should also note, as examples, that

$$f_{AB}(t) = 1 + 2t - 4t^2 + 16t^3 - \cdots \notin \mathbb{N}$$

and

$$f_{ABAA}(t) = 3 + \frac{6}{5}t - \frac{36}{125}t^2 + \frac{432}{3125}t^3 - \cdots \notin \mathbb{Z}[[t]]. \qquad \square$$

By Theorem 1, our algebraic invariant $f_w(t)$ of a word w can differentiate these automata $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ according to the hierarchy of numbers appeared in coefficients. We note that we are just interested in the growth along one direction if we imagine crystals or quasicrystals. The above proof of Theorem 1 looks very elementary, which is derived from our simple combinatorics. However, it seems to be not so easy to find such a result at the beginning. One believes that it is much more important to observe a new fact in this kind of study.

5 Motivation and Review

For a one dimensional tiling $\mathcal{T}$, we define $\Omega(\mathcal{T})$ to be the set of lengths of all intervals as tiles in $\mathcal{T}$. Here, we assume that $\Omega(\mathcal{T})$ is finite. We choose a set Ω of letters with $\sharp\Omega = \sharp\Omega(\mathcal{T})$, and fix a bijection from $\Omega(\mathcal{T})$ to Ω. Then, we shall identify $\Omega(\mathcal{T})$ with Ω, and a one dimensional tiling with a bi-infinite sequence of letters, using such a bijection. Hence, we shall consider $\mathcal{T}$ as

$$\cdots\ X_{-2}X_{-1}X_0X_1X_2\ \cdots \qquad \big(X_i \in \Omega = \Omega(\mathcal{T})\big).$$

A subword of $\mathcal{T}$ is a finite successive subsequence $X_iX_{i+1}X_{i+2}\cdots X_j$ for some integers $i \le j$. Let $S(\mathcal{T})$ be the set of subwords of $\mathcal{T}$, and put $S^*(\mathcal{T}) = S(\mathcal{T}) \cup \{\phi\}$, where ϕ may be viewed as an empty word. Two tilings $\mathcal{T}$ and $\mathcal{T}'$ are said to be locally indistinguishable if there is a bijection $\pi : \Omega(\mathcal{T}) \longrightarrow \Omega(\mathcal{T}')$ such that $S(\mathcal{T}') = \{\pi(w) \mid w \in S(\mathcal{T})\}$, where $\pi(w) = \pi(X_1)\pi(X_2)\cdots\pi(X_r)$ for $w = X_1X_2\cdots X_r \in S(\mathcal{T})$.

Let $\mathcal{T}$ be a one dimensional tiling. Let (i, a, j) be a triplet of $a \in S(\mathcal{T})$ and $1 \le i, j \le \ell(a)$, and set

$$\mathfrak{M} = \mathfrak{M}(\mathcal{T}) = \{\boldsymbol{z}, \boldsymbol{e}\} \cup \big\{(i, a, j) \mid a \in S(\mathcal{T}), 1 \le i, j \le \ell(a)\big\},$$

where $\boldsymbol{z}$ and $\boldsymbol{e}$ are new abstract independent symbols.

We recall a matrix unit, called E_{ij}. One can imagine that (i, a, j) is corresponding to $E_{ij} = (i, E, j)$. Then, we remember the following rule:

$$E_{ij} \cdot E_{kl} = \delta_{jk}E_{il} = \begin{cases} E_{il} & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, in our case here, E itself is parametrized. That is, we have:

$$(i, a, j) \cdot (k, b, l) = \begin{cases} (p, c, q) & \text{if } \ldots, \\ z & \text{otherwise,} \end{cases}$$

as described below.

For $(i, a, j), (k, b, l) \in \mathfrak{M}$, we define a certain product, called the Kellendonk product, of (i, a, j) and (k, b, l) as follows. Pile up the j-th position of a and the k-th position of b. If one gets $c \in S(\mathcal{T})$ by this piling, then we define $(i, a, j) \cdot (k, b, l) = (p, c, q)$, where p is the position of c corresponding to i and q is the position of c corresponding to l satisfying $1 \leq p, q \leq \ell(c)$. Otherwise, we define $(i, a, j) \cdot (k, b, l) = z$. We also define $\boldsymbol{m} \cdot \boldsymbol{e} = \boldsymbol{e} \cdot \boldsymbol{m} = \boldsymbol{m}$ and $\boldsymbol{m} \cdot \boldsymbol{z} = \boldsymbol{z} \cdot \boldsymbol{m} = \boldsymbol{z}$ for all $\boldsymbol{m} \in \mathfrak{M}$. Then, $\mathfrak{M}$ becomes a monoid. Let $\mathfrak{A} = \mathbb{C}[\mathfrak{M}] = \bigoplus_{\boldsymbol{m} \in \mathfrak{M}} \mathbb{C}\boldsymbol{m}$ be the monoid algebra of $\mathfrak{M}$ over $\mathbb{C}$ (cf. [9–11]). Since $\mathfrak{M} = \mathfrak{M}(\mathcal{T})$ is a monoid, the monoid algebra $\mathfrak{A} = \mathfrak{A}(\mathcal{T}) = \mathbb{C}[\mathfrak{M}]$ becomes a bialgebra with a coalgebra map $\Delta : \boldsymbol{m} \mapsto \boldsymbol{m} \otimes \boldsymbol{m}$ and a counit map $\varepsilon : \boldsymbol{m} \mapsto 1$ for all $\boldsymbol{m} \in \mathfrak{M}$ (cf. [1]).

Theorem 2 (cf. [20]) *For a couple of one dimensional tilings $\mathcal{T}$ and $\mathcal{T}'$, the following two conditions are equivalent.*

(1) $\mathfrak{A}(\mathcal{T}) \simeq \mathfrak{A}(\mathcal{T}')$ *or* $\mathfrak{A}({}^t\mathcal{T}) \simeq \mathfrak{A}(\mathcal{T}')$ *as bialgebras.*
(2) $\mathcal{T}$ *and* $\mathcal{T}'$ *are locally indistinguishable, or* ${}^t\mathcal{T}$ *and* $\mathcal{T}'$ *are locally indistinguishable.*

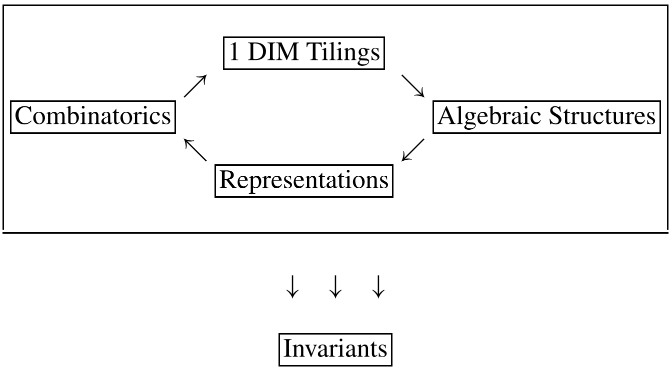
Note that ${}^t\mathcal{T}$ is the reverse of $\mathcal{T}$. That is, ${}^t\mathcal{T}$ is

$$\cdots X_2 X_1 X_0 X_{-1} X_{-2} \cdots$$

if $\mathcal{T}$ is

$$\cdots X_{-2} X_{-1} X_0 X_1 X_2 \cdots .$$

Therefore, $\mathcal{T} \mapsto \mathfrak{A}(\mathcal{T})$ is an invariant of one dimensional tilings. Let $\mathcal{T}$ be a one dimensional tiling, and put $\mathfrak{A} = \mathfrak{A}(\mathcal{T})$. To establish Theorem 2, it is important to study standard modules, which induces some combinatorics. We can draw the following global picture.



If we have objects which we want to study, then it may sometimes be a good idea to construct certain algebraic structures. Using such algebraic structures, there might be the corresponding representation theory, which produces several combinatorics. Usually such combinatorics may be very complicated, that is, more complicated than original objects. But, here we can obtain the above circle, which seems to be useful.

A left $\mathfrak{A}$-module V is called standard if the following two conditions are satisfied.

(S1) $\dim(V) < \infty$, $\boldsymbol{z}(V) = 0$
(S2) $\sharp\{\boldsymbol{x} \in \mathfrak{M} \mid \boldsymbol{x}(V) \neq 0\} < \infty$

Each $a \in S^*(\mathcal{T})$ induce an irreducible standard module called V_a (cf. [17, 20]).

Theorem 3 (cf. [20]) *Notation is as above.*

(1) $\{V_a \mid a \in S^*(\mathcal{T})\}$ *is a complete set of representatives for irreducible standard* $\mathfrak{A}$*-modules.*
(2) *Every standard* $\mathfrak{A}$*-module is completely reducible.*
(3) *Every standard* $\mathfrak{A}$*-module is isomorphic to* $V_{a_1} \oplus \cdots \oplus V_{a_r}$ *for some* $a_1, \ldots, a_r \in S^*(\mathcal{T})$.

There might be several ways to pick up certain combinatorics from given decomposition rules.

$$\boxed{\text{Tensor Product Decompositions}} \to \boxed{\text{Combinatorics}} \to \boxed{\text{Invariants}}$$

Here we choose the following type of decomposition:

$$V \otimes V = V \oplus (\text{Other Terms}),$$

which we understand as a quadratic equation in $x = g(t)$:

$$x^2 = x + (\text{Other Terms}).$$

There is another way, and then we obtain a higher degree equation. But, we really want to get concrete invariants. For that purpose, we should always solve our equation. Hence, it is better to choose quadratic equations.

For standard $\mathfrak{A}$-modules V_a, V_b with $a, b \in S^*(\mathcal{T})$, we see that $V_a \otimes V_b$ is also a standard $\mathfrak{A}$-module, which is given by $x(v) = \Delta(x)(v)$ for all $x \in \mathfrak{A}$ and $v \in V_a \otimes V_b$. Then, we obtain an irreducible decomposition:

$$V_a \otimes V_b = \bigoplus_{c \in S^*(\mathcal{T})} V_c^{\oplus \mu_c(a,b)},$$

where $\mu_c(a,b)$ is the multiplicity of V_c. We put $\mu_c(a) = \mu_c(a,a)$. Then, our quadratic equation (∗) is arising from this formula for $a = b$ as follows:

$$V_a \otimes V_a = \bigoplus_{c \in S^*(\mathcal{T})} V_c^{\oplus \mu_c(a)} = V_a \oplus \left(\bigoplus_{c \neq a} V_c^{\oplus \mu_c(a)} \right).$$

To construct associated groups and Lie algebras and to study them, we sometimes need to divide tiles into three pieces.

$$\boxed{\text{1 DIM Tilings}} \to \boxed{\text{Kellendonk Products}} \to \begin{cases} \boxed{\text{Lie Algebras}} \\ \boxed{\text{Groups}} \\ \boxed{\text{Bialgebras}} \end{cases}$$

For simple notation, we identify x in $\mathfrak{A}$ with $\bar{x} = x + \mathbb{C}z$ in $\mathfrak{A}/\mathbb{C}z$. If $\mathcal{T}$ is a one dimensional tiling, say

$$\mathcal{T} = \cdots X_{i-1} X_i X_{i+1} \cdots ,$$

then we need to change each letter X_i into $X_i' X_i'' X_i'''$ with totally new letters X_i', X_i'', X_i''' satisfying that $X_i', X_i'', X_i'', X_j', X_j'', X_j'''$ are all different if $X_i \neq X_j$, and that $X_i' = X_j', X_i'' = X_j'', X_i''' = X_j'''$ if $X_i = X_j$. By this rule, we obtain a new tiling

$$\tilde{\mathcal{T}} = \cdots X_{i-1}' X_{i-1}'' X_{i-1}''' X_i' X_i'' X_i''' X_{i+1}' X_{i+1}'' X_{i+1}''' \cdots .$$

This means $\Omega(\tilde{\mathcal{T}}) = \{X', X'', X''' \mid X \in \Omega(\mathcal{T})\}$. If we symbolically define $\theta : X \mapsto X' X'' X'''$, then we can write

$$\tilde{\mathcal{T}} = \cdots \theta(X_{i-1}) \theta(X_i) \theta(X_{i+1}) \cdots .$$

Put $S^\theta(\tilde{\mathcal{T}}) = \{\theta(w) \mid w \in S(\mathcal{T})\} \subset S(\tilde{\mathcal{T}})$, where $\theta(w) = \theta(X_1) \cdots \theta(X_r) = X_1' X_1'' X_1''' \cdots X_r' X_r'' X_r''' \in S^\theta(\tilde{\mathcal{T}})$ if $w = X_1 \cdots X_r \in S(\mathcal{T})$.

$$\boxed{\text{1 DIM Tilings}} \to \boxed{\text{Triple Substitution}} \to \begin{cases} \boxed{\mathfrak{sl}_2 \text{ Theory}} \\ \boxed{SL_2 \text{ Theory}} \end{cases}$$

We will see the simplest example, namely our tiling $\mathcal{T}$ is trivial:

$$\mathcal{T} = \cdots AAAAA \cdots .$$

We choose $\theta : A \mapsto BCD$. Then, we obtain

$$\tilde{\mathcal{T}} = \cdots BCDBCDBCDBCDBCD \cdots .$$

Therefore, we have

$$S(\mathcal{T}) = \{A, AA, AAA, \ldots\}$$

and

$$S^\theta(\tilde{\mathcal{T}}) = \{BCD, BCDBCD, BCDBCDBCD, \ldots\}.$$

By the definition of our product, $S(\mathcal{T})$ can not create $\mathfrak{sl}_2$ in general, but $S^\theta(\tilde{\mathcal{T}})$ can do. We may call the above map θ a triple substitution.

For a given one dimensional tiling $\mathcal{T}$, we use $\tilde{\mathcal{T}}$ to construct the associated group $\mathfrak{G} = \mathfrak{G}(\mathcal{T}, \theta)$. Put $\mathfrak{X} = \{(i, a, i+1) \mid a \in S^\theta(\tilde{\mathcal{T}}), 1 \le i < \ell(a)\}$ and $\mathfrak{Y} = \{(i+1, a, i) \mid a \in S^\theta(\tilde{\mathcal{T}}), 1 \le i < \ell(a)\}$. If $\xi = (i, a, j)$, then we put $\hat{\xi} = (j, a, i)$. For $\xi \in \mathfrak{X} \cup \mathfrak{Y}$ and $t \in \mathbb{C}$, we set $x_\xi(t) = 1 + t\xi$. Since $\xi^2 = 0$ and $x_\xi(t)^{-1} = x_\xi(-t)$, we see $x_\xi(t) \in (\mathfrak{A}(\tilde{\mathcal{T}})/\mathbb{C}z)^\times$, where $(\mathfrak{A}(\tilde{\mathcal{T}})/\mathbb{C}z)^\times$ is the multiplicative group of units in $\mathfrak{A}(\tilde{\mathcal{T}})/\mathbb{C}z$. Let $\mathfrak{G} = \mathfrak{G}(\mathcal{T}, \theta) = \langle x_\xi(t) \mid \xi \in \mathfrak{X} \cup \mathfrak{Y}, t \in \mathbb{C}\rangle \subset (\mathfrak{A}(\tilde{\mathcal{T}})/\mathbb{C}z)^\times$. We call $\mathfrak{G}$ the tiling group defined by $(\mathcal{T}, \theta)$. For $\xi \in \mathfrak{X} \cup \mathfrak{Y}$ and $u \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, we define $w_\xi(u) = x_\xi(u) x_{\hat{\xi}}(-u^{-1}) x_\xi(u)$ and $h_\xi(u) = w_\xi(u) w_\xi(-1)$. Let

$$\begin{aligned}
\mathfrak{G}_+ &= \left\langle x_\xi(t) \mid \xi \in \mathfrak{X}, t \in \mathbb{C}\right\rangle,\\
\mathfrak{G}_- &= \left\langle x_\xi(t) \mid \xi \in \mathfrak{Y}, t \in \mathbb{C}\right\rangle,\\
\mathfrak{G}_0 &= \left\langle h_\xi(u) \mid \xi \in \mathfrak{X} \cup \mathfrak{Y}, u \in \mathbb{C}^\times\right\rangle.
\end{aligned}$$

For each $\xi \in \mathfrak{X}$, we define

$$\mathfrak{G}_\xi = \left\langle x_\xi(t), x_{\hat{\xi}}(t) \mid t \in \mathbb{C}\right\rangle \subset \mathfrak{G}.$$

Then we have the following properties.

Theorem 4 (cf. [4, 5]) *Notation is as above.*

(1) $\mathfrak{G}_\xi \simeq SL_2(\mathbb{C})$.
(2) $\mathfrak{G} = \mathfrak{G}_\pm \mathfrak{G}_\mp \mathfrak{G}_0 \mathfrak{G}_\pm$ (Gauss Decomposition).

Using Gauss decompositions, we obtain:

$$\mathfrak{G} = \bigcup_{g \in \mathfrak{G}_\pm} g(\mathfrak{G}_\mp \mathfrak{G}_0 \mathfrak{G}_\pm) g^{-1} = \bigcup_{g \in \mathfrak{G}} g(\mathfrak{G}_\mp \mathfrak{G}_0 \mathfrak{G}_\pm) g^{-1}.$$

Here the universal enveloping algebra of a Lie algebra L is denoted by $U(L)$. We also use $\tilde{\mathcal{T}}$ to construct the Lie algebra associated with a given one dimensional tiling $\mathcal{T}$. Let $\mathfrak{L} = \mathfrak{L}(\mathcal{T}, \theta)$ be the Lie subalgebra of $\mathfrak{A}(\tilde{\mathcal{T}})/\mathbb{C}z$ generated by $\mathfrak{X} \cup \mathfrak{Y}$, where the corresponding Lie bracket is given by $[x, y] = xy - yx$ in $\mathfrak{A}(\tilde{\mathcal{T}})/\mathbb{C}z$. Then, $\mathfrak{L}$ is called the tiling Lie algebra defined by $(\mathcal{T}, \theta)$. We define

$$\begin{aligned}
\mathfrak{L}_+ &= \langle \xi \mid \xi \in \mathfrak{X}\rangle,\\
\mathfrak{L}_- &= \langle \xi \mid \xi \in \mathfrak{Y}\rangle,\\
\mathfrak{L}_0 &= \left\langle [\xi, \eta] \mid \xi \in \mathfrak{X}, \eta \in \mathfrak{Y}\right\rangle.
\end{aligned}$$

For each $\xi \in \mathfrak{X}$, we define

$$\mathfrak{L}_\xi = \langle \xi, \hat{\xi}\rangle = \mathbb{C}\xi \oplus \mathbb{C}[\xi, \hat{\xi}] \oplus \mathbb{C}\hat{\xi} \subset \mathfrak{L}.$$

Then we have the following properties.

Theorem 5 (cf. [3–5]) *Notation is as above.*

(1) $\mathfrak{L}_\xi \simeq \mathfrak{sl}_2(\mathbb{C})$.
(2) $\mathfrak{L} = \mathfrak{L}_- \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_+$ (Triangular Decomposition).
(3) $U(\mathfrak{L}) = U(\mathfrak{L}_\pm)U(\mathfrak{L}_\mp)U(\mathfrak{L}_\pm)$ (Additive Gauss Decomposition).

As above, we have $\mathfrak{G} = \langle \mathfrak{G}_\xi \mid \xi \in \mathfrak{X}\rangle$ and $\mathfrak{L} = \langle \mathfrak{L}_\xi \mid \xi \in \mathfrak{X}\rangle$ with $\mathfrak{G}_\xi \simeq SL_2(\mathbb{C})$ and $\mathfrak{L}_\xi \simeq \mathfrak{sl}_2(\mathbb{C})$. Using Gauss decompositions and additive Gauss decompositions, it is interesting to study $\mathfrak{G}$ and $\mathfrak{L}$ as algebraic invariants of tilings.

We may say that our triple substitution is, philosophically speaking, corresponding to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & 0 & b & 0 & 0 \\ 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & a & 0 & 0 & b \\ c & 0 & 0 & d & 0 & 0 \\ 0 & c & 0 & 0 & d & 0 \\ 0 & 0 & c & 0 & 0 & d \end{pmatrix}.$$

This may not exactly be fit to our original triple substitution, but may be a good explanation philosophically. Honestly saying, our triple substitution has formally imitated the so-called quark model, which says that every baryon (a kind of hadron or a certain particle) consists of three quarks.

We now have three objects as invariants of one dimensional tilings, namely bialgebras, Lia algebras and groups. According to Theorem 2, bialgebras are advantaged here compared with Lie algebras and groups. However, Lie algebras and groups are much more familiar. Hence, it would be better to obtain Theorem 2 type results for Lie algebras and groups. To show such a result or to create the corresponding representation theory, Gauss decompositions and additive Gauss decompositions seem to be useful. In fact, $\mathfrak{sl}_2$-theory and SL_2-theory are very important in Lie theory. Hence, also in our case, we hope that such fundamental sub-objects could work well. Furthermore, to study Lie algebras and groups corresponding to higher dimensional tilings in general, we need to create a new and wide setting including (locally) extended affine Lie algebras probably (cf. [2, 21]). In higher dimensional case, bialgebras might also be advantaged (cf. [20]). But, we hope that Lie algebras and groups would be strong tools to develop further. In fact, even in case of bialgebras, we could NEVER expect the results like Theorems 1 and 2 before. Therefore, we would like to expect such a new development using Lie algebras and groups. We stand and walk on the way toward our invisible goal. Generally speaking, for a tiling $\mathcal{T}$, we can define the associated (bi-)algebra $\mathfrak{A}(\mathcal{T})$, the associated Lie algebra $\mathfrak{L}(\mathcal{T})$ and the associated group $\mathfrak{G}(\mathcal{T})$. Then, we would like to handle $\mathfrak{A}$, $\mathfrak{L}$ and $\mathfrak{G}$ as functors from the category of tilings to the category of algebras $\mathcal{T} \mapsto \mathfrak{A}(\mathcal{T})$, Lie algebras $\mathcal{T} \mapsto \mathfrak{L}(\mathcal{T})$ and groups $\mathcal{T} \mapsto \mathfrak{G}(\mathcal{T})$ respectively. Then, we could show how aperiodic structures can be understood in algebra theory, in Lie algebra theory and in group theory respectively. For that purpose, we need to create the category of tilings at least. Especially we should produce suitable morphisms between tilings. But there seems to be no good reference on this topic yet. It is very important not only to study each tiling itself as a local picture, but also to study the category of tilings as a global picture. Finally we should point out that there might be another choice to define the corresponding algebras, Lie algebras and groups. There must be lots of ways to approach to aperiodic orders from algebraic side.

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Toward Berenstein-Zelevinsky Data in Affine Type A, Part III: Proof of the Connectedness

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Abstract We prove the connectedness of the crystal $(\mathcal{BZ}_{\mathbb{Z}}^{\sigma}; \mathrm{wt}, \widehat{\varepsilon}_p, \widehat{\varphi}_p, \widehat{e}_p, \widehat{f}_p)$, which we introduced in Contemp. Math. 565, 143–184 (2012).

1 Introduction

This paper is a continuation of our previous works ([8] and [9]). In [8], motivated by the works ([5] and [4]) of Kamnitzer on Mirković-Vilonen polytopes in finite types, we introduced an affine analog of Berenstein-Zelevinsky datum (BZ datum for short) in type $A_{l-1}^{(1)}$. Let us recall its construction briefly. For a finite interval I in $\mathbb{Z}$, we denote by $\mathcal{BZ}_I$ the set of those BZ data of type $A_{|I|}$ which satisfy a certain normalization condition, called the w_0-normalization condition in [9]. The family $\{\mathcal{BZ}_I \,|\, I \text{ is a finite interval in } \mathbb{Z}\}$ forms a projective system, and hence the set $\mathcal{BZ}_{\mathbb{Z}}$ of BZ data of type A_∞ is defined to be a kind of projective limit of this projective system. Furthermore, for $l \geq 3$, we define the set $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ of BZ data of type $A_{l-1}^{(1)}$ to be the fixed point subset of $\mathcal{BZ}_{\mathbb{Z}}$ under a natural action of the automorphism $\sigma : \mathbb{Z} \to \mathbb{Z}$ given by $\sigma(j) = j + l$ for $j \in \mathbb{Z}$. Note that a BZ datum of type $A_{l-1}^{(1)}$ is realized as a collection of those integers, indexed by the set of infinite Maya diagrams, which satisfy the "*edge inequalities*", "*tropical Plücker relations*", and some additional

Dedicated to Professor Michio Jimbo on the occasion of his sixtieth birthday.

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conditions (see Definitions 4 and 5 for details). The set $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ has a $U_q(\widehat{\mathfrak{sl}}_l)$-crystal structure, which is naturally induced by that on $\mathcal{BZ}_I$. In [8], we proved that there exists a distinguished connected component $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O})$ of $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$, which is isomorphic as a crystal to the crystal basis $B(\infty)$ of the negative part $U_q^-(\widehat{\mathfrak{sl}}_l)$ of $U_q(\widehat{\mathfrak{sl}}_l)$. We anticipated that the connected component $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O})$ is identical to the whole of $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$, but we could not prove it in [8]. The purpose of this paper is prove the anticipated identity, that is, to prove the connectedness of the crystal graph of $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$.

In [9], we introduced the notion of e-BZ data of type $A_{l-1}^{(1)}$, which are defined in the same way as BZ data with another normalization condition, called the e-normalization condition in [9]. In this paper, we mainly treat e-BZ data instead of BZ data for the following reasons. First, it is known that the set $(\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}$ of e-BZ data of type $A_{l-1}^{(1)}$ is isomorphic as a crystal to $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$, and hence the connectedness of the crystal graph of $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ is equivalent to that of $(\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}$. Second, in [9], we showed that there is a natural correspondence between e-BZ data and (certain) limits of irreducible Lagrangians of the varieties associated to quivers of finite type A. Thus, we can use geometrical (or quiver-theoretical) methods for the study of e-BZ data. This is an advantage of e-BZ data.

Our main result (Theorem 5) states that the crystal $(\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}$ is isomorphic to $B(\infty)$. Because we already know that a distinguished connected component of $(\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}$ is isomorphic to $B(\infty)$, Theorem 5 tells us that this connected component is identical to the whole of $(\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}$. In other words, we obtain a new explicit realization of $B(\infty)$ in terms of an affine analog of a BZ datum. Our strategy for proving Theorem 5 is as follows. In [6], Kashiwara and the third author gave those conditions which characterize $B(\infty)$ uniquely (see Theorem 6 for details). We will establish Theorem 5 by verifying that the $(\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}$ indeed satisfies these conditions. In particular, we will construct a strict embedding $\Psi_p^* : (\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma} \to (\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma} \otimes B_p^*$, called the *Kashiwara embedding* (see condition (5) of Theorem 6). In order to construct such an embedding, we define another crystal structure, which we call the *ordinary crystal structure*, on $(\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}$ via a certain involution (denoted by $\sharp$) on the crystal $(\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}$. For this purpose, we take advantage of e-BZ data mentioned above. First, we consider a given e-BZ datum in $(\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}$ as a (certain) limit of irreducible Lagrangians of the varieties associated to quivers of finite type A. Second, we take the images of these irreducible Lagrangians under the (so-called) $*$-operation. Here we note that for an irreducible Lagrangian, the $*$-operation is described explicitly in terms of transposition of matrices (see Sect. 2.5). Finally, by taking a limit of these images under the $*$-operation, we obtain the involution $\sharp$ on $(\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}$. This construction plays a crucial role in our proof.

This paper is organized as follows. In Sect. 2, we give a quick review of results in our previous works. In Sect. 3, we introduce a new crystal structure, called the *ordinary crystal structure*, on $\mathcal{BZ}_I^e$. Here, $\mathcal{BZ}_I^e$ is the set of e-BZ data associated to a finite interval I. Since $(\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}$ is the set of σ-fixed points of a kind of projective limit of $\mathcal{BZ}_I^e$'s, we can define the ordinary crystal structure on $(\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}$ induced naturally by that of $\mathcal{BZ}_I^e$'s. However, in order to overcome some technical difficulties in following this procedure, we need a quiver-theoretical interpretation of $\mathcal{BZ}_I^e$. We treat these technicalities in Sect. 4. In Sect. 5, we prove our main result (Theorem 5) by checking the conditions in Theorem 6 for the $(\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}$.

Finally, let us mention some related works, which appeared recently. The first one is by Muthiah [7]. In [3], Braverman, Finkelberg, and Gaitsgory introduced analogs of Mirković-Vilonen cycles in the case of an affine Kac-Moody group, and defined a crystal structure on the set of those cycles. After that, Muthiah studied the crystal structure of those cycles in an explicit way, and proved that it is isomorphic to the crystal $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ in affine type A. The second one is by Baumann, Kamnitzer, and Tingley [1]. Let $\mathfrak{g}$ be a symmetric affine Kac-Moody Lie algebra. In [1], they introduced the notion of affine Mirković-Vilonen polytopes by using the theory of preprojective algebras of the same type as $\mathfrak{g}$, and showed that there exists a bijection between the set of affine Mirković-Vilonen polytopes and the crystal basis $B(-\infty)$ of the positive part $U_q^+(\mathfrak{g})$ of $U_q(\mathfrak{g})$. It seems to us that these works are closely related to results in this paper. However, an explicit relationship between them is still unclear; this is our future problem.

2 Review of Known Results

2.1 Preliminaries on Root Data

Let $\mathfrak{t}$ be a vector space over $\mathbb{C}$ with basis $\{\epsilon_i\}_{i\in\mathbb{Z}}$; we set $h_i := \epsilon_i - \epsilon_{i+1}$, $i \in \mathbb{Z}$. We define $\Lambda_i, \Lambda_i^c \in \mathfrak{t}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$, $i \in \mathbb{Z}$ by

$$\langle \epsilon_j, \Lambda_i \rangle_{\mathbb{Z}} := \begin{cases} 1 & \text{if } j \le i, \\ 0 & \text{if } j > i, \end{cases} \qquad \langle \epsilon_j, \Lambda_i^c \rangle_{\mathbb{Z}} = \begin{cases} 0 & \text{if } j \le i, \\ 1 & \text{if } j > i, \end{cases}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{Z}} : \mathfrak{t} \times \mathfrak{t}^* \to \mathbb{C}$ is the canonical pairing, and set $\alpha_i := -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}$, $i \in \mathbb{Z}$. Let $W_{\mathbb{Z}} := \langle \sigma_i | i \in I \rangle (\subset GL(\mathfrak{t}))$ be the Weyl group of type A_∞, where σ_i is the automorphism of $\mathfrak{t}$ defined by $\sigma_i(t) = t - \langle t, \alpha_i \rangle_{\mathbb{Z}} h_i$, $t \in \mathfrak{t}$; the group $W_{\mathbb{Z}}$ also acts on $\mathfrak{t}^*$ by $\sigma_i(\lambda) = \lambda - \langle h_i, \lambda \rangle_{\mathbb{Z}} \alpha_i$, $\lambda \in \mathfrak{t}^*$.

Let $I = [n+1, n+m]$ be a finite interval in $\mathbb{Z}$ whose cardinality is equal to m, and consider a finite-dimensional subspace $\mathfrak{h}_I := \bigoplus_{i\in I} \mathbb{C} h_i$ of $\mathfrak{t}^*$. For each $i \in I$, set $\alpha_i^I := \pi_I(\alpha_i)$ and $\varpi_i^I := \pi_I(\Lambda_i)$, where $\pi_I : \mathfrak{t}^* \to \mathfrak{h}_I^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_I, \mathbb{C})$ is the natural projection; we denote by $\langle \cdot, \cdot \rangle_I$ the canonical pairing between $\mathfrak{h}_I$ and $\mathfrak{h}_I^*$. Then we can regard $(\{\alpha_i^I\}_{i\in I}, \{h_i\}_{i\in I}, \mathfrak{h}_I^*, \mathfrak{h}_I)$ as the root datum of type A_m. Also, the set $\{\varpi_i^I\}_{i\in I}$ can be regarded as the set of fundamental weights. Let W_I be the subgroup of $W_{\mathbb{Z}}$ generated by σ_i, $i \in I$. Since σ_i stabilizes the subspace $\mathfrak{h}_I$ of $\mathfrak{t}$ for all $i \in I$, we can regard W_I as a subgroup of $GL(\mathfrak{h}_I)$; the group W_I acts on $\mathfrak{h}_I^*$ in a usual way.

2.2 BZ Data Associated to a Finite Interval

Set $\widetilde{I} := I \cup \{n+m+1\}$. A subset $\mathbf{k} \subset \widetilde{I}$ is called a Maya diagram associated to I; we denote by $\mathcal{M}_I$ the set of all Maya diagrams associated to I, and set $\mathcal{M}_I^\times := \mathcal{M}_I \setminus \{\phi, \widetilde{I}\}$. We identify $\mathcal{M}_I^\times$ with $\Gamma_I := \bigsqcup_{i \in I} W_I \varpi_i^I$ via the bijection $[n+1, i] \leftrightarrow \varpi_i^I$. Under this identification, $\langle \cdot, \cdot \rangle_I$ induces a pairing between $\mathfrak{h}_I$ and $\mathcal{M}_I^\times$, which is given explicitly as follows:

$$\langle h_i, \mathbf{k} \rangle_I = \begin{cases} 1 & \text{if } i \in \mathbf{k} \text{ and } i+1 \notin \mathbf{k}, \\ -1 & \text{if } i \notin \mathbf{k} \text{ and } i+1 \in \mathbf{k}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Let $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_I^\times}$ be a collection of integers indexed by $\mathcal{M}_I^\times$. For each $\mathbf{k} \in \mathcal{M}_I^\times$, we call $M_{\mathbf{k}}$ the $\mathbf{k}$-component of $\mathbf{M}$, and denote it by $(\mathbf{M})_{\mathbf{k}}$.

Definition 1 (1) A collection $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_I^\times}$ of integers indexed by $\mathcal{M}_I^\times$ is called a Berenstein-Zelevinsky datum (BZ datum for short) associated to I if it satisfies the following conditions:

(BZ-1) for all indices $i \neq j$ in $\widetilde{I}$ and all $\mathbf{k} \in \mathcal{M}_I$ such that $\mathbf{k} \cap \{i, j\} = \phi$,

$$M_{\mathbf{k} \cup \{i\}} + M_{\mathbf{k} \cup \{j\}} \leq M_{\mathbf{k} \cup \{i,j\}} + M_{\mathbf{k}};$$

(BZ-2) for all indices $i < j < k$ in $\widetilde{I}$ and all $\mathbf{k} \in \mathcal{M}_I$ such that $\mathbf{k} \cap \{i, j, k\} = \phi$,

$$M_{\mathbf{k} \cup \{i,k\}} + M_{\mathbf{k} \cup \{j\}} = \min\{M_{\mathbf{k} \cup \{i,j\}} + M_{\mathbf{k} \cup \{k\}}, M_{\mathbf{k} \cup \{j,k\}} + M_{\mathbf{k} \cup \{i\}}\}.$$

Here, $M_\phi = M_{\widetilde{I}} = 0$ by convention.

(2) A BZ datum $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_I^\times}$ is called a w_0-BZ (*resp.*, e-BZ) datum if it satisfies the following normalization condition:

(BZ-0) for every $i \in I$, $M_{[i+1, n+m+1]} = 0$ (*resp.*, $M_{[n+1, i]} = 0$).

We denote by $\mathcal{BZ}_I$ (*resp.*, $\mathcal{BZ}_I^e$) the set of all w_0-BZ (*resp.*, e-BZ) data.

For $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_I^\times} \in \mathcal{BZ}_I$, define a new collection $\mathbf{M}^* = (M_{\mathbf{k}}^*)_{\mathbf{k} \in \mathcal{M}_I^\times}$ of integers by

$$M_{\mathbf{k}}^* := M_{\mathbf{k}^c},$$

where $\mathbf{k}^c := \widetilde{I} \setminus \mathbf{k}$ is the complement of $\mathbf{k}$ in $\widetilde{I}$. Then, $\mathbf{M}^*$ is an element of $\mathcal{BZ}_I^e$, and the map $* : \mathbf{M} \mapsto \mathbf{M}^*$ gives a bijection from $\mathcal{BZ}_I$ to $\mathcal{BZ}_I^e$. We also denote its inverse by $*$.

Let $K = [n'+1, n'+m']$ be a subinterval of I, and define

$$\mathcal{M}_I^\times(K) := \left\{\mathbf{k} \in \mathcal{M}_I^\times \,|\, \mathbf{k} = \left[n+1, n'\right] \cup \mathbf{k}' \text{ for some } \mathbf{k}' \in \mathcal{M}_K^\times\right\}.$$

Then, $\mathcal{M}_K^\times$ is naturally identified with $\mathcal{M}_I^\times(K)$ via the bijection $\mathbf{k}' \mapsto [n+1, n'] \cup \mathbf{k}'$. We denote its inverse by $\mathrm{res}_K^I : \mathcal{M}_I^\times(K) \xrightarrow{\sim} \mathcal{M}_K^\times$.

For $\mathbf{M} = (M_\mathbf{k})_{\mathbf{k} \in \mathcal{M}_I^\times} \in \mathcal{BZ}_I^e$, we define a new collection $\mathbf{M}_K = (M'_\mathbf{m})_{\mathbf{m} \in \mathcal{M}_K^\times}$ of integers indexed by $\mathcal{M}_K^\times$ by

$$M'_\mathbf{m} := M_{(\mathrm{res}_K^I)^{-1}(\mathbf{m})}.$$

Then, $\mathbf{M}_K$ is an e-BZ datum associated to K.

2.3 Crystal Structure on BZ Data Associated to a Finite Interval

First, we recall the crystal structure on $\mathcal{BZ}_I$. For $\mathbf{M} = (M_\mathbf{k})_{\mathbf{k} \in \mathcal{M}_I^\times} \in \mathcal{BZ}_I$ and $i \in I$, define

$$\mathrm{wt}(\mathbf{M}) := \sum_{i \in I} M_{[n+1,i]} \alpha_i^I,$$

$$\varepsilon_i(\mathbf{M}) := -(M_{[n+1,i]} + M_{[n+1,i-1] \cup \{i+1\}} - M_{[n+1,i-1]} - M_{[n+1,i+1]}),$$

$$\varphi_i(\mathbf{M}) := \varepsilon_i(\mathbf{M}) + \langle h_i, \mathrm{wt}(\mathbf{M}) \rangle_I.$$

Proposition 1 (1) *Let* $\mathbf{M} = (M_\mathbf{k})_{\mathbf{k} \in \mathcal{M}_I^\times} \in \mathcal{BZ}_I$. *If* $\varepsilon_i(\mathbf{M}) > 0$, *then there exits a unique* w_0*-BZ datum* $\mathbf{M}' = (M'_\mathbf{k})_{\mathbf{k} \in \mathcal{M}_I^\times}$ *such that*

(i) $M'_{[n+1,i]} = M_{[n+1,i]} + 1$,
(ii) $M'_\mathbf{k} = M_\mathbf{k}$ *for all* $\mathbf{k} \in \mathcal{M}_I^\times \setminus \mathcal{M}_I^\times(i)$, *where* $\mathcal{M}_I^\times(i) := \{\mathbf{k} \in \mathcal{M}_I^\times | i \in \mathbf{k}$ *and* $i+1 \notin \mathbf{k}\}$.

(2) *There exits a unique* w_0*-BZ datum* $\mathbf{M}'' = (M''_\mathbf{k})_{\mathbf{k} \in \mathcal{M}_I^\times}$ *such that*

(iii) $M''_{[n+1,i]} = M_{[n+1,i]} - 1$,
(iv) $M''_\mathbf{k} = M_\mathbf{k}$ *for all* $\mathbf{k} \in \mathcal{M}_I^\times \setminus \mathcal{M}_I^\times(i)$.

We set

$$\widetilde{e}_i\mathbf{M} := \begin{cases} \mathbf{M}' & \text{if } \varepsilon_i(\mathbf{M}) > 0, \\ 0 & \text{if } \varepsilon_i(\mathbf{M}) = 0, \end{cases} \quad \text{and} \quad \widetilde{f}_i\mathbf{M} := \mathbf{M}''.$$

Proposition 2 *The set* $\mathcal{BZ}_I$, *equipped with the maps* wt, $\varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i$, *is a crystal, which is isomorphic to* $(B(\infty); \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i)$.

The explicit form of the action of the lowering Kashiwara operator $\widetilde{f}_i$ on $\mathcal{BZ}_I$ is given by the following:

Proposition 3 *For* $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_I^\times} \in \mathcal{BZ}_I$, *we have*

$$(\widetilde{f}_i \mathbf{M})_{\mathbf{k}} = \begin{cases} \min\{M_{\mathbf{k}}, M_{s_i \mathbf{k}} + c_i(\mathbf{M})\} & \textit{if } \mathbf{k} \in \mathcal{M}_I^\times(i), \\ M_{\mathbf{k}} & \textit{otherwise}, \end{cases} \tag{2.2}$$

where $c_i(\mathbf{M}) = \langle h_i, \mathrm{wt}(\mathbf{M}) \rangle_I + \varepsilon_i(\mathbf{M}) - 1$.

Through the bijection $* : \mathcal{BZ}_I \xrightarrow{\sim} \mathcal{BZ}_I^e$, we can define the $*$-crystal structure on $\mathcal{BZ}_I^e$. Namely, for $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_I^\times} \in \mathcal{BZ}_I^e$, we set

$$\mathrm{wt}(\mathbf{M}) := \mathrm{wt}\big(\mathbf{M}^*\big), \qquad \varepsilon_i^*(\mathbf{M}) := \varepsilon_i\big(\mathbf{M}^*\big), \qquad \varphi_i^*(\mathbf{M}) := \varphi_i\big(\mathbf{M}^*\big),$$

$$\widetilde{e}_i^* := * \circ \widetilde{e}_i \circ *, \quad \text{and} \quad \widetilde{f}_i^* := * \circ \widetilde{f}_i \circ *.$$

It is easy to obtain the following corollaries.

Corollary 1 *Let* $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_I^\times} \in \mathcal{BZ}_I^e$.

(1) *If* $\varepsilon_i^*(\mathbf{M}) > 0$, *then* $\widetilde{e}_i^* \mathbf{M}$ *is a unique e-BZ datum such that*
 (i) $(\widetilde{e}_i^* \mathbf{M})_{[i+1, n+m+1]} = M_{[i+1, n+m+1]} + 1$,
 (ii) $(\widetilde{e}_i^* \mathbf{M})_{\mathbf{k}} = M_{\mathbf{k}}$ *for all* $\mathbf{k} \in \mathcal{M}_I^\times \setminus \mathcal{M}_I^\times(i)^*$, *where* $\mathcal{M}_I^\times(i)^* := \{\mathbf{k} \in \mathcal{M}_I^\times \,|\, i \notin \mathbf{k}$ *and* $i + 1 \in \mathbf{k}\}$.

(2) $\widetilde{f}_i^* \mathbf{M}$ *is a unique e-BZ datum such that*
 (iii) $(\widetilde{f}_i^* \mathbf{M})_{[i+1, n+m+1]} = M_{[i+1, n+m+1]} - 1$,
 (iv) $(\widetilde{f}_i^* \mathbf{M})_{\mathbf{k}} = M_{\mathbf{k}}$ *for all* $\mathbf{k} \in \mathcal{M}_I^\times \setminus \mathcal{M}_I^\times(i)^*$.

(3) *For* $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_I^\times} \in \mathcal{BZ}_I^e$, *we have*

$$\big(\widetilde{f}_i^* \mathbf{M}\big)_{\mathbf{k}} = \begin{cases} \min\{M_{\mathbf{k}}, M_{s_i \mathbf{k}} + c_i^*(\mathbf{M})\} & \textit{if } \mathbf{k} \in \mathcal{M}_n^\times(i)^*, \\ M_{\mathbf{k}} & \textit{otherwise}, \end{cases} \tag{2.3}$$

where $c_i^*(\mathbf{M}) := \langle h_i, \mathrm{wt}(\mathbf{M}) \rangle_I + \varepsilon_i^*(\mathbf{M}) - 1$.

Corollary 2 *The set* $\mathcal{BZ}_I^e$, *equipped with the maps* wt, $\varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*$, *is a crystal, which is isomorphic to* $(B(\infty); \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*)$.

2.4 Lusztig Data vs. BZ Data

Let $\Delta_I^+ = \{(i, j) | i, j \in \widetilde{I} \text{ with } i < j\}$, and set

$$\mathcal{B}_I := \big\{\mathbf{a} = (a_{i,j})_{(i,j) \in \Delta_I^+} | a_{i,j} \in \mathbb{Z}_{\geq 0} \text{ for any } (i, j) \in \Delta_I^+\big\},$$

which is just the set of all $m(m+1)/2$-tuples of nonnegative integers indexed by Δ_I^+. Here, m is the cardinality of the interval I. An element of $\mathcal{B}_I$ is called a Lusztig datum associated to I.

We define two crystal structures on $\mathcal{B}_I$ (see [10] and [9] for details). For $\mathbf{a} \in \mathcal{B}_I$, set

$$\mathrm{wt}(\mathbf{a}) := -\sum_{i\in I} r_i \alpha_i^I, \quad \text{where } r_i := \sum_{k=n+1}^{i} \sum_{l=i+1}^{n+m+1} a_{k,l},\ i \in I.$$

For $i \in I$, we set

$$A_k^{(i)}(\mathbf{a}) := \sum_{s=n+1}^{k} (a_{s,i+1} - a_{s-1,i}), \quad n+1 \le k \le i,$$

$$A_l^{*(i)}(\mathbf{a}) := \sum_{t=l+1}^{n+m+1} (a_{i,t} - a_{i+1,t+1}), \quad i \le l \le n+m+1,$$

where $a_{n,i} = 0$ and $a_{i+1,n+m+2} = 0$ by convention, and define

$$\varepsilon_i(\mathbf{a}) := \max\{A_{n+1}^{(i)}(\mathbf{a}), \ldots, A_i^{(i)}(\mathbf{a})\}, \qquad \varphi_i(\mathbf{a}) := \varepsilon_i(\mathbf{a}) + \langle h_i, \mathrm{wt}(\mathbf{a})\rangle,$$

$$\varepsilon_i^*(\mathbf{a}) := \max\{A_i^{*(i)}(\mathbf{a}), \ldots, A_{n+m}^{*(i)}(\mathbf{a})\}, \qquad \varphi_i^*(\mathbf{a}) := \varepsilon_i^*(\mathbf{a}) + \langle h_i, \mathrm{wt}(\mathbf{a})\rangle.$$

Also, set

$$k_e := \min\{n+1 \le k \le i \,|\, \varepsilon_i(\mathbf{a}) = A_k^{(i)}(\mathbf{a})\},$$

$$k_f := \max\{n+1 \le k \le i \,|\, \varepsilon_i(\mathbf{a}) = A_k^{(i)}(\mathbf{a})\},$$

$$l_e := \max\{i \le l \le n+m \,|\, \varepsilon_i^*(\mathbf{a}) = A_l^{*(i)}(\mathbf{a})\},$$

$$l_f := \min\{i \le l \le n+m \,|\, \varepsilon_i^*(\mathbf{a}) = A_l^{*(i)}(\mathbf{a})\}.$$

For a given $\mathbf{a} \in \mathcal{B}_I$, we define four $m(m+1)/2$-tuples of integers $\mathbf{a}^{(p)} = (a_{k,l}^{(p)})$, $p = 1, 2, 3, 4$, by

$$a_{k,l}^{(1)} := \begin{cases} a_{k_e,i} + 1 & \text{if } k = k_e, l = i, \\ a_{k_e,i+1} - 1 & \text{if } k = k_e, l = i+1, \\ a_{k,l} & \text{otherwise}, \end{cases}$$

$$a_{k,l}^{(2)} := \begin{cases} a_{k_f,i} - 1 & \text{if } k = k_f, l = i, \\ a_{k_f,i+1} + 1 & \text{if } k = k_f, l = i+1, \\ a_{k,l} & \text{otherwise}, \end{cases}$$

$$a_{k,l}^{(3)} := \begin{cases} a_{i,l_e+1} - 1 & \text{if } k = i, l = l_e+1, \\ a_{i+1,l_e+1} + 1 & \text{if } k = i+1, l = l_e+1, \\ a_{k,l} & \text{otherwise}, \end{cases}$$

$$a^{(4)}_{k,l} := \begin{cases} a_{i,l_f+1}+1 & \text{if } k=i, l=l_f+1, \\ a_{i+1,l_f+1}-1 & \text{if } k=i+1, l=l_f+1, \\ a_{k,l} & \text{otherwise.} \end{cases}$$

Now, we define Kashiwara operators on $\mathcal{B}_I$ as follows:

$$\widetilde{e}_i\mathbf{a} := \begin{cases} \mathbf{0} & \text{if } \varepsilon_i(\mathbf{a})=0, \\ \mathbf{a}^{(1)} & \text{if } \varepsilon_i(\mathbf{a})>0, \end{cases} \quad \text{and} \quad \widetilde{f}_i\mathbf{a} := \mathbf{a}^{(2)},$$

$$\widetilde{e}^*_i\mathbf{a} := \begin{cases} \mathbf{0} & \text{if } \varepsilon^*_i(\mathbf{a})=0, \\ \mathbf{a}^{(3)} & \text{if } \varepsilon^*_i(\mathbf{a})>0, \end{cases} \quad \text{and} \quad \widetilde{f}^*_i\mathbf{a} := \mathbf{a}^{(4)}.$$

Proposition 4 ([10]) *Each of* $(\mathcal{B}_I, \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i)$ *and* $(\mathcal{B}_I, \mathrm{wt}, \varepsilon^*_i, \varphi^*_i, \widetilde{e}^*_i, \widetilde{f}^*_i)$ *is a crystal, which is isomorphic to* $B(\infty)$.

Following [9], we call the first one the ordinary crystal structure on $\mathcal{B}_I$; the second one is called the $*$-crystal structure on $\mathcal{B}_I$.

Definition 2 ([2]) Let $\mathbf{k} = \{k_{n+1} < k_{n+2} < \cdots < k_{n+u}\} \in \mathcal{M}^\times_I$. For such a $\mathbf{k}$, a $\mathbf{k}$-tableau is an upper-triangular matrix $C = (c_{p,q})_{n+1\le p\le q\le n+u}$, with integer entries, satisfying the condition

$$c_{p,p} = k_p, \quad n+1 \le p \le n+u,$$

and the usual monotonicity condition for semi-standard tableaux:

$$c_{p,q} \le c_{p,q+1}, \qquad c_{p,q} < c_{p+1,q}.$$

For $\mathbf{a} = (a_{i,j}) \in \mathcal{B}_I$, define a collection $\mathbf{M}(\mathbf{a}) = (M_{\mathbf{k}}(\mathbf{a}))_{\mathbf{k}\in\mathcal{M}^\times_I}$ of integers by

$$M_{\mathbf{k}}(\mathbf{a}) := -\sum_{j=n+1}^{n+u}\sum_{i=n+1}^{k_j-1} a_{i,k_j} + \min\left\{ \sum_{n+1\le p<q\le n+u} a_{c_{p,q},c_{p,q}+(q-p)} \,\middle|\, \begin{array}{l} C=(c_{p,q}) \text{ is} \\ \text{a } \mathbf{k}\text{-tableau} \end{array} \right\}.$$

The following lemma is verified easily by direct calculation.

Lemma 1 *Let* $\mathbf{k} = \{k_{n+1} < k_{n+2} < \cdots < k_{n+u}\}$ *be a Maya diagram associated to* I.

(1) *If there exists* s *such that* $k_l = l$ *for all* $n+1 \le l \le s$, *then we have*

$$M_{\mathbf{k}}(\mathbf{a}) = -\sum_{j=s+1}^{n+u}\sum_{i=n+1}^{k_j-1} a_{i,k_j} + \min\left\{ \sum_{q=s+1}^{n+u}\sum_{p=n+1}^{q-1} a_{c_{p,q},c_{p,q}+(q-p)} \,\middle|\, \begin{array}{c} C=(c_{p,q}) \textit{ is} \\ \textit{a } \mathbf{k}\textit{-tableau} \end{array} \right\}.$$

In particular, $M_{\mathbf{k}}(\mathbf{a})$ *depends only on* $a_{i,j}$ *with* $j \geq s+1$.

(2) *If there exists* t *such that* $k_{l-m+u-1} = l$ *for all* $t+1 \leq l \leq n+m+1$, *then we have*

$$M_{\mathbf{k}}(\mathbf{a}) = -\sum_{j=n+1}^{t-m+u-1} \sum_{i=n+1}^{k_j-1} a_{i,k_j} - \sum_{j=t-m+u}^{n+u} \sum_{i=n+1}^{t} a_{i,j+m-u+1}$$
$$+ \min\left\{ \sum_{p=n+1}^{t-m+u-1} \sum_{q=p+1}^{n+u} a_{c_{p,q},c_{p,q}+(q-p)} \,\middle|\, \begin{array}{c} C = (c_{p,q}) \text{ is} \\ \text{a } \mathbf{k}\text{-tableau} \end{array} \right\}.$$

In particular, $M_{\mathbf{k}}(\mathbf{a})$ *depends only on* $a_{i,j}$ *with* $i \leq t$.

Theorem 1 ([2, 10]) *Let* Ψ_I *denote the map* $\mathbf{a} \mapsto \mathbf{M}(\mathbf{a})$. *For every* $\mathbf{a} \in \mathcal{B}_I$, $\Psi_I(\mathbf{a}) = \mathbf{M}(\mathbf{a})$ *is an e-BZ datum. Moreover,* $\Psi_I : \mathcal{B}_I \to \mathcal{BZ}_I^e$ *is an isomorphism of crystals with respect to the* $*$*-crystal structures on* $\mathcal{B}_I$ *and* $\mathcal{BZ}_I^e$.

2.5 BZ Data Arising from the Lagrangian Construction of $B(\infty)$

Let (I, H) be the double quiver of type A_m. Here, the finite interval $I = [n+1, n+m]$ in $\mathbb{Z}$ is considered as the set of vertices, and H as the set of arrows. Let $\mathrm{out}(\tau)$ (*resp.*, $\mathrm{in}(\tau)$) denote the outgoing (*resp.*, incoming) vertex of $\tau \in H$. For a given $\tau \in H$, we denote by $\overline{\tau}$ the same edge as τ with the reverse orientation. Then, the map $\tau \mapsto \overline{\tau}$ defines an involution of H. An orientation Ω is a subset of H such that $\Omega \cap \overline{\Omega} = \phi$ and $\Omega \cup \overline{\Omega} = H$. Note that (I, Ω) is a Dynkin quiver of type A_m.

Let $\nu = (\nu_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I$. In the following, we regard ν as an element of $Q_+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i^I$ via the map $\nu \mapsto \sum_{i \in I} \nu_i \alpha_i^I$. Let $V(\nu) = \bigoplus_{i \in I} V(\nu)_i$ be an I-graded complex vector space with dimension vector $\underline{\dim}\, V(\nu) = \nu$. Set

$$E_{V(\nu),\Omega} := \bigoplus_{\tau \in \Omega} \mathrm{Hom}_{\mathbb{C}}\big(V(\nu)_{\mathrm{out}(\tau)}, V(\nu)_{\mathrm{in}(\tau)}\big),$$
$$X(\nu) := \bigoplus_{\tau \in H} \mathrm{Hom}_{\mathbb{C}}\big(V(\nu)_{\mathrm{out}(\tau)}, V(\nu)_{\mathrm{in}(\tau)}\big).$$

We will write an element of $E_{V(\nu),\Omega}$ or $X(\nu)$ as $B = (B_\tau)$, where B_τ is an element of $\mathrm{Hom}_{\mathbb{C}}(V(\nu)_{\mathrm{out}(\tau)}, V(\nu)_{\mathrm{in}(\tau)})$. Define a symplectic form ω on $X(\nu)$ by

$$\omega\big(B, B'\big) := \sum_{\tau \in H} \varepsilon(\tau) \mathrm{tr}\big(B_{\overline{\tau}} B'_\tau\big),$$

where $\varepsilon(\tau) = 1$ for $\tau \in \Omega$ and $\varepsilon(\tau) = -1$ for $\tau \in \overline{\Omega}$, and regard $X(\nu)$ as the cotangent bundle $T^* E_{V(\nu),\Omega}$ of $E_{V(\nu),\Omega}$ via the symplectic form ω.

Also, the reductive group $G(\nu) := \prod_{i\in I} GL(V(\nu)_i)$ acts on $E_{V(\nu),\Omega}$ and $X(\nu)$ by: $(B_\tau) \mapsto (g_{\mathrm{in}(\tau)} B_\tau g_{\mathrm{out}(\tau)}^{-1})$ for $g = (g_i) \in G(\nu)$. Since the action of $G(\nu)$ on $X(\nu)$ preserves the symplectic form ω, we can consider the corresponding moment map $\mu : X(\nu) \to (\mathfrak{g}(\nu))^* \cong \mathfrak{g}(\nu)$. Here $\mathfrak{g}(\nu) = \mathrm{Lie}G(\nu)$, and we identify $\mathfrak{g}(\nu)$ with its dual via the Killing form. We set

$$\Lambda(\nu) := \mu^{-1}(0).$$

Then, $\Lambda(\nu)$ is a $G(\nu)$-invariant closed Lagrangian subvariety of $X(\nu)$. We denote by $\mathrm{Irr}\,\Lambda(\nu)$ the set of all irreducible components of $\Lambda(\nu)$.

Let $\nu, \nu', \overline{\nu} \in Q_+$, with $\nu = \nu' + \overline{\nu}$. Consider the diagram

$$\Lambda(\nu') \times \Lambda(\overline{\nu}) \xleftarrow{q_1} \Lambda(\nu', \overline{\nu}) \xrightarrow{q_2} \Lambda(\nu). \tag{2.4}$$

Here, $\Lambda(\nu', \overline{\nu})$ denotes the variety of those $(B, \phi', \overline{\phi})$ for which $B \in \Lambda(\nu)$, and $\phi' = (\phi'_i), \overline{\phi} = (\overline{\phi}_i)$ give an exact sequence of I-graded complex vector spaces

$$0 \longrightarrow V(\nu')_i \xrightarrow{\phi'_i} V(\nu) \xrightarrow{\overline{\phi}_i} V(\overline{\nu}) \longrightarrow 0$$

such that $\mathrm{Im}\,\phi'$ is stable by B; note that B induces $B' : V(\nu') \to V(\nu')$ and $\overline{B} : V(\overline{\nu}) \to V(\overline{\nu})$. The maps q_1 and q_2 are defined by $q_1(B, \phi', \overline{\phi}) := (B', \overline{B})$ and $q_2(B, \phi', \overline{\phi}) := B$, respectively. For $i \in I$ and $\Lambda \in \mathrm{Irr}\,\Lambda(\nu)$, we set

$$\varepsilon_i(\Lambda) := \varepsilon_i(B) \quad \text{and} \quad \varepsilon_i^*(\Lambda) := \varepsilon_i^*(B),$$

where B is a general point of Λ, and

$$\varepsilon_i(B) := \dim_{\mathbb{C}} \mathrm{Coker}\Big(\bigoplus_{\tau;\mathrm{in}(\tau)=i} V(\nu)_{\mathrm{out}(\tau)} \xrightarrow{\oplus B_\tau} V(\nu)_i \Big),$$

$$\varepsilon_i^*(B) := \dim_{\mathbb{C}} \mathrm{Ker}\Big(V(\nu)_i \xrightarrow{\oplus B_\tau} \bigoplus_{\tau;\mathrm{out}(\tau)=i} V(\nu)_{\mathrm{in}(\tau)} \Big);$$

also for $k, l \in \mathbb{Z}_{\geq 0}$, we set

$$\big(\mathrm{Irr}\,\Lambda(\nu)\big)_{i,k} := \big\{\Lambda \in \mathrm{Irr}\,\Lambda(\nu) | \varepsilon_i(\Lambda) = k\big\} \quad \text{and}$$

$$\big(\mathrm{Irr}\,\Lambda(\nu)\big)_i^l := \big\{\Lambda \in \mathrm{Irr}\,\Lambda(\nu) | \varepsilon_i^*(\Lambda) = l\big\}.$$

Suppose now that $\overline{\nu} = c\alpha_i$ (*resp.*, $\nu' = c\alpha_i$) for $c \in \mathbb{Z}_{\geq 0}$. Since $\Lambda(c\alpha_i) = \{0\}$, we have the following diagrams as special cases of (2.4):

$$\Lambda(\nu') \cong \Lambda(\nu') \times \Lambda(c\alpha_i) \xleftarrow{q_1} \Lambda(\nu', c\alpha_i) \xrightarrow{q_2} \Lambda(\nu), \tag{2.5}$$

$$\Lambda(\overline{\nu}) \cong \Lambda(c\alpha_i) \times \Lambda(\overline{\nu}) \xleftarrow{q_1} \Lambda(c\alpha_i, \overline{\nu}) \xrightarrow{q_2} \Lambda(\nu). \tag{2.6}$$

Diagrams (2.5) and (2.6) induce bijections

$$\widetilde{e}_i^{\max} : \left(\operatorname{Irr} \Lambda(\nu)\right)_{i,c} \xrightarrow{\sim} \left(\operatorname{Irr} \Lambda\left(\nu'\right)\right)_{i,0} \quad \text{and} \quad \widetilde{e}_i^{*\max} : \left(\operatorname{Irr} \Lambda(\nu)\right)_i^c \xrightarrow{\sim} \left(\operatorname{Irr} \Lambda(\overline{\nu})\right)_i^0,$$

respectively. Then, we define maps

$$\widetilde{e}_i, \widetilde{e}_i^* : \bigsqcup_{\nu \in Q_+} \operatorname{Irr} \Lambda(\nu) \to \bigsqcup_{\nu \in Q_+} \operatorname{Irr} \Lambda(\nu) \sqcup \{0\} \quad \text{and}$$
$$\widetilde{f}_i, \widetilde{f}_i^* : \bigsqcup_{\nu \in Q_+} \operatorname{Irr} \Lambda(\nu) \to \bigsqcup_{\nu \in Q_+} \operatorname{Irr} \Lambda(\nu)$$

as follows. If $c > 0$, then

$$\begin{aligned}\widetilde{e}_i : \left(\operatorname{Irr} \Lambda(\nu)\right)_{i,c} &\xrightarrow{\sim} \left(\operatorname{Irr} \Lambda\left(\nu'\right)\right)_{i,0} \xrightarrow{\sim} \left(\operatorname{Irr} \Lambda(\nu + \alpha_i)\right)_{i,c-1},\\ \widetilde{e}_i^* : \left(\operatorname{Irr} \Lambda(\nu)\right)_i^c &\xrightarrow{\sim} \left(\operatorname{Irr} \Lambda(\overline{\nu})\right)_i^0 \xrightarrow{\sim} \left(\operatorname{Irr} \Lambda(\nu + \alpha_i)\right)_i^{c-1};\end{aligned}$$

and $\widetilde{e}_i \Lambda = 0$, $\widetilde{e}_i^* \Lambda' = 0$ for $\Lambda \in (\operatorname{Irr} \Lambda(\nu))_{i,0}$, $\Lambda' \in (\operatorname{Irr} \Lambda(\nu))_i^0$, respectively. Also, we define

$$\begin{aligned}\widetilde{f}_i : \left(\operatorname{Irr} \Lambda(\nu)\right)_{i,c} &\xrightarrow{\sim} \left(\operatorname{Irr} \Lambda\left(\nu'\right)\right)_{i,0} \xrightarrow{\sim} \left(\operatorname{Irr} \Lambda(\nu - \alpha_i)\right)_{i,c+1},\\ \widetilde{f}_i^* : \left(\operatorname{Irr} \Lambda(\nu)\right)_i^c &\xrightarrow{\sim} \left(\operatorname{Irr} \Lambda(\overline{\nu})\right)_i^0 \xrightarrow{\sim} \left(\operatorname{Irr} \Lambda(\nu - \alpha_i)\right)_i^{c+1}.\end{aligned}$$

Let $* : B \mapsto {}^t B$ be an automorphism of $X(\nu)$, where ${}^t B$ is the transpose of $B \in X(\nu)$. Then, $\Lambda(\nu)$ is stable under $*$, and it induces an automorphism of $\operatorname{Irr} \Lambda(\nu)$.

Lemma 2 ([6]) *We have* $\widetilde{e}_i^* = * \circ \widetilde{e}_i \circ *$ *and* $\widetilde{f}_i^* = * \circ \widetilde{f}_i \circ *$.

Theorem 2 ([6]) (1) *For* $\Lambda \in \operatorname{Irr} \Lambda(\nu)$, *we set* $\operatorname{wt} \Lambda := -\nu$, $\varphi_i(\Lambda) := \varepsilon_i(\Lambda) + \langle h_i, \operatorname{wt} \Lambda \rangle$. *Then,* $(\bigsqcup_{\nu \in Q_+} \operatorname{Irr} \Lambda(\nu); \operatorname{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i)$ *is a crystal, which is isomorphic to* $(B(\infty); \operatorname{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i)$.

(2) *Set* $\varphi_i^*(\Lambda) = \varepsilon_i^*(\Lambda) + \langle h_i, \operatorname{wt} \Lambda \rangle_I$. *Then,* $(\bigsqcup_{\nu \in Q_+} \operatorname{Irr} \Lambda(\nu); \operatorname{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*)$ *is a crystal, which is isomorphic to* $(B(\infty); \operatorname{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*)$.

A Maya diagram $\mathbf{k} \in \mathcal{M}_I^\times$ can be written as a disjoint union of intervals:

$$\mathbf{k} = [s_1 + 1, t_1] \sqcup [s_2 + 1, t_2] \sqcup \cdots \sqcup [s_l + 1, t_l],$$
$$\text{where } n \le s_1 < t_1 < s_2 < t_2 < \cdots < s_l < t_l \le n + m + 1;$$

the interval $K_p = [s_p + 1, t_p]$ is called the p-th component of $\mathbf{k}$ for $1 \le p \le l$. Define two subsets $\operatorname{out}(\mathbf{k})$ and $\operatorname{in}(\mathbf{k})$ of I by

$$\operatorname{out}(\mathbf{k}) := \{t_p | 1 \le p \le l\} \cap I, \qquad \operatorname{in}(\mathbf{k}) := \{s_p | 1 \le p \le l\} \cap I.$$

Also, we define two subsets I_t and I_s of I by

$$I_t := \begin{cases} \mathrm{out}(\mathbf{k}) \cup \{n+1, n+m\} & \text{if } s_1 \geq n+2, t_l = n+m+1, \\ \mathrm{out}(\mathbf{k}) \cup \{n+1\} & \text{if } s_1 \geq n+2, t_l \leq n+m, \\ \mathrm{out}(\mathbf{k}) \cup \{n+m\} & \text{if } s_1 \leq n+1, t_l = n+m+1, \\ \mathrm{out}(\mathbf{k}) \cup \{n+1, n+m\} & \text{if } s_1 \geq n+1, t_l \leq n+m, \end{cases}$$

$$I_s := \begin{cases} \mathrm{in}(\mathbf{k}) \cup \{n+1, n+m\} & \text{if } s_1 = n, t_l \leq n+m-1, \\ \mathrm{in}(\mathbf{k}) \cup \{n+1\} & \text{if } s_1 = n, t_l \geq n+m, \\ \mathrm{in}(\mathbf{k}) \cup \{n+m\} & \text{if } s_1 \geq n+1, t_l \leq n+m-1, \\ \mathrm{in}(\mathbf{k}) \cup \{n+1, n+m\} & \text{if } s_1 \geq n+1, t_l \geq n+m. \end{cases}$$

Then, there exists a unique orientation $\Omega(\mathbf{k})$ such that I_t is identical to the set of source vertices of the quiver $(I, \Omega(\mathbf{k}))$, and I_s is identical to the set of sink vertices of this quiver.

For $B = (B_\tau)_{\tau \in H} \in X(\nu)$, we set

$$M_{\mathbf{k}}(B) := -\dim_{\mathbb{C}} \mathrm{Coker}\Big(\bigoplus_{k \in \mathrm{out}(\mathbf{k})} V(\nu)_k \xrightarrow{\oplus B_\mu} \bigoplus_{l \in \mathrm{in}(\mathbf{k})} V(\nu)_l \Big),$$

where $\mu := \tau_{i_1} \tau_{i_2} \cdots \tau_{l_q}$ is a path from $k \in \mathrm{out}(\mathbf{k})$ to $l \in \mathrm{in}(\mathbf{k})$ under the orientation $\Omega(\mathbf{k})$, and $B_\mu := B_{\tau_{i_1}} B_{\tau_{i_2}} \cdots B_{\tau_{i_q}}$ is the corresponding composite of linear maps. For $\Lambda \in \mathrm{Irr}\,\Lambda(\nu)$ and $\mathbf{k} \in \mathcal{M}_I^\times$, define

$$M_{\mathbf{k}}(\Lambda) := M_{\mathbf{k}}(B),$$

where $B = (B_\tau)_{\tau \in H}$ is a general point of Λ.

Proposition 5 ([10]) (1) *For each $\Lambda \in \mathrm{Irr}\,\Lambda(\nu)$, a collection* $\mathbf{M}(\Lambda) := (M_{\mathbf{k}}(\Lambda))_{\mathbf{k} \in \mathcal{M}_I^\times}$ *of integers is an e-BZ datum.*

(2) *The map $\Psi_I : \bigsqcup_{\nu \in Q_+} \mathrm{Irr}\,\Lambda(\nu) \to \mathcal{BZ}_I^e$, defined by $\Lambda \mapsto \mathbf{M}(\Lambda)$, gives rise to an isomorphism of crystals* $(\bigsqcup_{\nu \in Q_+} \mathrm{Irr}\,\Lambda(\nu); \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*) \xrightarrow{\sim} (\mathcal{BZ}_I^e; \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*)$.

2.6 BZ Data Associated to $\mathbb{Z}$

Definition 3 (1) For a given integer $r \in \mathbb{Z}$, a subset $\mathbf{k}$ of $\mathbb{Z}$ is called a Maya diagram of charge r if it satisfies the following condition: there exist nonnegative integers p and q such that

$$\mathbb{Z}_{\leq r-p} \subset \mathbf{k} \subset \mathbb{Z}_{\leq r+q}, \quad |\mathbf{k} \cap \mathbb{Z}_{>r-p}| = p, \tag{2.7}$$

where $|\mathbf{k} \cap \mathbb{Z}_{>r-p}|$ denotes the cardinality of the finite set $\mathbf{k} \cap \mathbb{Z}_{>r-p}$. We denote by $\mathcal{M}_{\mathbb{Z}}^{(r)}$ the set of all Maya diagrams of charge r, and set $\mathcal{M}_{\mathbb{Z}} := \bigcup_{r \in \mathbb{Z}} \mathcal{M}_{\mathbb{Z}}^{(r)}$.

(2) For a Maya diagram $\mathbf{k}$ of charge r, let $\mathbf{k}^c := \mathbb{Z} \setminus \mathbf{k}$ be the complement of $\mathbf{k}$ in $\mathbb{Z}$. We call $\mathbf{k}^c$ the complementary Maya diagram of charge r associated to $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{(r)}$. We denote by $\mathcal{M}_{\mathbb{Z}}^{(r),c}$ the set of all complementary Maya diagrams of charge r, and set $\mathcal{M}_{\mathbb{Z}}^{c} := \bigcup_{r\in\mathbb{Z}} \mathcal{M}_{\mathbb{Z}}^{(r),c}$.

A map $c : \mathcal{M}_{\mathbb{Z}} \to \mathcal{M}_{\mathbb{Z}}^{c}$ defined by $\mathbf{k} \mapsto \mathbf{k}^c$ is a bijection; the inverse of this map is also denoted by c.

We identify $\Xi_{\mathbb{Z}} := \bigsqcup_{r\in\mathbb{Z}} W_{\mathbb{Z}} \Lambda_r$ (*resp.*, $\Gamma_{\mathbb{Z}} := \bigsqcup_{r\in\mathbb{Z}} W_{\mathbb{Z}} \Lambda_r^c$) with $\mathcal{M}_{\mathbb{Z}}$ (*resp.*, $\mathcal{M}_{\mathbb{Z}}^{c}$) via the bijection $\Lambda_r \leftrightarrow \mathbb{Z}_{\le r}$ (*resp.*, $\Lambda_r^c \leftrightarrow \mathbb{Z}_{>r}$). Under the identification $\Xi_{\mathbb{Z}} \cong \mathcal{M}_{\mathbb{Z}}$ (*resp.*, $\Gamma_{\mathbb{Z}} \cong \mathcal{M}_{\mathbb{Z}}^{c}$), there is an induced action of $\sigma_i \in W_{\mathbb{Z}}$ on $\mathcal{M}_{\mathbb{Z}}$ (*resp.*, $\mathcal{M}_{\mathbb{Z}}^{c}$). It is easy to see that the explicit form of this action is just the transposition $(i, i+1)$ of $\mathbb{Z}$. For $\xi \in \Xi_{\mathbb{Z}}$ (*resp.*, $\gamma \in \Gamma_{\mathbb{Z}}$), we denote by $\mathbf{k}(\xi)$ (*resp.*, $\mathbf{k}(\gamma)$) the corresponding Maya diagram.

Let I be a finite interval in $\mathbb{Z}$, and $\mathrm{res}_I : \mathcal{M}_{\mathbb{Z}} \to \mathcal{M}_I$ a map defined by $\mathrm{res}_I(\mathbf{k}) = \mathbf{k} \cap \widetilde{I}$ for $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$. Set $\mathcal{M}_{\mathbb{Z}}(I) := \{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}} | \mathbf{k} = \mathbb{Z}_{\le n} \cup \mathbf{k}_I \text{ for some } \mathbf{k}_I \in \mathcal{M}_I^{\times}\}$. Then the map res_I induces a bijection from $\mathcal{M}_{\mathbb{Z}}(I)$ to $\mathcal{M}_I^{\times}$. For $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(I)$, if we set $\Omega_I(\mathbf{k}) := (\mathrm{res}_I)^{-1}(\widetilde{I} \setminus \mathrm{res}_I(\mathbf{k}))$ for $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(I)$, then $\Omega_I(\mathbf{k}) \in \mathcal{M}_{\mathbb{Z}}(I)$ and the map $\Omega_I : \mathcal{M}_{\mathbb{Z}}(I) \to \mathcal{M}_{\mathbb{Z}}(I)$ is a bijection. Also, if we define $\mathrm{res}_I^c := \mathrm{res}_I \circ c : \mathcal{M}_{\mathbb{Z}}^{c} \to \mathcal{M}_I$, then it induces bijections $\mathrm{res}_I^c : \mathcal{M}_{\mathbb{Z}}^{c}(I) := (\mathcal{M}_{\mathbb{Z}}(I))^c \xrightarrow{\sim} \mathcal{M}_I^{\times}$ and $\Omega_I^c : \mathcal{M}_{\mathbb{Z}}^{c}(I) \xrightarrow{\sim} \mathcal{M}_{\mathbb{Z}}^{c}(I)$ in a similar way.

Let $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}}$ be a collection of integers indexed by $\mathcal{M}_{\mathbb{Z}}$. For such an $\mathbf{M}$, we define $\mathbf{M}_I := (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}(I)}$. By the bijection $\mathrm{res}_I : \mathcal{M}_{\mathbb{Z}}(I) \xrightarrow{\sim} \mathcal{M}_I^{\times}$, $\mathbf{M}_I$ can be regarded as a collection of integers indexed by $\mathcal{M}_I^{\times}$. Similarly, for $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}^{c}}$, we define $\mathbf{M}_I := (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}^{c}(I)}$, which is regarded as a collection of integers indexed by $\mathcal{M}_I^{\times}$.

Definition 4 (1) A collection $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}^{c}}$ of integers is called a complementary BZ (c-BZ for short) datum associated to $\mathbb{Z}$ if it satisfies the following conditions:

(1-a) For each finite interval K in $\mathbb{Z}$, $\mathbf{M}_K = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_K^{\times}}$ is an element of $\mathcal{BZ}_K$.
(1-b) For each $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{c}$, there exists a finite interval I in $\mathbb{Z}$ such that

- (1-i) $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{c}(I)$,
- (1-ii) for every finite interval $J \supset I$, $M_{\Omega_J^c(\mathbf{k})} = M_{\Omega_I^c(\mathbf{k})}$.

(2) A collection $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}}$ of integers is called an e-BZ datum associated to $\mathbb{Z}$ if it satisfies the following conditions:

(2-a) For each finite interval K in $\mathbb{Z}$, $\mathbf{M}_K = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_K^{\times}}$ is an element of $\mathcal{BZ}_K^e$.
(2-b) For each $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$, there exists a finite interval I in $\mathbb{Z}$ such that

- (2-i) $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(I)$,
- (2-ii) for every finite interval $J \supset I$, $M_{\Omega_J(\mathbf{k})} = M_{\Omega_I(\mathbf{k})}$.

We denote by $\mathcal{BZ}_{\mathbb{Z}}$ (*resp.*, $\mathcal{BZ}_{\mathbb{Z}}^e$) the set of all c-BZ (*resp.*, e-BZ) data associated to $\mathbb{Z}$.

For a given c-BZ datum $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}^c} \in \mathcal{BZ}_{\mathbb{Z}}$, we define a new collection $\mathbf{M}^* = (M_{\mathbf{k}}^*)_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}}$ of integers by $M_{\mathbf{k}}^* := M_{\mathbf{k}^c}$. As in the case of finite intervals, $\mathbf{M}^*$ is an element of $\mathcal{BZ}_{\mathbb{Z}}^e$ and the map $* : \mathcal{BZ}_{\mathbb{Z}} \to \mathcal{BZ}_{\mathbb{Z}}^e$ is a bijection. The inverse of this bijection is also denoted by $*$. We note that $*^2 = \mathrm{id}$.

Let $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}^c} \in \mathcal{BZ}_{\mathbb{Z}}$ be a c-BZ datum. For each complementary Maya diagram $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^c$, we denote by $\mathrm{Int}^c(\mathbf{M}; \mathbf{k})$ the set of all finite intervals I in $\mathbb{Z}$ satisfying condition (1-b) in the definition above.

For $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$, we define another collection $\Theta(\mathbf{M}) = (\Theta(M)_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}}$ of integers as follows. Fix $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$ and take the complement $\mathbf{k}^c \in \mathcal{M}_{\mathbb{Z}}^c$ of $\mathbf{k}$. Since $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$, there exists a finite interval $I \in \mathrm{Int}^c(\mathbf{M}; \mathbf{k}^c)$. Then we define $\Theta(M)_{\mathbf{k}} := M_{(\mathrm{res}_I^c)^{-1}(\mathrm{res}_I(\mathbf{k}))}$; this definition does not depend on the choice of I.

Now, let $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}} \in \mathcal{BZ}_{\mathbb{Z}}^e$. Note that $\mathbf{M}^* \in \mathcal{BZ}_{\mathbb{Z}}$. We set

$$\mathrm{Int}^e(\mathbf{M}; \mathbf{k}) := \mathrm{Int}^c\big(\mathbf{M}^*; \mathbf{k}^c\big). \tag{2.8}$$

Lemma 3 *The set* $\mathrm{Int}^e(\mathbf{M}; \mathbf{k})$ *is identical to the set of all finite intervals* I *in* $\mathbb{Z}$ *satisfying condition* (2-*b*) *in Definition* 4.

Proof It suffices to show that $I \in \mathrm{Int}^e(\mathbf{M}; \mathbf{k}) = \mathrm{Int}^c(\mathbf{M}^*; \mathbf{k}^c)$ if and only if I satisfies condition (2-b). By the definition of $\mathcal{M}_{\mathbb{Z}}^c(I)$, the condition $\mathbf{k}^c \in \mathcal{M}_{\mathbb{Z}}^c(I)$ is equivalent to the condition $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(I)$. Suppose that this condition is satisfied. Recall the following equation in Lemma 3.3.1 of [9]: $\Omega_I^c(\mathbf{k}^c) = (\Omega_I(\mathbf{k}))^c$ for $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(I)$. From this, we deduce that

$$M^*_{\Omega_I^c(\mathbf{k}^c)} = M_{(\Omega_I^c(\mathbf{k}^c))^c} = M_{\Omega_I(\mathbf{k})}.$$

Thus, condition (1-ii) for $\mathbf{M}^*$ and $\mathbf{k}^c$ is equivalent to condition (2-ii) for $\mathbf{M}$ and $\mathbf{k}$. This proves the lemma. □

2.7 Action of Kashiwara Operators

First, we define the action of the raising Kashiwara operators $\widetilde{e}_p$, $p \in \mathbb{Z}$, on $\mathcal{BZ}_{\mathbb{Z}}$. For $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}^c} \in \mathcal{BZ}_{\mathbb{Z}}$ and $p \in \mathbb{Z}$, set

$$\varepsilon_p(\mathbf{M}) := -\big(\Theta(M)_{\mathbf{k}(\Lambda_p)} + \Theta(M)_{\mathbf{k}(\sigma_p\Lambda_p)} - \Theta(M)_{\mathbf{k}(\Lambda_{p+1})} - \Theta(M)_{\mathbf{k}(\Lambda_{p-1})}\big).$$

Let $I \in \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\Lambda_p)^c) \cap \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\sigma_p\Lambda_p)^c) \cap \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\Lambda_{p+1})^c) \cap \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\Lambda_{p-1})^c)$. Then it is known that $\varepsilon_p(\mathbf{M}) = \varepsilon_p(\mathbf{M}_I)$, and hence this is a nonnegative integer.

If $\varepsilon_p(\mathbf{M}) = 0$, then we set $\widetilde{e}_p\mathbf{M} = 0$. Suppose that $\varepsilon_p(\mathbf{M}) > 0$. Then we define $\widetilde{e}_p\mathbf{M} = (M'_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}^c}$ as follows. For $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^c$, take a finite interval I in $\mathbb{Z}$ such that $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^c(I)$ and $I \in \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\Lambda_p)^c) \cap \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\sigma_p\Lambda_p)^c) \cap \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\Lambda_{p+1})^c) \cap \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\Lambda_{p-1})^c)$. Set

$$M'_{\mathbf{k}} := (\widetilde{e}_p\mathbf{M}_I)_{\mathrm{res}_I^c(\mathbf{k})}.$$

Here we note that $\widetilde{e}_p\mathbf{M}_I$ is defined since $\mathbf{M}_I \in \mathcal{BZ}_I$.

Second, let us define the action of the lowering Kashiwara operators $\widetilde{f}_p$, $p \in \mathbb{Z}$, on $\mathcal{BZ}_\mathbb{Z}$. For $\mathbf{M} = (M_\mathbf{k})_{\mathbf{k}\in\mathcal{M}^c_\mathbb{Z}} \in \mathcal{BZ}_\mathbb{Z}$ and $p \in \mathbb{Z}$, we define $\widetilde{f}_p\mathbf{M} = (M''_\mathbf{k})_{\mathbf{k}\in\mathcal{M}^c_\mathbb{Z}}$ as follows. For $\mathbf{k} \in \mathcal{M}^c_\mathbb{Z}$, take a finite interval I in $\mathbb{Z}$ such that $\mathbf{k} \in \mathcal{M}^c_\mathbb{Z}(I)$ and $I \in \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\Lambda_p)^c) \cap \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\sigma_p\Lambda_p)^c)$. Set

$$M''_\mathbf{k} := (\widetilde{f}_p\mathbf{M}_I)_{\mathrm{res}^c_I(\mathbf{k})}.$$

Proposition 6 ([8]) (1) *The definition above of $M'_\mathbf{k}$ (resp., $M''_\mathbf{k}$) does not depend on the choice of I.*

(2) *For each $\mathbf{M} = (M_\mathbf{k})_{\mathbf{k}\in\mathcal{M}^c_\mathbb{Z}} \in \mathcal{BZ}_\mathbb{Z}$ and $p \in \mathbb{Z}$, $\widetilde{e}_p\mathbf{M}$ (resp., $\widetilde{f}_p\mathbf{M}$) is contained in $\mathcal{BZ}_\mathbb{Z} \cup \{0\}$ (resp., $\mathcal{BZ}_\mathbb{Z}$).*

For $\mathbf{M} \in \mathcal{BZ}^e_\mathbb{Z}$, set $\varepsilon^*_p(\mathbf{M}) := \varepsilon_p(\mathbf{M}^*)$, $p \in \mathbb{Z}$. We define the Kashiwara operators $\widetilde{e}^*_p$ and $\widetilde{f}^*_p$ on $\mathcal{BZ}^e_\mathbb{Z}$ by

$$\widetilde{e}^*_p\mathbf{M} := \begin{cases} (\widetilde{e}_p(\mathbf{M}^*))^* & \text{if } \varepsilon^*_p(\mathbf{M}) > 0, \\ 0 & \text{if } \varepsilon^*_p(\mathbf{M}) = 0, \end{cases} \quad \text{and} \quad \widetilde{f}^*_p\mathbf{M} := \big(\widetilde{f}_p\big(\mathbf{M}^*\big)\big)^*.$$

The following corollary is easily obtained from the proposition above.

Corollary 3 *For each $\mathbf{M} \in \mathcal{BZ}^e_\mathbb{Z}$ and $p \in \mathbb{Z}$, $\widetilde{e}^*_p\mathbf{M}$ (resp., $\widetilde{f}^*_p\mathbf{M}$) is contained in $\mathcal{BZ}^e_\mathbb{Z} \cup \{0\}$ (resp., $\mathcal{BZ}^e_\mathbb{Z}$).*

2.8 BZ Data of Type $A^{(1)}_{l-1}$

Fix $l \in \mathbb{Z}_{\geq 3}$. Let $\widehat{\mathfrak{g}}$ be the affine Lie algebra of type $A^{(1)}_{l-1}$, $\widehat{\mathfrak{h}}$ the Cartan subalgebra of $\widehat{\mathfrak{g}}$, $\widehat{h}_i \in \widehat{\mathfrak{h}}$, $i \in \widehat{I} := \{0, 1, \dots, l-1\}$, the simple coroots of $\widehat{\mathfrak{g}}$, and $\widehat{\alpha}_i \in \widehat{\mathfrak{h}}^* := \mathrm{Hom}_\mathbb{C}(\widehat{\mathfrak{h}}, \mathbb{C})$, $i \in \widehat{I}$, the simple roots of $\widehat{\mathfrak{g}}$. We set $\widehat{Q}^+ := \sum_i \mathbb{Z}_{\geq 0}\widehat{\alpha}_i$ and $\widehat{Q}^- := -\widehat{Q}^+$. Note that $\langle \widehat{h}_i, \widehat{\alpha}_j\rangle = \widehat{a}_{ij}$ for $i, j \in \widehat{I}$. Here, $\langle\cdot,\cdot\rangle : \widehat{\mathfrak{h}} \times \widehat{\mathfrak{h}}^* \to \mathbb{C}$ is the canonical pairing, and $\widehat{A} = (\widehat{a}_{ij})_{i,j\in\widehat{I}}$ is the Cartan matrix of type $A^{(1)}_{l-1}$ with index set $\widehat{I}$; the entries $\widehat{a}_{ij}$ are given by

$$\widehat{a}_{ij} := \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1 \text{ or } l-1, \\ 0 & \text{otherwise.} \end{cases}$$

Now, consider a bijection $\tau : \mathbb{Z} \to \mathbb{Z}$ given by $\tau(j) := j + 1$ for $j \in \mathbb{Z}$. It induces an automorphism $\tau : \mathfrak{t}^* \xrightarrow{\sim} \mathfrak{t}^*$ such that $\tau(\Lambda_j) = \Lambda_{j+1}$ and $\tau(\Lambda^c_j) = \Lambda^c_{j+1}$ for all $j \in \mathbb{Z}$. It follows that $\tau \circ \sigma_j = \sigma_{j+1} \circ \tau$. Also, for $i \in \widehat{I}$, define a family S_i of automorphism of $\mathfrak{t}^*$ by

$$S_i := \{\sigma_{i+al} | a \in \mathbb{Z}\}.$$

Since $l \geq 3$, $\sigma_{j_1}\sigma_{j_2} = \sigma_{j_2}\sigma_{j_1}$ for all $\sigma_{j_1}, \sigma_{j_2} \in S_i$, and for a fixed $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$ or $\mathcal{M}_{\mathbb{Z}}^c$, there exists a finite subset $S_i(\mathbf{k}) \subset S_i$ such that $\sigma_j(\mathbf{k}) = \mathbf{k}$ for every $\sigma_j \in S_i \setminus S_i(\mathbf{k})$. Therefore, we can define an infinite product $\widehat{\sigma_i} := \prod_{\sigma_j \in S_i} \sigma_j$ of operators acting on $\mathcal{M}_{\mathbb{Z}}$ and $\mathcal{M}_{\mathbb{Z}}^c$. Note that we have $\tau \circ \widehat{\sigma_i} = \widehat{\sigma}_{i+1} \circ \tau$, where we regard $i \in \widehat{I}$ as an element of $\mathbb{Z}/l\mathbb{Z}$.

Set $\sigma := \tau^l$. For $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$, we define new collections $\sigma(\mathbf{M})$ and $\sigma^{-1}(\mathbf{M})$ of integers indexed by $\mathcal{M}_{\mathbb{Z}}^c$ by $\sigma(\mathbf{M})_{\mathbf{k}} := \mathbf{M}_{\sigma^{-1}(\mathbf{k})}$ and $\sigma^{-1}(\mathbf{M})_{\mathbf{k}} := \mathbf{M}_{\sigma(\mathbf{k})}$ for each $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^c$, respectively. It is shown in [8] that $\sigma(\mathbf{M})$ and $\sigma^{-1}(\mathbf{M})$ are both elements of $\mathcal{BZ}_{\mathbb{Z}}$.

Similarly, for $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^e$, we can define new collections $\sigma^{\pm}(\mathbf{M})$, and prove that they are both elements of $\mathcal{BZ}_{\mathbb{Z}}^e$.

Lemma 4 ([8]) (1) *On $\mathcal{BZ}_{\mathbb{Z}}$, we have $\Theta \circ \sigma = \sigma \circ \Theta$.*

(2) *For $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$ and $p \in \mathbb{Z}$, $\varepsilon_p(\sigma(\mathbf{M})) = \varepsilon_{\sigma^{-1}(p)}(\mathbf{M})$.*

(3) *The equalities $\sigma \circ \widetilde{e}_p = \widetilde{e}_{\sigma(p)} \circ \sigma$ and $\sigma \circ \widetilde{f}_p = \widetilde{f}_{\sigma(p)} \circ \sigma$ hold on $\mathcal{BZ}_{\mathbb{Z}} \cup \{0\}$ for all $p \in \mathbb{Z}$. Here it is understood that $\sigma(0) = 0$.*

Definition 5 Set

$$\mathcal{BZ}_{\mathbb{Z}}^{\sigma} := \left\{\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}} | \sigma(\mathbf{M}) = \mathbf{M}\right\} \quad \text{and} \quad \left(\mathcal{BZ}_{\mathbb{Z}}^e\right)^{\sigma} := \left\{\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^e | \sigma(\mathbf{M}) = \mathbf{M}\right\}.$$

An element $\mathbf{M}$ of $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ (*resp*., $(\mathcal{BZ}_{\mathbb{Z}}^e)^{\sigma}$) is called a c-BZ (*resp*., e-BZ) datum of type $A_{l-1}^{(1)}$.

2.9 Crystal Structure on $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$

Now we define a crystal structure on $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$, following [8]. For $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ and $p \in \widehat{I}$, we set

$$\mathrm{wt}(\mathbf{M}) := \sum_{p \in \widehat{I}} \Theta(\mathbf{M})_{\mathbf{k}(\Lambda_p)} \widehat{\alpha}_p, \qquad \widehat{\varepsilon}_p(\mathbf{M}) := \varepsilon_p(\mathbf{M}),$$

$$\widehat{\varphi}_p(\mathbf{M}) := \widehat{\varepsilon}_p(\mathbf{M}) + \left\langle \widehat{h}_p, \mathrm{wt}(\mathbf{M}) \right\rangle.$$

In order to define the action of Kashiwara operators, we need the following.

Lemma 5 ([8]) *Let $q, q' \in \mathbb{Z}$, with $|q - q'| \geq 2$. Then, we have $\widetilde{e}_q \widetilde{e}_{q'} = \widetilde{e}_{q'} \widetilde{e}_q$, $\widetilde{f}_q \widetilde{f}_{q'} = \widetilde{f}_{q'} \widetilde{f}_q$, and $\widetilde{e}_q \widetilde{f}_{q'} = \widetilde{f}_{q'} \widetilde{e}_q$, as operators on $\mathcal{BZ}_{\mathbb{Z}} \cup \{0\}$.*

For $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ and $p \in \widehat{I}$, we define $\widehat{e}_p\mathbf{M}$ and $\widehat{f}_p\mathbf{M}$ as follows. If $\widehat{\varepsilon}_p(\mathbf{M}) = 0$, then we set $\widehat{e}_p\mathbf{M} := 0$. If $\widehat{\varepsilon}_p(\mathbf{M}) > 0$, then we define a new collection $\widehat{e}_p\mathbf{M} = (M'_{\mathbf{k}})$ of integers indexed by $\mathcal{M}_{\mathbb{Z}}^c$ by

$$M'_{\mathbf{k}} := (e_{L(\mathbf{k},p)}\mathbf{M})_{\mathbf{k}} \quad \text{for each } \mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^c.$$

Here, $L(\mathbf{k}, p) := \{q \in p + l\mathbb{Z} | q \in \mathbf{k} \text{ and } q+1 \notin \mathbf{k}\}$ and $e_{L(\mathbf{k},p)} := \prod_{q \in L(\mathbf{k},p)} \widetilde{e}_q$. By the definition, $L(\mathbf{k}, p)$ is a finite set such that $|q - q'| > 2$ for all $q, q' \in L(\mathbf{k}, p)$ with $q \neq q'$. Therefore, by Lemma 5, $e_{L(\mathbf{k},p)}$ is a well-defined operator on $\mathcal{BZ}_{\mathbb{Z}}$.

A collection $\widehat{f}_p\mathbf{M} = (M''_{\mathbf{k}})$ of integers indexed by $\mathcal{M}^c_{\mathbb{Z}}$ is defined by

$$M''_{\mathbf{k}} := (f_{L(\mathbf{k},p)}\mathbf{M})_{\mathbf{k}} \quad \text{for each } \mathbf{k} \in \mathcal{M}^c_{\mathbb{Z}},$$

where $f_{L(\mathbf{k},p)} := \prod_{q \in L(\mathbf{k},p)} \widetilde{f}_q$. By the same reasoning as above, we see that $f_{L(\mathbf{k},p)}$ is a well-defined operator on $\mathcal{BZ}_{\mathbb{Z}}$.

Proposition 7 ([8]) (1) *We have* $\widehat{e}_p\mathbf{M} \in \mathcal{BZ}^\sigma_{\mathbb{Z}} \cup \{0\}$ *and* $\widehat{f}_p\mathbf{M} \in \mathcal{BZ}^\sigma_{\mathbb{Z}}$.

(2) *The set* $\mathcal{BZ}^\sigma_{\mathbb{Z}}$, *equipped with the maps* wt, $\widehat{\varepsilon}_p, \widehat{\varphi}_p, \widehat{e}_p, \widehat{f}_p$, *is a* $U_q(\widehat{\mathfrak{sl}}_l)$*-crystal.*

Let $\mathbf{O}$ be a collection of integers indexed by $\mathcal{M}^c_{\mathbb{Z}}$ whose $\mathbf{k}$-component is equal to 0 for all $\mathbf{k} \in \mathcal{M}^c_{\mathbb{Z}}$. It is obvious that $\mathbf{O} \in \mathcal{BZ}^\sigma_{\mathbb{Z}}$. Let $\mathcal{BZ}^\sigma_{\mathbb{Z}}(\mathbf{O})$ denote the connected component of the crystal $\mathcal{BZ}^\sigma_{\mathbb{Z}}$ containing $\mathbf{O}$. The following is the main result of [8].

Theorem 3 ([8]) *As a crystal,* $(\mathcal{BZ}^\sigma_{\mathbb{Z}}(\mathbf{O}); \text{wt}, \widehat{\varepsilon}_p, \widehat{\varphi}_p, \widehat{e}_p, \widehat{f}_p)$ *is isomorphic to* $B(\infty)$ *for* $U_q(\widehat{\mathfrak{sl}}_l)$.

In a manner similar to the one in [8], we can define a crystal structure on $(\mathcal{BZ}^e_{\mathbb{Z}})^\sigma$. By the construction, it is easy to see that $* \circ \sigma = \sigma \circ *$. Therefore, the restriction of $* : \mathcal{BZ}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{BZ}^e_{\mathbb{Z}}$ to the subset $\mathcal{BZ}^\sigma_{\mathbb{Z}}$ gives rise to a bijection $* : \mathcal{BZ}^\sigma_{\mathbb{Z}} \xrightarrow{\sim} (\mathcal{BZ}^e_{\mathbb{Z}})^\sigma$. We denote by $\mathbf{O}^*$ the image of $\mathbf{O} \in \mathcal{BZ}^\sigma_{\mathbb{Z}}$ under the bijection $*$. Then, $\mathbf{O}^*$ is a collection of integers indexed by $\mathcal{M}_{\mathbb{Z}}$ whose $\mathbf{k}$-component is equal to 0 for all $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$.

For $\mathbf{M} \in (\mathcal{BZ}^e_{\mathbb{Z}})^\sigma$ and $p \in \mathbb{Z}$, we define

$$\text{wt}(\mathbf{M}) := \text{wt}\big(\mathbf{M}^*\big), \qquad \widehat{\varepsilon}^*_p(\mathbf{M}) := \widehat{\varepsilon}_p\big(\mathbf{M}^*\big), \qquad \widehat{\varphi}^*_p(\mathbf{M}) := \widehat{\varepsilon}^*_p(\mathbf{M}) + \big\langle \widehat{h}_p, \text{wt}(\mathbf{M}) \big\rangle,$$

and

$$\widehat{e}^*_p\mathbf{M} := \begin{cases} (\widehat{e}_p(\mathbf{M}^*))^* & \text{if } \widehat{\varepsilon}^*_p(\mathbf{M}) > 0, \\ 0 & \text{if } \widehat{\varepsilon}^*_p(\mathbf{M}) = 0, \end{cases} \qquad \widehat{f}^*_p := \big(\widehat{f}_p\big(\mathbf{M}^*\big)\big)^*.$$

The following corollary is an easy consequence of Theorem 3.

Corollary 4 (1) *The set* $(\mathcal{BZ}^e_{\mathbb{Z}})^\sigma$, *equipped with the maps* wt, $\widehat{\varepsilon}^*_p, \widehat{\varphi}^*_p, \widehat{e}^*_p, \widehat{f}^*_p$, *is a* $U_q(\widehat{\mathfrak{sl}}_l)$*-crystal.*

(2) *Let* $(\mathcal{BZ}^e_{\mathbb{Z}})^\sigma(\mathbf{O}^*)$ *be the connected component of the crystal* $(\mathcal{BZ}^e_{\mathbb{Z}})^\sigma$ *containing* $\mathbf{O}^* \in (\mathcal{BZ}^e_{\mathbb{Z}})^\sigma$. *Then,* $((\mathcal{BZ}^e_{\mathbb{Z}})^\sigma(\mathbf{O}^*); \text{wt}, \widehat{\varepsilon}^*_p, \widehat{\varphi}^*_p, \widehat{e}^*_p, \widehat{f}^*_p)$ *is isomorphic as a crystal to* $B(\infty)$ *for* $U_q(\widehat{\mathfrak{sl}}_l)$.

3 Ordinary Crystal Structure on $\mathcal{BZ}_I^e$

3.1 The Operator $\sharp$

Let $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_I^\times}$ be an e-BZ datum associated to a finite interval $I = [n+1, n+m]$. Set $\mathrm{wt}^\vee(\mathbf{M}) := \sum_{i \in I} M_{[i+1,n+m+1]} h_i$. Then the following equality holds:

$$\left\langle \mathrm{wt}^\vee(\mathbf{M}), \alpha_i^I \right\rangle_I = \left\langle h_i, \mathrm{wt}(\mathbf{M}) \right\rangle_I.$$

Definition 6 For each $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_I} \in \mathcal{BZ}_I^e$, we define a new collection $\mathbf{M}^\sharp = (M_{\mathbf{k}}^\sharp)_{\mathbf{k} \in \mathcal{M}_I^\times}$ of integers by

$$M_{\mathbf{k}}^\sharp := M_{\mathbf{k}^c} - \left\langle \mathrm{wt}^\vee(\mathbf{M}), \mathbf{k} \right\rangle_I.$$

It is easy to verify the following lemma.

Lemma 6 (1) $\mathrm{wt}(\mathbf{M}^\sharp) = \mathrm{wt}(\mathbf{M})$ *and* $\mathrm{wt}^\vee(\mathbf{M}^\sharp) = \mathrm{wt}^\vee(\mathbf{M})$.
(2) $(\mathbf{M}^\sharp)^\sharp = \mathbf{M}$.

Lemma 7 $\mathbf{M}^\sharp \in \mathcal{BZ}_I^e$.

Proof It suffices to check conditions (BZ-0), (BZ-1), and (BZ-2). Condition (BZ-0) is checked by an easy calculation. Let us check (BZ-1): for $\mathbf{k} \cap \{i, j\} = \phi$,

$$M_{\mathbf{k}\cup\{i\}}^\sharp + M_{\mathbf{k}\cup\{j\}}^\sharp \le M_{\mathbf{k}\cup\{i,j\}}^\sharp + M_{\mathbf{k}}^\sharp.$$

Since

$$\left\langle h_k, \mathbf{k} \cup \{i\} \right\rangle_I = \begin{cases} 1 & \text{if } k = i, \\ -1 & \text{if } k = i-1, \\ \langle h_k, \mathbf{k} \rangle_I & \text{otherwise,} \end{cases}$$

$$\left\langle h_k, \mathbf{k} \cup \{j\} \right\rangle_I = \begin{cases} 1 & \text{if } k = j, \\ -1 & \text{if } k = j-1, \\ \langle h_k, \mathbf{k} \rangle_I & \text{otherwise,} \end{cases}$$

and

$$\left\langle h_k, \mathbf{k} \cup \{i, j\} \right\rangle_I = \begin{cases} 1 & \text{if } k = i \text{ or } j, \\ -1 & \text{if } k = i-1 \text{ or } j-1, \\ \langle h_k, \mathbf{k} \rangle_I & \text{otherwise,} \end{cases}$$

we obtain the following equalities:

$$\left\langle h_k, \mathbf{k} \cup \{i\} \right\rangle_I + \left\langle h_k, \mathbf{k} \cup \{j\} \right\rangle_I = \left\langle h_k, \mathbf{k} \cup \{i, j\} \right\rangle_I + \langle h_k, \mathbf{k} \rangle_I \quad \text{for all } k \in I.$$

From this, we deduce that

$$\left\langle \mathrm{wt}^{\vee}(\mathbf{M}), \mathbf{k} \cup \{i\}\right\rangle_I + \left\langle \mathrm{wt}^{\vee}(\mathbf{M}), \mathbf{k} \cup \{j\}\right\rangle_I = \left\langle \mathrm{wt}^{\vee}(\mathbf{M}), \mathbf{k} \cup \{i, j\}\right\rangle_I + \left\langle \mathrm{wt}^{\vee}(\mathbf{M}), \mathbf{k}\right\rangle_I.$$

Since $M_{(\mathbf{k}\cup\{i\})^c} + M_{(\mathbf{k}\cup\{j\})^c} \le M_{(\mathbf{k}\cup\{i,j\})^c} + M_{\mathbf{k}^c}$, condition (BZ-1) is satisfied for $\mathbf{M}^\sharp$.

Now, suppose that $\mathbf{k} \cap \{i, j, k\} = \phi$ with $i < j < k$. Then, for every $l \in I$, we have

$$\begin{aligned}\left\langle h_l, \mathbf{k} \cup \{i, k\}\right\rangle_I + \left\langle h_l, \mathbf{k} \cup \{j\}\right\rangle_I &= \left\langle h_l, \mathbf{k} \cup \{j, k\}\right\rangle_I + \langle h_l, \mathbf{k}\rangle_I \\ &= \left\langle h_l, \mathbf{k} \cup \{i, j\}\right\rangle_I + \langle h_l, \mathbf{k}\rangle_I.\end{aligned}$$

From this equality, we see that condition (BZ-2) is satisfied for $\mathbf{M}^\sharp$ by the same argument as for condition (BZ-1). □

For $\mathbf{M} \in \mathcal{BZ}_I^e$, set $\varepsilon_i(\mathbf{M}) := -M_{[n+1,i-1]\cup\{i+1\}}$.

Lemma 8 $\varepsilon_i(\mathbf{M}) = \varepsilon_i^*(\mathbf{M}^\sharp)$.

Proof By Lemma 6(2), it suffices to show that $\varepsilon_i(\mathbf{M}^\sharp) = \varepsilon_i^*(\mathbf{M})$. By the definitions, we have

$$\varepsilon_i^*(\mathbf{M}) = -M_{[i+1,n+m+1]} - M_{\{i\}\cup[i+2,n+m+1]} + M_{[i+2,n+m+1]} + M_{[i,n+m+1]}.$$

Also, we compute:

$$\begin{aligned}\varepsilon_i\left(\mathbf{M}^\sharp\right) &= -M^\sharp_{[n+1,i-1]\cup\{i+1\}} \\ &= -M_{([n+1,i-1]\cup\{i+1\})^c} + \left\langle \mathrm{wt}^{\vee}(\mathbf{M}), [n+1, i-1] \cup \{i+1\}\right\rangle_I \\ &= -M_{\{i\}\cup[i+2,n+m+1]} + \sum_{l\in I} M_{[l+1,n+m+1]}\left\langle h_l, [n+1, i-1] \cup \{i+1\}\right\rangle_I \\ &= -M_{\{i\}\cup[i+2,n+m+1]} - M_{[i+1,n+m+1]} + M_{[i+2,n+m+1]} + M_{[i,n+m+1]}.\end{aligned}$$

Thus, we obtain the desired equality. □

Lemma 9 (1) *If* $\varepsilon_i(\mathbf{M}) > 0$, *then*

(a) $(\widetilde{e}_i^*(\mathbf{M}^\sharp))^\sharp_{\mathbf{k}} = M_{\mathbf{k}} + 1$ *for* $\mathbf{k} \in \mathcal{M}_I^\times(i)^*$,
(b) $(\widetilde{e}_i^*(\mathbf{M}^\sharp))^\sharp_{\mathbf{k}} = M_{\mathbf{k}}$ *for* $\mathbf{k} \in \mathcal{M}_I^\times \setminus (\mathcal{M}_I^\times(i) \cup \mathcal{M}_I^\times(i)^*)$.

(2) *For every* $\mathbf{M} \in \mathcal{BZ}_I^e$,

(a) $(\widetilde{f}_i^*(\mathbf{M}^\sharp))^\sharp_{\mathbf{k}} = M_{\mathbf{k}} - 1$ *for* $\mathbf{k} \in \mathcal{M}_I^\times(i)^*$,
(b) $(\widetilde{f}_i^*(\mathbf{M}^\sharp))^\sharp_{\mathbf{k}} = M_{\mathbf{k}}$ *for* $\mathbf{k} \in \mathcal{M}_I^\times \setminus (\mathcal{M}_I^\times(i) \cup \mathcal{M}_I^\times(i)^*)$.

Proof Since part (2) is proved in a similar way, we only give a proof of part (1). Suppose that $\mathbf{k} \in \mathcal{M}_I^\times(i)^*$ or $\mathbf{k} \in \mathcal{M}_I^\times \setminus (\mathcal{M}_I^\times(i) \cup \mathcal{M}_I^\times(i)^*)$. Then, $\mathbf{k}^c \in \mathcal{M}_I^\times \setminus \mathcal{M}_I^\times(i)^*$. Also, since $\varepsilon_i^*(\mathbf{M}^\sharp) = \varepsilon_i(\mathbf{M}) > 0$, it follows that

$$\begin{aligned}\widetilde{e}_i^*\left(\mathbf{M}^\sharp\right)_{\mathbf{k}^c} &= \mathbf{M}^\sharp_{\mathbf{k}^c} \quad \text{by Proposition 1}\\ &= M_{\mathbf{k}} - \left\langle \mathrm{wt}^\vee(\mathbf{M}), \mathbf{k}^c \right\rangle_I\\ &= M_{\mathbf{k}} + \left\langle \mathrm{wt}^\vee(\mathbf{M}), \mathbf{k} \right\rangle_I.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\left(\widetilde{e}_i^*\left(\mathbf{M}^\sharp\right)\right)^\sharp_{\mathbf{k}} &= \widetilde{e}_i^*\left(\mathbf{M}^\sharp\right)_{\mathbf{k}^c} - \left\langle \mathrm{wt}^\vee\left(\widetilde{e}_i^*\left(\mathbf{M}^\sharp\right)\right), \mathbf{k}\right\rangle_I\\ &= M_{\mathbf{k}} + \left\langle \mathrm{wt}^\vee(\mathbf{M}), \mathbf{k}\right\rangle_I - \left\langle \mathrm{wt}^\vee(\mathbf{M}), \mathbf{k}\right\rangle_I - \langle h_i, \mathbf{k}\rangle_I \quad \text{by Lemma 6(1)}\\ &= M_{\mathbf{k}} - \langle h_i, \mathbf{k}\rangle_I\\ &= \begin{cases} M_{\mathbf{k}} + 1 & \text{if } \mathbf{k} \in \mathcal{M}_I^\times(i)^*,\\ M_{\mathbf{k}} & \text{if } \mathcal{M}_I^\times \setminus (\mathcal{M}_I^\times(i) \cup \mathcal{M}_I^\times(i)^*).\end{cases}\end{aligned}$$

This proves the lemma. □

Proposition 8 (1) *Assume that* $\varepsilon_i(\mathbf{M}) > 0$. *Then, there exists a unique e-BZ datum* $\mathbf{M}^{[1]}$ *such that*

(a) $(\mathbf{M}^{[1]})_{\mathbf{k}} = M_{\mathbf{k}} + 1$ *for* $\mathbf{k} \in \mathcal{M}_I^\times(i)^*$,
(b) $(\mathbf{M}^{[1]})_{\mathbf{k}} = M_{\mathbf{k}}$ *for* $\mathbf{k} \in \mathcal{M}_I^\times \setminus (\mathcal{M}_I^\times(i) \cup \mathcal{M}_I^\times(i)^*)$.

(2) *There exists a unique e-BZ datum* $\mathbf{M}^{[2]}$ *such that*

(a) $(\mathbf{M}^{[2]})_{\mathbf{k}} = M_{\mathbf{k}} - 1$ *for* $\mathbf{k} \in \mathcal{M}_I^\times(i)^*$,
(b) $(\mathbf{M}^{[2]})_{\mathbf{k}} = M_{\mathbf{k}}$ *for* $\mathbf{k} \in \mathcal{M}_I^\times \setminus (\mathcal{M}_I^\times(i) \cup \mathcal{M}_I^\times(i)^*)$.

Proof Since part (2) is proved in a similar way, we only give a proof of part (1). The existence of the required $\mathbf{M}^{[1]}$ is already proved in Lemma 9. Let $\mathbf{N}^{[1]}$ be another *e*-BZ datum which satisfy conditions (a) and (b). For the uniqueness, it suffices to show that $\mathbf{M}^{[1]}_{\mathbf{k}} = \mathbf{N}^{[1]}_{\mathbf{k}}$ for an arbitrary subinterval $\mathbf{k} = [s+1, t]$ of $\widetilde{I}$, where $\widetilde{I} = [n+1, n+m+1]$. If $[s+1, t] \in \mathcal{M}_I^\times \setminus \mathcal{M}_I^\times(i)$, then the assertion is obvious from conditions (a) and (b). Assume that $[s+1, t] \in \mathcal{M}_I^\times(i)$. Here we note that such an interval $[s+1, t]$ has the following form:

$$[s+1, i], \quad n \le s \le i-1.$$

If $s = n$, then we have $(\mathbf{M}^{[1]})_{[n+1,i]} = (\mathbf{N}^{[1]})_{[n+1,i]} = 0$ by the normalization condition. Now, suppose that $(\mathbf{M}^{[1]})_{[s,i]} = (\mathbf{N}^{[1]})_{[s,i]}$. Then, by the tropical Plücker relation for $\mathbf{k} = [s+1, i-1]$ and $s < i < i+1$, we have

$$\begin{aligned}&\big(\mathbf{M}^{[1]}\big)_{[s+1,i]}+\big(\mathbf{M}^{[1]}\big)_{[s,i-1]\cup\{i+1\}}\\&\quad+\min\big\{\big(\mathbf{M}^{[1]}\big)_{[s,i-1]}+\big(\mathbf{M}^{[1]}\big)_{[s+1,i+1]},\big(\mathbf{M}^{[1]}\big)_{[s+1,i-1]\cup\{i+1\}}+\big(\mathbf{M}^{[1]}\big)_{[s,i]}\big\}.\end{aligned}$$

Also, by conditions (a) and (b), we have

$$\big(\mathbf{M}^{[1]}\big)_{[s,i-1]\cup\{i+1\}}=M_{[s,i-1]\cup\{i+1\}}+1,$$
$$\big(\mathbf{M}^{[1]}\big)_{[s+1,i-1]\cup\{i+1\}}=M_{[s+1,i-1]\cup\{i+1\}}+1,$$

$$\big(\mathbf{M}^{[1]}\big)_{[s,i-1]}=M_{[s,i-1]},\qquad \big(\mathbf{M}^{[1]}\big)_{[s+1,i+1]}=M_{[s+1,i+1]}.$$

Therefore, we deduce that

$$\begin{aligned}\big(\mathbf{M}^{[1]}\big)_{[s+1,i]}&=-M_{[s,i-1]\cup\{i+1\}}-1\\&\quad+\min\big\{M_{[s,i-1]}+M_{[s+1,i+1]},M_{[s+1,i-1]\cup\{i+1\}}+1+\big(\mathbf{M}^{[1]}\big)_{[s,i]}\big\}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\big(\mathbf{N}^{[1]}\big)_{[s+1,i]}&=-M_{[s,i-1]\cup\{i+1\}}-1\\&\quad+\min\big\{M_{[s,i-1]}+M_{[s+1,i+1]},M_{[s+1,i-1]\cup\{i+1\}}+1+\big(\mathbf{N}^{[1]}\big)_{[s,i]}\big\}.\end{aligned}$$

Consequently, we obtain $(\mathbf{M}^{[1]})_{[s+1,i]}=(\mathbf{N}^{[1]})_{[s+1,i]}$. This proves the proposition. □

Corollary 5 *For* $\mathbf{k}\in\mathcal{M}_I^\times(i)$, *we have*

$$\big(\mathbf{M}^{[2]}\big)_{\mathbf{k}}=\min\big\{M_{\mathbf{k}}+1,M_{\sigma_i\mathbf{k}}+\varepsilon_i(\mathbf{M})\big\}.$$

Proof From the uniqueness of $\mathbf{M}^{[2]}$, it follows that $\mathbf{M}^{[2]}=(\widetilde{f}_i^*(\mathbf{M}^\sharp))^\sharp$. Therefore,

$$\begin{aligned}\big(\mathbf{M}^{[2]}\big)_{\mathbf{k}}&=\big(\widetilde{f}_i^*\big(\mathbf{M}^\sharp\big)\big)^\sharp_{\mathbf{k}}\\&=\big(\widetilde{f}_i^*\big(\mathbf{M}^\sharp\big)\big)_{\mathbf{k}^c}-\big\langle\mathrm{wt}^\vee\big(\widetilde{f}_i^*\big(\mathbf{M}^\sharp\big)\big),\mathbf{k}\big\rangle_I\\&=\min\big\{\big(\mathbf{M}^\sharp\big)_{\mathbf{k}^c},\big(\mathbf{M}^\sharp\big)_{\sigma_i\mathbf{k}^c}+c_i^*\big(\mathbf{M}^\sharp\big)\big\}-\big\langle\mathrm{wt}^\vee(\mathbf{M}),\mathbf{k}\big\rangle_I+\langle h_i,\mathbf{k}\rangle_I\\&=\min\big\{M_{\mathbf{k}}+1,M_{\sigma_i\mathbf{k}}+\big\langle\mathrm{wt}^\vee(\mathbf{M}),\sigma_i\mathbf{k}-\mathbf{k}\big\rangle_I+c_i^*\big(\mathbf{M}^\sharp\big)+1\big\}.\end{aligned}$$

Here, we remark that $\langle h_i,\mathbf{k}\rangle=1$ since $\mathbf{k}\in\mathcal{M}_I^\times(i)$. Let us compute the second term on the right-hand side of the last equality. Note that $\sigma_i\mathbf{k}-\mathbf{k}=-\langle h_i,\mathbf{k}\rangle_I\alpha_i^I=-\alpha_i^I$. Hence we deduce that

$$\begin{aligned}\text{the second term}&=M_{\sigma_i\mathbf{k}}-\big\langle\mathrm{wt}^\vee(\mathbf{M}),\alpha_i^I\big\rangle_I+\big\langle h_i,\mathrm{wt}\big(\mathbf{M}^\sharp\big)\big\rangle_I+\varepsilon_i^*\big(\mathbf{M}^\sharp\big)-1+1\\&=M_{\sigma_i\mathbf{k}}+\varepsilon_i(\mathbf{M}).\end{aligned}$$

This proves the corollary. □

3.2 Ordinary Crystal Structure on $\mathcal{BZ}_I^e$

We define another crystal structure on $\mathcal{BZ}_I^e$ via the bijections $\mathcal{B}_I \xrightarrow{\sim} \bigsqcup_{\nu \in Q_+} \operatorname{Irr} \Lambda(\nu) \xrightarrow{\sim} \mathcal{BZ}_I^e$. Let $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_I^\times}$ be an e-BZ datum. Then, there exists a unique Lusztig datum $\mathbf{a}$ (or equivalently, a unique irreducible Lagrangian $\Lambda_{\mathbf{a}}$) such that $\mathbf{M} = \mathbf{M}(\mathbf{a})$. Now we define

$$\varepsilon_i(\mathbf{M}) := \varepsilon_i(\mathbf{a}) = \varepsilon_i(\Lambda_{\mathbf{a}}), \qquad \varphi_i(\mathbf{M}) := \varphi_i(\mathbf{a}) = \varphi_i(\Lambda_{\mathbf{a}}),$$

$$\widetilde{e}_i\mathbf{M} := \begin{cases} \mathbf{M}(\widetilde{e}_i\mathbf{a}) = \mathbf{M}(\widetilde{e}_i\Lambda_{\mathbf{a}}) & \text{if } \varepsilon_i(\mathbf{a}) > 0, \\ 0 & \text{if } \varepsilon_i(\mathbf{a}) = 0, \end{cases} \quad \text{and} \quad \widetilde{f}_i\mathbf{M} := \mathbf{M}(\widetilde{f}_i\mathbf{a}) = \mathbf{M}(\widetilde{f}_i\Lambda_{\mathbf{a}}).$$

By the definitions, it is obvious that the set $\mathcal{BZ}_I^e$, equipped with the maps wt, ε_i, φ_i, $\widetilde{e}_i$, $\widetilde{f}_i$, is a $U_q(\mathfrak{sl}_{m+1})$-crystal, and the bijections above give rise to isomorphisms of crystals

$$(\mathcal{B}_I; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i) \xrightarrow{\sim} \Big(\bigsqcup_{\nu \in Q_+} \operatorname{Irr} \Lambda(\nu); \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\Big)$$
$$\xrightarrow{\sim} \big(\mathcal{BZ}_I^e; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\big).$$

We call this crystal structure the ordinary crystal structure on $\mathcal{BZ}_I^e$.

Lemma 10 *For $\Lambda \in \operatorname{Irr} \Lambda(\nu)$, we have*

$$\mathbf{M}\big(\Lambda^*\big) = \mathbf{M}(\Lambda)^\sharp.$$

Proof We write $\nu = \sum_{i \in I} \nu_i \alpha_i^I$. Let B be a general point of Λ. Then its transpose ${}^tB$ is also a general point of Λ^*. Therefore, we compute:

$$\begin{aligned}
M_{\mathbf{k}}\big(\Lambda^*\big) &= M_{\mathbf{k}}\big({}^tB\big) \\
&= -\dim_{\mathbb{C}} \operatorname{Coker}\Big(\bigoplus_{k \in \mathrm{out}(\mathbf{k})} V(\nu)_k \xrightarrow{\oplus {}^tB_\mu} \bigoplus_{l \in \mathrm{in}(\mathbf{k})} V(\nu)_l\Big) \\
&= -\dim_{\mathbb{C}} \operatorname{Ker}\Big(\bigoplus_{l \in \mathrm{in}(\mathbf{k})} V(\nu)_l \xrightarrow{\oplus B_\mu} \bigoplus_{k \in \mathrm{out}(\mathbf{k})} V(\nu)_k\Big) \\
&= -\dim_{\mathbb{C}} \operatorname{Coker}\Big(\bigoplus_{l \in \mathrm{out}(\mathbf{k}^c)} V(\nu)_l \xrightarrow{\bigoplus B_\mu} \bigoplus_{k \in \mathrm{in}(\mathbf{k}^c)} V(\nu)_k\Big) \\
&\quad + \sum_{k \in \mathrm{in}(\mathbf{k}^c)} \dim_{\mathbb{C}} V(\nu)_k - \sum_{l \in \mathrm{out}(\mathbf{k}^c)} \dim_{\mathbb{C}} V(\nu)_l \\
&= M_{\mathbf{k}^c}(\Lambda) + \sum_{k \in \mathrm{out}(\mathbf{k})} \nu_k - \sum_{l \in \mathrm{in}(\mathbf{k})} \nu_l.
\end{aligned}$$

Here, for the third equality, we take the transpose ${}^t({}^tB) = B$ of ${}^tB$. By the definitions of $\mathrm{out}(\mathbf{k})$ and $\mathrm{in}(\mathbf{k})$, we have

$$\langle h_p, \mathbf{k}\rangle_I = \begin{cases} 1 & \text{if } p \in \mathrm{out}(\mathbf{k}), \\ -1 & \text{if } p \in \mathrm{in}(\mathbf{k}), \\ 0 & \text{otherwise,} \end{cases}$$

and hence

$$\left\langle \mathrm{wt}^\vee\big(\mathbf{M}(\Lambda)\big), \mathbf{k}\right\rangle_I = -\sum_{k \in \mathrm{out}(\mathbf{k})} \nu_k + \sum_{l \in \mathrm{in}(\mathbf{k})} \nu_l.$$

From these, it follows that

$$M_{\mathbf{k}}(\Lambda) = M_{\mathbf{k}^c}(\Lambda) - \left\langle \mathrm{wt}^\vee\big(\mathbf{M}(\Lambda)\big), \mathbf{k}\right\rangle_I = \big(\mathbf{M}(\Lambda)^\sharp\big)_{\mathbf{k}}.$$

This proves the lemma. □

Proposition 9 *As operators on $\mathcal{BZ}_I^e$,*

$$\widetilde{e}_i = \sharp \circ \widetilde{e}_i^* \circ \sharp \quad \textit{and} \quad \widetilde{f}_i = \sharp \circ \widetilde{f}_i^* \circ \sharp.$$

Proof Let $\mathbf{M} \in \mathcal{BZ}_I^e$. We only give a proof of the first equality, since the proof of the second one is similar.

If $\varepsilon_i(\mathbf{M}) = \varepsilon_i^*(\mathbf{M}^\sharp) = 0$, then $\widetilde{e}_i\mathbf{M} = (\widetilde{e}_i^*(\mathbf{M}^\sharp))^\sharp = 0$ by the definitions. So, assume that $\varepsilon_i(\mathbf{M}) = \varepsilon_i^*(\mathbf{M}^\sharp) > 0$. Let Λ be a unique irreducible Lagrangian such that $\mathbf{M} = \mathbf{M}(\Lambda)$. Since the bijection $\Xi_I : \Lambda \mapsto \mathbf{M}(\Lambda)$ is an isomorphism with respect to both the ordinary and $*$-crystal structures, we have

$$\begin{aligned}\big(\widetilde{e}_i^*(\mathbf{M}^\sharp)\big)^\sharp &= \big(\widetilde{e}_i^*\big(\mathbf{M}(\Lambda)^\sharp\big)\big)^\sharp = \big(\widetilde{e}_i^*\big(\mathbf{M}\big(\Lambda^*\big)\big)\big)^\sharp = \mathbf{M}\big(\widetilde{e}_i^*\Lambda^*\big)^\sharp = \mathbf{M}\big((\widetilde{e}_i\Lambda)^*\big)^\sharp \\ &= \mathbf{M}(\widetilde{e}_i\Lambda) = \widetilde{e}_i\mathbf{M}(\Lambda) = \widetilde{e}_i\mathbf{M},\end{aligned}$$

as desired. □

The following corollary is obvious by the consideration above.

Corollary 6 (1) *Let $\mathbf{M} \in \mathcal{BZ}_I^e$, and assume that $\varepsilon_i(\mathbf{M}) > 0$. Then, $\widetilde{e}_i\mathbf{M}$ is a unique e-BZ datum such that*

(a) $(\widetilde{e}_iM)_{\mathbf{k}} = M_{\mathbf{k}} + 1$ *for* $\mathbf{k} \in \mathcal{M}_I^\times(i)^*$,
(b) $(\widetilde{e}_iM)_{\mathbf{k}} = M_{\mathbf{k}}$ *for* $\mathbf{k} \in \mathcal{M}_I^\times \setminus (\mathcal{M}_I^\times(i) \cup \mathcal{M}_I^\times(i)^*)$.

(2) *For every* $\mathbf{M} \in \mathcal{BZ}_I^e$,

$$(\widetilde{f}_i\mathbf{M})_{\mathbf{k}} = \begin{cases} \min\{M_{\mathbf{k}} + 1, M_{\sigma_i\mathbf{k}} + \varepsilon_i(\mathbf{M})\} & \textit{if } \mathbf{k} \in \mathcal{M}_I^\times(i). \\ M_{\mathbf{k}} - 1 & \textit{if } \mathbf{k} \in \mathcal{M}_I^\times(i)^*, \\ M_{\mathbf{k}} & \textit{if } \mathbf{k} \in \mathcal{M}_I^\times \setminus (\mathcal{M}_I^\times(i) \cup \mathcal{M}_I^\times(i)^*). \end{cases}$$

Proposition 10 *Let* $i, j \in I$, *and* $\mathbf{M} \in \mathcal{BZ}_I^e$. *Set* $c := \varepsilon_i(\mathbf{M})$ *and* $\mathbf{M}' = \widetilde{e}_p^c \mathbf{M}$.

(1) *We have*

$$\varepsilon_i^*(\mathbf{M}) = \max\left\{\varepsilon_i^*\left(\mathbf{M}'\right), c - \left\langle h_i, \mathrm{wt}\left(\mathbf{M}'\right)\right\rangle_I\right\}.$$

(2) *If* $i \neq j$ *and* $\varepsilon_j^*(\mathbf{M}) > 0$, *then*

$$\varepsilon_j\left(\widetilde{e}_p^*\mathbf{M}\right) = c, \qquad \widetilde{e}_j^c\left(\widetilde{e}_i^*\mathbf{M}\right) = \widetilde{e}_i^*\mathbf{M}'.$$

(3) *If* $\varepsilon_i^*(\mathbf{M}) > 0$, *then we have*

$$\varepsilon_i\left(\widetilde{e}_i^*\mathbf{M}\right) = \begin{cases} \varepsilon_i(\mathbf{M}) & \text{if } \varepsilon_i^*(\mathbf{M}') \geq c - \langle h_i, \mathrm{wt}(\mathbf{M}')\rangle_I, \\ \varepsilon_i(\mathbf{M}) - 1 & \text{if } \varepsilon_i^*(\mathbf{M}') < c - \langle h_i, \mathrm{wt}(\mathbf{M}')\rangle_I, \end{cases}$$

and

$$\widetilde{e}_i^{c'}\left(\widetilde{e}_i^*\mathbf{M}\right) = \begin{cases} \widetilde{e}_i^*\mathbf{M}' & \text{if } \varepsilon_i^*(\mathbf{M}') \geq c - \langle h_i, \mathrm{wt}(\mathbf{M}')\rangle_I, \\ \mathbf{M}' & \text{if } \varepsilon_i^*(\mathbf{M}') < c - \langle h_i, \mathrm{wt}(\mathbf{M}')\rangle_I. \end{cases}$$

Here, we set $c' := \varepsilon_i(\widetilde{e}_i^*\mathbf{M})$.

Proof Recall that the bijection $\Xi_I : \Lambda \mapsto \mathbf{M}(\Lambda)$ is an isomorphism with respect to both the ordinary and $*$-crystal structures. Therefore, all of the desired equations follow immediately from the corresponding ones, which hold in $\bigsqcup_{\nu \in Q_+} \mathrm{Irr}\,\Lambda(\nu)$ (see [6]). This proves the proposition. □

4 Ordinary Crystal Structure on $(\mathcal{BZ}_{\mathbb{Z}}^e)^\sigma$

4.1 Definition of Ordinary Kashiwara Operators on $\mathcal{BZ}_{\mathbb{Z}}^e$

For $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}} \in \mathcal{BZ}_{\mathbb{Z}}^e$ and $p \in \mathbb{Z}$, we set

$$\varepsilon_p(\mathbf{M}) := -M_{\mathbf{k}(\sigma_p \Lambda_p)}.$$

Observe that if $\mathbf{k}(\sigma_p \Lambda_p) \in \mathcal{M}_{\mathbb{Z}}(I)$, then

$$\varepsilon_p(\mathbf{M}) = -M_{\mathbf{k}(\sigma_p \Lambda_p)} = -(\mathbf{M}_I)_{\mathrm{res}_I(\mathbf{k}(\sigma_p \Lambda_p))} = -(\mathbf{M}_I)_{\mathbf{k}(\sigma_p \varpi_p^I)} = \varepsilon_p(\mathbf{M}_I).$$

First, let us define the ordinary raising Kashiwara operators on $\mathcal{BZ}_{\mathbb{Z}}^e$. If $\varepsilon_p(\mathbf{M}) > 0$, the we define a new collection $\mathbf{M}^{[1]} = (M_{\mathbf{k}}^{[1]})_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}}$ of integers as follows. For a given $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$, take a finite interval I in $\mathbb{Z}$ such that $\mathbf{k}, \sigma_p \mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(I)$, and $I \in \mathrm{Int}^e(\mathbf{M}; \mathbf{k}(\Lambda_p)) \cap \mathrm{Int}^e(\mathbf{M}; \mathbf{k}(\sigma_p \Lambda_p)) \cap \mathrm{Int}^e(\mathbf{M}; \mathbf{k}(\Lambda_{p+1})) \cap \mathrm{Int}^e(\mathbf{M}; \mathbf{k}(\Lambda_{p-1}))$. Then, we set

$$M_{\mathbf{k}}^{[1]} := (\widetilde{e}_p \mathbf{M}_I)_{\mathrm{res}_I(\mathbf{k})}.$$

Here, $\widetilde{e}_p$ is the ordinary raising Kashiwara operator on $\mathcal{BZ}_I^e$ defined in the previous section. Now, we define the action of $\widetilde{e}_p$ on $\mathcal{BZ}_{\mathbb{Z}}^e$ by

$$\widetilde{e}_p\mathbf{M} := \begin{cases} \mathbf{M}^{[1]} & \text{if } \varepsilon_p(\mathbf{M}) > 0, \\ 0 & \text{if } \varepsilon_p(\mathbf{M}) = 0. \end{cases}$$

Note that the definition above does not depend on the choice of I.

Next, let us define the ordinary lowering Kashiwara operators on $\mathcal{BZ}_{\mathbb{Z}}^e$. For $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}} \in \mathcal{BZ}_{\mathbb{Z}}^e$ and $p \in \mathbb{Z}$, we define a new collection $\widetilde{f}_p\mathbf{M} = (M_{\mathbf{k}}^{[2]})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}}$ of integers as follows. For a given $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$, take a finite interval I in $\mathbb{Z}$ such that $\mathbf{k}, \sigma_p\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(I)$, and $I \in \mathrm{Int}^e(\mathbf{M}; \mathbf{k}(\Lambda_p)) \cap \mathrm{Int}^e(\mathbf{M}; \mathbf{k}(\sigma_p\Lambda_p))$. Then we set

$$M_{\mathbf{k}}^{[2]} := (\widetilde{f}_p\mathbf{M}_I)_{\mathrm{res}_I(\mathbf{k})}.$$

Here, $\widetilde{f}_p$ is the ordinary lowering Kashiwara operator on $\mathcal{BZ}_I^e$ defined in the previous section.

For $p \in \mathbb{Z}$, we set

$$\mathcal{M}_{\mathbb{Z}}(p) := \{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}} \mid p \in \mathbf{k},\ p+1 \notin \mathbf{k}\},$$
$$\mathcal{M}_{\mathbb{Z}}(p)^* := \{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}} \mid p \notin \mathbf{k},\ p+1 \in \mathbf{k}\}.$$

The following lemma follows easily from the definitions.

Lemma 11 *Let* $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}} \in \mathcal{BZ}_{\mathbb{Z}}^e$.

(1) *If* $\varepsilon_p(\mathbf{M}) > 0$, *then*
  (a) $(\widetilde{e}_pM)_{\mathbf{k}} = M_{\mathbf{k}} + 1$ *for* $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{\times}(p)^*$,
  (b) $(\widetilde{e}_pM)_{\mathbf{k}} = M_{\mathbf{k}}$ *for* $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{\times} \setminus (\mathcal{M}_{\mathbb{Z}}^{\times}(p) \cup \mathcal{M}_{\mathbb{Z}}^{\times}(p)^*)$.

(2) *For each* $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^e$, *we have*

$$(\widetilde{f}_p\mathbf{M})_{\mathbf{k}} = \begin{cases} \min\{M_{\mathbf{k}} + 1, M_{\sigma_p\mathbf{k}} + \varepsilon_p(\mathbf{M})\} & \textit{if } \mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{\times}(p), \\ M_{\mathbf{k}} - 1 & \textit{if } \mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{\times}(p)^*, \\ M_{\mathbf{k}} & \textit{if } \mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{\times} \setminus (\mathcal{M}_{\mathbb{Z}}^{\times}(p) \cup \mathcal{M}_{\mathbb{Z}}^{\times}(p)^*). \end{cases}$$

Proposition 11 (1) *If* $\varepsilon_p(\mathbf{M}) > 0$, *then* $\widetilde{e}_p\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^e$.
(2) *For every* $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^e$ *and* $p \in \mathbb{Z}$, *we have* $\widetilde{f}_p\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^e$.

In the next subsection, we give a proof of this proposition.

4.2 Proof of Proposition 11

Since part (2) is obtained in a similar way, we only give a proof of part (1). We will only verify that condition (2-b) in Definition 4 is satisfied for $\widetilde{e}_p\mathbf{M}$ with $\varepsilon_p(\mathbf{M}) > 0$, since the remaining ones are easily verified. Namely, we will prove the following:

Claim 1 *Assume that* $\varepsilon_p(\mathbf{M}) > 0$, *and let* $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$. *Then, there exists a finite interval* I *in* $\mathbb{Z}$ *such that for every* $J \supset I$,

$$(\widetilde{e}_p\mathbf{M})_{\Omega_I(\mathbf{k})} = (\widetilde{e}_p\mathbf{M})_{\Omega_J(\mathbf{k})}. \tag{4.1}$$

Take a finite interval $K = [n_K + 1, n_K + m_K]$ in $\mathbb{Z}$ such that $\mathbf{k}, \sigma_p\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(K)$, and $K \in \mathrm{Int}^e(\mathbf{M}; \mathbf{k}(\Lambda_p)) \cap \mathrm{Int}^e(\mathbf{M}; \mathbf{k}(\sigma_p\Lambda_p)) \cap \mathrm{Int}^e(\mathbf{M}; \mathbf{k}(\Lambda_{p+1})) \cap \mathrm{Int}^e(\mathbf{M}; \mathbf{k}(\Lambda_{p-1}))$. Set $K' := [n_K, n_k + m_K + 1]$. Since $\mathcal{M}_{\mathbb{Z}}(K')$ is a finite set, we can take a finite interval $I = [n_I + 1, n_I + m_I]$ in $\mathbb{Z}$ with the following properties:

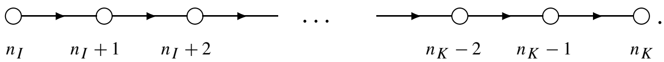
$$I \supset K, \quad \text{and} \quad I \in \mathrm{Int}^e(\mathbf{M}; \mathbf{n}) \quad \text{for all } \mathbf{n} \in \mathcal{M}_{\mathbb{Z}}\big(K'\big). \tag{4.2}$$

In the following, we will show that such an interval I satisfies the condition in Claim 1. We may assume that $J = \{n_I\} \cup I$ (case (i)) or $J = I \cup \{n_I + m_I + 1\}$ (case (ii)). We show Eq. (4.1) only in case (i); the assertion in case (ii) follows by a similar (and easier) argument.

Before starting a proof, we give some lemmas. We set $\mathbf{a} = (a_{i,j})_{(i,j)\in\Delta_I^+} := \Psi_I^{-1}(\mathbf{M}_I) \in \mathcal{B}_I$, $\mathbf{b} = (b_{k,l})_{(k,l)\in\Delta_J^+} := \Psi_J^{-1}(\mathbf{M}_J) \in \mathcal{B}_J$, $\Lambda_{\mathbf{a}} := \Xi_I^{-1}(\mathbf{M}_I) \in \mathrm{Irr}\,\Lambda(\nu_I)$, and $\Lambda_{\mathbf{b}} := \Xi_J^{-1}(\mathbf{M}_J) \in \mathrm{Irr}\,\Lambda(\nu_J)$, where $\nu_I = \mathrm{wt}\,(\mathbf{M}_I)$ and $\nu_J = \mathrm{wt}\,(\mathbf{M}_J)$. Let $B^I = (B^I_\tau) \in \Lambda_{\mathbf{a}}$ and $B^J = (B^J_\tau) \in \Lambda_{\mathbf{b}}$ be general points.

Lemma 12 *Let* $\sigma(n_I \to n_K)$ *be the path from* n_I *to* n_K *defined as follows*:

$\sigma(n_I \to n_K)$:



Then, the corresponding composite map $B^J_{\sigma(n_I\to n_K)} : V(\nu_J)_{n_I} \to V(\nu_J)_{n_K}$ *is a zero map*.

Proof Since $\Lambda_{n_K} \in \mathcal{M}_{\mathbb{Z}}(K')$, we have

$$\begin{aligned}(\mathbf{M}_I)_{[n_K+1, n_I+m_I+1]} &= M_{\Omega_I(\Lambda_{n_K})} \\ &= M_{\Omega_J(\Lambda_{n_K})} = (\mathbf{M}_J)_{[n_K+1, n_I+m_I+1]} = -(\nu_J)_{n_K}.\end{aligned}$$

Also, by the definition, we have

$$\begin{aligned}(\mathbf{M}_I)_{[n_K+1, n_I+m_I+1]} &= (\mathbf{M}_J)_{\{n_I\}\cup[n_K+1, n_I+m_I+1]} \\ &= -\dim_{\mathbb{C}} \mathrm{Coker}\big(V(\nu_J)_{n_I} \xrightarrow{B^J} V(\nu_J)_{n_K}\big).\end{aligned}$$

From these, we obtain

$$(\nu_J)_{n_K} = \dim_{\mathbb{C}} \mathrm{Coker}\big(V(\nu_J)_{n_I} \xrightarrow{B^J} V(\nu_J)_{n_K}\big).$$

This shows that the map $B^J_{\sigma(n_I\to n_K)}$ is a zero map, as desired. □

Let $\mathbf{a}^* = (a^*_{i,j})_{(i,j)\in\Delta_I^+} := \Psi_I^{-1}(\mathbf{M}_I^\sharp)$ and $\mathbf{b}^* = (b^*_{k,l})_{(k,l)\in\Delta_J^+} := \Psi_J^{-1}(\mathbf{M}_J^\sharp)$.

Lemma 13 *We have* $b^*_{n_I,l} = 0$ *for* $n_K + 1 \le l \le n_I + m_I + 1$.

Proof First, note that

$$\big(\mathbf{M}_J^\sharp\big)_{[n_I+1,n_I+m_I+1]} = -\sum_{l=n_I+1}^{n_I+m_I+1} b^*_{n_I,l} \quad \text{and} \quad \big(\mathbf{M}_J^\sharp\big)_{[n_I+1,n_K]} = -\sum_{l=n_I+1}^{n_K} b^*_{n_I,l}.$$

Next, by Lemma 10, we have

$$\big(\mathbf{M}_J^\sharp\big)_{[n_I+1,n_K]} = M_{[n_I+1,n_K]}\big(\Lambda^*_{\mathbf{b}}\big) = -\dim_{\mathbb{C}} \mathrm{Coker}\big(V(\nu_J)_{n_K} \xrightarrow{{}^t(B_J)} V(\nu_J)_{n_I}\big).$$

Since $({}^t(B^J))_{\sigma(n_K\to n_I)} = {}^t(B^J_{\sigma(n_I\to n_K)}) = 0$ by Lemma 12, we deduce that

$$\big(\mathbf{M}_J^\sharp\big)_{[n_I+1,n_K]} = -(\nu_J)_{n_I}.$$

Therefore, we deduce that

$$\sum_{l=n_K+1}^{n_I+m_I+1} b^*_{n_I,l} = -\big(\mathbf{M}_J^\sharp\big)_{[n_I+1,n_I+m_I+1]} + \big(\mathbf{M}_J^\sharp\big)_{[n_I+1,n_K]} = 0.$$

Since $b^*_{n_I,l}$ is nonnegative for all l, we obtain the desired equality. □

Lemma 14 *For every* $n_I + 1 \le s \le t \le n_I + m_I + 1$, *we have*

$$\big(\mathbf{M}_J^\sharp\big)_{[s,t]} = \big(\mathbf{M}_I^\sharp\big)_{[s,t]} + (\nu_I)_{s-1} - (\nu_J)_{s-1} - (\nu_I)_t + (\nu_J)_t.$$

Here, by convention, $(\nu_I)_{s-1} = 0$ *for* $s = n_I + 1$ *and* $(\nu_I)_t = (\nu_J)_t = 0$ *for* $t = n_I + m_I + 1$.

Proof We assume that $n_I + 1 < s \le t < n_I + m_I + 1$; in the remaining case, the desired equation follows by a similar (and easier) argument.

Write $\mathbf{n}_I = [s,t] \in \mathcal{M}_I^\times$. Then we have

$$\begin{aligned}
\big(\mathbf{M}_I^\sharp\big)_{\mathbf{n}_I} &= -\dim_{\mathbb{C}} \mathrm{Coker}\big(V(\nu_I)_t \xrightarrow{{}^t(B_I)} V(\nu_I)_{s-1}\big) \\
&= -\dim_{\mathbb{C}} \mathrm{Ker}\big(V(\nu_I)_{s-1} \xrightarrow{B_I} V(\nu_I)_t\big) \\
&= -\dim_{\mathbb{C}} \mathrm{Coker}\big(V(\nu_I)_{s-} \xrightarrow{B_I} V(\nu_I)_t\big) - (\nu_I)_{s-1} + (\nu_I)_t \\
&= (\mathbf{M}_I)_{\mathbf{n}_I^c} - (\nu_I)_{s-1} + (\nu_I)_t.
\end{aligned}$$

Similarly, we have

$$\big(\mathbf{M}_J^\sharp\big)_{\mathbf{n}_J} = (\mathbf{M}_J)_{\mathbf{n}_J^c} - (\nu_J)_{s-1} + (\nu_J)_t.$$

Here we set $\mathbf{n}_J := [s, t] \in \mathcal{M}_J^{\times}$. Since $\mathbf{n}_J^c = \{n_I\} \cup \mathbf{n}_I^c$, we obtain

$$(\mathbf{M}_I)_{\mathbf{n}_I^c} = (\mathbf{M}_I)_{\mathbf{n}_J^c}.$$

Therefore, we deduce that

$$\begin{aligned}\left(\mathbf{M}_J^{\sharp}\right)_{\mathbf{n}_J} &= (\mathbf{M}_I)_{\mathbf{n}_I^c} - (\nu_J)_{s-1} + (\nu_J)_t \\ &= \left(\mathbf{M}_I^{\sharp}\right)_{\mathbf{n}_I} + (\nu_I)_{s-1} - (\nu_I)_t - (\nu_J)_{s-1} + (\nu_J)_t.\end{aligned}$$

This proves the lemma. □

Corollary 7 *For every $n_I + 1 \le i < j \le n_I + m_I + 1$, we have*

$$b_{i,j}^* = a_{i,j}^*.$$

Proof The desired equality follows easily from Lemma 14 and the chamber ansatz maps (see [2]):

$$b_{i,j}^* = \left(\mathbf{M}_J^{\sharp}\right)_{[i,j]} + \left(\mathbf{M}_J^{\sharp}\right)_{[i+1,j-1]} - \left(\mathbf{M}_J^{\sharp}\right)_{[i+,j]} - \left(\mathbf{M}_J^{\sharp}\right)_{[i,j-1]},$$

$$a_{i,j}^* = \left(\mathbf{M}_I^{\sharp}\right)_{[i,j]} + \left(\mathbf{M}_I^{\sharp}\right)_{[i+1,j-1]} - \left(\mathbf{M}_I^{\sharp}\right)_{[i+,j]} - \left(\mathbf{M}_I^{\sharp}\right)_{[i,j-1]}.$$

□

Proposition 12 *We have*

$$\left(\left(\widetilde{e}_p^*(\mathbf{M}_I^{\sharp})\right)_K\right)_{[p+1,n_K+m_K+1]} = \left(\left(\widetilde{e}_p^*(\mathbf{M}_J^{\sharp})\right)_K\right)_{[p+1,n_K+m_K+1]}.$$

Proof Note that the desired equation is equivalent to the following:

$$\left(\widetilde{e}_p^*(\mathbf{M}_I^{\sharp})\right)_{[n_I+1,n_K]\cup[p+1,n_K+m_K+1]} = \left(\widetilde{e}_p^*(\mathbf{M}_J^{\sharp})\right)_{[n_I,n_K]\cup[p+1,n_K+m_K+1]}. \tag{4.3}$$

Let $\mathbf{a}' = (a'_{i,j})_{(i,j)\in\Delta_I^+} := \Psi_I^{-1}(\widetilde{e}_p^*(\mathbf{M}_I^{\sharp}))$ and $\mathbf{b}' = (b'_{k,l})_{(k,l)\in\Delta_J^+} := \Psi_J^{-1}(\widetilde{e}_p^*(\mathbf{M}_J^{\sharp}))$, and set $\mathbf{l} := [n_I + 1, n_K] \cup [p + 1, n_K + m_K + 1]$ and $\mathbf{m} := [n_I, n_K] \cup [p + 1, n_K + m_K + 1]$. Then, Eq. (4.3) is equivalent to the following:

$$M_{\mathbf{l}}\left(\mathbf{a}'\right) = M_{\mathbf{m}}\left(\mathbf{b}'\right). \tag{4.4}$$

Observe that by Lemma 13, Corollary 7, and the definition of the action of $\widetilde{e}_p^*$,

$$a'_{i,j} = b'_{i,j} \quad \left((i,j) \in \Delta_I^+\right) \quad \text{and} \quad b'_{n_I,l} = 0 \quad (n_K + 1 \le l \le n_I + m_I + 1). \tag{4.5}$$

Since the Maya diagrams $\mathbf{l}$ and $\mathbf{m}$ satisfy the condition of Lemma 1(1) with $s = n_K$, we have

$$M_{\mathbf{l}}(\mathbf{a}') = -\sum_{j=p+1}^{n_K+m_K+1}\sum_{i=n_I+1}^{l-1} a'_{i,j} + \min\left\{ \sum_{t=n_K+1}^{2n_K+m_K-p+1}\sum_{s=n_I+1}^{t-1} a'_{c_{s,t},c_{s,t}+(t-s)} \,\middle|\, \begin{array}{l} C=(c_{s,t}) \text{ is} \\ \text{an } \mathbf{l}\text{-tableau} \end{array} \right\},$$

and

$$M_{\mathbf{m}}(\mathbf{b}') = -\sum_{l=p+1}^{n_K+m_K+1}\sum_{k=n_I}^{l-1} b'_{k,l} + \min\left\{ \sum_{t=n_K+1}^{2n_K+m_K-p+1}\sum_{s=n_I}^{t-1} b'_{d_{s,t},d_{s,t}+(t-s)} \,\middle|\, \begin{array}{l} D=(d_{s,t}) \text{ is} \\ \text{an } \mathbf{m}\text{-tableau} \end{array} \right\}.$$

Here, by (4.5),

$$\sum_{j=p+1}^{n_K+m_K+1}\sum_{i=n_I+1}^{l-1} a'_{i,j} = \sum_{l=p+1}^{n_K+m_K+1}\sum_{k=n_I}^{l-1} b'_{k,l}.$$

Now, let $D = (d_{s,t})_{n_I \le s \le t \le 2n_k+m_k-p+1}$ be an **m**-tableau, and define two upper-triangular matrices

$$D' = (d'_{s,t})_{n_I \le s \le t \le 2n_k+m_k-p+1} \quad \text{and} \quad D'' = (d''_{s,t})_{n_I+1 \le s \le t \le 2n_k+m_k-p+1}$$

by

$$d'_{s,t} := \begin{cases} n_I & \text{if } s = n_I, \\ d_{p,q} & \text{otherwise,} \end{cases} \quad \text{and} \quad d''_{s,t} := d_{s,t}.$$

Then, D' is an **m**-tableau and D'' is an **l**-tableau. From the definitions and (4.5), we see that

$$\begin{aligned} \sum_{t=n_K+1}^{2n_K+m_K-p+1}\sum_{s=n_I}^{t-1} b'_{d_{s,t},d_{s,t}+(t-s)} &\ge \sum_{t=n_K+1}^{2n_K+m_K-p+1}\sum_{s=n_I}^{t-1} b'_{d'_{s,t},d'_{s,t}+(t-s)} \\ &= \sum_{t=n_K+1}^{2n_K+m_K-p+1}\sum_{s=n_I+1}^{t-1} b'_{d''_{s,t},d''_{s,t}+(t-s)} \\ &= \sum_{t=n_K+1}^{2n_K+m_K-p+1}\sum_{s=n_I+1}^{t-1} a'_{d''_{s,t},d''_{s,t}+(t-s)}. \end{aligned}$$

Therefore, we obtain

$$\min\left\{\sum_{t=n_K+1}^{2n_K+m_K-p+1}\sum_{s=n_I+1}^{t-1} a'_{c_{s,t},c_{s,t}+(t-s)} \,\middle|\, \begin{array}{l} C=(c_{s,t}) \text{ is} \\ \text{an } \mathbf{l}\text{-tableau}\end{array}\right\}$$
$$=\min\left\{\sum_{t=n_K+1}^{2n_K+m_K-p+1}\sum_{s=n_I}^{t-1} b'_{d_{s,t},d_{s,t}+(t-s)} \,\middle|\, \begin{array}{l} D=(d_{s,t}) \text{ is} \\ \text{an } \mathbf{m}\text{-tableau}\end{array}\right\},$$

and hence (4.4). □

Now let us start the proof of Eq. (4.1). By (4.2) and Proposition 9, we have

$$\begin{aligned}(\widetilde{e}_p\mathbf{M})_{\Omega_I(\mathbf{k})} &= (\widetilde{e}_p\mathbf{M}_I)_{\mathrm{res}_I(\Omega_I(\mathbf{k}))} = (\widetilde{e}_p\mathbf{M}_I)_{\mathbf{k}_I^c} = \left(\left(\widetilde{e}_p^*(\mathbf{M}_I^\sharp)\right)^\sharp\right)_{\mathbf{k}_I^c} \\ &= \left(\widetilde{e}_p^*(\mathbf{M}_I^\sharp)\right)_{\mathbf{k}_I} - \left\langle \mathrm{wt}^\vee\left(\widetilde{e}_p^*(\mathbf{M}_I^\sharp)\right), \mathbf{k}_I^c\right\rangle_I \\ &= \left(\widetilde{e}_p^*(\mathbf{M}_I^\sharp)\right)_{\mathbf{k}_I} + \left\langle \mathrm{wt}^\vee(\mathbf{M}_I), \mathbf{k}_I\right\rangle_I + \langle h_p, \mathbf{k}_I\rangle_I.\end{aligned}$$

Here, $\mathbf{k}_I := \mathrm{res}_I(\mathbf{k})$ and $\mathbf{k}_I^c := \widetilde{I}\setminus\mathbf{k}_I$. Similarly, we have

$$(\widetilde{e}_p\mathbf{M})_{\Omega_J(\mathbf{k})} = \left(\widetilde{e}_p^*(\mathbf{M}_J^\sharp)\right)_{\mathbf{k}_J} + \left\langle \mathrm{wt}^\vee(\mathbf{M}_J), \mathbf{k}_J\right\rangle_J + \langle h_p, \mathbf{k}_J\rangle_J.$$

By these, the proof of Eq. (4.1) is reduced to showing:

(a) $\langle h_q, \mathbf{k}_I\rangle_I = \langle h_q, \mathbf{k}_J\rangle_J$ for all $q \in K'$;
(b) $\langle \mathrm{wt}^\vee(\mathbf{M}_I), \mathbf{k}_I\rangle_I = \langle \mathrm{wt}^\vee(\mathbf{M}_J), \mathbf{k}_J\rangle_J$;
(c) $(\widetilde{e}_p^*(\mathbf{M}_I^\sharp))_{\mathbf{k}_I} = (\widetilde{e}_p^*(\mathbf{M}_J^\sharp))_{\mathbf{k}_J}$.

By the definitions, (a) is easily shown. Let us show (b). Since $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(K')$, we have $\langle h_q, \mathbf{k}_I\rangle_I = 0$ for $q \notin K'$. Therefore, we see that

$$\begin{aligned}\left\langle \mathrm{wt}^\vee(\mathbf{M}_I), \mathbf{k}_I\right\rangle_I &= \sum_{q\in I}\langle h_q, \mathbf{k}_I\rangle_I (\mathbf{M}_I)_{[q+1,n_I+m_I+1]} = \sum_{q\in I}\langle h_q, \mathbf{k}_I\rangle_I M_{\Omega_I(\mathbf{k}(\Lambda_q))} \\ &= \sum_{q\in K'}\langle h_q, \mathbf{k}_I\rangle_I M_{\Omega_I(\mathbf{k}(\Lambda_q))}.\end{aligned}$$

Similarly, we see that

$$\left\langle \mathrm{wt}^\vee(\mathbf{M}_J), \mathbf{k}_J\right\rangle_J = \sum_{q\in K'}\langle h_q, \mathbf{k}_J\rangle_J M_{\Omega_J(\mathbf{k}(\Lambda_p))}.$$

Consequently, in view of (a), it suffices to show that $M_{\Omega_I(\mathbf{k}(\Lambda_q))} = M_{\Omega_J(\mathbf{k}(\Lambda_q))}$ for all $q \in K'$, which follows from (4.1). Thus, we have shown (b). For (c), it suffices to show the following proposition.

Proposition 13

$$\big(\widetilde{e}_p^*(\mathbf{M}_I^\sharp)\big)_K = \big(\widetilde{e}_p^*(\mathbf{M}_J^\sharp)\big)_K.$$

Proof We remark that $(\widetilde{e}_p^*(\mathbf{M}_I^\sharp))_K$ and $(\widetilde{e}_p^*(\mathbf{M}_J^\sharp))_K$ are both elements of $\mathcal{BZ}_K^e$. Hence it suffices to show the following:

(d) $((\widetilde{e}_p^*(\mathbf{M}_I^\sharp))_K)_{[p+1,n_K+m_K+1]} = ((\widetilde{e}_p^*(\mathbf{M}_J^\sharp))_K)_{[p+1,n_K+m_K+1]}$;
(e) $((\widetilde{e}_p^*(\mathbf{M}_I^\sharp))_K)_{\mathbf{m}} = ((\widetilde{e}_p^*(\mathbf{M}_J^\sharp))_K)_{\mathbf{m}}$ for all $\mathbf{m} \in \mathcal{M}_K^\times \setminus \mathcal{M}_K^\times(p)^*$.

Because (d) is already shown in Proposition 12, the remaining task is to show (e). We set $\mathbf{m}^I := (\mathrm{res}_K^I)^{-1}(\mathbf{m})$. Since $\mathbf{m}^I \in \mathcal{M}_I^\times \setminus \mathcal{M}_I^\times(p)^*$, we have

$$\big((\widetilde{e}_p^*(\mathbf{M}_I^\sharp))_K\big)_{\mathbf{m}} = \big(\widetilde{e}_p^*(\mathbf{M}_I^\sharp)\big)_{\mathbf{m}^I} = \big(\mathbf{M}_I^\sharp\big)_{\mathbf{m}^I} = (\mathbf{M}_I)_{(\mathbf{m}^I)^c} - \big\langle \mathrm{wt}^\vee(\mathbf{M}_I), \mathbf{m}^I\big\rangle_I.$$

We set $\mathbf{m}^{\mathbb{Z}} := \mathrm{res}_K^{-1}(\mathbf{m}) \in \mathcal{M}_{\mathbb{Z}}(K) \subset \mathcal{M}_{\mathbb{Z}}$. Then we have $\mathbf{m}^I = \mathrm{res}_I(\mathbf{m}^{\mathbb{Z}})$ and $(\mathbf{m}^I)^c = \mathrm{res}_I(\Omega_I(\mathbf{m}^{\mathbb{Z}}))$. Therefore, we obtain

$$\big((\widetilde{e}_p^*(\mathbf{M}_I^\sharp))_K\big)_{\mathbf{m}} = M_{\Omega_I(\mathbf{m}^{\mathbb{Z}})} - \big\langle \mathrm{wt}^\vee(\mathbf{M}_I), \mathrm{res}_I\big(\mathbf{m}^{\mathbb{Z}}\big)\big\rangle_I.$$

Also, we obtain a similar equation, with I replaced by J. By the same argument as in the proof of (b), we deduce that $\langle \mathrm{wt}^\vee(\mathbf{M}_I), \mathrm{res}_I(\mathbf{m}^{\mathbb{Z}})\rangle_I = \langle \mathrm{wt}^\vee(\mathbf{M}_J), \mathrm{res}_J(\mathbf{m}^{\mathbb{Z}})\rangle_J$. Now it remains to verify that $M_{\Omega_I(\mathbf{m}^{\mathbb{Z}})} = M_{\Omega_J(\mathbf{m}^{\mathbb{Z}})}$, which follows easily from (4.1). Thus, we have shown (e). This proves the proposition. □

4.3 Ordinary Crystal Structure on $(\mathcal{BZ}_{\mathbb{Z}}^e)^\sigma$

First, we give some properties of ordinary Kashiwara operators on $\mathcal{BZ}_{\mathbb{Z}}^e$. Because all of those are obtained by the same argument as in [8], we omit the proofs of them.

Lemma 15 (1) *Let* $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^e$ *and* $p \in \mathbb{Z}$. *Then,* $\widetilde{e}_p \widetilde{f}_p \mathbf{M} = \mathbf{M}$. *Also, if* $\varepsilon_p(\mathbf{M}) \neq 0$, *then* $\widetilde{f}_p \widetilde{e}_p \mathbf{M} = \mathbf{M}$.

(2) *For* $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^e$ *and* $p, q \in \mathbb{Z}$ *with* $|p-q| \geq 2$, *we have* $\varepsilon_p(\widetilde{f}_p\mathbf{M}) = \varepsilon_p(\mathbf{M}) + 1$ *and* $\varepsilon_q(\widetilde{f}_p\mathbf{M}) = \varepsilon_q(\mathbf{M})$. *Also, if* $\varepsilon_p(\mathbf{M}) \neq 0$, *then* $\varepsilon_p(\widetilde{e}_p\mathbf{M}) = \varepsilon_p(\mathbf{M}) - 1$ *and* $\varepsilon_q(\widetilde{e}_p\mathbf{M}) = \varepsilon_q(\mathbf{M})$.

(3) *For* $q, q' \in \mathbb{Z}$ *with* $|q-q'| \geq 2$, *we have* $\widetilde{e}_q\widetilde{e}_{q'} = \widetilde{e}_{q'}\widetilde{e}_q$, $\widetilde{f}_q\widetilde{f}_{q'} = \widetilde{f}_{q'}\widetilde{f}_q$ *and* $\widetilde{e}_q\widetilde{f}_{q'} = \widetilde{f}_{q'}\widetilde{e}_q$ *as operators on* $\mathcal{BZ}_{\mathbb{Z}}^e \cup \{0\}$.

(4) *For* $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^e$, *we have* $\varepsilon_p(\sigma(\mathbf{M})) = \varepsilon_{\sigma^{-1}(p)}(\mathbf{M})$.

(5) *The equalities* $\sigma \circ \widetilde{e}_p = \widetilde{e}_{\sigma(p)} \circ \sigma$ *and* $\sigma \circ \widetilde{f}_p = \widetilde{f}_{\sigma(p)} \circ \sigma$ *hold on* $\mathcal{BZ}_{\mathbb{Z}}^e \cup \{0\}$.

Next, let us define the ordinary $U_q(\widehat{\mathfrak{sl}}_l)$-crystal structure on $(\mathcal{BZ}_{\mathbb{Z}}^e)^\sigma$. Recall that the map $\mathrm{wt} : (\mathcal{BZ}_{\mathbb{Z}}^e)^\sigma \to \widehat{P}$ is already defined. Here, $\widehat{P}$ is the weight lattice for $\widehat{\mathfrak{sl}}_l$. For $\mathbf{M} \in (\mathcal{BZ}_{\mathbb{Z}}^e)^\sigma$ and $p \in \widehat{I}$, we define

$$\widehat{\varepsilon}_p(\mathbf{M}) := \varepsilon_p(\mathbf{M}), \qquad \widehat{\varphi}_p(\mathbf{M}) := \widehat{\varepsilon}_p(\mathbf{M}) + \big\langle \widehat{h}_p, \mathrm{wt}(\mathbf{M})\big\rangle.$$

For given $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$ and $p \in \widehat{I}$, we set $L^e(\mathbf{k}, p) := \{q \in p + l\mathbb{Z} | \langle h_q, \mathbf{k}\rangle_{\mathbb{Z}} \neq 0\}$; note that $L^e(\mathbf{k}, p)$ is a finite set. For $\mathbf{M} \in (\mathcal{BZ}^e_{\mathbb{Z}})^\sigma$, we define

$$\widehat{e}_p\mathbf{M} := \begin{cases} \mathbf{M}^{(1)} & \text{if } \widehat{\varepsilon}_p(\mathbf{M}) > 0, \\ 0 & \text{if } \widehat{\varepsilon}_p(\mathbf{M}) = 0, \end{cases} \quad \text{and} \quad \widehat{f}_p\mathbf{M} = \mathbf{M}^{(2)},$$

where $\mathbf{M}^{(i)} = (M^{(i)}_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}}$, $i = 1, 2$, are the collections of integers defined by

$$M^{(1)}_{\mathbf{k}} := (e_{L^e(\mathbf{k},p)}\mathbf{M})_{\mathbf{k}}, \qquad M^{(2)}_{\mathbf{k}} := (f_{L^e(\mathbf{k},p)}\mathbf{M})_{\mathbf{k}} \quad \text{for each } \mathbf{k} \in \mathcal{M}_{\mathbb{Z}}.$$

Proposition 14 *Let* $\mathbf{M} \in (\mathcal{BZ}^e_{\mathbb{Z}})^\sigma$ *and* $p \in \widehat{I}$. *Then, we have* $\widehat{e}_p\mathbf{M} \in (\mathcal{BZ}^e_{\mathbb{Z}})^\sigma \cup \{0\}$ *and* $\widehat{f}_p\mathbf{M} \in (\mathcal{BZ}^e_{\mathbb{Z}})^\sigma$.

In order to prove the proposition above, we need the next lemma.

Lemma 16 *For given* $\mathbf{M} \in \mathcal{BZ}^e_{\mathbb{Z}}$ *and* $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$, *there exists a finite interval* $I = [n_I + 1, n_I + m_I]$ *such that for every* $J = [n_J + 1, n_J + m_J]$ *with* $n_J < n_I$ *and* $n_J + m_J > n_I + m_I$,

(1) $(\widetilde{e}_{n_J}\mathbf{M})_{\Omega_J(\mathbf{k})} = (\widetilde{e}_{n_J+m_J+1}\mathbf{M})_{\Omega_J(\mathbf{k})} = (\widetilde{e}_{n_J}\widetilde{e}_{n_J+m_J+1}\mathbf{M})_{\Omega_J(\mathbf{k})} = M_{\Omega_J(\mathbf{k})}$;
(2) $(\widetilde{f}_{n_J}\mathbf{M})_{\Omega_J(\mathbf{k})} = (\widetilde{f}_{n_J+m_J+1}\mathbf{M})_{\Omega_J(\mathbf{k})} = (\widetilde{f}_{n_J}\widetilde{f}_{n_J+m_J+1}\mathbf{M})_{\Omega_J(\mathbf{k})} = M_{\Omega_J(\mathbf{k})}$.

Here we remark that the equalities in part (1) *hold under the assumption that* $\widetilde{e}_{n_J}\mathbf{M} \neq \{0\}$ *and* $\widetilde{e}_{n_J+m_J+1}\mathbf{M} \neq \{0\}$.

Proof We only prove that

$$(\widetilde{e}_{n_J}\widetilde{e}_{n_J+m_J+1}\mathbf{M})_{\Omega_J(\mathbf{k})} = M_{\Omega_J(\mathbf{k})} \tag{4.6}$$

under the condition that $\widetilde{e}_{n_J}\mathbf{M} \neq \{0\}$ and $\widetilde{e}_{n_J+m_J+1}\mathbf{M} \neq \{0\}$; the other equalities can be proved by a similar (and easier) argument.

For the $\mathbf{M}$ and $\mathbf{k}$ above, take finite intervals $K = [n_K + 1, n_K + m_K]$ and $I = [n_I + 1, n_I + m_J]$ as in (4.2). Let $J = [n_J + 1, n_J + m_J] \supsetneq I$, with $n_J < n_I$ and $n_J + m_J > n_I + m_I$. Take another finite interval $L = [n_L + 1, n_L + m_L] \supset J$ such that

$$M_{\Omega_J(\mathbf{k})} = (\mathbf{M}_L)_{\mathrm{res}_L(\Omega_J(\mathbf{k}))},$$
$$(\widetilde{e}_{n_J}\widetilde{e}_{n_J+m_J+1}\mathbf{M})_{\Omega_J(\mathbf{k})} = (\widetilde{e}_{n_J}\widetilde{e}_{n_J+m_J+1}\mathbf{M}_L)_{\mathrm{res}_L(\Omega_J(\mathbf{k}))},$$

$$\varepsilon_{n_J}(\mathbf{M}) = \varepsilon_{n_J}(\mathbf{M}_L), \quad \text{and} \quad \varepsilon_{n_J+m_J+1}(\mathbf{M}) = \varepsilon_{n_J+m_J+1}(\mathbf{M}_L).$$

Note that such an interval L always exists. Hence Eq. (4.6) is equivalent to

$$(\widetilde{e}_{n_J}\widetilde{e}_{n_J+m_J+1}\mathbf{M}_L)_{\mathrm{res}_L(\Omega_J(\mathbf{k}))} = (\mathbf{M}_L)_{\mathrm{res}_L(\Omega_J(\mathbf{k}))}. \tag{4.7}$$

In what follows, we use the notation of Subsect. 4.2. Namely, set $\mathbf{b} = (b_{k,l})_{(k,l)\in\Delta_L^+} := \Psi_L^{-1}(\mathbf{M}_L) \in \mathcal{B}_L$, and $\Lambda_{\mathbf{b}} := \Xi_L^{-1}(\mathbf{M}_L) \in \operatorname{Irr}\Lambda(\nu_L)$, where $\nu_L = \operatorname{wt}(\mathbf{M}_L)$. Let $B^L = (B_\tau^L) \in \Lambda_{\mathbf{b}}$ be a general point.

Since $L \supsetneq I$, we can show the following claim by an argument similar to the one for Lemma 12:

Claim 2 *Both of the composite maps*

$$B^L_{\sigma(n_I \to n_K)} : V(\nu_J)_{n_I} \to V(\nu_J)_{n_K} \quad \textit{and}$$

$$B^L_{\sigma(n_I+m_J+1 \to n_K+m_K+1)} : V(\nu_J)_{n_I+m_I+1} \to V(\nu_J)_{n_K+m_K+1}$$

are zero maps.

Write $\operatorname{res}_L(\Omega_J(\mathbf{k}))$ as a disjoint union of finite intervals:

$$\operatorname{res}_L\big(\Omega_J(\mathbf{k})\big) = [s_1+1, t_1] \sqcup [s_2+1, t_2] \sqcup \cdots \sqcup [s_l+1, t_l].$$

Then, by the construction,

$$s_1 = n_L, \qquad t_1 = n_J, \qquad s_2 = \min\{q \in \mathbb{Z} | q \notin \mathbf{k}\} - 1,$$

$$s_l = \max\{q \in \mathbb{Z} | q \in \mathbf{k}\}, \qquad t_l = n_J + m_J + 1,$$

and

$$s_1 + 1 = n_L + 1 < t_1 = n_J < n_I < n_K < s_2, \tag{4.8}$$

$$s_l < n_K + m_K + 1 < n_I + m_I + 1 < n_J + m_J + 1 = t_l < n_L + m_L + 1. \tag{4.9}$$

From these, we deduce that

$$\operatorname{out}\big(\operatorname{res}_L\big(\Omega_J(\mathbf{k})\big)\big) = \{t_1, t_2, \ldots, t_l\}, \qquad \operatorname{in}\big(\operatorname{res}_L\big(\Omega_J(\mathbf{k})\big)\big) = \{s_2, s_3, \ldots, s_l\}.$$

Because

$$B^L_{\sigma(t_1 \to s_2)} = B^L_{\sigma(n_K \to s_2)} \circ B^L_{\sigma(n_I \to n_K)} \circ B^L_{\sigma(t_1 \to n_I)} = 0,$$

$$B^L_{\sigma(t_l \to s_l)} = B^L_{\sigma(n_K+m_K+1 \to s_l)} \circ B^L_{\sigma(n_I+m_I+1 \to n_K+m_K+1)} \circ B^L_{\sigma(t_l \to n_I+m_I+1)} = 0$$

by Claim 2, (4.8), and (4.9), we see that

$$\begin{aligned}(\mathbf{M}_L)_{\operatorname{res}_L(\Omega_J(\mathbf{k}))} &= -\dim_{\mathbb{C}} \operatorname{Coker}\Bigg(\bigoplus_{1\le u\le l} V(\nu_L)_{t_u} \xrightarrow{\oplus B^L_\sigma} \bigoplus_{2\le v\le l} V(\nu_L)_{s_v}\Bigg) \\ &= -\dim_{\mathbb{C}} \operatorname{Coker}\Bigg(\bigoplus_{2\le u\le l-1} V(\nu_L)_{t_u} \xrightarrow{\oplus B^L_\sigma} \bigoplus_{2\le v\le l} V(\nu_L)_{s_v}\Bigg).\end{aligned} \tag{4.10}$$

Let $\mathbf{b}_1 := \widetilde{e}_{n_J}\widetilde{e}_{n_J+m_J+1}\mathbf{b}$, and consider the corresponding irreducible Lagrangian $\Lambda_{\mathbf{b}_1} \in \mathrm{Irr}\,\Lambda(\nu_L^1)$. Here we write $\nu_L^1 = \nu_L - \alpha_{n_J} - \alpha_{n_J+m_L+1}$. Let $B_1^L = ((B_1^L)_\tau) \in \Lambda_{\mathbf{b}_1}$ be a general point. By the definitions of $\widetilde{e}_{n_J}$ and $\widetilde{e}_{n_J+m_J+1}$ (see Subsect. 2.5), we may assume that

$$\left(B_1^L\right)_\tau = B_\tau^L \quad \text{if } \mathrm{out}(\tau) \neq n_J, n_J+m_J+1 \text{ and if } \mathrm{in}(\tau) \neq n_J, n_J, n_J+m_J+1. \tag{4.11}$$

Then, by Claim 2,

$$\left(B_1^L\right)_{\sigma(n_I\to n_K)} = 0, \qquad \left(B_1^L\right)_{\sigma(n_I+m_J+1\to n_K+m_K+1)} = 0.$$

Therefore, by the same argument as for $(\mathbf{M}_L)_{\mathrm{res}_L(\Omega_J(\mathbf{k}))}$, we deduce that

$$\begin{aligned}&(\widetilde{e}_{n_J}\widetilde{e}_{n_J+m_J+1}\mathbf{M}_L)_{\mathrm{res}_L(\Omega_J(\mathbf{k}))}\\ &\quad = -\dim_{\mathbb{C}}\mathrm{Coker}\left(\bigoplus_{2\le u\le l-1} V\left(\nu_L^1\right)_{t_u} \xrightarrow{\oplus(B_1^L)_\sigma} \bigoplus_{2\le v\le l} V\left(\nu_L^1\right)_{s_v}\right).\end{aligned} \tag{4.12}$$

Since $n_J < s_2 < t_2 < \cdots < s_{l-1} < t_{l-1} < s_l < n_J+m_J+1$, the right-hand side of the last equality in (4.10) is equal to that of (4.12). Thus, we have proved Eq. (4.7). This completes the proof of the lemma. □

Proof of Proposition 14 We only prove that $\widehat{e}_p\mathbf{M} \in (\mathcal{BZ}_{\mathbb{Z}}^e)^\sigma \cup \{0\}$. If $\widehat{\varepsilon}_p(\mathbf{M}) = 0$, then the assertion is obvious. So, we assume that $\widehat{\varepsilon}_p(\mathbf{M}) > 0$.

First, we prove that $\widehat{e}_p\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^e$. Condition (2-a) in Definition 4 can be checked by the same argument as in [8]. Let us show that condition (2-b) is satisfied. Fix $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$ and take a finite interval $I = [n_I+1, n_I+m_I]$ satisfying condition (4.2), with $\mathbf{M}$ replaced by $\widetilde{e}_{L^e(\mathbf{k},p)}\mathbf{M}$. Let $I' = [n_{I'}+1, n_{I'}+m_{I'}]$ be an interval such that $n_{I'} < n_I$, $n_{I'}+m_{I'} > n_I+m_I$, and $I' \in \mathrm{Int}^e(\widetilde{e}_{L^e(\mathbf{k},p)}\mathbf{M}, \mathbf{k})$. We will show that this I' satisfies condition (2-b).

Take $J = [n_J+1, n_J+m_J] \supset I'$. By the definitions, we have

$$L^e\left(\Omega_J(\mathbf{k}), p\right) = L^e(\mathbf{k}, p) \cup \delta(J, p; l).$$

Here,

$$\delta(J, p; l) := \begin{cases} \{n_J, n_J+m_J+1\} & \text{if } n_J \equiv p, n_J+m_J+1 \equiv p \pmod{l},\\ \{n_J\} & \text{if } n_J \equiv p, n_J+m_J+1 \not\equiv p \pmod{l},\\ \{n_J+m_J+1\} & \text{if } n_J \not\equiv p, n_J+m_J+1 \equiv p \pmod{l},\\ \phi & \text{if } n_J \not\equiv p, n_{I'}+m_J+1 \not\equiv p \pmod{l}.\end{cases}$$

From this, we deduce by Lemma 16 that

$$(\widehat{e}_p\mathbf{M})_{\Omega_J(\mathbf{k})} = (\widetilde{e}_{L^e(\Omega_J(\mathbf{k}),p)}\mathbf{M})_{\Omega_J(\mathbf{k})} = (\widetilde{e}_{L^e(\mathbf{k},p)}\mathbf{M})_{\Omega_{I'}(\mathbf{k})} \quad \text{for every } J \supset I'.$$

Note that by Lemma 15, $\varepsilon_q(\mathbf{M}) = \widehat{\varepsilon}_p(\mathbf{M}) > 0$ for every $q \in p + l\mathbb{Z}$. Since $I' \in \mathrm{Int}^e(\widetilde{e}_{L^e(\mathbf{k},p)}\mathbf{M}, \mathbf{k})$, we conclude that

$$(\widetilde{e}_{L^e(\mathbf{k},p)}\mathbf{M})_{\Omega_J(\mathbf{k})} = (\widetilde{e}_{L^e(\mathbf{k},p)}\mathbf{M})_{\Omega_{I'}(\mathbf{k})}.$$

This shows that condition (2-b) is satisfied.

It remains to show that $\widehat{e}_p\mathbf{M}$ is σ-invariant. However, this follows easily from Lemma 15. This proves the proposition. □

Now we are ready to state one of the main results of this paper.

Theorem 4 *$((\mathcal{BZ}_{\mathbb{Z}}^e)^\sigma; \mathrm{wt}, \widehat{\varepsilon}_p, \widehat{\varphi}_p, \widehat{e}_p, \widehat{f}_p)$ is a $U_q(\widehat{\mathfrak{sl}}_l)$-crystal.*

Lemma 17 *Let $\mathbf{M} \in (\mathcal{BZ}_{\mathbb{Z}}^e)^\sigma$ and $p \in \widehat{I}$. Then, the following hold:*

(1) $\mathrm{wt}\,(\widehat{e}_p\mathbf{M}) = \mathrm{wt}\,(\mathbf{M}) + \widehat{\alpha}_p$ *if* $\widehat{\varepsilon}_p(\mathbf{M}) > 0$, *and* $\mathrm{wt}\,(\widehat{f}_p\mathbf{M}) = \mathrm{wt}\,(\mathbf{M}) - \widehat{\alpha}_p$;
(2) $\widehat{\varepsilon}_p(\widehat{e}_p\mathbf{M}) = \widehat{\varepsilon}_p(\mathbf{M}) - 1$ *if* $\widehat{\varepsilon}_p(\mathbf{M}) > 0$, *and* $\widehat{\varepsilon}_p(\widehat{f}_p\mathbf{M}) = \widehat{\varepsilon}_p(\mathbf{M}) + 1$.

Proof We only prove the first equation of part (1), since the other ones follow by a similar (and easier) argument.

Let $\mathbf{M} \in (\mathcal{BZ}_{\mathbb{Z}}^e)^\sigma$, with $\widehat{\varepsilon}_p(\mathbf{M}) > 0$, and $J = [n_J+1, n_J+m_J] \in \bigcap_{q\in\widehat{I}} \mathrm{Int}^e(\widehat{e}_p\mathbf{M}, \mathbf{k}(\Lambda_q))$. Then,

$$\begin{aligned}\mathrm{wt}\,(\widehat{e}_p\mathbf{M}) &= \sum_{q\in\widehat{I}} \Theta(\widehat{e}_p\mathbf{M})_{\mathbf{k}(\Lambda_q)}\widehat{\alpha}_q \\ &= \sum_{q\in\widehat{I}} (\widehat{e}_p\mathbf{M})_{\Omega_J(\mathbf{k}(\Lambda_q))}\widehat{\alpha}_q \\ &= \sum_{q\in\widehat{I}} (\widetilde{e}_{L^e(\Omega_J(\mathbf{k}(\Lambda_q)),p)}\mathbf{M})_{\Omega_J(\mathbf{k}(\Lambda_q))}\widehat{\alpha}_q.\end{aligned}$$

Here we note that

$$L^e\big(\Omega_J\big(\mathbf{k}(\Lambda_q)\big), p\big) = L^e\big(\mathbf{k}(\Lambda_q), p\big) \cup \delta(J, p; l) = \begin{cases} \{q\} \cup \delta(J, p; l) & \text{if } p = q, \\ \delta(J, p; l) & \text{if } p \neq q. \end{cases}$$

Now we assume that J is sufficiently large. More precisely, for each $q \in \widehat{I}$, let us take an interval $I' = I'_q$ as in the proof of Proposition 14, and then take J in such a way that $J \supset \bigcup_{q\in\widehat{I}} I'_q$. Then, by Lemma 16, we deduce that

$$(\widetilde{e}_{L^e(\Omega_J(\mathbf{k}(\Lambda_q)),p)}\mathbf{M})_{\Omega_J(\mathbf{k}(\Lambda_q))} = \begin{cases} (\mathbf{M})_{\Omega_J(\mathbf{k}(\Lambda_p))} + 1 & \text{if } p = q, \\ (\mathbf{M})_{\Omega_J(\mathbf{k}(\Lambda_p))} & \text{if } p \neq q. \end{cases}$$

Therefore, we obtain

$$\mathrm{wt}\,(\widehat{e}_p\mathbf{M}) = \mathrm{wt}\,(\mathbf{M}) + \widehat{\alpha}_p.$$

□

Proof of Theorem 4 By Lemma 17, it suffices to prove the following:

$$\widehat{e}_p\widehat{f}_p\mathbf{M} = \mathbf{M} \quad \text{for every } \mathbf{M} \in \big(\mathcal{BZ}_{\mathbb{Z}}^e\big)^\sigma \text{ and } p \in \widehat{I}.$$

Since this follows easily from Lemmas 15 and 16, we omit the details of its proof. □

4.4 Uniqueness of an Element of Weight Zero

It is easy to show the following lemma.

Lemma 18 *Let* $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}} \in \mathcal{BZ}_{\mathbb{Z}}^{e}$. *Then, each component* $M_{\mathbf{k}}$ *for* $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$ *is a nonpositive integer.*

The next corollary is a direct consequence of this lemma.

Corollary 8 *Let* $\mathbf{M} \in (\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}$. *Then,* $\mathrm{wt}\,(\mathbf{M}) \in \widehat{Q}^{-}$.

Proposition 15 *For* $\mathbf{M} \in (\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}$, *the following are equivalent.*

(a) $\widehat{\varepsilon}_p(\mathbf{M}) = 0$ *for every* $p \in \widehat{I}$.
(b) $\mathbf{M} = \mathbf{O}^{*}$.

Proof Since (b) $\Rightarrow$ (a) is obvious, we will prove that $M_{\mathbf{k}} = 0$ for all $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$ under the assumption that $\widehat{\varepsilon}_p(\mathbf{M}) = 0$ for every $p \in \widehat{I}$.

We note that $M_{\mathbf{k}(\Lambda_q)} = 0$ for every $q \in \mathbb{Z}$ by the normalization condition. Let $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}} \setminus (\bigcup_{q \in \mathbb{Z}} \mathbf{k}(\Lambda_q))$. Then there exists the smallest finite interval $I_{\mathbf{k}}$ such that $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(I_{\mathbf{k}})$. We prove the assertion above by induction on $t := |I_{\mathbf{k}}| \geq 1$.

Assume that $t = 1$. Then, $\mathbf{k} = \sigma_q \mathbf{k}(\Lambda_q) = \mathbb{Z}_{\leq q-1} \cup \{q+1\}$ for some $q \in \mathbb{Z}$. If we take $q' \in \widehat{I}$ such that $q \equiv q' \pmod{l}$, then we have

$$M_{\sigma_q \mathbf{k}(\Lambda_q)} = \varepsilon_q(\mathbf{M}) = \widehat{\varepsilon}_{q'}(\mathbf{M}) = 0.$$

Now, we assume that $t > 1$, and

(i) the assertion holds for every $\mathbf{m} \in \mathcal{M}_{\mathbb{Z}}$ with $|I_{\mathbf{m}}| < t$.

Step 1. Let $\mathbf{k} = \mathbb{Z}_{\leq n} \cup \{n+t+1\}$ for some $n \in \mathbb{Z}$. We use the tropical Plücker relation for $i = n+1$, $j = n+t$, $k = n+t+1$:

$$\begin{aligned} &M_{\mathbb{Z}_{\leq n} \cup \{n+t\}} + M_{\mathbb{Z}_{\leq n} \cup \{n+1, n+t+1\}} \\ &\quad = \min\{M_{\mathbb{Z}_{\leq n} \cup \{n+1\}} + M_{\mathbb{Z}_{\leq n} \cup \{n+t, n+t+1\}}, M_{\mathbb{Z}_{\leq n} \cup \{n+t+1\}} + M_{\mathbb{Z}_{\leq n} \cup \{n+1, n+t\}}\}. \end{aligned}$$

By the assumption (i), we see that

$$M_{\mathbb{Z}_{\leq n} \cup \{n+t\}} = M_{\mathbb{Z}_{\leq n} \cup \{n+1, n+t+1\}} = M_{\mathbb{Z}_{\leq n} \cup \{n+1\}} = M_{\mathbb{Z}_{\leq n} \cup \{n+1, n+t\}} = 0.$$

Since $M_{\mathbb{Z}_{\leq n} \cup \{n+t+1\}} = M_{\mathbf{k}}$ and $M_{\mathbb{Z}_{\leq n} \cup \{n+t, n+t+1\}} = M_{\mathbf{k} \cup \{n+t\}}$ are both nonpositive integers, we obtain

$$M_{\mathbf{k}} = M_{\mathbf{k} \cup \{n+t\}} = 0.$$

Step 2. Let $\mathbf{k} = \mathbb{Z}_{\leq n} \cup \{k_1 < \cdots < k_r\}$, with $k_1 = n+s+1$ $(1 < s \leq t)$, and $k_r = n+t+1$. We prove the assertion by descending induction on s. If $s = t$, then $r = 1$ and the assertion is already proved in Step 1. Assume that

(ii) the assertion holds for every $\mathbf{m} = \mathbb{Z}_{\leq n} \cup \{m_1 < \cdots < m_{r'}\}$, with $m_1 = n + s' + 1 > k_1$, and $m_{r'} = n + t + 1$.

Set $\mathbf{k}' := \mathbf{k} \setminus \{n + s + 1, n + t + 1\}$, and use the tropical Plücker relation for $i = n + 1$, $j = n + s + 1$, $k = n + t + 1$, and $\mathbf{k}'$:

$$\begin{aligned} &M_{\mathbf{k}' \cup \{n+s+1\}} + M_{\mathbf{k}' \cup \{n+1,n+t+1\}} \\ &\quad = \min\{M_{\mathbf{k}' \cup \{n+1\}} + M_{\mathbf{k}' \cup \{n+s+1,n+t+1\}}, M_{\mathbf{k}' \cup \{n+t+1\}} + M_{\mathbf{k}' \cup \{n+1,n+s+1\}}\}. \end{aligned}$$

By the assumption (i), we obtain

$$M_{\mathbf{k}' \cup \{n+s+1\}} = M_{\mathbf{k}' \cup \{n+1,n+t+1\}} = M_{\mathbf{k}' \cup \{n+1\}} = M_{\mathbf{k}' \cup \{n+1,n+s+1\}} = 0.$$

Also, we have

$$M_{\mathbf{k}' \cup \{n+t+1\}} = 0$$

by the assumption (ii). Therefore, by Lemma 18, we conclude that

$$M_{\mathbf{k}} = M_{\mathbf{k}' \cup \{n+s+1,n+t+1\}} = 0.$$

This proves the proposition. □

The following corollary is a key to the proof of the connectedness of the crystal graph of the $U_q(\widehat{\mathfrak{sl}}_l)$-crystal $((\mathcal{BZ}_{\mathbb{Z}}^e)^\sigma; \mathrm{wt}, \widehat{\varepsilon}_p^*, \widehat{\varphi}_p^*, \widehat{e}_p^*, \widehat{f}_p^*)$, which will be given in the next section.

Corollary 9 $\mathbf{O}^*$ *is the unique element of* $(\mathcal{BZ}_{\mathbb{Z}}^e)^\sigma$ *of weight zero.*

Proof It suffices to show the following:

$$\text{If} \quad \mathbf{M} \neq \mathbf{O}^*, \quad \text{then} \quad \mathrm{wt}\,(\mathbf{M}) \neq 0.$$

Let $\mathbf{M} \neq \mathbf{O}^*$. By Proposition 15, there exists $p \in \widehat{I}$ such that $\widehat{\varepsilon}_p(\mathbf{M}) > 0$. This implies that $\widehat{e}_p\mathbf{M} \in (\mathcal{BZ}_{\mathbb{Z}}^e)^\sigma$. Therefore, by Corollary 8, we have

$$\mathrm{wt}\,(\widehat{e}_p\mathbf{M}) \in \widehat{Q}^-. \tag{4.13}$$

Also, because $((\mathcal{BZ}_{\mathbb{Z}}^e)^\sigma; \mathrm{wt}, \widehat{\varepsilon}_p, \widehat{\varphi}_p, \widehat{e}_p, \widehat{f}_p)$ is a $U_q(\widehat{\mathfrak{sl}}_l)$-crystal (Theorem 4), we have

$$\mathrm{wt}\,(\widehat{e}_p\mathbf{M}) = \mathrm{wt}\,(\mathbf{M}) + \widehat{\alpha}_p. \tag{4.14}$$

Now, suppose that $\mathrm{wt}\,(\mathbf{M}) = 0$. Then, by (4.14), we obtain

$$\mathrm{wt}\,(\widehat{e}_p\mathbf{M}) = \widehat{\alpha}_p,$$

which contradicts (4.13). Thus, we conclude that $\mathrm{wt}\,(\mathbf{M}) \neq 0$. This proves the corollary. □

4.5 Some Other Properties

The results of this subsection will be used in the next section.

Lemma 19 *Let* $p, q \in \mathbb{Z}$ *with* $p \neq q$, *and* $\mathbf{M} \in \mathcal{BZ}^e_{\mathbb{Z}}$.

(1) *If* $\varepsilon_p(\mathbf{M}) > 0$, *then* $\varepsilon^*_q(\widetilde{e}_p\mathbf{M}) = \varepsilon^*_q(\mathbf{M})$.
(2) *If* $\varepsilon^*_p(\mathbf{M}) > 0$, *then* $\varepsilon_q(\widetilde{e}^*_p\mathbf{M}) = \varepsilon_q(\mathbf{M})$.

Proof Because part (2) can be proved by a similar (and easier) argument, we only give a proof of part (1). By the definitions, we have

$$\begin{aligned}\varepsilon^*_q(\widetilde{e}_p\mathbf{M}) = &-\Theta\big((\widetilde{e}_p\mathbf{M})^*\big)_{\mathbf{k}(\Lambda_q)} - \Theta\big((\widetilde{e}_p\mathbf{M})^*\big)_{\mathbf{k}(\sigma_q\Lambda_q)} \\ &+ \Theta\big((\widetilde{e}_p\mathbf{M})^*\big)_{\mathbf{k}(\Lambda_{q+1})} + \Theta\big((\widetilde{e}_p\mathbf{M})^*\big)_{\mathbf{k}(\Lambda_{q-1})}. \end{aligned} \tag{4.15}$$

For simplicity of notation, we write $\mathbf{k}_1 = \mathbf{k}(\Lambda_q), \mathbf{k}_2 = \mathbf{k}(\sigma_q\Lambda_q), \mathbf{k}_3 = \mathbf{k}(\Lambda_{q+1})$, $\mathbf{k}_4 = \mathbf{k}(\Lambda_{q-1})$. Take a finite interval I such that $p \in I$ and $I \in \bigcap_{k=1}^4 \mathrm{Int}^c((\widetilde{e}_p\mathbf{M})^*, \mathbf{k}_k)$.

Let us compute the second term on the right-hand side of (4.15).

$$\Theta\big((\widetilde{e}_p\mathbf{M})^*\big)_{\mathbf{k}(\sigma_q\Lambda_q)} = \big((\widetilde{e}_p\mathbf{M})^*\big)_{\Omega^c_I(\mathbf{k}(\sigma_q\Lambda_q)^c)} = (\widetilde{e}_p\mathbf{M})_{\Omega_I(\mathbf{k}(\sigma_q\Lambda_q))}.$$

Since $p \neq q$, the following two cases occur:

case (a): $p = q \pm 1$ ($\Leftrightarrow \Omega_I(\mathbf{k}(\sigma_q\Lambda_q)) \in \mathcal{M}_{\mathbb{Z}}(p)^*$),
case (b): $|p - q| \geq 2$ ($\Leftrightarrow \Omega_I(\mathbf{k}(\sigma_q\Lambda_q)) \in \mathcal{M}_{\mathbb{Z}} \setminus (\mathcal{M}_{\mathbb{Z}}(p) \cup \mathcal{M}_{\mathbb{Z}}(p)^*)$).

By Lemma 11, we have

$$(\widetilde{e}_p\mathbf{M})_{\Omega_I(\mathbf{k}(\sigma_q\Lambda_q))} = \begin{cases} M_{\Omega_I(\mathbf{k}(\sigma_q\Lambda_q))} + 1 & \text{in case (a),} \\ M_{\Omega_I(\mathbf{k}(\sigma_q\Lambda_q))} & \text{in case (b).} \end{cases}$$

By a similar computation, we obtain

$$\Theta\big((\widetilde{e}_p\mathbf{M})^*\big)_{\mathbf{k}(\Lambda_q)} = M_{\Omega_I(\mathbf{k}(\Lambda_q))},$$

$$(\widetilde{e}_p\mathbf{M})_{\Omega_I(\mathbf{k}(\Lambda_{q+1}))} = \begin{cases} M_{\Omega_I(\mathbf{k}(\Lambda_{q+1}))} + 1 & \text{if } p = q+1, \\ M_{\Omega_I(\mathbf{k}(\Lambda_{q+1}))} & \text{otherwise,} \end{cases}$$

$$(\widetilde{e}_p\mathbf{M})_{\Omega_I(\mathbf{k}(\Lambda_{q-1}))} = \begin{cases} M_{\Omega_I(\mathbf{k}(\Lambda_{q-1}))} + 1 & \text{if } p = q-1, \\ M_{\Omega_I(\mathbf{k}(\Lambda_{q-1}))} & \text{otherwise.} \end{cases}$$

Combining the above, we deduce that

$$\begin{aligned}\varepsilon^*_q(\widetilde{e}_p\mathbf{M}) &= -M_{\Omega_I(\mathbf{k}(\Lambda_q))} - M_{\Omega_I(\mathbf{k}(\sigma_q\Lambda_q))} + M_{\Omega_I(\mathbf{k}(\Lambda_{q+1}))} + M_{\Omega_I(\mathbf{k}(\Lambda_{q-1}))} \\ &= \varepsilon^*_q(\mathbf{M}).\end{aligned}$$

This proves the lemma. □

Proposition 16 *Let* $p, q \in \widehat{I}$, *and* $\mathbf{M} \in (\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}$. *Set* $c := \widehat{\varepsilon}_p(\mathbf{M})$ *and* $\mathbf{M}' := \widehat{e}_p^{c}\mathbf{M}$.

(1) *We have*

$$\widehat{\varepsilon}_p^*(\mathbf{M}) = \max\{\widehat{\varepsilon}_p^*(\mathbf{M}'), c - \langle \widehat{h}_p, \mathrm{wt}(\mathbf{M}')\rangle\}.$$

(2) *If* $p \neq q$ *and* $\widehat{\varepsilon}_q^*(\mathbf{M}) > 0$, *then*

$$\widehat{\varepsilon}_q(\widehat{e}_p^*\mathbf{M}) = c, \qquad \widehat{e}_q^{c}(\widehat{e}_p^*\mathbf{M}) = \widehat{e}_p\mathbf{M}'.$$

(3) *If* $\widehat{\varepsilon}_p^*(\mathbf{M}) > 0$, *then*

$$\widehat{\varepsilon}_p(\widehat{e}_p^*\mathbf{M}) = \begin{cases} \widehat{\varepsilon}_p(\mathbf{M}) & \text{if } \widehat{\varepsilon}_p^*(\mathbf{M}') \geq c - \langle \widehat{h}_p, \mathrm{wt}(\mathbf{M}')\rangle, \\ \widehat{\varepsilon}_p(\mathbf{M}) - 1 & \text{if } \widehat{\varepsilon}_p^*(\mathbf{M}') < c - \langle \widehat{h}_p, \mathrm{wt}(\mathbf{M}')\rangle, \end{cases}$$

and

$$\widehat{e}_p^{c'}(\widehat{e}_p^*\mathbf{M}) = \begin{cases} \widehat{e}_p^*\mathbf{M}' & \text{if } \widehat{\varepsilon}_p^*(\mathbf{M}') \geq c - \langle \widehat{h}_p, \mathrm{wt}(\mathbf{M}')\rangle, \\ \mathbf{M}' & \text{if } \widehat{\varepsilon}_p^*(\mathbf{M}') < c - \langle \widehat{h}_p, \mathrm{wt}(\mathbf{M}')\rangle. \end{cases}$$

Here, we set $c' := \widehat{\varepsilon}_p(\widehat{e}_p^*\mathbf{M})$.

Proof By taking a sufficiently large finite interval I, each of the equations above follows from the corresponding one in the case of finite intervals (Proposition 10).

As an example, let us show part (1). By the definitions and Lemma 19, it suffices to show that

$$\varepsilon_p^*(\mathbf{M}) = \max\{\varepsilon_p^*(\widetilde{e}_p^{c}\mathbf{M}), -c - \langle \widehat{h}_p, \mathrm{wt}(\mathbf{M})\rangle\}.$$

Let $\mathbf{k}_k$, $k = 1, 2, 3, 4$, be the Maya diagrams which we introduced in the proof of Lemma 19. Note that there exists a finite interval I such that

(a) $I \in (\bigcap_{k=1}^{4} \mathrm{Int}^c(\mathbf{M}^*, \mathbf{k}_k)) \cap (\bigcap_{k=1}^{4} \mathrm{Int}^c((\widetilde{e}_p^{c}\mathbf{M})^*, \mathbf{k}_k))$,
(b) $(\widetilde{e}_p^{c}\mathbf{M})_I = \widetilde{e}_p^{c}\mathbf{M}_I$,
(c) $c = \varepsilon_p(\mathbf{M}) = \varepsilon_p(\mathbf{M}_I)$,
(d) $\langle \widehat{h}_p, \mathrm{wt}(\mathbf{M})\rangle = \langle h_p, \mathrm{wt}(\mathbf{M}_I)\rangle_I$.

For such an interval I, we have

$$\begin{aligned} \varepsilon_p^*(\widetilde{e}_p^{c}\mathbf{M}) &= \varepsilon_p((\widetilde{e}_p^{c}\mathbf{M})^*) \\ &= \varepsilon_p(((\widetilde{e}_p^{c}\mathbf{M})^*)_I) \quad \text{by (a)} \\ &= \varepsilon_p(((\widetilde{e}_p^{c}\mathbf{M})_I)^*) \\ &= \varepsilon_p((\widetilde{e}_p^{c}\mathbf{M}_I)^*) \quad \text{by (b)} \\ &= \varepsilon_p^*(\widetilde{e}_p^{\varepsilon_p(\mathbf{M}_I)}\mathbf{M}_I) \quad \text{by (c)}. \end{aligned}$$

Similarly, we obtain $\varepsilon_p^*(\mathbf{M}) = \varepsilon_p^*(\mathbf{M}_I)$. Therefore, it suffices to show that

$$\varepsilon_p^*(\mathbf{M}_I) = \max\{\varepsilon_p^*(\mathbf{M}_I'), \varepsilon_p(\mathbf{M}_I) - \langle h_p, \mathrm{wt}(\mathbf{M}_I')\rangle_I\}, \tag{4.16}$$

where we set $\mathbf{M}'_I := \widetilde{e}_p^{\,\varepsilon_p(\mathbf{M}_I)}\mathbf{M}_I$. Here, we note that Eq. (4.15) is just the equation in part (1) of Proposition 10. Thus, we have shown part (1).

Since the other equations are shown in a similar way, we omit the details of their proofs. □

5 Proof of the Connectedness of $((\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}; \mathrm{wt}, \widehat{\varepsilon}_p^*, \widehat{\varphi}_p^*, \widehat{e}_p^*, \widehat{f}_p^*)$

5.1 Strategy

The aim of this section is to prove the following theorem.

Theorem 5 (Main theorem) *As a crystal,* $((\mathcal{BZ}_{\mathbb{Z}}^{e})^{\sigma}; \mathrm{wt}, \widehat{\varepsilon}_p^*, \widehat{\varphi}_p^*, \widehat{e}_p^*, \widehat{f}_p^*)$ *is isomorphic to* $B(\infty)$ *for* $U_q(\widehat{\mathfrak{sl}}_l)$. *In particular, the crystal graph of this crystal is connected.*

In order to prove this theorem, we use a characterization of $B(\infty)$, which was obtained in [6]; although it is valid for an arbitrary symmetrizable Kac-Moody Lie algebra, we restrict ourselves to the case of type $A_{l-1}^{(1)}$.

For $p \in \widehat{I}$, we define a crystal $(B_p^*; \mathrm{wt}, \widehat{\varepsilon}_p^*, \widehat{\varphi}_p^*, \widehat{e}_p^*, \widehat{f}_p^*)$ as follows.

$$B_p^* := \{b_p^*(n) | n \in \mathbb{Z}\},$$

$$\mathrm{wt}\big(b_p^*(n)\big) := n\widehat{\alpha}_p, \qquad \widehat{\varepsilon}_q^*\big(b_p^*(n)\big) := \begin{cases} -n & \text{if } q = p, \\ -\infty & \text{if } q \neq p, \end{cases}$$

$$\widehat{\varphi}_q^*\big(b_p^*(n)\big) := \begin{cases} n & \text{if } q = p, \\ -\infty & \text{if } q \neq p, \end{cases}$$

$$\widehat{e}_q^*\big(b_p^*(n)\big) := \begin{cases} b_p^*(n+1) & \text{if } q = p, \\ 0 & \text{if } q \neq p, \end{cases} \qquad \widehat{f}_q^*\big(b_p^*(n)\big) := \begin{cases} b_p^*(n-1) & \text{if } q = p, \\ 0 & \text{if } q \neq p. \end{cases}$$

For simplicity of notation, we set $b_p^* := b_p^*(0)$.

Theorem 6 ([6]) *Let* $(B; \mathrm{wt}, \widehat{\varepsilon}_p^*, \widehat{\varphi}_p^*, \widehat{e}_p^*, \widehat{f}_p^*)$ *be a* $U_q(\widehat{\mathfrak{sl}}_l)$*-crystal, and let* b_∞^* *be an element of* B *of weight zero. We assume that the following seven conditions are satisfied.*

(1) $\mathrm{wt}(B) \subset \widehat{Q}^-$.
(2) b_∞^* *is the unique element of* B *of weight zero.*
(3) $\widehat{\varepsilon}_p^*(b_\infty^*) = 0$ *for all* $p \in \widehat{I}$.
(4) $\widehat{\varepsilon}_p^*(b) \in \mathbb{Z}$ *for all* $p \in \widehat{I}$ *and* $b \in B$.
(5) *For each* $p \in \widehat{I}$, *there exists a strict embedding* $\Psi_p^* : B \to B \otimes B_p^*$.

(6) $\Psi_p^*(B) \subset B \otimes \{(\widehat{f}_p^*)^n b_p^* | n \geq 0\}$ *for all* $p \in \widehat{I}$.
(7) *For every* $b \in B$ *such that* $b \neq b_\infty^*$, *there exists* $p \in \widehat{I}$ *such that* $\Psi_p^*(b) = b' \otimes (\widehat{f}_p^*)^n b_p^*$ *with* $n > 0$.

Then, $(B; \mathrm{wt}, \widehat{\varepsilon}_p^*, \widehat{\varphi}_p^*, \widehat{e}_p^*, \widehat{f}_p^*)$ *is isomorphic as a crystal to* $B(\infty)$.

Let us check the seven conditions above for the crystal $((\mathcal{BZ}_\mathbb{Z}^e)^\sigma; \mathrm{wt}, \widehat{\varepsilon}_p^*, \widehat{\varphi}_p^*, \widehat{e}_p^*, \widehat{f}_p^*)$, with $b_\infty^* = \mathbf{O}^*$. Conditions (1)~(4) are obvious from the definitions. In the next subsection, we construct a strict embedding $\Psi_p^* : (\mathcal{BZ}_\mathbb{Z}^e)^\sigma \to (\mathcal{BZ}_\mathbb{Z}^e)^\sigma \otimes B_p^*$ for each $p \in \widehat{I}$, and check conditions (6) and (7).

Remark 1 Since our aim is to prove Theorem 5, we consider the $*$-crystal structure on $(\mathcal{BZ}_\mathbb{Z}^e)^\sigma$, not the ordinary crystal structure.

5.2 *Proof of Theorem 5 and the Connectedness*

Definition 7 Let $p \in \widehat{I}$. We define a map $\Psi_p^* : (\mathcal{BZ}_\mathbb{Z}^e)^\sigma \to (\mathcal{BZ}_\mathbb{Z}^e)^\sigma \otimes B_p^*$ by $\Psi_p(\mathbf{M}) := \mathbf{M}' \otimes (\widehat{f}_p^*)^c b_p^*$. Here, $c := \widehat{\varepsilon}_p(\mathbf{M})$ and $\mathbf{M}' := \widehat{e}_p^c \mathbf{M}$.

The following lemma is obvious from the definitions.

Lemma 20 (1) Ψ_p *is an injective map*.
(2) *For every* $\mathbf{M} \in (\mathcal{BZ}_\mathbb{Z}^e)^\sigma$, *we have* $\mathrm{wt}(\Psi_p^*(\mathbf{M})) = \mathrm{wt}(\mathbf{M})$.

Proof of Theorem 5 If condition (5) is satisfied for the Ψ_p^* above, then conditions (6) and (7) are automatically satisfied by the definitions. Therefore, the remaining task is to check condition (5). However, by an argument similar to the one in [6], this follows from Proposition 16. Thus, we have established the theorem. □

Corollary 10 (1) $(\mathcal{BZ}_\mathbb{Z}^e)^\sigma(\mathbf{O}^*) = (\mathcal{BZ}_\mathbb{Z}^e)^\sigma$.
(2) $\mathcal{BZ}_\mathbb{Z}^\sigma(\mathbf{O}) = \mathcal{BZ}_\mathbb{Z}^\sigma$.

Proof (1) is a direct consequence of the main theorem. Applying the map $*$ on both sides of (1), we obtain (2). □

The second equality above is what we announced in the "note added in proof" of [8].

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Quiver Varieties and Tensor Products, II

Hiraku Nakajima

Abstract We define a family of homomorphisms on a collection of convolution algebras associated with quiver varieties, which gives a kind of coproduct on the Yangian associated with a symmetric Kac-Moody Lie algebra. We study its property using perverse sheaves.

1 Introduction

In the conference the author explained his joint work with Guay on a construction of a coproduct on the Yangian $Y(\mathfrak{g})$ associated with an affine Kac-Moody Lie algebra $\mathfrak{g}$. It is a natural generalization of the coproduct on the usual Yangian $Y(\mathfrak{g})$ for a finite dimensional complex simple Lie algebra $\mathfrak{g}$ given by Drinfeld [7]. Its definition is motivated also by a recent work of Maulik and Okounkov [12] on a geometric construction of a tensor product structure on equivariant homology groups of holomorphic symplectic varieties, in particular of quiver varieties. The purpose of this paper is to explain this geometric background.

For quiver varieties of finite type, the geometric coproduct corresponding to the Drinfeld coproduct on Yangian $Y(\mathfrak{g})$ was studied in [17, 21, 22]. (And one corresponding to the coproduct on $\mathfrak{g}$ was studied also in [11].) But the results depend on the algebraic definition of the coproduct. As it is not known how to define a coproduct on $Y(\mathfrak{g})$ for an arbitrary Kac-Moody Lie algebra $\mathfrak{g}$, the results cannot be generalized to other types.

In this paper, we take a geometric approach and define a kind of a coproduct on convolution algebras associated with quiver varieties together with a $\mathbb{C}^*$-action preserving the holomorphic symplectic form, and study its properties using perverse sheaves.

In fact, we have an ambiguity in the definition of the coproduct, and we have a family of coproducts Δ_c, parametrized by c in a certain affine space. This ambiguity of the coproduct was already noticed in [22, Remark in §5.2]. Maulik-Okounkov

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theory gives a canonical choice of c for a quiver variety of an arbitrary type, and gives the formula of Δ_c on standard generators of $Y(\mathfrak{g})$. Therefore we can take the formula as a definition of the coproduct and check its compatibility with the defining relations of $Y(\mathfrak{g})$. This will be done for an affine Kac-Moody Lie algebra $\mathfrak{g}$ as we explained in the conference. (The formula is a consequence of results in [12], and hence is not explained here.)

Although there is a natural choice, the author hopes that our framework, considering also other possibilities for Δ, is suitable for a modification to other examples of convolution algebras when geometry does not give us such a canonical choice. (For example, the AGT conjecture for a general group. See [20].)

Remark also that our construction is specific for $Y(\mathfrak{g})$, and is not clear how to apply for a quantum loop algebra $U_q(\mathbf{L}\mathfrak{g})$. We need to replace cohomology groups by K groups to deal with the latter, but many of our arguments work only for cohomology groups.

Finally let us comment on a difference on the coproduct for quiver varieties of finite type and other types. A coproduct on an algebra A usually means an algebra homomorphism $\Delta\colon A \to A \otimes A$ satisfying the coassociativity. In our setting the algebra A depends on the dimension vector, or equivalently dominant weight $\mathbf{w}$. Hence Δ is supposed to be a homomorphism from the algebra $A(\mathbf{w})$ for $\mathbf{w}$ to the tensor product $A(\mathbf{w}^1) \otimes A(\mathbf{w}^2)$ with $\mathbf{w} = \mathbf{w}^1 + \mathbf{w}^2$. For a quiver of type ADE, this is true, but not in general. See Remark 1 for the crucial point. The target of Δ is, in general, larger than $A(\mathbf{w}^1) \otimes A(\mathbf{w}^2)$. Fortunately this difference is not essential, for example, study of tensor product structures of representations of Yangians.

1.1 Notations

The definition and notation of quiver varieties related to a coproduct are as in [17], except the followings:

- Linear maps i, j are denoted by a, b here.
- A quiver possibly contains edge loops. Roots are defined as in [6, §2]. They are obtained from coordinate vectors at loop free vertices or $\pm$ elements in the fundamental region by applying some sequences of reflections at loop free vertices.
- Varieties $\mathfrak{Z}, \widetilde{\mathfrak{Z}}$ are denote by $\mathfrak{T}, \widetilde{\mathfrak{T}}$ here.

We say a quiver is of *finite type*, if its underlying graph is of type ADE. We say it is of *affine type*, if it is Jordan quiver or its underlying graph is an extended Dynkin diagram of type ADE.

For $\mathbf{v} = (v_i)$, $\mathbf{v}' = (v_i') \in \mathbb{Z}^I$, we say $\mathbf{v} \le \mathbf{v}'$ if $v_i \le v_i'$ for any $i \in I$.

For a variety X, $H_*(X)$ denote its Borel-Moore homology group. It is the dual to $H_c^*(X)$ the cohomology group with compact support.

We will use the homology group $H_*(L)$ of a closed variety L in a smooth variety M in several contexts. There is often a preferred degree in the context, which is written as 'top' below. For example, if L is Lagrangian, it is $\dim_{\mathbb{C}} M$. If M has several components M_α of various dimensions, we mean $H_{\mathrm{top}}(L)$ to be the direct sum of $H_{\mathrm{top}}(L \cap M_\alpha)$, though the degree 'top' changes for each $L \cap M_\alpha$.

Let $D(X)$ denote the bounded derived category of complexes of constructible $\mathbb{C}$-sheaves on X. When X is smooth, $\mathcal{C}_X \in D(X)$ denote the constant sheaf on X shifted by $\dim X$. If X is a disjoint union of smooth varieties X_α with various dimensions, we understand $\mathcal{C}_X$ as the direct sum of $\mathcal{C}_{X_\alpha}$.

The intersection cohomology (IC for short) complex associated with a smooth locally closed subvariety $Y \subset X$ and a local system ρ on Y is denoted by $IC(Y, \rho)$ or $IC(\overline{Y}, \rho)$. If ρ is the trivial rank 1 local system, we simply denote it by $IC(Y)$ or $IC(\overline{Y})$.

2 Quiver Varieties

In this section we fix the notation for quiver varieties. See [13, 14] for detail.

Suppose that a finite graph is given. Let I be the set of vertices and E the set of edges. In [13, 14] the author assumed that the graph does not contain edge loops (i.e., no edges joining a vertex with itself), but most of results (in particular definitions, natural morphisms, etc.) hold without this assumption.

Let H be the set of pairs consisting of an edge together with its orientation. So we have $\#H = 2\#E$. For $h \in H$, we denote by $\mathrm{i}(h)$ (resp. $\mathrm{o}(h)$) the incoming (resp. outgoing) vertex of h. For $h \in H$ we denote by $\overline{h}$ the same edge as h with the reverse orientation. Choose and fix an orientation $\varOmega$ of the graph, i.e., a subset $\varOmega \subset H$ such that $\overline{\varOmega} \cup \varOmega = H$, $\varOmega \cap \overline{\varOmega} = \emptyset$. The pair $(I, \varOmega)$ is called a *quiver*.

Let $V = (V_i)_{i \in I}$ be a finite dimensional I-graded vector space over $\mathbb{C}$. The dimension of V is a vector

$$\dim V = (\dim V_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I.$$

If V^1 and V^2 are I-graded vector spaces, we define vector spaces by

$$\mathrm{L}\big(V^1, V^2\big) \stackrel{\text{def.}}{=} \bigoplus_{i \in I} \mathrm{Hom}\big(V_i^1, V_i^2\big), \qquad \mathrm{E}\big(V^1, V^2\big) \stackrel{\text{def.}}{=} \bigoplus_{h \in H} \mathrm{Hom}\big(V_{\mathrm{o}(h)}^1, V_{\mathrm{i}(h)}^2\big).$$

For $B = (B_h) \in \mathrm{E}(V^1, V^2)$ and $C = (C_h) \in \mathrm{E}(V^2, V^3)$, let us define a multiplication of B and C by

$$CB \stackrel{\text{def.}}{=} \left(\sum_{\mathrm{i}(h)=i} C_h B_{\overline{h}}\right)_i \in \mathrm{L}\big(V^1, V^3\big).$$

Multiplications ba, Ba of $a \in \mathrm{L}(V^1, V^2)$, $b \in \mathrm{L}(V^2, V^3)$, $B \in \mathrm{E}(V^2, V^3)$ are defined in the obvious manner. If $a \in \mathrm{L}(V^1, V^1)$, its trace $\mathrm{tr}(a)$ is understood as $\sum_i \mathrm{tr}(a_i)$.

For two I-graded vector spaces V, W with $\mathbf{v} = \dim V$, $\mathbf{w} = \dim W$, we consider the vector space given by

$$\mathbf{M} \equiv \mathbf{M}(\mathbf{v}, \mathbf{w}) \stackrel{\text{def.}}{=} \mathrm{E}(V, V) \oplus \mathrm{L}(W, V) \oplus \mathrm{L}(V, W),$$

where we use the notation $\mathbf{M}$ when $\mathbf{v}$, $\mathbf{w}$ are clear in the context. The above three components for an element of $\mathbf{M}$ will be denoted by $B = \bigoplus B_h$, $a = \bigoplus a_i$, $b = \bigoplus b_i$ respectively.

The orientation Ω defines a function $\varepsilon\colon H \to \{\pm 1\}$ by $\varepsilon(h) = 1$ if $h \in \Omega$, $\varepsilon(h) = -1$ if $h \in \overline{\Omega}$. We consider ε as an element of $\mathrm{L}(V, V)$. Let us define a symplectic form ω on $\mathbf{M}$ by

$$\omega\big((B, a, b), (B', a', b')\big) \stackrel{\text{def.}}{=} \operatorname{tr}\big(\varepsilon B\, B'\big) + \operatorname{tr}\big(ab' - a'b\big).$$

Let $G \equiv G_{\mathbf{v}}$ be an algebraic group defined by

$$G \equiv G_{\mathbf{v}} \stackrel{\text{def.}}{=} \prod_i \mathrm{GL}(V_i).$$

Its Lie algebra is the direct sum $\bigoplus_i \mathfrak{gl}(V_i)$. The group G acts on $\mathbf{M}$ by

$$(B, a, b) \mapsto g \cdot (B, a, b) \stackrel{\text{def.}}{=} \big(gBg^{-1}, ga, bg^{-1}\big)$$

preserving the symplectic structure.

The moment map vanishing at the origin is given by

$$\mu(B, a, b) = \varepsilon B\, B + ab \in \mathrm{L}(V, V),$$

where the dual of the Lie algebra of G is identified with $\mathrm{L}(V, V)$ via the trace.

We would like to consider a 'symplectic quotient' of $\mu^{-1}(0)$ divided by G. However we cannot expect the set-theoretical quotient to have a good property. Therefore we consider the quotient using the geometric invariant theory. Then the quotient depends on an additional parameter $\zeta = (\zeta_i)_{i \in I} \in \mathbb{Z}^I$ as follows: Let us define a character of G by

$$\chi_\zeta(g) \stackrel{\text{def.}}{=} \prod_{i \in I} (\det g_i)^{-\zeta_i}.$$

Let $A(\mu^{-1}(0))$ be the coordinate ring of the affine variety $\mu^{-1}(0)$. Set

$$A\big(\mu^{-1}(0)\big)^{G, \chi_\zeta^n} \stackrel{\text{def.}}{=} \big\{ f \in A\big(\mu^{-1}(0)\big) \mid f\big(g \cdot (B, a, b)\big) = \chi_\zeta(g)^n f\big((B, a, b)\big) \big\}.$$

The direct sum with respect to $n \in \mathbb{Z}_{\geq 0}$ is a graded algebra, hence we can define

$$\mathfrak{M}_\zeta \equiv \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \equiv \mathfrak{M}_\zeta(V, W) \stackrel{\text{def.}}{=} \operatorname{Proj}\Big(\bigoplus_{n \geq 0} A\big(\mu^{-1}(0)\big)^{G, \chi_\zeta^n} \Big).$$

This is the *quiver variety* introduced in [13]. Since this space is unchanged when we replace χ by a positive power χ^N ($N > 0$), this space is well-defined for $\zeta \in \mathbb{Q}^I$. We call ζ a *stability parameter*.

We use two special stability parameters in this paper. When $\zeta = 0$, the corresponding $\mathfrak{M}_0$ is an affine algebraic variety whose coordinate ring consists of the G-invariant functions on $\mu^{-1}(0)$.

Another choice is $\zeta_i = 1$ for all i. In this case, we denote the corresponding variety simply by $\mathfrak{M}$. The corresponding stability condition is that an I-graded subspace V' of V invariant under B and contained in $\operatorname{Ker} b$ is 0 [14, Lemma 3.8]. The stability and semistability are equivalent in this case, and the action of G on the set $\mu^{-1}(0)^{\mathrm{s}}$ of stable points is free, and $\mathfrak{M}$ is the quotient $\mu^{-1}(0)^{\mathrm{s}}/G$. In particular $\mathfrak{M}$ is nonsingular.

3 Tensor Product Varieties

Let $W^2 \subset W$ be an I-graded subspace and $W^1 = W/W^2$ be the quotient. We fix an isomorphism $W \cong W^1 \oplus W^2$. We define a one parameter subgroup $\lambda\colon \mathbb{C}^* \to G_W$ by $\lambda(t) = \mathrm{id}_{W^1} \oplus t\,\mathrm{id}_{W^2}$. Then $\mathbb{C}^*$ acts on $\mathfrak{M}$, $\mathfrak{M}_0$ through λ.

We fix $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{w}^1 = \dim W^1$, $\mathbf{w}^2 = \dim W^2$ throughout this paper. Since we use several quiver varieties with different dimension vectors, let us use the notation $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1)$, etc. for those, while the notation $\mathfrak{M}$ means the original $\mathfrak{M}(\mathbf{v}, \mathbf{w})$.

3.1 Fixed Points

We consider the fixed point loci $\mathfrak{M}^{\mathbb{C}^*}$, $\mathfrak{M}_0^{\mathbb{C}^*}$. The former decomposes as

$$\mathfrak{M}^{\mathbb{C}^*} = \bigsqcup_{\mathbf{v}=\mathbf{v}^1+\mathbf{v}^2} \mathfrak{M}\bigl(\mathbf{v}^1, \mathbf{w}^1\bigr) \times \mathfrak{M}\bigl(\mathbf{v}^2, \mathbf{w}^2\bigr) \tag{1}$$

(see [17, Lemma 3.2]). The isomorphism is given by considering the direct sum of $[B^1, a^1, b^1] \in \mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1)$ and $[B^2, a^2, b^2] \in \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$ as a point in $\mathfrak{M}$. Since quiver varieties $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1)$, $\mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$ are connected, this is a decomposition of $\mathfrak{M}^{\mathbb{C}^*}$ into connected components.

Let us study the second fixed point locus $\mathfrak{M}_0^{\mathbb{C}^*}$. We have a morphism

$$\sigma\colon \bigsqcup_{\mathbf{v}=\mathbf{v}^1+\mathbf{v}^2} \mathfrak{M}_0\bigl(\mathbf{v}^1, \mathbf{w}^1\bigr) \times \mathfrak{M}_0\bigl(\mathbf{v}^2, \mathbf{w}^2\bigr) \to \mathfrak{M}_0^{\mathbb{C}^*}$$

given by the direct sum as above. This cannot be an isomorphism unless $\mathbf{v} = 0$ as the inverse image of 0 consists of several points corresponding to various decomposition $\mathbf{v} = \mathbf{v}^1 + \mathbf{v}^2$. This is compensated by considering the direct limit $\mathfrak{M}_0(\mathbf{w}) = \bigcup_{\mathbf{v}} \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ if the underlying graph is of type ADE. But this trick does not solve the problem yet in general. For example, if the quiver is the Jordan quiver, and $\mathbf{v}^1 = \mathbf{w}^1 = \mathbf{v}^2 = \mathbf{w}^2 = 1$, we have $\mathfrak{M}_0(\mathbf{v}^1, \mathbf{w}^1) = \mathfrak{M}_0(\mathbf{v}^2, \mathbf{w}^2) = \mathbb{C}^2$, while $\mathfrak{M}_0^{\mathbb{C}^*} = S^2(\mathbb{C}^2)$. The morphism σ is the quotient map $\mathbb{C}^2 \times \mathbb{C}^2 \to S^2(\mathbb{C}^2) = (\mathbb{C}^2 \times \mathbb{C}^2)/S_2$. Let us study σ further.

Using the stratification [14, Lemma 3.27] we decompose $\mathfrak{M}_0 = \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ as

$$\mathfrak{M}_0 = \bigsqcup_{\mathbf{v}^0} \mathfrak{M}_0^{\mathrm{reg}}\bigl(\mathbf{v}^0, \mathbf{w}\bigr) \times \mathfrak{M}_0\bigl(\mathbf{v} - \mathbf{v}^0, 0\bigr), \tag{2}$$

where $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w})$ is the open subvariety of $\mathfrak{M}_0(\mathbf{v}^0, \mathbf{w})$ consisting of closed free orbits, and $\mathfrak{M}_0(\mathbf{v} - \mathbf{v}^0, 0)$ is the quiver variety associated with $W = 0$. For quiver varieties of type ADE, the factor $\mathfrak{M}_0(\mathbf{v} - \mathbf{v}^0, 0)$ is a single point 0. It is nontrivial in general. For example, if the quiver is the Jordan quiver, we have $\mathfrak{M}_0(\mathbf{v} - \mathbf{v}^0, 0) = S^n(\mathbb{C}^2)$ where $n = \mathbf{v} - \mathbf{v}^0$. Then

Lemma 1 (1) *The above stratification induces a stratification*

$$\mathfrak{M}_0^{\mathbb{C}^*} = \bigsqcup_{\substack{\mathbf{v}^0, {}^1\mathbf{v}, {}^2\mathbf{v} \\ \mathbf{v}^0 = {}^1\mathbf{v} + {}^2\mathbf{v}}} \mathfrak{M}_0^{\mathrm{reg}}\bigl({}^1\mathbf{v}, \mathbf{w}^1\bigr) \times \mathfrak{M}_0^{\mathrm{reg}}\bigl({}^2\mathbf{v}, \mathbf{w}^2\bigr) \times \mathfrak{M}_0\bigl(\mathbf{v} - \mathbf{v}^0, 0\bigr).$$

(2) σ *is a surjective finite morphism.*

Thus the factor with $W = 0$ appears twice in $\mathfrak{M}_0(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}_0(\mathbf{v}^2, \mathbf{w}^2)$ while it appears only once in $\mathfrak{M}_0^{\mathbb{C}^*}$.

Proof (1) We consider $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w})$ as an open subvariety in $\mathfrak{M}(\mathbf{v}^0, \mathbf{w})$ and restrict the decomposition (1). Then it is easy to check that $(x, y) \in \mathfrak{M}({}^1\mathbf{v}, \mathbf{w}^1) \times \mathfrak{M}({}^2\mathbf{v}, \mathbf{w}^2)$ is contained in $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w})$ if and only if x, y are in $\mathfrak{M}_0^{\mathrm{reg}}({}^1\mathbf{v}, \mathbf{w}^1)$, $\mathfrak{M}_0^{\mathrm{reg}}({}^2\mathbf{v}, \mathbf{w}^2)$ respectively. Thus $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w})^{\mathbb{C}^*} = \mathfrak{M}_0^{\mathrm{reg}}({}^1\mathbf{v}, \mathbf{w}^1) \times \mathfrak{M}_0^{\mathrm{reg}}({}^2\mathbf{v}, \mathbf{w}^2)$. Now the assertion is clear as $\mathbb{C}^*$ acts trivially on the factor $\mathfrak{M}_0(\mathbf{v} - \mathbf{v}^0, 0)$.

(2) The coordinate ring of $\mathfrak{M}_0$ is generated by the following two types of functions:

- $\mathrm{tr}(B_{h_N} B_{h_{N-1}} \cdots B_{h_1} \colon V_{\mathrm{o}(h_1)} \to V_{\mathrm{i}(h_N)} = V_{\mathrm{o}(h_1)})$, where $h_1, \dots, h_N$ is a cycle in our graph.
- $\chi(b_{\mathrm{i}(h_N)} B_{h_N} B_{h_{N-1}} \cdots B_{h_1} a_{\mathrm{o}(h_1)})$, where $h_1, \dots, h_N$ is a path in our graph, and χ is a linear form on $\mathrm{Hom}(W_{\mathrm{o}(h_1)}, W_{\mathrm{i}(h_N)})$.

Then the generators for $\mathfrak{M}_0^{\mathbb{C}^*}$ are the first type functions and second type functions with $\chi = (\chi_1, \chi_2) \in \mathrm{Hom}(W^1_{\mathrm{o}(h_1)}, W^1_{\mathrm{i}(h_N)}) \oplus \mathrm{Hom}(W^2_{\mathrm{o}(h_1)}, W^2_{\mathrm{i}(h_N)})$.

If we pull back these functions by σ, they become sums of the same types of functions for $\mathfrak{M}_0(\mathbf{v}^1, \mathbf{w}^1)$ and $\mathfrak{M}_0(\mathbf{v}^2, \mathbf{w}^2)$. From this observation, we can easily see that σ is a finite morphism. From (1) it is clearly surjective. □

Remark 1 Let $Z(\mathbf{v}^a, \mathbf{w}^a)$ be the fiber product $\mathfrak{M}(\mathbf{v}^a, \mathbf{w}^a) \times_{\mathfrak{M}_0(\mathbf{v}^a,\mathbf{w}^a)} \mathfrak{M}(\mathbf{v}^a, \mathbf{w}^a)$ for $a = 1, 2$. The fiber product $\mathfrak{M}^{\mathbb{C}^*} \times_{\mathfrak{M}_0^{\mathbb{C}^*}} \mathfrak{M}^{\mathbb{C}^*}$ is larger than the union of the products $Z(\mathbf{v}^1, \mathbf{w}^1) \times Z(\mathbf{v}^2, \mathbf{w}^2)$ in general. For example, consider the Jordan quiver variety with $\mathbf{v}^1 = \mathbf{v}^2 = \mathbf{w}^1 = \mathbf{w}^2 = 1$. Then $\mathfrak{M}(\mathbf{v}^a, \mathbf{w}^a)$ is $\mathbb{C}^2$. The product $Z(\mathbf{v}^1, \mathbf{w}^1) \times Z(\mathbf{v}^2, \mathbf{w}^2)$ is consisting of points (p_1, q_1, p_2, q_2) with $p_1 = q_1$, $p_2 = q_2$. On the other hand, $\mathfrak{M}^{\mathbb{C}^*} \times_{\mathfrak{M}_0^{\mathbb{C}^*}} \mathfrak{M}^{\mathbb{C}^*}$ contains also points with $p_1 = q_2$, $p_2 = q_1$.

On the other hand, if the quiver is of type ADE, we do not have the factor $\mathfrak{M}_0(\mathbf{v} - \mathbf{v}^0, 0)$, and they are the same.

3.2 Review of [17]

In this subsection we recall results in [17, §3], with emphasis on subvarieties in the affine quotient $\mathfrak{M}_0$.

We first define the following varieties which were implicitly introduced in [17, §3]:

$$\mathfrak{T}_0 \stackrel{\text{def.}}{=} \left\{ x \in \mathfrak{M}_0 \mid \lim_{t\to 0} \lambda(t)x \text{ exists} \right\},$$

$$\widetilde{\mathfrak{T}}_0 \stackrel{\text{def.}}{=} \left\{ x \in \mathfrak{M}_0 \mid \lim_{t\to 0} \lambda(t)x = 0 \right\}.$$

By the proof of [17, Lemma 3.6] we have the following: $x = [B, a, b]$ is in $\mathfrak{T}_0$ (resp. $\widetilde{\mathfrak{T}}_0$) if and only if

- $b_{\mathrm{i}(h_N)} B_{h_N} B_{h_{N-1}} \cdots B_{h_1} a_{\mathrm{o}(h_1)}$ maps $W^2_{\mathrm{o}(h_1)}$ to $W^2_{\mathrm{i}(h_N)}$ (resp. $W^2_{\mathrm{o}(h_1)}$ to 0 and the whole $W_{\mathrm{o}(h_1)}$ to $W^2_{\mathrm{i}(h_N)}$) for any path in the doubled quiver.

From this description it also follows that $\mathfrak{T}_0$, $\widetilde{\mathfrak{T}}_0$ are closed subvarieties in $\mathfrak{M}_0$.

We have the inclusion $i\colon \mathfrak{T}_0 \to \mathfrak{M}_0$ and the projection $p\colon \mathfrak{T}_0 \to \mathfrak{M}_0^{\mathbb{C}^*}$ defined by taking $\lim_{t\to 0} \lambda(t)x$. The latter is defined as $\mathfrak{M}_0$ is affine.

We define $\mathfrak{T} \stackrel{\text{def.}}{=} \pi^{-1}(\mathfrak{T}_0)$, $\widetilde{\mathfrak{T}} \stackrel{\text{def.}}{=} \pi^{-1}(\widetilde{\mathfrak{T}}_0)$. These definitions coincide with ones in [17, §3]. Note that we do not have an analog of $p\colon \mathfrak{T}_0 \to \mathfrak{M}_0^{\mathbb{C}^*}$ for $\mathfrak{T}$. Instead we have a decomposition

$$\mathfrak{T} = \bigsqcup_{\mathbf{v}=\mathbf{v}^1+\mathbf{v}^2} \mathfrak{T}\big(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2\big) \tag{3}$$

into locally closed subvarieties, and the projection

$$p_{(\mathbf{v}^1,\mathbf{v}^2)}\colon \mathfrak{T}\big(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2\big) \to \mathfrak{M}\big(\mathbf{v}^1, \mathbf{w}^1\big) \times \mathfrak{M}\big(\mathbf{v}^2, \mathbf{w}^2\big), \tag{4}$$

which is a vector bundle. These are defined by considering the limit $\lim_{t\to 0} \lambda(t)x$. Note that they intersect in their closures, contrary to (1), which was the decomposition into connected components. Since pieces in (3) are mapped to different components, $p_{(\mathbf{v}^1,\mathbf{v}^2)}$'s do not give a morphism defined on $\mathfrak{T}$.

As a vector bundle, $\mathfrak{T}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2)$ is the subbundle of the normal bundle of $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$ in $\mathfrak{M}$ consisting of positive weight spaces. Its rank is half of the codimension of $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$. In fact, the restriction of the tangent space of $\mathfrak{M}$ to $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$ decomposes into weight ± 1 and 0 spaces such that

- the weight 0 subspace gives the tangent bundle of $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$,
- the weight 1 and -1 subspaces are dual to each other with respect to the symplectic form on $\mathfrak{M}$.

We define a partial order $<$ on the set $\{(\mathbf{v}^1, \mathbf{v}^2) \mid \mathbf{v}^1 + \mathbf{v}^2 = \mathbf{v}\}$ defined by $(\mathbf{v}^1, \mathbf{v}^2) \le (\mathbf{v}'^1, \mathbf{v}'^2)$ if and only if $\mathbf{v}^1 \le \mathbf{v}'^1$. We extend it to a total order and denote it also by $<$. Let

$$\mathfrak{T}_{\le(\mathbf{v}^1,\mathbf{v}^2)} \stackrel{\text{def.}}{=} \bigcup_{(\mathbf{v}'^1,\mathbf{v}'^2)\le(\mathbf{v}^1,\mathbf{v}^2)} \mathfrak{T}\big(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2\big),$$

and let $\mathfrak{T}_{<(\mathbf{v}^1,\mathbf{v}^2)}$ be the union obtained similarly by replacing $\le$ by $<$. Then $\mathfrak{T}_{\le(\mathbf{v}^1,\mathbf{v}^2)}$, $\mathfrak{T}_{<(\mathbf{v}^1,\mathbf{v}^2)}$ are closed subvarieties in $\mathfrak{T}$.

3.3 The Fiber Product $Z_{\mathfrak{T}}$

We introduce one more variety, following [12]. Let us consider $\mathfrak{T}_0$ as a subvariety in $\mathfrak{M}_0 \times \mathfrak{M}_0^{\mathbb{C}^*}$, where the projection to the second factor is given by p. Let $Z_{\mathfrak{T}} \subset \mathfrak{M} \times \mathfrak{M}^{\mathbb{C}^*}$ be the inverse image of $\mathfrak{T}_0$ under the restriction of the projective morphism $\pi \times \pi$. This variety is an analog of the variety $Z = \mathfrak{M} \times_{\mathfrak{M}_0} \mathfrak{M}$ introduced in [14, §7]. Note that $Z_{\mathfrak{T}}$ is also given as a fiber product $\mathfrak{T} \times_{\mathfrak{M}_0^{\mathbb{C}^*}} \mathfrak{M}^{\mathbb{C}^*}$, where $\mathfrak{T} \to \mathfrak{M}_0^{\mathbb{C}^*}$ is given by the composition of $\pi : \mathfrak{T} \to \mathfrak{T}_0$ and $p : \mathfrak{T}_0 \to \mathfrak{M}_0^{\mathbb{C}^*}$. We will consider a cycle in $Z_{\mathfrak{T}}$ as a correspondence in $\mathfrak{M} \times \mathfrak{M}^{\mathbb{C}^*}$ later. Note that the projection $p_1 : Z_{\mathfrak{T}} \to \mathfrak{M}$ is proper, but $p_2 : Z_{\mathfrak{T}} \to \mathfrak{M}^{\mathbb{C}^*}$ is not.

We consider the two decompositions (1, 3). For brevity, we change the notation as

$$\mathfrak{M}^{\mathbb{C}^*} = \bigsqcup_{\alpha} \mathfrak{M}_\alpha, \qquad \mathfrak{T} = \bigsqcup_{\alpha} \mathfrak{T}_\alpha.$$

We also recall

$$\mathfrak{T}_{\le\alpha} = \bigsqcup_{\beta\le\alpha} \mathfrak{T}_\beta, \qquad \mathfrak{T}_{<\alpha} = \bigsqcup_{\beta<\alpha} \mathfrak{T}_\beta.$$

These are closed subvarieties in $\mathfrak{T}$.

Then they induce a decomposition

$$Z_{\mathfrak{T}} = \bigsqcup_{\alpha,\beta} Z_{\mathfrak{T},\alpha,\beta},$$

with

$$Z_{\mathfrak{T},\alpha,\beta} \stackrel{\text{def.}}{=} Z_{\mathfrak{T}} \cap (\mathfrak{T}_\alpha \times \mathfrak{M}_\beta).$$

We have the corresponding decomposition

$$Z^{\mathbb{C}^*} = \mathfrak{M}^{\mathbb{C}^*} \times_{\mathfrak{M}_0^{\mathbb{C}^*}} \mathfrak{M}^{\mathbb{C}^*} = \bigsqcup_{\alpha,\beta} Z_{\alpha,\beta}$$

induced from the decomposition of the first and second factors.

We also define

$$Z_{\mathfrak{T},\leq\alpha,\beta} \stackrel{\text{def.}}{=} Z_{\mathfrak{T}} \cap (\mathfrak{T}_{\leq\alpha} \times \mathfrak{M}_\beta), \qquad Z_{\mathfrak{T},<\alpha,\beta} \stackrel{\text{def.}}{=} Z_{\mathfrak{T}} \cap (\mathfrak{T}_{<\alpha} \times \mathfrak{M}_\beta).$$

They are closed subvarieties in $Z_{\mathfrak{T}}$ and $Z_{\mathfrak{T},\alpha,\beta}$ is an open subvariety in $Z_{\mathfrak{T},\leq\alpha,\beta}$. On the other hand, each $Z_{\alpha,\beta}$ is a closed subvariety in $\mathfrak{M}_\alpha \times \mathfrak{M}_\beta$.

Each piece $Z_{\mathfrak{T},\alpha,\beta}$ is a vector bundle over $Z_{\alpha,\beta}$, which is pull-back of $\mathfrak{T}_\alpha \to \mathfrak{M}_\alpha$. Therefore its rank is half of the codimension of $\mathfrak{M}_\alpha$ in $\mathfrak{M}$.

Proposition 1 (1) *Each irreducible component of $Z_{\mathfrak{T},\alpha,\beta}$ is at most half dimensional in $\mathfrak{M} \times \mathfrak{M}^{\mathbb{C}^*}$, and hence the same is true for $Z_{\mathfrak{T}}$.*

(2) *Irreducible components of $Z_{\mathfrak{T}}$ of half dimension are Lagrangian subvarieties in $\mathfrak{M} \times \mathfrak{M}^{\mathbb{C}^*}$.*

Here $\mathfrak{M}^{\mathbb{C}^*}$ has several connected components of various dimensions, so the above more precisely meant half dimensional in each component $\mathfrak{M} \times \mathfrak{M}_\beta$.

Proof (1) It is known that $\pi : \mathfrak{M} \to \mathfrak{M}_0$ is semismall, if we replace the target by the image $\pi(\mathfrak{M}_0)$. (This is a consequence of [13, 6.11] as explained in [19, 2.23].) Therefore irreducible components of $Z = \mathfrak{M} \times_{\mathfrak{M}_0} \mathfrak{M}$ are at most half dimensional in $\mathfrak{M} \times \mathfrak{M}$. As σ is a finite morphism, the same is true for $Z^{\mathbb{C}^*}$. Now the assertion for $Z_{\mathfrak{T},\alpha,\beta}$ follows as it is a vector bundle over $Z_{\alpha,\beta}$ whose rank is equal to the half of codimension of $\mathfrak{M}_\alpha$.

(2) This follows from the local description of π in [16, Theorem 3.3.2] which respects the symplectic form from its proof, together with the fact that $\pi^{-1}(0)$ is isotropic by the proof of [13, Theorem 5.8]. □

4 Coproduct

In this section we define a kind of a coproduct on the convolution algebra $H_*(Z)$. The target of Δ is, in general, larger than the tensor product $H_*(Z(\mathbf{w}^1)) \otimes H_*(Z(\mathbf{w}^2))$ as we mentioned in the introduction.

4.1 Convolution Algebras

Recall the fiber product $Z = \mathfrak{M} \times_{\mathfrak{M}_0} \mathfrak{M}$. The convolution product defines an algebra structure on $H_*(Z)$:

$$a * b \stackrel{\text{def.}}{=} p_{13*}\big(p_{12}^*(a) \cap p_{23}^*(b)\big), \quad a, b \in H_*(Z),$$

where p_{ij} is the projection from $\mathfrak{M} \times \mathfrak{M} \times \mathfrak{M}$ to the product of the i^{th} and j^{th}-factors, and Z is viewed as a subvariety in $\mathfrak{M} \times \mathfrak{M}$ for the cap product. (See [5, §2.7] for more detail.)

As $\pi : \mathfrak{M} \to \pi(\mathfrak{M})$ is a semismall morphism, the top degree component $H_{\text{top}}(Z)$ is a subalgebra, where 'top' is equal to the complex dimension of $\mathfrak{M} \times \mathfrak{M}$. Moreover $H_*(Z)$ is a graded algebra, where the degree p elements are in $H_{\text{top}-p}(Z)$. (See [5, §8.9].)

Take $x \in \mathfrak{M}_0$. We consider the inverse image $\pi^{-1}(x) \subset \mathfrak{M}$ and denote it by $\mathfrak{M}_x$. (When $x = 0$, this is denoted by $\mathfrak{L}$ usually.) Then the convolution gives $\bigoplus H_{\text{top}-p}(\mathfrak{M}_x)$ a structure of a module of $H_*(Z)$. Here 'top' is the difference of complex dimensions of $\mathfrak{M}$ and the stratum containing x.

Similarly we can define a graded algebra structure on

$$H_*\big(Z^{\mathbb{C}^*}\big) = \bigoplus H_{\text{top}-p}\big(Z^{\mathbb{C}^*}\big),$$

where 'top' means the complex dimension of $\mathfrak{M}^{\mathbb{C}^*} \times \mathfrak{M}^{\mathbb{C}^*}$, possibly different on various connected components. By Sect. 3.1 it is close to

$$\bigoplus_{\mathbf{v}^1+\mathbf{v}^2=\mathbf{v}^2} H_*\big(Z\big(\mathbf{v}^1, \mathbf{w}^1\big)\big) \otimes H_*\big(Z\big(\mathbf{v}^2, \mathbf{w}^2\big)\big),$$

but is different in general, as explained in Remark 1.

We denote by $\mathfrak{M}_x^{\mathbb{C}^*}$ the inverse image $(\pi^{\mathbb{C}^*})^{-1}(x)$ in $\mathfrak{M}^{\mathbb{C}^*}$ for $x \in \mathfrak{M}_0^{\mathbb{C}^*}$. Its homology $\bigoplus H_{\text{top}-p}(\mathfrak{M}_x^{\mathbb{C}^*})$ is a graded module of $H_*(Z^{\mathbb{C}^*})$. Here 'top' is the difference of complex dimensions of $\mathfrak{M}^{\mathbb{C}^*}$ (resp. $\mathfrak{M}$) and the stratum containing x.

4.2 Convolution by $Z_{\mathfrak{T}}$

Take $x \in \mathfrak{M}_0^{\mathbb{C}^*}$. We consider the inverse image $(p \circ \pi_{\mathfrak{T}})^{-1}(x) \subset \mathfrak{T} \subset \mathfrak{M}$ and denote it by $\mathfrak{T}_x$. (When $x = 0$, this is denoted by $\widetilde{\mathfrak{T}}$ in Sect. 3.2.) By the convolution product its homology $H_*(\mathfrak{T}_x) = \bigoplus H_{\text{top}-p}(\mathfrak{T}_x)$ is a graded module of $H_*(Z)$. Here 'top' is the difference of complex dimensions of $\mathfrak{M}$ and the stratum containing x.

Let $\mathfrak{T}_{\alpha,x}, \mathfrak{T}_{\le\alpha,x}, \mathfrak{T}_{<\alpha,x}$ be the intersection of $\mathfrak{T}_x$ with $\mathfrak{T}_\alpha, \mathfrak{T}_{\le\alpha}, \mathfrak{T}_{<\alpha}$ respectively. We have a short exact sequence

$$0 \to H_{\text{top}-p}(\mathfrak{T}_{<\alpha,x}) \to H_{\text{top}-p}(\mathfrak{T}_{\le\alpha,x}) \to H_{\text{top}-p}(\mathfrak{T}_{\alpha,x}) \to 0.$$

(See [17, §3] and (11) below.)

Let us restrict the projection $p_\alpha\colon \mathfrak{T}_\alpha \to \mathfrak{M}_\alpha$ in (4) to $\mathfrak{T}_{\alpha,x}$. As $\pi^{\mathbb{C}^*} \circ p_\alpha = p \circ \pi$, it identifies $\mathfrak{T}_{\alpha,x}$ with its inverse image of $\mathfrak{M}_{\alpha,x} \stackrel{\text{def.}}{=} \mathfrak{M}_\alpha \cap \mathfrak{M}_x^{\mathbb{C}^*}$. Therefore we can replace the last term of the short exact sequence by $H_{\mathrm{top}-p}(\mathfrak{M}_{\alpha,x})$ thanks to the Thom isomorphism:

$$0 \to H_{\mathrm{top}-p}(\mathfrak{T}_{<\alpha,x}) \to H_{\mathrm{top}-p}(\mathfrak{T}_{\le\alpha,x}) \to H_{\mathrm{top}-p}(\mathfrak{M}_{\alpha,x}) \to 0. \tag{5}$$

Our convention of ‘top’ is compatible for $\mathfrak{T}_x$ and $\mathfrak{M}_x$ as the rank of the vector bundle is the half of codimension of $\mathfrak{M}_\alpha$ in $\mathfrak{M}$. Since $\mathfrak{T}_{\le\alpha} = \mathfrak{T}$ when α is the maximal element, we get

Lemma 2 *$H_{\mathrm{top}-p}(\mathfrak{T}_x)$ has a filtration whose associated graded is isomorphic to $H_{\mathrm{top}-p}(\mathfrak{M}_x^{\mathbb{C}^*})$.*

Choice of splittings $H_{\mathrm{top}-p}(\mathfrak{T}_{\le\alpha,x}) \leftarrow H_{\mathrm{top}-p}(\mathfrak{M}_{\alpha,x})$ in (5) for all α gives an isomorphism $H_{\mathrm{top}-p}(\mathfrak{T}_x) \cong H_{\mathrm{top}-p}(\mathfrak{M}_x^{\mathbb{C}^*})$. Our next goal is to understand the space of all splittings in a geometric way.

For this purpose we consider the top degree homology group $H_{\mathrm{top}}(Z_{\mathfrak{T}})$. They are spanned by Lagrangian irreducible components of $Z_{\mathfrak{T}}$ by Proposition 1.

Let $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ and $p \in \mathbb{Z}$. The convolution product

$$a \mapsto c * a \stackrel{\text{def.}}{=} p_{1*}\bigl(c \cap p_2^*(a)\bigr)$$

defines an operator

$$c*\colon H_{\mathrm{top}-p}\bigl(\mathfrak{M}_x^{\mathbb{C}^*}\bigr) \to H_{\mathrm{top}-p}(\mathfrak{T}_x), \tag{6}$$

where p_1, p_2 are projections from $\mathfrak{M} \times \mathfrak{M}^{\mathbb{C}^*}$ to the first and second factors. The degree shift p is preserved by the argument in [5, §8.9]. (If we choose c from $H_{\mathrm{top}-k}(Z_{\mathfrak{T}})$, the convolution maps $H_{\mathrm{top}-p}$ to $H_{\mathrm{top}-p-k}$.) Note also that the above operation is well-defined as p_1 is proper, while the operator $p_{2*}(c \cap p_1^*(-))$ is not in this setting.

An arbitrary class $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ does not give a splitting of (5), as it is nothing to do with the decomposition $\mathfrak{T} = \bigsqcup \mathfrak{T}_\alpha$. Let us write down a sufficient condition to give a splitting.

Since c is in the top degree, it is a linear combination of fundamental classes of Lagrangian irreducible components of $Z_{\mathfrak{T}}$. From Proposition 1(1), half-dimensional irreducible components are closures of half-dimensional irreducible components of $Z_{\mathfrak{T},\beta,\alpha}$ for some pair α, β. Therefore we can write

$$c = \sum_{\alpha,\beta} c_{\beta,\alpha}.$$

Moreover its proof there, the latter are pull-backs of half-dimensional irreducible components of $Z_{\beta,\alpha}$ under the projection $p_\beta \times \mathrm{id}_{\mathfrak{M}_\alpha}$.

We impose the following conditions on c:

$$c_{\beta,\alpha} = 0 \text{ unless } \alpha \geq \beta, \tag{7a}$$

$$c_{\alpha,\alpha} = \left[\overline{(p_\alpha \times \mathrm{id}_{\mathfrak{M}_\alpha})^{-1}(\Delta_{\mathfrak{M}_\alpha})}\right]. \tag{7b}$$

The first condition also means that c is in the image of $\bigoplus_\alpha H_{\mathrm{top}}(Z_{\mathfrak{T},\leq\alpha,\alpha}) \to H_{\mathrm{top}}(Z_{\mathfrak{T}})$. Note that $\bigsqcup_\alpha Z_{\mathfrak{T},\leq\alpha,\alpha}$ is a disjoint union of closed subvarieties in $Z_{\mathfrak{T}}$, and hence the push-forward homomorphism is defined.

Proposition 2 *Let $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ with the conditions* (7a), (7b). *Then $c*$ is an isomorphism and gives a splitting of* (5) *for all α.*

We will show the converse in Sect. 5.2: $c*$ gives a splitting if and only if c satisfies (7a), (7b).

Proof By the first condition the operator $c*$ restricts to $H_{\mathrm{top}-p}(\mathfrak{M}_{\alpha,x}) \to H_{\mathrm{top}-p}(\mathfrak{T}_{\leq\alpha,x})$. And by the second condition it gives the identity if we compose $H_{\mathrm{top}-p}(\mathfrak{T}_{\leq\alpha,x}) \to H_{\mathrm{top}-p}(\mathfrak{M}_{\alpha,x})$. Thus $c*$ gives a splitting of (5). □

Next we construct the inverse of $c*$ also by a convolution product. We consider

$$\mathfrak{T}_0^- \stackrel{\text{def.}}{=} \left\{x \in \mathfrak{M}_0 \mid \lim_{t\to\infty} \lambda(t)x \text{ exists}\right\},$$

and the similarly defined variety $\mathfrak{T}^-$ also by replacing $t \to 0$ by $t \to \infty$. We have the inclusion $i^-\colon \mathfrak{T}_0^- \to \mathfrak{M}_0$ and the projection $p^-\colon \mathfrak{T}_0^- \to \mathfrak{M}_0^{\mathbb{C}^*}$. Note also that $\mathfrak{T}_0 \cap \mathfrak{T}_0^- = \mathfrak{M}_0^{\mathbb{C}^*}$.

Let us define $Z_{\mathfrak{T}^-}$ as the fiber product $\mathfrak{M}^{\mathbb{C}^*} \times_{\mathfrak{M}_0^{\mathbb{C}^*}} \mathfrak{T}^-$, and consider it as a subvariety in $\mathfrak{M}^{\mathbb{C}^*} \times \mathfrak{M}$. We swap the first and second factors from $Z_{\mathfrak{T}}$ as it becomes more natural when we consider a composite of correspondences.

Since p_1 is proper on $Z_{\mathfrak{T}^-} \cap p_2^{-1}(\mathfrak{T}_x)$, a class $c^- \in H_{\mathrm{top}}(Z_{\mathfrak{T}^-})$ defines the well-defined convolution product $c^- * a = p_{1*}(c^- \cap p_2^*(a))$ for $a \in H_{\mathrm{top}-p}(\mathfrak{T}_x)$, and defines an operator

$$c^- *\colon H_{\mathrm{top}-p}(\mathfrak{T}_x) \to H_{\mathrm{top}-p}\left(\mathfrak{M}_x^{\mathbb{C}^*}\right). \tag{8}$$

By the associativity of the convolution product, the composite $c^- * (c * \bullet) \in \mathrm{End}(H_{\mathrm{top}-p}(\mathfrak{M}_x^{\mathbb{C}^*}))$ is given by the convolution of

$$c^- * c = p_{13*}\left(p_{12}^*\left(c^-\right) \cap p_{23}^*(c)\right),$$

where p_{ij} is the projection from $\mathfrak{M}^{\mathbb{C}^*} \times \mathfrak{M} \times \mathfrak{M}^{\mathbb{C}^*}$ to the product of the i^{th} and j^{th}-factors.

Proposition 3 *Suppose that $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ satisfies the conditions* (7a), (7b). *Then there exists a class $c^{-1} \in H_{\mathrm{top}}(Z_{\mathfrak{T}^-})$ such that $c^{-1} * c$ is equal to $[\Delta_{\mathfrak{M}^{\mathbb{C}^*}}]$.*

Proof We have decomposition $\mathfrak{T}^- = \bigsqcup_\alpha \mathfrak{T}^-_\alpha$, and the projection $p^-_\alpha : \mathfrak{T}^-_\alpha \to \mathfrak{M}_\alpha$. The index set $\{\alpha\}$ is the same as before, as it parametrizes the connected components of $\mathfrak{M}^{\mathbb{C}^*}$.

Since the order $<$ plays the opposite role for $\mathfrak{T}^-$,

$$\mathfrak{T}^-_{\geq\alpha} \stackrel{\text{def.}}{=} \bigsqcup_{\beta\geq\alpha} \mathfrak{T}^-_\beta, \qquad \mathfrak{T}^-_{>\alpha} \stackrel{\text{def.}}{=} \bigsqcup_{\beta>\alpha} \mathfrak{T}^-_\beta,$$

are closed subvarieties in $\mathfrak{T}^-$.

We define $Z_{\mathfrak{T}^-,\gamma,\beta} \stackrel{\text{def.}}{=} Z_{\mathfrak{T}^-} \cap (\mathfrak{M}_\gamma \cap \mathfrak{T}^-_\beta)$ and $Z_{\mathfrak{T}^-,\gamma,\geq\beta}$ as above. We then impose the following conditions on $c^- = \sum c^-_{\gamma,\beta}$:

$$c^-_{\gamma,\beta} = 0 \text{ unless } \gamma \leq \beta,$$

$$c^-_{\gamma,\gamma} = \left[\overline{\left(\mathrm{id}_{\mathfrak{M}_\gamma} \times p^-_\gamma\right)^{-1}(\Delta_{\mathfrak{M}_\gamma})}\right].$$

These conditions imply that $c^- * c$ is unipotent, more precisely is upper triangular with respect to the block decomposition $H_{\mathrm{top}}(Z^{\mathbb{C}^*}) = \bigoplus_{\gamma,\alpha} H_{\mathrm{top}}(Z_{\gamma,\alpha})$, and the $H_{\mathrm{top}}(Z_{\alpha,\alpha})$ component is $[\Delta_{\mathfrak{M}_\alpha}]$ for all α. Noticing that we can represent $(c^- * c)^{-1}$ as a class in $H_{\mathrm{top}}(Z^{\mathbb{C}^*})$ by the convolution product, we define $c^{-1} = (c^- * c)^{-1} * c^- \in H_{\mathrm{top}}(Z_{\mathfrak{T}^-})$ to get $c^{-1} * c = [\Delta_{Z^{\mathbb{C}^*}}]$. □

Remark 2 If we consider the convolution product in the opposite order, we get

$$c * c^{-1} \in H_{\mathrm{top}}\left(\mathfrak{T} \times_{\mathfrak{M}_0^{\mathbb{C}^*}} \mathfrak{T}^-\right),$$

where $\mathfrak{T} \to \mathfrak{M}_0^{\mathbb{C}^*}$ (resp. $\mathfrak{T} \to \mathfrak{M}_0^{\mathbb{C}^*}$) is $p \circ \pi$ (resp. $p^- \circ \pi$). In general, there are no inclusion relations between $\mathfrak{T} \times_{\mathfrak{M}_0^{\mathbb{C}^*}} \mathfrak{T}^-$ and $Z = \mathfrak{M} \times_{\mathfrak{M}_0} \mathfrak{M}$. Therefore the equality $c * c^{-1} = [\Delta_{\mathfrak{M}}]$ does not make sense at the first sight. However the actual thing we need is the operator $c^- *$ in (8). Proposition 3 implies that the composite $c^{-1} * (c*)$ of the operator is the identity on $H_{\mathrm{top}-p}(\mathfrak{M}_x^{\mathbb{C}^*})$ for each x. Then we have $c * (c^{-1} *)$ is also the identity on $H_{\mathrm{top}-p}(\mathfrak{T}_x)$, as both $H_{\mathrm{top}-p}(\mathfrak{M}_x^{\mathbb{C}^*})$ and $H_{\mathrm{top}-p}(\mathfrak{T}_x)$ are vector spaces of same dimension.

Later we will see that we do not loose any information when we consider c^{-1} as such an operator. In particular, we will see that c^{-1} is uniquely determined by c, i.e., we will prove the uniqueness of the left inverse in the proof of Theorem 1.

4.3 Coproduct by Convolution

We define a coproduct using the convolution in this subsection.

Let $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ be a class satisfying the conditions (7a), (7b). We take the class $c^{-1} \in H_{\mathrm{top}}(Z_{\mathfrak{T}^-})$ as in Proposition 3. We define a homomorphism $\Delta_c : H_*(Z) \to$

$H_*(Z^{\mathbb{C}^*})$ by

$$\Delta_c(\bullet) = c^{-1} * \bullet * c = p_{14*}\big(p_{12}^*\big(c^{-1}\big) \cap p_{23}^*(\bullet) \cap p_{34}^*(c)\big), \tag{10}$$

where we consider the convolution product in $\mathfrak{M}^{\mathbb{C}^*} \times \mathfrak{M} \times \mathfrak{M} \times \mathfrak{M}^{\mathbb{C}^*}$. This preserves the grading.

Since $c^{-1} * c = 1$, we have $\Delta_c(1) = 1$. But it is not clear at this moment that Δ_c is an algebra homomorphism since we do not know $c * c^{-1} = 1$, as we mentioned in Remark 2. The proof is postponed until the next subsection.

4.4 Sheaf-Theoretic Analysis

In this subsection, we reformulate the result in the previous subsection using perverse sheaves.

By [5, §8.9] we have a natural graded algebra isomorphism

$$H_*(Z) \cong \operatorname{Ext}^\bullet_{D(\mathfrak{M}_0)}(\pi_! \mathcal{C}_{\mathfrak{M}}, \pi_! \mathcal{C}_{\mathfrak{M}}),$$

where the multiplication on the right hand side is given by the Yoneda product and the grading is the natural one. Here the semismallness of π guarantees that the grading is preserved.

We have similarly

$$H_*\big(Z^{\mathbb{C}^*}\big) \cong \operatorname{Ext}^\bullet_{D(\mathfrak{M}_0^{\mathbb{C}^*})}\big(\pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}, \pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}\big).$$

In this subsection we define a functor sending $\pi_! \mathbb{C}_{\mathfrak{M}}$ to $\pi_!^{\mathbb{C}^*} \mathbb{C}_{\mathfrak{M}^{\mathbb{C}^*}}$ to give a homomorphism $H_*(Z) \to H_*(Z^{\mathbb{C}^*})$ which coincides with Δ_c.

For a later purpose, we slightly generalize the setting from the previous subsection. If $\mathbf{v}' \le \mathbf{v}$, we have a closed embedding $\mathfrak{M}_0(\mathbf{v}', \mathbf{w}) \subset \mathfrak{M}_0 = \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$, given by adding the trivial representation with dimension $\mathbf{v} - \mathbf{v}'$.

We consider the push-forward $\pi_! \mathcal{C}_{\mathfrak{M}(\mathbf{v}', \mathbf{w})}$ as a complex in $D(\mathfrak{M}_0)$. By the decomposition theorem [1] it is a semisimple complex. Furthermore $\pi_! \mathcal{C}_{\mathfrak{M}(\mathbf{v}', \mathbf{w})}$ is a perverse sheaf, as $\pi : \mathfrak{M}(\mathbf{v}', \mathbf{w}) \to \pi(\mathfrak{M}(\mathbf{v}', \mathbf{w}))$ is semismall [3]. Let $P(\mathfrak{M}_0)$ denote the full subcategory of $D(\mathfrak{M}_0)$ consisting of all perverse sheaves that are finite direct sums of perverse sheaves L, which are isomorphic to direct summand of $\pi_! \mathcal{C}_{\mathfrak{M}(\mathbf{v}', \mathbf{w})}$ with various $\mathbf{v}'$.

Replacing $\mathfrak{M}_0$, $\mathfrak{M}(\mathbf{v}', \mathbf{w})$ by $\mathfrak{M}_0^{\mathbb{C}^*}$, $\mathfrak{M}(\mathbf{v}', \mathbf{w})^{\mathbb{C}^*}$ respectively, we introduce the full subcategory $P(\mathfrak{M}_0^{\mathbb{C}^*})$ of $D(\mathfrak{M}_0^{\mathbb{C}^*})$ as above. Here we replace π by $\pi^{\mathbb{C}^*} : \mathfrak{M}(\mathbf{v}', \mathbf{w})^{\mathbb{C}^*} \to \mathfrak{M}_0(\mathbf{v}', \mathbf{w})^{\mathbb{C}^*}$, which is the restriction of π.

Let $i : \mathfrak{T}_0 \to \mathfrak{M}_0$ and $p : \mathfrak{T}_0 \to \mathfrak{M}_0^{\mathbb{C}^*}$ as in Sect. 3.2. We consider $p_! i^* : D(\mathfrak{M}_0) \to D(\mathfrak{M}_0^{\mathbb{C}^*})$. This is an analog of the restriction functor in [8, §4], [9, §9.2], and was introduced in the quiver variety setting in [22, §5]. It is an example of the hyperbolic localization.

Lemma 3 (1) *The functor* $p_! i^*$ *sends* $P(\mathfrak{M}_0)$ *to* $P(\mathfrak{M}_0^{\mathbb{C}^*})$.

(2) *Let* $\mathbf{v}' \le \mathbf{v}$. *The complex* $p_! i^* \pi_! \mathcal{C}_{\mathfrak{M}(\mathbf{v}',\mathbf{w})}$ *has a canonical filtration whose associated graded is canonically identified with* $\pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}(\mathbf{v}',\mathbf{w})^{\mathbb{C}^*}}$.

This was proved in [22, Lemma 5.1] for quiver varieties of finite type, but the proof actually gives the above statements for general types.

Let us recall how the filtration is defined. Let us assume $\mathbf{v}' = \mathbf{v}$ for brevity. Consider the diagram

$$\begin{array}{ccccc} \mathfrak{M} & \xleftarrow{\ i'\ } & \mathfrak{T} = \bigsqcup \mathfrak{T}_\alpha & \xrightarrow{\ \bigsqcup p_\alpha\ } & \bigsqcup \mathfrak{M}_\alpha = \mathfrak{M}^{\mathbb{C}^*} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_{\mathfrak{T}}} & & \downarrow{\scriptstyle \pi^{\mathbb{C}^*}} \\ \mathfrak{M}_0 & \xleftarrow[\ \ ]{\ i\ } & \mathfrak{T}_0 & \xrightarrow[\ p\ ]{} & \mathfrak{M}_0^{\mathbb{C}^*} \end{array}$$

where i' is the inclusion, $\pi_{\mathfrak{T}}$ is the restriction of π to $\mathfrak{T}$, and p_α is the projection of the vector bundle (4). Note that each p_α is a morphism, but the union $\bigsqcup p_\alpha$ does *not* gives a morphism $\mathfrak{T} \to \mathfrak{M}^{\mathbb{C}^*}$.

Recall the order $<$ on the set $\{\alpha\}$ of fixed point components, and closed subvarieties $\mathfrak{T}_{\le\alpha}$, $\mathfrak{T}_{<\alpha}$ in Sect. 3.2. Let $\pi_{\le\alpha}$, $\pi_{<\alpha}$ be the restrictions of $\pi_{\mathfrak{T}}$ to $\mathfrak{T}_{\le\alpha}$, $\mathfrak{T}_{<\alpha}$ respectively. Then the main point in [22, Lemma 5.1] (based on [8, §4]) was to note that there is the canonical short exact sequence

$$0 \to \pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}_\alpha} \to (p \circ \pi_{\le\alpha})_! \mathcal{C}_{\mathfrak{T}_{\le\alpha}} \to (p \circ \pi_{<\alpha})_! \mathcal{C}_{\mathfrak{T}_{<\alpha}} \to 0. \tag{11}$$

Since $\mathfrak{T}_{\le\alpha} = \mathfrak{T}$ for the maximal element α and we have $i^* \pi_! \mathcal{C}_{\mathfrak{M}} = \pi_{\mathfrak{T}!} i'^* \mathcal{C}_{\mathfrak{M}}$, this gives the desired filtration.

During the proof it was also shown that $(p \circ \pi_{\le\alpha})_! \mathcal{C}_{\mathfrak{T}_{\le\alpha}}$, $(p \circ \pi_{<\alpha})_! \mathcal{C}_{\mathfrak{T}_{<\alpha}}$ are semisimple. (It is not stated explicitly in [22], but comes from [8, 4.7].) Therefore the short exact sequence (11) splits, and hence $p_! i^* \pi_! \mathcal{C}_{\mathfrak{M}}$ and $\bigoplus_\alpha \pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}_\alpha} = \pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}$ is isomorphic. The choice of an isomorphism depends on the choice of splittings of the above short exact sequences for all α.

The exact sequence (11) is the sheaf theoretic counterpart of (5). More precisely it is more natural to consider the transpose of (5):

$$0 \to (p \circ \pi_{<\alpha})_* \mathcal{C}_{\mathfrak{T}_{<\alpha}} \to (p \circ \pi_{\le\alpha})_* \mathcal{C}_{\mathfrak{T}_{\le\alpha}} \to \pi_*^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}_\alpha} \to 0, \tag{12}$$

obtained by applying the Verdier duality.

Recall that we study $H_{\mathrm{top}}(Z_{\mathfrak{T}})$ in order to describe a splitting of (5) by convolution.

Lemma 4 *We have a natural isomorphism*

$$H_{\mathrm{top}}(Z_{\mathfrak{T}}) \cong \mathrm{Hom}_{D(\mathfrak{M}_0^{\mathbb{C}^*})}\left(p_! i^* \pi_! \mathcal{C}_{\mathfrak{M}}, \pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}\right).$$

The proof is exactly the same as [5, Lemma 8.6.1], once we use the base change $i^*\pi_!\mathcal{C}_{\mathfrak{M}} = \pi_{\mathfrak{T}!}i'^*\mathcal{C}_{\mathfrak{M}}$.

This isomorphism is compatible with the convolution operator (6) in the following way: Let i_x denote the inclusion $\{x\} \to \mathfrak{M}_0^{\mathbb{C}^*}$. Then an element c in $\operatorname{Hom}_{D(\mathfrak{M}_0^{\mathbb{C}^*})}(p_!i^*\pi_!\mathcal{C}_{\mathfrak{M}}, \pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}) \cong \operatorname{Hom}_{D(\mathfrak{M}_0^{\mathbb{C}^*})}(\pi_*^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}, p_*i^!\pi_*\mathcal{C}_{\mathfrak{M}})$ defines an operator

$$H^p\big(i_x^!\pi_*^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}\big) \to H^p\big(i_x^! p_* i^! \pi_* \mathcal{C}_{\mathfrak{M}}\big) \tag{13}$$

by the Yoneda product. (See [5, 8.6.13].) We have

$$H^p\big(i_x^! p_* i^! \pi_* \mathcal{C}_{\mathfrak{M}}\big) \cong H^p\big(i_x^!(p\circ\pi_{\mathfrak{T}})_* i'^! \mathcal{C}_{\mathfrak{M}}\big) \cong H^p\big((p\circ\pi_{\mathfrak{T}})_* i_x'^! \mathcal{C}_{\mathfrak{M}}\big),$$

where i'_x is the inclusion of $\mathfrak{T}_x$ in $\mathfrak{M}$. The last one is nothing but $H_{\mathrm{top}-p}(\mathfrak{T}_x)$. Similarly $H^p(i_x^!\pi_*^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}})$ is naturally isomorphic to $H_{\mathrm{top}-p}(\mathfrak{M}_x^{\mathbb{C}^*})$. Then we have

Lemma 5 *Under the isomorphism in Lemma 4, the operator (13) given by $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ is equal to one in (6).*

The proof is the same as in [5, §8.6].

The conditions (7a), (7b) on $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ is translated into a language for the right hand side. We have the following equivalent to the condition (7a), (7b):

$$c \text{ maps } (p\circ\pi_{\le\alpha})_*\mathcal{C}_{\mathfrak{T}_{\le\alpha}} \text{ to } \bigoplus_{\beta\le\alpha}\pi_*^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}_\beta}, \tag{14a}$$

$$c\colon (p\circ\pi_{\le\alpha})_*\mathcal{C}_{\mathfrak{T}_{\le\alpha}}/(p\circ\pi_{<\alpha})_*\mathcal{C}_{\mathfrak{T}_{<\alpha}} \to \pi_*^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}_\alpha} \quad \text{is the identity.} \tag{14b}$$

Here the identity means the natural homomorphism given by (12).

Thus c satisfying (14a), (14b) gives a splitting of (11) and hence an isomorphism $p_!i^*\pi_!\mathcal{C}_{\mathfrak{M}} \cong \pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}$. Therefore we have a graded algebra homomorphism

$$\begin{aligned}\operatorname{Ext}^\bullet_{D(\mathfrak{M}_0)}(\pi_!\mathcal{C}_{\mathfrak{M}}, \pi_!\mathcal{C}_{\mathfrak{M}}) &\xrightarrow{p_!i^*} \operatorname{Ext}^\bullet_{D(\mathfrak{M}_0^{\mathbb{C}^*})}\big(p_!i^*\pi_!\mathcal{C}_{\mathfrak{M}}, p_!i^*\pi_!\mathcal{C}_{\mathfrak{M}}\big)\\ &\xrightarrow[\cong]{\operatorname{Ad}(c)} \operatorname{Ext}^\bullet_{D(\mathfrak{M}_0^{\mathbb{C}^*})}\big(\pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}, \pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}\big).\end{aligned} \tag{15}$$

It is compatible with (13), i.e.,

$$\begin{array}{ccc} H^p(i_x^!\pi_*^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}) & \xrightarrow{c} & H^p(i_x^! p_* i^! \pi_* \mathcal{C}_{\mathfrak{M}}) \\ \Big\downarrow a & & \Big\downarrow \operatorname{Ad}(c)p_*i^!(a) \\ H^p(i_x^!\pi_*^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}) & \xrightarrow{c} & H^p(i_x^! p_* i^! \pi_* \mathcal{C}_{\mathfrak{M}}) \end{array}$$

is commutative.

For $Z_{\mathfrak{T}^-}$ we have the following:

Lemma 6 *We have natural isomorphisms*

$$H_{\mathrm{top}}(Z_{\mathfrak{T}^-}) \cong \mathrm{Hom}_{D(\mathfrak{M}_0^{\mathbb{C}^*})}\big(\pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}, p_*^- i^{-!}\pi_!\mathcal{C}_{\mathfrak{M}}\big)$$
$$\cong \mathrm{Hom}_{D(\mathfrak{M}_0^{\mathbb{C}^*})}\big(\pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}, p_! i^*\pi_!\mathcal{C}_{\mathfrak{M}}\big).$$

The first isomorphism is one as in Lemma 4. We exchange the first and second factors, as we have changed the order of factors $\mathfrak{M}^{\mathbb{C}^*}$ and $\mathfrak{M}$ containing $Z_{\mathfrak{T}^-}$. The sheaves are replaced by their Verdier dual. The second isomorphism is induced by

$$p_*^- i^{-!}\pi_!\mathcal{C}_{\mathfrak{M}} \cong p_! i^*\pi_!\mathcal{C}_{\mathfrak{M}},$$

proved by Braden [4] (see Theorem 1 and the Eq. (1) at the end of Sect. 3).

We now have

Theorem 1 *The coproduct Δ_c in (10) is equal to (15). In particular, Δ_c is an algebra homomorphism.*

Proof The isomorphisms in Lemmas 4, 6 are compatible with the product. Therefore, $c^{-1} * c = [\Delta_{\mathfrak{M}^{\mathbb{C}^*}}]$ means that the composite

$$\pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}} \xrightarrow{c^{-1}} p_! i^*\pi_!\mathcal{C}_{\mathfrak{M}} \xrightarrow{c} \pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}$$

is the identity. (Note that the order of c, c^{-1} is swapped as we need to consider the transpose of homomorphisms for convolution.)

As $\pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}$ and $p_! i^*\pi_!\mathcal{C}_{\mathfrak{M}}$ are semisimple, c, c^{-1} can be considered as linear maps between isotypic components. (See Sect. 5.2 for explicit descriptions of isotypic components.) Therefore $c \circ c^{-1} = \mathrm{id}$ implies $c^{-1} \circ c = \mathrm{id}$ also. This, in particular, shows the uniqueness of c^{-1} mentioned in Remark 2. Moreover this c^{-1} is the inverse of c used in (15). Therefore Δ_c coincides with (15) again thanks to the compatibility between the convolution and Yoneda products. □

4.5 Coassociativity

Since Δ_c depends on the choice of the class c, the coassociativity does not hold in general. We give a sufficient condition on c (in fact, various c's) to have the coassociativity in this subsection.

Let $W = W^1 \oplus W^2 \oplus W^3$ be a decomposition of the I-graded vector space. Let $\mathbf{w} = \mathbf{w}^1 + \mathbf{w}^2 + \mathbf{w}^3$ be the corresponding dimension vectors. Setting $W^{23} = W^2 \oplus W^3$, we have a flag $W^3 \subset W^{23} \subset W$ with $W^3/W^{23} \cong W^2$, $W/W^{23} = W^1$. This gives us a preferred order among factors generalizing to $W^2 \subset W$ in the previous setting.

The two dimensional torus $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathfrak{M} = \mathfrak{M}(\mathbf{v}, \mathbf{w})$ through the homomorphism $\lambda\colon T \to G_W$ defined by $\lambda(t_2, t_3) = \mathrm{id}_{W^1} \oplus t_2\,\mathrm{id}_{W^2} \oplus t_3\,\mathrm{id}_{W^3}$.

We have two ways of putting braces for the sum $\mathbf{w} = (\mathbf{w}^1 + \mathbf{w}^2) + \mathbf{w}^3 = \mathbf{w}^1 + (\mathbf{w}^2 + \mathbf{w}^3)$ respecting the order. We have corresponding two $\mathbb{C}^*$'s in T given by $\{(1, t_3)\}$ and $\{(t_2, t_2)\}$. We denote the former by $\mathbb{C}^*_{12,3}$ and the latter by $\mathbb{C}^*_{1,23}$. We then consider fixed points varieties, tensor product varieties, and fiber products for both $\mathbb{C}^*$'s. We denote them by $\mathfrak{M}^{12,3}$, $\mathfrak{T}^{12,3}$, $Z_{\mathfrak{T}^{12,3}}$, $\mathfrak{M}^{1,23}$, $\mathfrak{T}^{1,23}$, $Z_{\mathfrak{T}^{1,23}}$, etc. They correspond to block matrices $\begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix}$ and $\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$ respectively.

On these varieties, we have the action of the remaining $\mathbb{C}^* = T/\mathbb{C}^*_{12,3}$ and $T/\mathbb{C}^*_{1,23}$ respectively. Then we can consider the fixed point sets $(\mathfrak{M}^{12,3})^{\mathbb{C}^*}$, $(\mathfrak{M}^{12,3})^{\mathbb{C}^*}$. Both are nothing but the torus fixed points $\mathfrak{M}^T$. We denote it by $\mathfrak{M}^{1,2,3}$. We denote the corresponding fiber product by $Z_{1,2,3}$. In $\mathfrak{T}_0^{12,3}$, $\mathfrak{T}_0^{1,23}$, we consider subvarieties consisting of points $\lim_{t\to 0}$ exists as before. They can be described as the variety consisting of points $x = [B, a, b]$ such that $b_{\mathrm{i}(h_N)} B_{h_N} B_{h_{N-1}} \cdots B_{h_1} a_{\mathrm{o}(h_1)}$ preserves the flag $W^3 \subset W^{23} \subset W$, i.e., $\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$. In particular, the variety is the same for one defined in $\mathfrak{T}_0^{12,3}$ and in $\mathfrak{T}_0^{1,23}$. Therefore it is safe to write both by $\mathfrak{T}_0^{1,2,3}$. We have the corresponding fiber product $Z_{\mathfrak{T}^{1,2,3}} \overset{\text{def.}}{=} \mathfrak{T}^{1,2,3} \times_{\mathfrak{M}_0^{1,2,3}} \mathfrak{M}^{1,2,3}$.

We need two more classes of varieties corresponding to $\begin{bmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$ and $\begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$ respectively. Tensor product varieties are

$$\mathfrak{T}_0^{(1,2),3} \overset{\text{def.}}{=} \mathfrak{T}_0^{1,2,3} \cap \mathfrak{M}_0^{12,3}, \qquad \mathfrak{T}_0^{1,(2,3)} \overset{\text{def.}}{=} \mathfrak{T}_0^{1,2,3} \cap \mathfrak{M}_0^{1,23}$$

respectively. We define the fiber products $Z_{\mathfrak{T}^{(1,2),3}} = \mathfrak{T}^{(1,2),3} \times_{\mathfrak{M}_0^{1,2,3}} \mathfrak{M}^{1,2,3}$, $Z_{\mathfrak{T}^{1,(2,3)}} = \mathfrak{T}^{1,(2,3)} \times_{\mathfrak{M}_0^{1,2,3}} \mathfrak{M}^{1,2,3}$.

A class $c^{12,3} \in H_{\mathrm{top}}(Z_{\mathfrak{T}^{12,3}})$ gives the coproduct

$$\Delta_{c^{12,3}} \colon H_*(Z) \to H_*(Z_{12,3}),$$

and similarly $c^{1,23} \in H_{\mathrm{top}}(Z_{\mathfrak{T}^{12,3}})$ gives $\Delta_{c^{1,23}}$. These correspond to $\Delta \otimes 1$ and $1 \otimes \Delta$ for the usual coproduct respectively.

A class $c^{(1,2),3} \in H_{\mathrm{top}}(Z_{\mathfrak{T}^{(1,2),3}})$ gives

$$\Delta_{c^{(1,2),3}} \colon H_*(Z_{12,3}) \to H_*(Z_{1,2,3}),$$

and similarly $c^{1,(2,3)} \in H_{\mathrm{top}}(Z_{\mathfrak{T}^{(1,2),3}})$ gives $\Delta_{c^{1,(2,3)}}$. Thus we have two ways going from $H_*(Z)$ to $H_*(Z_{1,2,3})$:

$$\begin{array}{ccc} H_*(Z) & \xrightarrow{\Delta_{c^{12,3}}} & H_*(Z_{12,3}) \\ {\scriptstyle \Delta_{c^{1,23}}}\downarrow & & \downarrow{\scriptstyle \Delta_{c^{(1,2),3}}} \\ H_*(Z_{1,23}) & \xrightarrow[\Delta_{c^{1,(2,3)}}]{} & H_*(Z_{1,2,3}) \end{array} \tag{16}$$

The commutativity of this diagram means the coassociativity of our coproduct.

Proposition 4 *The diagram (16) is commutative if*

$$c^{12,3} * c^{(1,2),3} = c^{1,(2,3)} * c^{1,23}$$

holds in $H_{\mathrm{top}}(Z_{\mathfrak{T}^{1,2,3}})$.

The proof is obvious.

4.6 Equivariant Homology Version

Let $G = \prod_i \mathrm{GL}(W_i^1) \times \mathrm{GL}(W_i^2)$. The group G acts on $\mathfrak{M}$, $\mathfrak{M}^{\mathbb{C}^*}$ and various other varieties considered in the previous subsections.

We consider a $\mathbb{C}^* \times \mathbb{C}^*$-action on $\mathfrak{M}$ defined by

$$(t_1, t_2) \cdot B_h = \begin{cases} t_1 B_h & \text{if } h \in \Omega, \\ t_2 B_h & \text{if } h \in \overline{\Omega}, \end{cases} \qquad (t_1, t_2) \cdot a = a, \qquad (t_1, t_2) \cdot b = t_1 t_2 b.$$

Let $\mathbb{G} = \mathbb{C}^* \times \mathbb{C}^* \times G$.

Remark 3 When the graph does not contain a cycle, the action of a factor $\mathbb{C}^*$ of $\mathbb{C}^* \times \mathbb{C}^*$, lifted to the double cover, can be move to an action through $\mathbb{C}^* \to G$. Therefore we only have an action of $\mathbb{C}^* \times G$ essentially in this case.

The results in the previous subsections hold in the equivariant category: we replace the homology $H_*(X)$ by the equivariant homology $H_*^{\mathbb{G}}(X)$. For the derived category $D(X)$ of complexes of constructible sheaves, we use their equivariant version $D_{\mathbb{G}}(X)$, considered in [2, 10].

The following observations are obvious, but useful. Top degree components of Z give a base for both $H_{\mathrm{top}}(Z)$ and $H_{\mathrm{top}}^{\mathbb{G}}(Z)$. Therefore we have a natural isomorphism

$$H_{\mathrm{top}}(Z_{\mathfrak{T}}) \cong H_{\mathrm{top}}^{\mathbb{G}}(Z_{\mathfrak{T}}).$$

The corresponding statement for the right hand side of Lemma 4 is

$$\mathrm{Hom}_{D(\mathfrak{M}_0^{\mathbb{C}^*})}\big(p_! i^* \pi_! \mathcal{C}_{\mathfrak{M}}, \pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}\big) \cong \mathrm{Hom}_{D_{\mathbb{G}}(\mathfrak{M}_0^{\mathbb{C}^*})}\big(p_! i^* \pi_! \mathcal{C}_{\mathfrak{M}}, \pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}\big).$$

This is also true as $p_! i^* \pi_! \mathcal{C}_{\mathfrak{M}}$, $\pi_!^{\mathbb{C}^*} \mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}$ are $\mathbb{G}$-equivariant perverse sheaves. (See [10, 1.16(a)].)

In particular, $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ defines the coproduct Δ_c for the equivariant version $\Delta_c \colon H_*^{\mathbb{G}}(Z) \to H_*^{\mathbb{G}}(Z^{\mathbb{C}^*})$. Also to check the coassociativity of the coproduct, we only need to check the condition in Proposition 4 for the *non*-equivariant homology.

Remark 4 In a wider framework of a holomorphic symplectic manifold with torus action satisfying certain conditions, Maulik and Okounkov [12] give a 'canonical' element c. It is called the *stable envelop*. It is defined first on the analog of $Z_{\mathfrak{T}}$ for the quiver varieties with generic complex parameters (deformations of $\mathfrak{M}$, $\mathfrak{M}^{\mathbb{C}^*}$), and then as the limit when parameters go to 0. It satisfies (7a), (7b) and the condition in Proposition 4. Therefore their stable envelop together with the construction in this section gives a canonical coproduct, satisfying the coassociativity.

5 Tensor Product Multiplicities

In this section, we give the formula of tensor product multiplicities with respect to the coproduct Δ_c in terms of IC sheaves.

5.1 Decomposition of the Direct Image Sheaf

We give the decomposition of $\pi_!(\mathcal{C}_{\mathfrak{M}})$ in this subsection. For this purpose, we introduce a refinement of the stratification (2). We do not need to worry about the first factor $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w})$ as it cannot be decomposed further. On the other hand the second factor $\mathfrak{M}_0(\mathbf{v} - \mathbf{v}^0, 0)$ parametrizes isomorphism classes of semisimple modules M of the preprojective algebra corresponding to the quiver. They decompose into direct sum of simple modules as

$$M = M_1^{\oplus n_1} \oplus M_2^{\oplus n_2} \oplus \cdots \oplus M_N^{\oplus n_N}.$$

Dimension vectors of all simple modules have been classified by Crawley-Boevey [6, Th. 1.2]. (In fact, he also classifies pairs $(\mathbf{v}^0, \mathbf{w})$ with $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0, \mathbf{w}) \neq \emptyset$.) Let $\delta_1, \delta_2, \dots, \delta_N$ be such vectors which are $\leq \mathbf{v}$. They are all positive roots satisfying certain conditions. For example, for a quiver of type ADE, they are simple roots. For a quiver of affine type ADE, they are simple roots and the positive generator δ of imaginary roots. For a Jordan quiver, it is the vector $1 \in \mathbb{Z} = \mathbb{Z}^I$.

We then have

$$\mathfrak{M}_0\bigl(\mathbf{v} - \mathbf{v}^0, 0\bigr) = S^{n_1}\mathfrak{M}_0^{\mathrm{reg}}(\delta_1, 0) \times S^{n_2}\mathfrak{M}_0^{\mathrm{reg}}(\delta_2, 0) \times \cdots \times S^{n_N}\mathfrak{M}_0^{\mathrm{reg}}(\delta_N, 0),$$

with $\mathbf{v}^0 + n_1\delta_1 + \cdots + n_N\delta_N = \mathbf{v}$. Here $\mathfrak{M}_0^{\mathrm{reg}}(\delta_k, 0)$ parametrizes simple modules with dimension vector δ_k, or equivalently points in $\mathfrak{M}_0(\delta_k, 0)$ whose stabilizers are nonzero scalars times the identity. Its symmetric power $S^{n_k}\mathfrak{M}_0^{\mathrm{reg}}(\delta_k, 0)$ parametrizes semisimple modules

$$M_1^{\oplus m_1} \oplus M_2^{\oplus m_2} \oplus \cdots$$

such that $M_1, M_2, \dots$ are distinct simple modules with dimension δ_k and the total number of simple factors is n_k.

The symmetric power $S^{n_k}\mathfrak{M}_0^{\mathrm{reg}}(\delta_k, 0)$ decomposes further according to multiplicities $m_1, m_2, \ldots$. As we may assume $m_1 \geq m_2 \geq \cdots$, they define partition λ_k of n_k. Let us denote by $S_{\lambda_k}\mathfrak{M}_0^{\mathrm{reg}}(\delta_k, 0)$ the space parametrizing semisimple modules having multiplicities λ_k.

Thus we have

$$\mathfrak{M}_0 = \bigsqcup \mathfrak{M}_0^{\mathrm{reg}}\bigl(\mathbf{v}^0, \mathbf{w}\bigr) \times \mathfrak{M}_0(\boldsymbol{\lambda}) \tag{17}$$

with $\mathbf{v}^0 + |\lambda_1|\delta_1 + \cdots + |\lambda_N|\delta_N = \mathbf{v}$, where

$$\mathfrak{M}_0(\boldsymbol{\lambda}) \stackrel{\text{def.}}{=} S_{\lambda_1}\mathfrak{M}_0^{\mathrm{reg}}(\delta_1, 0) \times S_{\lambda_2}\mathfrak{M}_0^{\mathrm{reg}}(\delta_2, 0) \times \cdots \times S_{\lambda_N}\mathfrak{M}_0^{\mathrm{reg}}(\delta_N, 0).$$

This is nothing but the decomposition given in [13, 6.5], [14, 3.27].

This stratification has a simple form when the quiver is of type ADE. Each δ_k is a simple root α_i, and $\mathfrak{M}_0^{\mathrm{reg}}(\delta_k, 0)$ is a one point given by the simple module S_i. The symmetric product $S^{n_k}\mathfrak{M}_0^{\mathrm{reg}}(\delta_k, 0)$ is also a one point $S_i^{\oplus n_k}$, and hence we do not need to consider the partition λ_k. Thus we can safely forget factors $S_{\lambda_k}\mathfrak{M}_0^{\mathrm{reg}}(\delta_k, 0)$ and get

$$\mathfrak{M}_0 = \bigsqcup \mathfrak{M}_0^{\mathrm{reg}}\bigl(\mathbf{v}^0, \mathbf{w}\bigr),$$

with $\mathbf{v}^0 \leq \mathbf{v}$.

For the affine case δ_k is either simple root or δ, as we mentioned above. If δ_k is a simple root, we can forget the factor $S^{n_k}\mathfrak{M}_0^{\mathrm{reg}}(\delta_k, 0)$ as in the ADE cases. If $\delta_k = \delta$, then $\mathfrak{M}_0^{\mathrm{reg}}(\delta, 0)$ is $\mathbb{C}^2$ for the Jordan quiver or $\mathbb{C}^2 \setminus \{0\}/\varGamma$ for the affine quiver corresponding to a finite subgroup $\varGamma \subset \mathrm{SU}(2)$ via the McKay correspondence. Therefore we have

$$\mathfrak{M}_0 = \bigsqcup \mathfrak{M}_0^{\mathrm{reg}}\bigl(\mathbf{v}^0, \mathbf{w}\bigr) \times \bigl(S_\lambda\mathbb{C}^2 \text{ or } S_\lambda\bigl(\mathbb{C}^2 \setminus \{0\}\bigr)/\varGamma\bigr). \tag{18}$$

Return back to a general quiver. We denote each stratum in (17) by $\mathfrak{M}_0(\mathbf{v}^0; \boldsymbol{\lambda})$ for brevity. Here $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$. For a simple local system ρ on this stratum, we consider the corresponding IC sheaf

$$IC\bigl(\mathfrak{M}_0\bigl(\mathbf{v}^0; \boldsymbol{\lambda}\bigr), \rho\bigr).$$

Then the decomposition theorem for a semismall projective morphism [3] implies a canonical direct sum decomposition

$$\pi_!\mathcal{C}_{\mathfrak{M}} \cong \bigoplus IC\bigl(\mathfrak{M}_0\bigl(\mathbf{v}^0; \boldsymbol{\lambda}\bigr), \rho\bigr) \otimes H_{\mathrm{top}}(\mathfrak{M}_{x_{\mathbf{v}^0;\boldsymbol{\lambda}}})_\rho. \tag{19}$$

Here $x_{\mathbf{v}^0;\boldsymbol{\lambda}}$ is a point in the stratum $\mathfrak{M}_0(\mathbf{v}^0; \boldsymbol{\lambda})$ and $\mathfrak{M}_{x_{\mathbf{v}^0;\boldsymbol{\lambda}}} = \pi^{-1}(x_{\mathbf{v}^0;\boldsymbol{\lambda}})$ as before. Then $H_{\mathrm{top}}(\mathfrak{M}_{x_{\mathbf{v}^0;\boldsymbol{\lambda}}})_\rho$ denotes the isotypic component of ρ in the homology group $H_{\mathrm{top}}(\mathfrak{M}_{x_{\mathbf{v}^0;\boldsymbol{\lambda}}})$ of the fiber with respect to the monodromy action.

This decomposition determines representations of the convolution algebra $H_{\mathrm{top}}(Z) = \mathrm{End}_{D(\mathfrak{M}_0)}(\pi_!\mathcal{C}_{\mathfrak{M}})$ (see [5, §8.9]):

Theorem 2 (1) $\{H_{\text{top}}(\mathfrak{M}_{x_{\mathbf{v}^0;\boldsymbol{\lambda}}})_\rho\}$ *is the set of isomorphism classes of simple modules of* $H_{\text{top}}(Z)$.

(2) *We have*

$$H_{\text{top}}(Z) \cong \bigoplus \operatorname{End}\bigl(H_{\text{top}}(\mathfrak{M}_{x_{\mathbf{v}^0;\boldsymbol{\lambda}}})_\rho\bigr).$$

When the quiver is of type ADE, it was proved that only trivial local systems on strata appear [16, §15] in the direct summand of $\pi_!\mathcal{C}_{\mathfrak{M}}$, and hence we have

$$\pi_!(\mathcal{C}_{\mathfrak{M}}) \cong \bigoplus IC\bigl(\mathfrak{M}_0\bigl(\mathbf{v}^0,\mathbf{w}\bigr)\bigr) \otimes H_{\text{top}}(\mathfrak{M}_{x_{\mathbf{v}^0}}),$$

where we remove the local system ρ from the notation for the IC sheaves.

For a quiver of general type, the argument used in [16, §15] implies that the simple local system ρ is trivial on the factor $\mathfrak{M}_0^{\text{reg}}(\mathbf{v}^0,\mathbf{w})$, i.e., all simple modules $M_1, M_2, \ldots$ are of the form S_i. In general, the author does not know what kind of local system ρ can appear on these factors. But we can show that only trivial local system appears for an affine quiver:

Lemma 7 *Suppose that the quiver is of affine type. Then*

$$\pi_!(\mathcal{C}_{\mathfrak{M}}) \cong \bigoplus_{\mathbf{v}^0,\lambda} IC\bigl(\mathfrak{M}_0\bigl(\mathbf{v}^0,\mathbf{w}\bigr)\bigr) \boxtimes (\mathcal{C}_{\overline{S_\lambda(\mathbb{C}^2)}} \textit{ or } \mathcal{C}_{\overline{S_\lambda(\mathbb{C}^2/\Gamma)}}) \otimes H_{\text{top}}(\mathfrak{M}_{x_{\mathbf{v}^0;\lambda}}).$$

Proof By the argument in [16, §15], it is enough to assume $\mathbf{v}^0 = 0$ and hence $\mathfrak{M}_0^{\text{reg}}(\mathbf{v}^0,\mathbf{w})$ is a single point. Then a point in the stratum $x_{\mathbf{v}^0;\lambda}$ is a point in $S_\lambda\mathbb{C}^2$ or $S_\lambda((\mathbb{C}^2\setminus\{0\})/\Gamma)$, and hence is written as $m_1x_1 + m_2x_2 + \cdots$, where x_1, x_2 are distinct points in $\mathbb{C}^2$ or $(\mathbb{C}^2\setminus\{0\})/\Gamma$. Then the fiber $\mathfrak{M}_{x_{\mathbf{v}^0;\lambda}}$ is the product of punctual Quot schemes parametrizing quotients Q of the trivial rank r sheaf $\mathcal{O}_{\mathbb{C}^2}^{\oplus r}$ over $\mathbb{C}^2$ such that Q is supported at 0 and the length is m_i. Here r is given by $\langle \mathbf{w}, c\rangle$, where c is the central element of the affine Lie algebra or $\mathbf{w}$ itself for the Jordan quiver. This follows from the alternative description of quiver varieties of affine types, explained in [18]. (Remark: In [18, §4], it was written that the fiber is the product of punctual Hilbert schemes, but it is wrong.) It is known that top degree part H_{top} of a punctual Quot scheme is 1-dimensional (see [15, Ex. 5.15]). Therefore the monodromy action is trivial. Moreover $\overline{S_\lambda(\mathbb{C}^2)}$ and $\overline{S_\lambda(\mathbb{C}^2/\Gamma)}$ only have finite quotient singularities, and hence are rationally smooth. Therefore the intersection complexes are constant sheaves, shifted by dimensions. □

5.2 A Description of $H_{\text{top}}(Z_{\mathfrak{T}})$

As in Theorem 2 we have a natural isomorphism

$$H_{\text{top}}(Z_{\mathfrak{T}}) \cong \bigoplus_{\mathbf{v}^1,\mathbf{v}^2,\boldsymbol{\lambda},\rho} \operatorname{Hom}\bigl(H_{\text{top}}\bigl(\mathfrak{M}^{\mathbb{C}^*}_{x_{\mathbf{v}^1,\mathbf{v}^2;\boldsymbol{\lambda}}}\bigr)_\rho, H_{\text{top}}(\mathfrak{T}_{x_{\mathbf{v}^1,\mathbf{v}^2;\boldsymbol{\lambda}}})_\rho\bigr) \tag{20}$$

from Lemma 4 and the above decomposition.

Thus $c \in H_{\mathrm{top}}(Z_{\mathfrak{T}})$ is determined by its convolution action $H_{\mathrm{top}}(\mathfrak{M}_x^{\mathbb{C}^*}) \to H_{\mathrm{top}}(\mathfrak{T}_x)$ for $x = x_{\mathbf{v}^1,\mathbf{v}^2;\boldsymbol{\lambda}}$ in each stratum. Then the converse of Proposition 2 is clear.

5.3 Tensor Product Multiplicities in Terms of *IC* Sheaves

As in the previous subsection, we also refine the stratification in Lemma 1 as

$$\mathfrak{M}_0^{\mathbb{C}^*} = \bigsqcup \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^2, \mathbf{w}^2) \times \mathfrak{M}_0(\boldsymbol{\lambda}),$$

where

$$\mathfrak{M}_0(\boldsymbol{\lambda}) = S_{\lambda_1}\mathfrak{M}_0^{\mathrm{reg}}(\delta_1, 0) \times S_{\lambda_2}\mathfrak{M}_0^{\mathrm{reg}}(\delta_2, 0) \times \cdots \times S_{\lambda_N}\mathfrak{M}_0^{\mathrm{reg}}(\delta_N, 0)$$

as before. For a simple local system ρ on $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^2, \mathbf{w}^2) \times \mathfrak{M}_0(\boldsymbol{\lambda})$, we consider the corresponding IC sheaf. We then have

$$\begin{aligned}\pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}} = \bigoplus IC\big(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^2, \mathbf{w}^2) \times \mathfrak{M}_0(\boldsymbol{\lambda}), \rho\big) \\ \otimes H_{\mathrm{top}}\big(\mathfrak{M}^{\mathbb{C}^*}_{x_{\mathbf{v}^1,\mathbf{v}^2;\boldsymbol{\lambda}}}\big)_\rho,\end{aligned}$$

where $x_{\mathbf{v}^1,\mathbf{v}^2;\boldsymbol{\lambda}}$ is a point in the stratum $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^2, \mathbf{w}^2) \times \mathfrak{M}_0(\boldsymbol{\lambda})$. Then $H_{\mathrm{top}}(\mathfrak{M}^{\mathbb{C}^*}_{x_{\mathbf{v}^1,\mathbf{v}^2;\boldsymbol{\lambda}}})_\rho$ is a simple module of $H_{\mathrm{top}}(Z^{\mathbb{C}^*})$, and any simple module is isomorphic to a module of this form as before.

By Δ_c in (10) we consider $H_{\mathrm{top}}(\mathfrak{M}^{\mathbb{C}^*}_{x_{\mathbf{v}^1,\mathbf{v}^2;\boldsymbol{\lambda}}})_\rho$ as a module over $H_{\mathrm{top}}(Z)$. Since $H_{\mathrm{top}}(Z)$ is semisimple, it decomposes into a direct sum of $H_{\mathrm{top}}(\mathfrak{M}_{x_{\mathbf{v}^0;\boldsymbol{\lambda}}})_{\rho'}$ with various $\mathbf{v}^0$, $\boldsymbol{\lambda}'$, ρ'. Let us define the 'tensor product multiplicity' by

$$n^{\mathbf{v}^0;\boldsymbol{\lambda}',\rho'}_{\mathbf{v}^1,\mathbf{v}^2;\boldsymbol{\lambda},\rho} \overset{\text{def.}}{=} \big[H_{\mathrm{top}}\big(\mathfrak{M}^{\mathbb{C}^*}_{x_{\mathbf{v}^1,\mathbf{v}^2;\boldsymbol{\lambda}}}\big)_\rho : H_{\mathrm{top}}(\mathfrak{M}_{x_{\mathbf{v}^0;\boldsymbol{\lambda}'}})_{\rho'}\big]. \tag{21}$$

These multiplicity has a geometric description:

Theorem 3 *The multiplicity* $n^{\mathbf{v}^0;\boldsymbol{\lambda}',\rho'}_{\mathbf{v}^1,\mathbf{v}^2;\boldsymbol{\lambda},\rho}$ *is equal to*

$$\big[p_! i^* IC\big(\mathfrak{M}_0(\mathbf{v}^0;\boldsymbol{\lambda}'), \rho'\big) : IC\big(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^2, \mathbf{w}^2) \times \mathfrak{M}_0(\boldsymbol{\lambda}), \rho\big)\big].$$

Recall that $p_! i^* IC(\mathfrak{M}_0(\mathbf{v}^0;\boldsymbol{\lambda}), \rho)$ is a direct sum of $IC(\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^2, \mathbf{w}^2) \times \mathfrak{M}_0(\boldsymbol{\lambda}'), \rho')$ with various $\mathbf{v}^1$, $\mathbf{v}^2$, $\boldsymbol{\lambda}'$, ρ' by Lemma 3. The right hand side of the above formula denote the decomposition multiplicity.

This formula is a direct consequence of decompositions of $\pi_!\mathcal{C}_{\mathfrak{M}}$, $\pi_!^{\mathbb{C}^*}\mathcal{C}_{\mathfrak{M}^{\mathbb{C}^*}}$ and the identification of Δ_c with $\mathrm{Ad}(c)p_! i^*$ in (15). (See also [22, Th. 5.1].)

Remark 5 For a quiver of type ADE, we do not have data $\boldsymbol{\lambda}$, ρ, $\boldsymbol{\lambda}'$, ρ', and multiplicities $n^{\mathbf{v}^0}_{\mathbf{v}^1,\mathbf{v}^2}$ is nothing but the usual tensor product multiplicity of finite dimensional representations of the Lie algebra $\mathfrak{g}$ of type ADE [22, Th. 5.1].

In general, the author does not know how to understand the behavior of $IC(\mathfrak{M}_0(\mathbf{v}^0;\boldsymbol{\lambda}),\rho)$ under $p_! i^*$. For affine types, only constant sheaves $\mathcal{C}_{\overline{S_\lambda(\mathbb{C}^2/\Gamma)}}$ appear in $\pi_!\mathcal{C}_{\mathfrak{M}}$, and local systems on $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^1,\mathbf{w}^1)\times\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^2,\mathbf{w}^2)\times S_\lambda(\mathbb{C}^2\backslash\{0\}/\Gamma)$ can be determined. It should be possible to determine multiplicities from the tensor product multiplicity for the affine Lie algebra. But it is yet to be clarified.

5.4 Fixed Point Version

Let a be a semisimple element in the Lie algebra of $\mathbb{G}$. Then it defines a homomorphism

$$\rho_a\colon H^*_{\mathbb{G}}(\mathrm{pt})\to\mathbb{C}.$$

Let A be the smallest torus whose Lie algebra contains a. Let Z^A be the fixed point set. Then we have a homomorphism

$$r_a\colon H^{\mathbb{G}}_*(Z)\otimes_{H^*_{\mathbb{G}}(\mathrm{pt})}\mathbb{C}\to H_*\big(Z^A\big)$$

as the composite of the pull back and the multiplication of $1\otimes\rho_a(e(N))^{-1}$, where N is the normal bundle of $\mathfrak{M}^A$ in $\mathfrak{M}$, and $e(N)$ is its A-equivariant Euler class. (See [5, §5.11].) Then r_a is an algebra isomorphism. Similarly we have

$$r_a\colon H^{\mathbb{G}}_*\big(Z^{\mathbb{C}^*}\big)\otimes_{H^*_{\mathbb{G}}(\mathrm{pt})}\mathbb{C}\to H_*\big(\big(Z^{\mathbb{C}^*}\big)^A\big).$$

We then have a specialized coproduct

$$\Delta_c\colon H_*\big(Z^A\big)\to H_*\big(\big(Z^{\mathbb{C}^*}\big)^A\big).$$

Those convolution algebras can be studied in terms of perverse sheaves appearing whose shifts appear in direct summand in $\pi^A\mathbb{C}_{\mathfrak{M}^A}$, $(\pi^{\mathbb{C}^*})^A_!\mathbb{C}_{(\mathfrak{M}^{\mathbb{C}^*})^A}$, where π^A, $(\pi^{\mathbb{C}^*})^A$ are restrictions of π and $\pi^{\mathbb{C}^*}$ to A-fixed point sets $\mathfrak{M}^A$ and $(\mathfrak{M}^{\mathbb{C}^*})^A$. See [5, §8.6] for detail.

The tensor product multiplicities with respect to the specialized Δ_c are described by the functor $p^A_!(i^A)^*$, where p^A, i^A are restrictions of p and i to A-fixed point sets. Since the result is almost the same as Theorem 3, we omit the detail. The difference is that the algebra is not semisimple in general, and multiplicities are considered in the Grothendieck group of the category of modules of convolution algebras. In geometric side, perverse sheaves are *not* preserved by the functor $p^A_!(i^A)^*$. They are sent to direct sums of *shifts* of perverse sheaves in general.

As we mentioned in the introduction, the target of Δ_c in (10) is $H_*(Z^{\mathbb{C}^*})$, which is larger than the tensor product of the corresponding algebra for $\mathbf{w}^1$, $\mathbf{w}^2$ in general.

This is because of the existence of the third factor in Lemma 1(1). To avoid this, we assume that generators $\operatorname{tr}(B_{h_N} B_{h_{N-1}} \cdots B_{h_1} \colon V_{o(h_1)} \to V_{i(h_N)} = V_{o(h_1)})$ have non-trivial weights with respect to A. Then the A-fixed point set in the third factor $\mathfrak{M}_0(\mathbf{v} - \mathbf{v}^0, 0)$ is automatically trivial, and hence we have

$$\left(Z^{\mathbb{C}^*}\right)^A = \bigsqcup_{\mathbf{v}^1+\mathbf{v}^2=\mathbf{v}} Z\left(\mathbf{v}^1, \mathbf{w}^1\right)^A \times Z\left(\mathbf{v}^2, \mathbf{w}^2\right)^A.$$

This assumption is rather mild and satisfied for example if the compositions of $A \to \mathbb{G}$ with the projections $\mathbb{G} \to \mathbb{C}^*$ to the first and second factor of $\mathbb{G}$ both have positive weights. This condition occurs when we study modules of $Y(\mathfrak{g})$ for example, as both are identities in that case.

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Derivatives of Schur, Tau and Sigma Functions on Abel-Jacobi Images

Atsushi Nakayashiki and Keijiro Yori

Abstract We study derivatives of Schur and tau functions from the view point of the Abel-Jacobi map. We apply the results to establish several properties of derivatives of the sigma function of an (n, s) curve. As byproducts we have an expression of the prime form in terms of derivatives of the sigma function and addition formulae which generalize those of Onishi for hyperelliptic sigma functions.

1 Introduction

The Riemann's theta function of an algebraic curve X of genus g can be considered, through the Abel-Jacobi map, as a multivalued multiplicative analytic function on X^g. The Riemann's vanishing theorem tells that the theta function shifted by the Riemann's constant vanishes identically on X^{g-1}. However it is possible to find certain derivatives of the theta function such that they become multivalued multiplicative analytic functions on X^{g-1}. Onishi [13] found such derivatives explicitly in the case of hyperelliptic curves. The extension of the results to the curve $y^n = f(x)$ is given in [9]. These explicit derivatives of the theta function are used to construct certain addition formulae in [13]. The aim of this paper is to generalize and clarify the structure of the results on derivatives and addition formulae in [13] by studying Schur and tau functions.

We consider a certain plane algebraic curve X, called an (n, s) curve [2], which contains curves $y^n = f(x)$ as a special case. As in [13] we study sigma functions [1, 11] rather than Riemann's theta function since it is simpler to describe derivatives. Sigma functions can be expressed by the tau function of the KP-hierarchy

Dedicated to Michio Jimbo on his sixtieth birthday.

A part of the results in the present paper is reported in [16].

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[4, 5, 12]. The expansion of the tau function with respect to Schur functions is known very explicitly due to Sato's theory of universal Grassmann manifold (UGM) [14, 15]. In the case corresponding to the sigma function of an (n,s) curve the expansion of the tau function begins from the Schur function $s_\lambda(t)$ corresponding to the partition λ determined from the gap sequence at ∞ of X. Notice that Schur functions themselves can be considered as a special case of tau functions [3].

For a theta function solution of the KP-hierarchy the image of the Abel-Jacobi map of a point on a Riemann surface is transformed, in the tau function, to the vector of the form

$$[z] = {}^t\big(z, z^2/2, z^3/3, \ldots\big), \tag{1}$$

where z being a local coordinate at a base point. Being motivated by this we consider, in general, the map $z \mapsto [z]$ as an analogue of the Abel-Jacobi map for Schur and tau functions. For the Schur function corresponding to an (n,s) curve a similar map is considered in [2] as the rational limit of the Abel-Jacobi map.

The Schur function $s_\lambda(t)$, $t = (t_1, t_2, \ldots)$, corresponding to a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ is the polynomial in $t_1, t_2, \ldots$ defined by

$$s_\lambda(t) = \det\big(p_{\lambda_i - i + j}(t)\big)_{1 \le i,j \le l}, \qquad \exp\left(\sum_{i=1}^{\infty} t_i k^i\right) = \sum_{i=1}^{\infty} p_i(t) k^i.$$

We firstly study, for each k satisfying $k \le g$, the condition under which a derivative

$$\partial^\alpha s_\lambda\big([z_1] + \cdots + [z_k]\big), \tag{2}$$

vanishes identically, where, for $\alpha = (\alpha_1, \alpha_2, \ldots)$, ∂^α denote $\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots$ and $\partial_i = \partial/\partial t_i$. A sufficient condition can easily be found. Let us define the weight of α by $\mathrm{wt}\,\alpha = \sum_{i=1}^{\infty} i\alpha_i$ and set $N_{\lambda,k} = \lambda_{k+1} + \cdots + \lambda_l$. Then the derivative (2) vanishes, if $\mathrm{wt}\,\alpha < N_{\lambda,k}$.

Concerning to derivatives such that (2) does not vanish identically we have found two kinds of α satisfying $\mathrm{wt}\,\alpha = N_{\lambda,k}$. One is $\alpha = (N_{\lambda,k}, 0, 0, \ldots)$ for which the following recursive relation holds:

$$\partial_1^{N_{\lambda,k}} s_\lambda\left(\sum_{i=1}^{k}[z_i]\right) = \frac{c'_{\lambda,k}}{c'_{\lambda,k-1}} \partial_1^{N_{\lambda,k-1}} s_\lambda\left(\sum_{i=1}^{k-1}[z_i]\right) z^{\lambda_k} + O\big(z_k^{\lambda_k+1}\big), \tag{3}$$

where $c'_{\lambda,k}$ is a certain constant (Theorem 4).

The other kind of derivatives exist only for λ corresponding to a gap sequence. A gap sequence of genus g is a sequence of positive integers $w_1 < \cdots < w_g$ such that its complement in the set of non-negative integers $\mathbb{Z}_{\ge 0}$ is a semi-group. To each gap sequence a partition $\lambda = (\lambda_1, \ldots, \lambda_g)$ is associated by

$$\lambda = (w_g, \ldots, w_2, w_1) - (g-1, \ldots, 1, 0).$$

Let $w_1^* < w_2^* < \cdots$ be the complement of $\{w_i\}$ in $\mathbb{Z}_{\geq 0}$. For each k the number m_k and the sequence $a_j^{(k)}$, $1 \leq j \leq m_k$, are defined by

$$m_k = \sharp\{i | w_i^* < g - k\},$$
$$\left(a_1^{(k)}, \ldots, a_{m_k}^{(k)}\right) = (w_{g-k}, w_{g-k-1}, \ldots, w_{g-k-m_k+1}) - \left(w_1^*, \ldots, w_{m_k}^*\right).$$

Then $\sum_{j=1}^{m_k} a_j^{(k)} = N_{\lambda,k}$ and the following relation is valid:

$$\partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} s_\lambda\left(\sum_{i=1}^{k}[z_i]\right)$$
$$= \pm\partial_{a_1^{(k-1)}} \cdots \partial_{a_{m_k}^{(k-1)}} s_\lambda\left(\sum_{i=1}^{k-1}[z_i]\right) z_k^{\lambda_k} + O\left(z_k^{\lambda_k+1}\right). \tag{4}$$

These derivatives generalize those of [9, 13]. Our construction here clarifies the condition under which extensions of derivatives in [13] exist.

The tau function corresponding to a point of the cell UGM^λ of UGM specified by a partition λ has the expansion of the form

$$\tau(t) = s_\lambda(t) + \sum_{\lambda<\mu} \xi_\mu s_\mu(t). \tag{5}$$

We show that the vanishing property and the Eqs. (3), (4) for Schur functions hold without any change if Schur functions are replaced by tau functions. To this end we need to study derivatives of Schur functions $s_\mu(t)$ corresponding to partitions μ satisfying $\lambda \leq \mu$ simultaneously. For example we have to study properties of "$a_j^{(k)}$-derivatives" of $s_\mu(t)$ where $a_j^{(k)}$ are determined from λ.

In the case corresponding to (n, s) curves all the properties of tau functions established in this way are transplanted to sigma functions without much difficulty using the relation of the sigma function with the tau function.

For applications to addition formulae we need to study derivatives of Schur functions not only at $[z_1] + \cdots + [z_k]$ but at $[z_1] - [z_2]$. In this case we have

$$\partial_1^{N'_{\lambda,1}} s_\lambda\left([z_1] - [z_2]\right) = (-1)^{l-1} \frac{c_\lambda}{c'_{\lambda,1}} \partial_1^{N_{\lambda,1}} s_\lambda\left([z_1]\right) z_2^{l-1} + O\left(z_2^l\right), \tag{6}$$

where $N'_{\lambda,1} = \lambda_2 + \cdots + \lambda_l - l + 1$ and c_λ is the constant given in Theorem 2. It can be proved using the rational analogue of the Riemann's vanishing theorem for Schur functions [2]. Again (6) and related properties are valid for tau and sigma functions without any change. As a corollary we obtain the expression of the prime form [6, 10, 11] in terms of a certain derivative of the sigma function and consequently closed addition formulae for sigma functions. Here "closed" means "without using prime form". The simplest example of the addition formula in the case of an (n, s) curve $X : y^n - x^s - \sum \lambda_{ij} x^i y^j = 0$, is

$$\frac{\partial_{u_1}^{N_{\lambda,2}} \sigma(p_2 + p_1) \partial_{u_1}^{N'_{\lambda,1}} \sigma(p_2 - p_1)}{(\partial_{u_1}^{N_{\lambda,1}} \sigma(p_1))^2 (\partial_{u_1}^{N_{\lambda,1}} \sigma(p_2))^2} = (-1)^g c_\lambda \left(c'_{\lambda,1}\right)^{-4} c'_{\lambda,2}(x_2 - x_1), \qquad (7)$$

where $p_i \in X$ is identified with its Abel-Jacobi image, $x_i = x(p_i)$ and λ is the partition corresponding to the gap sequence at ∞ of X. It generalizes the famous addition formula for Weierstrass' sigma function

$$\frac{\sigma(u_1 + u_2)\sigma(u_1 - u_2)}{\sigma(u_1)^2 \sigma(u_2)^2} = \wp(u_2) - \wp(u_1),$$

since $(x_i, y_i) = (\wp(u_i), \wp'(u_i))$, $i = 1, 2$, are two points on $y^2 = 4x^3 - g_2 x - g_3$ and the right hand side can be written as $x_2 - x_1$. The formulae in [13] for hyperelliptic sigma functions are recovered if we use "$a_j^{(k)}$-derivatives" instead of u_1-derivative (see the remark after Corollary 10).

The present paper is organized as follows. In section two properties of derivatives of Schur functions are studied. The notion of gap sequence and the sequence $a_i^{(k)}$ are introduced. We lift the properties of Schur function in section two to functions satisfying similar expansion to the tau functions of the KP-hierarchy in Sect. 3. In Sect. 4 the properties on derivatives of the sigma function are proved using the sigma function expression of the tau function. The expression of the prime form in terms of a derivative of the sigma function of an (n, s) curve is given in Sect. 5. Addition formulae for sigma functions are proved.

2 Schur Function

A sequence of non-negative integers $\lambda = (\lambda_1, \ldots, \lambda_l)$ satisfying $\lambda_1 \geq \cdots \geq \lambda_l$ is called a partition. The number of non-zero elements in λ is called the length of λ and is denoted by $l(\lambda)$. We identify λ with partitions which are obtained from λ by adding arbitrary number of 0's, i.e. $(\lambda_1, \ldots, \lambda_l, 0, \ldots, 0)$. We set $|\lambda| = \lambda_1 + \cdots + \lambda_l$.

Let $t = (t_1, t_2, t_3, \ldots)$ and $p_n(t)$ the polynomial in t defined by

$$\exp\left(\sum_{n=1}^{\infty} t_n k^n\right) = \sum_{n=0}^{\infty} p_n(t) k^n. \qquad (8)$$

We set $p_n(t) = 0$ for $n < 0$.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ Schur functions $s_\lambda(t)$ and $S_\lambda(x)$ are defined by

$$s_\lambda(t) = \det\left(p_{\lambda_i - i + j}(t)\right)_{1 \leq i, j \leq l},$$
$$S_\lambda(x) = \frac{\det(x_j^{\lambda_i + l - i})_{1 \leq i, j \leq l}}{\prod_{i<j}(x_i - x_j)}. \qquad (9)$$

The function $S_\lambda(x)$ is a symmetric polynomial of $x_1, \ldots, x_l$ which is homogeneous of degree $|\lambda|$.

We introduce the symbol $[x]$ by

$$[x] = {}^t\left(x, \frac{x^2}{2}, \frac{x^3}{3}, \ldots\right),$$

which is an analogue of Abel-Jacobi map in the theory of Schur functions. With this symbol, $s_\lambda(t)$ and $S_\lambda(x)$ are related by

$$s_\lambda\left(\sum_{i=1}^{n}[x_i]\right) = S_\lambda(x),$$

for $n \geq l(\lambda)$. From this relation we have

Proposition 1 *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition of length l. Then*

(i) $s_\lambda(\sum_{i=1}^{l}[x_i]) = s_{(\lambda_1,\ldots,\lambda_{l-1})}(\sum_{i=1}^{l-1}[x_i])x_l^{\lambda_l} + O(x_l^{\lambda_l+1})$.
(ii) *If $k < l$,* $s_\lambda(\sum_{i=1}^{k}[x_i]) = 0$.

Proof (i) It immediately follows from the definition of $S_\lambda(x)$.

(ii) We have

$$s_\lambda\left(\sum_{i=1}^{k}[x_i]\right) = s_\lambda\left(\sum_{i=1}^{k}[x_i] + [0] + \cdots + [0]\right).$$

The right hand side is zero by (i) since $\lambda_l \geq 1$. □

Let G^c be a subset of the set of non-negative integers $\mathbb{Z}_{\geq 0}$. We assume that G^c is a semi-group, that is, it is closed under addition and contains 0. Set $G = \mathbb{Z}_{\geq 0} \backslash G^c$.

Definition 1 Let g be a positive integer. G is called a gap sequence of genus g, if $\sharp G = g$. Elements of G and G^c are called gaps and non-gaps respectively.

For a gap sequence of genus g enumerate elements of G and G^c respectively as

$$w_1 < w_2 < \cdots < w_g,$$
$$0 = w_1^* < w_2^* < w_3^* < \cdots .$$

Then $w_1 = 1$. For, otherwise G^c contains 1 and $G^c = \mathbb{Z}_{\geq 0}$ which is impossible due to $g \geq 1$. With this notation in mind we sometimes use $(w_1, \ldots, w_g)$ to denote a gap sequence instead of $\{w_1, \ldots, w_g\}$.

Example 1 Let (n, s) be a pair of relatively prime integers such that $n, s \geq 2$. We set

$$G^c = \{in + js | i, j \geq 0\}.$$

Then G is a gap sequence of genus $g = 1/2(n-1)(s-1)$ [2]. We call G the gap sequence of type (n, s). It is characterized by the condition that G^c is generated by two elements.

Example 2 Let $G = \{1, 2, 3, 7\}$ and $G^c = \mathbb{Z}_{\geq 0} \backslash G$. Then G is a gap sequence of genus four. In this case G^c is generated by 4, 5, 6. Therefore G is not of type (n, s) for any (n, s).

In this way the gap sequences are classified by the minimum number of generators of G^c.

For a gap sequence $\{w_1, \ldots, w_g\}$ we associate a partition λ by

$$\lambda = (w_g, \ldots, w_1) - (g-1, \ldots, 1, 0).$$

A special property of the partition determined from a gap sequence is the following.

Proposition 2 *If λ is determined from a gap sequence $(w_1, \ldots, w_g)$, then $s_\lambda(t)$ does not depend on t_i, $i \notin \{w_1, \ldots, w_g\}$.*

In order to prove the proposition we introduce some notation.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ we associate a strictly decreasing sequence of numbers $\bar{w}_i$ by

$$(\bar{w}_1, \ldots, \bar{w}_l) = (\lambda_1, \ldots, \lambda_l) + (l-1, l-2, \ldots, 0).$$

By this correspondence the set of partitions of length at most l bijectively corresponds to the set of strictly decreasing sequence non-negative integers $\bar{w}_1 > \cdots > \bar{w}_l \geq 0$.

For $(\bar{w}_1, \ldots, \bar{w}_l)$ we set

$$(w_l, \ldots, w_1) = (\bar{w}_1, \ldots, \bar{w}_l).$$

The introduction of the notation $\bar{w}_i$ is for the sake of simplicity in proofs and that of w_i is for the sake of being consistent with the notation of gap sequence.

For integers $i_1, \ldots, i_l$ define the symbol $[i_1, \ldots, i_l]$ as the determinant of the $l \times l$ matrix whose j-th row is

$$\big(\ldots, p_{i_j - 1}(t), p_{i_j}(t)\big).$$

We write $[i_1, \ldots, i_l](t)$ if it is necessary to write explicitly the dependence on t.

By the definition, $[i_1, \ldots, i_l]$ is skew symmetric in the numbers $i_1, \ldots, i_l$ and becomes zero if two numbers coincide or some number is negative.

With this notation

$$s_\lambda(t) = [\bar{w}_1, \ldots, \bar{w}_l].$$

Differentiating (8) by t_i we have

$$\partial_i p_n(t) = p_{n-i}(t), \quad \partial_i = \frac{\partial}{\partial t_i}.$$

Therefore we have

$$\partial_i s_\lambda(t) = \sum_{j=1}^{l} [\bar{w}_1, \ldots, \bar{w}_j - i, \ldots, \bar{w}_l].$$

Proof of Proposition 2 We have to show, for $i \geq 2$,

$$\partial_{w_i^*} s_\lambda(t) = \sum_{j=1}^{l} D_j = 0, \quad D_j = \left[w_l, \ldots, w_j - w_i^*, \ldots, w_1\right]. \tag{10}$$

If $w_j - w_i^* < 0$, obviously $D_j = 0$. Suppose that $w_j - w_i^* > 0$. Let $G = \{w_1, \ldots, w_g\}$. Then $w_j - w_i^* \in G$. For, if $w_j - w_i^* \in G^c$ then $w_j \in G^c + w_i^* \subset G^c$ which is absurd. Thus $w_j - w_i^* = w_k$ for some k. Notice that $w_i^* \geq 1$ and $k \neq j$, since $i \geq 2$. Therefore $D_j = 0$ because two rows coincide. Consequently (10) is proved. □

Definition 2 Let G be a gap sequence of genus g. For $0 \leq k \leq g-1$ we define a positive integer m_k and a sequence of integers $a_i^{(k)}$, $1 \leq i \leq m_k$ by

$$m_k = \sharp\left\{i | w_i^* < g - k\right\},$$
$$\left(a_1^{(k)}, \ldots, a_{m_k}^{(k)}\right) = (w_{g-k}, w_{g-k-1}, \ldots, w_{g-k-m_k+1}) - \left(w_1^*, \ldots, w_{m_k}^*\right).$$

Example 3 For the gap sequence of type $(2, 2g+1)$ we have

$$(w_1, w_2, \ldots, w_g) = (1, 3, \ldots, 2g-1), \qquad \left(w_1^*, w_2^*, w_3^*, \ldots\right) = (0, 2, 4, \ldots).$$

Then

$$m_k = \sharp\{i | 2i - 2 < g - k\} = \left[\frac{g-k+1}{2}\right],$$
$$\left(a_1^{(k)}, a_2^{(k)}, \ldots\right) = (2g - 2k - 1, 2g - 2k - 5, 2g - 2k - 9, \ldots).$$

This sequence recovers the rule for derivatives in [13].

For a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ and a number k such that $0 \leq k \leq l-1$ we set

$$N_{\lambda,k} = \lambda_{k+1} + \cdots + \lambda_l. \tag{11}$$

Lemma 1 (i) $a_1^{(k)} > \cdots > a_{m_k}^{(k)} \geq 1$.

(ii) *Each* $a_i^{(k)}$ *belongs to* G.

(iii) *Let* λ *be the partition determined from* G *then*

$$\sum_{i=1}^{m_k} a_i^{(k)} = N_{\lambda,k}.$$

Proof (i) Notice that $(w_{g-k}, w_{g-k-1}, \ldots)$ is strictly decreasing and $(w_1^*, w_2^*, \ldots)$ is increasing. Therefore $\{a_i^{(k)}\}$ is strictly decreasing. Since G and G^c are complement to each other we have

$$\{0, 1, \ldots, g-k-1\} = \left\{w_1^*, \ldots, w_{m_k}^*\right\} \sqcup \{w_1, \ldots, w_{g-k-m_k}\}. \tag{12}$$

Then, by the definition of the number m_k,

$$\begin{aligned} &w_1^* < \cdots < w_{m_k}^* < g-k \le w_{m_k+1}^* < \cdots, \\ &w_1 < \cdots < w_{g-k-m_k} < g-k \le w_{g-k-m_k+1} < \cdots < w_{g-k} < \cdots. \end{aligned} \tag{13}$$

In particular $a_{m_k}^{(k)} = w_{g-k-m_k+1} - w_{m_k}^* \ge 1$.

(ii) Suppose that $a_j^{(k)} \in G^c$. Since G^c is a semi-group we have

$$w_{g-k-j+1} = a_j^{(k)} + w_j^* \in G^c,$$

which is absurd. Thus $a_j^{(k)} \in G$.

(iii) By (12) we have

$$\begin{aligned} \sum_{i=1}^{m_k} a_i^{(k)} &= \sum_{i=g-k-m_k+1}^{g-k} w_i - \sum_{i=1}^{m_k} w_i^* \\ &= \sum_{i=g-k-m_k+1}^{g-k} w_i - \left(\sum_{i=1}^{g-k-1} i - \sum_{i=1}^{g-k-m_k} w_i \right) \\ &= \sum_{i=1}^{g-k} w_i - \sum_{i=1}^{g-k-1} i = \sum_{i=k+1}^{g} \lambda_i. \end{aligned}$$

□

For $\alpha = (\alpha_1, \alpha_2, \ldots)$ with finite number of non-zero components we define the weight of α and the symbol ∂^α by

$$\mathrm{wt}\,\alpha = \sum_{i=1}^{\infty} i\alpha_i, \quad \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots.$$

The weight of ∂^α is defined to be the weight of α.

Proposition 3 *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition and $0 \le k \le l-1$. If $\mathrm{wt}\,\alpha < N_{\lambda,k}$ we have*

$$\partial^\alpha s_\lambda\left(\sum_{i=1}^{k}[x_i]\right) = 0.$$

For $k = 0$ the right hand side should be understood as $\partial^\alpha s_\lambda(0)$.

Proof Notice that $\partial^\alpha s_\lambda(t)$ is a linear combination of determinants of the form

$$[\bar{w}_1 - r_1, \ldots, \bar{w}_l - r_l], \quad r_1 + \cdots + r_l = \mathrm{wt}\,\alpha. \tag{14}$$

If (14) is not zero, $\bar{w}_i - r_i$ are all non-negative and different. Thus there exists a permutation $(i_1, \ldots, i_l)$ of $(1, \ldots, l)$ such that

$$\bar{w}_{i_1} - r_{i_1} > \cdots > \bar{w}_{i_l} - r_{i_l} \ge 0.$$

Let μ be the partition corresponding to this strictly decreasing sequence. Then

$$s_\mu(t) = [\bar{w}_{i_1} - r_{i_1}, \ldots, \bar{w}_{i_l} - r_{i_l}].$$

If $l(\mu) > k$, $s_\mu(\sum_{i=1}^{k}[x_i]) = 0$ by (ii) of Proposition 1.

We prove that $l(\mu) \le k$ is impossible if $\mathrm{wt}\,\alpha < N_{\lambda,k}$. Suppose that $l(\mu) \le k$. Then $\mu = (\mu_1, \ldots, \mu_k, 0, \ldots, 0)$ and

$$\bar{w}_{i_l} - r_{i_l} = 0, \quad \bar{w}_{i_{l-1}} - r_{i_{l-1}} = 1, \quad \ldots, \quad \bar{w}_{i_{k+1}} - r_{i_{k+1}} = l-k-1.$$

Therefore

$$r_{i_l} = \bar{w}_{i_l}, \quad r_{i_{l-1}} = \bar{w}_{i_{l-1}} - 1, \quad \ldots, \quad r_{i_{k+1}} = \bar{w}_{i_{k+1}} - (l-k-1),$$

and we have

$$\begin{aligned} r_{i_l} + \cdots + r_{i_{k+1}} &= \bar{w}_{i_l} + \cdots + \bar{w}_{i_{k+1}} - (1 + 2 + \cdots + l-k-1) \\ &\ge \bar{w}_l + \cdots + \bar{w}_{k+1} - (1 + 2 + \cdots + l-k-1). \end{aligned}$$

On the other hand

$$\begin{aligned} r_{i_l} + \cdots + r_{i_{k+1}} &\le r_l + \cdots + r_1 = \mathrm{wt}\,\alpha \\ &< \lambda_{k+1} + \cdots + \lambda_l \\ &= \bar{w}_{k+1} + \cdots + \bar{w}_l - (1 + 2 + \cdots + l-k-1), \end{aligned}$$

which is a contradiction. Thus Proposition 3 is proved. □

Theorem 1 *Let $\lambda = (\lambda_1, \ldots, \lambda_g)$ be the partition determined from a gap sequence of genus g, $0 \le k \le g$ and $a_j^{(k)}$ the associated sequence of numbers for $k \ne g$. We set $s_{(\lambda_1,\ldots,\lambda_k)}(\sum_{i=1}^k [x_i]) = 1$ for $k = 0$ and $\partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} = 1$ for $k = g$.*

(i) *We have*

$$\partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} s_\lambda \left(\sum_{i=1}^k [x_i] \right) = c_k s_{(\lambda_1,\ldots,\lambda_k)} \left(\sum_{i=1}^k [x_i] \right),$$

where $c_k = \pm 1$, $k \ne g$ is given by the sign of the permutation

$$c_k = \operatorname{sgn} \begin{pmatrix} w_1^* & \ldots & w_{m_k}^* & w_{g-k-m_k} & \ldots & w_1 \\ g-k-1 & g-k-2 & \ldots & \ldots & 1 & 0 \end{pmatrix},$$

and $c_g = 1$.

(ii) *Let $\mu = (\mu_1, \ldots, \mu_g)$ be a partition such that $\mu_i = \lambda_i$ for $k+1 \le i \le g$. Then*

$$\partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} s_\mu \left(\sum_{i=1}^k [x_i] \right) = c_k s_{(\mu_1,\ldots,\mu_k)} \left(\sum_{i=1}^k [x_i] \right),$$

where c_k is the same as in (i).

Remark 1 For the gap sequence of type (n, s) it can be checked that the derivative determined from the sequence $a_j^{(k)}$ is the same as that found in [9]. In that case (i) of Theorem 1 is proved in that paper.

Lemma 2 *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition, $0 \le k \le l-1$ and $r_1, \ldots, r_l$ non-negative integers. Suppose that the following conditions*:

$$\sum_{i=1}^l r_i = N_{\lambda,k}, \tag{15}$$

$$[\bar{w}_1 - r_1, \ldots, \bar{w}_l - r_l] \left(\sum_{i=1}^k [x_i] \right) \ne 0. \tag{16}$$

Then

(i) *We have $r_i = 0$ for $1 \le i \le k$.*
(ii) *The sequence $(\bar{w}_{k+1} - r_{k+1}, \ldots, \bar{w}_l - r_l)$ is a permutation of $(l-k-1, \ldots, 1, 0)$.*
(iii) *We have*

$$[\bar{w}_1 - r_1, \ldots, \bar{w}_l - r_l] \left(\sum_{i=1}^k [x_i] \right) = c s_{(\lambda_1,\ldots,\lambda_k)} \left(\sum_{i=1}^k [x_i] \right),$$

where $c = \pm 1$.

Proof By the assumption (16) there exists a permutation $(i_1, \ldots, i_l)$ of $(1, \ldots, l)$ and a partition $\mu = (\mu_1, \ldots, \mu_l)$ such that

$$\bar{w}_{i_1} - r_{i_1} > \cdots > \bar{w}_{i_l} - r_{i_l} \geq 0,$$
$$s_\mu(t) = [\bar{w}_{i_1} - r_{i_1}, \ldots, \bar{w}_{i_l} - r_{i_l}],$$

and $l(\mu) \leq k$ as in the proof of Proposition 3. In particular $\mu_i = 0$ for $i \geq k+1$ which means

$$\bar{w}_{i_l} - r_{i_l} = 0, \quad \ldots, \quad \bar{w}_{i_{k+1}} - r_{i_{k+1}} = l - k - 1.$$

By a similar calculation to that in the proof of Proposition 3 we have

$$\begin{aligned} r_{i_{k+1}} + \cdots + r_{i_l} &= \bar{w}_{i_{k+1}} + \cdots + \bar{w}_{i_l} - (1 + 2 + \cdots + l - k - 1) \\ &\geq \bar{w}_{k+1} + \cdots + \bar{w}_l - (1 + 2 + \cdots + l - k - 1), \end{aligned} \tag{17}$$

and

$$\begin{aligned} r_{i_{k+1}} + \cdots + r_{i_l} &\leq r_1 + \cdots + r_l \\ &= \lambda_{k+1} + \cdots + \lambda_l \\ &= \bar{w}_{k+1} + \cdots + \bar{w}_l - (1 + 2 + \cdots + l - k - 1), \end{aligned} \tag{18}$$

where we use (15). Therefore every inequalities in (17) and (18) are equalities. Then $r_{i_1} = \cdots = r_{i_k} = 0$ by (18) and $(i_{k+1}, \ldots, i_l)$ is a permutation of $(k+1, \ldots, l)$ by (17). It, then, implies that $(i_1, \ldots, i_k)$ is a permutation of $(1, \ldots, k)$.

Since

$$(\bar{w}_{i_1} - r_{i_1}, \ldots, \bar{w}_{i_l} - r_{i_l}) = (\bar{w}_{i_1}, \ldots, \bar{w}_{i_k}, \bar{w}_{i_{k+1}} - r_{i_{k+1}}, \ldots, \bar{w}_{i_l} - r_{i_l})$$

and it is strictly decreasing, $(i_1, \ldots, i_k) = (1, \ldots, k)$. Thus

$$[\bar{w}_{i_1} - r_{i_1}, \ldots, \bar{w}_{i_l} - r_{i_l}] = [\bar{w}_1, \ldots, \bar{w}_k, l - k - 1, \ldots, 1, 0]. \tag{19}$$

□

Lemma 3 *For a positive integer m and a set of integers $i_1, \ldots, i_k$ we have*

$$[i_1, \ldots, i_k, m - 1, \ldots, 1, 0] = [i_1 - m, \ldots, i_k - m].$$

Proof Expand the determinant at $m + k$-th row, $m + k - 1$-th row, …, until $k + 1$-st row successively and get the result. □

Applying the lemma to (19) we have

$$[\bar{w}_{i_1} - r_{i_1}, \ldots, \bar{w}_{i_l} - r_{i_l}] = \left[\bar{w}_1 - (l - k), \ldots, \bar{w}_k - (l - k)\right]$$

$$= s_{(\lambda_1,\ldots,\lambda_k)}(t).$$

Since $(i_1,\ldots,i_l)$ is a permutation of $(1,\ldots,l)$,

$$[\bar{w}_1 - r_1,\ldots,\bar{w}_l - r_l] = \pm s_{(\lambda_1,\ldots,\lambda_k)}(t).$$

Proof of Theorem 1 In this proof we fix k and denote $a_j^{(k)}$ simply by a_j. Recall that

$$s_\lambda(t) = [w_g,\ldots,w_1].$$

We compute the value of $\partial_{a_1}\cdots\partial_{a_{m_k}} s_\lambda(t)$ at $t = t^{(k)} := [x_1] + \cdots + [x_k]$.

Step 1. We first consider the term for which the row labeled by $w_{g-k-(i-1)}$ is differentiated by ∂_{a_i} for $1 \le i \le m_k$. It is of the form

$$A := [w_g,\ldots,w_{g-k+1}, w_{g-k} - a_1,\ldots,w_{g-k-(m_k-1)} - a_{m_k}, w_{g-k-m_k},\ldots,w_1].$$

By the definition of a_i

$$w_{g-k-(i-1)} - a_i = w_i^*.$$

Therefore

$$A = \left[w_g,\ldots,w_{g-k+1}, w_1^*,\ldots,w_{m_k}^*, w_{g-k-m_k},\ldots,w_1\right].$$

Using (12) we have

$$\begin{aligned} A &= c_k[w_g,\ldots,w_{g-k+1}, g-k-1,\ldots,1,0] \\ &= c_k s_{(\lambda_1,\ldots,\lambda_k)}(t). \end{aligned}$$

Step 2. We prove that the terms differentiated in a different way from that in Step 1 are zero at $t = t^{(k)}$.

By Lemma 1(iii) and Lemma 2(i) the term is zero at $t^{(k)}$ if some row corresponding to w_i, $g-k+1 \le i \le g$, is differentiated. Therefore, for non-zero terms, only the last $g-k$ rows are differentiated.

So let us consider a term for which only some of last $g-k$ rows are differentiated. Notice that a term is zero if some row is differentiated more than once. In fact some row corresponding to w_j with $g-k-m_k+1 \le j \le g-k$ is not differentiated in this case. By (13) $w_j \ge g-k$. Consequently it is impossible for the sequence $(w_{g-k},\ldots,w_1)$ to be a permutation of $(g-k-1,\ldots,1,0)$. Then this term is zero at $t^{(k)}$ by Lemma 2(ii).

As a consequence of the above argument we know that a term is zero if some row labeled by w_j with $g-k-m_k+1 \le j \le g-k$ is not differentiated. So let us consider a term for which each row corresponding to w_j with $g-k-m_k+1 \le j \le g-k$ is differentiated exactly once. We assume that the row corresponding $w_{g-k-(i-1)}$ is differentiated by ∂_{a_i} for $1 \le i < j$ with some $j \le m_k$ and ∂_{a_j} differentiates the row corresponding to $w_{g-k-(j'-1)}$ for some j' with $j < j'$. We have

$$w_{g-k} - a_1 = w_1^*, \quad \ldots, \quad w_{g-k-(j-2)} - a_{j-1} = w_{j-1}^*,$$

and

$$\begin{aligned} w_{g-k-(j'-1)} - a_j &= w_{g-k-(j'-1)} - \left(w_{g-k-(j-1)} - w_j^*\right) \\ &= w_j^* - (w_{g-k-(j-1)} - w_{g-k-(j'-1)}) < w_j^*. \end{aligned} \tag{20}$$

If $w_{g-k-(j'-1)} - a_j$ belongs to G^c, we have

$$w_{g-k-(j'-1)} - a_j \in \left\{w_1^*, \ldots, w_{j-1}^*\right\},$$

by (20). Thus the term is zero since two rows coincide.

Suppose that $w_{g-k-(j'-1)} - a_j$ belongs to G. Then

$$w_{g-k-(j'-1)} - a_j \in \{w_1, \ldots, w_{g-k-m_k}\},$$

since $w_j^* < g-k$ and (13). In this case the term in consideration is zero since again two rows coincide. Thus (i) of Theorem 1 is proved.

Step 3. We prove (ii) of Theorem 1. Let $w'_g > \cdots > w'_1$ be the strictly decreasing sequence corresponding to μ, that is,

$$\left(w'_g, \ldots, w'_1\right) = (\mu_1, \ldots, \mu_g) + (g-1, \ldots, 1, 0).$$

By assumption $w_i = w'_i$ for $1 \le i \le g-k$. Define $w_i'^*$, $i \ge 0$ by

$$\begin{aligned} &\left\{w_i'^* | i \ge 0\right\} = \mathbb{Z}_{\ge 0} \backslash \left\{w'_i\right\}, \\ &0 = w_1'^* < w_2'^* < \cdots. \end{aligned}$$

Then $w_i^* = w_i'^*$ for $1 \le i \le m_k$, since

$$\begin{aligned} \left\{w_1^*, \ldots, w_{m_k}^*\right\} \sqcup \left\{w'_1, \ldots, w'_{g-k-m_k}\right\} &= \left\{w_1^*, \ldots, w_{m_k}^*\right\} \sqcup \{w_1, \ldots, w_{g-k-m_k}\} \\ &= \{0, 1, \ldots, g-k-1\}. \end{aligned}$$

As a consequence the arguments in step 1 and step 2 are valid without any change if w_i, w_i^* are replaced by w'_i, $w_i'^*$ respectively. □

Next we study properties of Schur functions with respect to t_1 derivative.

Theorem 2 *Let* $\lambda = (\lambda_1, \ldots, \lambda_l)$ *be a partition*, $(w_l, \ldots, w_1)$ *the corresponding strictly decreasing sequence and* $0 \le k \le l$. *Then*

$$\partial_1^{N_{\lambda,k}} s_\lambda\left(\sum_{i=1}^{k}[x_i]\right) = c'_{\lambda,k} s_{(\lambda_1,\ldots,\lambda_k)}\left(\sum_{i=1}^{k}[x_i]\right),$$

where

$$c'_{\lambda,k} = \frac{N_{\lambda,k}!}{\prod_{i=1}^{l-k} w_i!} \prod_{i<j}^{l-k} (w_j - w_i).$$

Proof We have

$$s_\lambda(t)=[w_l,\ldots,w_1].$$

By Leibniz's rule

$$\partial_1^{N_{\lambda,k}} s_\lambda(t)=\sum_{r_1+\cdots+r_l=N_{\lambda,k}} \frac{N_{\lambda,k}!}{r_1!\cdots r_l!}[w_l-r_l,\ldots,w_1-r_1]. \tag{21}$$

By Lemma 2, if $[w_l-r_l,\ldots,w_1-r_1](t^{(k)})\neq 0$ then $r_i=0$ for $l-k+1\le i\le l$, $(w_{l-k},\ldots,w_1-r_1)$ is a permutation of $(l-k-1,\ldots,1,0)$ and

$$[w_l-r_l,\ldots,w_1-r_1]\big(t^{(k)}\big)=\operatorname{sgn}\begin{pmatrix} w_{l-k} & \cdots & \cdots & w_1-r_1\\ l-k-1 & \cdots & 1 & 0\end{pmatrix}.$$

In this case we can write

$$w_i-r_i=\sigma(i-1),\quad 1\le i\le l-k,$$

for some σ of an element of the symmetric group S_{l-k} which acts on $\{0,1,\ldots,l-k-1\}$. We define $1/n!=0$ for $n<0$ for the sake of convenience. Then

$$\partial_1^{N_{\lambda,k}} s_\lambda\big(t^{(k)}\big)=A_{\lambda,k}s_{(\lambda_1,\ldots,\lambda_k)}\big(t^{(k)}\big),$$

where

$$A_{\lambda,k}=\sum_{\sigma\in S_{l-k}}\operatorname{sgn}\sigma\frac{N_{\lambda,k}!}{(w_1-\sigma(0))!\cdots(w_{l-k}-\sigma(l-k-1))!}.$$

We have

$$\frac{A_{\lambda,k}}{N_{\lambda,k}!}=\det\left(\frac{1}{(w_i-(j-1))!}\right)_{1\le i,j\le l-k} \tag{22}$$

$$=\prod_{i=1}^{l-k}\frac{1}{w_i!}\det\left(\prod_{m=0}^{j-2}(w_i-m)\right)_{1\le i,j\le l-k}, \tag{23}$$

where we set $\prod_{m=0}^{j-2}(w_i-m)=1$ for $j=1$. Notice that the rule $1/n!=0$ for $n<0$ is taken into account in rewriting (22) to (23), since, if $w_i-(j-1)<0$ then $\prod_{m=0}^{j-2}(w_i-m)=0$.

Let us set

$$D=\det\left(\prod_{m=0}^{j-2}(w_i-m)\right)_{1\le i,j\le l-k}.$$

Expanding $\prod_{m=0}^{j-2}(w_i - m)$ in w_i we easily have

$$D = \det\big(w_i^{j-1}\big)_{1\le i,j\le l-k} = \prod_{i<j}^{l-k}(w_j - w_i),$$

and consequently

$$\frac{A_{\lambda,k}}{N_{\lambda,k}!} = \frac{\prod_{i<j}^{l-k}(w_j - w_i)}{\prod_{i=1}^{l-k} w_i!}.$$

□

In order to study addition formulae of sigma functions we need to study properties of Schur functions at $t = [x_1] - [x_2]$.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ let $\lambda' = (\lambda'_1, \ldots, \lambda'_{l'})$ be the conjugate of λ, i.e. $\lambda'_i = \sharp\{j|\lambda_j \ge i\}$.

Theorem 3 *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition of length l, $\lambda' = (\lambda'_1, \ldots, \lambda'_{l'})$ and $\tilde{\lambda}' = (\lambda'_1 - 1, \ldots, \lambda'_{l'} - 1)$. Then*

$$s_\lambda\left([x] - \sum_{i=1}^{l'}[x_i]\right) = (-1)^{N_{\lambda,1}} s_{\tilde{\lambda}'}\left(\sum_{i=1}^{l'}[x_i]\right)\prod_{j=1}^{l'}(x - x_j).$$

Proof This theorem is essentially proved in the proof of Theorem 5.5 in [2]. In [2] λ is assumed to be the partition corresponding to the gap sequence of type (n, s). In that case $\lambda = \lambda'$ and the assertion in this theorem is not stated. Here we give a proof since it is a key theorem for applications to addition formulae. For the notational simplicity we prove the assertion by interchanging λ and λ'. All facts and notation concerning Schur and symmetric functions used in this proof can be found in [8].

Let $e_i = e_i(x_1, \ldots, x_m)$ be the elementary symmetric function:

$$\prod_{i=1}^{m}(t + x_i) = \sum_{i=0}^{m} e_i t^{m-i}. \tag{24}$$

They satisfy the relation

$$e_i(x_1, \ldots, x_m) = e_i(x_1, \ldots, x_{m-1}) + x_m e_{i-1}(x_1, \ldots, x_{m-1}). \tag{25}$$

In general, for a partition $\mu = (\mu_1, \ldots, \mu_m)$, the following equation holds:

$$S_{\mu'}(x_1, \ldots, x_m) = \det(e_{\mu_i - i + j})_{1\le i,j\le m}. \tag{26}$$

Let $\mathbf{a}_j$ be the column vector defined by

$$\mathbf{a}_j = {}^t(e_{\lambda_1 - 1 + j}, e_{\lambda_2 - 2 + j}, \ldots, e_{\lambda_l - l + j}),$$

where $e_r = e_r(x, x_1, \ldots, x_l)$.

By (25), (26) we have

$$
\begin{aligned}
s_{\lambda'}\big([x]+[x_1]+\cdots+[x_l]\big) &= S_{\lambda'}(x,x_1,\ldots,x_l)\\
&= \det(e_{\lambda_i-i+j})_{1\le i,j\le l}\\
&= \det(\mathbf{a}_1+x\mathbf{a}_0,\mathbf{a}_2+x\mathbf{a}_1,\ldots,\mathbf{a}_{l-1}+x\mathbf{a}_l)\\
&= \sum_{j=0}^{l} x^j \det(\mathbf{a}_0,\ldots,\mathbf{a}_{j-1},\mathbf{a}_{j+1},\ldots,\mathbf{a}_l)\\
&= \det\begin{pmatrix} 1 & -x & \cdots & (-x)^l \\ \mathbf{a}_0 & \mathbf{a}_1 & \cdots & \mathbf{a}_l \end{pmatrix}. \qquad (27)
\end{aligned}
$$

Let $p_r=\sum_{i=1}^{l} x_i^k$ be the power sum symmetric function, ω, $\widehat{\omega}$ and ι the automorphisms of the ring of symmetric polynomials in $x_1,\ldots,x_l$ defined by

$$\widehat{\omega}(p_r)=(-1)^r p_r, \qquad \iota(p_r)=-p_r, \qquad \omega=\iota\circ\widehat{\omega}. \qquad (28)$$

Notice that $\widehat{\omega}$ is, in terms of x_j, the map sending x_j to $-x_j$ for $1\le j\le l$. Then

$$s_{\lambda'}\Bigg([x]-\sum_{i=1}^{l}[x_i]\Bigg)=(-1)^{|\lambda|}\omega\Bigg(s_{\lambda'}\Bigg([-x]+\sum_{i=1}^{l}[x_i]\Bigg)\Bigg). \qquad (29)$$

It can be checked by computing the right hand side using (28) and the relation $S_\mu(-x_1,\ldots,-x_m)=(-1)^{|\mu|}S_\mu(x_1,\ldots,x_m)$.

Let $h_i=h_i(x_1,\ldots,x_l)$ be the complete symmetric function:

$$\frac{1}{\prod_{i=1}^{l}(1-tx_i)}=\sum_{i=0}^{\infty}h_i x^i.$$

Then $\omega(e_i)=h_i$ and

$$\omega(\mathbf{a}_j)={}^t(h_{\lambda_1-1+j},\ldots,h_{\lambda_l-l+j}). \qquad (30)$$

By (27) and (29) we have

$$s_{\lambda'}\big([x]-[x_1]-\cdots-[x_l]\big)=(-1)^{|\lambda|}\det\begin{pmatrix} 1 & x & \cdots & x^l \\ \omega(\mathbf{a}_0) & \omega(\mathbf{a}_1) & \cdots & \omega(\mathbf{a}_l)\end{pmatrix}. \qquad (31)$$

Using the relation,

$$\sum_{j=0}^{k}(-1)^j e_j h_{k-j}=0, \quad k\ge 1, \qquad (32)$$

we have

$$\sum_{j=0}^{k}(-1)^j e_j \omega(\mathbf{a}_{l-j}) = \mathbf{o}. \tag{33}$$

By (24), (30), (31), (33) we obtain

$$s_{\lambda'}\left([x]-\sum_{i=1}^{l}[x_i]\right) = (-1)^{l+|\lambda|}\det\big(\omega(\mathbf{a}_0),\ldots,\omega(\mathbf{a}_{l-1})\big)\prod_{j=1}^{l}(x-x_j)$$
$$= (-1)^{N_{\lambda',1}}\det(h_{\lambda_i-1-i+j})_{1\le i,j\le l}\prod_{j=1}^{l}(x-x_j).$$

Then the theorem follows from

$$S_{(\mu_1,\ldots,\mu_m)}(x_1,\ldots,x_m) = \det(h_{\mu_i-i+j})_{1\le i,j\le m}. \qquad \square$$

Corollary 1 *Let $\lambda=(\lambda_1,\ldots,\lambda_l)$ be a partition of length l. Then $s_\lambda([x_1]-[x_2])$ is not identically zero if and only if $\lambda_i = 1$ for $2\le i\le l$, that is, λ is a hook.*

Proof Setting $x_i = 0$ for $2 \le i \le l'$ in Theorem 3 we have

$$s_\lambda\big([x]-[x_1]\big) = (-1)^{N_{\lambda,1}} s_{\tilde{\lambda}'}\big([x_1]\big)x^{l'-1}(x-x_1). \tag{34}$$

Thus $s_\lambda([x]-[x_1]) \neq 0$ is equivalent to $s_{\tilde{\lambda}'}([x_1]) \neq 0$. The latter is equivalent to the condition that the length of $\tilde{\lambda}'$ is one. It means that $\lambda' = (\lambda'_1, 1^{l'-1})$ which is equivalent to that λ is a hook. $\square$

Theorem 4 *Let $\lambda=(\lambda_1,\ldots,\lambda_l)$ be a partition of length l, $(w_l,\ldots,w_1)$ the corresponding sequence and $N'_{\lambda,1} = \sum_{i=2}^{l}\lambda_i - l + 1$.*

(i) *If $n < N'_{\lambda,1}$*

$$\partial_1^n s_\lambda\big([x_1]-[x_2]\big) = 0.$$

(ii) *We have*

$$\partial_1^{N'_{\lambda,1}} s_\lambda\big([x_1]-[x_2]\big) = c_\lambda s_{(\lambda_1,1^{l-1})}\big([x_1]-[x_2]\big),$$

where

$$c_\lambda = \frac{N'_{\lambda,1}!}{\prod_{i=1}^{l-1}(w_i-1)!}\prod_{i<j}^{l-1}(w_j-w_i).$$

(iii) *Let* $\mu = (\mu_1, \ldots, \mu_{l'})$ *be a partition of length* $l' \geq l$ *such that* $\mu_i = \lambda_i$ *for* $2 \leq i \leq l$ *and* $\mu_i = 1$ *for* $i > l$. *Then*

$$\partial_1^{N'_{\lambda,1}} s_\mu\big([x_1] - [x_2]\big) = c_\lambda s_{(\mu_1, 1^{l'-1})}\big([x_1] - [x_2]\big).$$

(iv) *For* $m, n \geq 1$ *we have*

$$a_{(m,1^{n-1})}\big([x_1] - [x_2]\big) = (-1)^{n-1} x_1^{m-1} x_2^{n-1}(x_1 - x_2).$$

Proof Notice that

$$\partial_1 s_\lambda(t) = \sum_{i=1}^{l} [w_l, \ldots, w_i - 1, \ldots, w_1].$$

In the right hand side $[w_l, \ldots, w_i - 1, \ldots, w_1] \neq 0$ if and only if all its components are different. In terms of the diagram of λ, $\partial_1 s_\lambda(t)$ is a sum of $s_\mu(t)$ with μ being the diagram obtained from λ by removing one box. For example

$$\partial_1 s_{(2,2,1)}(t) = s_{(2,1,1)}(t) + s_{(2,2)}(t).$$

(i) Notice that $N'_{\lambda,1}$ is a number of boxes on second to l-th rows of the diagram of λ which are on the right of the first column. Thus if $n < N'_{\lambda,1}$ it is impossible to get the hook diagram by removing n boxes from λ. Then the assertion of (i) follows from Corollary 1.

(ii) There is only one hook diagram in diagrams obtained from λ by removing $N'_{\lambda,1}$ boxes. It is $\mu := (\lambda_1, 1^{l-1})$. Let us compute the coefficient c of $s_\mu(t)$ in $\partial_1^{N'_{\lambda,1}} s_\lambda(t)$. Consider Eq. (21) with $N_{\lambda,k}$ being replaced by $N'_{\lambda,1}$. In the right hand side, $s_\mu(t)$ appears only as a term such that $r_l = 0$ and $(w_{l-1} - r_{l-1}, \ldots, w_1 - r_1)$ is a permutation of $(l-1, \ldots, 2, 1)$. Let us write, for $1 \leq i \leq l-1$,

$$w_i - r_i = \sigma(i), \quad \sigma \in S_{l-1}.$$

Then by a similar calculation to that in the proof of Theorem 2 we have

$$\frac{c}{N'_{\lambda,1}!} = \sum_{\sigma \in S_{l-1}} \frac{\operatorname{sgn}\sigma}{(w_1 - \sigma(1))! \cdots (w_{l-1} - \sigma(l-1))!} = \frac{\prod_{i<j}^{l-1}(w_j - w_i)}{\prod_{i=1}^{l-1}(w_i - 1)!}.$$

(iii) Similarly to the proof of (ii) the only Schur function appearing in $\partial_1^{N'_{\lambda,1}} s_\mu(t)$ which does not vanish at $t = [x_1] - [x_2]$ is $s_\nu(t)$, $\nu = (\mu_1, 1^{l'-1})$. Let us compute the coefficient c' of $s_\nu(t)$ in $\partial_1^{N'_{\lambda,1}} s_\mu(t)$.

Let $(w'_{l'}, \ldots, w'_1)$ be the strictly decreasing sequence corresponding to μ. Then

$$\partial_1^{N'_{\lambda,1}} s_\mu(t) = \sum_{r_1 + \cdots + r_{l'} = N'_{\lambda,1}} \frac{N'_{\lambda,1}!}{r_1! \cdots r_{l'}!} \big[w'_{l'} - r_{l'}, \ldots, w'_1 - r_1\big]. \tag{35}$$

In the right hand side $[w'_{l'}-r_l,\dots,w'_1-r_1]$ is proportional to $s_\nu(t)$ if and only if $r_i=0$ for $i=l'$ or $i<l'-l$, and $(w'_{l'-1}-r_{l'-1},\dots,w'_{l'-l+1}-r_{l'-l+1})$ is a permutation of $(l'-1,l'-2,\dots,l'-l+1)$. Let us write

$$w'_i-r_i=\sigma(i),\quad l'-l+1\le i\le l'-1,\ \sigma\in S_{l-1}.$$

Then

$$\begin{aligned}\frac{c'}{N'_{\lambda,1}!}&=\sum_{\sigma\in S_{l-1}}\frac{\operatorname{sgn}\sigma}{(w'_{l'-l+1}-\sigma(l'-l+1))!\cdots(w'_{l'-1}-\sigma(l'-1))!}\\&=\frac{\prod_{l'-l+1\le i<j\le l'-1}(w'_j-w'_i)}{\prod_{i=l'-l+1}^{l'-1}(w'_i-l'+l-1)!}.\end{aligned}\tag{36}$$

Let us rewrite c' in terms of λ_j. By assumption $\mu_i=\lambda_i$ for $2\le i\le l$ which implies

$$w'_i=\mu_{l'+1-i}+i-1=\lambda_{l'+1-i}+i-1,\quad l'-l+1\le i\le l'-1.$$

Substitute it into (36) and get

$$c'=\frac{N'_{\lambda,1}!}{\prod_{i=2}^{l}(\lambda_i+l-1-i)!}\prod_{2\le i<j\le l}(\lambda_i-\lambda_j+j-i),$$

which equals to c_λ.

(iv) Set $\lambda=(m,1^{n-1})$ in (34). Then, using $s_{(r)}([x])=x^r$, we get the assertion of (iv). □

3 τ-Function

In this section we lift the properties of Schur functions which have been proved in the previous section to τ-functions.

Let $\le$ be the partial order on the set of partitions defined as follows. For two partitions $\lambda=(\lambda_1,\dots,\lambda_l)$, $\mu=(\mu_1,\dots,\mu_{l'})$, $\lambda\le\mu$ if and only if $\lambda_i\le\mu_i$ for all i.

For a partition $\lambda=(\lambda_1,\dots,\lambda_l)$ we consider a function $\tau(t)$ given as a series of the form

$$\tau(t)=s_\lambda(t)+\sum_{\lambda<\mu}\xi_\mu s_\mu(t),\tag{37}$$

where $\xi_u\in\mathbb{C}$.

Example Let X be a compact Riemann surface of genus $g\ge1$, p_∞ a point of X, $w_1<\cdots<w_g$ the gap sequence at p_∞ and z a local coordinate at p_∞. Embed

the affine ring of $X\backslash\{p_\infty\}$ into Sato's universal Grassmann manifold (UGM) as in the paper [12]. Then the tau function corresponding to this point of UGM has the expansion of the form (37).

Proposition 4 *Let* $\lambda = (\lambda_1, \ldots, \lambda_l)$ *a partition,* $\tau(t)$ *be a function of the form* (37) *and* $0 \le k \le l-1$. *Then, if* $\mathrm{wt}\,\alpha < N_{\lambda,k}$

$$\partial^\alpha \tau\left(\sum_{i=1}^{k}[x_i]\right) = 0.$$

Proof For $\mu = (\mu_1, \ldots, \mu_{l'})$ satisfying $\lambda \le \mu$ we have

$$\mathrm{wt}\,\alpha < \sum_{i=k+1}^{l} \lambda_i \le \sum_{i=k+1}^{l'} \mu_i .$$

Thus

$$\partial^\alpha s_\mu\left(\sum_{i=1}^{k}[x_i]\right) = 0,$$

by Proposition 3. The assertion of the proposition follows from (37). □

For a function $\tau(t)$ of the form (37) and $1 \le k \le l$ let $\tau^{(k)}(t)$ be the function defined by

$$\tau^{(k)}(t) = s_{(\lambda_1,\ldots,\lambda_k)}(t) + \sum_{\mu} \xi_\mu s_{(\mu_1,\ldots,\mu_k)}(t),$$

where the sum in the right hand side is over all partitions $\mu = (\mu_1, \ldots, \mu_l)$ such that $\lambda < \mu$ and $\mu_i = \lambda_i$ for $k+1 \le i \le l$. In particular $\tau^{(l)}(\sum_{i=1}^{k}[x_i]) = \tau(\sum_{i=1}^{k}[x_i])$. We set $\tau^{(0)}(t) = 1$.

Theorem 5 *Let* $\lambda = (\lambda_1, \ldots, \lambda_g)$ *be the partition determined from a gap sequence of genus* g, $\tau(t)$ *a function of the form* (37).

(i) *We have, for* $0 \le k \le g$,

$$\partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} \tau\left(\sum_{i=1}^{k}[x_i]\right) = c_k \tau^{(k)}\left(\sum_{i=1}^{k}[x_i]\right),$$

where c_k *is the same as in Theorem* 1.

(ii) *We have, for* $k \ge 1$,

$$\tau^{(k)}\left(\sum_{i=1}^{k}[x_i]\right) = \tau^{(k-1)}\left(\sum_{i=1}^{k-1}[x_i]\right) x_k^{\lambda_k} + O\left(x_k^{\lambda_k+1}\right).$$

Proof Let $\mu=(\mu_1,\ldots,\mu_l)$ be a partition of length l such that $\lambda\le\mu$. Then $l\ge g$ and

$$\mathrm{wt}\,(\partial_{a_1^{(k)}}\cdots\partial_{a_{m_k}^{(k)}})=\sum_{i=1}^{m_k}a_i^{(k)}=\sum_{i=k+1}^{g}\lambda_i\le\sum_{i=k+1}^{l}\mu_i. \tag{38}$$

If the inequality in the right hand side is not an equality,

$$\partial_{a_1^{(k)}}\cdots\partial_{a_{m_k}^{(k)}}s_\mu\left(\sum_{i=1}^{k}[x_i]\right)=0, \tag{39}$$

by Proposition 3. Therefore, if the left hand side of (39) does not vanish, $l=g$ and

$$\sum_{i=k+1}^{g}\mu_i=\sum_{i=k+1}^{g}\lambda_i.$$

Since $\lambda_i\le\mu_i$ for any i, it implies $\mu_i=\lambda_i$ for $k+1\le i\le g$. For such μ we have, by Theorem 1,

$$\partial_{a_1^{(k)}}\cdots\partial_{a_{m_k}^{(k)}}s_\mu\left(\sum_{i=1}^{k}[x_i]\right)=c_k s_{(\mu_1,\ldots,\mu_k)}\left(\sum_{i=1}^{k}[x_i]\right).$$

The assertion (i) follows from this.

(ii) The assertion easily follows from (i) of Proposition 1 and the definition of $\tau^{(k)}(t)$. □

Combining (i) and (ii) of Theorem 5 we have

Corollary 2 *Under the same assumption as in Theorem 5 we have, for* $1\le k\le g$,

$$\partial_{a_1^{(k)}}\cdots\partial_{a_{m_k}^{(k)}}\tau\left(\sum_{i=1}^{k}[x_i]\right)=\frac{c_k}{c_{k-1}}\partial_{a_1^{(k-1)}}\cdots\partial_{a_{m_{k-1}}^{(k-1)}}\tau\left(\sum_{i=1}^{k-1}[x_i]\right)x_k^{\lambda_k}+O\big(x_k^{\lambda_k+1}\big).$$

Corresponding to Theorem 2 we have

Theorem 6 *Let* $\lambda=(\lambda_1,\ldots,\lambda_l)$ *be a partition of length* l, $\tau(t)$ *a function of the form* (37), $0\le k\le l$. *Then*

$$\partial_1^{N_{\lambda,k}}\tau\left(\sum_{i=1}^{k}[x_i]\right)=c'_{\lambda,k}\tau^{(k)}\left(\sum_{i=1}^{k}[x_i]\right),$$

where $c'_{\lambda,k}$ *is the same as in Theorem 2.*

Proof The theorem can be proved in a similar manner to Theorem 5 using Theorem 2. □

Corollary 3 *Under the same assumption as in Theorem 6 we have, for* $1 \leq k \leq l$,

$$\partial_1^{N_{\lambda,k}} \tau\left(\sum_{i=1}^{k}[x_i]\right) = \frac{c'_{\lambda,k}}{c'_{\lambda,k-1}} \partial_1^{N_{\lambda,k-1}} \tau\left(\sum_{i=1}^{k-1}[x_i]\right) x_k^{\lambda_k} + O\left(x_k^{\lambda_k+1}\right).$$

In order to state the properties for $\tau(t)$ corresponding to Theorem 4 let us introduce one more function $\tau_2(t)$ associated with $\tau(t)$ by

$$\tau_2(t) = s_{(\lambda_1,1^{l-1})}(t) + \sum_{\mu} \xi_\mu s_{(\mu_1,1^{l'-1})}(t), \tag{40}$$

where the sum in the right hand side is over all partitions $\mu = (\mu_1, \ldots, \mu_{l'})$ of length $l' \geq l$ satisfying $\lambda < \mu$, $\mu_i = \lambda_i$ for $2 \leq i \leq l$ and $\mu_i = 1$ for $i > l$.

Theorem 7 *Let* $\lambda = (\lambda_1, \ldots, \lambda_l)$ *be a partition of length* l *and* $\tau(t)$ *a function of the form* (37).

(i) *If* $n < N'_{\lambda,1}$

$$\partial_1^n \tau\left([x_1] - [x_2]\right) = 0.$$

(ii) *We have*

$$\partial_1^{N'_{\lambda,1}} \tau\left([x_1] - [x_2]\right) = c_\lambda \tau_2\left([x_1] - [x_2]\right),$$

where c_λ *is the same as in Theorem* 4.

(iii) *We have*

$$\tau_2\left([x_1] - [x_2]\right) = (-1)^{\lambda_1-1} x_1^{\lambda_1-1} x_2^{l-1} (x_1 - x_2)(1 + \cdots),$$

where $\cdots$ *part is a series in* x_1, x_2 *containing only terms proportional to* $x_1^i x_2^j$ *with* $i + j > 0$.

(iv) *We have the expansion*

$$\tau_2\left([x_1] - [x_2]\right) = (-1)^{l-1} \tau^{(1)}\left([x_1]\right) x_2^{l-1} + O\left(x_2^l\right).$$

Proof (i) By (i) of Theorem 4 we have $\partial_1^n s_\lambda([x_1] - [x_2]) = 0$.

Suppose that $\lambda < \mu$ and $\mu = (\mu_1, \ldots, \mu_{l'})$ is of length l'. Then, $l' \geq l$ and

$$n < \sum_{i=2}^{l} \lambda_i - (l-1) \leq \sum_{i=2}^{l} \mu_i + \sum_{i=l+1}^{l'} (\mu_i - 1) - (l-1) = \sum_{i=2}^{l'} \mu_i - \left(l' - 1\right).$$

Thus $\partial_1^n s_\mu([x_1] - [x_2]) = 0$ by Theorem 4(i) and the assertion (i) is proved.

(ii) By (ii) of Theorem 4 we have

$$\partial_1^{N'_{\lambda,1}} \tau\big([x_1]-[x_2]\big) = c_\lambda s_{(\lambda_1,1^{l-1})}\big([x_1]-[x_2]\big) + \sum_\mu \xi_\mu \partial_1^{N'_{\lambda,1}} s_\mu\big([x_1]-[x_2]\big). \quad (41)$$

Let us compute the second term in the right hand side of (41).

Suppose that $\mu > \lambda$, $\mu = (\mu_1, \dots, \mu_{l'})$ is of length l' and $\partial_1^{N'_{\lambda,1}} s_\mu([x_1]-[x_2]) \neq 0$. In such a case, similarly to the proof of Theorem 4(ii), it can be shown that μ should be of the form $\mu = (\mu_1, \lambda_2, \dots, \lambda_l, 1^{l'-l})$. Then

$$\partial_1^{N'_{\lambda,1}} s_\mu\big([x_1]-[x_2]\big) = c_\lambda s_{(\mu_1,1^{l'-1})}\big([x_1]-[x_2]\big),$$

by (iii) of Theorem 4. Thus the right hand side of (41) becomes $c_\lambda \tau_2([x_1]-[x_2])$.

(iii) This is a direct consequence of Theorem 4(iv).

(iv) Substituting $t = [x_1]-[x_2]$ in $\tau_2(t)$ we have, by (iv) of Theorem 4,

$$\begin{aligned} &\tau_2\big([x_1]-[x_2]\big) \\ &\quad = (-1)^{l-1} x_1^{\lambda_1-1} x_2^{l-1}(x_1-x_2) + \sum \xi_\mu (-1)^{l'-1} x_1^{\mu_1-1} x_2^{l'-1}(x_1-x_2) \\ &\quad = (-1)^{l-1}\Big(x_1^{\lambda_1} + \sum_\mu \xi_\mu x_1^{\mu_1}\Big) x_2^{l-1} + O\big(x_2^l\big), \end{aligned} \quad (42)$$

where the sum in μ in the right hand side is over all partitions μ of the form $\mu = (\mu_1, \lambda_2, \dots, \lambda_l)$ with $\mu_1 > \lambda_1$. Then the term in the bracket in the right hand side of (42) is $\tau^{(1)}([x_1])$. Thus (iv) is proved. □

4 σ-Function

In this section we deduce properties of sigma functions from those of tau functions established in the previous section. To this end we briefly recall the definitions and properties of sigma functions.

Let (n, s) be a pair of relatively prime integers satisfying $2 \le n < s$ and X the compact Riemann surface corresponding to the algebraic curve defined by

$$f(x,y) = 0, \qquad f(x,y) = y^n - x^s - \sum_{ni+sj<ns} \lambda_{ij} x^i y^j. \quad (43)$$

We assume that the affine curve (43) is nonsingular. Then the genus of X is $g = 1/2(n-1)(s-1)$. The Riemann surface X is called an (n, s) curve [2]. It has a point ∞ over the point $x = \infty$.

For a meromorphic function F on X we denote by $\mathrm{ord}_\infty F$ the order of a pole at ∞. The variables x and y can be considered as meromorphic functions on X which satisfy

$$\mathrm{ord}_\infty x = n, \qquad \mathrm{ord}_\infty y = s.$$

Let φ_i, $i \geq 1$, be monomials of x and y satisfying the conditions

$$\{\varphi_i | i \geq 1\} = \{x^i y^j | i \geq 0, n > j \geq 0\},$$
$$\mathrm{ord}_\infty \varphi_i < \mathrm{ord}_\infty \varphi_{i+1}, \quad i \geq 1. \tag{44}$$

For example $\varphi_1 = 1$, $\varphi_2 = x$.

The gap sequence $w_1 < \cdots < w_g$ at ∞ of X is defined by

$$\{w_i\} = \mathbb{Z}_{\geq 0} \setminus \{\mathrm{ord}_\infty \varphi_i | i \geq 1\}.$$

It becomes a gap sequence of type (n, s) defined in of Example 1 in Sect. 2.

A basis of holomorphic one forms on X is given by

$$du_{w_i} := -\frac{\varphi_{g+1-i} dx}{f_y}, \quad 1 \leq i \leq g. \tag{45}$$

Let z be the local coordinate at ∞ such that

$$x = \frac{1}{z^n}, \qquad y = \frac{1}{z^s}\big(1 + O(z)\big). \tag{46}$$

Then we have

$$du_{w_i} = z^{w_i - 1}\big(1 + O(z)\big)dz. \tag{47}$$

We fix an algebraic fundamental form $\widehat{\omega}(p_1, p_2)$ on X [11] and decompose it as

$$\widehat{\omega}(p_1, p_2) = d_{p_2}\Omega(p_1, p_2) + \sum_{i=1}^{g} du_{w_i}(p_1) dr_i(p_2),$$

where

$$\Omega(p_1, p_2) = \frac{\sum_{i=0}^{n-1} y_1^i \left[\frac{f(z,w)}{w^{i+1}}\right]_+ |_{(z,w)=(x_2,y_2)}}{(x_1 - x_2) f_y(x_1, y_1)} dx_1,$$
$$\left[\sum_{m=-\infty}^{\infty} a_m w^m\right]_+ = \sum_{m=0}^{\infty} a_m w^m.$$

Then dr_i automatically becomes a differential of the second kind whose only singularity is ∞ and $\{du_{w_i}, dr_i\}$ is a symplectic basis of $H^1(X, \mathbb{C})$ [11].

We take a symplectic basis of the homology group $H_1(X, \mathbb{Z})$ and define period matrices ω_i, η_i, $i = 1, 2$ by

$$2\omega_1 = \left(\int_{\alpha_j} du_{w_i}\right), \qquad 2\omega_2 = \left(\int_{\beta_j} du_{w_i}\right),$$
$$-2\eta_1 = \left(\int_{\alpha_j} dr_i\right), \qquad -2\eta_2 = \left(\int_{\beta_j} dr_i\right).$$

The normalized period matrix τ is given by $\tau = \omega_1^{-1}\omega_2$.

Let $\tau\,{}^t\delta' + {}^t\delta''$, $\delta', \delta'' \in \mathbb{R}^g$ be the Riemann's constant with respect to the choice $(\{\alpha_i, \beta_i\}, \infty)$, $\delta = {}^t(\delta', \delta'')$ and $\theta[\delta](z, \tau)$ the Riemann's theta function with the characteristic δ. The sigma function for these data is defined in [1] (see also [11]).

Definition 3 The sigma function $\sigma(u)$, $u = {}^t(u_{w_1}, \ldots, u_{w_g})$ of an (n, s) curve X is defined by

$$\sigma(u) = C \exp\left(\frac{1}{2}{}^t u \eta_1 \omega_1^{-1} u\right)\theta[\delta]\big((2\omega_1)^{-1}u, \tau\big),$$

where C is a certain constant.

Let $\lambda = (\lambda_1, \ldots, \lambda_g)$ be the partition corresponding to the gap sequence at ∞ of X. Then the constant C is specified by the condition that the expansion of $\sigma(u)$ at the origin is of the form

$$\sigma(u) = s_\lambda(t)|_{t_{w_i} = u_{w_i}} + \cdots,$$

where $\cdots$ part is a series in u_{w_i} only containing terms proportional to $\prod u_{w_i}^{\alpha_i}$ with $\sum \alpha_i w_i > |\lambda|$.

For $m_i \in \mathbb{Z}^g$, $i = 1, 2$, the sigma function obeys the following transformation rule:

$$\sigma\left(u + 2\sum_{i=1}^{2}\omega_i m_i\right) = (-1)^{{}^t m_1 m_2 + 2(\delta' m_1 - \delta'' m_2)} \exp\left(2\sum_{i=1}^{2}{}^t(\eta_i m_i)\left(u + \sum_{i=1}^{2}\omega_i m_i\right)\right)\sigma(u). \quad (48)$$

Let A be the affine ring of $X\backslash\{\infty\}$. As a vector space $\{\varphi_i | i \geq 1\}$ is a basis of A. We embed A into UGM using the local coordinate z as in [12]. Then the tau function $\tau(t)$ of the KP-hierarchy corresponding to this point of UGM has the form

$$\tau(t) = s_\lambda(t) + \sum_{\lambda < \mu} \xi_\mu s_\mu(t).$$

It can be expressed in terms of the sigma function as

$$\tau(t) = \exp\left(-\sum_{i=1}^{\infty} c_i t_i + \frac{1}{2}\widehat{q}(t)\right)\sigma(Bt), \quad (49)$$

where $\widehat{q}(t) = \sum \widehat{q}_{ij} t_i t_j$, $B = (b_{ij})_{1 \leq i \leq g, 1 \leq j}$ a certain $g \times \infty$ matrix satisfying the condition

$$b_{ij} = \begin{cases} 0 & \text{if } j < w_i, \\ 1 & \text{if } j = w_i, \end{cases} \quad (50)$$

and $c_i, \widehat{q}_{ij}, b_{ij}$ are certain constants [12],[1] [4, 5]. The constant c_i are irrelevant to c_k in Theorem 1 and is not used in other parts of this paper.

In this section ∂_i is used for $\partial/\partial t_i$ as in the previous section and ∂_{u_i} is used for $\partial/\partial u_i$.

A point $p \in X$ is identified with its Abel-Jacobi image $\int_\infty^p d\mathbf{u}$, where $d\mathbf{u} = {}^t(du_{w_1}, \dots, du_{w_g})$. By the definition of the matrix B, for $p \in X$, the following equation is valid:

$$B\big[z(p)\big] = p, \tag{51}$$

where $z(p)$ is the value of the local coordinate z at p and

$$\big[z(p)\big] = {}^t\big[z(p), z(p)^2/2, \dots\big]$$

as before.

Corresponding to Proposition 4 we have

Theorem 8 *Let* $0 \le k \le g-1$. *If* $\sum_{i=1}^g \alpha_i w_i < N_{\lambda,k}$ *then*

$$\partial_{u_{w_1}}^{\alpha_1} \cdots \partial_{u_{w_g}}^{\alpha_g} \sigma\left(\sum_{i=1}^k p_i\right) = 0,$$

for $p_1, \dots, p_k \in X$.

Remark 2 In the case of the curve $y^n = f(x)$ Theorem 8 is proved in [9].

Lemma 4 *Let* $0 \le k \le g-1$. *If* $\operatorname{wt}\alpha < N_{\lambda,k}$

$$\partial^\alpha \sigma(Bt)|_{t=t^{(k)}} = 0,$$

where $t^{(k)} = \sum_{i=1}^k [z_i]$, $z_i = z(p_i)$ *and* $p_1, \dots, p_k \in X$.

Proof The assertion easily follows from (49) and Proposition 4. □

Proof of Theorem 8 We introduce the lexicographical order on $\mathbb{Z}_{\ge 0}^g$ comparing from the right. Namely define $(\alpha_1, \dots, \alpha_g) < (\beta_1, \dots, \beta_g)$ if there exists $1 \le i \le g$ such that $\alpha_g = \beta_g, \dots, \alpha_{i+1} = \beta_{i+1}$ and $\alpha_i < \beta_i$.

We prove

$$\partial_{u_{w_1}}^{\beta_1} \cdots \partial_{u_{w_g}}^{\beta_g} \sigma\big(Bt^{(k)}\big) = 0, \qquad \sum_{i=1}^g \beta_i w_i < N_{\lambda,k}, \tag{52}$$

by induction on the order of $(\beta_1, \dots, \beta_g)$.

[1] In the defining equation of c_i in [12] c_i should be corrected to c_i/i.

The case $(\beta_1, \dots, \beta_g) = (0, \dots, 0)$ is obvious by Lemma 4.

Take $(\beta_1, \dots, \beta_g) > (0, \dots, 0)$ such that $\sum_{i=1}^{g} \beta_i w_i < N_{\lambda,k}$. Suppose that (52) is valid for any $(\beta'_1, \dots, \beta'_g)$ satisfying $(\beta'_1, \dots, \beta'_g) < (\beta_1, \dots, \beta_g)$ and $\sum_{i=1}^{g} \beta'_i w_i < N_{\lambda,k}$.

Notice that $\sigma(Bt)$ is a composition of $\sigma(u)$ with

$$u_{w_i} = t_{w_i} + \sum_{w_i < j} b_{ij} t_j, \quad 1 \le i \le g. \tag{53}$$

By the chain rule,

$$\partial_{w_i} = \partial_{u_{w_i}} + \sum_{l<i} b_{l w_i} \partial_{u_{w_l}}. \tag{54}$$

Let j be the maximum number such that $\beta_j \ne 0$. Then

$$\begin{aligned}
&\partial_{w_1}^{\beta_1} \cdots \partial_{w_g}^{\beta_g} \sigma(Bt) \\
&\quad = \partial_{u_1}^{\beta_1} (\partial_{u_{w_2}} + b_{1 w_2} \partial_{u_1})^{\beta_2} \cdots \left(\partial_{u_{w_j}} + \sum_{l<j} b_{l w_j} \partial_{u_{w_l}} \right)^{\beta_j} \sigma(Bt) \\
&\quad = \partial_{u_1}^{\beta_1} \partial_{u_{w_2}}^{\beta_2} \cdots \partial_{u_{w_j}}^{\beta_j} \sigma(Bt) + \cdots ,
\end{aligned} \tag{55}$$

where $\cdots$ part contains terms of the form

$$\partial_{u_1}^{\gamma_1} \cdots \partial_{u_{w_g}}^{\gamma_g} \sigma(Bt), \quad \sum_{i=1}^{g} \gamma_i w_i < N_{\lambda,k}, \quad (\gamma_1, \dots, \gamma_g) < (\beta_1, \dots, \beta_g).$$

At $t = t^{(k)}$ the left hand side of (55) vanishes by Lemma 4 and $\cdots$ part in the right hand side of (55) vanishes by the assumption of induction. Thus (52) is proved. □

Corresponding to Theorem 5 and Corollary 2 we have

Corollary 4 *Let $1 \le k \le g$ and $p_1, \dots, p_k \in X$. Then*

(i) *We have*

$$\partial_{u_{a_1^{(k)}}} \cdots \partial_{u_{a_{m_k}^{(k)}}} \sigma\left(\sum_{i=1}^{k} p_i \right) = c_k S_{(\lambda_1, \dots, \lambda_k)}(z_1, \dots, z_k) + \cdots ,$$

where $\cdots$ part is a series in z_i containing only terms proportional to $\prod_{i=1}^{k} z_i^{\alpha_i}$ with $\sum_{i=1}^{k} \alpha_i > \sum_{i=1}^{k} \lambda_i$.

(ii) *The following expansion is valid*:

$$\partial_{u_{a_1^{(k)}}} \cdots \partial_{u_{a_{m_k}^{(k)}}} \sigma\left(\sum_{i=1}^{k} p_i\right) = \frac{c_k}{c_{k-1}} \partial_{u_{a_1^{(k-1)}}} \cdots \partial_{u_{a_{m_{k-1}}^{(k-1)}}} \sigma\left(\sum_{i=1}^{k-1} p_i\right) z_k^{\lambda_k} + O\big(z_k^{\lambda_k+1}\big).$$

Proof By Theorem 8 and (54) we have

$$\partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} \sigma(Bt)|_{t=t^{(k)}} = \partial_{u_{a_1^{(k)}}} \cdots \partial_{u_{a_{m_k}^{(k)}}} \sigma\left(\sum_{i=1}^{k} p_i\right). \tag{56}$$

Let us write (49) as $\sigma(Bt) = \varepsilon(t)\tau(t)$ with

$$\varepsilon(t) = \exp\left(\sum_{i=1}^{\infty} c_i t_i - \frac{1}{2}\widehat{q}(t)\right).$$

By Proposition 4 and Corollary 2 we have

$$\begin{aligned}
&\partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} \sigma(Bt)|_{t=t^{(k)}} \\
&\quad = \varepsilon\big(t^{(k)}\big) \partial_{a_1^{(k)}} \cdots \partial_{a_{m_k}^{(k)}} \tau\big(t^{(k)}\big) \\
&\quad = c_{k-1}^{-1} c_k \varepsilon\big(t^{(k)}\big) \partial_{a_1^{(k-1)}} \cdots \partial_{a_{m_{k-1}}^{(k-1)}} \tau\big(t^{(k-1)}\big) z_k^{\lambda_k} + O\big(z_k^{\lambda_k+1}\big) \\
&\quad = c_{k-1}^{-1} c_k \partial_{a_1^{(k-1)}} \cdots \partial_{a_{m_{k-1}}^{(k-1)}} \sigma(Bt)|_{t=t^{(k-1)}} z_k^{\lambda_k} + O\big(z_k^{\lambda_k+1}\big).
\end{aligned} \tag{57}$$

Then the assertion (ii) follows from (56) and the assertion (i) follows from the second line of (57), Theorem 5(i) and the definition of $\tau^{(k)}(t)$. □

The following corollary can similarly be proved using Theorem 6 and Corollary 3.

Corollary 5 *Let* $1 \le k \le g$ *and* $p_1, \ldots, p_k \in X$. *Then*

(i) *We have*

$$\partial_{u_1}^{N_{\lambda,k}} \sigma\left(\sum_{i=1}^{k} p_i\right) = c'_{\lambda,k} S_{(\lambda_1,\ldots,\lambda_k)}(z_1, \ldots, z_k) + \cdots,$$

where $\cdots$ *part is a series in* z_i *containing only terms proportional to* $\prod_{i=1}^{k} z_i^{\alpha_i}$ *with* $\sum_{i=1}^{k} \alpha_i > \sum_{i=1}^{k} \lambda_i$.

(ii) *The following expansion holds*:

$$\partial_{u_1}^{N_{\lambda,k}}\sigma\left(\sum_{i=1}^{k} p_i\right) = \frac{c'_{\lambda,k}}{c'_{\lambda,k-1}}\partial_{u_1}^{N_{\lambda,k-1}}\sigma\left(\sum_{i=1}^{k-1} p_i\right) z_k^{\lambda_k} + O\big(z_k^{\lambda_k+1}\big).$$

Corresponding to Theorem 7 we have

Theorem 9 (i) *If $n < N'_{\lambda,1}$ we have, for $p_1, p_2 \in X$,*

$$\partial_{u_1}^{n}\sigma(p_1 - p_2) = 0.$$

(ii) *The following expansion with respect to $z_i = z(p_i)$, $i = 1, 2$ is valid*:

$$\partial_{u_1}^{N'_{\lambda,1}}\sigma(p_1 - p_2) = (-1)^{g-1}c_\lambda (z_1 z_2)^{g-1}(z_1 - z_2)(1 + \cdots),$$

where $\cdots$ part is a series in z_1, z_2 which contains only terms proportional to $z_1^i z_2^j$ with $i + j > 0$.

(iii) *We have*

$$\partial_{u_1}^{N'_{\lambda,1}}\sigma(p_1 - p_2) = (-1)^{g-1}\frac{c_\lambda}{c'_{\lambda,1}}\partial_{u_1}^{N_{\lambda,1}}\sigma(p_1) z_2^{g-1} + O\big(z_2^{g}\big).$$

Proof (i) Notice that

$$\partial_1^{m}\sigma(Bt) = \partial_{u_1}^{m}\sigma(Bt) \tag{58}$$

for any m. Differentiating $\sigma(Bt) = \varepsilon(t)\tau(t)$ and using (58) and (i) of Theorem 7 we have the assertion.

(ii) We have, by (i), (ii), (iii) of Theorem 7,

$$\begin{aligned}\partial_{u_1}^{N'_{\lambda,1}}\sigma(p_1 - p_2) &= \partial_1^{N'_{\lambda,1}}\sigma(Bt)|_{t=[z_1]-[z_2]} \\ &= \varepsilon\big([z_1] - [z_2]\big)\partial_1^{N'_{\lambda,1}}\tau\big([z_1] - [z_2]\big) \\ &= c_\lambda \varepsilon\big([z_1] - [z_2]\big)\tau_2\big([z_1] - [z_2]\big) \\ &= c_\lambda(-1)^{g-1}(z_1 z_2)^{g-1}(z_1 - z_2)(1 + \cdots).\end{aligned}$$

(iii) In the computation in (ii) we have

$$\begin{aligned}&c_\lambda \varepsilon\big([z_1] - [z_2]\big)\tau_2\big([z_1] - [z_2]\big) \\ &\quad = (-1)^{g-1}c_\lambda\big(c'_{\lambda,1}\big)^{-1}\varepsilon\big([z_1] - [z_2]\big)\partial_1^{N_{\lambda,1}}\tau\big([z_1]\big) z_2^{g-1} + O\big(z_2^{g}\big) \\ &\quad = (-1)^{g-1}c_\lambda\big(c'_{\lambda,1}\big)^{-1}\partial_1^{N_{\lambda,1}}\sigma(Bt)|_{t=[z_1]} z_2^{g-1} + O\big(z_2^{g}\big),\end{aligned}$$

by Theorem 7(i), (ii), (iv) and Theorem 6. Then the assertion follows from (58). □

5 Addition Formulae

Let $E(p_1, p_2)$ be the prime form [6, 10] of an (n,s) curve X. In [11] we have introduced the prime function $\tilde{E}(p_1, p_2)$ by

$$\tilde{E}(p_1,p_2) = -E(p_1,p_2)\prod_{i=1}^{2}\sqrt{du_{w_g}(p_i)}\exp\left(\frac{1}{2}\int_{p_1}^{p_2}{}^t d\mathbf{u}\eta_1\omega_1^{-1}\int_{p_1}^{p_2}d\mathbf{u}\right). \quad (59)$$

Notice that $\tilde{E}(p_1, p_2)$ is not a $(-1/2, -1/2)$ form but a multi-valued analytic function on X and thus it has a sense to talk about the transformation rule if p_i goes around a cycle of X.

The prime function has the following properties.

(i) $\tilde{E}(p_2, p_1) = -\tilde{E}(p_1, p_2)$.
(ii) As a function of p_1, the zero divisor of $\tilde{E}(p_1, p_2)$ is $p_2 + (g-1)\infty$.
(iii) Let $m_i = {}^t(m_{i1}, \ldots, m_{ig}) \in \mathbb{Z}^g$. If p_2 goes round the cycle $\gamma = \sum_{i=1}^{g}(m_{1i}\alpha_i + m_{2i}\beta_i)$, $\tilde{E}(p_1, p_2)$ transforms as

$$\tilde{E}(p_1,\gamma p_2) = T\left(m_1, m_2\middle|\int_{p_1}^{p_2}d\mathbf{u}\right)\tilde{E}(p_1,p_2), \quad (60)$$

with

$$T(m_1,m_2|u) = (-1)^{{}^t m_1 m_2 + 2(\delta' m_1 - \delta'' m_2)}\exp\left(2\sum_{i=1}^{2}{}^t(\eta_i m_i)\left(u + \sum_{i=1}^{2}\omega_i m_i\right)\right).$$

(iv) At (∞, ∞), $\tilde{E}(p_1, p_2)$ has the expansion of the form

$$\tilde{E}(p_1,p_2) = (z_1 z_2)^{g-1}(z_1 - z_2)\left(1 + \sum_{i+j\geq 1} c_{ij} z_1^i z_2^j\right), \quad (61)$$

where $z_i = z(p_i)$.

The specialization $\tilde{E}(\infty, p)$ of $\tilde{E}(p_1, p_2)$ is defined by

$$-\tilde{E}(p_1,p_2) = \tilde{E}(\infty,p_2)z_1^{g-1} + O\left(z_1^g\right). \quad (62)$$

It has the following properties corresponding to (iii) and (iv) above.

(iii)′ Under the same notation as in (iii) for $\tilde{E}(p_1, p_2)$ we have

$$\tilde{E}(\infty,\gamma p_2) = T\left(m_1, m_2\middle|\int_{\infty}^{p_2}d\mathbf{u}\right)\tilde{E}(\infty,p_2). \quad (63)$$

(iv)′ $\tilde{E}(\infty, p_2) = z_2^g + O(z_2^{g+1})$.

The following theorem gives the expression of the prime function in terms of a derivative of the sigma function.

Theorem 10 *Let* $\lambda = (\lambda_1, \ldots, \lambda_g)$ *be the partition corresponding to the gap sequence at* ∞ *of an* (n, s) *curve* X *. Then*

$$\tilde{E}(p_1, p_2) = (-1)^{g-1} c_\lambda^{-1} \partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - p_2).$$

Lemma 5 *Under the same notation as in* (60) *we have*

$$\partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - \gamma p_2) = T\left(m_1, m_2 \middle| \int_{p_1}^{p_2} d\mathbf{u}\right) \partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - p_2).$$

Proof Notice that $\gamma p_2 = p_2 + 2\omega_1 m_1 + 2\omega_2 m_2$ and

$$\sigma\left(u - 2\sum_{i=1}^{2} \omega_i m_i\right) = T(-m_1, -m_2|u)\sigma(u). \tag{64}$$

Applying $\partial_{u_1}^{N'_{\lambda,1}}$ to (64) and setting $u = p_1 - p_2$, we get, by Theorem 9(i) we have

$$\begin{aligned} &\partial_{u_1}^{N'_{\lambda,1}} \sigma\left(p_1 - p_2 - 2\sum_{i=1}^{2} \omega_i m_i\right) \\ &= T\left(-m_1, -m_2 \middle| \int_{p_2}^{p_1} d\mathbf{u}\right) \partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - p_2). \end{aligned} \tag{65}$$

Then the assertion follows from $T(-m_1, -m_2|u) = T(m_1, m_2| - u)$. □

Proof of Theorem 10 Notice that

$$\partial_{u_1}^{N'_{\lambda,1}} \sigma(-u) = -\partial_{u_1}^{N'_{\lambda,1}} \sigma(u), \tag{66}$$

since $\sigma(-u) = (-1)^{|\lambda|}\sigma(u)$ [11] and $N'_{\lambda,1} = |\lambda| - 2g + 1$.

Consider the function

$$F(p_1, p_2) = \frac{\partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - p_2)}{\tilde{E}(p_1, p_2)}. \tag{67}$$

It is symmetric in p_1 and p_2 by (66), (i) of properties of $\tilde{E}(p_1, p_2)$ and is a meromorphic function on $X \times X$ by Lemma 5. Fix p_1 near ∞. As a function of p_2 $F(p_1, p_2)$ has no singularity by Theorem 8, Theorem 9(ii) and the property (ii) of $\tilde{E}(p_1, p_2)$. Therefore it does not depend on p_2. It means that, for some non-empty open neighborhood U of ∞, $F(p_1, p_2)$ does not depend on p_2 on $U \times X$. Since $F(p_1, p_2)$ is symmetric, it is a constant on $U \times U$. Thus it is a constant on $X \times X$ because it is meromorphic. The constant can be determined by comparing the expansion using Theorem 9(ii) and the property (iv) of $\tilde{E}(p_1, p_2)$. □

Corollary 6 *For $p \in X$ we have*

$$\tilde{E}(\infty, p) = {c'_{\lambda,1}}^{-1} \partial_{u_1}^{N_{\lambda,1}} \sigma(p).$$

Proof Compare the expansion in the equation of Theorem 10 using (iii) of Theorem 9. □

Remark 3 In the case of hyperelliptic curves the prime function can be given using the derivative determined from the sequence $a_j^{(2)}$. This is because $p_1 - p_2$ can be written as a sum $p_1 + p_2^*$ where *denoting the hyperelliptic involution. Such expression is given in [7].

The following theorem had been proved in [11].

Theorem 11 ([11]) *For $n \geq g$ and $p_i \in X$, $1 \leq i \leq n$,*

$$\sigma\left(\sum_{i=1}^{n} p_i\right) = \frac{\prod_{i=1}^{n} \tilde{E}(\infty, p_i)^n}{\prod_{i<j}^{n} \tilde{E}(p_i, p_j)} \det\bigl(\varphi_i(p_j)\bigr)_{1 \leq i,j \leq n}.$$

By comparing the top term of the series expansion in $z(p_n)$, using Theorem 2 and Corollary 5, beginning from $n = g$ successively in the equation of this theorem we get

Corollary 7 *For $n < g$ we have*

$$\partial_{u_1}^{N_{\lambda,n}} \sigma\left(\sum_{i=1}^{n} p_i\right) = c'_{\lambda,n} \frac{\prod_{i=1}^{n} \tilde{E}(\infty, p_i)^n}{\prod_{i<j}^{n} \tilde{E}(p_i, p_j)} \det\bigl(\varphi_i(p_j)\bigr)_{1 \leq i,j \leq n}.$$

Combining Theorem 10 and Corollaries 6 and 7 we have the following addition formulae for sigma functions.

Corollary 8 (i) *For $n \geq g$ and $p_i \in X$, $1 \leq i \leq n$,*

$$\frac{\sigma(\sum_{i=1}^{n} p_i) \prod_{i<j} \partial_{u_1}^{N'_{\lambda,1}} \sigma(p_j - p_i)}{\prod_{i=1}^{n} (\partial_{u_1}^{N_{\lambda,1}} \sigma(p_i))^n} = b_{\lambda,n} \det\bigl(\varphi_i(p_j)\bigr)_{1 \leq i,j \leq n},$$

with

$$b_{\lambda,n} = (-1)^{\frac{1}{2}gn(n-1)} c_{\lambda}^{\frac{1}{2}n(n-1)} \bigl(c'_{\lambda,1}\bigr)^{-n^2}.$$

(ii) *For* $n < g$

$$\frac{\partial_{u_1}^{N_{\lambda,n}} \sigma(\sum_{i=1}^n p_i) \prod_{i<j} \partial_{u_1}^{N'_{\lambda,1}} \sigma(p_j - p_i)}{\prod_{i=1}^n (\partial_{u_1}^{N_{\lambda,1}} \sigma(p_i))^n} = b_{\lambda,n} \det\big(\varphi_i(p_j)\big)_{1\le i,j\le n},$$

with

$$b_{\lambda,n} = (-1)^{\frac{1}{2}gn(n-1)} c_\lambda^{\frac{1}{2}n(n-1)} \big(c'_{\lambda,1}\big)^{-n^2} c'_{\lambda,n}.$$

Similarly, using Theorem 11, Theorem 1 and Corollary 4, we have

Corollary 9 *For* $n < g$ *and* $p_i \in X$, $1 \le i \le n$, *we have*

$$\partial_{u_{a_1^{(n)}}} \cdots \partial_{u_{a_{m_n}^{(n)}}} \sigma\left(\sum_{i=1}^n p_i\right) = c_n \frac{\prod_{i=1}^n \tilde{E}(\infty, p_i)^n}{\prod_{i<j}^n \tilde{E}(p_i, p_j)} \det\big(\varphi_i(p_j)\big)_{1\le i,j\le n}.$$

Corollary 10 *For* $n < g$ *and* $p_i \in X$, $1 \le i \le n$, *we have*

$$\frac{\partial_{u_{a_1^{(n)}}} \cdots \partial_{u_{a_{m_n}^{(n)}}} \sigma(\sum_{i=1}^n p_i) \prod_{i<j} \partial_{u_1}^{N'_{\lambda,1}} \sigma(p_j - p_i)}{\prod_{i=1}^n (\partial_{u_1}^{N_{\lambda,1}} \sigma(p_i))^n} = b'_{\lambda,n} \det\big(\varphi_i(p_j)\big)_{1\le i,j\le n},$$

with

$$b'_{\lambda,n} = (-1)^{\frac{1}{2}gn(n-1)} c_\lambda^{\frac{1}{2}n(n-1)} \big(c'_{\lambda,1}\big)^{-n^2} c_n.$$

In the case of hyperelliptic curves $\partial_{u_1}^{N'_{\lambda,1}} \sigma(p_j - p_i)$ can be replaced by a constant multiple of "$a_j^{(2)}$-derivative" as remarked before (Remark after Corollary 6). Then Corollaries 8, 10 recovers the formulae in [13].

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Padé Interpolation for Elliptic Painlevé Equation

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Abstract An interpolation problem related to the elliptic Painlevé equation is formulated and solved. A simple form of the elliptic Painlevé equation and the Lax pair are obtained. Explicit determinant formulae of special solutions are also given.

1 Introduction

There exists a close connection between the Painlevé equations and the Padé approximations (e.g. [6, 19]). An interesting feature of the Padé approach to Painlevé equation is that we can obtain Painlevé equations, its Lax formalism and special solutions simultaneously once we set up a suitable Padé problem. This method is applicable also for discrete cases and it gave a hint for a Lax pair [20] for the elliptic difference Painlevé equation [14].

In this paper, we analyze the elliptic Painlevé equation, its Lax pair and special solutions, by using the Padé approach. In particular, we study the discrete deformation along one special direction.[1] As a result, we obtain a remarkably simple form of the elliptic Painlevé equation (39), (40) and its Lax pair (46), (14) or (15), together with their explicit special solutions given by Eqs. (36), (57) and (70).

This paper is organized as follows. In Sect. 2, we set up the interpolation problem. In Sect. 3, we derive two fundamental contiguity relations satisfied by the interpolating functions. In Sect. 4, we show that the variables f, g appearing in the contiguity

[1]Though all the directions are equivalent due to the Bäcklund transformations, there exists one special direction in the formulation on $\mathbb{P}^1 \times \mathbb{P}^1$ for which the equation takes a simple form like QRT system [11]. Jimbo-Sakai's q-Painlevé six equation [3] is a typical example of such beautiful equations. For various q-difference cases, the Lax formalisms for such direction were studied in [21].

Dedicated to Professor Michio Jimbo on his 60th birthday.

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relations satisfy the elliptic Painlevé equation. Interpretation of the contiguity relations as the Lax pair for elliptic Painlevé equation is given in Sect. 5. In Sect. 6, explicit determinant formulae for the interpolation problem are given. Derivation of the Painlevé equation (39), (40) based on affine Weyl group action is given in Appendix.

2 The Interpolation Problem

In this section, we will set up an interpolation problem which we study in this paper.

Notations Let p, q be two base variables satisfying constraints $|p|, |q| < 1$. We denote by $\vartheta_p(x)$ the Jacobi theta function with base p:

$$\vartheta_p(x) = \prod_{i=0}^{\infty}\left(1 - xp^i\right)\left(1 - x^{-1}p^{i+1}\right), \qquad \vartheta_p(px) = \vartheta_p\left(x^{-1}\right) = -x^{-1}\vartheta_p(x). \quad (1)$$

The elliptic Gamma function [13] and Pochhammer symbol are defined as

$$\begin{aligned} \Gamma(x; p, q) &= \prod_{i,j=0}^{\infty} \frac{(1 - x^{-1}p^{i+1}q^{j+1})}{(1 - xp^i q^j)}, \\ \vartheta_p(x)_s &= \frac{\Gamma(q^s x; p, q)}{\Gamma(x; p, q)} = \prod_{i=0}^{s-1} \vartheta_p\left(q^i x\right), \end{aligned} \quad (2)$$

where the last equality holds for $s \in \mathbb{Z}_{\geq 0}$. We shall use the standard convention

$$\begin{aligned} \Gamma(x_1, \ldots, x_\ell; p, q) &= \Gamma(x_1; p, q) \cdots \Gamma(x_\ell; p, q), \\ \vartheta_p(x_1, \ldots, x_\ell)_s &= \vartheta_p(x_1)_s \cdots \vartheta_p(x_\ell)_s. \end{aligned} \quad (3)$$

Padé Problem Let $m, n \in \mathbb{Z}_{\geq 0}$, and let $a_1, \ldots, a_6, k$ be complex parameters with a constraint:

$$\prod_{i=1}^{6} a_i = k^3. \quad (4)$$

In this paper we consider the following interpolation problem:

$$Y_s = \frac{V(q^{-s})}{U(q^{-s})}, \qquad (s = 0, 1, \ldots, N = m + n), \quad (5)$$

specified by the following data:

- The interpolated sequence Y_s is given by

$$Y_s = Y\big(q^{-s}\big) = \prod_{i=1}^{6} \frac{\vartheta_p(a_i)_s}{\vartheta_p(\frac{k}{a_i})_s}, \qquad Y(x) = \prod_{i=1}^{6} \frac{\Gamma(\frac{a_i}{x}, \frac{k}{a_i}; p, q)}{\Gamma(\frac{k}{a_i x}, a_i; p, q)}. \tag{6}$$

- The interpolating functions $U(x)$, $V(x)$ are defined as

$$U(x) = \sum_{i=0}^{n} u_i \phi_i(x), \qquad V(x) = \sum_{i=0}^{m} v_i \chi_i(x), \tag{7}$$

with basis

$$\begin{aligned}
\phi_i(x) &= \frac{T_{a_2}^{-i} T_{a_4}^{i} Y(x)}{Y(x)} = \frac{\vartheta_p(\frac{a_4}{x}, \frac{k}{q^i a_4 x})_i}{\vartheta_p(\frac{a_2}{q^i x}, \frac{k}{a_2 x})_i} \frac{\vartheta_p(\frac{a_2}{q^i}, \frac{k}{a_2})_i}{\vartheta_p(a_4, \frac{k}{q^i a_4})_i}, \\
\chi_i(x) &= \frac{Y(x)}{T_{a_1}^{i} T_{a_3}^{-i} Y(x)} = \frac{\vartheta_p(\frac{a_3}{q^i x}, \frac{k}{a_3 x})_i}{\vartheta_p(\frac{a_1}{x}, \frac{k}{q^i a_1 x})_i} \frac{\vartheta_p(a_1, \frac{k}{q^i a_1})_i}{\vartheta_p(\frac{a_3}{q^i}, \frac{k}{a_3})_i},
\end{aligned} \tag{8}$$

where $T_a : f(a) \mapsto f(qa)$.

The coefficients u_i, v_i are determined by Eq. (5) which is a system of linear homogeneous equations. We normalize them as $u_0 = 1$.

Remark on the Choice of the Bases $\phi_i(x)$, $\chi_i(x)$ The problem we are considering is a version of PPZ scheme (interpolation with prescribed poles and zeros) [22]. Note that

$$\begin{aligned}
U(x) &= \frac{U_{\mathrm{num}}(x)}{U_{\mathrm{den}}(x)}, \quad U_{\mathrm{den}}(x) = \vartheta_p\left(\frac{a_2}{q^n x}, \frac{k}{a_2 x}\right)_n, \\
V(x) &= \frac{V_{\mathrm{num}}(x)}{V_{\mathrm{den}}(x)}, \quad V_{\mathrm{den}}(x) = \vartheta_p\left(\frac{a_1}{x}, \frac{k}{q^m a_1 x}\right)_m,
\end{aligned} \tag{9}$$

where $U_{\mathrm{num}}(x)$, $U_{\mathrm{den}}(x)$ (resp. $V_{\mathrm{num}}(x)$, $V_{\mathrm{den}}(x)$) are theta functions of order $2n$ (resp. $2m$). Furthermore, the functions $x^m U_{\mathrm{num}}(x)$, $x^n V_{\mathrm{num}}(x)$, $x^m U_{\mathrm{den}}(x)$, $x^n V_{\mathrm{den}}(x)$ (and hence $U(x)$, $V(x)$, $\phi_i(x)$, $\chi_i(x)$ also) are "symmetric": $F(k/qx) = F(x)$. We will fix the denominator U_{den} (resp. V_{den}) as above in order to specify the prescribed zeros (resp. poles). For the numerator U_{num} (resp. V_{num}), contrarily, one may take any basis of theta functions as far as they have the same order, same quasi p-periodicity, and same symmetry under $x \leftrightarrow \frac{k}{qx}$ as U_{den} (resp. V_{den}). In this sense, the choice of the basis ϕ_i, χ_i in Eq. (8) is not so essential for general argument, however, we will see that it is convenient for explicit expression of the functions $U(x)$, $V(x)$ in Sect. 6.

Parameters of the Elliptic Painlevé Equation The elliptic Painlevé equation is specified by a generic configuration of 8 points on $\mathbb{P}^1 \times \mathbb{P}^1$. We parametrize them as $(f_*(\xi_i), g_*(\xi_i))_{i=1,\dots,8}$, where

$$f_*(x) = \frac{\vartheta_p(\frac{c_2}{x}, \frac{\kappa_1}{c_2 x})}{\vartheta_p(\frac{c_1}{x}, \frac{\kappa_1}{c_1 x})}, \qquad g_*(x) = \frac{\vartheta_p(\frac{c_4}{x}, \frac{\kappa_2}{c_4 x})}{\vartheta_p(\frac{c_3}{x}, \frac{\kappa_2}{c_3 x})}, \tag{10}$$

and c_i are parameters independent of x. The functions $f_*(x), g_*(x)$ satisfy $f_*(x) = f_*(\frac{\kappa_1}{x})$, $g_*(x) = g_*(\frac{\kappa_2}{x})$, and they give a parametrization of an elliptic curve of degree (2, 2).[2] We define functions $F_f(x)$ and $G_g(x)$ as

$$\begin{aligned} F_f(x) &= \vartheta_p\left(\frac{c_1}{x}, \frac{\kappa_1}{c_1 x}\right) f - \vartheta_p\left(\frac{c_2}{x}, \frac{\kappa_1}{c_2 x}\right), \\ G_g(x) &= \vartheta_p\left(\frac{c_3}{x}, \frac{\kappa_2}{c_3 x}\right) g - \vartheta_p\left(\frac{c_4}{x}, \frac{\kappa_2}{c_4 x}\right). \end{aligned} \tag{11}$$

Note that $F_f(x) = 0 \Leftrightarrow f = f_*(x)$ and $G_g(x) = 0 \Leftrightarrow g = g_*(x)$.

In this paper, the Painlevé equation appears with the following parameters

$$(\kappa_1, \kappa_2) = \left(k, \frac{k^2}{a_1}\right), \qquad (\xi_1, \dots, \xi_8) = \left(\frac{k}{q}, kq^{m+n}, \frac{k}{a_1 q^m}, \frac{a_2}{q^n}, a_3, a_4, a_5, a_6\right). \tag{12}$$

Note that $\kappa_1^2 \kappa_2^2 = q \xi_1 \cdots \xi_8$ due to the constraint (4).

3 Contiguity Relations

Here, we will derive two fundamental contiguity relations[3] satisfied by the functions $V(x), Y(x)U(x)$.

Special Direction *T* of Deformation For any quantity (or function) F depending on variables $k, a_1, \dots, a_6, m, n, \dots$, we denote by $\overline{F} = T(F)$ its parameter shift along a special direction T:

$$T : (k, a_1, \dots, a_6, m, n) \mapsto \left(kq, \frac{a_1}{q}, a_2, a_3 q, \dots, a_6 q, m+1, n-1\right). \tag{13}$$

This special direction is chosen so that $T : (\kappa_1, \kappa_2, \xi_i) \mapsto (\kappa_1 q, \kappa_2 q^3, \xi_i q)$ and the corresponding elliptic Painlevé equation will take a simple form.

[2] The choice of parameters $c_1, \dots, c_4$ (and over all normalization of $f_*(x), g_*(x)$) is related to the fractional linear transformations on $\mathbb{P}^1 \times \mathbb{P}^1$.

[3] Since the contiguity relations (14), (15) are similar to the linear relations of the R_{II} chains [17], it may be possible to derive them as a reduction of three discrete-time non-autonomous Toda chain by using the method in [18].

Proposition 1 *The functions $y(x) = V(x), Y(x)U(x)$ satisfy the following contiguity relations*:

$$L_2: \quad \frac{G_g(\frac{kx}{a_1})\prod_{i=1}^{8}\vartheta_p(\frac{\xi_i}{x})}{\vartheta_p(\frac{k}{a_1x},\frac{k}{qx})}y(x) - \frac{G_g(x)\prod_{i=1}^{8}\vartheta_p(\frac{k}{x\xi_i})}{\vartheta_p(\frac{a_1}{x},\frac{q}{x})}y\left(\frac{x}{q}\right)$$
$$-\frac{C_0F_f(x)\vartheta_p(\frac{k}{x^2},\frac{a_1}{qx},\frac{kq}{a_1x})}{x}\overline{y}(x) = 0, \tag{14}$$

$$L_3: \quad G_g\left(\frac{kqx}{a_1}\right)\vartheta_p\left(\frac{k}{qx},\frac{kq}{a_1x}\right)\overline{y}(x) - G_g(x)\vartheta_p\left(\frac{1}{x},\frac{a_1}{q^2x}\right)\overline{y}(qx)$$
$$-\frac{C_1\overline{F_f(qx)}\vartheta_p(\frac{k}{qx^2})}{x\vartheta_p(\frac{k}{a_1x},\frac{a_1}{qx})}y(x) = 0, \tag{15}$$

where C_0, C_1, f, g are some constants w.r.t. x.

Proof We put $\mathbf{y}(x) = \left[\begin{smallmatrix} V(x) \\ Y(x)U(x)\end{smallmatrix}\right]$ and define the Casorati determinants D_i as

$$\begin{aligned} D_1(x) &:= \det\left[\mathbf{y}(x), \mathbf{y}\left(\frac{x}{q}\right)\right], \\ D_2(x) &:= \det\left[\mathbf{y}(qx), \mathbf{y}(x)\right], \\ D_3(x) &:= \det\left[\overline{\mathbf{y}}(x), \mathbf{y}(x)\right], \\ D_4(x) &:= \det\left[\overline{\mathbf{y}}(x), \mathbf{y}\left(\frac{x}{q}\right)\right]. \end{aligned} \tag{16}$$

Then the desired contiguity relations are obtained from the identity

$$\begin{aligned} &D_1(x)\overline{y}(x) - D_4(x)y(x) + D_3(x)y\left(\frac{x}{q}\right) = 0, \\ &D_4(qx)\overline{y}(x) - D_3(x)\overline{y}(qx) - \overline{D_2(x)}y(x) = 0, \end{aligned} \tag{17}$$

by using the formulae for D_i given in the next Lemma 1. □

Lemma 1 *The determinants* (16) *take the following form*:

$$\begin{aligned} D_1(x) &= \mathcal{N}(x)Y(x)c\frac{\vartheta_p(\frac{k}{x^2},\frac{q}{x},\frac{a_1}{x})F_f(x)}{x\vartheta_p(\frac{k}{qx},\frac{k}{xa_1})\prod_{i=1}^{8}\vartheta_p(\frac{k}{x\xi_i})}, \\ D_2(x) &= \mathcal{N}(x)Y(x)c\frac{\vartheta_p(\frac{k}{q^2x^2},\frac{k}{q^2x},\frac{k}{qxa_1})F_f(qx)}{qx\vartheta_p(\frac{1}{x},\frac{a_1}{qx})\prod_{i=1}^{8}\vartheta_p(\frac{\xi_i}{qx})}, \end{aligned} \tag{18}$$

$$D_3(x) = \mathcal{N}(x)Y(x)c' \frac{G_g(x)}{\vartheta_p(\frac{k}{qx}, \frac{k}{xa_1}, \frac{kq}{xa_1}, \frac{a_1}{qx})},$$

$$D_4(x) = \mathcal{N}(x)Y(x)c' \frac{\vartheta_p(\frac{q}{x}, \frac{a_1}{x})G_g(\frac{kx}{a_1})}{\vartheta_p(\frac{k}{qx}, \frac{k}{qx}, \frac{k}{xa_1}, \frac{k}{xa_1}, \frac{kq}{xa_1}, \frac{a_1}{qx})} \prod_{i=1}^{8} \frac{\vartheta_p(\frac{\xi_i}{x})}{\vartheta_p(\frac{k}{x\xi_i})},$$

where

$$\mathcal{N}(x) = \frac{\vartheta_p(\frac{1}{q^{m+n}x}, \frac{k}{qx})_{m+n+1}}{U_{\rm den}(x)V_{\rm den}(x)}. \tag{19}$$

Proof The functions $U(x)$, $V(x)$ and, due to the constraint (4), the function $Y(x)$ are elliptic (p-periodic) functions in x. Hence the ratios $\frac{D_i(x)}{Y(x)}$ are also elliptic. They are of order $2m + 2n +$ (small corrections) and have sequences of zeros and poles represented as $\vartheta_p(\frac{1}{q^{m+n}x}, \frac{k}{qx})_{m+n+1}$ and $U_{\rm den}V_{\rm den}$ modulo corrections at the boundaries of the sequence. Then we can compute the ratios $\frac{D_i(x)}{Y(x)}$, and each of them are determined up to 2 unknown constants. In the computation, the following relations are useful (they are derived by a straightforward computation)

$$G(x) := \frac{Y(qx)}{Y(x)} = \prod_{i=1}^{6} \frac{\vartheta_p(\frac{k}{a_i qx})}{\vartheta_p(\frac{a_i}{qx})}, \tag{20}$$

$$K(x) := \frac{\overline{Y}(x)}{Y(x)} = \frac{\vartheta_p(\frac{k}{a_1}, \frac{k}{a_2}, \frac{a_1}{q}, \frac{kq}{a_1})}{\vartheta_p(\frac{k}{a_1 x}, \frac{k}{a_2 x}, \frac{a_1}{qx}, \frac{kq}{a_1 x})} \prod_{i=3}^{6} \frac{\vartheta_p(\frac{a_i}{x})}{\vartheta_p(a_i)}, \tag{21}$$

$$\mathcal{N}\left(\frac{k}{qx}\right) = \frac{qx^2}{k}\mathcal{N}(x), \tag{22}$$

and

$$\frac{\mathcal{N}(qx)}{\mathcal{N}(x)} = \frac{\vartheta_p(\frac{q}{x}, \frac{q^N k}{x}, \frac{a_1}{x}, \frac{k}{q^m a_1 x}, \frac{k}{a_2 x}, \frac{a_2}{q^n x})}{\vartheta_p(\frac{1}{q^{N+1}x}, \frac{k}{qx}, \frac{q^m a_1}{x}, \frac{k}{a_1 x}, \frac{q^n k}{a_2 x}, \frac{a_2}{x})}. \tag{23}$$

• *Computation of* $D_1(x)$, $D_2(x)$: First, we count the degree of the elliptic function

$$\frac{D_1(x)}{Y(x)} = \frac{1}{G(\frac{x}{q})} V(x)U\left(\frac{x}{q}\right) - V\left(\frac{x}{q}\right)U(x). \tag{24}$$

Substituting

$$U\left(\frac{x}{q}\right)=\frac{U_{\mathrm{num}}(\frac{x}{q})}{U_{\mathrm{den}}(\frac{x}{q})}=\frac{\vartheta_p(\frac{k}{a_2x},\frac{a_2}{q^nx})}{\vartheta_p(\frac{q^nk}{a_2x},\frac{a_2}{x})}\frac{U_{\mathrm{num}}(\frac{x}{q})}{U_{\mathrm{den}}(x)},$$
$$V\left(\frac{x}{q}\right)=\frac{V_{\mathrm{num}}(\frac{x}{q})}{V_{\mathrm{den}}(\frac{x}{q})}=\frac{\vartheta_p(\frac{k}{q^ma_2x},\frac{a_1}{x})}{\vartheta_p(\frac{k}{a_1x},\frac{q^ma_1}{x})}\frac{V_{\mathrm{num}}(\frac{x}{q})}{V_{\mathrm{den}}(x)}, \tag{25}$$

we have

$$\begin{aligned}\frac{D_1(x)}{Y(x)}=&\frac{1}{U_{\mathrm{den}}(x)V_{\mathrm{den}}(x)}\frac{\vartheta_p(\frac{a_1}{x})}{\vartheta_p(\frac{k}{a_1x})}\\&\times\left\{\frac{\vartheta_p(\frac{a_2}{q^nx})}{\vartheta_p(\frac{q^nk}{a_2x})}\prod_{i=3}^{6}\frac{\vartheta_p(\frac{a_i}{x})}{\vartheta_p(\frac{k}{a_ix})}V_{\mathrm{num}}(x)U_{\mathrm{num}}\left(\frac{x}{q}\right)\right.\\&\left.-\frac{\vartheta_p(\frac{k}{q^ma_1x})}{\vartheta_p(\frac{q^ma_1}{x})}U_{\mathrm{num}}(x)V_{\mathrm{num}}\left(\frac{x}{q}\right)\right\}.\end{aligned} \tag{26}$$

The function $D_1(x)/Y(x)$ is p-periodic function of order $2m+2n+6$ with denominator

$$U_{\mathrm{den}}(x)\left\{V_{\mathrm{den}}(x)\frac{\vartheta_p(\frac{k}{a_1x})}{\vartheta_p(\frac{a_1}{x})}\right\}\vartheta_p\left(\frac{q^ma_1}{x}\right)\vartheta_p\left(\frac{q^nk}{a_2x}\right)\prod_{i=3}^{6}\vartheta_p\left(\frac{k}{a_ix}\right). \tag{27}$$

Next, we study the zeros. When x and $\frac{x}{q}$ are both in the Padé interpolation grid (i.e. for $x=1,q^{-1},\dots,q^{-N+1}$), it follows obviously that $D_1(x)=0$. Noting the symmetry properties

$$U\left(\frac{k}{qx}\right)=U(x),\qquad V\left(\frac{k}{qx}\right)=V(x),\qquad G\left(\frac{k}{qx}\right)=\frac{1}{G(\frac{x}{q})}, \tag{28}$$

we have

$$\frac{D_1(\frac{k}{x})}{Y(\frac{k}{x})}=G\left(\frac{x}{q}\right)U(x)V\left(\frac{x}{q}\right)-U\left(\frac{x}{q}\right)V(x)=-G\left(\frac{x}{q}\right)\frac{D_1(x)}{Y(x)}. \tag{29}$$

Then it follows that $D_1(x)=0$ at $x=k,kq,\dots,kq^{N-1}$ and furthermore, due to the relation $\mathbf{y}(x)=\mathbf{y}(\frac{x}{q})$ for $x^2=k$, we have $D_1(x)=0$ at $x^2=k$ (i.e. $x=\pm\sqrt{k},\pm\sqrt{kp}$). As a result, the function $X(x)$ defined by

$$D_1(x)=\mathcal{N}(x)Y(x)\frac{\vartheta_p(\frac{k}{x^2},\frac{q}{x},\frac{a_1}{x})}{x\vartheta_p(\frac{k}{qx},\frac{k}{xa_1})\prod_{i=1}^{8}\vartheta_p(\frac{k}{x\xi_i})}X(x) \tag{30}$$

is a theta function of degree 2 such that $X(\frac{x}{p}) = X(\frac{k}{x}) = \frac{x^2}{k}X(x)$, hence it can be written as $X(x) = cF_f(x)$ by suitable constants c, f. D_2 is easily obtained since $D_2(x) = D_1(qx)$.

• *Computation of* $D_3(x), D_4(x)$: First we note a relation between $D_3(x)$ and $D_4(x)$. Using $U(\frac{k}{qx}) = U(x)$, $\overline{U}(\frac{k}{x}) = \overline{U}(x)$ and similar relations for $V(x)$ we have

$$\begin{aligned}
\frac{D_3(\frac{k}{qx})}{Y(\frac{k}{qx})} &= U\left(\frac{k}{qx}\right)\overline{V}\left(\frac{k}{qx}\right) - K\left(\frac{k}{qx}\right)\overline{U}\left(\frac{k}{qx}\right)V\left(\frac{k}{qx}\right)\\
&= U(x)\overline{V}(qx) - K\left(\frac{k}{qx}\right)\overline{U}(qx)V(x)\\
&= \frac{G(x)}{Y(qx)}\left\{Y(x)U(x)\overline{V}(qx) - \frac{K(\frac{k}{qx})}{G(x)}Y(qx)\overline{U}(qx)V(x)\right\}\\
&= G(x)\frac{D_4(qx)}{Y(qx)},
\end{aligned} \tag{31}$$

where we have used the relation $\frac{K(\frac{k}{qx})}{G(x)} = K(qx)$ at the last step.

Let us compute $D_3(x)$. Substituting the relation

$$\begin{aligned}
\overline{U}(x) &= \frac{\overline{U_{\text{num}}}(x)}{\overline{U_{\text{den}}}(x)} = \vartheta_p\left(\frac{k}{a_2x}, \frac{a_2}{q^nx}\right)\frac{\overline{U_{\text{num}}}(x)}{U_{\text{den}}(x)},\\
\overline{V}(x) &= \frac{\overline{V_{\text{num}}}(x)}{\overline{V_{\text{den}}}(x)} = \frac{\vartheta_p(\frac{k}{q^ma_1x})}{\vartheta_p(\frac{a_1}{qx}, \frac{k}{a_1x}, \frac{qk}{a_1x})}\frac{\overline{V_{\text{num}}}(x)}{V_{\text{den}}(x)},
\end{aligned} \tag{32}$$

into

$$\frac{D_3(x)}{Y(x)} = U(x)\overline{V}(x) - K(x)\overline{U}(x)V(x), \tag{33}$$

we have

$$\begin{aligned}
\frac{D_3(x)}{Y(x)} = {} & \frac{1}{U_{\text{den}}(x)V_{\text{den}}(x)}\frac{1}{\vartheta_p(\frac{k}{a_1x}, \frac{qk}{a_1x}, \frac{a_1}{qx})}\\
&\times\left\{\vartheta_p\left(\frac{a_2}{q^nx}, \frac{a_3}{x}, \ldots, \frac{a_6}{x}\right)V_{\text{num}}(x)\overline{U_{\text{num}}}(x)\right.\\
&\left. - \vartheta_p\left(\frac{k}{q^ma_1x}\right)\overline{V_{\text{num}}}(x)U_{\text{num}}(x)\right\}.
\end{aligned} \tag{34}$$

Hence, $\frac{D_3(x)}{Y(x)}$ is of degree $2m + 2n + 3$.

$D_3(x)$ has zeros at $x = 1, q^{-1}, \dots, q^{-N}$ and $x = k, qk, \dots, q^{N-1}k$, where the latter zeros follow from those of $D_4(x)$ through Eq. (31). Hence, we obtain

$$D_3(x) = \mathcal{N}(x)Y(x)\frac{1}{\vartheta_p(\frac{k}{qx}, \frac{k}{xa_1}, \frac{kq}{xa_1}, \frac{a_1}{qx})}Z(x), \tag{35}$$

where $Z(x)$ is a theta function of degree 2 such as $Z(\frac{x}{p}) = Z(\frac{k^2}{a_1 x}) = \frac{a_1 x^2}{k^2}Z(x)$, namely $Z(x) = c' G_g(x)$ for some c' and g as desired. $D_4(x)$ is derived by the relation (31). □

Corollary 1 *For any pair $i, j \in \{3, 4, 5, 6\}$ we have*

$$\frac{\alpha(a_i)}{\alpha(a_j)}\frac{F_f(a_i)}{F_f(a_j)} = \frac{U(a_i)V(a_i/q)}{U(a_j)V(a_j/q)}, \qquad \frac{\beta(a_i)}{\beta(a_j)}\frac{G_g(a_i)}{Gg(a_j)} = \frac{U(a_i)\overline{V}(a_i)}{U(a_j)\overline{V}(a_j)}, \tag{36}$$

where

$$\begin{aligned} \alpha(x) &= \mathcal{N}(x)\frac{\vartheta_p(\frac{k}{x^2}, \frac{q}{x}, \frac{a_1}{x})}{x\vartheta_p(\frac{k}{qx}, \frac{k}{xa_1})\prod_{i=1}^{8}\vartheta_p(\frac{k}{x\xi_i})}, \\ \beta(x) &= \mathcal{N}(x)\frac{1}{\vartheta_p(\frac{k}{qx}, \frac{k}{xa_1}, \frac{kq}{xa_1}, \frac{a_1}{qx})}. \end{aligned} \tag{37}$$

Proof By the definition of D_1, D_3, we have for $x = a_i$ $(i = 3, 4, 5, 6)$

$$\begin{aligned} \frac{D_1(x)}{Y(x)} &= \frac{1}{G(x/q)}V(x)U\left(\frac{x}{q}\right) - U(x)V\left(\frac{x}{q}\right) = -U(x)V\left(\frac{x}{q}\right), \\ \frac{D_3(x)}{Y(x)} &= \overline{V}(x)U(x) - K(x)\overline{U}(x)V(x) = U(x)\overline{V}(x). \end{aligned} \tag{38}$$

Then, from the first and the third equation of (18), one has Eq. (36). □

The formulae (36) are convenient in order to obtain f, g from $U(x), V(x)$.

4 Elliptic Painlevé Equation

In this section, we study the Eqs. (14), (15) for generic variables f, g apart from the Padé problem, and prove that the variables f, g satisfy the elliptic Painlevé equation.

Theorem 1 *If the Eqs.* (14), (15) *are compatible, then the variables f, g and $\overline{f}, \overline{g}$ should be related by*

$$\frac{F_f(x)\overline{F_f(qx)}}{F_f(\frac{xa_1}{k})\overline{F_f(\frac{q^2xa_1}{k})}} = \prod_{i=1}^{8}\frac{\vartheta_p(\frac{\xi_i}{x})}{\vartheta_p(\frac{k^2}{x\xi_i a_1})}, \quad \textit{for } g = g_*(x), \tag{39}$$

and

$$\frac{G_g(x)\overline{G_g(qx)}}{G_g(\frac{kqx}{a_1})\overline{G_g(\frac{kqx}{a_1})}} = \prod_{i=1}^{8} \frac{\vartheta_p(\frac{\xi_i}{x})}{\vartheta_p(\frac{k}{qx\xi_i})}, \quad \textit{for } \overline{f} = \overline{f_*}(qx). \tag{40}$$

Proof From equations $\overline{L_2}|_{x\to qx}$ (14) and L_3 (15) we have

$$\begin{aligned} &\frac{\overline{G_g(\frac{kqx}{a_1})}\prod_{i=1}^{8}\vartheta_p(\frac{\xi_i}{x})}{\vartheta_p(\frac{kq}{a_1x},\frac{k}{qx})}\overline{y}(qx) = \frac{\overline{G_g(qx)}\prod_{i=1}^{8}\vartheta_p(\frac{k}{qx\xi_i})}{\vartheta_p(\frac{a_1}{q^2x},\frac{1}{x})}\overline{y}(x),\\ &G_g\left(\frac{kqx}{a_1}\right)\vartheta_p\left(\frac{k}{qx},\frac{kq}{a_1x}\right)\overline{y}(x) = G_g(x)\vartheta_p\left(\frac{1}{x},\frac{a_1}{q^2x}\right)\overline{y}(qx), \end{aligned} \tag{41}$$

for $\overline{f} = \overline{f_*}(qx)$, hence we have Eq. (40).

For $g = g_*(x)$, we have from Eqs. (14), (15) that

$$\begin{aligned} &\frac{G_g(\frac{kx}{a_1})\prod_{i=1}^{8}\vartheta_p(\frac{\xi_i}{x})}{\vartheta_p(\frac{k}{a_1x},\frac{k}{qx})}y(x) = \frac{C_0F_f(x)\vartheta_p(\frac{k}{x^2},\frac{a_1}{qx},\frac{kq}{a_1x})}{x}\overline{y}(x),\\ &G_g\left(\frac{kqx}{a_1}\right)\vartheta_p\left(\frac{k}{qx},\frac{kq}{a_1x}\right)\overline{y}(x) = \frac{C_1\overline{F_f(qx)}\vartheta_p(\frac{k}{qx^2})}{x\vartheta_p(\frac{k}{a_1x},\frac{a_1}{qx})}y(x), \end{aligned} \tag{42}$$

hence

$$G_g\left(\frac{kx}{a_1}\right)G_g\left(\frac{kqx}{a_1}\right)\prod_{i=1}^{8}\vartheta_p\left(\frac{\xi_i}{x}\right) = \frac{w}{x^2}F_f(x)\overline{F_f(qx)}\vartheta_p\left(\frac{k}{x^2},\frac{k}{qx^2}\right), \tag{43}$$

where $w = C_0C_1$. The Eq. (43) holds also by replacing $x \to \frac{k^2}{a_1x}$ since $g_*(x) = g_*(\frac{k^2}{a_1x})$. Taking a ratio Eq. (43) with Eq. $(43)|_{x\to\frac{k^2}{a_1x}}$ we have Eq. (39). □

The next Lemma 2 shows that the relations (39), (40) are equivalent to the time evolution equation for the elliptic Painlevé.[4]

Lemma 2 *The solution* $\overline{f}$ *of Eq.* (39) *is a rational function of* (f, g) *of degree* $(1, 4)$, *which is characterized by the following conditions*: (i) *its numerator and denominator have* 8 *zeros at* $f = f_*(\xi), g = g_*(\xi)$, (ii) *if* $f = f_*(u)$, $g = g_*(u)$ $(u \neq \xi)$ *then* $\overline{f} = \overline{f_*}(\frac{a_1u}{k})$. *Similarly, by Eq.* (40), $\overline{g}$ *is uniquely given as a rational function of* $(\overline{f}, g)$ *of degree* (4, 1), *satisfying the conditions* (i′) *it has* 8 *points of*

[4]Since the elliptic Painlevé equation [14] is rather complicated, its concise expressions have been pursued by several authors (e.g. [8–10]). The system (39), (40) is supposed to be the simplest one.

indeterminacy at $\overline{f} = \overline{f_*}(q\xi)$, $g = g_*(\xi)$, (ii′) *if* $\overline{f} = \overline{f_*}(qu)$, $g = g_*(u)$ $(u \neq \xi)$ *then* $\overline{g} = \overline{g_*}(\frac{q^3ku}{a_1}) = g_*(\frac{qku}{a_1})$.

Proof Written in the form

$$F_f(x)\overline{F_f(qx)}\prod_{i=1}^{8}\vartheta_p\left(\frac{k^2}{x\xi_i a_1}\right) = F_f\left(\frac{xa_1}{k}\right)\overline{F_f\left(\frac{q^2xa_1}{k}\right)}\prod_{i=1}^{8}\vartheta_p\left(\frac{\xi_i}{x}\right), \tag{44}$$

the Eq. (39) is quasi p-periodic in x of degree (apparently) 12 with symmetry under $x \leftrightarrow \frac{k^2}{a_1x}$. Since it is divisible by a factor $\vartheta_p(\frac{k^2}{a_1x^2})$, it is effectively of degree 8. Then the solution $\overline{f}$ of this equation takes the form

$$\overline{f} = \frac{A(x)f + B(x)}{C(x)f + D(x)}, \tag{45}$$

where the coefficients $A(x), \ldots, D(x)$ are $x \leftrightarrow \frac{k^2}{a_1x}$-symmetric p-periodic functions of degree 8, namely polynomials of $g = g_*(x)$ of degree 4. Hence $\overline{f}$ is a rational function of (f, g) of degree $(1, 4)$. The conditions (i), (ii) are obvious by the form of Eq. (39). The structure of the solution $\overline{g} = \overline{g}(\overline{f}, g)$ of the Eq. (40) is similar. □

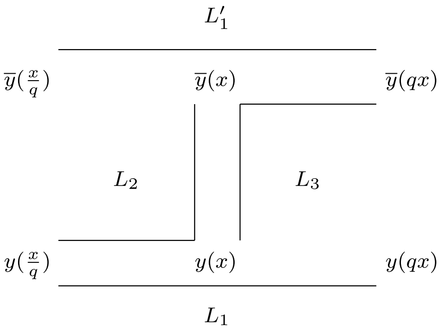
Remark on the Geometric Characterization of the Solutions f, g As a consequence of the above results, the variables f, g obtained from the Padé problem give special solutions of the elliptic Painlevé equation. Since they are (Bäcklund transformations of) the terminating hypergeometric solution [4, 5], they have the following geometric characterization. Let C_1 be a curve of degree $(2n, 2n + 1)$ passing through the 8 points $(f_*(\xi_i), g_*(\xi_i))_{i=1}^8$ in Eqs. (10), (12) with multiplicity $n(1^8) + (0, 1, 1, 0, 0, 0, 0, 0)$. Similarly, let C_2 be a curve of degree $(2m + 2, 2m + 1)$ passing through the 8 points with multiplicity $m(1^8) + (0, 1, 0, 1, 1, 1, 1, 1)$. C_1 and C_2 are unique rational curves. Except for the assigned 8 points, there exists unique unassigned intersection point $(f, g) \in C_1 \cap C_2$ which is the solution.

5 Lax Formalism

In this section, we prove that the elliptic Painlevé equation (39), (40) are sufficient for the compatibility of Eqs. (14), (15).

Solving $\overline{y}(x)$ and $\overline{y}(qx)$ from eqs. L_2, $L_2|_{x\to qx}$ and plugging them into L_3, one has the following difference equation (Fig. 1):

Fig. 1 Lax equations



$$
\begin{aligned}
L_1: \quad & \frac{\vartheta_p(\frac{k}{a_1 x}, \frac{k}{qx}) \prod_{i=1}^{8} \vartheta_p(\frac{k}{x\xi_i})}{F_f(x)\vartheta_p(\frac{k}{x^2}, \frac{a_1}{x}, \frac{q}{x})} y\left(\frac{x}{q}\right) + \frac{q\vartheta_p(\frac{1}{x}, \frac{a_1}{qx}) \prod_{i=1}^{8} \vartheta_p(\frac{\xi_i}{qx})}{F_f(qx)\vartheta_p(\frac{k}{q^2x^2}, \frac{k}{q^2x}, \frac{k}{a_1 qx})} y(qx) \\
& + \left\{ \frac{w\overline{F_f(qx)}\vartheta_p(\frac{k}{qx^2})}{x^2 G_g(x) G_g(\frac{kqx}{a_1})} - \frac{qG_g(qx)\prod_{i=1}^{8}\vartheta_p(\frac{k}{qx\xi_i})}{F_f(qx)G_g(\frac{kqx}{a_1})\vartheta_p(\frac{k}{q^2x^2})} \right. \\
& \left. - \frac{G_g(\frac{kx}{a_1})\prod_{i=1}^{8}\vartheta_p(\frac{\xi_i}{x})}{F_f(x)G_g(x)\vartheta_p(\frac{k}{x^2})} \right\} y(x) = 0. \qquad (46)
\end{aligned}
$$

The pairs of equations $\{L_1, L_2\}$, $\{L_1, L_3\}$ and $\{L_2, L_3\}$ are equivalent with each other.

The above expression L_1 (46) contains variables $f, g, \overline{f}, w$. We will rewrite and characterize it in terms of f, g only. This characterization is a key of the proof of the compatibility. To do this, we first note the following

Lemma 3 *The factor w satisfying the relation* (43) *is explicitly given by* (f, g) *as*

$$
w = C\frac{\overline{f}_{\text{den}}(f, g)}{\varphi(f, g)}, \qquad (47)
$$

where $\overline{f}_{\text{den}}(f, g)$ is a polynomial of degree $(1, 4)$ defined as the denominator of the rational function $\overline{f} = \overline{f}(f, g)$, and $\varphi(f, g)$ is the defining polynomial of the degree $(2, 2)$ curve parametrized by $f_*(x), g_*(x)$, and C is a constant independent of f, g, x.

Proof The relation (43) follows from Eq. (47) by using

$$
\varphi|_{g=g_*(x)} = C' \frac{F_f(x)F_f(\frac{a_1 x}{k})}{g_{*\text{den}}(x)^2}, \qquad (48)
$$

$$
\left(\overline{f}_{\text{den}}\overline{f}_{*\text{num}}(qx) - \overline{f}_{\text{num}}\overline{f}_{*\text{den}}(qx)\right)\Big|_{g=g_*(x)} = C'' \frac{F_f(\frac{a_1 x}{k})\prod_{i=1}^{8}\vartheta_p(\frac{\xi_i}{x})}{g_{*\text{den}}(x)^4}, \qquad (49)
$$

where C', C'' are constants, $g_{*\mathrm{den}}(x)=\vartheta_p(\frac{c_3}{x},\frac{k^2}{a_1c_3x})$ is the denominator of $g_*(x)$, and similarly $\overline{f_{*}}_{\mathrm{den}}(x)=\vartheta_p(\frac{c_1}{x},\frac{kq}{c_1x})$, $\overline{f_{*}}_{\mathrm{num}}(x)=\vartheta_p(\frac{c_2}{x},\frac{kq}{c_2x})$. □

Lemma 4 *In terms of variables* f,g, *the Eq.* (46) *is represented as a polynomial equation* $L_1(f,g)=0$ *of degree* $(3,2)$ *characterized*[5] *by the following vanishing conditions at*: (1) 10 *points* $(f_*(u),g_*(u))$ *where* $u=\xi, qx$ *and* $\frac{k}{x}$, (2) 2 *more points* (f,g) *such as*

$$f=f_*(x),\qquad \frac{y(x)}{y(\frac{x}{q})}\frac{G_g(\frac{kx}{a_1})}{G_g(x)}=\frac{\vartheta_p(\frac{k}{a_1x},\frac{k}{qx})}{\vartheta_p(\frac{a_1}{x},\frac{q}{x})}\prod_{i=1}^{8}\frac{\vartheta_p(\frac{k}{\xi_ix})}{\vartheta_p(\frac{\xi_i}{x})}, \tag{50}$$

and

$$f=f_*(qx),\qquad \frac{y(qx)}{y(x)}\frac{G_g(\frac{kqx}{a_1})}{G_g(qx)}=\frac{\vartheta_p(\frac{k}{a_1qx},\frac{k}{q^2x})}{\vartheta_p(\frac{a_1}{qx},\frac{1}{x})}\prod_{i=1}^{8}\frac{\vartheta_p(\frac{k}{\xi_iqx})}{\vartheta_p(\frac{\xi_i}{qx})}. \tag{51}$$

Proof Due to the Eq. (43), the residue of L_1 at the apparent pole $g=g_*(x)$ vanishes. Replacing x with $\frac{k}{qx}$ in Eq. (43) and using the relations $F_f(\frac{k}{x})=\frac{x^2}{k}F_f(x)$ and $G_g(\frac{k^2}{a_1x})=\frac{a_1x^2}{k^2}G_g(x)$, we have

$$qx^2G_g(x)G_g(qx)\prod_{i=1}^{8}\vartheta_p\left(\frac{k}{q\xi_ix}\right)=wF_f(qx)\overline{F_f(qx)}\vartheta_p\left(\frac{k}{qx^2},\frac{k}{q^2x^2}\right), \tag{52}$$

hence, the residue of L_1 at $g=g_*(\frac{kqx}{a_1})=g_*(\frac{k}{qx})$ also vanishes. From these vanishing of residues and the Eq. (47), the L.H.S of Eq. (46) turns out to be a polynomial in (f,g) of degree $(3,2)$, after multiplying by $F_f(x)F_f(qx)\varphi$. Check of the vanishing conditions (1), (2) are easy. □

In a similar way, solving $y(\frac{x}{q})$, $y(x)$ form L_3, $L_3|_{x\to x/q}$ and substituting them into L_2, one has

$$\begin{aligned} L_1':\quad &\frac{\vartheta_p(\frac{1}{x},\frac{a_1}{q^2x})\prod_{i=1}^{8}\vartheta_p(\frac{\xi_i}{x})}{\vartheta_p(\frac{k}{qx^2},\frac{k}{qx},\frac{kq}{xa_1})\overline{F_f(qx)}}\overline{y}(qx)+\frac{\vartheta_p(\frac{k}{x},\frac{kq^2}{xa_1})\prod_{i=1}^{8}\vartheta_p(\frac{k}{x\xi_i})}{q\vartheta_p(\frac{kq}{x^2},\frac{q}{x},\frac{a_1}{qx})\overline{F_f(x)}}\overline{y}\left(\frac{x}{q}\right)\\ &+\left\{\frac{w\vartheta_p(\frac{k}{x^2})F_f(x)}{x^2G_g(x)G_g(\frac{kx}{a_1})}-\frac{G_g(\frac{x}{q})\prod_{i=1}^{8}\vartheta_p(\frac{k}{x\xi_i})}{q\vartheta_p(\frac{kq}{x^2})\overline{F_f(x)}G_g(\frac{kx}{a_1})}\right.\\ &\left.-\frac{G_g(\frac{kqx}{a_1})\prod_{i=1}^{8}\vartheta_p(\frac{\xi_i}{x})}{\vartheta_p(\frac{k}{qx^2})\overline{F_f(qx)}G_g(x)}\right\}\overline{y}(x)=0. \end{aligned} \tag{53}$$

[5] This geometric characterization of the difference equation L_1 is essentially the same as that in [20].

By the similar analysis as L_1, we have the following

Lemma 5 *In terms of variables* $\overline{f}, g$, *the Eq.* (53) *is represented as a polynomial equation* $L_1'(\overline{f}, g) = 0$ *of degree* $(3, 2)$ *characterized by the following vanishing conditions at*: (1) 10 *points* $(\overline{f_*}(qu), g_*(u))$ *where* $u = \xi, \frac{x}{q}$ *and* $\frac{k}{qx}$. (2) 2 *more points* $(\overline{f}, g)$ *such as*

$$\overline{f} = \overline{f_*(x)}, \qquad \frac{\overline{y}(x)}{\overline{y}(\frac{x}{q})} \frac{G_g(\frac{x}{q})}{G_g(\frac{kx}{a_1})} = \frac{\vartheta_p(\frac{kq^2}{a_1 x}, \frac{k}{x})}{\vartheta_p(\frac{a_1}{qx}, \frac{q}{x})}, \tag{54}$$

and

$$\overline{f} = \overline{f_*(qx)}, \qquad \frac{\overline{y}(qx)}{\overline{y}(x)} \frac{G_g(x)}{G_g(\frac{kqx}{a_1})} = \frac{\vartheta_p(\frac{kq}{a_1 x}, \frac{k}{qx})}{\vartheta_p(\frac{a_1}{qx}, \frac{1}{x})}. \tag{55}$$

Proof In terms of $(\overline{f}, g)$, the gauge factor w (47) is written as

$$w = C''' \frac{f_{\rm den}(\overline{f}, g)}{\overline{\varphi}(\overline{f}, g)}, \tag{56}$$

where $f_{\rm den}(\overline{f}, g)$ is the denominator of the rational function $f = f(\overline{f}, g)$, and $\overline{\varphi}(\overline{f}, g)$ is the defining polynomial of the curve parametrized by $\overline{f}_*(qx), g_*(x)$, and C''' is a constant. Then the proof of the Lemma is the same as the proof of the Lemma 4. □

Proposition 2 *The Eq.* (53) *expressed in terms of* $(\overline{f}, \overline{g})$ *is equivalent with the transformation* $T(L_1) = \overline{L_1}$ *of Eq.* (46).

Proof This fact is a consequence of Lemmas 2, 4 and 5. The geometric proof in the q-difference case [21] is also available here (see Lemmas 4.2–4.6 in [21]). □

6 Determinant Formulae

In this section, we present explicit determinant formulae for the solutions $U(x)$, $V(x)$ of the interpolation problem (5).

Theorem 2 *Interpolating rational functions $U(x)$, $V(x)$ have the following determinant expressions*:

$$U(x) = \text{const.} \begin{vmatrix} m^U_{0,0} & \cdots & m^U_{0,n} \\ \vdots & \ddots & \vdots \\ m^U_{n-1,0} & \cdots & m^U_{n-1,n} \\ \phi_0(x) & \cdots & \phi_n(x) \end{vmatrix}, \qquad (57)$$

$$V(x) = \text{const.} \begin{vmatrix} m^V_{0,0} & \cdots & m^V_{0,m} \\ \vdots & \ddots & \vdots \\ m^V_{m-1,0} & \cdots & m^V_{m-1,m} \\ \chi_0(x) & \cdots & \chi_m(x) \end{vmatrix},$$

where

$$m^U_{ij} = {}_{12}V_{11}\big(q^{-1}k; q^{-N}, q^{N-i-1}a_1, q^{-j}a_2, q^i a_3, q^j a_4, a_5, a_6; q\big),$$
$$m^V_{ij} = {}_{12}V_{11}\left(q^{-1}k; q^{-N}, q^{-j}\frac{k}{a_1}, q^{N-i-1}\frac{k}{a_2}, q^j\frac{k}{a_3}, q^i\frac{k}{a_4}, \frac{k}{a_5}, \frac{k}{a_6}; q\right), \qquad (58)$$

and ${}_{n+5}V_{n+4}$ (${}_{n+3}E_{n+2}$ *in convention of* [4]) *is the very-well poised*, *balanced elliptic hypergeometric series* [1, 15, 16]

$${}_{n+5}V_{n+4}(u_0; u_1, \ldots, u_n; z) = \sum_{s=0}^{\infty} \frac{\vartheta_p(u_0 q^{2s})}{\vartheta_p(u_0)} \prod_{j=0}^{n} \frac{\vartheta_p(u_j)_s}{\vartheta_p(qu_0/u_j)_s} z^s. \qquad (59)$$

Proof In general, the solution of interpolation problem

$$V(x_s) = Y_s U(x_s), \quad s = 0, \ldots, N \qquad (60)$$

is written by the following determinants:

$$U(x) = \begin{vmatrix} \chi_0(x_0) & \cdots & \chi_m(x_0) & Y_0\phi_0(x_0) & \cdots & Y_0\phi_n(x_0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \chi_0(x_N) & \cdots & \chi_m(x_N) & Y_N\phi_0(x_N) & \cdots & Y_N\phi_n(x_N) \\ 0 & \cdots & 0 & \phi_0(x) & \cdots & \phi_n(x) \end{vmatrix}, \qquad (61)$$

and

$$V(x) = \begin{vmatrix} \chi_0(x_0) & \cdots & \chi_m(x_0) & Y_0\phi_0(x_0) & \cdots & Y_0\phi_n(x_0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \chi_0(x_N) & \cdots & \chi_m(x_N) & Y_N\phi_0(x_N) & \cdots & Y_N\phi_n(x_N) \\ \chi_0(x) & \cdots & \chi_m(x) & 0 & \cdots & 0 \end{vmatrix}. \qquad (62)$$

We apply these formulae for Y_s, $\phi_i(x)$, $\chi_i(x)$ given by (6), (8) and $x_s = q^{-s}$. Note that $\phi_i(x_s)$, $\chi_i(x_s)$ can be written as

$$\begin{aligned} \phi_i(x_s) &= \frac{\vartheta_p(\frac{k}{a_2}, \frac{k}{a_4}, q^{-i}a_2, q^i a_4)_s}{\vartheta_p(a_2, a_4, q^i\frac{k}{a_2}, q^{-i}\frac{k}{a_4})_s}, \\ \chi_i(x_s) &= \frac{\vartheta_p(a_1, a_3, q^{-i}\frac{k}{a_1}, q^i\frac{k}{a_3})_s}{\vartheta_p(\frac{k}{a_1}, \frac{k}{a_3}, q^i a_1, q^{-i}a_3)_s}. \end{aligned} \tag{63}$$

To rewrite the determinant in Eq. (61), we use the multiplication by a matrix

$$L = \left[\begin{array}{c|c} (L_{ij})_{i,j=0}^N & \\ \hline & 1 \end{array}\right] \tag{64}$$

from the left, where

$$L_{ij} = \frac{\vartheta_p(q^{2j-1}k)}{\vartheta_p(q^{-1}k)} \frac{\vartheta_p(q^{-1}k, q^{-N}, q^{N-i-1}a_1, \frac{k}{a_1}, q^i a_3, \frac{k}{a_3})_j}{\vartheta_p(q, q^N k, q^{-N+i+1}\frac{k}{a_1}, a_1, q^{-i}\frac{k}{a_3}, a_3)_j} q^j. \tag{65}$$

For the last $n+1$ columns, we have

$$\begin{aligned} \sum_{s=0}^N L_{is} Y_s \phi_j(x_s) &= {}_{12}V_{11}\big(q^{-1}k; q^{-N}, q^{N-i-1}a_1, q^{-j}a_2, q^i a_3, q^j a_4, a_5, a_6; q\big) \\ &= m_{ij}^U. \end{aligned} \tag{66}$$

For the first $m+1$ columns, we have

$$\sum_{s=0}^N L_{is}\chi_j(x_s) = {}_{10}V_9\left(q^{-1}k; q^{-N}, q^{N-i-1}a_1, q^{-j}\frac{k}{a_1}, q^i a_3, q^j\frac{k}{a_3}; q\right). \tag{67}$$

Using the Frenkel-Turaev summation formula ($u_1 \cdots u_5 = qu_0^2$, $u_5 = q^{-n}$) [2, 15, 16]:

$${}_{10}V_9(u_0; u_1, \ldots, u_5; q) = \frac{\vartheta_p(qu_0, \frac{qu_0}{u_1u_2}, \frac{qu_0}{u_1u_3}, \frac{qu_0}{u_2u_3})_n}{\vartheta_p(\frac{qu_0}{u_1}, \frac{qu_0}{u_2}, \frac{qu_0}{u_3}, \frac{qu_0}{u_1u_2u_3})_n}, \tag{68}$$

the expression (67) can be evaluated as

$$\frac{\vartheta_p(k, q^{-N+i+j+1}, q^{-N+1}\frac{k}{a_1a_3}, q^{j-i}\frac{a_1}{a_3})_N}{\vartheta_p(q^{-N+j+1}\frac{1}{a_3}, q^{-i}\frac{k}{a_3}, q^j a_1, q^{-N+i+1}\frac{k}{a_1})_N}, \tag{69}$$

and it vanishes for $0 \le i+j < N$. Hence, we obtain the formula for $U(x)$ in (57) by Laplace expansion. The case for function $V(x)$ is similar. □

Theorem 2 supplies also formulae for special solutions f, g of the elliptic Painlevé equation through Eq. (36). Moreover we have

Lemma 6 *For $i, j \in \{3,4,5,6\}$, the ratios in Eq. (36) have following simple form*

$$\frac{U(a_i)}{U(a_j)} = \frac{c_i T_{a_2}^{-1} T_{a_i}(\tau^U)}{c_j T_{a_2}^{-1} T_{a_j}(\tau^U)}, \qquad \frac{V(a_i/q)}{V(a_j/q)} = \frac{c'_i T_{a_1} T_{a_i}^{-1}(\tau^V)}{c'_j T_{a_1} T_{a_j}^{-1}(\tau^V)}, \tag{70}$$

where $\tau^U = \det(m^U_{i,j})_{i,j=0}^{n-1}$ *and* $\tau^V = \det(m^V_{i,j})_{i,j=0}^{m-1}$,

$$c_3 = q^{\frac{n(n-1)}{2}} \frac{(q^{-n}\frac{k}{a_3}, q)_n (a_3, q)_n (q^{-m-n+1}\frac{a_3}{a_1}, q)_n (q^{m+1}\frac{a_1 a_3}{k}, q)_n}{(\frac{k}{a_2 a_3}, q)_n (\frac{q a_3}{a_2}, q)_n (q^{-m-n+1}\frac{a_3}{a_1}, q^2)_n (q^{m+n}\frac{a_1 a_3}{k}, 1)_n},$$

$$c_4 = \frac{(q^{-n}\frac{k}{a_4}, q)_n (a_4, q)_n}{(\frac{k}{a_2 a_4}, 1)_n (\frac{q a_4}{a_2}, q^2)_n}, \qquad c_i = \frac{(\frac{k}{q a_i}, 1)_n (a_i, 1)_n}{(\frac{k}{a_2 a_i}, q)_n (\frac{q a_i}{a_2}, q)_n}, \qquad (i = 5, 6),$$

$(x, v)_n = \prod_{i=0}^{n-1} \vartheta_p(x v^i)$ *and*

$$\left(c'_3, c'_4, c'_5, c'_6\right) = (c_4, c_3, c_5, c_6)|_{(m,n,a_1,\ldots,a_6)\mapsto(n,m,\frac{k}{a_2},\frac{k}{a_1},\frac{k}{a_4},\frac{k}{a_3},\frac{k}{a_5},\frac{k}{a_6})}.$$

Proof Since $\phi_i(a_4) = \delta_{i,0}$ $(i \geq 0)$, we have

$$\frac{U(a_4)}{\text{const.}} = \det\left(m^U_{i,j+1}\right)_{i,j=0}^{n-1} = T_{a_2}^{-1} T_{a_4}\left(\tau^U\right). \tag{71}$$

Using the symmetry of $U(x)$ in parameters $a_3, \ldots, a_6$, the first relation of Eq. (70) follows. The second relation is similar. □

The determinant expressions for the special solutions have been known for various (discrete) Painlevé equations (see [7, 12] for example). Our method using Padé interpolation gives a simple and direct way to obtain them.

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Appendix: Affine Weyl Group Actions

Here we give a derivation of the Painlevé equation (39), (40) from the affine Weyl group actions [9, 21].

Define multiplicative transformations s_{ij}, c, μ_{ij}, ν_{ij} $(1 \le i \ne j \le 8)$ acting on variables $h_1, h_2, u_1, \ldots, u_8$ as

$$\begin{aligned} s_{ij} &= \{u_i \leftrightarrow u_j\}, \qquad c = \{h_1 \leftrightarrow h_2\}, \\ \mu_{ij} &= \left\{h_1 \mapsto \frac{h_1 h_2}{u_i u_j},\ u_i \mapsto \frac{h_2}{u_j},\ u_j \mapsto \frac{h_2}{u_i}\right\}, \\ \nu_{ij} &= \left\{h_2 \mapsto \frac{h_1 h_2}{u_i u_j},\ u_i \mapsto \frac{h_1}{u_j},\ u_j \mapsto \frac{h_1}{u_i}\right\}. \end{aligned} \tag{72}$$

These actions generate the affine Weyl group of type $E_8^{(1)}$ with the following simple reflections:

$$\begin{array}{c} \qquad\qquad\qquad s_{12} \\ \qquad\qquad\qquad | \\ c - \mu_{12} - s_{23} - s_{34} - \cdots - s_{78}. \end{array} \tag{73}$$

We extend the actions bi-rationally on variables (f, g). The nontrivial actions are as follows:

$$c(f) = g, \qquad c(g) = f, \qquad \mu_{ij}(f) = \tilde{f}, \qquad \nu_{ij}(g) = \tilde{g}, \tag{74}$$

where, $\tilde{f} = \tilde{f}_{ij}$ and $\tilde{g} = \tilde{g}_{ij}$ are rational functions in (f, g) defined by

$$\frac{\tilde{f} - \mu_{ij}(f_i)}{\tilde{f} - \mu_{ij}(f_j)} = \frac{(f - f_i)(g - g_j)}{(f - f_j)(g - g_i)}, \qquad \frac{\tilde{g} - \nu_{ij}(g_i)}{\tilde{g} - \nu_{ij}(g_j)} = \frac{(g - g_i)(f - f_j)}{(g - g_j)(f - f_i)}, \tag{75}$$

$(f_i, g_i) = (f_\star(u_i), g_\star(u_i))$, and

$$f_\star(z) = \frac{\vartheta_p(\frac{d_2}{z}, \frac{h_1}{d_2 z})}{\vartheta_p(\frac{d_1}{z}, \frac{h_1}{d_1 z})}, \qquad g_\star(z) = \frac{\vartheta_p(\frac{d_2}{z}, \frac{h_2}{d_2 z})}{\vartheta_p(\frac{d_1}{z}, \frac{h_2}{d_1 z})}, \tag{76}$$

as in Eq. (10). As a rational function of (f, g), $\tilde{f}$ is characterized by the following properties: (i) it is of degree $(1, 1)$ with indeterminate points (f_i, g_i), (f_j, g_j), (ii) it maps generic points on the elliptic curve $(f_\star(z), g_\star(z))$ to $\frac{\vartheta_p(\frac{d_2}{z}, \frac{h_1 h_2}{d_2 z u_1 u_2})}{\vartheta_p(\frac{d_1}{z}, \frac{h_1 h_2}{d_1 z u_i u_j})}$. Using this geometric characterization, we have

$$\mu_{ij}\left\{\frac{\mathcal{F}_f(\frac{h_1 z}{h_2})}{\mathcal{F}_f(z)}\right\} = \frac{\vartheta_p(\frac{u_i}{z}, \frac{u_j}{z})}{\vartheta_p(\frac{h_2}{u_i z}, \frac{h_2}{u_j z})} \frac{\mathcal{F}_f(\frac{h_1 z}{h_2})}{\mathcal{F}_f(z)}, \quad \text{for } g = g_\star(z), \tag{77}$$

where the functions $\mathcal{F}_f(z)$ (and $\mathcal{G}_g(z)$) are defined in a similar way as Eq. (11)

$$\begin{aligned}\mathcal{F}_f(z) &= \vartheta_p\left(\frac{d_1}{z}, \frac{h_1}{d_1 z}\right) f - \vartheta_p\left(\frac{d_2}{z}, \frac{h_1}{d_2 z}\right),\\ \mathcal{G}_g(z) &= \vartheta_p\left(\frac{d_1}{z}, \frac{h_2}{d_1 z}\right) g - \vartheta_p\left(\frac{d_2}{z}, \frac{h_2}{d_2 z}\right).\end{aligned} \tag{78}$$

Let us consider the following compositions [9]

$$r = s_{12}\mu_{12}s_{34}\mu_{34}s_{56}\mu_{56}s_{78}\mu_{78}, \qquad T = rcrc. \tag{79}$$

Their actions on variables (h_i, u_i) are given by

$$\begin{aligned} &r(h_1) = v h_2, \qquad r(h_2) = h_2, \qquad r(u_i) = \frac{h_2}{u_i},\\ &T(h_1) = q h_1 v^2, \qquad T(h_2) = q^{-1} h_2 v^2, \qquad T(u_i) = u_i v,\end{aligned} \tag{80}$$

where $v = qh_2/h_1$, $q = h_1^2 h_2^2/(u_1 \cdots u_8)$. From Eq. (77) and $r(\frac{h_1}{h_2}) = \frac{qh_2}{h_1}$, the evolution $T(f) = rcrc(f) = r(f)$ is determined as

$$\frac{\mathcal{F}_f(z)}{\mathcal{F}_f(\frac{h_1 z}{h_2})} \frac{T(\mathcal{F}_f)(\frac{qh_2 z}{h_1})}{T(\mathcal{F}_f)(z)} = \prod_{i=1}^{8} \frac{\vartheta_p(\frac{u_i}{z})}{\vartheta_p(\frac{h_2}{u_i z})}, \qquad \text{for } g = g_\star(z). \tag{81}$$

Similarly, since $cTc = T^{-1}$, $T^{-1}(g)$ is determined by

$$\frac{\mathcal{G}_g(z)}{\mathcal{G}_g(\frac{h_2 z}{h_1})} \frac{T^{-1}(\mathcal{G}_g)(\frac{qh_1 z}{h_2})}{T^{-1}(\mathcal{G}_g)(z)} = \prod_{i=1}^{8} \frac{\vartheta_p(\frac{u_i}{z})}{\vartheta_p(\frac{h_1}{u_i z})}, \qquad \text{for } f = f_\star(z). \tag{82}$$

By a re-scaling of variables $(h_i, u_i, d_i) = (\kappa_i \lambda^2, \xi_i \lambda, c_i \lambda)$ with $\lambda = (h_1^3 h_2^{-1})^{\frac{1}{4}}$, we have $\mathcal{F}_f(z) = F_f(\frac{z}{\lambda})$, $T(\mathcal{F}_f)(z) = T(F_f)(\frac{\kappa_1}{\kappa_2}\frac{z}{\lambda})$ and so on, since $T(\lambda) = \frac{h_2}{h_1}\lambda$. Then the above equations take the form (39), (40), by putting $z = \lambda x$.

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Non-commutative Harmonic Oscillators

Hiroyuki Ochiai

Abstract This is a survey on the non-commutative harmonic oscillator, which is a generalization of usual (scalar) harmonic oscillators to the system introduced by Parmeggiani and Wakayama. With the definitions and the basic properties, we summarize the positivity of several related operators with $\mathfrak{sl}_2$ interpretations. We also mention some unsolved questions, in order to clarify the current status of the problems and expected further development.

1 Introduction

A *non-commutative harmonic oscillator* Q is the Weyl quantization of a matrix-valued quadratic forms in (x, ξ). This is a generalization of the usual (scalar) harmonic oscillator to the system introduced by A. Parmeggiani and M. Wakayama [21]. The adjective "non-commutative" originates from the two kinds of non-commutativity: one comes from the system, the other is due to the Weyl quantization. The main concern to this system has been devoted to the spectral problems, especially, the explicit determination of eigenvalues and eigenstates for the discrete spectrum, and their generating function, so called, spectral zeta functions.

In this paper, we deal with the case the system of *ordinary* differential operators of the system size *two*, following the original work by Parmeggiani and Wakayama [20]. (We note that little is known in the case that the size of the system is greater than two.) The operator Q is defined by

$$A\left(-\partial_x^2 + x^2\right)/2 + B(x\partial_x + 1/2), \tag{1}$$

where $x \in \mathbb{R}$, $\partial_x = d/dx$, A, B are real constant matrices of size two, A is symmetric, and B is skew-symmetric; $A^T = A$, $B^T = -B$. This operator Q is densely defined on the space $L^2(\mathbb{R}, \mathbb{C}^2)$ of $\mathbb{C}^2$-valued square integrable functions on the real

In honor of Professor Jimbo's 60th birthday.

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line $\mathbb{R}$. This is the Weyl quantization of

$$A(\xi^2 + x^2) + \sqrt{-1}Bx\xi, \tag{2}$$

which is a positive definite hermitian matrix if the constant matrix $A + \sqrt{-1}B$ is positive definite. In such a case, the system Q is positive elliptic, unbounded, self-adjoint. So it has only a discrete spectrum with finite multiplicities: $0 < \lambda_1 \le \lambda_2 \le \cdots (\to +\infty)$. In considering the spectral problem, there is no harm by the orthogonal change of coordinates of $\mathbb{C}^2$, so we may assume that A is a real diagonal matrix. Since we have assumed the system size is two, the matrix B is a constant multiple of the standard one $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

Remark 1 In the case that the size is greater than two, the orbit decomposition of the simultaneous action of the orthogonal group on the pair of symmetric and skew-symmetric matrices is considered by [2], and it shows that the representatives of isomorphism classes are rather complicated in general.

We continue the case that the system size is two. This means that the system Q is

$$Q = Q_{\alpha,\beta} = \begin{bmatrix} \frac{\alpha}{2}(-\partial_x^2 + x^2) & -(x\partial + \frac{1}{2}) \\ x\partial + \frac{1}{2} & \frac{\beta}{2}(-\partial_x^2 + x^2) \end{bmatrix}, \tag{3}$$

where $\alpha, \beta \in \mathbb{R}$ are the parameters specifying the operator Q.

In the case $\alpha = \beta$, the system $Q_{\alpha,\alpha}$ is unitarily equivalent to the scalar operator $((\partial_x^2 + (\alpha^2 - 1)x^2)/2)I_2$, where I_2 is the identity matrix of size two. Hence the corresponding spectral problem is solved for $Q_{\alpha,\alpha}$. For example, if $\alpha > 1$, then the spectrum is of the form $\sqrt{\alpha^2 - 1}(N + 1/2)$ with a non-negative integer N, and the eigenfunction is written in terms of Hermite functions. If $\alpha = 1$, then the set of spectra is the real half line $\{\lambda \ge 0\}$, and if $0 < \alpha < 1$, then the set of spectra is the whole real line $\mathbb{R}$. This unitary equivalence is first obtained in [21, 22] by using the oscillator representation ϖ of $\mathfrak{sl}_2$ (see Sect. 2), and is later obtained from the Malliavin calculus [26]. Both approaches use the unitary operators involving the function $e^{\sqrt{-1}x^2} = \cos(x^2) + \sqrt{-1}\sin(x^2)$, which is highly oscillating as $x \to \pm\infty$. This suggests the difficulty of the naive numerical computation of eigenfunctions. The general case $\alpha \ne \beta$ is considered to be the perturbation of the 'diagonal case' $\alpha = \beta$. From this point of view, Parmeggiani ([19, 20]) gives some "clustering theorems" for the spectrum.

If α and β is large enough, then the system Q is also considered to be the perturbation of the two split scalar harmonic oscillators. To be more precise, let us write

$$\frac{1}{\sqrt{\alpha\beta}} Q_{\alpha,\beta} = \begin{bmatrix} \sqrt{\alpha/\beta} & 0 \\ 0 & \sqrt{\beta/\alpha} \end{bmatrix} (-\partial_x^2 + x^2)/2 + \frac{1}{\sqrt{\alpha\beta}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} (x\partial_x + 1/2). \tag{4}$$

For a fixed ratio α/β and taking the limit $1/\sqrt{\alpha\beta} \to 0$, the system goes to the direct sum of two independent harmonic oscillators; $(\alpha/\beta)^{\pm 1/2}(-\partial_x^2 + x^2)/2$. Since the perturbation term is bounded with respect to the elliptic first term, we can apply Rellich's theory to obtain a qualitative property [18, 20].

Apart from these special cases of parameters, the spectrum does not seem to have a simple behavior, such as an arithmetic progression as in the harmonic oscillators. A numerical verification [13] with an accuracy supports this observation. In order to describe the whole structure of the spectra, we need some 'new' functions.

In what follows in this paper, we always assume that the system is elliptic, that is, the positivity assumption $A+\sqrt{-1}B>0$ is now translated as $\alpha, \beta > 0$ and $\alpha\beta > 1$.

Remark 2 Contrary to the case $\alpha\beta > 1$, nothing is known for $\alpha\beta \le 1$, except that it seems that Parmeggiani has recently proved that the system Q has only the continuous spectrum in the case $\alpha\beta = 1$.

2 Inequalities

If the scalar symbol is non-negative, then its Weyl quantization gives a positive operator. However, this is not true for a system. A counter example is given by Hörmander [4]

$$Q=\begin{bmatrix} x^2 & -\sqrt{-1}(x\partial_x+1/2) \\ -\sqrt{-1}(x\partial_x+1/2) & -\partial_x^2 \end{bmatrix}. \tag{5}$$

The symbol of Q is positive semi-definite, while the system Q is not positive, i.e., there exists an $u \in L^2(\mathbb{R}) \otimes \mathbb{C}^2$ such that $\langle Qu, u\rangle < 0$.

We may recall the oscillator representation ϖ of $\mathfrak{sl}_2$. Let

$$H=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad X^+=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad X^-=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{6}$$

be the standard basis of the three-dimensional simple Lie algebra $\mathfrak{sl}_2(\mathbb{R}) = \{A \in M(2,\mathbb{R}) \mid \operatorname{tr} A = 0\}$. The Lie bracket $[A,B]=AB-BA$ is given as

$$[H,X^+]=2X^+, \qquad [H,X^-]=-2X^-, \qquad [X^+,X^-]=H. \tag{7}$$

We recall (e.g., [5]) the representation ϖ of $\mathfrak{sl}_2$ realized in Weyl algebra $\mathbb{C}[x,\partial_x]$ by

$$\varpi(H)=x\partial_x+1/2, \varpi\left(X^+\right)=x^2/2, \varpi\left(X^-\right)=-\partial_x^2/2. \tag{8}$$

Then the operator Q in (5) is unitarily equivalent to

$$\begin{bmatrix} 2\varpi(X^+) & \varpi(H) \\ -\varpi(H) & 2\varpi(X^-) \end{bmatrix} = \varpi\begin{bmatrix} 2X^+ & H \\ -H & 2X^- \end{bmatrix}. \tag{9}$$

The last matrix is considered to be an element in $M(2, U(\mathfrak{sl}_2))$, where $U(\mathfrak{sl}_2)$ denotes the universal enveloping algebra of $\mathfrak{sl}_2$. This matrix has the following relation with the Capelli matrix:

$$\begin{bmatrix} 2X^+ & H \\ -H & 2X^- \end{bmatrix}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = 2\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{10}$$

where E_{ij} is the matrix unit, and I is the identity matrix, living in $\mathfrak{gl}_2 = M(2,\mathbb{R})$. The column determinant

$$\det \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22}+I \end{bmatrix} \tag{11}$$

of the Capelli matrix with an appropriate shift in the diagonal gives a central element of $U(\mathfrak{gl}_2)$. By Schur's lemma, we know every central element acts by a scalar on each irreducible representation, and its scalar has been determined for the Capelli elements, not only for the size two but for general matrix size. We may pose a question: can we diagonalize the Capelli matrix rather than Capelli element (= its determinant)? This question may have a relation with the spectral problem of non-commutative harmonic oscillators.

The non-commutative harmonic oscillator Q in (1) is expressed in terms of $\mathfrak{sl}_2$, or rather $\mathfrak{gl}_2 \otimes \mathfrak{sl}_2$ by

$$A \otimes \varpi\big(X^+ + X^-\big) + B \otimes \varpi(H) = (\iota \otimes \varpi)\big(A \otimes \big(X^+ + X^-\big) + B \otimes H\big), \tag{12}$$

where ι is a natural representation of $\mathfrak{gl}_2$ on $\mathbb{C}^2$. We have not yet found an intimate relation between Capelli elements and non-commutative harmonic oscillators. However, both operators share the two kinds of non-commutativity (in the sense of [21]), the matrix system as well as Weyl quantization and the Lie algebraic symmetry. Note that the positivity of the operator is related to the estimate of the lowest eigenvalues. This is rather different a question than analyzing all the eigenvalues. In the case of non-commutative harmonic oscillators, [18] and [20] give some estimates of the lowest eigenvalues, and it still requires an improvement.

The positivity of another operator

$$Q = \begin{bmatrix} a_1x^2 - a_2\partial_x^2 & -(x\partial + 1/2) \\ x\partial + 1/2 & a_3x^2 - a_4\partial_x^2 \end{bmatrix} \tag{13}$$

with real parameters a_1, a_2, a_3, and a_4 is considered in [23]. The answer is given as follows; there exists a real-valued real analytic function $\Phi(s_1,s_2)$ such that $\Phi(a_1a_4, a_2a_3) > 0$ if and only if Q is positive. Although this function Φ is explicitly defined by the determinant of the matrix of infinite size, the nature of the function Φ is still unclear. For example, we do not know whether this function Φ arises elsewhere. Note that the operator (13) is written in terms of $\mathfrak{sl}_2$ as

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\varpi(H) + \begin{bmatrix} 2a_1 & 0 \\ 0 & 2a_3 \end{bmatrix}\varpi\big(X^+\big) + \begin{bmatrix} 2a_2 & 0 \\ 0 & 2a_4 \end{bmatrix}\varpi\big(X^-\big), \tag{14}$$

or as $\iota \otimes \varpi$ on $\mathfrak{gl}_2 \otimes \mathfrak{sl}_2$, again.

A system of *partial* differential operators related with the non-commutative harmonic oscillator has been considered prior by [1], in the occasion to give a counter example of the Fefferman-Phong inequality for systems.

3 Special Values of Spectral Zeta Functions

Although we can not write up each spectrum, its generating function is proved to be manageable. The spectral zeta function, denoted by $\zeta_Q(s)$, is defined by

$$\zeta_Q(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}. \tag{15}$$

The general property, such as an absolute convergence on the right half plane, the analytic continuation to whole s-plane has been established in [6, 7]. They also give an expression of the special value $\zeta_Q(2)$ by using the confluent Heun function, and later it is simplified in [16] as

$$\zeta_Q(s) = \frac{3}{2}\zeta(2)\frac{(\alpha+\beta)^2}{(\alpha\beta-1)\alpha\beta}\left(1+\frac{\alpha-\beta}{\alpha+\beta}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4},1;\frac{1}{1-\alpha\beta}\right)^2\right), \tag{16}$$

where ${}_2F_1$ is the Gauss hypergeometric function. In these special exponents, the Gauss hypergeometric function reduces to the complete elliptic integral. This enables us to give a connection with the modular forms [11]. This argument can be generalized for a general special values $\zeta_Q(s)$ at positive integers $s \geq 2$ as in [9, 10]. The related series arising in the Taylor expansion of such hypergeometric-like functions are examined also in [8, 12].

4 Real Picture Versus Complex Picture

The description of the spectrum, which is the main concern for the positive non-commutative harmonic oscillators, is first given in [22] by the infinite continued fractions, and later in [14] by the monodromy representations and the connection coefficients of the Heun differential equations. The former has a *real* nature, while the latter has a *complex* one. These two approaches have their own advantage: taking a truncation of the infinite continued fraction, we obtain an approximation of each eigenvalues. This enables us to give, e.g., the estimate of the lowest eigenvalues. By the connection coefficients or monodromy in the complex domain, such an estimate is rather difficult. On the other hand, the description of spectra and the eigenfunctions in terms of the monodromy of the Heun equation, e.g., enables us to control the multiplicity of the spectrum. We will explain this feature in more detail.

Heun equation is the second order ordinary differential equation on the complex projective line with four regular singular points [3]. It has one accessory parameter, by definition, a parameter undetermined by the local exponents. The dependence of the monodromy, the global behavior of the analytic continuation of any holomorphic solution at a regular point, on the accessory parameter is believed to be a 'difficult function'. In a special case, apart from Heun, this dependence is controlled by the Painlevé equation, which is sufficiently non-trivial. In representation theory of finite-dimensional Lie groups and Lie algebras, special functions such as hypergeometric functions, Beta integral, Gamma functions often arises, in the case of non-commutative harmonic oscillators we have in [14] and encounter with the Fuchsian ordinary differential equation on the complex line with an accessory parameter. Recently, another example is obtained in a different context [17] in the restriction of Heckman-Opdam hypergeometric function to the one-dimensional singular locus.

In our case, Heun's operator is of the form:

$$H = \frac{d^2}{dz} + \left(\frac{1-n}{z} + \frac{-n}{z-1} + \frac{n+(3/2)}{z-\alpha\beta}\right) + \frac{-(3/2)nz - q}{z(z-1)(z-\alpha\beta)}, \tag{17}$$

where n corresponds to the eigenvalues of Q with some normalization, and the accessory parameter q is explicitly written as a rational function in α, β and n. The Riemann scheme of this operator H is

$$\begin{Bmatrix} w=0 & 1 & \alpha\beta & \infty \\ 0 & 0 & 0 & 3/2 \\ n & n+1 & -n-(1/2) & -n \end{Bmatrix}. \tag{18}$$

Note that in the case $n \in \mathbb{R}$ is an integer or a half integer, the differences of exponents at four singular points consist of exactly two integers and two half integers (= integer + 1/2). If all the four differences are half integers [25] or integers [24] simultaneously, then there is an integral expression of a solution of such a Heun equation, but we have not yet obtained such a concise expression in our case (18).

We call a function $u \in L^2(\mathbb{R}) \otimes \mathbb{C}^2$ *even* if $u(-x) = u(x)$ and *odd* if $u(-x) = -u(x)$. Since the operator Q preserves the parity, an eigenfunction is a sum of the even and odd eigenfunctions. In the complex picture we use the different ordinary differential operators H's corresponding to the even/odd cases. One of the corollaries of the main theorem in [15] is that the following four conditions are equivalent:

(i) The multiplicity of odd eigenfunction of Q with the (normalized) spectrum n is greater than 1.
(ii) The Heun equation H has a two-dimensional holomorphic solution on the disk containing 0 and 1.
(iii) The Heun equation H has a non-zero rational solution.
(iv) The Heun equation H has a solution of the form $\sqrt{z-\alpha\beta}\times$ (a non-zero rational function).

Moreover, (i) to (iv) occur only for $n \in \mathbb{Z}_{>0}$, and in the cases (iii) and (iv), we can specify by n the locations and the possible orders of poles for such rational functions. This means that the condition (iii) to (iv) are a finite condition.

We note that we have not yet established the equivalence for even eigenfunctions, because the corresponding result in [14] requires the inhomogenous Heun equation, while the odd case corresponds to the homogeneous Heun equation. It should be improved.

In our complex picture, even eigenfunctions and odd eigenfunctions correspond to the different Fuchsian equations, so that the interaction between even and odd eigenfunctions is not controlled well. Is there any intimate relation between these two Fuchsian equations?

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The Inversion Formula of Polylogarithms and the Riemann-Hilbert Problem

Shu Oi and Kimio Ueno

Abstract In this article, we set up a method of reconstructing the polylogarithms $\mathrm{Li}_k(z)$ from zeta values $\zeta(k)$ via the Riemann-Hilbert problem. This is referred to as "a recursive Riemann-Hilbert problem of additive type." Moreover, we suggest a framework of interpreting the connection problem of the Knizhnik-Zamolodchikov equation of one variable as a Riemann-Hilbert problem.

1 Introduction

Polylogarithms $\mathrm{Li}_k(z)$ $(k \geq 2)$ satisfy *the inversion formula*

$$\mathrm{Li}_k(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \mathrm{Li}_{k-j}(z) + \mathrm{Li}_{2,\underbrace{1,\ldots,1}_{k-2}}(1-z) = \zeta(k).$$

Applying the Riemann-Hilbert problem of additive type (alternatively, Plemelj-Birkhoff decomposition) [1, 2, 4] to this inversion formula, we show that $\mathrm{Li}_k(z)$ can be reconstructed from boundary values $\zeta(k)$. We prove this by using the Riemann-Hilbert problem recursively so that we refer to this method as *a recursive Riemann-Hilbert problem of additive type*.

As a generalization of this method, we can reconstruct multiple polylogarithms $\mathrm{Li}_{k_1,\ldots,k_r}(z)$ from multiple zeta values $\zeta(k_1,\ldots,k_r)$. This is nothing but interpreting

Dedicated to Professor Michio Jimbo.

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the connection relation [3]

$$\mathcal{L}(z) = \mathcal{L}^{(1)}(z)\,\Phi_{\mathrm{KZ}}$$

between the fundamental solutions of *the Knizhnik-Zamolodchikov equation of one variable* (KZ equation, for short)

$$\frac{dG}{dz} = \left(\frac{X_0}{z} + \frac{X_1}{1-z}\right)G$$

as a Riemann-Hilbert problem. Here Φ_{KZ} is *Drinfel'd associator* and $\mathcal{L}(z)$ (resp. $\mathcal{L}^{(1)}(z)$) is the fundamental solution of KZ equation normalized at $z = 0$ (resp. $z = 1$). We have completely solved this problem and a preprint is now in preparation.

2 The Inversion Formula of Polylogarithms

For positive integers k, polylogarithms $\mathrm{Li}_k(z)$ are introduced as follows: First we set $\mathrm{Li}_1(z) = -\log(1-z)$. In the domain $D = \mathbf{C} \setminus \{z = x | 1 \le x\}$, $\mathrm{Li}_1(z)$ has a branch such that $\mathrm{Li}_1(0) = 1$ (the principal value of $\mathrm{Li}_1(z)$). Starting from the principal value of $\mathrm{Li}_1(z)$, we introduce $\mathrm{Li}_k(z)$, which are holomorphic on D, recursively by

$$\mathrm{Li}_k(z) = \int_0^z \frac{\mathrm{Li}_{k-1}(t)}{t}dt \quad (k \ge 2) \tag{1}$$

where the integral contour is assumed to be in D. Then $\mathrm{Li}_k(z)$ has a Taylor expansion

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} \tag{2}$$

on $|z| < 1$. We obtain, for $k \ge 2$,

$$\lim_{z \to 1, z \in D} \mathrm{Li}_k(z) = \zeta(k), \tag{3}$$

where $\zeta(k)$ is the Riemann zeta value $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$.

From (1), we have differential recursive relations:

$$\frac{d}{dz}\mathrm{Li}_1(z) = \frac{1}{1-z}, \qquad \frac{d}{dz}\mathrm{Li}_k(z) = \frac{\mathrm{Li}_{k-1}(z)}{z} \quad (k \ge 2). \tag{4}$$

By virtue of (1), $\mathrm{Li}_k(z)$ is analytically continued to a many-valued analytic function on $\mathbf{P}^1 \setminus \{0, 1, \infty\}$. However, in this article, we will use the notation $\mathrm{Li}_k(z)$ as the principal value stated previously.

We also define multiple polylogarithms $\mathrm{Li}_{2,1,\ldots,1}(z)$ $(k \ge 2)$ as

$$\mathrm{Li}_{2,\underbrace{1,\ldots,1}_{k-2}}(z) = \int_0^z \frac{(-1)^{k-1}}{(k-1)!}\frac{\log^{k-1}(1-t)}{t}dt. \tag{5}$$

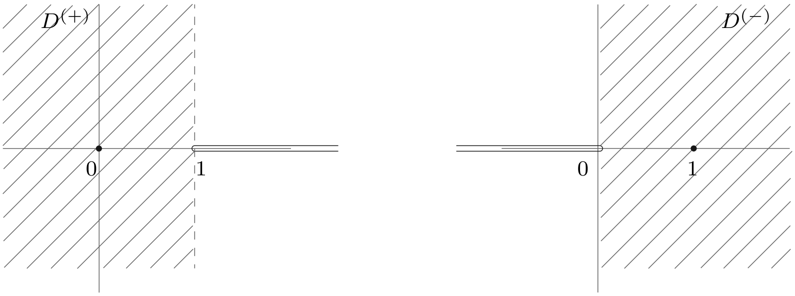

Fig. 1 The domains $D^{(+)}$, $D^{(-)}$

By using these relations and (3), one can obtain easily *the inversion formula* of polylogarithms.

Proposition 1 (the inversion formula of polylogarithms) *For* $k \geq 2$, *the following functional relation holds.*

$$\mathrm{Li}_k(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \mathrm{Li}_{k-j}(z) + \mathrm{Li}_{2,\underbrace{1,\ldots,1}_{k-2}}(1-z) = \zeta(k). \tag{6}$$

Proof Differentiating the left hand side of the equation (6), we have

$$\frac{d}{dz}\left(\mathrm{Li}_k(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \mathrm{Li}_{k-j}(z) + \mathrm{Li}_{2,\underbrace{1,\ldots,1}_{k-2}}(1-z)\right) = 0.$$

Therefore the left hand side of (6) is a constant. Taking the limit of the left hand side of (6) as $z \in D$ tends to 1 and using (3), we see that the constant is equal to $\zeta(k)$. □

The branch of $\mathrm{Li}_{2,\underbrace{1,\ldots,1}_{k-2}}(1-z)$ on the domain $D' = \mathbf{C} \setminus \{z = x | x \leq 0\}$ is determined from the principal value of $\log z$.

3 The Recursive Riemann-Hilbert Problem of Additive Type

Let $D^{(+)}$, $D^{(-)}$ (Fig. 1) be domains of $\mathbf{C}$ defined by

$$D^{(+)} = \{z = x + yi | x < 1, -\infty < y < \infty\} \subset D,$$
$$D^{(-)} = \{z = x + yi | 0 < x, -\infty < y < \infty\} \subset D'.$$

The following theorem says that polylogarithms $\mathrm{Li}_k(z)$ are characterized by the inversion formula.

Theorem 1 *Put* $f_1^{(+)}(z) = \mathrm{Li}_1(z)$. *For* $k \geq 2$, *we assume that* $f_k^{(\pm)}(z)$ *are holomorphic functions on* $D^{(\pm)}$ *satisfying the functional relation*

$$f_k^{(+)}(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} f_{k-j}^{(+)}(z) + f_k^{(-)}(z) = \zeta(k) \quad \left(z \in D^{(+)} \cap D^{(-)}\right), \tag{7}$$

the asymptotic conditions

$$\frac{d}{dz} f_k^{(\pm)}(z) \to 0 \quad \left(z \to \infty, z \in D^{(\pm)}\right), \tag{8}$$

and the normalization condition

$$f_k^{(+)}(0) = 0. \tag{9}$$

Then we have

$$f_k^{(+)}(z) = \mathrm{Li}_k(z), \qquad f_k^{(-)}(z) = \mathrm{Li}_{2,\underbrace{1,\ldots,1}_{k-2}}(1-z) \quad (k \geq 2).$$

Proof We prove the theorem by induction on $k \geq 2$. For the case $k = 2$, the proof can be done in the same manner as the case $k > 2$ from the definition of $f_1^{(+)}(z)$. So we assume that $f_j^{(+)}(z) = \mathrm{Li}_j(z)$ and $f_j^{(-)}(z) = \mathrm{Li}_{2,\underbrace{1,\ldots,1}_{j-2}}(1-z)$ for $2 \leq j \leq k-1$. Now we show that $f_k^{(+)}(z) = \mathrm{Li}_k(z)$, $f_k^{(-)}(z) = \mathrm{Li}_{2,\underbrace{1,\ldots,1}_{k-2}}(1-z)$. From the assumption, the Eq. (7) becomes

$$f_k^{(+)}(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \mathrm{Li}_{k-j}(z) + f_k^{(-)}(z) = \zeta(k). \tag{10}$$

Differentiating this equation, we have

$$\begin{aligned}
0 &= \frac{d}{dz}\left(f_k^{(+)}(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \mathrm{Li}_{k-j}(z) + f_k^{(-)}(z) \right) \\
&= \frac{d}{dz} f_k^{(+)}(z) + \sum_{j=1}^{k-2} \left(\frac{1}{z} \frac{(-1)^j \log^{j-1} z}{(j-1)!} \mathrm{Li}_{k-j}(z) + \frac{(-1)^j \log^j z}{j!} \frac{\mathrm{Li}_{k-j-1}(z)}{z} \right) \\
&\quad + \frac{1}{z} \frac{(-1)^{k-1} \log^{k-2} z}{(k-2)!} \mathrm{Li}_1(z) + \frac{(-1)^{k-1} \log^{k-1} z}{(k-1)!} \frac{1}{1-z} \\
&\quad + \frac{d}{dz} f_k^{(-)}(z)
\end{aligned}$$

$$= \frac{d}{dz} f_k^{(+)}(z) - \frac{\mathrm{Li}_{k-1}(z)}{z} + \frac{1}{1-z} \frac{(-1)^{k-1} \log^{k-1} z}{(k-1)!} + \frac{d}{dz} f_k^{(-)}(z).$$

Thus we obtain

$$\frac{d}{dz} f_k^{(+)}(z) - \frac{\mathrm{Li}_{k-1}(z)}{z} = -\frac{1}{1-z} \frac{(-1)^{k-1} \log^{k-1} z}{(k-1)!} - \frac{d}{dz} f_k^{(-)}(z) \qquad (11)$$

on $z \in D^{(+)} \cap D^{(-)}$. Here, the left hand side of (11) is holomorphic on $D^{(+)}$ and the right hand side of (11) is holomorphic on $D^{(-)}$. Therefore the both sides of (11) are entire functions. Using the asymptotic condition (8) and

$$\frac{\mathrm{Li}_{k-1}(z)}{z} \to 0 \quad \left(z \to \infty, z \in D^{(+)}\right), \qquad \frac{\log^{k-1} z}{1-z} \to 0 \quad \left(z \to \infty, z \in D^{(-)}\right),$$

we have that both sides of (11) are 0 by virtue of Liouville's theorem. Therefore we have

$$f_k^{(+)}(z) = \int^z \frac{\mathrm{Li}_{k-1}(z)}{z} dz = \mathrm{Li}_k(z) + c_k^{(+)},$$

$$f_k^{(-)}(z) = \int^z -\frac{1}{1-z} \frac{(-1)^{k-1} \log^{k-1} z}{(k-1)!} dz = \mathrm{Li}_{2,\underbrace{1,\ldots,1}_{k-2}}(1-z) + c_k^{(-)},$$

where $c_k^{(+)}, c_k^{(-)}$ are integral constants. From the normalization condition (9), it is clear that $c_k^{(+)}$ is equal to 0. Finally, substituting $f_k^{(+)}(z)$ and $f_k^{(-)}(z)$ in (7), we obtain

$$\mathrm{Li}_k(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \mathrm{Li}_{k-j}(z) + \mathrm{Li}_{2,\underbrace{1,\ldots,1}_{k-2}}(1-z) + c_k^{(-)} = \zeta(k). \qquad (12)$$

Comparing the inversion formula (6), we have $c_k^{(-)} = 0$. This concludes the proof. □

The Eq. (10) is interpreted as the decomposition of the holomorphic function

$$\sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \mathrm{Li}_{k-j}(z)$$

on $z \in D^{(+)} \cap D^{(-)}$ to a sum of a function $f_k^{(+)}(z)$, which is holomorphic on $D^{(+)}$, and a function $f_k^{(-)}(z)$, which is holomorphic on $D^{(-)}$. This decomposition is nothing but a Riemann-Hilbert problem of additive type. The theorem says that polylogarithms $\mathrm{Li}_k(z)$ can be constructed from the boundary value $\zeta(k)$ by applying this Riemann-Hilbert problem recursively. In this sense, we call (7) *the recursive Riemann-Hilbert problem of additive type*.

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Some Remarks on the Quantum Hall Effect

Vincent Pasquier

Abstract This review is destinated to an integrable systems community and attempts to motivate the Quantum Hall Effect as a possible field of application. We review some of its aspects using a microscopic (wave function) point of view, and we describe it as an incompressible quantum fluid droplet deforming under external perturbations. In particular, we discuss some well known open and closed geometries. We attempt to relate the Q.H.E. hydrodynamics to the Calogero-Sutherland model and Benjamin Ono equation from the bulk and boundary point of view. Finally, as an illustration, we discuss the so called non dissipative viscosity.

1 Introduction

These notes were written at the occasion of a celebration in honor of Professor Michio Jimbo.

They are intended for nonspecialists, the aim being to indicate a few research directions. I gather a few topics in the Quantum Hall Effect (Q.H.E.) (see [16, 30] for comprehensive reviews), limiting myself to the microscopic approach via wave functions and concentrating on how the system adjusts to geometric deformations.

I review the well known bulk [37] edge correspondence from that point of view. One way to connect the bulk aspects of the Q.H.E. described by a topologically protected ground state and the edge state physic is by probing the bulk with outside perturbations. This can be effectively modeled by a boundary conformal field theory restricted to the outside region and the bulk is represented by an appropriate edge state [11, 37]. I limit myself to characterize the bulk by a Laughlin wave function [24] and attempt to motivate the Calogero-Sutherland model [35] as a tool to describe the edge. Since is not possible relate the dynamics of the Calogero-Sutherland model to the Quantum Hall interactions, I try to bypass this difficulty by

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establishing a relation through the area preserving diffeomorphisms. The arguments I give for this relation are not rigorous and should be understood as an advocacy rather than a scientific justification.

I also consider wave functions on surfaces without edge such as the torus or the sphere and recover the ground state and its quasihole excitations on the torus as eigenstates of a modified Calogero Sutherland model.

Finally, I discuss the non dissipative viscosity [4, 32, 33, 36] which arose interest recently and is related to the way the system reacts to an adiabatic deformation.

The content of the following sections can be briefly described as follows. The second part reviews the correspondence between bulk and edge. The third part is devoted to study the relations of the Hall effect with the Calogero-Sutherland model. The fourth part studies the Hall effect on surfaces without boundaries such as the sphere or the torus. Finally, the last part is devoted to make some remarks on the viscosity.

Most of the material presented is motivated by a collaboration with Benoit Estienne, Raoul Santachiara and Didina Serban. I also wish to thank Michel Bergere, Jerome Dubail, Gregoire Misguich and J. Shiraishi for sharing their insights with me.

2 Bulk and Edge

To illustrate the emergence of the edge physics, let us consider the problem of evaluating the norm of a Laughlin wave function.

We recall that in the symmetric gauge, the Lowest Landau Level (LLL) wave functions are holomorphic polynomials in the variable z times a non holomorphic exponential factor $e^{-\frac{1}{2l^2}\bar{z}z}$. l is the magnetic length proportional to the inverse magnetic field $l^2 = h/eB$. The fractional quantum Hall effect wave functions we shall consider here are obtained as a linear combination of multi-particle Lowest Landau Level wave functions. The constraint of incompressibility is put by hand by requiring certain vanishing conditions and one looks for wave functions with the minimal degree in z_i compatible with the vanishing conditions. This minimal degree condition can easily be understood as follows: Imagine the particles are confined in a disc shaped box of radius R. since $z^n e^{-\frac{1}{2l^2}\bar{z}z}$ is mostly concentrated on a circle of radius $\sqrt{n}l$, the degree is bounded by $(R/l)^2$, and we are interested in putting as many particles as we can inside the box.

We can also use coherent states localized on patches of area $2\pi l^2$. The repulsive interactions induce a hard core repulsion between these patches. We shall see that each particle occupies a extended area $2\pi l^2/\nu$ so that the maximal density is $\nu/2\pi l^2$. The quantity ν, called the filling fraction turns out to be a rational (in all the experimental known situations) and is the most important parameter characterizing the properties of a Hall state.

One requires that the Laughlin wave function vanishes as $(z_i - z_j)^m$ at short distance and so $\nu = 1/m$. For N_e particles, it is obtained by raising to the power m

the Slater determinant (antisymmetrized product) of LLL orbitals z^{k-1}, $1 \le k \le N_e$ (we omit the nonholomorphic factor which can be incorporated in the definition of the measure):

$$\psi_m(z_i) = \prod_{i<j}^{N_e} (z_i - z_j)^m \tag{1}$$

Note that m must be an integer for the wave function to be polynomial.

Its squared norm is therefore given by:

$$Z_m = \int \prod_{i=1}^{N_e} d^2 z_i e^{-\frac{1}{2l^2}\bar{z}_i z_i} \prod_{i<j} |z_i - z_j|^{2m} \tag{2}$$

It can be interpreted as the partition function of a Coulomb gas at an inverse temperature m. The electrostatic charges located at z_i, $1 \le i \le N_e$ repel each other with a logarithmic potential. The partition function is obtained by integrating over the positions of the charges, e^{-mE} with E given by:

$$\begin{aligned} E(z_i) &= -2 \sum_{i<j=1}^{N_e} \log|z_i - z_j| + \sum_{i=1}^{N_e} \frac{1}{2ml^2} z_i \bar{z}_i \\ &= -\int d^2 z \rho(z) \int d^2 w \rho(w) \log|z-w| + \frac{1}{2ml^2} \int d^2 z \rho(z) z\bar{z} \end{aligned} \tag{3}$$

where $\rho(z)$ is the density of charges.

In the saddle point approximation, $Z_m \sim e^{-mE_0}$, with E_0 the minimum of the energy. The minimization consists in canceling the electric field at the position of the charges:

$$\frac{1}{2ml^2}\bar{z} - \int d^2 w \rho(w) \frac{1}{z-w} = 0 \tag{4}$$

Differentiating with respect to $\bar{z}$ using $\partial_{\bar{z}} z^{-1} = \pi \delta^2(z)$, we obtain a constant density inside the domain occupied by the charges: $\rho = 1/2\pi m l^2$. By rotational invariance the boundary of the domain must be a disc with a radius such that the integrated density is the total number of particles N_e: $R^2 = 2mN_e l^2$.[1]

Notice that the generic $m \neq 1$ case can be brought back to $m = 1$ by rescaling $\rho_m = \rho_1/m$, $z = \sqrt{m} z_1$ in (3), (4). So, semi-classically, the effect of m is simply to rescale by m the area without changing the shape.

We expect the physical properties of the system to be independent of the magnetic length l provided we keep the radius of the disc fixed. In other words, we are interested in the large N_e limit, keeping $l^2 N_e$ fixed (Some physical effects may depend on N though, we expect they can be apprehended in a $1/N$ expansion). In the

[1] The reader can verify that to the leading order in $1/N_e$, the energy is given by $N_e^2(3/4 - 1/2 \log m)$ [15].

following, we set $l^2 = 1/2N$, N being the magnetic flux through the disc and let $N \to \infty$ keeping $N_e/N = \nu$ fixed.

Consider now the partition function (2) in presence of a perturbation. Let us add a quadrupole to the background potential:

$$Z_m(t) = \int \prod_{i=1}^{N_e} d^2 z_i e^{-N(\bar{z}_i z_i + \frac{t}{2}(z_i^2 + \bar{z}_i^2))} \prod_{i<j} |z_i - z_j|^{2m} \tag{5}$$

This partition function was computed exactly for $m = 1$ in [15] with orthogonal polynomials (in Sect. 5 these polynomials will be described explicitly). Here, in order to illustrate the emergence of the edge physics we look at it from a semi-classical point of view and limit ourselves to $t \ll 1$.

Let us see how the quadrupole modifies the shape of the boundary. The change of shape can be modeled by a surface density proportional to the normal displacement. The electrostatic potential induced by this surface density must have the correct behavior $tz^2/4 + h.c.$ at infinity and vanish at the boundary $|z| = 1$, (the normalization is such that the potential between 2 unit charges at z, w is $\log|z - w|$, so, there is an additional factor $1/2$ with respect to (5)). This fully determines it to be twice the real part of $\phi(z) = t/4(z^2 - 1/z^2)$. The change of radius is then equal to the normal electric field at the boundary: $z\partial_z\phi(z) = t\cos(2\theta)$. To this approximation, it approaches the exact result, an ellipse [15] (we rederive this result in the Appendix) such that the ratio of its large axis over the small axis is $1 + 2t + O(t^2)$. $\partial_z\phi(z)$ having no simple poles inside the disc, the area is not modified by this deformation. Notice that this ellipse does *not* coincide with an equipotential of the background potential of (5) for which the ratio of the large over the small axis is $1 + t$.

Following [37], let us now consider the partition function (2) in presence of L external sources:

$$\frac{Z_m^L}{Z_m} = \frac{1}{Z_m} \prod_i^{N_e} \int d^2 z_i e^{-z_i \bar{z}_i N} \prod_{k=1}^{L} |z_i - v_k|^{2q_k} \prod_{i<j} |z_i - z_j|^{2m} \tag{6}$$

If all external charges q_k equal m,up to a global normalization, we are considering the diagonal part of the L-particles density matrix of a system of $N + L$ particles. Setting $q_k = m$ fixes the position of one of the particles at $z = v_k$ and so, m is the charge of the particles at positions z_k. Setting $q_k = 1$ creates a quasi-hole of fractional charge $1/m$.

To go beyond the semi-classical result we consider the case $m = 1$ with all external charges q_k equal to one. The correlation can be obtained exactly using orthogonal polynomial techniques [8, 26] with the following outcome. Define the kernel:

$$K_N(\bar{x}, y) = \sum_{p=0}^{N} (\bar{x}y)^p / h_p \tag{7}$$

with $h_n = n!/N^n$. The correlator (6) is given by:

$$\frac{Z_1^L}{Z_1} = \frac{h_N \dots h_{N+L-1}}{\prod_{i<j}(v_i - v_j)(\bar{v}_i - \bar{v}_j)} \left| K_{N+L-1}(\bar{v}_k, v_l) \right| \tag{8}$$

In the case where the charges are located outside the droplet, $|v_i| > 1$, the series (7) is dominated by the large powers of p and can be expanded in $1/N$ powers:

$$K_N(x, \bar{y}) = \frac{(x\bar{y})^{N-1}}{h_N} \left(\frac{1}{x\bar{y} - 1} - \frac{1}{N} \frac{1}{(x\bar{y} - 1)^3} + \cdots \right) \tag{9}$$

So, to the leading order, the partition functions is given by:

$$\frac{Z_1^L}{Z_1} = \prod_{i=1}^{L} (v_i \bar{v}_i)^N \prod_{i,j} \frac{1}{1 - (v_i \bar{v}_j)^{-1}} \tag{10}$$

To this order, it coincides with the Dyson gas correlator obtained by substituting in (6): $d\mu(z) = d^2 z \; e^{-z\bar{z}N} \to d\mu(z) = d\theta$ and integrating over the boundary of the disc.

By fusion, integer charges q_k can be deduced from this case, and the power -1 in each factor of the product must be replaced by $-q_i q_j$. Since this result coincides with the saddle point approximation, we can deduce from (3) the leading order in $1/N$ for m generic in (6) by multiplying the energy by the effective inverse temperature $E \to mE$ and rescaling the charges $q_i \to q_i/m$:

$$\frac{Z_m^L}{Z_m} = \prod_i (v_i \bar{v}_i)^{Nq_i} \prod_{i,j} \left(1 - \frac{R^2}{\bar{v}_i v_j}\right)^{-\frac{q_i q_j}{m}} \tag{11}$$

Here $R^2 = m$ is the square radius of charged disc when the filling fraction is $1/m$.

At the leading order we can approximate the Hall system by a perfect conductor interacting with the outside charges through its boundary only.[2] This suggests a connection between the Quantum Hall effect and boundary CFT which is further explored in [11].

[2]Setting $R = 1$, one can present the second factor in (11) as the expectation value of a product of Vertex operators:

$$\begin{aligned} \frac{Z_m^L}{Z_m} &= \langle 0| \prod_{k,i} \left(1 - \frac{1}{\bar{v}_k z_i}\right)^{q_k} \prod_{l,j} \left(1 - \frac{z_j}{v_l}\right)^{q_l} |0\rangle \\ &= \left\langle \prod_k e^{q_i \phi_+(\bar{v}_k^{-1})} \prod_l e^{q_j \phi_-(v_l)} \right\rangle \end{aligned} \tag{12}$$

In the second line, $\phi_\pm(z)$ are the positive and negative modes of the field $\phi(z) = \sum_{n\neq 0} z^n k \partial_{p_n}/n$. It is obtained by substituting $\sum_i z_i^k = p_k$ into the first line. To obtain the correlation, we use the commutation relations $[p_k, p_l] = k/m\delta_{k+l}$ to eliminate the modes p_k, $k < 0, k > 0$ which respectively annihilate the left and right vacuum.

3 Calogero-Sutherland Hamiltonian

An immediate consequence of what precedes is that if we use the boundary limit (leading $1/N$ behavior) of the measure (2) to define a scalar product on symmetric polynomials by:

$$\langle P, Q\rangle = \prod_i^N \int d\mu(z_i) \prod_{i<j} |z_i - z_j|^{2m} \bar{P}(z_i) Q(z_i) \tag{13}$$

and use the partition order of the basis m_λ to construct an orthogonal basis by a Gram-Schmidt procedure, we recover the Jack polynomials $J_\lambda^{1/m}$ in the large N limit.[3]

Another way to see the Jack polynomials emerge is to obtain the Laughlin wave function as an eigenstate of a Calogero-Sutherland (C.S.) Hamiltonian [9, 22]. Setting $m = \beta$ (real not necessarily integer) it is easy to verify that:

$$\left(\sum_i (z_i \partial_{z_i})^2 - 2\beta(\beta-1)\sum_{i<j} \frac{z_i z_j}{(z_i - z_j)^2} - E_0\right)\Psi_\beta(z_i) = 0 \tag{15}$$

with $E_0 = \beta^2 N(N-1)(2N-1)/6$.

This Hamiltonian acts on factorized wave functions $m(z_i)\Psi_\beta(z_i)$ with $m(z_i)$ a polynomial. Its energy eigenvalues are given by $\sum_{j=0}^{N-1} k_j^2$ with

$$k_j = \beta j + \lambda_{N-j} \tag{16}$$

where $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_N$ is a partition. The important property is that although individually k_i can take any integer values, collectively they are constrained to be separated by at least β.

At this point, we are not in position to assert that the Calogero-Sutherland Hamiltonian (15) can be regarded as an effective Hamiltonian representing the Coulomb interaction in the fractional QHE. Nevertheless, the collective behavior of the momenta described above is completely analogous to the vanishing conditions obeyed by the quantum Hall effect wave functions. For this reason mainly we argue that the Calogero-Sutherland model may be a good tool to study the fractional Hall effect.

---

[3] Alternatively, we could have defined our scalar product by:

$$\langle P, Q\rangle = \prod_i^N \left(\int d\mu(z_i)\right) \bar{P}(z_i) Q(z_i) \tag{14}$$

restricted to polynomials vanishing as $(z-w)^m$ when 2 variables approach each other. Such polynomials are known to generate an ideal of the Jack polynomials $J^{-2/(m-1)}(z_i)$ and it is clear that $J_\lambda^{1/m} \prod_{i<j}(z_i - z_j)^m$ is an ordered orthonormal basis of this ideal.

Other Q.H.E. wave functions (so called Rezayi-Read wave functions related to parafermions) are obtained by requiring that the polynomial vanishes as $(z-w)^m$ when $k+1$ particle approach each other. Such polynomials are known to generate an ideal of the Jack polynomials $J^{-(k+1)/(m-1)}(z_i)$ [14], I do not know of a description of their norm in the limit of large N.

To see this property, it is convenient to conjugate it (with the energy term substracted) by $\Psi_\beta(z_i)$ to obtain its action on the prefactor:

$$\mathcal{H}^\beta = \sum_i (z_i \partial_{z_i})^2 + 2\beta \sum_{i \neq j} \frac{z_i^2}{z_i - z_j} \partial_{z_i} = D_{0,2} + 2N\beta D_{0,1}, \tag{17}$$

where $D_{0,1} = \sum_i z_i \partial_{z_i}$ is the angular momentum which commutes with $\mathcal{H}^\beta$ and $D_{0,2}$ can be diagonalized in each sector of angular momentum. Consider the following basis indexed by partitions $m_\lambda = \sum_\sigma z_1^{\lambda_{\sigma_1}} \dots z_N^{\lambda_{\sigma_N}}$, where the sum is restricted to distinct permutations σ of λ_i [25]. In this basis, ($D_{0,1} = |\lambda|$), there is a partial order on partitions for which $D_{0,2}$ is triangular [35]. Its eigenvalues are the diagonal matrix elements given by $\sum_k \lambda_k(\lambda_k - 2\beta k)$ and the corresponding eigenvectors are proportional to the Jack polynomials: $J^\alpha(z_i)$ with $\alpha = 1/\beta$.[4]

In (17), it is convenient to expand $\mathcal{H}$ in powers of $1/N$ so that the coefficient in the expansion depend on the partition λ and are independent of N.

One remarkable property of the Calogero-Sutherland model is that there exists an infinite set of commuting quantities E_n with eigenvalues $\sum_{j=0}^{N-1} k_j^n$, with k_j given by (16). The N^0 coefficient of these conserved quantities denoted $D_{0,n}$ are a $\beta \neq 1$ deformation of the generators $\sum_i (z_i \partial_i)^N$ of the W_∞ algebra [21].[5] The deformation of W_∞ related to the Calogero-Sutherland model has been pursued by Shiraishi and his collaborators [5, 6], see also [34] for a more mathematical approach. I have borrowed from this last paper the notation of the generators $D_{m,n}$ which can be thought as the quantization of $z^m(z\partial_z)^n$.

At this point, we make a digression to interpret β as a Luttiger parameter. For a free field compactified on a circle of radius 2π with the action $\frac{g}{4\pi}\int |\partial\phi|^2 dz$, g is called the Luttinger parameter and has a deep physical meaning. Its partition function on a cylinder with aspect ratio τ is due to the classical contribution of harmonic periodic fields times the quantum fluctuations: $Z_c = \frac{1}{\eta\bar{\eta}} \sum_{n,m} q^{\Delta_{n,m}} \bar{q}^{\Delta'_{n,m}}$ with $q = e^{i\pi\tau}$ and $\Delta_{n,m} = g(n + m/g)^2/4$, $\Delta'_{n,m} = g(n - m/g)^2/4$. One can compare this spectrum with the Calogero spectrum, if one identifies the winding number n with the number N of particles and the momentum m with the current $\sum_i z_i \partial_i$. To match the macroscopic dispersion relations around the Fermi momentum $k_f = N\beta$

[4]Mathematicians use the inverse convention as physicists $\alpha = 1/\beta$, and the Jack polynomials eigenstates of $\mathcal{H}^\beta$ are denoted $J^{1/\beta}(z_i)$.

[5]Let us give a qualitative explanation of why we expect the W_∞ algebra (also called Girvin-Macdonald-Plazman algebra in this context) to arise in the QHE [10, 17]. If we couple the system to an external probe $V(x)$, in degenerate perturbation theory, one needs to project this potential into the LLL. Denoting the lowest Landau level orbitals u_α $1 \leq \alpha \leq d_N$ (d_N the dimension of the LLL Hilbert space), the effect of this projection is to convert the potential into a matrix: $V(x) \to V_{\alpha\beta} = \langle\alpha|V|\beta\rangle = \int \bar{u}_\alpha(x) u_\beta(x) V(x)$. In this way, by an appropriate choice of V's, we generate the space of $d_N \times d_N$ matrices acting inside the LLL which we view as the Lie algebra is $gl(d_N)$. The W_∞ algebra occurs as the limit of $gl(d_N)$ when we let N tend to infinity. So for example, in the symmetric gauge, we can take the $V = \bar{z}^n z^m$ $m, n \geq 0$ which after integrating (2) by part become $(2l^2\partial_z)^n z^m$.

when we modify the particle number and the current with the linearized free field classical contribution ($\delta_{n,m} \propto n + m/g$), the comparison with (16) imposes to identify g with β.

We can allow the area (equivalently the degree of the polynomial) to increase while keeping the number of particles fixed. This is realized by placing voids where vanishing condition are imposed to the wave function compatible with its polynomial nature. As we saw above, it can be achieved by inserting a fractional $1/\beta$ charges at positions x_i^{-1}, equivalently, by multiplying Ψ_β by the kernel:

$$K(x_i, z_k) = \prod_{i=1}^{M}\prod_{i=1}^{N}(1 - x_i z_k) \tag{18}$$

The kernel K obeys the duality equation:

$$\mathcal{H}^{\beta}(z) + \beta\mathcal{H}^{\beta^{-1}}(x)K(x, z) = 0, \tag{19}$$

enabling to separate the variables:

$$K(x, z) = \sum_{\lambda} N_\lambda J_\lambda^\alpha(z) J_{\lambda'}^{\alpha^{-1}}(x) \tag{20}$$

with λ, λ' two conjugated partitions and λ contained in a rectangle with N rows M columns. Therefore, the quasi-hole dynamics decouples from the particles and is governed by a $1/\beta$ C.S. Hamiltonian.

Let us rewrite (17) in the collective variables coordinates introduced above. We set $\sum_i z_i^k = p_k$, $k \geq 1$, then [2, 5, 20]:

$$\begin{aligned} D_{0,1} &= \sum_{n\geq 1} p_n n\partial_{p_n} \\ D_{0,2} &= (1-\beta)\sum_{n\geq 1} np_n n\partial_{p_n} + \sum_{n,l\geq 1} p_{n+l} n\partial_{p_n} l\partial_{p_l} + \beta p_l p_n (n+l)\partial_{p_{l+n}} \end{aligned} \tag{21}$$

If we introduce the potential ϕ with $y = z\partial_z\phi(z)$ and $\phi(z) = p_0 + N\log(z) + \sum_n z^n p_n/n$, ($p_{-n}$ defined by: $p_{-n} = \beta^{-1}z^{-n}\partial_{p_n}$ for $n > 0$), and the zero mode p_0 is conjugated to N, $e^{p_0}N = (N+1)e^{p_0}$. The conserved quantities can be expressed in terms of the electric field $y(z) = z\partial_z\phi(z)$

$$\begin{aligned} \mathcal{L} &= \frac{\beta}{2}\int d\theta : y^2 : \\ \mathcal{H}^\beta &= \beta^2\int d\theta : \frac{\beta^{-1}-1}{2} y H(\partial_\theta y) + \frac{1}{3}y^3 : \end{aligned} \tag{22}$$

$z = e^{i\theta}$ and $H(z^n) = \epsilon(n)z^n/i$ is the Hilbert transform. The normal order symbol means that ∂_{p_k} sits at the right of p_l for $k, l > 0$.

This Hamiltonian is hermitian for $p_n^+ = \beta^{-1}n\partial_{p_n}$, $n\partial_{p_n}^+ = \beta p_n$, so ϕ is antihermitian $\phi^+ = -\phi$. Moreover, it obeys a duality relation reflecting (19):

$$\mathcal{H}^\beta(p_n, n\partial_{p_n}) + \beta\mathcal{H}^{\beta^{-1}}\left(-\beta p_n, -\beta^{-1}n\partial_{p_n}\right) = 0 \tag{23}$$

It can be diagonalized in each angular momentum sector $|\lambda|$ and its eigenvectors are the Jack polynomials J_λ^α expressed in the p_n basis.

In particular, if we take one quasihole $M = 1$, the kernel $K(x, z_i)$ is identified with vertex operator creating a hole: $V(x,t) = e^{-\phi_+(x)}|0\rangle$ [5]. The coefficient of x^k displaces the k outer orbitals by one unit, and therefore create a quasihole of charge $-\beta^{-1}$ inside the sea. By duality, the coefficients of the expansion of $e^{\beta\phi_+(x)}|0\rangle$ are eigenstates of charge 1.

In the classical limit $\beta \to \infty$ and in the flat limit where sums over n are replaced by integrals over k, the classical Hamiltonian is:

$$\mathcal{H}_c = \int -\frac{1}{2} y H y_x + \frac{1}{3} y^3 dx \tag{24}$$

$H(e^{ikx}) = \epsilon(k)e^{ikx}/i$ being the Hilbert transform. This Hamiltonian was introduced by Benjamin and Ono (B.O.) [7, 29] to describe the dynamics of waves in deep water. From what preceeds, it should also describe the dynamics of the edge deformations (Nl playing the role of the depth).[6] Namely, we expect the field $y(\theta)$ to describe the fluctuations of the disc boundary, or if we conformally map the disc to the lower half plane, the fluctuations about $y(x) = 0$ (see [1] for a more complete discussion and possible generalizations. Recently P. Wiegmann [38] has given arguments to justify the occurrence of this equation in the Hall effect context).

The Poisson bracket of the field y is given by:

$$\{y(x), y(x')\} = -\delta'(x - x') \tag{25}$$

from which we obtain the dynamics it satisfies:

$$\dot{y} - 2yy_x + Hy_{xx} = 0 \tag{26}$$

This equation can be interpreted as an Euler equation ($\dot{y} + v(y)y_x = -p_x$), with a speed $v(y)$ varying linearly with the density y through $v(y) = V_0 - 2y$ where V_0 can be eliminated by a Galilean transformation ($x \to x + V_0 t,\ v(y) \to v(y) - V_0$). The pressure term Hy_x is nonlocal (without the Hilbert transform, the corresponding term in (24) would be a pure derivative). It is the normal derivative of a harmonic function in the lower half plane equal to y at the boundary.

The (imaginary) potential $\phi(x) = -i \int^x y(u)du$ satisfies $i\dot{\phi} = y^2 - Hy_x$. If we forget about the Hilbert transform, denoting $\phi_\pm$ the projection of ϕ on its positive or negative Fourier modes, the hole vertex operator $V_\pm(x,t) = e^{\pm\phi_+}$ satisfies the free equation: $\dot{V} = \pm i V_{xx}$ which has the Galilean solution: $V_\pm = e^{i(kx \mp k^2 t)}$.

[6] In [12], we extended these observations to bulk wave functions described by a correlation function of ϕ_{12} of the Ising conformal field theory which describes the Pfaffian state [27] vanishing when three particles approach the same point [18]. The primary fields $\phi_{21}(x_k)$ represents the quasiparticle [28]. Both the particle and the quasiparticle fields need to be dressed by a bosonic vertex operator in order to make them mutually local (Their OPE needs to be regular instead of the square root singularity of $\phi_{12}.\phi_{21}$). The kernel K is then given a correlator of dressed $\phi_{12}(z_i), \phi_{21}(x_k)$ fields and is also a solution of (19) giving rise to a rank two generalization of the Benjamin Ono Hamiltonian also appearing in [3].

A stationary real solution is given by:

$$\phi = \log \frac{x - vt - 2i/v}{x - vt + 2i/v} \tag{27}$$

The Lorentzian shape of this soliton should be contrasted with the exponential tail of the KdV soliton which occurs for the local dispersion $p = -y_{xx}$.

Let us attempt to make a connection to the edge excitations in the Hall effect. Approach a unit charge at distance v^{-1} from the boundary $y = 0$. Together with its mirror image, it creates a dipole moment $2v^{-1}$ and the induced potential is equal to (27) at $t = 0$. Since the electric attraction between the charge and its image is v, assuming there is a constant magnetic field of unit strength, the dipole must move at speed v for the electromagnetic repulsion to compensate the attraction in agreement with (27).

4 On the Sphere and on the Torus

It can be useful to consider Quantum Hall wave functions on surfaces without boundary such as the sphere (CP^1) or the torus.

4.1 Sphere

We place N magnetic fluxes at the center of the unit sphere, and construct our wave functions as a product of LLL orbitals.

The sphere is a Kähler manifold with metric $\mu dz d\bar{z}/\pi$ with $\mu = 1/(1 + z\bar{z})^2$. We consider the line bundle (L, h), h being the fiber metric with curvature equal to the area form: $\mu = -\partial_z \partial_{\bar{z}} \log(h)$, $h = 1/(1 + z\bar{z})$.

Putting N fluxes amounts to raise L to the power N. In this way, we get an inner product on holomorphic sections of L^N (the LLL wave functions) which are polynomials of degree N:

$$\langle \psi | \psi' \rangle = \int \frac{dz d\bar{z}}{\pi (1 + \bar{z}_i z_i)^{2+N}} \bar{\psi}(z) \psi'(z) \tag{28}$$

An orthonormal basis is given by the orbitals:

$$\psi_n = \frac{1}{N+1} \binom{N}{n} z^n, \quad 0 \le n \le N \tag{29}$$

As for the plane, the Laughlin wave function is obtained by raising the Slater determinant of the $N + 1$ LLL orbitals to the power m. On the other hand, the partition function at inverse temperature m of a Coulomb gas of N_e particles interacting with an electrostatic potential equal to the logarithm of the cord distance is given by:

$$Z_m = \prod_{i=1}^{N_e} \int \frac{d^2 z_i}{(1 + \bar{z}_i z_i)^{2+m(N_e - 1)}} \prod_{i<j} |z_i - z_j|^{2m} \tag{30}$$

To identify this partition function with the squared norm of the Laughlin wave function, we must have $m(N_e-1)=N$.

The shift S ($S=m$ for the Laughlin wave function) relates the magnetic flux to the number of particles through the formula:

$$\frac{N_e}{\nu}=N+S \tag{31}$$

It can be thought as the residual area left when the sphere is maximally occupied. Recently, it has been recognized as an important quantity [33], related to the Hall viscosity [4] see Sect. 5.

In the simple case $m=1$, the wave function is the antisymmetrized product of the LLL orbitals and the shift relates the dimension of the LLL Hilbert space $N_e=\dim H^0(CP^1,T^N)$ to the flux through the sphere. This relation also results from the expansion of the density in powers of $1/N$:

$$\rho(z,\bar{z})=N+\frac{R}{2} \tag{32}$$

where $R=2$ is the curvature of the sphere, and in that case the shift coincides with the first Chern number of (L,h). For a Hall effect defined on a Riemann surfaces, (32) has an obvious generalization, and it should be possible to give S defined by (31) a more local characterization.

Taking a product of two wave functions amounts to construct a composite particle by fusionning the particles of each wave function. The inverse filling fraction has the interpretation of the charge (amount of area carried by a particle) and half the shift is the spin. Under composition, these quantities add up. So, taking the product of wave functions, the shift and the inverse filling fraction behave additively:

$$\begin{aligned}\nu^{-1}(\psi_1\psi_2)&=\nu^{-1}(\psi_1)+\nu^{-1}(\psi_2)\\ S(\psi_1\psi_2)&=S(\psi_1)+S(\psi_2)\end{aligned} \tag{33}$$

4.2 Cylinder and Torus

The cylinder is parametrized by $z=x+\tau y$, with $\tau=\tau_1+i\tau_2,\ \tau_2>0, 0\le x\le 2\pi$. The metric is $(dx+\tau dy)(dx+\bar{\tau}dy)$. From it, we obtain the Hamiltonian with N fluxes per unit area (measured with $dx\wedge dy$):

$$\mathcal{H}=g^{ij}\pi_i\pi_j=|\tau|^2\pi_x^2+\pi_y^2-\tau_1(\pi_x\pi_y+\pi_y\pi_x) \tag{34}$$

where (in the Landau gauge) $\pi_x=-i\partial_x+Ny$, $\pi_y=-i\partial_y$ and g^{ij} is the inverse metric matrix. The LLL eigenfunctions on the infinite cylinder periodic in $x\to x+2\pi$ are defined for k integer. They factorize into a nonholomorphic part independent of k equal to $e^{i\frac{N\tau}{2}y^2}$ (the square of which defines the fiber metric h^N), and a holomorphic part:

$$\psi_k(z)=e^{i\frac{k^2\tau}{2N}}e^{ikz} \tag{35}$$

We construct a basis of N LLL wave functions $0 \le k_0 \le N-1$ on the torus $y = y+1$ by summing ψ_k over $k = k_0$ modulo N:

$$\psi_{k_0}(z) = \sum_{k=k_0[N]} e^{i\frac{k^2\tau}{2N}} e^{ikz} \tag{36}$$

The torus Hilbert space is characterized by the quasi-periodicity conditions:

$$\begin{aligned} \psi(z+2\pi) &= \psi(z) \\ \psi(z+\tau) &= e^{-iNz} e^{-iN\frac{\tau}{2}} \psi(z) \end{aligned} \tag{37}$$

Multi-particle states are obtained as linear combinations of a products of LLL single particle wave functions. The Laughlin wave functions with filling fraction $1/m$ are the multi-particle states with the maximal number of variables which vanish as $(z_i - z_j)^m$ at short distance. For N a multiple of m, $N = mN_e$ ($S = 0$ in (31) as expected) this space is m dimensional. An exact expression can be given [19]:

$$\Psi_m(z_i) = F\left(\sum_{i=1}^{N_e} z_i\right) \prod_{i<j}^{N_e} \theta_1(z_i - z_j)^m \tag{38}$$

where,

$$\theta_1(z) = \sum_{n \in Z+1/2} (-)^{n+1/2} q^{n^2} e^{inz} \tag{39}$$

with $q = e^{i\frac{\tau}{2}}$. It obeys the quasi-periodicity conditions:

$$\begin{aligned} \theta_1(z+2\pi) &= -\theta_1(z) \\ \theta_1(z+\tau) &= -q^{-1} e^{-iz} \theta_1(z) \end{aligned} \tag{40}$$

F belongs to the m dimensional space of sections characterized by the quasi-periodicity conditions:

$$\begin{aligned} F(z+2\pi) &= (-)^{N-m} F(z) \\ F(z+\tau) &= (-)^{N-m} e^{-imz} e^{-im\frac{\tau}{2}} F(z) \end{aligned} \tag{41}$$

Let us see that as in the case of the plane, the wave function can be set as the solution of a Hamiltonian. The $m = 1$ case is instructive and shows that it a the solution of a modified Calogero-Sutherland Hamiltonian. From (36), we see that all ψ_k are solutions of the same heat equation: $(Nqd/dq + d/dz^2)\psi(z) = 0$. Therefore, their exterior product is a solution of the modified Hamiltonian:

$$\left(Nq\partial_q + \sum_i \partial_{z_i}^2\right) \Psi_1(z_i) = 0 \tag{42}$$

Clearly, the C.S. must be modified by a derivative with respect to τ. Introducing the periodic potential with a double pole at the origin:

$$V(z) = -\partial_z^2 \log \theta_1(z) \tag{43}$$

one can show that Ψ_m is an eigenstate of $\mathcal{H}^\beta \Psi_\beta(z_i) = 0$:

$$\mathcal{H}^\beta = -N\beta q \partial_q - \sum_i \partial_{z_i}^2 + \beta(\beta-1)\sum_{i\neq j} V(z_i - z_j) \tag{44}$$

The modified Hamiltonian (44) was considered in [13, 23], and the fact that Ψ_m is an eigenstate of (44) essentially results from the following identity [23]: Define $\phi(z) = \partial_z \log\theta_1(z)$, then:

$$2\phi(z_1 - z_2)\phi(z_2 - z_3) + \frac{q\partial_q \theta_1(z_3 - z_1)}{\theta_1(z_1 - z_2)} + \text{cycl} = 3c \tag{45}$$

Using the heat equation $-q\partial_q\theta_1(z) = \partial_z^2\theta_1(z)$ (which follows from (39)), one is led to show that

$$\begin{aligned} &\big(\phi(z_1 - z_2) + \phi(z_2 - z_3)\phi(z_3 - z_1)\big)^2 \\ &\quad + \big(\phi'(z_1 - z_2) + \phi'(z_2 - z_3) + \phi'(z_3 - z_1)\big) = 3c. \end{aligned} \tag{46}$$

Both terms of the sum (46) are doubly periodic in z_1 and have a residue respectively equal to ± 1 at the double poles $z_1 = z_2, z_3$. ϕ being odd, there are no single poles, so, the sum must be a constant and c is the z coefficient of ϕ: $c = -1/12 + 2\sum_1^\infty q^{2n}/(1-q^{2n})^2$.

Observe that the factor F does not change the argument because it obeys the heat equation, but is crucial to insure that the wave function obeys the proper quasiperiodicity conditions.

Remarkably, as in the case of the plane, quasihole excitations can be also recast as solutions of the C.S. Hamiltonian. The identity similar to (19) [23]:

$$\big(\mathcal{H}^\beta(z) + \beta\mathcal{H}^{\beta^{-1}}(x)\big)K(x,z) = E_{N,M}K(x,z) \tag{47}$$

with the kernel now given by:

$$K(x,z) = \prod_{i<j}^{N_e}\theta_1(z_i - z_j)^\beta \prod_{k<l}^{M}\theta_1(x_k - x_l)^{\frac{1}{\beta}} \prod_{i=1}^{N_e}\prod_{k=1}^{M}\theta_1(x_k - z_j) \tag{48}$$

5 Some Remarks on Viscosity

Let us briefly review some recent results the viscosity mostly following [4, 32, 33, 36] which relates a purely geometric aspect, the Berry curvature, to the Coulomb partition function (2) when it is reinterpreted as the norm of a Laughlin wave function.

The viscosity tensor expresses the linear response of the stress tensor to a deformation of the substrate parametrized by its moduli. For example, the Q.H.E. ground state torus wave function $\Psi_m(z_i)$ is also a holomorphic section of a line bundle on the moduli space $(H^+/SL_2(Z))$ parametrized by τ. According to Avron Seiler and Zograff, the (non dissipative) viscosity is related to the Berry curvature:

$$\mu = -\partial_\tau \partial_{\bar{\tau}} \log Z_m d\tau \wedge d\bar{\tau}. \tag{49}$$

In the $m = 1$ case, Ψ_1 is the exterior product of LLL sections (37) which are normalized with $(\tau_2 N)^{1/4}$, where $\tau_2 = Im(\tau)$ is the area of the torus. Therefore, the area dependence is $Z_1 \propto \tau_2^{N_e/2}$, and $\mu = N_e/2$, times the curvature of the upper half plane H^+.

For $m \neq 1$ it turns out that μ remains unchanged. The argument can be summarized as follow, keeping the area fixed $N_e \to N'_e = N_e/m$, and from (38) $Z_m \propto \tau_2^{mN'_e/2} = Z_1$.

Geometrically, we can think of the wave function bundle over the torus moduli space as closely related to the bundle $SL_2(R) \to H^+ = SL_2(R)/SO(2)$ raised to the power $mN_e/2$. Each particle takes a phase $e^{i\theta m/2}$ under the $SO(2)$ rotation which gives to $m/2$ the interpretation of a spin. In general, Read argues [32] that μ scales as $N_e S/2$ where S is the shift, which gives the shift S the status of a spin which characterizes a Hall state.

To confirm the universality of this result, let us repeat the argument for the droplet and verify that the same conclusions apply to an open system. We deform the complex structure $z \to x = z - t\bar{z}$ and consider the Lagrangian:

$$\mathcal{L} = |\dot{z} - t\dot{\bar{z}}|^2 - i(\bar{z}\dot{z} - z\dot{\bar{z}}) \tag{50}$$

Let us see that the Q.H.E. droplet ground state wave function $\Psi(z_i)$ is a holomorphic section over the Poincaré t-disc. In terms of deformed oscillators:

$$a_t = a - ta^+, \qquad a_t^+ = a^+ - \bar{t}a \tag{51}$$

where $a = \partial_{\bar{z}} + z$, $a^+ = -\partial_z + \bar{z}$, the Hamiltonian is a harmonic oscillator:

$$\mathcal{H} = \frac{1}{(1 - t\bar{t})^2} a_t^+ a_t \tag{52}$$

The undeformed oscillators $b = \partial_z + \bar{z}$, $b^+ = -\partial_{\bar{z}} + z$ commute with a_t, a_t^+. From the ground state annihilated by a_t and b:

$$\psi = e^{-z\bar{z} + t\bar{z}^2} \tag{53}$$

one generates the lowest Landau orbitals by acting with b^+. Remarkably, after factorizing ψ, they are polynomials in the orthogonal variables $x = z - t\bar{z}$ generated from 1 (the 0^{th} orbital) by the action of:

$$B^+ = x + \frac{t}{2}\partial_x. \tag{54}$$

In this way, we obtain a basis of orthogonal polynomials for the measure

$$|\psi|^2 d^2 z = e^{-\frac{2x\bar{x} + t\bar{x}^2 + \bar{t}x^2}{1 - t\bar{t}}} \frac{d^2 x}{(1 - t\bar{t})}, \tag{55}$$

which are given by Hermite polynomials, thus, recovering the result of [15].

Following [15], we can use a saddle point approximation to estimate the shape of the Landau orbitals. The outcome is that the droplet shape equation is not modified in the z variable and given by: $|z|^2 = N_e/2$. It corresponds to an ellipse in the orthogonal variables $x, \bar{x}$. For the sake of completeness, we derive this result in the Appendix.

To obtain the curvature, we observe that the LLL orbitals are holomorphic in t, the square of their norm is $n!Z_t$ where Z_t is the 0^{th} orbital square norm. So, the total curvature is N times the contribution of each orbital. We have:

$$Z_t = \int |\psi|^2 d^2 z = Z_0 \sqrt{1 - t\bar{t}} \tag{56}$$

From which we deduce that the Berry curvature is $N_e/2$ times the curvature of the Poincaré t-disc.

The argument can be generalized to a Laughlin wave function if we assume that the deformed wave function is given by $\prod_{i<j}(x_i - x_j)^m \prod_i \psi(z_i)$ which is the lowest degree polynomial in x_i vanishing as ϵ^m when two variables approach each other. Its square norm coincides with the Coulomb plasma at temperature $1/m$ in presence of a quadrupole field at infinity. Although quite natural, this form of the wave function has been questioned recently [31].

Appendix: Shape of the Droplet

Let us obtain the shape of the droplet with a potential given by (55) using orthogonal polynomials. Here we take t to be real: $t = \tanh(\mu)$. Making the rescaling:

$$y = \cosh(\mu)x \tag{57}$$

the measure (55) simplifies to:

$$e^{-2y\bar{y} + t(\bar{y}^2 + y^2)} d^2 y \tag{58}$$

and the creation operator (54) becomes:

$$\begin{aligned} b^+ &= y + \alpha \partial_y \\ b &= \partial_y \end{aligned} \tag{59}$$

with $\alpha = \sinh(2\mu)/4$. The LLL orbitals obey $b^+ b \psi_n = n\psi_n$. In the WKB approximation, setting $\psi_n = e^{\int^y p dy}$, one obtains for p:

$$\alpha p^2 + yp - n = 0 \tag{60}$$

Putting

$$y = 2\sqrt{n\alpha} \sinh \phi \tag{61}$$

the admissible solution is given by $p = \sqrt{n/\alpha} e^{-\phi}$, $\psi_n = e^{n(\phi - \frac{e^{-2\phi}}{2})}$.

The square norm of the orbitals becomes:

$$\int e^{-2y\bar{y} + t(\bar{y}^2 + y^2) + 2n\Re(\phi - \frac{e^{-2\phi}}{2})} d^2 y \tag{62}$$

Setting $e^{-\phi} = u$, $e^{-\bar{\phi}} = v$, one obtains the saddle point equations:

$$u - u^{-1} + v/t - tv^{-1} = 0$$
$$v - v^{-1} + u/t - tu^{-1} = 0$$

with the solution $uv = t$, or $|e^{-\phi}|^2 = \tanh(\mu)$.

From (61), the orbital maxima lie on the curves:

$$y_n(\theta) = \sqrt{\frac{n}{2}}\big(e^{-\mu}\cos(\theta) + e^{\mu}\sin(\theta)\big) \tag{63}$$

So, the droplet has a constant density and is bounded by the domain delimited by y_{N_e}.

Since $y = \cosh(\mu)z - \sinh(\mu)\bar{z}$, in the z variables, the boundary is given by $|z| = \sqrt{N_e/2}$.

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Ordinary Differential Equations on Rational Elliptic Surfaces

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Abstract Corresponding to Oguiso-Shioda's classification of rational elliptic surfaces, we give 2nd order algebraic ordinary differential equations which can be solved by elliptic functions, in the form of the Hamiltonian system. There is a criterion for determining the types of rational elliptic surfaces from given biquadratic Hamiltonian systems. We also discuss about Bäcklund transformations which is different type from transformations that appear in a study of the Painlevé equations.

1 Introduction

The Painlevé equations were found by P. Painlevé and his coworkers through an effort to classify 2nd order algebraic ordinary differential equations of normal form with the Painlevé property (see [1, 11]). We say that the equation has the Painlevé property if it does not have movable singular points except poles. Their way of classification is as follows: In the first place, make a list of 50 equations possessing the Painlevé property by doing away with equations which has movable branch points; in the second place, remove equations which can be solved by quadrature in terms of elementary functions, or can be solved by using elliptic functions, or solutions of linear differential equations; then, there remain only new equations called the Painlevé equations.

In their list of 50 equations, there are many equations whose solutions can be expressed in the terms of elliptic functions, and the form of the equations looks like that of the Painlevé equations. Many of them can be obtained as autonomous limit from the Painlevé equations. On the other hand we know a correspondence between the Painlevé equations and some kind of rational surfaces. When we use the terminology, these equations in the Painlevé-Gambier's list, whose solutions are expressed by the term of elliptic functions, are found to correspond to rational elliptic surfaces.

Dedicated to Professor M. Jimbo on his 60th birthday.

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For example, the 8th equation in Gambier's paper [1] is written as

$$\frac{d^2y}{dt^2} = 2y^3 + \beta y + \gamma,$$

and it is an autonomous limit of the second Painlevé equation. This equation corresponds to the 43rd rational elliptic surface in the list of Oguiso-Shioda [6], except some particular parameters.

In the classification of Painlevé and Gambier the terminology of algebraic geometry is not used and they depend on the specific expression of differential equations for the classification. However a simple transformation of dependent variable make a change of the appearance of the equations, so an application of geometric point of view to a classification would be preferable. In fact a classifications of the six classical Painlevé equations was rewritten into a classification consist of eight types of Painlevé equations in the light of geometry. In this paper we shall give ordinary differential equations with the Painlevé property, whose solutions can be expressed by the terms of elliptic functions, in the form of the Hamiltonian system, on the basis of Oguiso-Shioda's classification of rational elliptic surfaces.

These ordinary differential equations with the Painlevé property, possessing elliptic functions solutions, did not spur wide interest of many researchers, since they can be solved simply by very well known functions. However, as in the paper of K. Kajiwara et al. [2], the Hamiltonian functions of the Painlevé equations are written in the terms of the elliptic curves which appear as fibers there, it shows that the terminology of rational elliptic surfaces is effective also for a study of the Painlevé equations. Besides, rational elliptic surfaces is richer in diversity than spaces of initial conditions for the Painlevé equations, and we can also see the intriguing phenomena about different types of Bäcklund transformations which do not appear in the case of the Painlevé equations.

The text is organized as follows: In the next section we will see a correspondence between rational elliptic surfaces and 9 points configurations, including infinitely near points, in $\mathbb{P}^2$, or 8 points configurations in $\mathbb{P}^1 \times \mathbb{P}^1$. To give such a points-configuration is the same thing as to give a rational elliptic surface. This section is a review on a result in the paper [13], which we need in this article. In the third section we construct Hamiltonian systems on the rational elliptic surfaces on the basis of these data. In the fourth section we will consider a normal form of Hamiltonian systems, using biquadratic forms. While we study a construction of equations from the data of surfaces in the previous section, in this section we will start from the normal forms of Hamiltonian and determine which type of a surface corresponds to the Hamiltonian inversely. As the biquadratic forms are expressed by 3×3 matrices, we will see a classification in the terms of 3×3 matrices. In the fifth section we see cases of plural singular fibers in detail, by taking $D_5^{(1)}$ as an example. In the sixth section we will see a kind of Bäcklund transformations which change the types of the normal form of the Hamiltonians. These types of Bäcklund transformations do not appear in a study of the Painlevé equations.

2 Review on Construction of Rational Elliptic Surfaces

K. Oguiso and T. Shioda classified rational elliptic surfaces in the paper [6]. In the wake of the classification by Oguiso and Shioda, we treat only rational elliptic surfaces which admit a section in this paper. On the other hand, each rational elliptic surface can be constructed by 9 points blowing-ups from $\mathbb{P}^2$. Hence to give a configuration of 9 points modulo the action of PGL_3, including the case of infinitely near points, is the same thing as to give a rational elliptic surface. We would like to give such data of configuration for each elliptic surface in this section.

In fact we already have such data for more general surfaces, which we call generalized Halphen surfaces, in a previous paper [13]. Only a specialization of parameters is enough to get information about all of rational elliptic surfaces.

However description of generalized Halphen surfaces is based on one fixed anti-canonical divisor. Therefore we need to fix one fiber among elliptic curves or its degenerations as a ruling divisor. Any rational elliptic surface has at least one singular fiber, so we fix one of singular fibers as a ruling divisor. When there are more than one singular fibers, we have different blowing-down structures depending on the choice of the ruling divisor.

A classification of singular fibers for elliptic surfaces are given by K. Kodaira [3, 4]. The list of singular fibers which appear on rational elliptic surfaces is as follows:

$A_0^{(1)*}$, $A_1^{(1)},\ldots,A_8^{(1)}$	$A_0^{(1)**}$, $A_1^{(1)*}$, $A_2^{(1)*}$	$D_4^{(1)},\ldots,D_8^{(1)}$	$E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$
I_1, I_2, …, I_9	II, III, IV	I_0^*, …, I_4^*	IV^*, III^*, II^*

The symbols in the lower row are Kodaira's original symbols, and the upper symbols denote Dynkin diagrams corresponding to the intersection form of the singular fiber. We will use the symbols of Dynkin diagrams mainly.

Here we pay attention to the fact that we have two different realizations of singular fibers of type $A_7^{(1)}$. The Picard lattice of a rational elliptic surface is isomorphic to Lorentzian lattice of rank 10, and the orthogonal complement of each irreducible components of the singular fiber of type $A_7^{(1)}$, become a sublattice in the Picard lattice. We denote it as $A_7^{(1)}$ when a generator of the sublattice has -8 as self-intersection, and we denote it as $A_7^{(1)\prime}$ when a generator has -2 as self-intersection.

Remark 1 We consider only one singular fiber above, but when we consider the list of Oguiso-Shioda's classification of rational elliptic surfaces, there are 5 cases that have two different embedding into the Picard lattice:

$$A_1^{\oplus 4}, \qquad A_3 \oplus A_1^{\oplus 2}, \qquad A_5 \oplus A_1, \qquad A_3^{\oplus 2}, \qquad A_7.$$

$A_1^{\oplus 4}$ corresponds to 13th and 14th surfaces; $A_3 \oplus A_1^{\oplus 2}$ corresponds to 21st and 22nd surfaces; $A_5 \oplus A_1$ corresponds to 28th and 29th surfaces; $A_3^{\oplus 2}$ corresponds to 35th and 36th surfaces; A_7 corresponds to 44th and 45th surfaces.

What it comes down to is that, for a given rational elliptic surface, we can construct a blowing-down to $\mathbb{P}^2$ when we fix a ruling divisor, and then we obtain 9 points configuration possibly including infinitely near points. To give such a 9 points configuration in $\mathbb{P}^2$ is the same thing as to give a rational elliptic surface, and we can use data of configurations already obtained in the previous paper [13].

Here we consider a blowing-down to $\mathbb{P}^2$, but we also consider a blowing-down to $\mathbb{P}^1 \times \mathbb{P}^1$ except $E_8^{(1)}$ type surface. It would be better because we can express an elliptic curve by a biquadratic form and then we can use calculation of 3×3 matrices. We will give data of 9 points configuration in $\mathbb{P}^2$ which is same as in the paper [13] and data of 8 points configuration in $\mathbb{P}^1 \times \mathbb{P}^1$ below. Both of them may include infinitely near points.

1. the case of $E_8^{(1)}$ type singular fiber

We give data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ as follows:

$$p_1 : (0:1:0) \leftarrow p_2 : (x/y, z/x) = (0,0) \leftarrow p_3 : \left(x/y, yz/x^2\right) = (0,0)$$

$$\leftarrow p_4 : \left(\frac{x}{y}, \frac{y^2 z}{x^3}\right) = (0,1) \leftarrow p_5 : \left(\frac{x}{y}, \frac{y(y^2 z - x^3)}{x^4}\right) = (0,0)$$

$$\leftarrow p_6 : \left(\frac{x}{y}, \frac{y^2(y^2 z - x^3)}{x^5}\right) = (0,0) \leftarrow p_7 : \left(\frac{x}{y}, \frac{y^3(y^2 z - x^3)}{x^6}\right) = (0,0)$$

$$\leftarrow p_8 : \left(\frac{x}{y}, \frac{y^4(y^2 z - x^3)}{x^7}\right) = (0,s) \leftarrow p_9 : \left(\frac{x}{y}, \frac{y(y^4(y^2 z - x^3) - s x^7)}{x^8}\right) = (0, \lambda).$$

Here an arrow means that the right hand side is infinitely near, namely, the point of the right hand side is in the exceptional divisor obtained by blowing-up of the left hand side. In this case, p_1 is the only point in $\mathbb{P}^2$ and the others are infinitely near points. We take x, y, z as a coordinate of $\mathbb{P}^2$, and infinitely near points are expressed by coordinate of blowing-up.

The data give a rational elliptic surface if and only if $\lambda = 0$. When $\lambda \neq 0$, we get a space of initial conditions of the first Painlevé equation. There remains an action of PGL_3 as

$$(x : y : z; \lambda; s) \sim \left(\nu^2 x : \nu^3 y : z; \nu^5 \lambda; \nu^4 s\right) \in \mathbb{P}^2 \times \mathbb{C}^2, \quad \nu \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\},$$

so we can normalize s as 0 or 1. When $s = 0$, there remains symmetry of $\nu \in \mathbb{C}^\times$. When $s = 1$, there remains symmetry of $\nu \in \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$. When a rational elliptic surface has a singular fiber of type $E_8^{(1)}$, then it is isomorphic to the surface obtained by blowing-up from $\mathbb{P}^2$ with the data $s = 0$ or the surface with the data $s = 1$.

The case of $E_8^{(1)}$ is the only one exception. We cannot blow down these surfaces to $\mathbb{P}^1 \times \mathbb{P}^1$.

2. $E_7^{(1)}$

We give data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ as follows:

$$
\begin{aligned}
&p_1 : (0:1:0) \leftarrow p_2 : (x/y, z/x) = (0,0) \leftarrow p_3 : \left(x/y, yz/x^2\right) = (0,0) \\
&\leftarrow p_6 : \left(\frac{x}{y}, \frac{y^2 z}{x^3}\right) = (0,1) \leftarrow p_7 : \left(\frac{x}{y}, \frac{y(y^2 z - x^3)}{x^4}\right) = (0,0) \\
&\leftarrow p_8 : \left(\frac{x}{y}, \frac{y^2(y^2 z - x^3)}{x^5}\right) = (0,-s) \leftarrow p_9 : \left(\frac{x}{y}, \frac{y(y^2(y^2 z - x^3) + s x^5)}{x^6}\right) = (0, -a_0), \\
&p_4 : (0:0:1), \qquad p_5 : (0:a_1:1).
\end{aligned}
$$

Here p_1, p_4, and p_5 are points in $\mathbb{P}^2$, and the others are infinitely near points.

The condition to be a rational elliptic surface is $\lambda = a_0 + a_1 = 0$. There remains an action of PGL_3 as

$$
(x:y:z; a_1, a_0; s) \sim \left(\nu^2 x : \nu^3 y : z; \nu^3 a_1, \nu^3 a_0; \nu^2 s\right) \in \mathbb{P}^2 \times \mathbb{C}^3, \quad \nu \in \mathbb{C}^\times,
$$

so we can normalize s as 0 or 1. When $s = 0$, there remains symmetry of $\nu \in \mathbb{C}^\times$. When $s = 1$, there remains symmetry of $\nu \in \{1, -1\}$.

We give a 8 points blowing-up from $\mathbb{P}^1 \times \mathbb{P}^1$:

$$
\begin{aligned}
&P_1 : (f,g) = (\infty,\infty) \leftarrow P_2 : (1/f, f/g) = (0,0) \leftarrow P_3 : \left(1/f, f^2/g\right) = (0,-1) \\
&\leftarrow P_4 : \left(1/f, f\left(f^2+g\right)/g\right) = (0,0) \leftarrow P_5 : \left(1/f, f^2\left(f^2+g\right)/g\right) = (0,s) \\
&\leftarrow P_6 : \left(1/f, f\left(f^2\left(f^2+g\right) - sg\right)/g\right) = (0, a_0), \\
&P_7 : (f,g) = (\infty, 0) \leftarrow P_8 : (1/f, fg) = (0, -a_1).
\end{aligned}
$$

Here we take a coordinate as $(f,g) \in \mathbb{P}^1 \times \mathbb{P}^1$, and correspondence with the previous construction is given by $(f,g) = (y/x, -x/z)$.

3. $E_6^{(1)}$

Data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ are given as follows:

$$
\begin{aligned}
&p_1 : (0:1:0) \leftarrow p_2 : (x/y, z/x) = (0,0) \leftarrow p_7 : \left(x/y, yz/x^2\right) = (0,1) \\
&\leftarrow p_8 : \left(\frac{x}{y}, \frac{y(yz - x^2)}{x^3}\right) = (0,s) \leftarrow p_9 : \left(\frac{x}{y}, \frac{y(y(yz - x^2) - s x^3)}{x^4}\right) = \left(0, -a_0 + s^2\right),
\end{aligned}
$$

$$p_4 \,:\, (0:0:1), \quad p_5 \,:\, (0:a_1:1), \quad p_3 \,:\, (1:0:0) \leftarrow p_6 \,:\, \begin{array}{c}(z/x, y/z)\\ = (0, -a_2).\end{array}$$

The surface is rational elliptic surface if and only if $\lambda = a_0 + a_1 + a_2 = 0$, and there remains an action of PGL_3 as

$$(x : y : z; a_i; s) \sim \big(\nu x : \nu^2 y : z; \nu^2 a_i; \nu s\big) \in \mathbb{P}^2 \times \mathbb{C}^4, \quad \nu \in \mathbb{C}^\times,$$

so we can normalize s as 0 or 1. When $s = 0$, there remains symmetry of $\nu \in \mathbb{C}^\times$.

Furthermore we give a 8 points blowing-up from $\mathbb{P}^1 \times \mathbb{P}^1$:

$$P_1 \,:\, (f, g) = (\infty, \infty) \,\leftarrow P_2 \,:\, (1/g, g/f) = (0, 1)$$

$$\leftarrow P_3 \,:\, \big(1/g, g(g-f)/f\big) = (0, s) \leftarrow P_4 \,:\, \begin{array}{c}\big(1/g, g\big(g(g-f) - sf\big)/f\big)\\ = \big(0, -a_0 + s^2\big),\end{array}$$

$$P_5 \,:\, (f, g) = (\infty, 0) \,\leftarrow P_6 \,:\, (1/f, fg) = (0, -a_2),$$

$$P_7 \,:\, (f, g) = (0, \infty) \,\leftarrow P_8 \,:\, (1/g, fg) = (0, a_1).$$

Here we put $(f, g)(= (x/z, y/x)) \in \mathbb{P}^1 \times \mathbb{P}^1$.

4. $D_8^{(1)}$

Data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ are given as follows:

$$p_1 \,:\, (0:1:0) \,\leftarrow p_2 \,:\, (x/y, z/x) = (0, 0) \,\leftarrow p_3 \,:\, \big(x/y, yz/x^2\big) = (0, 0)$$

$$\leftarrow p_8 \,:\, \left(\frac{x}{y}, \frac{y^2 z}{x^3}\right) = (0, s) \,\leftarrow p_9 \,:\, \left(\frac{x}{y}, \frac{y(y^2 z - s x^3)}{x^4}\right) = (0, -\lambda s),$$

$$p_4 \,:\, (0:0:1) \,\leftarrow p_5 \,:\, \left(\frac{y}{z}, \frac{x}{y}\right) = (0, 0) \,\leftarrow p_6 \,:\, \left(\frac{y}{z}, \frac{zx}{y^2}\right) = (0, 1)$$

$$\leftarrow p_7 \,:\, \left(\frac{y}{z}, \frac{z(zx - y^2)}{y^3}\right) = (0, 0).$$

The condition that the surface is a rational elliptic surface is $\lambda = 0$, and there remains an action of PGL_3 as

$$(x : y : z; \lambda; s) \sim \big(x : \nu y : \nu^2 z; \nu\lambda; \nu^4 s\big) \in \mathbb{P}^2 \times \mathbb{C} \times \mathbb{C}^\times, \quad \nu \in \mathbb{C}^\times,$$

so we can normalize $s = 1$, and symmetry of $\nu \in \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$ still remains.

Putting $(f, g)(= (x/z, y/x)) \in \mathbb{P}^1 \times \mathbb{P}^1$, we give a blowing-up from $\mathbb{P}^1 \times \mathbb{P}^1$:

$$P_1 \,:\, (f, g) = (\infty, \infty) \,\leftarrow P_2 \,:\, (1/g, g/f) = (0, 0)$$

$$\leftarrow P_3 \,:\, \big(1/g, g^2/f\big) = (0, s) \leftarrow P_4 \,:\, \big(1/g, g\big(g^2 - sf\big)/f\big) = (0, -\lambda s),$$
$$P_5 \,:\, (f, g) = (0, \infty) \;\leftarrow P_6 \,:\, (fg, 1/g) = (0, 0)$$
$$\leftarrow P_7 \,:\, \big(fg, 1/\big(fg^2\big)\big) = (0, 1) \leftarrow P_8 \,:\, \big(fg, \big(1 - fg^2\big)/\big(f^2g^3\big)\big) = (0, 0).$$

5. $D_7^{(1)}$

Data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ are given as follows:

$$p_1 \,:\, (0:1:0) \;\leftarrow p_2 \,:\, (x/y, z/x) = (0, 0) \;\leftarrow p_3 \,:\, \big(x/y, yz/x^2\big) = (0, 0)$$
$$\leftarrow p_8 \,:\, \left(\frac{x}{y}, \frac{y^2 z}{x^3}\right) = (0, s) \;\leftarrow p_9 \,:\, \left(\frac{x}{y}, \frac{y(y^2 z - s x^3)}{x^4}\right) = (0, -a_0 s),$$
$$p_4 \,:\, (0:0:1) \;\leftarrow p_6 \,:\, (x/z, y/x) = (0, 0),$$
$$p_5 \,:\, (0:1:1) \;\leftarrow p_7 \,:\, \left(\frac{x}{z}, \frac{y - z}{x}\right) = (0, a_1).$$

The surface is a rational elliptic surface if and only if $\lambda = a_0 + a_1 = 0$. Since an action of PGL_3

$$(x : y : z; a_i; s) \sim \big(x : \nu y : \nu z; \nu a_i; \nu^3 s\big) \in \mathbb{P}^2 \times \mathbb{C}^2 \times \mathbb{C}^\times, \quad \nu \in \mathbb{C}^\times$$

remains, we can normalize s as 1. There remains symmetry of $\nu \in \{1, \omega, \omega^2\}$, where $\omega^2 + \omega + 1 = 0$.

Putting $(f, g)(= ((yz)/x^2, -x/z)) \in \mathbb{P}^1 \times \mathbb{P}^1$, we give a blowing-up from $\mathbb{P}^1 \times \mathbb{P}^1$:

$$P_1 \,:\, (f, g) = (0, \infty) \;\leftarrow P_2 \,:\, \big(f, 1/(fg)\big) = (0, 0)$$
$$\leftarrow P_3 \,:\, \big(1/(fg), f^2 g\big) = (0, -s) \leftarrow P_4 \,:\, \big(1/(fg), fg\big(f^2 g + s\big)\big) = (0, -a_0 s),$$
$$P_5 \,:\, (f, g) = (\infty, 0) \;\leftarrow P_6 \,:\, \big(g, 1/(fg)\big) = (0, 0)$$
$$\leftarrow P_7 \,:\, \big(g, 1/\big(fg^2\big)\big) = (0, 1) \leftarrow P_8 \,:\, \left(g, \frac{1}{g}\left(\frac{1}{fg^2} - 1\right)\right) = (0, a_1).$$

6. $D_6^{(1)}$

Data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ are given as follows:

$$p_1 \,:\, (0:1:0) \;\leftarrow p_2 \,:\, \left(\frac{x}{y}, \frac{z}{x}\right) = (0, 1) \;\leftarrow p_3 \,:\, \left(\frac{x}{y}, \frac{y(z - x)}{x^2}\right) = (0, 0)$$
$$\leftarrow p_8 \,:\, \left(\frac{x}{y}, \frac{y^2(z - x)}{x^3}\right) = (0, s) \;\leftarrow p_9 \,:\, \left(\frac{x}{y}, \frac{y(y^2(z - x) - s x^3)}{x^4}\right) = \big(0, s(b_1 - a_0)\big),$$
$$p_4 \,:\, (0:0:1), \quad p_5 \,:\, (0:a_1:1), \quad p_6 \,:\, (1:0:0), \quad p_7 \,:\, (1:b_1:0).$$

The surface is a rational elliptic surface if and only if $\lambda = a_0 + a_1 = b_0 + b_1 = 0$. Since an action of PGL_3

$$(x : y : z; a_i, b_i; s) \sim \big(x : \nu y : z; \nu a_i, \nu b_i; \nu^2 s\big) \in \mathbb{P}^2 \times \mathbb{C}^4 \times \mathbb{C}^\times, \quad \nu \in \mathbb{C}^\times$$

remains, we can normalize s as 1. There remains symmetry of $\nu \in \{1, -1\}$.

Putting $(f, g)(= (\frac{y(z-x)}{xz}, \frac{x}{z-x})) \in \mathbb{P}^1 \times \mathbb{P}^1$, We give data of a blowing-up from $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\begin{aligned} &P_1 : (f, g) = (0, \infty) \leftarrow P_2 : \big(f, 1/(fg)\big) = (0, 0) \\ &\leftarrow P_3 : \big(1/(fg), f^2 g\big) = (0, s) \leftarrow P_4 : \begin{array}{c}\big(1/(fg), fg\big(f^2 g - s\big)\big) \\ = \big(0, s(b_1 - a_0)\big),\end{array} \\ &P_5 : (f, g) = (\infty, 0) \leftarrow P_6 : (1/f, fg) = (0, a_1), \\ &P_7 : (f, g) = (\infty, -1) \leftarrow P_8 : \big(1/f, f(g+1)\big) = (0, b_1). \end{aligned}$$

7. $D_5^{(1)}$

Data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ are given as follows:

$$\begin{aligned} &p_1 : (0:1:0) \leftarrow p_2 : \left(\frac{x}{y}, \frac{z}{x}\right) = (0, 1) \leftarrow p_8 : \left(\frac{x}{y}, \frac{y(z-x)}{x^2}\right) = (0, s) \\ &\leftarrow p_9 : \left(\frac{x}{y}, \frac{y(y(z-x) - sx^2)}{x^3}\right) = \big(0, s(s - a_0)\big), \\ &p_3 : (1 : a_2 : 1), \quad p_4 : (0:0:1), \quad p_5 : (0 : a_1 : 1), \\ &p_6 : (1:0:0), \quad p_7 : (1 : a_3 : 0). \end{aligned}$$

The condition that the surface is a rational elliptic surface is $\lambda = a_0 + a_1 + a_2 = 0$, and there remains an action of PGL_3 as

$$(x : y : z; a_i; s) \sim (x : \nu y : z; \nu a_i; \nu s) \in \mathbb{P}^2 \times \mathbb{C}^4 \times \mathbb{C}^\times, \quad \nu \in \mathbb{C}^\times,$$

so we can normalize s as 1.

Furthermore we give data of blowing-up form $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\begin{aligned} &P_1 : (f, g) = (0, \infty) \leftarrow P_2 : (1/g, fg) = (0, a_1), \\ &P_3 : (f, g) = (1, \infty) \leftarrow P_4 : \big(1/g, (f-1)g\big) = (0, a_3), \\ &P_5 : (f, g) = (\infty, 0) \leftarrow P_6 : (1/f, fg) = (0, -a_2), \\ &P_7 : (f, g) = (\infty, -s) \leftarrow P_8 : \big(1/f, f(g+s)\big) = (0, s - a_0). \end{aligned}$$

Here we put $(f, g)(= (\frac{x}{x-z}, \frac{y(x-z)}{xz})) \in \mathbb{P}^1 \times \mathbb{P}^1$.

8. $D_4^{(1)}$

Data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ are given as follows:

$$p_1 : (0:1:0) \leftarrow p_8 : (x/y, z/x) = \big(0, s/(s-1)\big)$$
$$\leftarrow p_9 : \left(\frac{x}{y}, \frac{y(sx+(1-s)y)}{x^2}\right) = \big(0, sa_0/(s-1)\big),$$
$$p_2 : (1:-a_2:1), \quad p_3 : (1:-a_1-a_2:1), \quad p_4 : (0:0:1),$$
$$p_5 : (0:a_3:1), \quad p_6 : (1:0:0), \quad p_7 : (1:a_4:0).$$

The condition that the surface is a rational elliptic surface is $\lambda = a_0 + a_1 + 2a_2 + a_3 + a_4 = 0$, and there remains an action of PGL_3 as

$$(x:y:z;a_i;s) \sim (x:\nu y:z;\nu a_i;s) \in \mathbb{P}^2 \times \mathbb{C}^5 \times \big(\mathbb{C}\setminus\{0,1\}\big), \quad \nu \in \mathbb{C}^\times.$$

Putting $(f,g)(= (z/(z-x), y/x)) \in \mathbb{P}^1 \times \mathbb{P}^1$, we give data of blowing-up from $\mathbb{P}^1 \times \mathbb{P}^1$:

$$P_1 : (f,g) = (0,0), \quad P_2 : (0,a_4), \quad P_3 : (\infty,-a_2), \quad P_4 : (\infty,-a_1-a_2),$$
$$P_5 : (1,\infty) \leftarrow P_9 : \big(1/g, g(f-1)\big) = (0,a_3),$$
$$P_7 : (s,\infty) \leftarrow P_8 : \big(1/g, g(f-s)\big) = (0,sa_0).$$

9. $A_2^{(1)*}$

Data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ are given as follows:

$$p_1 : (1:0:1), \quad p_2 : (1:-a_2:1), \quad p_3 : (1:-a_2-a_1:1),$$
$$p_4 : (0:0:1), \quad p_5 : (0:a_6:1), \quad p_6 : (0:a_0+a_6:1),$$
$$p_7 : (1:a_3:0), \quad p_8 : (1:a_3+a_4:0), \quad p_9 : (1:a_3+a_4+a_5:0).$$

The surface is a rational elliptic surface if and only if $\lambda = a_0 + a_1 + 2a_2 + 3a_3 + 4a_4 + a_5 + 2a_6 = 0$, and there remains an action of PGL_3 as

$$(x:y:z;a_i) \sim (x:\nu y:z;\nu a_i) \in \mathbb{P}^2 \times \mathbb{C}^7, \quad \nu \in \mathbb{C}^\times.$$

Putting $(f,g)(= (y/x, y/(z-x))) \in \mathbb{P}^1 \times \mathbb{P}^1$, we can also construct the surface by a blowing-up from $\mathbb{P}^1 \times \mathbb{P}^1$:

$$P_1 : (0,0), \quad P_2 : (a_3,-a_3), \quad P_3 : (a_3+a_4,-a_3-a_4),$$
$$P_4 : (a_3+a_4+a_5,-a_3-a_4-a_5), \quad P_5 : (-a_2,0), \quad P_6 : (-a_1-a_2,0),$$
$$P_7 : (0,a_6), \quad P_8 : (0,a_6+a_0).$$

10. $A_1^{(1)*}$

Data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ are given as follows:

$$p_1 \,:\, (1:0:a_2+a_7), \quad p_2 \,:\, (1:0:a_1+a_2+a_7),$$
$$p_3 \,:\, (1:0:a_0+a_1+a_2+a_7), \quad p_4 \,:\, \left(-a_7:1:a_7^2\right), \quad p_5 \,:\, (0:1:0),$$
$$p_6 \,:\, \left(a_3:1:a_3^2\right), \quad p_7 \,:\, \left(a_3+a_4:1:(a_3+a_4)^2\right),$$
$$p_8 \,:\, \left(a_3+a_4+a_5:1:(a_3+a_4+a_5)^2\right),$$
$$p_9 \,:\, \left(a_3+a_4+a_5+a_6:1:(a_3+a_4+a_5+a_6)^2\right).$$

The surface is a rational elliptic surface if and only if $\lambda = a_0+2a_1+3a_2+4a_3+3a_4+2a_5+a_6+2a_7 = 0$, and there remains an action of PGL_3 as

$$(x:y:z;a_i) \sim \left(\nu x:y:\nu^2 z;\nu a_i\right) \in \mathbb{P}^2 \times \mathbb{C}^8, \quad \nu \in \mathbb{C}^\times.$$

11. $A_0^{(1)**}$

Data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ are given as follows:

$$p_i \,:\, \left(c_i:2:c_i^3\right), \quad (i=1,\ldots,9),$$
$$c_1 = \frac{-2a_1-a_2+a_8}{3}, \qquad c_2 = \frac{a_1-a_2+a_8}{3}, \qquad c_3 = \frac{a_1+2a_2+a_8}{3},$$
$$c_4 = c_3+a_3, \qquad c_5 = c_4+a_4, \qquad c_6 = c_5+a_5, \qquad c_7 = c_6+a_6,$$
$$c_8 = c_7+a_7, \qquad c_9 = c_8+a_0.$$

The surface is a rational elliptic surface if and only if $\lambda = a_0+2a_1+4a_2+6a_3+5a_4+4a_5+3a_6+2a_7+3a_8 = 0$, and there remains an action of PGL_3 as

$$(x:y:z;a_i) \sim \left(\nu x:y:\nu^3 z;\nu a_i\right) \in \mathbb{P}^2 \times \mathbb{C}^9, \quad \nu \in \mathbb{C}^\times.$$

12. $A_8^{(1)}$

Data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ are given as follows:

$$p_1 \,:\, (1:0:0) \leftarrow p_2 \,:\, (z/x,y/z) = (0,0) \leftarrow p_6 \,:\, \left(z/x, xy/z^2\right) = (0,-1),$$
$$p_3 \,:\, (0:0:1) \leftarrow p_4 \,:\, (y/z,x/y) = (0,0) \leftarrow p_5 \,:\, \left(y/z, zx/y^2\right) = (0,-1),$$
$$p_7 \,:\, (0:1:0) \leftarrow p_8 \,:\, (x/y,z/x) = (0,0) \leftarrow p_9 \,:\, \left(x/y, yz/x^2\right) = (0,-1).$$

Here p_1, p_3, and p_7 are points in $\mathbb{P}^2$ and the others are infinitely near points.

There remains an action of PGL_3 as

$$(x:y:z) \sim \left(x:\nu y:\nu^2 z\right) \in \mathbb{P}^2, \quad \nu^3=1.$$

Putting $\mathbb{P}^1 \times \mathbb{P}^1$, we give data of 8 points configuration, which include infinitely near points, in $\mathbb{P}^1 \times \mathbb{P}^1$:

$$P_1 : (f,g)=(\infty,\infty) \leftarrow P_2 : (1/f, f/g)=(0,-1),$$
$$P_3 : (f,g)=(0,\infty) \leftarrow P_4 : (fg, 1/g)=(0,0) \leftarrow P_5 : \begin{array}{l}\left(fg^2, 1/g\right)\\ =(-1,0),\end{array}$$
$$P_6 : (f,g)=(\infty,0) \leftarrow P_7 : (1/f, fg)=(0,0) \leftarrow P_8 : \begin{array}{l}\left(1/f, f^2 g\right)\\ =(0,-1).\end{array}$$

Here P_1, P_3, and P_6 are points in $\mathbb{P}^1 \times \mathbb{P}^1$, and the others are infinitely near points.

13. $A_7^{(1)\prime}$

Data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ are given as follows:

$$p_1 : (0:0:1) \leftarrow p_3 : (x/z, y/z)=(0,0) \leftarrow p_5 : \left(y/x, x^2/yz\right)=(0,1),$$
$$p_2 : (1:0:0) \leftarrow p_6 : (z/x, y/z)=(0,a_1), \qquad p_4 : (0:1:1),$$
$$p_7 : (0:1:0) \leftarrow p_8 : (x/y, z/x)=(0,0) \leftarrow p_9 : \left(x/y, yz/x^2\right)=(0,a_0).$$

The surface is a rational elliptic surface if and only if $q=a_0 a_1=1$, and there remains symmetry:

$$(x:y:z) \sim (\nu x:y:z) \in \mathbb{P}^2, \quad \nu^2=1.$$

The surface is also constructed by 8 blowing-up from $\mathbb{P}^1 \times \mathbb{P}^1$ by using the data of configuration:

$$P_1 : (f,g)=(0,0) \leftarrow P_2 : (f, g/f)=(0,1),$$
$$P_3 : (f,g)=(\infty,0) \leftarrow P_4 : (1/f, fg)=(0,a_1),$$
$$P_5 : (f,g)=(0,\infty) \leftarrow P_6 : (fg, 1/g)=(1,0),$$
$$P_7 : (f,g)=(\infty,\infty) \leftarrow P_8 : (g/f, 1/g)=(a_0,0).$$

Here we put $(f,g)(=(\frac{x}{z}, \frac{y}{x})) \in \mathbb{P}^1 \times \mathbb{P}^1$.

14. $A_7^{(1)}$

We give data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ as follows:

$$p_1 : (1:0:1), \qquad p_2 : (1:0:0) \leftarrow p_6 : (z/x, y/z)=(0,1),$$

$$p_3 : (0:0:1) \leftarrow p_4 : (y/z, x/y) = (0,0) \leftarrow p_5 : \big(y/z, zx/y^2\big) = (0, a_0),$$
$$p_7 : (0:1:0) \leftarrow p_8 : (x/y, z/x) = (0,0) \leftarrow p_9 : \big(x/y, yz/x^2\big) = (0, a_1).$$

The condition that the surface is a rational elliptic surface is $q = a_0 a_1 = 1$.

Putting $(f,g)(=(\frac{x}{z},\frac{y}{x})) \in \mathbb{P}^1 \times \mathbb{P}^1$, we give data of 8 points configuration in $\mathbb{P}^1 \times \mathbb{P}^1$:

$$P_1 : (f,g) = (1,0), \qquad P_2 : (f,g) = (\infty, 0) \leftarrow P_3 : (1/f, fg) = (0,1),$$
$$P_4 : (f,g) = (0,\infty) \leftarrow P_5 : (fg, 1/g) = (0,0) \leftarrow P_6 : \begin{matrix}\big(fg, 1/\big(fg^2\big)\big)\\ = (0, a_0),\end{matrix}$$
$$P_7 : (f,g) = (\infty,\infty) \leftarrow P_8 : (g/f, 1/g) = (a_1, 0).$$

15. $A_6^{(1)}$

We give data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ as follows:

$$p_1 : (1:0:1), \qquad p_2 : (1:0:0) \leftarrow p_6 : (z/x, y/z) = (0, a_1),$$
$$p_3 : (0:0:1) \leftarrow p_5 : (y/z, x/y) = (0, a_0/b), \qquad p_4 : (0:1:1),$$
$$p_7 : (0:1:0) \leftarrow p_8 : (x/y, z/x) = (0,0) \leftarrow p_9 : \begin{matrix}\big(x/y, yz/x^2\big)\\ = (0, -b).\end{matrix}$$

The condition that the surface is a rational elliptic surface is $q = a_0 a_1 = 1$.

Putting $(f,g)(=(\frac{x}{z},\frac{y}{x})) \in \mathbb{P}^1 \times \mathbb{P}^1$, we give data of 8 points configuration in $\mathbb{P}^1 \times \mathbb{P}^1$:

$$P_1 : (f,g) = (1,0), \qquad P_2 : (f,g) = (\infty,0) \leftarrow P_3 : (1/f, fg) = (0, a_1),$$
$$P_4 : (f,g) = (0, b/a_0), \quad P_5 : (f,g) = (0,\infty) \leftarrow P_6 : (fg, 1/g) = (1,0),$$
$$P_7 : (f,g) = (\infty,\infty) \leftarrow P_8 : (g/f, 1/g) = (-b, 0).$$

16. $A_5^{(1)}$

We give data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ as follows:

$$p_1 : (1:0:1), \quad p_2 : (a_1:0:0), \quad p_4 : (0:1:1), \quad p_6 : (1:-a_2:0),$$
$$p_3 : (0:0:1) \leftarrow p_5 : (y/z, x/y) = (0, b_1/a_2),$$
$$p_7 : (0:1:0) \leftarrow p_8 : (x/y, z/x) = (0,0) \leftarrow p_9 : \begin{matrix}\big(x/y, yz/x^2\big)\\ = (0, -b_0/a_1).\end{matrix}$$

The surface is a rational elliptic surface if and only if $q = b_0 b_1 = 1$.

The surface is also constructed by 8 points blowing-up from $\mathbb{P}^1 \times \mathbb{P}^1$ by using data of configuration:

$$P_1 : (f,g) = (1,0), \quad P_2 : (f,g) = (a_1,0), \quad P_3 : (f,g) = (0, a_2/b_1),$$
$$P_4 : (f,g) = (\infty, -a_2), \quad P_5 : (f,g) = (0,\infty) \leftarrow P_6 : (fg, 1/g) = (1,0),$$
$$P_7 : (f,g) = (\infty,\infty) \leftarrow P_8 : (g/f, 1/g) = (-b_0/a_1, 0).$$

Here we put $(f,g)(= (\frac{x}{z}, \frac{y}{x})) \in \mathbb{P}^1 \times \mathbb{P}^1$.

17. $A_4^{(1)}$

We give data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ as follows:

$$p_1 : (1:0:1), \quad p_2 : (a_2:0:1), \quad p_3 : (a_1a_2:0:1),$$
$$p_4 : (0:1:1), \quad p_5 : (0:1:a_4), \quad p_6 : (1:-a_3:0),$$
$$p_7 : (0:1:0) \leftarrow p_8 : (x/y, z/x) = (0,0) \leftarrow p_9 : \begin{array}{l}(x/y, yz/x^2)\\ = (0, a_0/a_2).\end{array}$$

The surface is a rational elliptic surface if and only if $q = a_0a_1a_2a_3a_4 = 1$.

The surface is also constructed by 8 points blowing-up from $\mathbb{P}^1 \times \mathbb{P}^1$ by using data of configuration:

$$P_1 : (f,g) = (a_2,0), \quad P_2 : (f,g) = (a_1a_2,0), \quad P_3 : (f,g) = (1,\infty),$$
$$P_4 : (f,g) = (0,1), \quad P_5 : (f,g) = (0,1/a_4), \quad P_6 : (f,g) = (\infty, a_3),$$
$$P_7 : (f,g) = (\infty,\infty) \leftarrow P_8 : (g/f, 1/g) = (-a_0/a_2, 0).$$

Here we put $(f,g)(= (\frac{x}{z}, -\frac{y}{x-z})) \in \mathbb{P}^1 \times \mathbb{P}^1$.

18. $A_3^{(1)}$

We give data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ as follows:

$$p_1 : (1:0:1), \quad p_2 : (a_2:0:1), \quad p_3 : (a_1a_2:0:1),$$
$$p_4 : (0:1:1), \quad p_5 : (0:1:a_5), \quad p_6 : (1:-a_3:0),$$
$$p_7 : (1:-a_3a_4:0), \quad p_8 : (0:1:0) \leftarrow p_9 : (x/y, z/x) = (0, a_0).$$

The surface is a rational elliptic surface if and only if $q = a_0a_1a_2^2a_3^2a_4a_5 = 1$.

Putting $(f,g)(= (\frac{x}{z}, -\frac{y}{x-z})) \in \mathbb{P}^1 \times \mathbb{P}^1$, we give data of 8 points configuration in $\mathbb{P}^1 \times \mathbb{P}^1$:

$$P_1 : (f,g) = (a_2,0), \quad P_2 : (a_1a_2,0), \quad P_3 : (1,\infty), \quad P_4 : (1/a_0, \infty),$$

$$P_5 \ : \ (0,1), \qquad P_6 \ : \ (0,1/a_5), \qquad P_7 \ : \ (\infty, a_3), \qquad P_8 \ : \ (\infty, a_3 a_4).$$

Here we need no infinitely near points.

19. $A_2^{(1)}$

Data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ are given as follows:

$$\begin{aligned} &p_1 \ : \ (1:0:1), \qquad p_2 \ : \ (a_2:0:1), \qquad p_3 \ : \ (a_1 a_2:0:1),\\ &p_4 \ : \ (0:1:1), \qquad p_5 \ : \ (0:1:a_4), \qquad p_6 \ : \ (0:1:a_4 a_5),\\ &p_7 \ : \ (1:-a_3:0), \qquad p_8 \ : \ (1:-a_3 a_6:0), \qquad p_9 \ : \ (1:-a_0 a_3 a_6:0).\end{aligned}$$

The surface is a rational elliptic surface if and only if $q = a_0 a_1 a_2^2 a_3^3 a_4^2 a_5 a_6^2 = 1$.

Putting $(f,g)(= (\frac{z-x}{x}, \frac{z-x}{y})) \in \mathbb{P}^1 \times \mathbb{P}^1$, we can also construct the surface by a blowing-up from $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\begin{aligned} &P_1 \ : \ (f,g) = (1/a_2, 0), \qquad P_2 \ : \ \big(1/(a_1 a_2), 0\big), \qquad P_3 \ : \ (0, a_4),\\ &P_4 \ : \ (0, a_4 a_5), \qquad P_5 \ : \ (1,1), \qquad P_6 \ : \ (a_3, 1/a_3),\\ &P_7 \ : \ \big(a_3 a_6, 1/(a_3 a_6)\big), \qquad P_8 \ : \ \big(a_0 a_3 a_6, 1/(a_0 a_3 a_6)\big).\end{aligned}$$

20. $A_1^{(1)}$

Data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ are given as follows:

$$\begin{aligned} &p_1 \ : \ (a_2 a_7:0:1), \qquad p_2 \ : \ (a_1 a_2 a_7:0:1), \qquad p_3 \ : \ (a_0 a_1 a_2 a_7:0:1),\\ &p_4 \ : \ \big(1/a_7^2:1/a_7:1\big), \qquad p_5 \ : \ (1:1:1), \qquad p_6 \ : \ \big(a_3^2:a_3:1\big),\\ &p_7 \ : \ \big(a_3^2 a_4^2:a_3 a_4:1\big), \qquad p_8 \ : \ \big(a_3^2 a_4^2 a_5^2:a_3 a_4 a_5:1\big),\\ &p_9 \ : \ \big(a_3^2 a_4^2 a_5^2 a_6^2:a_3 a_4 a_5 a_6:1\big).\end{aligned}$$

The surface is a rational elliptic surface if and only if

21. $A_0^{(1)*}$

Data of 9 points configuration, which include infinitely near points, in $\mathbb{P}^2$ are given as follows:

$$\begin{aligned} &p_i \ : \ \left(\frac{1}{\sin^2 c_i} - \frac{1}{3} : -\frac{2\cos c_i}{\sin^3 c_i} : 1\right), \quad (i = 1, \dots, 9),\\ &c_1 = \frac{-2a_1 - a_2 + a_8}{3}, \qquad c_2 = \frac{a_1 - a_2 + a_8}{3}, \qquad c_3 = \frac{a_1 + 2a_2 + a_8}{3},\\ &c_4 = c_3 + a_3, \qquad c_5 = c_4 + a_4, \qquad c_6 = c_5 + a_5, \qquad c_7 = c_6 + a_6,\\ &c_8 = c_7 + a_7, \qquad c_9 = c_8 + a_0.\end{aligned}$$

The surface is a rational elliptic surface if and only if $\lambda = a_0 + 2a_1 + 4a_2 + 6a_3 + 5a_4 + 4a_5 + 3a_6 + 2a_7 + 3a_8 = 0$.

3 Ordinary Differential Equations on Rational Elliptic Surfaces

When a rational elliptic surface and its blowing-down are given, we obtain a pencil of cubic curves in $\mathbb{P}^2$ or a pencil of biquadratic forms on $\mathbb{P}^1 \times \mathbb{P}^1$. We see these pencils here. We follow a method which is treated in a paper of Kajiwara-Masuda-Noumi-Ohta-Yamada [2], although their interest is mainly put on Painlevé's cases.

Here the 62nd surface in the list of Oguiso-Shioda, which has a singular fiber of type $E_8^{(1)}$, is taken as an example, and we see the calculation with it in detail.

In general, a cubic curve in $\mathbb{P}^2$ is written as

$$F = \mu_0 x^3 + \mu_1 y^3 + \mu_2 z^3 + \mu_3 x^2 y + \mu_4 x^2 z + \mu_5 y^2 z + \mu_6 y^2 x + \mu_7 z^2 x + \mu_8 z^2 y + \mu_9 xyz = 0,$$

and it makes a 9 dimensional space modulo constant multiplication. It is enough to determine one dimensional pencil by putting the condition that curves pass through all of nine points. But, in this case, we must consider infinitely near points as well. The 9th point p_9 is expressed as $(x/y, y(y^4(y^2z - x^3) - sx^7)/x^8) = (0, \lambda)$, (we can take s as 0 or 1). Taking an inhomogeneous coordinate $(x/y, z/y)$, the point p_9 is written as $z/y = \epsilon^3 + s\epsilon^7 + \lambda\epsilon^8$ by setting $x/y = \epsilon$. Substituting this into F, consider the condition that the terms of degree less than 9 in ϵ vanish, namely,

$$F\left(\epsilon, 1, \epsilon^3 + s\epsilon^7 + \lambda\epsilon^8\right) = O\left(\epsilon^9\right).$$

This has not only solution $F = \mu_2 z^3 =: \mu_2 F_0$ but another solution $F = \mu_0(x^3 - y^2 z + sz^2 x) =: \mu_0 F_1$ when $\lambda = 0$. Combining two of them, we find a pencil $F = \mu_2 F_0 + \mu_0 F_1$. Here the condition $\lambda = 0$ is a condition for an elliptic surface.

The function which returns the value $(\mu_0 : \mu_2) \in \mathbb{P}^1$ for each points in the rational elliptic surface gives an elliptic fibration. When we put $H = -F_1/F_0$ and $(f, g) = (x/z, y/z)$, then

$$H = g^2 - f^3 - sf,$$

and the pencil is written as $\{H + \mu = 0\ ;\ \mu \in \mathbb{P}^1\}$.

The rational 2-form which has a pole only at the singular fiber $F_0 = z^3 = 0$ is determined up to constant, and is written as

$$\omega = \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{F_0} = dg \wedge df.$$

We can construct an algebraic Hamiltonian system which preserves the fibration as follows:

$$\frac{d}{dt} f = \rho(H) \frac{\partial H}{\partial g}, \qquad \frac{d}{dt} g = -\rho(H) \frac{\partial H}{\partial f}.$$

Here ρ is an arbitrary rational function in H, and the system is the Hamiltonian system with the Hamiltonian $\int \rho(H)dH$.

Similar calculation is applicable for the other cases. All cases except $E_8^{(1)}$ have a blowing-down to $\mathbb{P}^1 \times \mathbb{P}^1$, so H is written as a ratio of two biquadratic forms.

For $E_7^{(1)}$, $E_6^{(1)}$, $D_7^{(1)}$, $D_6^{(1)}$, $D_5^{(1)}$, we can take $F_0 = f_0^2 g_0^2$ as an image of the ruling singular fiber, when we take a coordinate $(f_0 : f_1; g_0 : g_1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Then, setting $(f, g) = (f_1/f_0, g_1/g_0)$ and $\omega = dg \wedge df$, and

$$H = \left(g^2, g, 1\right) M \begin{pmatrix} f^2 \\ f \\ 1 \end{pmatrix},$$

we can construct a similar system

$$\frac{d}{dt} f = \rho(H) \frac{\partial H}{\partial g}, \qquad \frac{d}{dt} g = -\rho(H) \frac{\partial H}{\partial f},$$

by using a 3×3 matrix M. With respect to each surfaces defined in the previous section, the matrix M can be written as follows:

$$M = M_{E_7} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & s \\ 0 & a_1 & 0 \end{pmatrix}, \qquad M_{E_6} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -s & -a_1 \\ 0 & -a_2 & 0 \end{pmatrix},$$

$$M_{D_7} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_1 & -1 \\ 0 & s & 0 \end{pmatrix}, \qquad M_{D_6} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -a_1 - b_1 & -s \\ 0 & -a_1 & 0 \end{pmatrix},$$

$$M_{D_5} = \begin{pmatrix} 1 & s & 0 \\ -1 & -(a_1 + a_3 + s) & s a_2 \\ 0 & a_1 & 0 \end{pmatrix}.$$

Here we can take s as 0 or 1 for $E_7^{(1)}, E_6^{(1)}$, and we can take s as 1 for $D_7^{(1)}, D_6^{(1)}, D_5^{(1)}$.

For $D_8^{(1)}$ and $D_4^{(1)}$, the image of the ruling singular fiber is $F_0 = f_0 f_1 g_0^2$, and $\omega = \frac{1}{f} dg \wedge df = dg \wedge d(\log f)$. So $\log f$ and g are canonical coordinates. When we write the system in a rational form,

$$\frac{d}{dt} f = \rho(H) f \frac{\partial H}{\partial g}, \qquad \frac{d}{dt} g = -\rho(H) f \frac{\partial H}{\partial f},$$

and H is expressed as

$$H = \frac{1}{f_0 f_1 g_0^2} \left(g_1^2, g_1 g_0, g_0^2\right) M \begin{pmatrix} f_1^2 \\ f_1 f_0 \\ f_0^2 \end{pmatrix} = \left(g^2, g, 1\right) M \begin{pmatrix} f \\ 1 \\ 1/f \end{pmatrix}.$$

Here the matrix M is

$$M = M_{D_8} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -s & 0 & -1 \end{pmatrix},$$

$$M_{D_4} = \begin{pmatrix} 1 & -1-s & s \\ a_1+2a_2 & -a_1-2a_2+(s-1)a_3+a_4 & sa_4 \\ a_2(a_1+a_2) & 0 & 0 \end{pmatrix}.$$

We can take s as 1 for $D_8^{(1)}$. For $D_4^{(1)}$, s is an element in $\mathbb{C} \setminus \{0, 1\}$. We can't normalize s, but we can multiply all a_i by $\nu \in \mathbb{C}^\times$ without changing surfaces.

As a matter of course, if we do not stick to biquadratic form, we can write them as usual rational Hamiltonian systems.

For $D_8^{(1)}$, putting $\hat{f} = 1/f$, $\hat{g} = -fg$, the symplectic form is written as $\omega = d\hat{g} \wedge d\hat{f}$, and the system is a usual Hamiltonian system with $H = \hat{f}^2\hat{g}^2 - \hat{f} - s/\hat{f}$.

For $D_4^{(1)}$, putting $\hat{g} = g/f$, the symplectic form is written as $\omega = d\hat{g} \wedge df$, and the system is a usual Hamiltonian system with a polynomial Hamiltonian $H = f(f-1)(f-s)g^2 + \{(a_1+2a_2)(f-1)f + a_3(s-1)f + a_4 s(f-1)\}g + a_2(a_1+a_2)f$. These are not biquadratic forms by using 3×3 matrices.

Remark 2 In a paper of Kajiwara-Masuda-Noumi-Ohta-Yamada [2], it is found that the Painlevé equations can be obtained as non-autonomous systems by taking s as an independent variable.

Although, in the case of $D_4^{(1)}$ and $D_8^{(1)}$, the Painlevé equations are not written by using matrices of size 3 usually, we can write them as algebraic ODE's by using matrices of size 3, following a manner which we saw here.

The Painlevé equations correspond to 8 Dynkin diagrams of types $D_4^{(1)}, \ldots, D_8^{(1)}$, and $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, and we saw analogous cases of rational elliptic surfaces so far. These systems on elliptic surfaces are obtained as an autonomous limit of the Painlevé equations. The correspondence between the Painlevé equations and the Dynkin are given as follows:

$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$
P_{VI}	P_{V}	$P_{\mathrm{III}}(D_6)$	$P_{\mathrm{III}}(D_7)$	$P_{\mathrm{III}}(D_8)$	P_{IV}	P_{II}	P_{I}

These systems on elliptic surfaces can be constructed for A types also, differently from the Painlevé's cases. Of A types, for types $A_3^{(1)}$, $A_4^{(1)}$, $A_5^{(1)}$, $A_6^{(1)}$, $A_7^{(1)}$, $A_7^{(1)\prime}$, and $A_8^{(1)}$, the image of the ruling singular fiber can be taken as $F_0 = f_0 f_1 g_0 g_1$, when we consider a blowing-down to $\mathbb{P}^1 \times \mathbb{P}^1$.

A symplectic form is written as $\omega = \frac{1}{gf} dg \wedge df = d(\log g) \wedge (\log f)$, and the system is

$$\frac{d}{dt} f = \rho(H) fg \frac{\partial H}{\partial g}, \qquad \frac{d}{dt} g = -\rho(H) fg \frac{\partial H}{\partial f},$$

and H is expressed as

$$H = \left(g, 1, \frac{1}{g}\right) M \begin{pmatrix} f \\ 1 \\ \frac{1}{f} \end{pmatrix}.$$

The matrix M is parameterized as follow:

$$M = M_{A_8} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_{A_7} = \begin{pmatrix} 0 & -a_0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

$$M_{A_7'} = \begin{pmatrix} 0 & 1 & 0 \\ -a_0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad M_{A_6} = \begin{pmatrix} 0 & 1/b & 0 \\ 1 & 0 & -1/b \\ 0 & -a_1 & a_1 \end{pmatrix},$$

$$M_{A_5} = \begin{pmatrix} 0 & b_1/a_2 & 0 \\ a_0 & 0 & -b_1/a_2 \\ 1/a_1 & -1-(1/a_1) & 1 \end{pmatrix},$$

$$M_{A_4} = \begin{pmatrix} 0 & 1 & -1 \\ a_0/a_2 & 0 & 1+(1/a_4) \\ -a_0a_3/a_2 & a_0a_3+(1/a_2a_4) & -1/a_4 \end{pmatrix},$$

$$M_{A_3} = \begin{pmatrix} a_0a_5 & -(1+a_0)a_5 & a_5 \\ -(1+a_4)a_0a_3a_5 & 0 & -1-a_5 \\ 1/(a_1a_2^2) & -(1+a_1)/a_1a_2 & 1 \end{pmatrix}.$$

For remaining $A_2^{(1)*}$, $A_1^{(1)*}$, $A_0^{(1)**}$ and $A_2^{(1)}$, $A_1^{(1)}$, $A_0^{(1)*}$, the expression is a little bit complicated. As the ruling divisor F_0, we can take

$$A_2^{(1)*} : F_0 = f_0g_0(f_0f_1 + g_0g_1),$$

$$A_1^{(1)*} : F_0 = (f_1g_0 + f_0g_1 - 2rf_0g_0)(f_1g_0 + f_0g_1),$$

$$A_0^{(1)**} : F_0 = (f_1g_0 - f_0g_1)^2 - 8r^2\big(f_0f_1g_0^2 + f_0^2g_0g_1\big) + 16r^4 f_0^2g_0^2,$$

$$A_2^{(1)} : F_0 = f_0g_0(f_1g_1 - f_0g_0), \quad A_1^{(1)} : F_0 = \big(f_1g_1 - r^2 f_0g_0\big)(f_1g_1 - f_0g_0),$$

$$A_0^{(1)*} : F_0 = f_1^2g_0^2 + f_0^2g_1^2 - \left(r^2 + \frac{1}{r^2}\right) f_0f_1g_0g_1 + \left(r^2 - \frac{1}{r^2}\right)^2 f_0^2g_0^2,$$

for each types of surfaces (cf. [5]). Here r is a suitable number except 1 or 0.

Symplectic form is given as $\omega = \frac{f_0^2 g_0^2}{F_0} dg \wedge df$.

We don't go into detail in the all cases, but, for $A_2^{(1)*}$, the symplectic form is written as $\omega = dg \wedge d\log(f+g)$. Putting $F = f + g$, the system is expressed as

$$\frac{dF}{dt} = \rho(H) F \frac{\partial H}{\partial g}, \qquad \frac{dg}{dt} = -\rho(H) F \frac{\partial H}{\partial F},$$

by using

$$H = \frac{1}{F}\big(g^2, g, 1\big) M \begin{pmatrix} (F-g)^2 \\ F-g \\ 1 \end{pmatrix}.$$

For $A_2^{(1)}$, the symplectic form is $\omega = d\log g \wedge d\log(fg-1)$, and when we put $F = fg - 1$, the system is

$$\frac{dF}{dt} = \rho(H) Fg \frac{\partial H}{\partial g}, \qquad \frac{dg}{dt} = -\rho(H) Fg \frac{\partial H}{\partial F},$$

with

$$H = \frac{1}{F}\big(g^2, g, 1\big) M \begin{pmatrix} (F-1)^2/g^2 \\ (F-1)/g \\ 1 \end{pmatrix}.$$

4 Canonical Forms of Biquadratic Hamiltonians

In the previous section we constructed a Hamiltonian system based on data of a rational elliptic surface. In this section we start from a Hamiltonian system and we tell the type of the surface associated to the system.

We concentrate on two types of Hamiltonian systems. The first one is biquadratic Hamiltonian and the biquadratic form is expressed as

$$H = \big(g^2, g, 1\big) M \begin{pmatrix} f^2 \\ f \\ 1 \end{pmatrix}$$

by using 3×3 matrix M. The system is

$$\frac{d}{dt} f = \rho(H) \frac{\partial H}{\partial g}, \qquad \frac{d}{dt} g = -\rho(H) \frac{\partial H}{\partial f}.$$

For the second one we put

$$H = \left(g, 1, \frac{1}{g}\right) M \begin{pmatrix} f \\ f \\ \frac{1}{f} \end{pmatrix}$$

by using 3×3 matrix M, and the system is given as

$$\frac{d}{dt} f = \rho(H) fg \frac{\partial H}{\partial g}, \qquad \frac{d}{dt} g = -\rho(H) fg \frac{\partial H}{\partial f}.$$

In this article we call the former a polynomial Hamiltonian system, and the latter a logarithmic Hamiltonian system.

In the previous section we gave a system on an elliptic surface which has a singular fiber of type $E_7^{(1)}$, $E_6^{(1)}$, $D_7^{(1)}$, $D_6^{(1)}$, $D_5^{(1)}$, respectively, and it was the polynomial type. A system on a surface of $A_8^{(1)}$, $A_7^{(1)\prime}$, $A_7^{(1)}$, $A_6^{(1)}$, $\ldots A_3^{(1)}$ was given in the form of logarithmic type. To tell a conclusion of this section in advance, all of biquadratic polynomial and logarithmic Hamiltonian systems in this sense coincide with one of these systems. Furthermore we give a classification of such systems.

We see the polynomial case first. Here the matrix M is not arbitrary. If M has all 0 in the first column or in the first row, then $H+\mu=0$ does not give an elliptic curve. Neither does it, when $a_{11}=a_{12}=a_{21}=0$. We do away with these cases.

There are the transformation $(f,g)\mapsto(g,f)$, and affine transformations $(f,g)\mapsto(f+a,g+b)$, and they don't change the type of systems. We see how they act on the matrix M.

The interchange of dependent variables makes the matrix M transposed. The affine transformations act on M as follows:

$$M\mapsto\begin{pmatrix}1&0&0\\2b&1&0\\b^2&b&1\end{pmatrix}M\begin{pmatrix}1&2a&a^2\\0&1&a\\0&0&1\end{pmatrix}.$$

Putting $M=(m_{i,j})_{i,j=1,2,3}$, the classification is described as follows:

$$m_{11}=0\to\quad\begin{cases}m_{12}m_{21}=0,\ (m_{12},m_{21})\neq(0,0)\to E_7^{(1)}\\ \qquad\qquad\qquad\qquad m_{12}m_{21}\neq 0\to E_6^{(1)}\end{cases}$$

$m_{11}\neq 0\to$ We can make $m_{13}=m_{31}=0$ by affine transformation.

$$\to\quad\begin{cases}\qquad\qquad\qquad m_{12}=m_{21}=0\to D_7^{(1)}\\ m_{12}m_{21}=0,\ (m_{12},m_{21})\neq(0,0)\to D_6^{(1)}\\ \qquad\qquad\qquad\qquad m_{12}m_{21}\neq 0\to D_5^{(1)}.\end{cases}$$

Here we omitted the case that $m_{11}=m_{12}=m_{21}=0$ because of elliptic curves' absence. Hence this completes the classification.

When $m_{11}\neq 0$, we can use solutions a,b to $a^2m_{11}+am_{12}+m_{13}=0$, $b^2m_{11}+bm_{21}+m_{31}=0$ for an affine transformation in order to make $m_{13}=m_{31}=0$.

In the next place we consider the logarithmic case. There are transformations of dependent variables $(f,g)\mapsto(g,f)$, $(f,g)\mapsto(1/f,g)$, and $(f,g)\mapsto(f,1/g)$, and they don't change the type of the systems.

Now we introduce notation. To express an entry of a matrix, $*$ is an element of $\mathbb{C}\setminus\{0\}$, and empty means that this entry is occupied by an arbitrary element of $\mathbb{C}$. Then biquadratic forms, from $A_8^{(1)}$ to $A_3^{(1)}$, are given as

$$A_8^{(1)}:\begin{pmatrix}*&0&0\\0&&*\\0&*&0\end{pmatrix},\qquad A_7^{(1)\prime}:\begin{pmatrix}0&*&0\\ *&&*\\0&*&0\end{pmatrix},\qquad A_7^{(1)}:\begin{pmatrix}*&0&0\\ *&&*\\0&*&0\end{pmatrix},$$

$$A_6^{(1)} : \begin{pmatrix} \ast & \ast & 0 \\ \ast & & \ast \\ 0 & \ast & 0 \end{pmatrix}, \qquad A_5^{(1)} : \begin{pmatrix} \ast & & \ast \\ \ast & & \ast \\ 0 & \ast & 0 \end{pmatrix},$$

$$A_4^{(1)} : \begin{pmatrix} \ast & & \ast \\ & & \ast \\ \ast & \ast & 0 \end{pmatrix}, \qquad A_3^{(1)} : \begin{pmatrix} \ast & \ast \\ & \\ \ast & \ast \end{pmatrix}.$$

Inversely, it is found that the biquadratic forms with these matrices correspond to these surfaces with suitable parameters.

If the matrix has all 0 in the 1st or 3rd column or row, then curves in the pencil does not give elliptic curves. The matrix with $m_{11} = m_{12} = m_{21} = 0$ does not give elliptic curves, too. Hence there remain only 7 types of matrices:

$$\begin{pmatrix} \ast & & \ast \\ 0 & & 0 \\ 0 & \ast & 0 \end{pmatrix}, \quad \begin{pmatrix} \ast & \ast & 0 \\ 0 & & 0 \\ 0 & \ast & \ast \end{pmatrix}, \quad \begin{pmatrix} \ast & 0 & 0 \\ 0 & & \ast \\ 0 & \ast & \ast \end{pmatrix}, \quad \begin{pmatrix} \ast & \ast & 0 \\ \ast & & 0 \\ 0 & \ast & \ast \end{pmatrix},$$

$$\begin{pmatrix} \ast & & \ast \\ 0 & & \ast \\ 0 & \ast & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \ast & \ast \\ \ast & & \ast \\ \ast & \ast & 0 \end{pmatrix}, \quad \begin{pmatrix} \ast & & \ast \\ 0 & & \\ 0 & \ast & \ast \end{pmatrix}.$$

These all can be reduced to the systems of $A_8^{(1)}, \ldots, A_3^{(1)}$ by simple transformations of dependent variables.

Remark 3 We take two of them as examples. We reduce 4th matrix to the matrix of $A_6^{(1)}$. It is enough to put $F = fg$:

$$\left(g, 1, \frac{1}{g}\right) \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & e & h \end{pmatrix} \begin{pmatrix} F/g \\ 1 \\ g/F \end{pmatrix} = \left(g, 1, \frac{1}{g}\right) \begin{pmatrix} 0 & b & 0 \\ a & d & h \\ c & e & 0 \end{pmatrix} \begin{pmatrix} F \\ 1 \\ 1/F \end{pmatrix}.$$

The next example is a transformation which change the 1st matrix into the matrix of $A_7^{(1)\prime}$. For the biquadratic form

$$H + \mu = \left(g, 1, \frac{1}{g}\right) \begin{pmatrix} a & b & c \\ 0 & \mu & 0 \\ 0 & d & 0 \end{pmatrix} \begin{pmatrix} f \\ 1 \\ 1/f \end{pmatrix},$$

we take α as a solution to $\alpha^2 - b\alpha + ac = 0$, and H can be written as $H = g(\frac{a}{\alpha} f + 1)(\frac{c}{f} + \alpha) + \frac{d}{g}$. Putting $G = g(\frac{1}{f} + \frac{\alpha}{c})$, $F = fG$, we get

$$H = \frac{ac}{\alpha} F + cG + \frac{d}{F} + \frac{d\alpha}{c} \frac{1}{G},$$

and this is the system of type $A_7^{(1)\prime}$.

We can reduce the others to known systems similarly.

Classification is given as follows:

$$A_8^{(1)}:\begin{pmatrix} * & 0 & 0 \\ 0 & & * \\ 0 & * & 0 \end{pmatrix}, \qquad A_7^{(1)\prime}:\begin{pmatrix} 0 & * & 0 \\ * & & * \\ 0 & * & 0 \end{pmatrix} \sim \begin{pmatrix} * & & * \\ 0 & & 0 \\ 0 & * & 0 \end{pmatrix} \sim \begin{pmatrix} * & * & 0 \\ 0 & & 0 \\ 0 & * & * \end{pmatrix},$$

$$A_7^{(1)}:\begin{pmatrix} * & 0 & 0 \\ * & & * \\ 0 & * & 0 \end{pmatrix} \sim \begin{pmatrix} * & 0 & 0 \\ 0 & & * \\ 0 & * & * \end{pmatrix},$$

$$A_6^{(1)}:\begin{pmatrix} * & * & 0 \\ * & & * \\ 0 & * & 0 \end{pmatrix} \sim \begin{pmatrix} * & * & 0 \\ * & & 0 \\ 0 & * & * \end{pmatrix} \sim \begin{pmatrix} * & & * \\ 0 & & * \\ 0 & * & 0 \end{pmatrix},$$

$$A_5^{(1)}:\begin{pmatrix} * & & * \\ * & & * \\ 0 & * & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & * & * \\ * & & * \\ * & * & 0 \end{pmatrix} \sim \begin{pmatrix} * & & * \\ 0 & & \\ 0 & * & * \end{pmatrix},$$

$$A_4^{(1)}:\begin{pmatrix} * & & * \\ & & * \\ * & * & 0 \end{pmatrix}, \qquad A_3^{(1)}:\begin{pmatrix} * & * \\ & \\ * & * \end{pmatrix}.$$

5 Oguiso-Shioda's Classification

We fixed one singular fiber as a ruling fiber so far for a study of dynamical systems on rational elliptic surfaces. We have 21 types of singular fibers, if we count two types for $A_7^{(1)}$. But there are 74 types of surfaces in the classification of Oguiso-Shioda ([6]). This is because a surface may have more than one singular fiber.

In Sect. 3 we constructed Hamilton systems on each surfaces. What we studied there includes the cases of plural singular fibers. In this section we see cases of plural singular fibers in detail, by taking $D_5^{(1)}$ as an example.

When we fix the fiber of type $D_5^{(1)}$ as the ruling fiber, the type of the surface is one of 16th, 30th, 50th, 52th, or 72th surface in the list of Oguiso-Shioda. Reducible singular fibers of each surfaces are described as

$$T = D_5, \qquad D_5 \oplus A_1, \qquad D_5 \oplus A_2, \qquad D_5 \oplus A_1^{\oplus 2}, \qquad D_5 \oplus A_3,$$

when we use Oguiso-Shioda's notation. Irreducible singular fibers of types $A_0^{(1)*}$, $A_0^{(1)**}$ may appear, but we don't consider about them. K. Oguiso and T. Shioda didn't use them for their classification.

Our aim is to determine the type of the surface at each value of the parameters. The matrix for type $D_5^{(1)}$ is

$$M_{D_5} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -a_1 - a_3 - 1 & a_2 \\ 0 & a_1 & 0 \end{pmatrix}.$$

If the condition $a_1=0$ $(a_2a_3a_0\neq 0)$, or $a_2=0$ $(a_3a_0a_1\neq 0)$, or $a_3=0$ $(a_0a_1a_2\neq 0)$, or $a_0=-a_1-a_2-a_3=0$ $(a_1a_2a_3\neq 0)$ holds, a fiber of type $A_1^{(1)}$ exists and the surface is 30th $(T=D_5\oplus A_1)$.

For the 30th surfaces, we see a calculation of singular fibers. In the pencil $\{H=\mu;\mu\in\mathbb{P}^1\}$, the fiber becomes type $A_1^{(1)}$ when the fiber is reducible. The curve $H+\mu=g^2f^2-g^2f+gf^2-(a_1+a_3+1)gf+a_1g+a_2f+\mu$ can be factorized as

$$H+\mu=\begin{cases} f(g^2f-g^2+gf-(a_3+1)g+a_2), & \text{if } a_1=0\ (a_2a_3a_0\neq 0),\ \mu=0,\\ g(gf^2-gf+f^2-(a_1+a_3+1)f+a_1), & \text{if } a_2=0\ (a_3a_0a_1\neq 0),\\ & \mu=0,\\ (f-1)(g^2f+gf-a_1g+a_2), & \text{if } a_3=0\ (a_0a_1a_2\neq 0),\ \mu=-a_2,\\ (g+1)(gf^2-gf+a_2f+a_1), & \text{if } a_0=a_1-a_2-a_3=0,\\ & (a_1a_2a_3\neq 0),\ \mu=a_1,\end{cases}$$

at each parameterizations, and they are the singular fibers of type $A_1^{(1)}$.

The 50th surface $(T=D_5\oplus A_2)$ appears when the parameters hold that $a_1=a_2=0$ $(a_3a_0\neq 0)$, or that $a_2=a_3=0$ $(a_1a_0\neq 0)$, or that $a_0=a_1=0$ $(a_2a_3\neq 0)$. The singular fibers of type $A_2^{(1)}$ are calculated as follows:

$$H+\mu=\begin{cases} gf(gf-g+f-a_3-1), & \text{if } a_1=a_2=0\ (a_3a_0\neq 0),\ \mu=0,\\ g(f-1)(gf+f-a_1), & \text{if } a_2=a_3=0\ (a_0a_1\neq 0),\ \mu=0,\\ (g+1)f(gf-g+a_2), & \text{if } a_0=a_1=0\ (a_2a_3\neq 0),\ \mu=0.\end{cases}$$

The 52th surface $(T=D_5\oplus A_1^{\oplus 2})$ appears at $a_1=a_3=0$ $(a_2a_0\neq 0)$, or $a_2=a_0=0$ $(a_1a_3\neq 0)$, or $a_0=a_3=0$ $(a_1a_2\neq 0)$. We can see the singular fiber of type $A_1^{(1)}$ as

$$H+\mu=\begin{cases} f(g^2f-g^2+fg-g+a_2), & \text{if } \mu=0,\\ (f-1)(g^2f+fg+a_2), & \text{if } \mu=-a_2,\end{cases}$$

when the parameters hold that $a_1=a_3=0$ $(a_2a_0\neq 0)$. The singular fiber is

$$H+\mu=\begin{cases} g(gf^2-gf+f^2-f+a_1), & \text{if } \mu=0,\\ (g+1)(gf^2-gf+a_1), & \text{if } \mu=a_1,\end{cases}$$

when $a_2=a_0=0$ $(a_1a_3\neq 0)$. The singular fiber is

$$H+\mu=\begin{cases} (g+1)(gf^2-gf+a_2f+a_1), & \text{if } \mu=a_1,\\ (f-1)(g^2f+gf-a_1g+a_2), & \text{if } \mu=-a_2,\end{cases}$$

when $a_3=a_0=0$ $(a_1a_2\neq 0)$.

The 72nd surface ($T = D_5 \oplus A_3$) appears at $a_1 = a_2 = a_3 = a_0 = 0$, and the singular fiber of type $A_3^{(1)}$ is expressed by

$$H + \mu = gf(g+1)(f-1)$$

with $\mu = 0$.

For the 30th, 50th, 52nd, and 72nd surfaces we can take the other singular fiber as the ruling fiber. For example we can take the singular fiber of type $A_3^{(1)}$ for the 72nd surface. In this case the correspondence between two coordinates gives us a transformation of our systems. We will see this later in the next section.

6 Bäcklund Transformations

We treat only birational transformations as Bäcklund transformations here. In a study of the Painlevé equations we know non-birational algebraic transformations called folding transformations at particular parameters. We have a classification of such transformations [15]. Although we are also able to consider such transformations for systems on rational elliptic surfaces, we concentrate on a study of birational transformations.

The systems on rational elliptic surfaces have Bäcklund transformations which is similar to these for the Painlevé equations. As is well known in a study of the Painlevé equations, the birational Bäcklund transformations are written in terms of affine Weyl groups symmetries. Let's see an example. In the case that the surface has a singular fiber of type $E_7^{(1)}$, the symmetry is written as affine Weyl group of type $A_1^{(1)}$ which is generated by transformations

$$\begin{aligned}
&s_1 : (f, g; \alpha_1) \mapsto \left(f + \frac{\alpha_1}{g}, g; -\alpha_1\right),\\
&\pi : (f, g; \alpha_1) \mapsto \left(-f, -g - f^2 - s; -\alpha_1\right),\\
&(s_0 = \pi \circ s_1 \circ \pi).
\end{aligned}$$

When the parameter α_1 is 0, then the surface is the 65th type in the Oguiso-Shioda's list. In that case a singular fiber of type $A_1^{(1)}$ appear as well as $E_7^{(1)}$, and the transformation s_1 becomes the identity.

We can use such data of the Painlevé equations for the rational elliptic surfaces (see [7–10, 13]).

Among these elements of Weyl group, a translation can be calculated as an automorphism of the surface. For the example above, $s_1 \circ \pi$ is a translation. These coincide with well known discrete integrable systems called QRT mappings ([12, 14]).

Remark 4 QRT mapping is defined as follows:

$$\bar{g} = \frac{u_1(f) - u_2(f)g}{u_2(f) - u_3(f)g}, \qquad \bar{f} = \frac{v_1(\bar{g}) - v_2(\bar{[g]})f}{v_2(\bar{g}) - v_3(\bar{g})f},$$

where

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = M_1 \begin{pmatrix} f^2 \\ f \\ 1 \end{pmatrix} \times M_0 \begin{pmatrix} f^2 \\ f \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = {}^t M_1 \begin{pmatrix} \bar{g}^2 \\ \bar{g} \\ 1 \end{pmatrix} \times {}^t M_0 \begin{pmatrix} \bar{g}^2 \\ \bar{g} \\ 1 \end{pmatrix}.$$

In order to make it suit our story, it is enough to set M_0 as the 3×3 matrix which define the ruling fiber, and put $M_1 = M$. Here the mapping $(f, g) \mapsto (\bar{f}, \bar{g})$ leaves the curve

$$\left(g^2, g, 1\right)(M_0 + \mu M_1) \begin{pmatrix} f^2 \\ f \\ 1 \end{pmatrix} = 0$$

invariant. Namely, the value of $H = -(g^2, g, 1)M_1{}^t(f^2, f, 1)/((g^2, g, 1)M_0{}^t(f^2, f, 1))$ is also invariant. The proof is not difficult (see Appendix A of [12]).

This mapping is a Bäcklund transformation of our system, because it is a canonical transformation, that is, $\bar{\omega} = \omega$. Here $\omega = dg \wedge df/((g^2, g, 1)M_0{}^t(f^2, f, 1))$. To prove this, we need the following calculation:

$$\begin{aligned}
&\frac{(v_2 - v_3 f)^2}{v_1 v_3 - v_2^2}\left(\bar{g}^2, \bar{g}, 1\right) M_0 \begin{pmatrix} \bar{f}^2 \\ \bar{f} \\ 1 \end{pmatrix} + \left(\bar{g}^2, \bar{g}, 1\right) M_0 \begin{pmatrix} f^2 \\ f \\ 1 \end{pmatrix} \\
&= \frac{1}{v_1 v_3 - v_2^2}\left(\bar{g}^2, \bar{g}, 1\right) M_0 \left[\begin{pmatrix} (v_1 - v_2 f)^2 \\ (v_1 - v_2 f)(v_2 - v_3 f) \\ (v_2 - v_3 f)^2 \end{pmatrix} + \left(v_1 v_3 - v_2^2\right) \begin{pmatrix} \bar{f}^2 \\ \bar{f} \\ 1 \end{pmatrix}\right] \\
&= \frac{1}{v_1 v_3 - v_2^2}\left(\bar{g}^2, \bar{g}, 1\right) M_0 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} (v_3, -2v_2, v_1) \begin{pmatrix} \bar{f}^2 \\ \bar{f} \\ 1 \end{pmatrix} = 0,
\end{aligned}$$

and so on. Using these, we get

$$\begin{aligned}
\bar{\omega} &= \frac{v_1 v_3 - v_2^2}{(v_2 - v_3 f)^2} \frac{d\bar{g} \wedge df}{(\bar{g}^2, \bar{g}, 1) M_0 \begin{pmatrix} \bar{f}^2 \\ \bar{f} \\ 1 \end{pmatrix}} \\
&= -\frac{d\bar{g} \wedge df}{(\bar{g}^2, \bar{g}, 1) M_0 \begin{pmatrix} f^2 \\ f \\ 1 \end{pmatrix}} = \frac{dg \wedge df}{(g^2, g, 1) M_0 \begin{pmatrix} f^2 \\ f \\ 1 \end{pmatrix}} = \omega.
\end{aligned}$$

On the other hand, when the surface has more than one singular fiber, we can construct a transformation which change the types of systems in the list. Such a transformation does not appear in the case of the Painlevé equations.

Let's look at an example of this kind of transformations. We see the surface of Oguiso-Shioda's 72nd type, which has singular fibers of type $D_5^{(1)}$ and of type $A_3^{(1)}$. If we set a singular fiber of type $D_5^{(1)}$ as the ruling fiber, then biquadratic form, in general, is given by the matrix

$$M=\begin{pmatrix}1 & -1 & 0\\ 1 & -(a_1+a_3+1) & a_1\\ 0 & a_2 & 0\end{pmatrix},\quad a_1+a_2+a_3=0.$$

When $a_1=a_2=a_3=0$, then a singular fiber of type $A_3^{(1)}$ also appears. Setting $H=(f^2,f,q)M^t(g^2,g,1)=fg(f-1)(g+1)$, the pencil $\{H=\mu\ ;\ \mu\in\mathbb{P}^1\}$ has the fiber of type $D_5^{(1)}$ at $\mu=\infty$, and the fiber of type $A_3^{(1)}$ at $\mu=0$. Our system can be given as

$$\frac{df}{dt}=\rho(H)\frac{\partial H}{\partial g},\qquad \frac{dg}{dt}=-\rho(H)\frac{\partial H}{\partial f},\quad H=fg(f-1)(g+1),$$

when we set the fiber of type $D_5^{(1)}$ as the ruling.

Our aim is to obtain a birational transformation which changes the form of the system from type $D_5^{(1)}$ into type $A_3^{(1)}$. The transformation of the independent variables is given by $(F,G)=(\frac{f-1}{f},\frac{g+1}{g})$. The Hamiltonian becomes $\widetilde{H}=1/H=(1-F)^2(1-G)^2/(FG)$, that is,

$$\widetilde{H}=\left(G,1,\frac{1}{G}\right)\begin{pmatrix}1 & -2 & 1\\ -2 & 4 & -2\\ 1 & -2 & 1\end{pmatrix}\begin{pmatrix}F\\ 1\\ \frac{1}{F}\end{pmatrix},$$

and this is a 3×3 matrix of type $A_3^{(1)}$. The differential system is written as

$$\frac{dF}{dt}=\widetilde{\rho}(\widetilde{H})FG\frac{\partial\widetilde{H}}{\partial G},\qquad \frac{dG}{dt}=-\widetilde{\rho}(\widetilde{H})FG\frac{\partial\widetilde{H}}{\partial F},\quad \widetilde{\rho}(\widetilde{H})=\rho(1/\widetilde{H})/\widetilde{H}.$$

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On the Spectral Gap of the Kac Walk and Other Binary Collision Processes on d-Dimensional Lattice

Makiko Sasada

Abstract We give a lower bound on the spectral gap for a class of binary collision processes. In ALEA Lat. Am. J. Probab. Math. Stat. 4, 205–222 (2008), Caputo showed that, for a class of binary collision processes given by simple averages on the complete graph, the analysis of the spectral gap of an N-component system is reduced to that of the same system for $N = 3$. In this paper, we give a comparison technique to reduce the analysis of the spectral gap of binary collision processes given by simple averages on d-dimensional lattice to that on the complete graph. We also give a comparison technique to reduce the analysis of the spectral gap of binary collision processes which are not given by simple averages to that given by simple averages. Combining them with Caputo's result, we give a new and elementary method to obtain spectral gap estimates. The method applies to a number of binary collision processes on the complete graph and also on d-dimensional lattice, including a class of energy exchange models which was recently introduced in arXiv:1109.2356, and zero-range processes.

1 Introduction

A sharp lower bound on the spectral gap of the process is essential to prove the hydrodynamic limit (cf. [8]). What is needed is that the gap, for the process confined to cubes of linear size N, shrinks at a rate N^{-2}. Up to constants, this is the best possible lower bound for a wide class of models discussed in the context of the study of the hydrodynamic limit.

Most of the techniques used to obtain the required lower bound rely on special features of the model, or a recursive approach due to Lu and Yau [10]. Recently, Caputo introduced a new and elementary method to obtain a lower bound on the spectral gap for some general class of binary collision processes which are reversible with respect to a family of product measures in [3]. In this paper, we extend his

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result in two ways. One way is that though in [3] only the process on the complete graph was considered, we consider the process on d-dimensional lattice where the interactions occur between nearest neighbor sites. We give a general method to compare the spectral gap of the local version on d-dimensional lattice and the original process on the complete graph. Secondly, we study a wider class of processes than the class studied in [3] and give a simple comparison technique between their spectral gaps. We emphasize that our technique can be applied to a wide class of processes which are reversible with respect to a family of product measures, and it allows to obtain the lower bound of the spectral gap easily. However, it is not necessarily sharp, so if the estimate given by our method is not enough sharp, then we need to try to use other techniques.

Following Caputo [3], we first consider the following energy conserving binary collision model introduced by M. Kac in [7], called Kac walk. Let $\nu = \nu_{N,\omega}$ denote the uniform probability measure on the $N-1$ dimensional sphere of the radius $\sqrt{\omega}$

$$S^{N-1}(\omega) = \left\{ \eta \in \mathbb{R}^N; \sum_{i=1}^{N} \eta_i^2 = \omega \right\},$$

and consider the ν-reversible Markov process on $S^{N-1}(\omega)$ with infinitesimal generator given by

$$\mathcal{L}^* f(\eta) = \frac{1}{2N} \sum_{i,j=1}^{N} D_{i,j} f(\eta)$$

where

$$D_{i,j} f(\eta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[f\left(R_\theta^{ij} \eta\right) - f(\eta) \right] d\theta,$$

and R_θ^{ij}, $i \neq j$ is a clockwise rotation of angle θ in the plane (η_i, η_j). As a convention, we take $R_\theta^{ii} = \mathrm{Id}$.

This Kac walk represents a system of N particles in one dimension evolving under a random collision mechanism. The state of the system is given by specifying the N velocities $\eta_1, \eta_2, \dots, \eta_N$. The random collision mechanism under which the state evolves is that at random times, a "pair collision" take place in such a way that the total energy $\sum_{i=1}^{N} \eta_i^2$ is conserved. Under the above dynamics, after the particles i and j collide, the distribution of the velocities (η_i, η_j) becomes uniform on the plane (η_i, η_j).

Note that $-\mathcal{L}^*$ is a non-negative, bounded self-adjoint operator on $L^2(\nu)$. Any constant is an eigenfunction with eigenvalue 0 and the spectral gap $\lambda^* = \lambda^*(N, \omega)$ is defined as

$$\lambda^*(N, \omega) := \inf \left\{ \frac{\nu(f(-\mathcal{L}^*) f)}{\nu(f^2)} \middle| \nu(f) = 0, \ f \in L^2(\nu) \right\} \tag{1}$$

where $\nu(f)$ stands for the expectation $\int f d\nu$. We define $\lambda^*(N) = \inf_{\omega>0} \lambda^*(N,\omega)$. For the Kac walk, by change of variables, it is easy to see that $\lambda^*(N) = \lambda^*(N,\omega)$ for all $\omega > 0$.

In [4], Carlen, Carvalho and Loss computed the exact value of $\lambda^*(N)$ for every N:

$$\lambda^*(N) = \frac{N+2}{4N}, \quad N \geq 2. \tag{2}$$

Caputo gave a simplified method to show this. Recall Theorem 1.1 in [3].

Theorem 1 (Caputo) *For $N \geq 3$,*

$$\lambda^*(N) = \bigl(3\lambda^*(3) - 1\bigr)\left(1 - \frac{2}{N}\right) + \frac{1}{N}. \tag{3}$$

In particular, (2) follows from (3) with $\lambda^*(3) = \frac{5}{12}$.

Now, we introduce the local version of the Kac walk. Fix $d \in \mathbb{N}$ and let Λ_N the d-dimensional cube of linear size N : $\Lambda_N = \{1, 2, \ldots, N\}^d$. The local version of the Kac walk is the $\nu = \nu_{|\Lambda_N|,\omega} = \nu_{N^d,\omega}$-reversible Markov process on $S^{|\Lambda_N|-1}(\omega)$ with infinitesimal generator given by

$$\mathcal{L}^{*,loc} f(\eta) = \frac{1}{2} \sum_{x\in\Lambda_N} \sum_{\substack{y\in\Lambda_N \\ \|x-y\|=1}} D_{x,y} f(\eta) \tag{4}$$

where $\|x - y\| = \sum_{i=1}^d |x_i - y_i|$. We define the spectral gap $\lambda^{*,loc}(N,\omega)$ by (1) with $-\mathcal{L}^*$ replaced by $-\mathcal{L}^{*,loc}$, and $\lambda^{*,loc}(N) := \inf_{\omega>0} \lambda^{*,loc}(N,\omega)$. It is also easy to see that $\lambda^{*,loc}(N) = \lambda^{*,loc}(N,\omega)$ for all $\omega > 0$.

We give a comparison theorem for $\lambda^{*,loc}(N)$ and $\lambda^*(N)$.

Theorem 2

$$\lambda^{*,loc}(N) \geq \frac{1}{96dN^2}\lambda^*\bigl(|\Lambda_N|\bigr).$$

In particular, since $\lambda^*(|\Lambda_N|) \geq \frac{1}{4}$ for all $N \geq 2$ by (2),

$$\lambda^{*,loc}(N) \geq \frac{1}{384dN^2}. \tag{5}$$

In the proof, we use the invariance of ν under the exchange of coordinates repeatedly, and the idea of "moving particle lemma" which was developed for the study of the spectral gap of interacting particle systems with discrete spins (cf. [12]). Generally, it is not easy to show the estimate corresponding to "moving particle lemma" for the systems with continuous spins. However, for the generator of the form (4), we show that the estimate can be established. A proof of Theorem 2 is given in the next subsection.

Next, we consider a generalization of Kac walk introduced in [4] by Carlen et al. Let $\rho(\theta)$ be a probability density on the circle, i.e.

$$\int_{-\pi}^{\pi} \rho(\theta) = 1.$$

Consider a $\nu = \nu_{N,\omega}$-reversible Markov process on $S^{N-1}(\omega)$ with infinitesimal generator given by

$$\mathcal{L} f(\eta) = \frac{1}{2N} \sum_{i,j=1}^{N} \int_{-\pi}^{\pi} \left[f\left(R_\theta^{ij} \eta\right) - f(\eta) \right] \rho(\theta) d\theta.$$

We define the spectral gap $\lambda(N, \omega)$ by (1) with $-\mathcal{L}^*$ replaced by $-\mathcal{L}$ and $\lambda(N) := \inf_{\omega>0} \lambda(N, \omega)$. For this generalization, $\lambda(N) = \lambda(N, \omega)$ holds for any $\omega > 0$ again, since $\mathcal{L}$ commutes with the unitary change of scale from $S^{N-1}(\omega)$ to $S^{N-1}(\omega')$, for any $\omega, \omega' > 0$. Indeed, $\nu_{N,\omega'}$ is the image of $\nu_{N,\omega}$ under the map $T : \eta \to \frac{\sqrt{\omega'}\eta}{\sqrt{\omega}}$ and if $f_T(\eta) = f(T\eta)$, then

$$\nu_{N,\omega}\left(f_T(-\mathcal{L}) f_T\right) = \nu_{N,\omega'}\left(f(-\mathcal{L}) f\right) \tag{6}$$

holds. Note that, to guarantee $\lambda(N) > 0$, we need some more assumptions on ρ.

We introduce the local version of this generalized Kac walk described by the infinitesimal generator

$$\mathcal{L}^{loc} f(\eta) = \frac{1}{2} \sum_{x \in \Lambda_N} \sum_{\substack{y \in \Lambda_N \\ \|x-y\|=1}} \int_{-\pi}^{\pi} \left[f\left(R_\theta^{xy} \eta\right) - f(\eta) \right] \rho(\theta) d\theta$$

and define the spectral gap $\lambda^{loc}(N, \omega)$ and $\lambda^{loc}(N)$, which satisfying $\lambda^{loc}(N) = \lambda^{loc}(N, \omega)$ for all $\omega > 0$, in the same manner as before. In [4], under the assumption that $\rho(\theta)$ is continuous and $\rho(0) > 0$, it is shown that

$$\lambda(N) \geq \lambda(2) \frac{N+2}{2N}, \quad N \geq 2. \tag{7}$$

Under their assumption on $\rho(\theta)$, it is also proved that $\lambda(2) > 0$ and therefore $\lambda(N) > 0$.

Our next result shows that the proof of (7) can be somewhat simplified, and we also have a lower bound on $\lambda^{loc}(N)$. Note that since we only assume that $\rho(\theta)$ is a probability density on the circle, $\lambda(2)$ is not necessarily positive.

Theorem 3

$$\lambda(N) \geq 2\lambda(2)\lambda^*(N), \qquad \lambda^{loc}(N) \geq 2\lambda(2)\lambda^{*,loc}(N).$$

In particular, with (2), *we have* (7) *and with* (5), *we have*

$$\lambda^{loc}(N) \geq \lambda(2) \frac{1}{192 d N^2}.$$

In the next result, we also give an upper bound of $\lambda(N)$ and $\lambda^{loc}(N)$. Denote the supremum of the spectral of $-\mathcal{L}$ for $N=2$ by κ:

$$\kappa := \sup_{\omega>0} \sup \left\{ \frac{\nu_{2,\omega}(f(-\mathcal{L})f)}{\nu_{2,\omega}(f^2)} \middle| \nu_{2,\omega}(f)=0,\ f \in L^2(\nu_{2,\omega}) \right\}.$$

Theorem 4

$$\lambda(N) \le 2\kappa\lambda^*(N), \qquad \lambda^{loc}(N) \le 2\kappa\lambda^{*,loc}(N).$$

In particular, since $\kappa \le 1$, *we have* $\lambda(N) \le 2\lambda^*(N)$ *and* $\lambda^{loc}(N) \le 2\lambda^{*,loc}(N)$.

The key of the proofs of the above comparison theorems is the fact that $\nu((D_{i,j}f)^2)$ is the expectation of the variance of f with respect to $\nu(\cdot|\mathcal{F}_{i,j})$ where $\mathcal{F}_{i,j}$ is the sigma algebra generated by the coordinates $\{\eta_k; k \neq i, j\}$, and therefore this variance can be estimated by the term of $\lambda(2)$ or κ and the corresponding Dirichlet form. Proofs of Theorem 3 and Theorem 4 are given in the Sect. 1.2.

In Sect. 2, we shall show that variants of the same methods can be used to obtain spectral gap estimates for several models sharing some of the features of the Kac walk or the generalization of Kac walk. In Sec. 3, we give two examples of such processes.

1.1 Proof of Theorem 2

We first introduce operators $E_{i,j}$ appearing in the definition of $\mathcal{L}^*$ and $\pi_{i,j}$ which represents the exchange of the velocity of particles i and j:

$$E_{i,j}f(\eta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\big(R_\theta^{ij}\eta\big)d\theta, \qquad \pi_{i,j}f(\eta) = f(\pi_{i,j}\eta)$$

where

$$(\pi_{i,j}\eta)_k = \begin{cases} \eta_k & \text{if } k \neq i, j, \\ \eta_j & \text{if } k = i, \\ \eta_i & \text{if } k = j. \end{cases}$$

As a convention, we take $\pi_{i,i}\eta = \eta$. Note that $E_{i,j}$ is a projection which coincides with ν-conditional expectation given σ-algebra $\mathcal{F}_{i,j}$ generated by variables $\{\eta_k; k \neq i, j\}$. In other words, $E_{i,j}f = \nu(f|\mathcal{F}_{i,j})$ is an average of f on the (η_i, η_j) plane with respect to ν. Therefore, we regard this model as a binary collision process given by simple averages. Note that, by the definition $D_{i,j} = E_{i,j} - \mathrm{Id}$.

To compare the Dirichlet form with respect to the long range operators with that of the local operators, we first prepare preliminary lemmas.

Lemma 1 *For any x, y and $z \in \Lambda_N$ satisfying $y \neq z$,*

$$\nu\big((D_{x,y}f)^2\big) \le 6\nu\big((\pi_{x,z}f - f)^2\big) + 3\nu\big((D_{z,y}f)^2\big)$$

for all $f \in L^2(\nu)$.

Proof If $x = y$, then $\nu((D_{x,y}f)^2) = 0$, so the inequality obviously holds. On the other hand, if $x \neq y$ and $y \neq z$, then for any η,

$$E_{x,y}f(\eta) = \pi_{x,z}\big(E_{z,y}(\pi_{x,z}f)\big)(\eta) = \big(E_{z,y}(\pi_{x,z}f)\big)(\pi_{x,z}\eta).$$

Therefore, by Schwarz's inequality and change of variables, we have

$$\begin{aligned}
\nu\big((D_{x,y}f)^2\big) &= \nu\big((E_{x,y}f - f)^2\big) = \nu\big(\big\{\big(E_{z,y}(\pi_{x,z}f)\big)(\pi_{x,z}\eta) - f(\eta)\big\}^2\big) \\
&= \nu\big(\big\{\big(E_{z,y}(\pi_{x,z}f)\big)(\eta) - (E_{z,y}f)(\eta) \\
&\quad + (E_{z,y}f)(\eta) - f(\eta) + f(\eta) - f(\pi_{x,z}\eta)\big\}^2\big) \\
&\le 3\nu\big(\big\{\big(E_{z,y}(\pi_{x,z}f)\big)(\eta) - (E_{z,y}f)(\eta)\big\}^2\big) + 3\nu\big((D_{z,y}f)^2\big) \\
&\quad + 3\nu\big((\pi_{x,z}f - f)^2\big).
\end{aligned}$$

Finally, since $\nu(\{(E_{z,y}(\pi_{x,z}f))(\eta) - (E_{z,y}f)(\eta)\}^2) = \nu(\{E_{z,y}(\pi_{x,z}f - f)\}^2) \le \nu(E_{z,y}(\pi_{x,z}f - f)^2) = \nu((\pi_{x,z}f - f)^2)$, we complete the proof. □

Lemma 2 *For any $x, y \in \Lambda_N$,*

$$\nu\big((\pi_{x,y}f - f)^2\big) \le 4\nu\big((D_{x,y}f)^2\big).$$

Proof Since $E_{x,y}f(\eta) = E_{x,y}f(\pi_{x,y}\eta)$, by Schwarz's inequality, we have

$$\nu\big((\pi_{x,y}f - f)^2\big) = \nu\big((\pi_{x,y}f - E_{x,y}f + E_{x,y}f - f)^2\big) \le 4\nu\big((D_{x,y}f)^2\big).$$

□

Proof of Theorem 2 For each pair $x, y \in \Lambda_N$ $(x \neq y)$, choose a canonical path $\Gamma(x, y) = (x = z_0, z_1, \dots, z_{n(x,y)} = y)$ where $n(x, y) \in \mathbb{N}$ and $\|z_i - z_{i+1}\| = 1$ for $0 \le i \le n(x, y) - 1$ by moving first in the first coordinate direction, then in the second coordinate direction, and so on. Then, by Lemma 1, we have

$$\nu\big((D_{x,y}f)^2\big) \le 6\nu\big((\pi_{x,z_{n(x,y)-1}}f - f)^2\big) + 3\nu\big((D_{z_{n(x,y)-1},y}f)^2\big). \tag{8}$$

On the other hand, since

$$\begin{aligned}
\pi_{x,z_{n(x,y)-1}} &= \pi_{z_0,z_1} \circ \pi_{z_1,z_2} \circ \cdots \circ \pi_{z_{n(x,y)-3},z_{n(x,y)-2}} \circ \pi_{z_{n(x,y)-2},z_{n(x,y)-1}} \\
&\quad \circ \pi_{z_{n(x,y)-3},z_{n(x,y)-2}} \circ \cdots \circ \pi_{z_1,z_2} \circ \pi_{z_0,z_1},
\end{aligned}$$

by Schwarz's inequality

$$\nu\big((\pi_{x,z_{n(x,y)-1}} f - f)^2\big) \le 4n(x,y) \sum_{i=0}^{n(x,y)-2} \nu\big((\pi_{z_i,z_{i+1}} f - f)^2\big). \tag{9}$$

Therefore, combining the inequalities (8), (9) and Lemma 2, we have

$$\nu\big((D_{x,y} f)^2\big) \le 96\, n(x,y) \sum_{i=0}^{n(x,y)-1} \nu\big((D_{z_i,z_{i+1}} f)^2\big).$$

Then, by the construction of canonical paths,

$$\begin{aligned}
\nu\big(f(-\mathcal{L}^*)f\big) &= \frac{1}{|\Lambda_N|} \sum_{x,y\in\Lambda_N} \nu\big((D_{x,y} f)^2\big) \\
&\le 96dN \frac{1}{|\Lambda_N|} \sum_{x,y\in\Lambda_N} \sum_{i=0}^{n(x,y)-1} \nu\big((D_{z_i,z_{i+1}} f)^2\big) \\
&\le 96dN^2 \sum_{\substack{x,y\in\Lambda_N \\ \|x-y\|=1}} \nu\big((D_{x,y} f)^2\big) = 96dN^2 \nu\big(f(-\mathcal{L}^{*,loc})f\big).
\end{aligned}$$

□

Remark 1 The key ideas of the proof of Theorem 2, Lemmas 1 and 2 were exactly same as the ideas presented in Sect. 2.5 of [1].

1.2 Proof of Theorem 3 and Theorem 4

We define an operator $\mathcal{L}_0$ on $L^2(\nu_{2,\omega})$ as

$$\mathcal{L}_0 f(\eta) = \frac{1}{2}\left\{ \int_{-\pi}^{\pi} \big[f\big(R_\theta^{12}\eta\big) - f(\eta)\big]\rho(\theta)d\theta + \int_{-\pi}^{\pi} \big[f\big(R_\theta^{21}\eta\big) - f(\eta)\big]\rho(\theta)d\theta \right\}$$

where $\eta \in \mathbb{R}^2$. For $N \ge 3$, $\eta \in \mathbb{R}^N$, $1 \le i < j \le N$ and $f : \mathbb{R}^N \to \mathbb{R}$, define $f_\eta^{i,j} : \mathbb{R}^2 \to \mathbb{R}$ as

$$f_\eta^{i,j}(p,q) = f(\eta_1, \eta_2, \ldots, \eta_{i-1}, p, \eta_{i+1}, \ldots, \eta_{j-1}, q, \eta_{j+1}, \ldots, \eta_N).$$

Then, we can rewrite the Markov generator as follows:

$$\mathcal{L} f(\eta) = \frac{1}{N} \sum_{i<j} \mathcal{L}_{i,j} f(\eta)$$

where $\mathcal{L}_{i,j} f(\eta) = (\mathcal{L}_0 f_\eta^{i,j})(\eta_i, \eta_j)$. Note that $f_\eta^{i,j}$ does not depend on η_i and η_j. Then, we have

$$\begin{aligned} \nu\big(f(-\mathcal{L}_{i,j})f\big) &= \nu\big(f_\eta^{i,j}(\eta_i,\eta_j)\big((-\mathcal{L}_0)f_\eta^{i,j}\big)(\eta_i,\eta_j)\big) \\ &= \nu\big(\nu_{2,\eta_i^2+\eta_j^2}\big(f_\eta^{i,j}(-\mathcal{L}_0)f_\eta^{i,j}\big)\big). \end{aligned} \tag{10}$$

Note that for $N=2$, $\mathcal{L} = \frac{1}{2}\mathcal{L}_0$. Therefore, by definition, we have for any $\omega > 0$ and $g \in L^2(\nu_{2,\omega})$,

$$2\lambda(2)\nu_{2,\omega}\big(\{g - \nu_{2,\omega}(g)\}^2\big) \le \nu_{2,\omega}\big(g(-\mathcal{L}_0)g\big) \le 2\kappa\nu_{2,\omega}\big(\{g-\nu_{2,\omega}(g)\}^2\big).$$

Since

$$\begin{aligned} \nu\big(\nu_{2,\eta_i^2+\eta_j^2}\big(\{f_\eta^{i,j} - \nu_{2,\eta_i^2+\eta_j^2}(f_\eta^{i,j})\}^2\big)\big) &= \nu\big(\{f_\eta^{i,j} - \nu_{2,\eta_i^2+\eta_j^2}(f_\eta^{i,j})\}^2\big) \\ &= \nu\big(\{f - E_{i,j}f\}^2\big), \end{aligned}$$

we have

$$2\lambda(2)\nu\big((D_{i,j}f)^2\big) \le \nu\big(f(-\mathcal{L}_{i,j})f\big) \le 2\kappa\nu\big((D_{i,j}f)^2\big).$$

Finally, it follows that

$$2\lambda(2)\nu\big(f\big(-\mathcal{L}^*\big)f\big) \le \nu\big(f(-\mathcal{L})f\big) \le 2\kappa\nu\big(f\big(-\mathcal{L}^*\big)f\big)$$

and therefore $2\lambda(2)\lambda^*(N) \le \lambda(N) \le 2\kappa\lambda^*(N)$. In the same way, $2\lambda(2)\lambda^{*,loc}(N) \le \lambda^{loc}(N) \le 2\kappa\lambda^{*,loc}(N)$ is shown.

Now, it remains to show that $\kappa \le 1$. This follows from this simple inequality obtained by Schwarz's inequality:

$$\begin{aligned} \nu_{2,\omega}\big(f(-\mathcal{L})f\big) &= \frac{1}{8}\nu_{2,\omega}\left(\int_{-\pi}^{\pi}\big[f\big(R_\theta^{12}\eta\big) - f(\eta)\big]^2\big(\rho(\theta)+\rho(-\theta)\big)d\theta\right) \\ &\le \frac{1}{4}\nu_{2,\omega}\left(\int_{-\pi}^{\pi}\big[f\big(R_\theta^{12}\eta\big)^2 + f(\eta)^2\big]\big(\rho(\theta)+\rho(-\theta)\big)d\theta\right) \\ &= \nu_{2,\omega}\big(f^2\big). \end{aligned}$$

We use only here the assumption that $\rho(\theta)$ is a probability density on the circle.

2 General Setting

The general setting can be described as follows. We consider a product space $\Omega = X^N$ where X, the single component space is a measurable space equipped with a probability measure μ. On Ω, we consider the product measure μ^N. Elements of

Ω will be denoted by $\eta = (\eta_1, \eta_2, \dots, \eta_N)$. Next, we take a measurable function $\xi : X \to \mathbb{R}^m$, for a given $m \geq 1$, and we define the probability measure $\nu = \nu_{N,\omega}$ on Ω as μ^N conditioned on the event

$$\Omega_{N,\omega} := \left\{ \eta \in \Omega; \sum_{i=1}^{N} \xi(\eta_i) = \omega \right\},$$

where $\omega \in \Theta_N$ is a given parameter and $\Theta_N := \{\sum_{i=1}^N \xi(\eta_i); \eta \in X^N\}$. We interpret the constraint on $\Omega_{N,\omega}$ as a conservation law.

In all the examples considered below there are no difficulties in defining the conditional probability ν, therefore we do not attempt here at a justification of this setting in full generality but rather refer to the examples for full rigor. As pointed out in [3], the crucial property of ν is that, for any set of indices A, conditioned on the σ-algebra $\mathcal{F}_A$ generated by variables $\eta_i, i \notin A$, ν becomes the μ-product law over η_j, $j \in A$, conditioned on the event

$$\sum_{j \in A} \xi(\eta_j) = \omega - \sum_{i \notin A} \xi(\eta_i).$$

We introduce some notations in analogy with the last section. For $N \geq 3$, $\eta \in X^N$, $1 \leq i < j \leq N$ and $f : X^N \to \mathbb{R}$, define $f_\eta^{i,j} : X^2 \to \mathbb{R}$ as

$$f_\eta^{i,j}(p, q) = f(\eta_1, \eta_2, \dots, \eta_{i-1}, p, \eta_{i+1}, \dots, \eta_{j-1}, q, \eta_{j+1}, \dots, \eta_N).$$

For each $\omega \in \Theta_2$, fix a well defined (possibly unbounded, with dense domain denoted by $\mathcal{D}(\mathcal{L}_0)$) nonnegative self-adjoint operator $\mathcal{L}_0 = \mathcal{L}_0^\omega$ defined on $L^2(\nu_{2,\omega})$ satisfying $\mathcal{L}_0 f = 0$ if f is a constant function. We are interested in the process on $\Omega_{N,\omega}$ described by the infinitesimal generator

$$\mathcal{L} f(\eta) = \frac{1}{N} \sum_{i<j} \mathcal{L}_{i,j} f(\eta) \tag{11}$$

where $\mathcal{L}_{i,j} f(\eta) = (\mathcal{L}_0 f_\eta^{i,j})(\eta_i, \eta_j) = (\mathcal{L}_0^{\xi(\eta_i)+\xi(\eta_j)} f_\eta^{i,j})(\eta_i, \eta_j)$. In all the examples considered below there are no difficulties to see that for each $\omega \in \Theta_N$ there exits a dense subset of $L^2(\nu_{N,\omega})$ denoted by $\mathcal{D}(\mathcal{L})$ such that for all $f \in \mathcal{D}(\mathcal{L})$, $\mathcal{L} f \in L^2(\nu_{N,\omega})$ is well defined, and $f_\eta^{i,j} \in \mathcal{D}(\mathcal{L}_0)$ for all $i < j$ and $\eta \in \Omega_{N,\omega}$. Moreover, by the construction, $\mathcal{L}$ is nonnegative self-adjoint operator on $\mathcal{D}(\mathcal{L})$. As before, we refer to the examples for fully rigorous formulations.

We also define the local version of the dynamics on $\Omega_{\Lambda_N,\omega}$ defined by

$$\mathcal{L}^{loc} f(\eta) = \sum_{\substack{x,y \in \Lambda_N \\ \|x-y\|=1}} \mathcal{L}_{x,y} f(\eta)$$

where $\mathcal{L}_{x,y} f(\eta) = (\mathcal{L}_0 f_\eta^{x,y})(\eta_x, \eta_y)$. Here $f_\eta^{x,y}$ is defined in the same way as $f_\eta^{i,j}$, and the sum runs over all unordered pairs $x, y \in \Lambda_N$ satisfying $\|x - y\| = 1$.

The spectral gap $\lambda(N,\omega)$ (resp. $\lambda^{loc}(N,\omega)$) is defined by (1) with $L^2(\nu)$ replaced by $\mathcal{D}(\mathcal{L})$ (resp. $\mathcal{D}(\mathcal{L}^{loc})$), and $-\mathcal{L}^*$ replaced by $-\mathcal{L}$ (resp. $-\mathcal{L}^{loc}$). As a convention, we may set $\lambda(N,\omega)=+\infty$ if ω is such that the measure ν becomes a Dirac delta. This convention shall apply also for $\lambda^{loc}(N,\omega)$, $\lambda^*(N,\omega)$ and $\lambda^{*,loc}(N,\omega)$ where the last two terms will be defined below.

To obtain lower and upper bounds on $\lambda(N,\omega)$ and $\lambda^{loc}(N,\omega)$, we consider a binary collision process given by simple averages, which was introduced by Caputo in [3]. This process is described by the infinitesimal generator

$$\mathcal{L}^* f(\eta)=\frac{1}{N}\sum_b\big\{\nu[f|\mathcal{F}_b]-f\big\} \tag{12}$$

where the sum runs over all $\binom{N}{2}$ unordered pairs $b=\{i,j\}$ and $\nu[f|\mathcal{F}_b]$ is the ν-conditional expectation of f given the variables $\eta_k,\ k\notin b$. Setting, as before, $D_{i,j}=D_b=\nu[\cdot|\mathcal{F}_b]-\mathrm{Id}$. As usual, we refer to the examples for fully rigorous formulations. As in the last section, we also consider the local version of the process described by the infinitesimal generator

$$\mathcal{L}^{*,loc} f(\eta)=\sum_{\substack{x,y\in\Lambda_N\\ \|x-y\|=1}} D_{x,y}f(\eta) \tag{13}$$

where $\|x-y\|=\sum_{i=1}^d|x_i-y_i|$. Note that in all the examples considered below, it is easy to check that $\mathcal{D}(\mathcal{L}^*)=\mathcal{D}(\mathcal{L}^{*,loc})=L^2(\nu)$. The spectral gap $\lambda^*(N,\omega)$ is defined by (1) and $\lambda^{*,loc}(N,\omega)$ is defined by (1) with $-\mathcal{L}^*$ replaced by $-\mathcal{L}^{*,loc}$.

Remark 2 $\mathcal{L}^*$ can be seen as a special case of $\mathcal{L}$ in the form (11) with $L_0^\omega f=\nu_{2,\omega}(f)-f$ for $f\in L^2(\nu_{2,\omega})$.

First, we show a comparison theorem between $\lambda^{*,loc}(N,\omega)$ and $\lambda^*(N,\omega)$.

Theorem 5 *For any $N\ge 2$ and $\omega\in\Theta_N$,*

$$\lambda^{*,loc}(N,\omega)\ge\frac{1}{96dN^2}\lambda^*\big(|\Lambda_N|,\omega\big).$$

In particular,

$$\inf_{N\ge2}\inf_{\omega\in\Theta_N}\lambda^*(N,\omega)>0 \tag{14}$$

implies

$$\inf_{N\ge2}\inf_{\omega\in\Theta_{|\Lambda_N|}}N^2\lambda^{*,loc}(N,\omega)>0. \tag{15}$$

Proof We repeat the proof of Theorem 2. Indeed we only used the property that the generators $\mathcal{L}^*$ and $\mathcal{L}^{*,loc}$ are described in the forms (12), (13) with the special operators $D_{i,j}$. □

Now, we give a comparison theorem between $\lambda(N,\omega)$ (resp. $\lambda^{loc}(N,\omega)$) and $\lambda^*(N,\omega)$ (resp. $\lambda^{*,loc}(N,\omega)$). Define $\lambda(2)=\inf_{\omega\in\Theta_2}\lambda(2,\omega)$ and κ as

$$\kappa := \sup_{\omega\in\Theta_2}\sup\left\{\frac{\nu_{2,\omega}(f(-\mathcal{L})f)}{\nu_{2,\omega}(f^2)}\middle|\nu_{2,\omega}(f)=0,\ f\in\mathcal{D}(\mathcal{L})\right\}$$

where $\mathcal{L}$ is the generator for $N=2$, namely $\frac{1}{2}\mathcal{L}_0$. Here, as a convention, we may set $\sup\{\frac{\nu_{2,\omega}(f(-\mathcal{L})f)}{\nu_{2,\omega}(f^2)}|\nu_{2,\omega}(f)=0,\ f\in\mathcal{D}(\mathcal{L})\}=-\infty$ if ω is such that the measure ν becomes a Dirac delta.

Theorem 6 *For any $N\geq 2$ and $\omega\in\Theta_N$,*

$$2\lambda(2)\lambda^*(N,\omega)\leq\lambda(N,\omega)\leq 2\kappa\lambda^*(N,\omega), \tag{16}$$

$$2\lambda(2)\lambda^{*,loc}(N,\omega)\leq\lambda^{loc}(N,\omega)\leq 2\kappa\lambda^{*,loc}(N,\omega). \tag{17}$$

In particular, if $\lambda(2)>0$, then (14) implies

$$\inf_{N\geq 2}\inf_{\omega\in\Theta_N}\lambda(N,\omega)>0 \tag{18}$$

$$\textit{and}\quad \inf_{N\geq 2}\inf_{\omega\in\Theta_{|\Lambda_N|}}N^2\lambda^{loc}(N,\omega)>0. \tag{19}$$

On the other hand, if $\kappa<\infty$, then (18) implies (14), (15) and (19).

Proof We repeat the steps of the proofs of Theorem 3 and 4 to show (16) and (17). Indeed, this is a simple consequence of (10) with $\xi(\eta_i)+\xi(\eta_j)$ in place of $\eta_i^2+\eta_j^2$, and the fact that $\mathcal{L}=\frac{1}{2}\mathcal{L}_0$ for $N=2$, which always hold under our general setting. Note that since we assume that $\mathcal{L}_0 f=0$ for any constant f, we have for any $\omega\in\Theta_2$ and $g\in L^2(\nu_{2,\omega})$,

$$\nu_{2,\omega}\big(g(-\mathcal{L}_0)g\big)=\nu_{2,\omega}\Big(\big\{g-\nu_{2,\omega}(g)\big\}(-\mathcal{L}_0)\big\{g-\nu_{2,\omega}(g)\big\}\Big),$$

and therefore,

$$2\lambda(2)\nu_{2,\omega}\Big(\big\{g-\nu_{2,\omega}(g)\big\}^2\Big)\leq\nu_{2,\omega}\big(g(-\mathcal{L}_0)g\big)\leq 2\kappa\nu_{2,\omega}\Big(\big\{g-\nu_{2,\omega}(g)\big\}^2\Big).$$

The latter part of the theorem follows from Theorem 5 and the former part of the theorem immediately, noting that (18) implies $\lambda(2)>0$. □

Remark 3 There exist many of models with the spectral gap satisfying $\lambda(2,\omega)>0$ for all $\omega\in\Theta_2$, but $\lambda(2)=0$. For these models, it is clear that the required lower bound (18) or (19) does not hold. For these models, we should give the estimate of $\lambda(N,\omega)$ not only in terms of N but also in ω (cf. [12, 13]).

Remark 4 By definition, $\lambda^*(2,\omega) = \frac{1}{2}$ for all ω except for ω such that $\lambda^*(2,\omega) = \lambda(2,\omega) = +\infty$. Therefore, for $N = 2$, (16) states that the following trivial relation holds:

$$\lambda(2) \le \lambda(2,\omega) \le \kappa.$$

Theorem 7 *Assume* $\lambda^*(3) := \inf_{\omega\in\Theta_3} \lambda^*(3,\omega) > \frac{1}{3}$ *and* $\lambda(2) > 0$. *Then*, (14), (15), (18) *and* (19) *hold.*

Proof Caputo proved in [3] that for $N \ge 2$ and $\omega \in \Theta_N$,

$$\lambda^*(N,\omega) \ge \big(3\lambda^*(3) - 1\big)\left(1 - \frac{2}{N}\right) + \frac{1}{N}$$

holds. Therefore, $\lambda^*(3) > \frac{1}{3}$ implies (14) holds, and therefore (15) also holds by Theorem 5. Then, since we assume $\lambda(2) > 0$, (18) and (19) also hold by Theorem 6. □

Remark 5 Whether the condition $\lambda^*(3) > \frac{1}{3}$ (or (14)) holds or not depends only on the triplet (X, ξ, μ). Namely the analysis of the spectral gap of the process described by the infinitesimal generator of the form (11) is reduced to the analysis of the property of the triplet, that is, the state space, the conservation law and the reversible measure, and the spectral gap of the same system for $N = 2$.

Remark 6 It is known that $\lambda^*(3) > \frac{1}{3}$ is not the necessary condition for (14). Indeed, Caputo showed in [3] that $\lambda^*(4) := \inf_{\omega\in\mathbb{R}^m} \lambda^*(4,\omega) > \frac{1}{4}$ and $\lambda^*(3) > 0$ also implies (14).

3 Examples

3.1 Kac Walk

The model discussed in the introduction can be seen as a special case of our general setting, so that Theorem 2, Theorem 3 and Theorem 4 become special cases of Theorem 5 and Theorem 6, respectively. Here $X = \mathbb{R}$, $\xi(\eta) = \eta^2$ (with $m = 1$) and μ is the centered Gaussian measure with variance $v > 0$. The choice of v does not influence the determination of $\nu_{N,\omega}$. As shown in [3], this model satisfies $\lambda^*(3) > \frac{1}{3}$ and therefore (14) holds.

3.2 Energy Exchange Model

Here we consider a special class of the energy exchange models introduced in [6] by Grigo et al. We refer [6] for background and motivation on the model. Let $X = \mathbb{R}_+$,

$\xi(\eta) = \eta$ (with $m = 1$), and μ be the Gamma distribution with a shape parameter $\gamma > 0$ and a scale parameter 1, i.e.

$$\mu(d\eta) = \eta^{\gamma-1}\frac{e^{-\eta}}{\Gamma(\gamma)}d\eta.$$

Note that the choice of the scale parameter does not influence the determination of $\nu_{N,\omega}$. We consider a Markov process defined by its infinitesimal generator $\mathcal{L}^{loc}$, and $\mathcal{L}$ given by

$$\mathcal{L}f(\eta) = \frac{1}{N}\sum_{i<j}\mathcal{L}_{i,j}f(\eta), \qquad \mathcal{L}^{loc}f(\eta) = \sum_{\substack{x,y\in\Lambda_N\\ \|x-y\|=1}}\mathcal{L}_{x,y}f(\eta)$$

where $\mathcal{L}_{i,j}f(\eta) = (\mathcal{L}_0 f_\eta^{i,j})(\eta_i, \eta_j)$, $\mathcal{L}_{x,y}f(\eta) = (\mathcal{L}_0 f_\eta^{x,y})(\eta_x, \eta_y)$,

$$\mathcal{L}_0 f(\eta) = \Lambda(\eta_1, \eta_2)\int_{[0,1]} P(\eta_1, \eta_2, d\alpha)\big[f(T_\alpha \eta) - f(\eta)\big].$$

Here, $\Lambda : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a continuous function and $P(\eta_1, \eta_2, d\alpha)$ is a probability measure on $[0, 1]$, which depends continuously on $(\eta_1, \eta_2) \in \mathbb{R}_+^2$. The maps T_α model the energy exchange between two sites, and are defined by

$$T_\alpha \eta = \eta + \big[\alpha\eta_2 - (1-\alpha)\eta_1\big][\mathfrak{e}_1 - \mathfrak{e}_2]$$

where $\mathfrak{e}_i$ denotes the i-th unit vector of $\mathbb{R}^2$. In words, the associated Markov process given by $\mathcal{L}^{loc}$ with $d = 1$ goes as follows: Consider the one-dimensional lattice $\{1, 2, \ldots, N\}$. To every site i of this lattice we associate an energy $\eta_i \in X = \mathbb{R}_+$. The collection of all the energies is denoted by $\eta = (\eta_1, \ldots, \eta_N) \in X^N$. To each nearest neighbor pair of the lattice we associate an independent exponential clock with a rate Λ that depends on the energies of this pair η_i, η_{i+1}. As soon as one of the $N - 1$ clocks rings, say for the pair $(i, i + 1)$, then a number $0 \le \alpha \le 1$ is drawn according to a distribution P, that only depends on the two energies η_i, η_{i+1}. Then, the updated configuration of the energies is such that the new energy at site i is $\alpha(\eta_i + \eta_{i+1})$, the new energy at site $i + 1$ is $(1 - \alpha)(\eta_i + \eta_{i+1})$, and all other energies remain unchanged.

To guarantee the reversibility of the process with respect to μ^N (or μ^{Λ_N}), we assume the following:

Assumption 1 The rate function Λ and the transition kernel P are of the form

$$\begin{aligned}\Lambda(\eta_1, \eta_2) &= \Lambda_s(\eta_1 + \eta_2)\Lambda_r\left(\frac{\eta_1}{\eta_1 + \eta_2}\right),\\ P(\eta_1, \eta_2, d\alpha) &= P\left(\frac{\eta_1}{\eta_1 + \eta_2}, d\alpha\right).\end{aligned} \tag{20}$$

Moreover, $\Lambda_s(\sigma)\Lambda_r(\beta) > 0$ for all $\sigma > 0$ and $0 < \beta < 1$, $\sup_{0<\beta<1} \Lambda_r(\beta) < \infty$, and the Markov chain on [0, 1] with transition kernel $P(\beta, d\alpha)$ has a unique invariant distribution $p(\cdot)$ given by

$$p(d\beta) = d\beta\big[\beta(1-\beta)\big]^{\gamma-1}\frac{\Gamma(2\gamma)}{\Gamma(\gamma)^2}\Lambda_r(\beta)\frac{1}{Z}$$

where Z is the normalizing constant, and p is a reversible measure for the Markov chain generated by P.

Remark 7 Grigo et al. pointed in [6] that the representation (20) naturally occurs in models originating from mechanical systems.

Under Assumption 1, Grigo et al. showed in [6] that $\mathcal{L}$ (resp. $\mathcal{L}^{loc}$) is reversible with respect to the product measure μ^N (resp. μ^{Λ_N}). Therefore, we define the spectral gap $\lambda(N,\omega)$ of $\mathcal{L}$ and $\lambda^{loc}(N,\omega)$ of $\mathcal{L}^{loc}$ for each $\omega > 0$ as before.

Theorem 8 *If* $\inf_{\sigma>0} \Lambda_s(\sigma) > 0$, *then*

$$\inf_{N\geq 2}\inf_{\omega>0}\lambda(N,\omega) > 0 \quad \textit{and} \quad \inf_{N\geq 2}\inf_{\omega>0} N^2\lambda^{loc}(N,\omega) > 0.$$

To prove the theorem, we first study the spectral gap for the generator $\mathcal{L}^*$ and $\mathcal{L}^{*,loc}$, which are the special case of the above model given by

$$\Lambda_s^*(\sigma) = 1, \qquad \Lambda_r^*(\beta) = 1, \qquad P^*(\beta, d\alpha) = \frac{\Gamma(2\gamma)}{\Gamma(\gamma)^2}\big\{\alpha(1-\alpha)\big\}^{d-1}d\alpha.$$

By definition, we can easily check that

$$\mathcal{L}^* f(\eta) = \frac{1}{N}\sum_{i<j} D_{i,j}f(\eta), \qquad \mathcal{L}^{*,loc} f(\eta) = \sum_{\substack{x,y\in\Lambda_N \\ \|x-y\|=1}} D_{x,y}f(\eta),$$

and, by the unitary change of scale from $\Omega_{N,\omega}$ to $\Omega_{N,1}$, we have

$$\lambda^*(N,\omega) = \lambda^*(N,1) := \lambda^*(N), \qquad \lambda^{*,loc}(N,\omega) = \lambda^{*,loc}(N,1) := \lambda^{*,loc}(N).$$

To obtain the exact value of $\lambda^*(N)$ for $N \geq 2$, we recall Theorem 1.1 in [3]:

$$\lambda^*(N) \geq \big(3\lambda^*(3) - 1\big)\left(1 - \frac{1}{N}\right) + \frac{2}{N}. \tag{21}$$

Moreover, if there exists $\psi : X \to \mathbb{R}$ such that the function

$$f_3(\eta_1, \eta_2, \eta_3) = \sum_{i=1}^{3}\psi(\eta_i)$$

satisfies, for $N=3$, $L^* f_3 = -\lambda^*(3) f_3 + \text{const.}$, regardless of the value of $\sum_{i=1}^3 \eta_i$ (although the constant may depend on this value), then (21) can be turned into an identity for each $N \geq 2$.

Next, we apply the method introduced by Carlen, et al. in [4] to solve the 3-dimensional problem. This approach was already used in [3] to show that $\lambda^*(N) = \frac{N+1}{3N}$ if $\gamma = 1$.

Theorem 9 *For any* $\gamma > 0$,

$$\lambda^*(N) = \frac{\gamma N + 1}{N(2\gamma+1)}. \tag{22}$$

Proof As same way in Examples 2.2 in [3], we observe that when $N=3$, then $\mathcal{L}^* + 1$ coincides with the average operator P introduced in [4]. Therefore we can apply the general analysis of Sect. 2 in [4]. The outcome is that

$$\lambda^*(3) \geq \frac{1}{3} \min\{2+\mu_1, 2-2\mu_2\} \tag{23}$$

where the parameters μ_1 and μ_2 are given by

$$\mu_1 = \inf_{\phi} \nu\big(\phi(\eta_1)\phi(\eta_2)\big), \qquad \mu_2 = \sup_{\phi} \nu\big(\phi(\eta_1)\phi(\eta_2)\big)$$

with ϕ chosen among all functions $\phi : X \to \mathbb{R}$ satisfying $\nu(\phi(\eta_1)^2) = 1$ and $\nu(\phi(\eta_1)) = 0$. Here ν stands for $\nu_{3,\omega}$ but we have removed the subscripts for simplicity. One checks that the parameters μ_1, μ_2 do not depend on ω. Write $\mathcal{K}\phi(\zeta) = \nu[\phi(\eta_2)|\eta_1 = \zeta], \zeta > 0$. This defines a self-adjoint Markov operator on $L^2(\nu_1)$, where ν_1 is the marginal on η_1 of ν. In particular, the spectrum $Sp(\mathcal{K})$ of $\mathcal{K}$ contains 1 (with eigen space given by the constants). Then μ_1, μ_2 are, respectively, the smallest and the largest value in $Sp(\mathcal{K}) \setminus \{1\}$, as we see by writing $\nu(\phi(\eta_1)\phi(\eta_2)) = \nu[\phi(\eta_1)\mathcal{K}\phi(\eta_1)]$. This is now a one-dimensional problem and μ_1, μ_2 can be computed as follows. To fix ideas we use the value $\omega = 1$ for the conservation law $\eta_1 + \eta_2 + \eta_3$. In this case ν_1 is the law on $[0,1]$ with density $\frac{\Gamma(3\gamma)}{\Gamma(2\gamma)\Gamma(\gamma)} \eta^{\gamma-1}(1-\eta)^{2\gamma-1}$. Moreover,

$$\mathcal{K}\phi(\eta_1) = \frac{\Gamma(2\gamma)}{\Gamma(\gamma)^2(1-\eta_1)^{2\gamma-1}} \int_0^{1-\eta_1} \phi(\eta_2)\big\{\eta_2(1-\eta_1-\eta_2)\big\}^{\gamma-1} d\eta_2.$$

In particular, $\phi(\eta) = \eta - \frac{1}{3}$ is an eigenfunction of $\mathcal{K}$ with eigenvalue $-\frac{1}{2}$. Moreover, $\mathcal{K}$ preserves the degree of polynomials so that if Q_n denotes the space of all polynomials of degree $d \leq n$ we have $\mathcal{K}Q_n \subset Q_n$. By induction we see that for each $n \geq 1$ the polynomial $\zeta^n + q_{n-1}(\zeta)$, for a suitable $q_{n-1} \in Q_{n-1}$, is an eigenfunction with eigenvalue $\mu_n = (-1)^n \frac{\Gamma(2\gamma)\Gamma(n+\gamma)}{\Gamma(\gamma)\Gamma(n+2\gamma)}$, and it is orthogonal to Q_{n-1} in $L^2(\nu_1)$. Since the union of $Q_n, n \geq 1$, is dense in $L^2(\nu_1)$ this shows that there is a complete orthonormal set of eigenfunctions ϕ_n, where ϕ_n is a polynomial of degree n with eigenvalue μ_n and $Sp(\mathcal{K}) = \{\mu_n, n = 0, 1, \ldots\}$. Therefore we can take $\mu_1 = -\frac{1}{2}$ and $\mu_2 = \frac{1+\gamma}{2(1+2\gamma)}$ in the formula (23) and we conclude that $\lambda^*(3) \geq \frac{1+3\gamma}{3(1+2\gamma)}$.

To end the proof, we take $f = \eta_1^2 + \eta_2^2 + \eta_3^2$ and, using $\nu[\eta_1^2|\eta_2] = \frac{1+\gamma}{2(1+2\gamma)}(\eta_2 - 1)^2$, we compute

$$\mathcal{L}^* f(\eta) = -\frac{1+3\gamma}{3(1+2\gamma)} f(\eta) + const.$$

Thus, $\lambda^*(3) = \frac{1+3\gamma}{3(1+2\gamma)}$. Clearly, the unitary change of scale does not alter the form of the eigenfunction so that (22) follows. □

Remark 8 The consequence of Theorem 9 was shown in [5] with a different proof.

Remark 9 By Theorem 2.12 in [6], for $d = 1$, $\lambda^{*,loc}(N) \geq \frac{\gamma}{2\gamma+1}\sin^2(\frac{\pi}{N+2})$ holds. However, to estimate the spectral gap with degenerate rate function, namely the case where $\inf_{\sigma>0} \Lambda_s(\sigma) = 0$, we need to estimate the spectral gap on the complete graph (see [13]).

Proof of Theorem 8 By Theorem 6, we only need to show that $\lambda(2) = \inf_{\omega>0} \lambda(2, \omega) > 0$. By the assumption, for $N = 2$,

$$\begin{aligned}&\nu\big(f(-\mathcal{L})f\big)\\&\quad= \Lambda_s(\omega)\nu\left(\Lambda_r\left(\frac{\eta_1}{\eta_1+\eta_2}\right)\int_{[0,1]}\big\{f(T_\alpha\eta) - f(\eta)\big\}^2 P\left(\frac{\eta_1}{\eta_1+\eta_2}, d\alpha\right)\right).\end{aligned}$$

Therefore, by the unitary change of scale from $\Omega_{2,\omega}$ to $\Omega_{2,1}$, we have

$$\lambda(2, \omega) = \frac{\Lambda_s(\omega)}{\Lambda_s(1)}\lambda(2, 1).$$

Then, by our assumption, $\lambda(2, 1) > 0$ and therefore $\lambda(2) > 0$. □

3.3 Zero-Range Processes

The class of zero-range processes is one of the well-studied interacting particle systems (cf. [8]). Though the process is of gradient type, the lower bound estimate of the spectral gap itself has been considered as an interesting problem and studied by several people ([2, 9, 11]). Here, we take $X = \mathbb{N} \cup \{0\}$, $\xi(\eta) = \eta$, and consider a partition function $Z(\cdot)$ on $\mathbb{R}_+$ by

$$Z(\alpha) = \sum_{k\geq 0} \frac{\alpha^k}{g(1)g(2)\ldots g(k)}$$

where $g : \mathbb{N} \to \mathbb{R}_+$ is a positive function. Let α^* denote the radius of convergence of Z:

$$\alpha^* = \sup\{\alpha \in \mathbb{R}_+; Z(\alpha) < \infty\}.$$

In order to avoid degeneracy, we assume that the partition function Z diverges at the boundary of its domain of definition:

$$\lim_{\alpha \uparrow \alpha^*} Z(\alpha) = \infty.$$

For $0 \le \alpha < \alpha^*$, let p_α be the probability measure on X given by

$$p_\alpha(\eta = k) = \frac{1}{Z(\alpha)} \frac{\alpha^k}{g(k)!}, \quad k \in X$$

where $g(k)! = g(1)g(2)\dots g(k)$. Note that the choice of $0 \le \alpha < \alpha^*$ does not influence the determination of $\nu = \nu_{N,\omega}$.

First, we consider $\mathcal{L}^*$ defined by (12) and study the value of $\lambda^*(3)$. Following the same argument of the computation of $\lambda^*(3)$ in Example 3.2, we can show that

$$\lambda^*(3) \ge \frac{1}{3} \min\{2 + \mu_1, 2 - 2\mu_2\}$$

where the parameters μ_1, μ_2 are, respectively, the smallest and the largest value in $\{Sp(\mathcal{K}_n) \setminus \{1\}; n \in \mathbb{N}\}$ and $\mathcal{K}_n = (\mathcal{K}_{ij}^{(n)})$ is the $n \times n$ matrix given by

$$\mathcal{K}_{ij}^{(n)} = \begin{cases} \frac{1}{g(n-j)!g(i-1-(n-j))!} (\sum_{l=0}^{i-1} \frac{1}{g(l)!g(i-1-l)!})^{-1} & \text{if } i > n-j \\ 0 & \text{if } i \le n-j. \end{cases}$$

By the Perron-Frobenius theorem, $\mu_1 > -1$. Therefore, $\mu_2 < \frac{1}{2}$ is a sufficient condition for $\lambda^*(3) > \frac{1}{3}$. In [14], the set $\{Sp(\mathcal{K}_n) \setminus \{1\}; n \in \mathbb{N}\}$ is completely determined for the cases where $g(k) = 1$ for all $k \in \mathbb{N}$ or $g(k) = k$ for all $k \in \mathbb{N}$. In the former case, $\mu_2 = \frac{1}{3}$ and in the latter case $\mu_2 = \frac{1}{4}$. It concludes that $\lambda^*(3) > \frac{1}{3}$ for both cases and therefore (14) and (15) hold.

Next, we consider the generator of zero-range processes defined by

$$\mathcal{L} f(\eta) = \frac{1}{N} \sum_{i<j} \mathcal{L}_{i,j} f(\eta), \qquad \mathcal{L}^{loc} f(\eta) = \sum_{\substack{x,y \in \Lambda_N \\ \|x-y\|=1}} \mathcal{L}_{x,y} f(\eta)$$

where $\mathcal{L}_{i,j} f(\eta) = (\mathcal{L}_0 f_\eta^{i,j})(\eta_i, \eta_j)$, $\mathcal{L}_{x,y} f(\eta) = (\mathcal{L}_0 f_\eta^{x,y})(\eta_x, \eta_y)$,

$$\begin{aligned} \mathcal{L}_0 f(\eta_1, \eta_2) = g(\eta_1)\{f(\eta_1 - 1, \eta_2 + 1) - f(\eta_1, \eta_2)\} \\ + g(\eta_2)\{f(\eta_1 + 1, \eta_2 - 1) - f(\eta_1, \eta_2)\}. \end{aligned}$$

As a convention, we take $g(0) = 0$. To apply Theorem 7, we need to study $\lambda(2) = \inf_{\omega \in \mathbb{N}_{\geq 0}} \lambda(2, \omega)$. For the choice $g(k) = 1$ for all $k \in \mathbb{N}$, it is known that $\lambda(2) = 0$. On the other hand, a sufficient condition for $\lambda(2) > 0$ was given in [9] as follows:

Proposition 1 *Assume that the following two conditions are satisfied*:

(i) $\sup_k |g(k+1) - g(k)| < \infty$,
(ii) *There exists* $k_0 \in \mathbb{N}$ *and* $C > 0$ *such that* $g(k) - g(j) \geq C$ *for all* $k \geq j + k_0$.

Then, *we have* $\lambda(2) > 0$.

Theorem 10 *Assume that the two conditions in Proposition 1 are satisfied*, *and* $\mu_2 < \frac{1}{2}$. *Then*, (18) *and* (19) *hold*.

Proof By the above argument, we can apply Theorem 7 straightforwardly. □

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A Restricted Sum Formula for a q-Analogue of Multiple Zeta Values

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Abstract We prove a new linear relation for a q-analogue of multiple zeta values. It is a q-extension of the restricted sum formula obtained by Eie, Liaw and Ong for multiple zeta values.

1 Introduction

Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a multi-index of positive integers. We call the values r and $\sum_{i=1}^{r} \alpha_i$ *depth* and *weight* of α, respectively. If $\alpha_1 \geq 2$, we say that α is *admissible*. For an admissible index $(\alpha_1, \ldots, \alpha_r)$, *multiple zeta value* (MZV) is defined by

$$\zeta(\alpha_1, \ldots, \alpha_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{\alpha_1} \cdots m_r^{\alpha_r}}.$$

Let $I_0(r, n)$ be the set of admissible indices of depth r and weight n. In [3], Eie, Liaw and Ong proved the following relation called a restricted sum formula:

$$\sum_{\alpha \in I_0(b,n)} \zeta\left(\alpha_1, \ldots, \alpha_b, 1^a\right) = \sum_{\beta \in I_0(a+1, a+b+1)} \zeta(\beta_1 + n - b - 1, \beta_2, \ldots, \beta_{a+1}), \tag{1}$$

where $a \geq 0, b \geq 1, n \geq b+1$ and 1^a is an abbreviation of the subsequence $(1, \ldots, 1)$ of length a. It is a generalization of the sum formula proved in [4, 9], which is the equality (1) with $a = 0$.

In this paper we prove a q-analogue of the restricted sum formula. Let $0 < q < 1$. For an admissible index $\alpha = (\alpha_1, \ldots, \alpha_r)$, a *$q$-analogue of multiple zeta value*

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(qMZV) [1, 5, 10] is defined by

$$\zeta_q(\alpha_1, \dots, \alpha_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{q^{(\alpha_1 - 1)m_1 + \cdots + (\alpha_r - 1)m_r}}{[m_1]^{\alpha_1} \cdots [m_r]^{\alpha_r}},$$

where $[n]$ is the q-integer $[n] := (1 - q^n)/(1 - q)$. In the limit $q \to 1$, qMZV converges to MZV. The main theorem of this article claims that qMZV's also satisfy the restricted sum formula:

Theorem 1 *For any integers $a \geq 0, b \geq 1$ and $n \geq b + 1$, it holds that*

$$\sum_{\alpha \in I_0(b,n)} \zeta_q\left(\alpha_1, \dots, \alpha_b, 1^a\right) = \sum_{\beta \in I_0(a+1,a+b+1)} \zeta_q(\beta_1 + n - b - 1, \beta_2, \dots, \beta_{a+1}). \tag{2}$$

Setting $a = 0$ we recover the sum formula for qMZV obtained by Bradley [2]. In [7] Okuda and the author proved a q-analogue of Ohno-Zagier's relation for MZV's [6]. It claims that the sum of qMZV's of fixed depth, weight and *height*, the number of elements α_i grater than 1, is written as a polynomial of the values $\zeta_q(n)$ $(n \in \mathbb{Z}_{\geq 2})$ with rational coefficients. The left hand side of (2) is a similar sum, which contains only the qMZV's with a tail 1^a.

The strategy to prove Theorem 1 is similar to that of the proof for MZV's. However we should overcome some new difficulties. In the calculation of the q-analogue case, some additional terms are of the form $\sum_{n=1}^{\infty} q^{kn}/[n]^k$ $(k \in \mathbb{Z}_{\geq 1})$. In the limit of $q \to 1$, it becomes a harmonic sum $\sum 1/n^k$, but it is presumably beyond the class of q-series described by qMZV's. To control such terms we make use of algebraic formulation of multiple harmonic series given in Sect. 2.2. We introduce a noncommutative polynomial algebra $\mathfrak{D}$ which is an extension of the algebra used in the proof of a q-analogue of Kawashima's relation for MZV [8]. Then the proof of Theorem 1 is reduced to an algebraic calculation in $\mathfrak{D}$ as will be seen in Sect. 2.3. We proceed the algebraic computation in Sect. 2.4 and finish the proof of Theorem 1.

Throughout this article we assume that $0 < q < 1$. We denote the set of multi-indices of positive integers, including non-admissible ones, of depth r and weight n by $I(r, n)$.

2 Proof

2.1 Summation over Indices

For $b \geq 1, n \geq 2$ and $M \in \mathbb{Z}_{\geq 1}$, define

$$K_{b,n}(M) := \sum_{\alpha \in I_0(b,n)} \sum_{m_1 > m_2 > \cdots > m_{b-1} > m_b = M} \prod_{j=1}^{b} \frac{q^{(\alpha_j - 1)m_j}}{[m_j]^{\alpha_j}}.$$

Since $\alpha_1 \geq 2$, the infinite sum in the right hand side is convergent. Note that $K_{1,n}(M) = q^{(n-1)M}/[M]^n$.

For positive integers ℓ, β, M and N ($N > M$), we set

$$f_\ell(N,M) := \sum_{N=k_1>k_2>\cdots>k_\ell>M} \frac{q^{k_1-M}}{[k_1-M]} \prod_{j=2}^{\ell} \frac{1}{[k_j-M]},$$

$$g_{\ell,\beta}(M) := \sum_{M=m_1\geq m_2\geq\cdots\geq m_\ell\geq 1} \frac{q^{(\beta-1)m_1}}{[m_1]^\beta} \prod_{j=2}^{\ell} \frac{q^{m_j}}{[m_j]}.$$

We set $f_\ell(N,M) = 0$ unless $N > M$. Note that $g_{1,\beta}(M) = K_{1,\beta}(M)$.

Lemma 1 *For $M \geq 1$, $b \geq 1$ and $n \geq 2$, it holds that*

$$K_{b,n}(M) = g_{b,n-b+1}(M) - \sum_{s=1}^{b-1} \sum_{N=M+1}^{\infty} K_{b-s,n-s}(N) f_s(N,M).$$

Proof For $k \geq 2$ and $m_1 > m_2$, it holds that

$$\begin{aligned}
&\sum_{\beta\in I(2,k)} \frac{q^{(\beta_1-1)m_1+(\beta_2-1)m_2}}{[m_1]^{\beta_1}[m_2]^{\beta_2}} \\
&= \frac{1}{[m_1][m_2]} \left(\left(\frac{q^{m_1}}{[m_1]} \right)^{k-1} - \left(\frac{q^{m_2}}{[m_2]} \right)^{k-1} \right) \Big/ \left(\frac{q^{m_1}}{[m_1]} - \frac{q^{m_2}}{[m_2]} \right) \\
&= \frac{q^{(k-2)m_2}}{[m_2]^{k-1}} \frac{1}{[m_1-m_2]} - \frac{q^{(k-2)m_1}}{[m_1]^{k-1}} \frac{q^{m_1-m_2}}{[m_1-m_2]}.
\end{aligned}$$

Using the above formula repeatedly we get

$$\begin{aligned}
K_{b,n}(M) &= \sum_{m_1>\cdots>m_{b-1}>m_b=M} \frac{q^{m_1}}{[m_1]} \sum_{\beta\in I(b,n-1)} \prod_{j=1}^{b} \frac{q^{(\beta_j-1)m_j}}{[m_j]^{\beta_j}} \\
&= \sum_{m_1>\cdots>m_{b-1}>m_b=M} \frac{q^{m_1}}{[m_1]} \left(\prod_{j=1}^{b-1} \frac{1}{[m_j-m_b]} \right) \frac{q^{(n-b-1)m_b}}{[m_b]^{n-b}} \\
&\quad - \sum_{s=1}^{b-1} \sum_{m_{b-s}=M+1}^{\infty} K_{b-s,n-s}(m_{b-s}) f_s(m_{b-s}, M).
\end{aligned}$$

The first term of the right hand side above is rewritten as follows. Setting $m_j = \ell_j + \cdots + \ell_{b-1} + M$ $(j = 1, \ldots, b-1)$, we have

$$\sum_{m_1 > \cdots > m_{b-1} > m_b = M} \frac{q^{m_1}}{[m_1]} \left(\prod_{j=1}^{b-1} \frac{1}{[m_j - m_b]} \right) \frac{q^{(n-b-1)m_b}}{[m_b]^{n-b}}$$
$$= \frac{q^{(n-b-1)M}}{[M]^{n-b}} \sum_{\ell_1, \ldots, \ell_{b-1}=1}^{\infty} \frac{q^{\ell_1 + \cdots + \ell_{b-1} + M}}{[\ell_1 + \cdots + \ell_{b-1} + M]} \prod_{j=1}^{b-1} \frac{1}{[\ell_j + \cdots + \ell_{b-1}]}.$$

Now take the sum with respect to $\ell_1, \ell_2, \ldots, \ell_{b-1}$ successively using the equality

$$\sum_{\ell=1}^{\infty} \frac{q^{\ell+m}}{[\ell+m]} \frac{1}{[\ell+n]} = \sum_{\ell=1}^{\infty} \left(\frac{q^{\ell+n}}{[\ell+n]} - \frac{q^{\ell+m}}{[\ell+m]} \right) \frac{q^{m-n}}{[m-n]} = \frac{q^{m-n}}{[m-n]} \sum_{\ell=1}^{m-n} \frac{q^{\ell+n}}{[\ell+n]}$$

which holds for any $m > n$. Then we obtain $g_{b,n-b+1}(M)$. □

Lemma 1 implies the following proposition, which can be proved by induction on b:

Proposition 1 *For positive integers r, ℓ and $N_1 > \cdots > N_r > M$, set*

$$h_{r,\ell}(N_1, \ldots, N_r, M) := \sum_{c \in I(r,\ell)} \left(\prod_{j=1}^{r-1} f_{c_j}(N_j, N_{j+1}) \right) f_{c_r}(N_r, M). \tag{3}$$

Then

$$K_{b,n}(M) = g_{b,n-b+1}(M) + \sum_{\ell=1}^{b-1} \sum_{r=1}^{\ell} (-1)^r \sum_{N_1 > N_2 > \cdots > N_r > M} g_{b-\ell,n-b+1}(N_1)\, h_{r,\ell}(N_1, \ldots, N_r, M) \tag{4}$$

for $b \geq 1, n \geq 2$ and $M \geq 1$.

Multiply $K_{b,n}(M)$ by the harmonic sum

$$\sum_{M > m_1 > \cdots > m_a > 0} \prod_{j=1}^{a} \frac{1}{[m_j]} \tag{5}$$

and take the sum over all $M \geq 1$. Then we get the left hand side of (2). In order to carry out the same calculation for the right hand side of (4), we prepare an algebraic formulation for multiple harmonic sums.

2.2 Algebraic Structure of Multiple Harmonic Sums

Denote by $\mathfrak{H}$ the non-commutative polynomial algebra over $\mathbb{Z}$ freely generated by the set of alphabets $S = \{z_k\}_{k=1}^{\infty} \cup \{\xi_k\}_{k=1}^{\infty}$. For a positive integer m, set

$$J_{z_k}(m) := \frac{q^{(k-1)m}}{[m]^k}, \qquad J_{\xi_k}(m) := \frac{q^{km}}{[m]^k}.$$

For a word $w = u_1 \cdots u_r \in \mathfrak{H}$ $(r \geq 1, u_i \in S)$ and $M \in \mathbb{Z}_{\geq 1}$, set

$$A_w(M) := \sum_{M > m_1 > \cdots > m_r > 0} J_{u_1}(m_1) \cdots J_{u_r}(m_r),$$

$$A^{\star}_w(M) := \sum_{M > m_1 \geq \cdots \geq m_r \geq 1} J_{u_1}(m_1) \cdots J_{u_r}(m_r).$$

We extend the maps $w \mapsto A_w(M)$ and $w \mapsto A^{\star}_w(M)$ to the $\mathbb{Z}$-module homomorphisms $A(M), A^{\star}(M) : \mathfrak{H} \to \mathbb{R}$ by $A_1(M) = 1$, $A^{\star}_1(M) = 1$ and $\mathbb{Z}$-linearity. Note that $A_{z_1^a}(M)$ is equal to the harmonic sum (5). If w is contained in the $\mathbb{Z}$-linear span of monomials $z_{i_1} \cdots z_{i_r}$ with $i_1 \geq 2$, $A_w(M)$ becomes a linear combination of qMZV's in the limit $M \to \infty$.

Denote by $\mathfrak{H}_\xi$ the $\mathbb{Z}$-subalgebra of $\mathfrak{H}$ generated by $\{\xi_k\}_{k=1}^{\infty}$. Define a $\mathbb{Z}$-bilinear map $\rho : \mathfrak{H}_\xi \times \mathfrak{H} \to \mathfrak{H}$ inductively by $\rho(1, w) = w (w \in \mathfrak{H})$, $\rho(v, 1) = v (v \in \mathfrak{H}_\xi)$ and

$$\rho(\xi_k v, z_\ell w) = \xi_k \rho(v, z_\ell w) + z_\ell \rho(\xi_k v, w) + z_{k+\ell} \rho(v, w),$$

$$\rho(\xi_k v, \xi_\ell w) = \xi_k \rho(v, z_\ell w) + \xi_\ell \rho(\xi_k v, w) + \xi_{k+\ell} \rho(v, w)$$

for $v \in \mathfrak{H}_\xi$ and $w \in \mathfrak{H}$.

Proposition 2 *For $v \in \mathfrak{H}_\xi, w \in \mathfrak{H}$ and $M \geq 1$, we have $A_v(M) A_w(M) = A_{\rho(v,w)}(M)$.*

Proof It is enough to consider the case where v and w are words. If $v = 1$ or $w = 1$, it is trivial. From the definition of $A(M)$, it holds that

$$A_{\xi_k w}(M) = \sum_{M > m > 0} \frac{q^{km}}{[m]^k} A_w(m), \qquad A_{z_\ell w}(M) = \sum_{M > n > 0} \frac{q^{(\ell-1)n}}{[n]^\ell} A_w(n). \tag{6}$$

Hence we find

$$A_{\xi_k v}(M) A_{z_\ell w}(M)$$
$$= \left(\sum_{M > m > n > 0} + \sum_{M > n > m > 0} + \sum_{M > m = n > 0} \right) \frac{q^{km}}{[m]^k} \frac{q^{(\ell-1)n}}{[n]^\ell} A_v(m) A_w(n)$$

$$= \sum_{M>m>0} \frac{q^{km}}{[m]^k} A_v(m) A_{z_\ell w}(m) + \sum_{M>n>0} \frac{q^{(\ell-1)n}}{[n]^\ell} A_{\xi_k v}(n) A_w(n)$$
$$+ \sum_{M>m>0} \frac{q^{(k+\ell-1)m}}{[m]^{k+\ell}} A_v(m) A_w(m)$$

and a similar formula for $A_{\xi_k v}(M) A_{\xi_\ell w}(M)$. Now the proposition follows from the induction on the sum of length of v and w. □

For $k \geq 1$, we define a $\mathbb{Z}$-linear map $\xi_k \circ \cdot : \mathfrak{d}_\xi \to \mathfrak{d}_\xi$ inductively by $\xi_k \circ 1 = 0$ and $\xi_k \circ (\xi_\ell v) = \xi_{k+\ell} v$ for $v \in \mathfrak{d}_\xi$. Now consider the $\mathbb{Z}$-linear map $d : \mathfrak{d}_\xi \to \mathfrak{d}_\xi$ defined by $d(1) = 1$ and $d(\xi_k v) = \xi_k d(v) + \xi_k \circ d(v)$ $(v \in \mathfrak{d}_\xi)$.

Proposition 3 *For any $v \in \mathfrak{d}_\xi$ and $M \geq 1$, it holds that $A^\star_v(M) = A_{d(v)}(M)$.*

Proof From the definition of $A(M)$ and $A^\star(M)$ we have

$$A^\star_{\xi_k v}(M) = \sum_{M>m>0} \frac{q^{km}}{[m]^k} A^\star_v(m+1), \qquad \sum_{M>m>0} \frac{q^{km}}{[m]^k} A_v(m+1) = A_{\xi_k v + \xi_k \circ v}(M).$$

To show the second formula, divide the sum $A_v(m+1)$ into the two parts with $m_1 = m$ and $m_1 < m$. Combining the two formulas above, we obtain the proposition by induction on length of v. □

2.3 Algebraic Formulation of the Main Theorem

To calculate the right hand side of (4) multiplied by the harmonic sum (5), we need the following formula:

Lemma 2 *For $n_1 > \cdots > n_s > n_{s+1} > 0$, set*

$$p(n_1, \ldots, n_s; n_{s+1}) := \frac{q^{n_1 - n_{s+1}}}{[n_1 - n_{s+1}]} \prod_{j=2}^{s} \frac{1}{[n_j - n_{s+1}]}. \tag{7}$$

Let $s \geq 1$, $v = z_1$ or ξ_1, and N and M be positive integers such that $N > M$. Then it holds that

$$\sum_{N>n_1>\cdots>n_{s+1}>M} p(n_1, \ldots, n_s; n_{s+1}) J_v(n_{s+1})$$
$$= \sum_{N>k_1>\cdots>k_{s+1}>M} J_v(k_1) p(k_2, \ldots, k_s, k_{s+1}; M)$$
$$+ \sum_{i=1}^{s} \sum_{N>k_1>\cdots>k_{s+1}>M} \frac{q^{k_1}}{[k_1]} \left(\prod_{j=2}^{i+1} \frac{1}{[k_j]} \right) p(k_{i+2}, \ldots, k_{s+1}; M),$$

where $p(\emptyset; M) = 1$.

Proof Here we prove the lemma in the case of $v = z_1$. The proof for $v = \xi_1$ is similar. Using

$$\frac{1}{[n_1 - n_{s+1}][n_{s+1}]} = \frac{1}{[n_1]}\left(\frac{1}{[n_1 - n_{s+1}]} + \frac{q^{n_{s+1}}}{[n_{s+1}]}\right),$$
$$\frac{1}{[n_j - n_{s+1}][n_{s+1}]} = \frac{1}{[n_j]}\left(\frac{q^{n_j - n_{s+1}}}{[n_j - n_{s+1}]} + \frac{1}{[n_{s+1}]}\right) \quad (j = 2, \dots, s),$$

we find that

$$p(n_1, \dots, n_s; n_{s+1}) J_v(n_{s+1}) = \sum_{i=0}^{s} \frac{q^{(1-\delta_{i,0})n_1}}{[n_1]} \left(\prod_{j=2}^{i+1} \frac{1}{[n_j]}\right) p(n_{i+1}, \dots, n_s; n_{s+1}).$$

Now take the sum of the both hand sides over $N > n_1 > \cdots > n_{s+1} > M$. In the right hand side, change the variables $n_1, \dots, n_{s+1}$ to $k_1, \dots, k_{s+1}$ by setting $n_t = k_t$ $(1 \le t \le i+1)$, $n_t = k_{i+1} - k_{i+2} + k_{t+1}$ $(i+2 \le t \le s)$ and $n_{s+1} = k_{i+1} - k_{i+2} + M$. Then we get the desired formula. □

Let $\mathfrak{d}_1$ be the $\mathbb{Z}$-subalgebra of $\mathfrak{d}$ generated by z_1 and ξ_1. Motivated by Lemma 2 we introduce the $\mathbb{Z}$-module homomorphism $\varphi_s : \mathfrak{d}_1 \to \mathfrak{d}_1$ $(s \in \mathbb{Z}_{\ge 0})$ defined in the following way. Determine $\varphi_s(w)$ for a word $w \in \mathfrak{d}_1$ inductively on s and length of w by $\varphi_0 = \mathrm{id}$, $\varphi_s(1) = \xi_1 z_1^{s-1}$ $(s \ge 1)$ and

$$\varphi_s(z_1 w) = z_1 \varphi_s(w) + \xi_1 \sum_{i=1}^{s} z_1^i \varphi_{s-i}(w), \qquad \varphi_s(\xi_1 w) = \xi_1 \sum_{i=0}^{s} z_1^i \varphi_{s-i}(w),$$

and extend it by $\mathbb{Z}$-linearity.

Proposition 4 *For $w \in \mathfrak{d}_1$ and any positive integers s, s', ℓ, β and N, we have*

$$\begin{aligned} \sum_{N > M_1 > M_2 > 0} f_{s'}(N, M_1) f_s(M_1, M_2) A_w(M_2) &= \sum_{N > M > 0} f_{s'}(N, M) A_{\varphi_s(w)}(M), \\ \sum_{M_1 > M_2 > 0} g_{\ell,\beta}(M_1) f_s(M_1, M_2) A_w(M_2) &= \sum_{M > 0} g_{\ell,\beta}(M) A_{\varphi_s(w)}(M). \end{aligned} \tag{8}$$

Proof Here we prove the first formula (8). The proof for the second is similar. It suffices to consider the case where $w = u_1 \cdots u_r$ $(r \ge 1, u_i \in S)$ is a word. The left hand side of (8) is equal to

$$\sum f_{s'}(N, M_1)\, p(M_1, k_1, \dots, k_{s-1}; M_2) \prod_{i=1}^{r} J_{u_i}(m_i),$$

where p is defined by (7) and the sum is over M_1, k_i $(1 \le i \le s-1)$, M_2, m_i $(1 \le i \le r)$ with the condition $N > M_1 > k_1 > \cdots > k_{s-1} > M_2 > m_1 > \cdots > m_r > 0$. Changing the variables $(k_1, \ldots, k_{s-1}, M_2)$ to $(n_1, \ldots, n_s)$ by $k_i = M_1 - n_1 + n_{i+1}$ $(1 \le i \le s-1)$ and $M_2 = M_1 - n_1 + m_1$, we obtain

$$\sum f_{s'}(N, M_1) p(n_1, \ldots, n_s; m_1) \prod_{i=1}^{r} J_{u_i}(m_i),$$

where the sum is over $N > M_1 > n_1 > \cdots > n_s > m_1 > \cdots > m_r > 0$. From Lemma 2 and the definition of φ_s, we see by induction on r that it is equal to the right hand side of (8). $\square$

We define the $\mathbb{Z}$-linear maps $\Phi_\ell : \mathfrak{d}_1 \to \mathfrak{d}_1$ $(\ell \ge 0)$ by $\Phi_0 :=$ id and

$$\Phi_\ell := \sum_{r=1}^{\ell} (-1)^r \sum_{c \in I(r,\ell)} \varphi_{c_1} \cdots \varphi_{c_r},$$

and $Z_s : \mathfrak{d}_1 \to \mathfrak{d}$ $(s \ge 0)$ by

$$Z_s(w) := \sum_{\ell=0}^{s} \rho\bigl(d\bigl(\xi_1^{s-\ell}\bigr), \Phi_\ell(w)\bigr).$$

Proposition 5 *For any integers $a \ge 0, b \ge 1$ and $n \ge b+1$, we have*

$$\sum_{\alpha \in I_0(b,n)} \zeta_q\bigl(\alpha_1, \ldots, \alpha_b, 1^a\bigr) = \sum_{s=0}^{b-1} \sum_{M>0} \frac{q^{(n-s-1)M}}{[M]^{n-s}} A_{Z_s(z_1^a)}(M). \tag{9}$$

Proof Using Proposition 4 repeatedly, we have

$$\begin{aligned} &\sum_{N_1 > N_2 > \cdots > N_r > M > 0} g_{b,m}(N_1) h_{r,\ell}(N_1, \ldots, N_r, M) A_{z_1^a}(M) \\ &= \sum_{c \in I(r,\ell)} \sum_{M>0} g_{b,m}(M) A_{\varphi_{c_1} \cdots \varphi_{c_r}(z_1^a)}(M), \end{aligned}$$

where $h_{r,\ell}$ is defined by (3). Hence Proposition 1 implies that

$$\begin{aligned} \sum_{\alpha \in I_0(b,m)} \zeta_q\bigl(\alpha_1, \ldots, \alpha_b, 1^a\bigr) &= \sum_{M>0} K_{b,m}(M) A_{z_1^a}(M) \\ &= \sum_{\ell=0}^{b-1} \sum_{M>0} g_{b-\ell,m-b+1}(M) A_{\Phi_\ell(z_1^a)}(M). \end{aligned}$$

Substituting

$$g_{j,n}(M)=\sum_{t=0}^{j-1}\frac{q^{(n+j-t-2)M}}{[M]^{n+j-t-1}}A^{\star}_{\xi_1^t}(M),$$

we get the desired formula from Propositions 2 and 3. □

As we will see in the next subsection, the elements $Z_s(z_1^a)$ $(s,a\geq 0)$ belong to the subalgebra of $\mathfrak{d}$ generated only by $\{z_k\}_{k=1}^{\infty}$ (see Propositions 6 and 8 below). Thus the right hand side of (9) will turn out to be a linear combination of qMZV's.

2.4 Proof of the Main Theorem

First we give a proof of Theorem 1 with $a=0$, that is, the sum formula for qMZV's. To this aim we prepare a recurrence relation of $d(\xi_1^k)$ $(k\geq 0)$.

Lemma 3 *Let $k\geq 1$. Then*

$$d\big(\xi_1^k\big)=\sum_{r=1}^{k}\sum_{c\in I(r,k)}\xi_{c_1}\cdots\xi_{c_r}.$$

Proof We prove the lemma by induction on k. The case of $k=1$ is trivial. Let $k\geq 2$. From the definition of d and the hypothesis of induction we see that

$$\begin{aligned}d\big(\xi_1^k\big)&=d\big(\xi_1\cdot\xi_1^{k-1}\big)=\xi_1\sum_{r=1}^{k-1}\sum_{c\in I(r,k-1)}\xi_{c_1}\cdots\xi_{c_r}+\xi_1\circ\left(\sum_{r=1}^{k-1}\sum_{c\in I(r,k-1)}\xi_{c_1}\cdots\xi_{c_r}\right)\\&=\sum_{r=2}^{k}\sum_{\substack{c\in I(r,k)\\c_1=1}}\xi_{c_1}\cdots\xi_{c_r}+\sum_{r=1}^{k-1}\sum_{\substack{c\in I(r,k)\\c_1\geq 2}}\xi_{c_1}\cdots\xi_{c_r}=\sum_{r=1}^{k}\sum_{c\in I(r,k)}\xi_{c_1}\cdots\xi_{c_r}.\end{aligned}$$

This completes the proof. □

Corollary 1 *For $k\geq 1$ it holds that*

$$d\big(\xi_1^k\big)=\sum_{a=1}^{k}\xi_a d\big(\xi_1^{k-a}\big). \tag{10}$$

The sum formula for qMZV's follows from the following proposition.

Proposition 6 $Z_s(1)=\delta_{s,0}$ $(s\geq 0)$.

Proof Using $\Phi_\ell = -\sum_{a=1}^{\ell} \varphi_a \Phi_{\ell-a}$ $(\ell \geq 1)$, we find that $\Phi_\ell(1) = (-\xi_1)^\ell$ $(\ell \geq 0)$ by induction on ℓ. Thus the proposition is reduced to the proof of

$$\sum_{\ell=0}^{s} (-1)^\ell \rho\big(d\big(\xi_1^{s-\ell}\big), \xi_1^\ell\big) = \delta_{s,0}.$$

Let us prove it by induction on s. Denote the left hand side above by T_s. It is trivial that $T_0 = 1$. Let $s \geq 1$. Divide T_s into the three parts

$$T_s = d\big(\xi_1^s\big) + \sum_{\ell=1}^{s-1} (-1)^\ell \rho\big(d\big(\xi_1^{s-\ell}\big), \xi_1^\ell\big) + (-1)^s \xi_1^s.$$

Rewrite the second part by using (10) and the definition of ρ and d. Then we get

$$\sum_{a=1}^{s-1} \xi_a \sum_{\ell=1}^{s-a} (-1)^\ell \rho\big(d\big(\xi_1^{s-a-\ell}\big), \xi_1^\ell\big) - \sum_{\ell=0}^{s-2} (-1)^\ell \xi_1 \rho\big(d\big(\xi_1^{s-1-\ell}\big), \xi_1^\ell\big)$$
$$- \sum_{a=1}^{s-1} \xi_{a+1} I_{s-a-1}. \tag{11}$$

From $(-1)^s \xi_1^s = -(-1)^{s-1} \xi_1 \rho(d(\xi_1^0), \xi_1^{s-1})$, which is the summand of the second term of (11) with $\ell = s-1$, and

$$d\big(\xi_1^s\big) = \sum_{a=1}^{s-1} \xi_a \rho\big(d\big(\xi_1^{s-a}\big), \xi_1^0\big) + \xi_s,$$

we obtain

$$T_s = \sum_{a=1}^{s-1} \xi_a T_{s-a} + \xi_s - \xi_1 T_{s-1} - \sum_{a=1}^{s-1} \xi_{a+1} T_{s-a-1}.$$

Therefore the induction hypothesis $T_a = \delta_{a,0}$ $(a < s)$ implies that $T_s = 0$. □

From Proposition 5 with $a = 0$ and Proposition 6, we see that

$$\sum_{\alpha \in I_0(b,n)} \zeta_q(\alpha_1, \ldots, \alpha_b) = \sum_{M>0} \frac{q^{(n-1)M}}{[M]^n} A_1(M) = \zeta_q(n).$$

Thus we get Theorem 1 in the case of $a = 0$. To complete the proof of Theorem 1, we should calculate $Z_s(z_1^a)$ for $a \geq 1$. For that purpose we prepare several lemmas.

Lemma 4 *For $\ell \geq 0$ and $w \in \mathfrak{d}_1$, it holds that*

$$\Phi_\ell(z_1 w) = \sum_{j=0}^{\ell} (-\xi_1)^{\ell-j} z_1 \Phi_j(w). \tag{12}$$

Proof For non-negative integers a and n, set $\eta_{0,n} = \delta_{n,0}$ and

$$\eta_{a,n} := \sum_{c \in I(a,n)} \xi_1 z_1^{c_1} \cdots \xi_1 z_1^{c_a} \quad (a \geq 1).$$

Then it holds that

$$\varphi_s\big(\xi_1^a z_1 w\big) = \sum_{t=0}^{s} (\eta_{a,s-t} + \eta_{a+1,s-t-1}) z_1 \varphi_t(w),$$

where $\eta_{a+1,-1} := 0$, for $a \geq 0$, $s \geq 0$ and $w \in \mathfrak{d}_1$. Using this formula we prove (12) by induction on ℓ. The case of $\ell = 0$ is trivial. Let $\ell \geq 1$. The induction hypothesis and the relation $\Phi_\ell = -\sum_{a=1}^{\ell} \varphi_a \Phi_{\ell-a}$ imply that

$$\Phi_\ell(z_1 w) = \sum_{j=0}^{\ell-1} \sum_{a=1}^{\ell-j} \sum_{t=0}^{a} (\eta_{\ell-a-j,a-t} + \eta_{\ell-a-j+1,a-t-1}) z_1 \varphi_t\big(\Phi_j(w)\big).$$

Divide the sum into the two parts with $t = 0$ and $t \geq 1$, and take the sum with respect to a. Then we obtain

$$\sum_{j=0}^{\ell-1} \left\{ \big(-\delta_{\ell-j,0} + (-1)^{\ell-j} \eta_{\ell-j,0}\big) z_1 \varphi_0\big(\Phi_j(w)\big) - \sum_{t=1}^{\ell-j} \delta_{\ell-j-t,0} z_1 \varphi_t\big(\Phi_j(w)\big) \right\}.$$

Since $\eta_{\ell-j,0} = \xi_1^{\ell-j}$, $\varphi_0 = \mathrm{id}$ and $-\sum_{j=0}^{\ell-1} \varphi_{\ell-j} \Phi_j = \varphi_\ell$, it is equal to the right hand side of (12). $\square$

Lemma 5 *For $k \geq 0$ and $w \in \mathfrak{d}_1$, it holds that*

$$\sum_{\ell=0}^{k} \rho\big(d\big(\xi_1^{k-\ell}\big), \xi_1^\ell z_1 w\big) = \sum_{\ell=0}^{k} z_{\ell+1}\, \rho\big(d\big(\xi_1^{k-\ell}\big), w\big). \tag{13}$$

Proof Denote the left hand side and the right hand side of (13) by L_k and R_k, respectively. The equality (13) holds when $k = 0$ because $L_0 = \rho(1, z_1 w) = z_1 w = z_1 \rho(1, w) = R_0$. Hereafter we assume that $k \geq 1$.

Divide L_k into the three parts

$$L_k = \rho\big(d\big(\xi_1^k\big), z_1 w\big) + \sum_{\ell=1}^{k-1} (-1)^\ell \rho\big(d\big(\xi_1^{k-\ell}\big), \xi_1^\ell z_1 w\big) + (-1)^k \xi_1^k z_1 w. \tag{14}$$

Let us rewrite the first part. Substitute (10) into $d(\xi_1^k)$. From the definition of ρ we see that the first part is equal to

$$\sum_{a=1}^{k} \Big(\xi_a \rho\big(d\big(\xi_1^{k-a}\big), z_1 w\big) + z_1 \rho\big(\xi_a d\big(\xi_1^{k-a}\big), w\big) + z_{a+1} \rho\big(d\big(\xi_1^{k-a}\big), w\big)\Big).$$

Note that the first term of the summand with $a=k$ is equal to $\xi_k z_1 w = \xi_k L_0$. Apply (10) again to the second term, and we see that the first part of the right hand side of (14) is equal to

$$\xi_k L_0 + \sum_{a=1}^{k-1} \xi_a \rho\bigl(d\bigl(\xi_1^{k-a}\bigr), z_1 w\bigr) + R_k. \tag{15}$$

We proceed the same calculation for the second part of (14). Here we decompose $\xi_1^\ell z_1 w = \xi_1 \cdot \xi_1^{\ell-1} z_1 w$ and use (10). As a result we get

$$\sum_{a=1}^{k-1}\sum_{\ell=1}^{k-a} (-1)^\ell \xi_a \rho\bigl(d\bigl(\xi_1^{k-\ell-a}\bigr), \xi_1^\ell z_1 w\bigr) - \sum_{a=1}^{k-1} \xi_{a+1} I_{k-a-1}$$
$$- \sum_{\ell=0}^{k-2} (-1)^l \xi_1 \rho\bigl(d\bigl(\xi_1^{k-1-\ell}\bigr), \xi_1^\ell z_1 w\bigr). \tag{16}$$

Note that the third part of (14) is equal to

$$-(-1)^{k-1} \xi_1 \rho\bigl(d\bigl(\xi_1^0\bigr), \xi_1^{k-1} z_1 w\bigr), \tag{17}$$

which is the summand of the third term of (16) with $\ell = k-1$. Hence the three parts (15), (16) and (17) add up to

$$\xi_k L_0 + \sum_{a=1}^{k-1} \xi_a L_{k-a} + R_k - \sum_{a=1}^{k-1} \xi_{a+1} L_{k-a-1} - \xi_1 L_{k-1} = R_k.$$

This completes the proof. □

Now we can prove the key formula to calculate $Z_s(z_1^a)$ for $a \ge 1$:

Proposition 7 *Let* $w \in \mathfrak{H}_1$ *and* $s \ge 0$. *Then* $Z_s(z_1 w) = \sum_{\ell=0}^{s} z_{\ell+1} Z_{s-\ell}(w)$.

Proof Using (12) we have

$$Z_s(z_1 w) = \sum_{\ell=0}^{s}\sum_{j=0}^{l} (-1)^{\ell-j} \rho\bigl(d\bigl(\xi_1^{s-\ell}\bigr), \xi_1^{\ell-j} z_1 \Phi_j(w)\bigr).$$

Because of Lemma 5 it is equal to

$$\sum_{j=0}^{s}\sum_{\ell=0}^{s-j} z_{\ell+1} \rho\bigl(d\bigl(\xi_1^{s-j-\ell}\bigr), \Phi_j(w)\bigr) = \sum_{\ell=0}^{s} z_{\ell+1} Z_{s-l}(w).$$

This completes the proof. □

Combining Proposition 6 and Proposition 7, we obtain the following formula:

Proposition 8 *For $s \geq 0$ and $a \geq 1$, it holds that*

$$Z_s\left(z_1^a\right) = \sum_{\gamma \in I(a,s+a)} z_{\gamma_1} \cdots z_{\gamma_a}.$$

At last let us prove Theorem 1 in the case of $a \geq 1$. From Proposition 5 and Proposition 8, it holds that

$$\sum_{\alpha \in I_0(b,n)} \zeta_q\left(\alpha_1, \ldots, \alpha_b, 1^a\right) = \sum_{s=0}^{b-1} \sum_{\gamma \in I(a,s+a)} \zeta_q(n - s - 1, \gamma_1, \ldots, \gamma_a).$$

Set $\beta_1 = b + 1 - s$. The right hand side becomes

$$\begin{aligned}
&\sum_{\beta_1=2}^{b+1} \sum_{\gamma \in I(a,a+b+1-\beta_1)} \zeta_q(\beta_1 + n - b - 1, \gamma_1, \ldots, \gamma_a) \\
&= \sum_{\beta \in I_0(a+1,a+b+1)} \zeta_q(\beta_1 + n - b - 1, \beta_2, \ldots, \beta_{a+1}).
\end{aligned}$$

This completes the proof of Theorem 1.

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A Trinity of the Borcherds Φ-Function

Ken-Ichi Yoshikawa

Abstract We discuss a trinity, i.e., three distinct expressions, of the Borcherds Φ-function on the analogy of the trinity of the Dedekind η-function.

1 Introduction—A Trinity of Dedekind η-Function

The Dedekind η-function is the holomorphic function on the complex upper half-plane $\mathfrak{H}$ defined as the infinite product

$$\eta(\tau) := q^{1/24} \prod_{n>0} \bigl(1 - q^n\bigr),$$

where $q := e^{2\pi i\tau}$. It is classical that $\eta(\tau)^{24}$ is a modular form for $SL_2(\mathbf{Z})$ of weight 12 vanishing at $+i\infty$ and this property characterizes the Dedekind η-function up to a constant.

Let us recall the trinity of the Dedekind η-function. Besides the definition as above, the Dedekind η-function admits at least two other distinct expressions, one analytic and the other algebro-geometric. Precisely speaking, we consider the Petersson norm

$$\bigl\|\eta(\tau)\bigr\| := (\Im\tau)^{1/4}\bigl|\eta(\tau)\bigr|$$

rather than the Dedekind η-function itself.

Let us explain an analytic counterpart of the Dedekind η-function. For $\tau \in \mathfrak{H}$, let E_τ be the elliptic curve defined by

$$E_\tau := \mathbf{C}/\mathbf{Z} + \tau\mathbf{Z},$$

which is equipped with the flat Kähler metric of normalized volume 1

$$g_\tau := dz \otimes d\bar{z}/\Im\tau.$$

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The Laplacian of (E_τ, g_τ) is the differential operator defined as

$$\square_\tau := -\Im\tau\, \frac{\partial^2}{\partial z \partial \bar{z}} = -\frac{\Im\tau}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

The set of eigenvalues of $\square_\tau$ is given by $\{\pi^2|m\tau+n|^2/\Im\tau\}_{(m,n)\in\mathbf{Z}^2}$ and hence the spectral zeta function of $\square_\tau$ is defined as

$$\zeta_\tau(s) := \sum_{(m,n)\neq(0,0)} \left(\frac{\Im\tau}{\pi^2\,|m\tau+n|^2}\right)^s.$$

It is classical that $\zeta_\tau(s)$ converges absolutely when $\Re s > 1$ and extends to a meromorphic function on $\mathbf{C}$. Moreover, $\zeta_\tau(s)$ is holomorphic at $s=0$. The value

$$\det{}^*\square_\tau := \exp\bigl(-\zeta'_\tau(0)\bigr)$$

is called the (regularized) determinant of $\square_\tau$ on the analogy of the identity for finite dimensional, non-degenerate, Hermitian matrices

$$\log\det H = -\frac{d}{ds}\Big|_{s=0} \operatorname{Tr} H^{-s}.$$

By Ray-Singer [29], the classical Kronecker limit formula can be stated as follows in this setting:

Theorem 1 *The following equality holds*

$$\det{}^*\square_\tau = 4\bigl\|\eta(\tau)\bigr\|^4.$$

Let us explain an algebro-geometric counterpart of the Dedekind η-function. Let $M_{m,n}(K)$ be the set of $m\times n$-matrices with entries in $K\subset\mathbf{C}$. Recall that every elliptic curve is expressed as the complete intersection of two quadrics of $\mathbf{P}^3$

$$E_A := \left\{[x]\in\mathbf{P}^3;\ \begin{array}{l} f_1(x) = a_{11}x_1^2 + a_{12}x_2^2 + a_{13}x_3^2 + a_{14}x_4^2 = 0 \\ f_2(x) = a_{21}x_1^2 + a_{22}x_2^2 + a_{23}x_3^2 + a_{24}x_4^2 = 0 \end{array}\right\},$$

where $A = (a_{ij}) = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) \in M_{2,4}(\mathbf{C})$. For $A\in M_{2,4}(\mathbf{C})$ and $1\le i<j\le 4$, we define

$$\Delta_{ij}(A) := \det(\mathbf{a}_i, \mathbf{a}_j).$$

Since the value $\|\eta(\tau)\|$ depends only on the isomorphism class of the elliptic curve E_τ, it makes sense to set $\|\eta(E_\tau)\| := \|\eta(\tau)\|$.

Theorem 2 *With the same notation as above, the following equality holds*

$$2^8\bigl\|\eta(E_A)\bigr\|^{24} = \prod_{1\le i<j\le 4} \bigl|\Delta_{ij}(A)\bigr|^2 \cdot \left(\frac{2\sqrt{-1}}{\pi^2}\int_{E_A} \alpha_A\wedge\overline{\alpha}_A\right)^6.$$

Here $\alpha_A \in H^0(E_A, \Omega^1_{E_A})$ *is defined as the residue of* f_1, f_2, *i.e.*,

$$\alpha_A := \Xi|_{E_A},$$

where Ξ *is a meromorphic* 1*-form on* $\mathbf{P}^3$ *satisfying the equation*

$$df_1 \wedge df_2 \wedge \Xi = \sum_{i=1}^{4} (-1)^{i-1} x_i dx_1 \wedge dx_{i-1} \wedge dx_{i+1} \wedge dx_4.$$

For $A = (a_{ij}) \in M_{2,4}(\mathbf{C})$, one can associate another elliptic curve

$$C_A := \left\{(x, y) \in \mathbf{C}^2;\ y^2 = 4(a_{11}x + a_{21})(a_{12}x + a_{22})(a_{13}x + a_{23})(a_{14}x + a_{24})\right\}.$$

Namely, C_A is the double covering of $\mathbf{P}^1$ with 4 branch points $(a_{11} : -a_{21})$, $(a_{12} : -a_{22})$, $(a_{13} : -a_{23})$, $(a_{14} : -a_{24})$. If $a_{11} = 0$ and $a_{12} = 1$, then C_A is an elliptic curve expressed by the Weierstrass equation. It is not difficult to see $C_A \cong E_A$ and

$$2^8 \left\|\eta(C_A)\right\|^{24} = \prod_{1 \le i < j \le 4} \left|\Delta_{ij}(A)\right|^2 \cdot \left(\frac{\sqrt{-1}}{2\pi^2} \int_{C_A} \frac{dx}{y} \wedge \overline{\frac{dx}{y}}\right)^6.$$

(We shall study an analogue of E_A and C_A for $K3$ surfaces later.)

Theorem 2 is easily verified when E_A is the projective embedding of E_τ by the linear system $|4\Theta|$. In this situation, the equations of E_A are the linear relations between the theta functions $\theta_{a,b}(z,\tau)$ $(a, b \in \{0, \frac{1}{2}\})$. General case of Theorem 2 follows from this special case by the invariance of the expression in Theorem 2 under the action of $GL_2(\mathbf{C}) \times (\mathbf{C}^*)^4$. See [16] for the details.

In this survey, we explain a generalization of the trinity of the Dedekind η-function as above to that of the Borcherds Φ-function. For this, we make the following replacements:

- elliptic curves $\Longrightarrow$ Enriques surfaces
- determinant of Laplacian $\Longrightarrow$ analytic torsion
- $\prod_{1 \le i < j \le 4} \Delta_{ij}(A) \Longrightarrow$ resultant of three quadratic forms in three variables

For the analytic aspect of the Borcherds Φ-function, our explanation is based on [34, 36], while for the algebro-geometric aspect of the Borcherds Φ-function, our explanation is based on [16]. In this survey, we will not give proofs. We refer the reader to these papers for the details.

2 Borcherds Φ-Function

In this section, we recall the Borcherds Φ-function.

2.1 Domains of Type IV and Its Realization as a Tube Domain

A free $\mathbf{Z}$-module of finite rank equipped with a non-degenerate, integral, symmetric bilinear form is called a lattice. The automorphism group of a lattice L is denoted by $O(L)$. For a lattice $L = (\mathbf{Z}^r, \langle\cdot,\cdot\rangle_L)$ and $k \in \mathbf{Q}$, we set $L(k) := (\mathbf{Z}^r, k\langle\cdot,\cdot\rangle_L)$. We define $\mathbb{U} := (\mathbf{Z}^2, \binom{0\,1}{1\,0})$. There exists a unique positive-definite, even, unimodular lattice of rank 8, up to an isometry. This lattice is denoted by $\mathbb{E}_8$.

Let Λ be a lattice of signature $(2, b^-)$. We define an open manifold Ω_Λ of dimension b^- as

$$\Omega_\Lambda := \left\{[Z] \in \mathbf{P}(\Lambda \otimes \mathbf{C});\ \langle Z, Z\rangle_\Lambda = 0,\ \langle Z, \bar{Z}\rangle_\Lambda > 0\right\}.$$

Then Ω_Λ is the set of maximal positive-definite subspaces of $\Lambda \otimes \mathbf{R}$ and is isomorphic to $SO(2, b^-)/SO(2) \times SO(b^-)$. Hence each connected component of Ω_Λ is isomorphic to a symmetric bounded domain of type IV of dimension b^-.

Assume that there exists $k \in \mathbf{Z}_{>0}$ and a lattice of signature $(1, b^- - 1)$ such that $\Lambda = \mathbb{U}(k) \oplus L$. Let $\{\mathbf{e}, \mathbf{f}\}$ be a basis of $\mathbb{U}(k)$ with $\mathbf{e}^2 = \mathbf{f}^2 = 0$, $\mathbf{e}\cdot\mathbf{f} = k$. We set $\mathbf{v} := \mathbf{e} \in \mathbb{U}(k)$ and $\mathbf{v}' := \mathbf{f}/k \in \mathbb{U}(k)^\vee$. Then we have an isomorphism of complex manifolds $L \otimes \mathbf{R} + i\mathcal{C}_L \cong \Omega_\Lambda$ given by the map

$$L \otimes \mathbf{R} + i\mathcal{C}_L \ni z \to Z = \left[\mathbf{v} - \frac{\langle z, z\rangle_L}{2}\mathbf{v}' + z\right] \in \Omega_\Lambda.$$

Here $\mathcal{C}_L := \{x \in L \otimes \mathbf{R};\ \langle x, x\rangle_L > 0\}$ is the positive cone of L. Since L is Lorentzian and hence $\mathcal{C}_L$ consists of two connected components, we choose one of them, say $\mathcal{C}_L^+$. Write Ω_Λ^+ for the component of Ω_Λ corresponding to $L \otimes \mathbf{R} + i\mathcal{C}_L^+$. Then we have the decomposition $\Omega_\Lambda = \Omega_\Lambda^+ \amalg \overline{\Omega_\Lambda^+}$. The subgroup of $O(\Lambda)$ preserving the connected components Ω_Λ^+, $\overline{\Omega_\Lambda^+}$ is denoted by $O^+(\Lambda)$. Clearly, $[O(\Lambda) : O^+(\Lambda)] = 2$.

2.2 Automorphic Forms over Domains of Type IV

Let us recall the notion of automorphic forms over Ω_Λ^+. There are several mutually equivalent definitions.

2.2.1 Automorphic Form as a Multicanonical Form on Ω_Λ^+

Let $\mathcal{L}$ be the tautological line bundle on Ω_Λ^+:

$$\mathcal{L} := \mathcal{O}_{\mathbf{P}(\Lambda\otimes\mathbf{C})}(-1)|_{\Omega_\Lambda^+} \subset \Omega_\Lambda^+ \times (\Lambda \otimes \mathbf{C}).$$

The natural action of $O^+(\Lambda)$ on $\Omega_\Lambda^+ \times (\Lambda \otimes \mathbf{C})$ induces the $O^+(\Lambda)$-action on $\mathcal{L}$. A holomorphic section $f \in H^0(\Omega_\Lambda^+, \mathcal{L}^k)$ is called an automorphic form for $\Gamma \subset O^+(\Lambda)$ of weight k with character χ if

$$f(\gamma Z) = \chi(\gamma)\,\gamma f(Z)$$

for all $Z \in \Omega_\Lambda^+$ and $\gamma \in \Gamma$, where $\chi : \Gamma \to \mathbf{C}^*$ is a finite character.

2.2.2 Automorphic Form as a Homogeneous Function on the Cone over Ω_Λ^+

Let $C_{\Omega_\Lambda^+}$ be the cone over Ω_Λ^+ obtained from $\mathcal{L}$ by contracting the zero section. Then a holomorphic function $F \in \mathcal{O}(C_{\Omega_\Lambda^+})$ is called an automorphic form on Ω_Λ^+ for $\Gamma \subset O^+(\Lambda)$ of weight k with character χ if

$$F\big(\gamma(\zeta)\big) = \chi(\gamma)\,F(\zeta), \qquad F(\lambda\,\zeta) = \lambda^{-k}\,F(\zeta)$$

for all $\zeta \in C_{\Omega_\Lambda^+}$, $\gamma \in \Gamma$ and $\lambda \in \mathbf{C}^*$.

2.2.3 Automorphic Form as a Function on Ω_Λ^+

Let $\ell \in \Lambda \otimes \mathbf{R}$ be such that $\langle \ell, \ell \rangle \geq 0$. Observe that

$$\sigma_\ell(Z) := \frac{Z}{\langle \ell, Z \rangle}, \quad Z \in \Omega_\Lambda^+$$

is a nowhere vanishing holomorphic section of $\mathcal{L}$. Via the assignment $f \mapsto f/\sigma_\ell^k$, we can define automorphic forms as follows: A holomorphic function $F(Z) \in \mathcal{O}(\Omega_\Lambda^+)$ is an automorphic form for Γ of weight k with character χ if for all $Z \in \Omega_\Lambda^+$ and $\gamma \in \Gamma$,

$$F(\gamma Z) = \chi(\gamma)\left(\frac{\langle \ell, \gamma Z \rangle}{\langle \ell, Z \rangle}\right)^k F(Z).$$

The choice of ℓ corresponds to the choice of a hyperplane at infinity of $\mathbf{P}(\Lambda \otimes \mathbf{C})$.

2.2.4 Automorphic Form as a Function on $L \otimes \mathbf{R} + i\mathcal{C}_L^+$

We have the $O^+(\Lambda)$-action on the tube domain $L \otimes \mathbf{R} + i\mathcal{C}_L^+$ via the identification $\Omega_\Lambda^+ \cong L \otimes \mathbf{R} + i\mathcal{C}_L^+$. Write $J(\gamma, y)$ for the Jacobian determinant of $\gamma \in O^+(\Lambda) \subset \mathrm{Aut}(L \otimes \mathbf{R} + i\mathcal{C}_L^+)$. By the relation between the canonical line bundle of Ω_Λ^+ and $\mathcal{L}$, there is a holomorphic function $j(\gamma, z)$ with

$$j(\gamma, z)^{\dim \Omega_\Lambda} = J(\gamma, z).$$

A holomorphic function $F(z) \in \mathcal{O}(L \otimes \mathbf{R} + i\mathcal{C}_L^+)$ is an automorphic form for Γ of weight k with character χ if for all $z \in L \otimes \mathbf{R} + i\mathcal{C}_L^+$ and $\gamma \in \Gamma$,

$$F(\gamma \cdot z) = \chi(\gamma)\, j(\gamma, z)^k\, F(z).$$

2.3 Borcherds Φ-Function

Define the Enriques lattice Λ as

$$\Lambda := \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(-2).$$

Then Λ is an even lattice of signature (2, 10). We define the discriminant divisor of Ω_Λ by

$$\mathcal{D}_\Lambda := \sum_{d \in \Lambda/\pm 1,\, d^2 = -2} d^\perp,$$

where $d^\perp := \{[Z] \in \Omega_\Lambda^+;\ \langle d, Z\rangle = 0\}$. Define $\{c(n)\}$ by the generating series:

$$\sum_{n \in \mathbf{Z}} c(n)\, q^n = \eta(\tau)^{-8}\eta(2\tau)^8\eta(4\tau)^{-8}.$$

2.3.1 Borcherds Φ-Function at the Level 1 Cusp

Let $\mathbf{v}$ be a primitive isotropic vector of $\mathbb{U} \subset \Lambda$ and set $L_1 := \mathbf{v}^\perp/\mathbf{v} \cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$. Then $L_1 \otimes \mathbf{R} + i\,\mathcal{C}_{L_1}^+ \cong \Omega_\Lambda^+$.

Definition 1 The Borcherds Φ-function is the formal Fourier series on the tube domain $L_1 \otimes \mathbf{R} + i\mathcal{C}_{L_1}^+$ defined as

$$\Phi_1(z) := \prod_{\lambda \in L_1 \cap \overline{\mathcal{C}_{L_1}^+} \setminus \{0\}} \left(\frac{1 - e^{\pi i \langle \lambda, z\rangle}}{1 + e^{\pi i \langle \lambda, z\rangle}}\right)^{c(\lambda^2/2)}.$$

2.3.2 Borcherds Φ-Function at the Level 2 Cusp

Let $\mathbf{v}$ be a primitive isotropic vector of $\mathbb{U}(2) \subset \Lambda$ and set $L_2 = \mathbf{v}^\perp/\mathbf{v} \cong \mathbb{U} \oplus \mathbb{E}_8(2)$. Then $L_2 \otimes \mathbf{R} + i\,\mathcal{C}_{L_2}^+ \cong \Omega_\Lambda^+$.

Definition 2 The Borcherds Φ-function is the formal Fourier series on the tube domain $L_2 \otimes \mathbf{R} + i\,\mathcal{C}^+_{L_2}$ defined as

$$\Phi_2(z) := 2^8 e^{2\pi i \langle \rho, z\rangle} \prod_{\lambda \in L_2,\, \langle \lambda, \rho\rangle > 0 \,\mathrm{or}\, \lambda \in \mathbf{N}\rho} \left(1 - e^{2\pi i \langle \lambda, z\rangle}\right)^{(-1)^{\langle \rho - \rho', \lambda\rangle} c(\lambda^2/2)},$$

where $\rho = ((0,1),0)$, $\rho' = ((1,0),0) \in L_2$.

Theorem 3 (Borcherds [8, 9]) *For $j = 1, 2$, the formal Fourier series $\Phi_j(z)$ as above converges absolutely for $z \in L_j \otimes \mathbf{R} + i\,\mathcal{C}^+_{L_j}$ with $\Im z \gg 0$ and extends to an automorphic form on $L_j \otimes \mathbf{R} + i\,\mathcal{C}^+_{L_j}$ for $O^+(\boldsymbol{\Lambda})$ of weight* 4. *Regarded as holomorphic functions on $\Omega^+_{\boldsymbol{\Lambda}}$, one has the equality up to a constant of modulus* 1

$$\Phi_1 = \Phi_2.$$

In what follows, we write $\Phi(z)$ for $\Phi_1(z)$ and $\Phi_2(z)$.

Definition 3 The Petersson norm of Φ is the C^∞ function on $L_j \otimes \mathbf{R} + i\,\mathcal{C}^+_{L_j}$ defined as

$$\left\|\Phi(z)\right\|^2 := \langle \Im z, \Im z\rangle^4 \left|\Phi_j(z)\right|^2.$$

Since the Petersson norm $\|\Phi(z)\|$ is $O^+(\boldsymbol{\Lambda})$-invariant, we regard $\|\Phi(z)\|$ as a function on the orthogonal modular variety $\Omega^+_{\boldsymbol{\Lambda}}/O^+(\boldsymbol{\Lambda})$.

By [9, Th. 13.3], $\log\|\Phi\|$ is defined as the finite part of the divergent integral:

$$-4\log\left\|\Phi(Z)\right\| - 8\left(\Gamma'(1) + \log(2\pi)\right) = \mathrm{Pf}\int_{SL_2(\mathbf{Z})\backslash\mathfrak{H}} F(\tau)\cdot \overline{\Theta_{\boldsymbol{\Lambda}}(\tau, Z)}\, y\, \frac{dx\,dy}{y^2},$$

where $F(\tau)$ is a certain vector-valued elliptic modular form for $Mp_2(\mathbf{Z})$ (cf. [36, Def. 7.6] with $\Lambda = \boldsymbol{\Lambda}$) and $\Theta_{\boldsymbol{\Lambda}}(\tau, Z)$ is the Siegel theta function [9] of the Enriques lattice $\boldsymbol{\Lambda}$. Then the expressions $\Phi_1(z)$ and $\Phi_2(z)$ are obtained by computing the above integral at the level 1 cusp and the level 2 cusp, respectively. For the necessity of the constant 2^8 in $\Phi_2(z)$, see [9, Th. 13.3 (5)] and [36, Eq. (7.9)].

Remark 1 One can rewrite the expression of $\Phi(z)$ using the dual lattice of $\boldsymbol{\Lambda}$. Set $L := \mathbb{U} \oplus \mathbb{E}_8(-1)$. Since the dual lattice of $\boldsymbol{\Lambda}$ is given by $\boldsymbol{\Lambda}^\vee = \mathbb{U} \oplus L(1/2)$, we get

$$\boldsymbol{\Lambda}^\vee(2) = \mathbb{U}(2) \oplus L.$$

Then the Borcherds Φ-function can be expressed as a function on $L \otimes \mathbf{R} + i\,\mathcal{C}^+_L$

$$\Phi(z) = \prod_{\lambda \in \mathbb{L} \cap \overline{\mathcal{C}^+_L} \setminus \{0\}} \left(\frac{1 - e^{2\pi i \langle \lambda, z\rangle}}{1 + e^{2\pi i \langle \lambda, z\rangle}}\right)^{c(\lambda^2/2)} = \sum_{\lambda \in \mathbb{L} \cap \overline{\mathcal{C}^+_L},\, \lambda^2 = 0,\, \text{primitive}} \frac{\eta(\langle \lambda, z\rangle)^{16}}{\eta(2\langle \lambda, z\rangle)^8}.$$

This identity is known as the denominator identity for the fake monster superalgebra. See [9, Example 13.7] and [30] for more details about the denominator identity for the fake monster superalgebra. See [7, 8] for the Fourier expansion of $\Phi_2(z)$.

3 Enriques Surfaces and Their Moduli Space

In this section, we recall Enriques surfaces.

3.1 K3 Surfaces

A compact connected complex surface X is a $K3$ surface if

$$H^1(X, \mathcal{O}_X) = 0, \qquad \Omega_X^2 \cong \mathcal{O}_X.$$

It is known that the diffeomorphism type underlying a $K3$ surface is unique. In particular, the second integral cohomology group of a $K3$ surface equipped with the cup-product pairing is isometric to the $K3$-lattice

$$\mathbb{L}_{K3} := \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8(-1) \oplus \mathbb{E}_8(-1).$$

For a $K3$ surface X, an isometry of lattices $\alpha\colon H^2(X, \mathbf{Z}) \cong \mathbb{L}_{K3}$ is called a marking.

Let X be a $K3$ surface and let $\alpha\colon H^2(X, \mathbf{Z}) \cong \mathbb{L}_{K3}$ be a marking. Since Ω_X^2 is trivial, there exists a unique nowhere vanishing holomorphic 2-form η on X, up to a non-zero constant. By the Hodge decomposition, we get the natural inclusion $H^0(X, \Omega_X^2) \subset H^2(X, \mathbf{Z}) \otimes \mathbf{C}$, so that the line $\mathbf{C}\eta \in \mathbf{P}(H^2(X, \mathbf{C}))$ is uniquely determined by X. The period of (X, α) is defined as the point of $\mathbf{P}(\mathbb{L}_{K3} \otimes \mathbf{C})$ corresponding to $\mathbf{C}\eta$ via the marking α:

$$\varpi(X, \alpha) := \left[\alpha(\eta)\right] \in \Omega_{\mathbb{L}_{K3}}.$$

Here we define $\Omega_{\mathbb{L}_{K3}} = \{[Z] \in \mathbf{P}(\mathbb{L}_{K3} \otimes \mathbf{C});\ \langle Z, Z\rangle = 0, \langle Z, \bar{Z}\rangle > 0\}$ as before. Notice that $[\alpha(\eta)] \in \Omega_{\mathbb{L}_{K3}}$ by the Riemann-Hodge bilinear relations $\int_X \eta \wedge \eta = 0$ and $\int_X \eta \wedge \overline{\eta} > 0$. For $K3$ surfaces and their moduli space, see [1] for more details.

3.2 Enriques Surfaces

A compact connected complex surface Y is an Enriques surface if

$$H^1(Y, \mathcal{O}_Y) = 0, \qquad \Omega_Y^2 \not\cong \mathcal{O}_Y, \qquad \left(\Omega_Y^2\right)^{\otimes 2} \cong \mathcal{O}_Y.$$

It is known that the universal covering of an Enriques surface is a $K3$ surface and an Enriques surface is obtained as the quotient of its universal covering by a fixed-point-free involution. Notice that a single $K3$ surface can cover many distinct Enriques surfaces (cf. [25–28] and Subsect. 5.3 below).

Let Y be an Enriques surface and let $\widetilde{Y} \to Y$ be the universal covering. Let $\iota\colon Y \to Y$ be the non-trivial covering transformation of $\widetilde{Y} \to Y$. Write $H^2(\widetilde{Y}, \mathbf{Z})_+$ and $H^2(\widetilde{Y}, \mathbf{Z})_-$ for the invariant and anti-invariant subspaces of $H^2(\widetilde{Y}, \mathbf{Z})$ with respect to the ι-action, respectively. Let $I : \mathbb{L}_{K3} \to \mathbb{L}_{K3}$ be the involution defined as

$$I(a, b, c, x, y) := (b, a, -c, y, x), \quad a, b, c \in \mathbb{U},\ x, y \in \mathbb{E}_8(-1).$$

By [13, 14], there exists a marking $\alpha\colon\ H^2(\widetilde{Y}, \mathbf{Z}) \cong \mathbb{L}_{K3}$ such that

$$\alpha \circ \iota^* \circ \alpha^{-1} = I.$$

Let $(\mathbb{L}_{K3})_+$ and $(\mathbb{L}_{K3})_-$ be the invariant and anti-invariant subspaces of $\mathbb{L}_{K3}$ with respect to the I-action, respectively. Then we have isometries of lattices

$$\alpha\bigl(H^2(\widetilde{Y}, \mathbf{Z})_+\bigr) = (\mathbb{L}_{K3})_+ \cong \mathbb{U}(2) \oplus \mathbb{E}_8(-2), \qquad \alpha\bigl(H^2(\widetilde{Y}, \mathbf{Z})_-\bigr) = (\mathbb{L}_{K3})_- \cong \boldsymbol{\Lambda}.$$

Since Y has no non-zero holomorphic 2-forms, we get $H^0(\widetilde{Y}, \Omega^2_{\widetilde{Y}}) \subset H^2(\widetilde{Y}, \mathbf{Z})_- \otimes \mathbf{C}$. Hence $\varpi(\widetilde{Y}, \alpha) \in \Omega_{\boldsymbol{\Lambda}}$ if α is a marking as above. The period of an Enriques surface $Y = \widetilde{Y}/\iota$ is defined as the period of its universal covering $\widetilde{Y}$, i.e.,

$$\varpi(Y) := \bigl[\varpi(\widetilde{Y}, \alpha)\bigr] \in \Omega^+_{\boldsymbol{\Lambda}}/O^+(\boldsymbol{\Lambda}),$$

where α is a marking satisfying $\alpha \circ \iota^* \circ \alpha^{-1} = I$ and $[\varpi(\widetilde{Y}, \alpha)]$ denotes the $O^+(\boldsymbol{\Lambda})$-orbit of $\varpi(\widetilde{Y}, \alpha)$. It is known that the isomorphism class of an Enriques surface is classified by its period:

Theorem 4 (Horikawa [13, 14]) *There exists a coarse moduli space of Enriques surfaces, denoted by $\mathcal{M}$. The period mapping induces an isomorphism between the analytic spaces*

$$\varpi : \mathcal{M} \ni [Y] \to \bigl[\varpi(Y)\bigr] \in \frac{\Omega^+_{\boldsymbol{\Lambda}} \setminus \mathcal{D}_{\boldsymbol{\Lambda}}}{O^+(\boldsymbol{\Lambda})}.$$

In what follows, we identify $\mathcal{M}$ with $(\Omega^+_{\boldsymbol{\Lambda}} \setminus \mathcal{D}_{\boldsymbol{\Lambda}})/O^+(\boldsymbol{\Lambda})$ by the map ϖ. We refer the reader to [1] for more details about Enriques surfaces and their moduli space. By Theorem 4, the period mapping for Enriques surfaces omit the discriminant locus. The Borcherds Φ-function characterize exactly the discriminant locus $\mathcal{D}_{\boldsymbol{\Lambda}}$.

Theorem 5 (Borcherds [8]) *The Borcherds Φ-function vanishes exactly on $\mathcal{D}_{\boldsymbol{\Lambda}}$ of order* 1. *In particular, Φ is a nowhere vanishing holomorphic section of the Hodge line bundle on $\mathcal{M}$.*

Since the line bundle of automorphic forms on an arithmetic quotient of a symmetric bounded domain is an ample line bundle by Baily-Borel, the moduli space of Enriques surfaces is quasi-affine by Theorem 5 [8]. In fact, the quasi-affinity of the moduli space holds for wider classes of $K3$ surfaces with involution. See [36].

4 Analytic Torsion and Borcherds Φ-Function: An Analytic Counterpart

The notion of holomorphic analytic torsion was introduced by Ray-Singer [29] in their works extending the classical notion of torsion in algebraic topology to certain analytic settings; they extended the construction of torsion of finite-dimensional acyclic complex to the setting of de Rham or Dolbeault complex, in which they replaced the usual finite-dimensional determinant of the combinatorial Laplacian to the regularized determinant of the Hodge-Kodaira Laplacian. In this section, we explain the construction of the Borcherds Φ-function via analytic torsion.

4.1 Analytic Torsion

Let (M, h^{TM}) be a compact connected Kähler manifold. Let $\Box_q = (\bar{\partial} + \bar{\partial}^*)^2$ be the Hodge-Kodaira Laplacian acting on $(0, q)$-forms on M. Since M is compact, the Hilbert space of square integrable $(0, q)$-forms on M splits into the direct sum $L^{0,q}_M = \bigoplus_{\lambda \in \sigma(\Box_q)} E(\lambda, \Box_q)$, where $\sigma(\Box_q) \subset \mathbf{R}_{\geq 0}$ is the spectrum of $\Box_q$ and $E(\lambda, \Box_q)$ is the eigenspace of $\Box_q$ with respect to the eigenvalue λ. Then $E(\lambda, \Box_q)$ is of finite-dimensional. The zeta function of $\Box_q$ is defined as

$$\zeta_q(s) := \sum_{\lambda \in \sigma(\Box_q) \setminus \{0\}} \lambda^{-s} \dim E(\lambda, \Box_q).$$

By the Weyl law of the asymptotic distribution of the eigenvalues of $\Box_q$, $\zeta_q(s)$ converges absolutely for $s \in \mathbf{C}$ with $\Im s > \dim M$. From the existence of the asymptotic expansion of the trace of the heat operator $e^{-t\Box_q}$ as $t \to 0$, it follows that $\zeta_q(s)$ extends to a meromorphic function on $\mathbf{C}$ and that $\zeta_q(s)$ is holomorphic at $s = 0$. After Ray-Singer [29], we make the following

Definition 4 The analytic torsion of (M, h^{TM}) is the real number defined as

$$\tau\bigl(M, h^{TM}\bigr) := \exp\Bigl[-\sum_{q \geq 0} (-1)^q q\, \zeta_q'(0)\Bigr].$$

When $\dim M = 1$, $\tau(M)^{-1}$ is exactly the determinant of Laplacian appearing in the formula for $\|\eta(\tau)\|$. After Theorem 1, it is natural to expect that the determinant of Laplacian or analytic torsion may produce a nice function on the moduli space. This is the main topic of this section.

One natural direction of such a generalization seems to be the study of the determinant of Laplacian for compact Riemann surfaces of higher genus $g > 1$. Among numbers of studies of the determinant of Laplacian for hyperbolic Riemann surfaces of genus $g > 1$, it is Zograf [37] and McIntyre-Takhtajan [24] who obtained a holomorphic function with infinite product expression on the Schottky space by using the determinant of Laplacian. On the other hand, Kokotov-Korotkin [17] considered the determinant of Laplacian with respect to the *flat* (but degenerate) Kähler metric $\omega \otimes \overline{\omega}$, where ω is an Abelian differential on a compact Riemann surface of genus $g > 1$. They proved that, as a function on the moduli space of pairs (C, ω), with C being a marked Riemann surfaces of genus $g > 1$ and ω being an Abelian differential on C, the determinant of Laplacian is expressed by using some classical quantities like prime forms, theta function and periods. Hence there are two different generalizations of Theorem 1 in higher genus $g > 1$.

Another direction of generalization is the study of analytic torsion for higher dimensional varieties. (For several reasons, in higher dimensions, analytic torsion seems to be more appropriate than a single determinant of Laplacian in considering a generalization of Theorem 1.) Among those varieties, we are interested in Enriques surfaces, since they can be regarded as one of the natural generalizations of elliptic curves in dimension 2. For other directions of generalization, we refer to [11, 33], where analytic torsion produces the Siegel modular form characterizing the Andreotti-Mayer locus and the section of certain line bundle on the moduli space of Calabi-Yau threefolds characterizing the discriminant locus.

4.2 Borcherds Φ-Function as the Analytic Torsion of Enriques Surface

As in the case of elliptic curves, we choose some special Kähler metric to construct an invariant of an Enriques surface. Since $c_1(Y)_{\mathbf{R}} = 0$ for an Enriques surface Y, there exists by Yau [31] a unique Ricci-flat Kähler form in each Kähler class on Y. In contrast to elliptic curves, the condition of Ricci-flatness with normalized volume 1 does not determine a unique Kähler form on Y, because the space of Kähler classes on Y has real dimension 10. Even though, we get the following:

Theorem 6 ([34]) *Let Y be an Enriques surface and let γ be a Ricci-flat Kähler metric on Y with normalized volume* 1. *Then the analytic torsion $\tau(Y, \gamma)$ is independent of the choice of such a Kähler metric γ. In particular, $\tau(Y, \gamma)$ is an invariant of Y.*

After Theorem 6, we may write $\tau(Y)$ for $\tau(Y, \gamma)$. Then the analytic torsion gives rise to the function on the moduli space of Enriques surfaces

$$\tau : \mathcal{M} \ni [Y] \to \tau(Y) \in \mathbf{R}.$$

Recall that the Petersson norm of the Borcherds $\varPhi$-function $\|\varPhi\|$ is $O^+(\boldsymbol{\Lambda})$-invariant and hence it descends to a function on $\mathcal{M}$. We write $\|\varPhi(Y)\|$ for $\|\varPhi(\varpi(\widetilde{Y},\alpha))\|$.

Theorem 7 ([34]) *There exists an absolute constant $C \neq 0$ such that for every Enriques surface Y, the following equality holds*

$$\tau(Y) = C \left\|\varPhi(Y)\right\|^{-1/4}.$$

The proofs of Theorems 6 and 7 are based on the curvature formula for (equivariant) Quillen metrics [4–6, 19] and the immersion formula for (equivariant) Quillen metrics [2, 3]. We compare the $\partial\bar{\partial}$ of $\log\tau$ and $\log\|\varPhi\|$ as currents on the Baily-Borel compactification of $\Omega^+_{\boldsymbol{\Lambda}}/O^+(\boldsymbol{\Lambda})$. For this, the curvature formula and the immersion formula for (equivariant) Quillen metrics play crucial roles. We refer the reader to [34] for the details of the proofs of Theorems 6 and 7.

As in the case of elliptic curves, we get an analytic expression of the Borcherds $\varPhi$-function by using analytic torsion. In fact, we can extend this result to arbitrary $K3$ surfaces with anti-symplectic involution. Namely, for a $K3$ surface X equipped with an involution $\iota\colon X \to X$ acting non-trivially on $H^0(X, \Omega^2_X)$, we can construct an invariant $\tau_M(X,\iota)$ by using the *equivariant* analytic torsion of (X,ι), the analytic torsion of the fixed-point-set of ι and a certain Bott-Chern secondary class. Here M refers to the isometry class of the invariant sublattice of $H^2(X,\mathbf{Z})$ with respect to the ι-action, which determines the topological type of ι. When $M = \mathbb{U}(2) \oplus \mathbb{E}_8(-2)$, we get the analytic torsion of Enriques surface τ as above. It is worth remarking that we can construct the invariant $\tau_M(X,\iota)$ without assuming the existence of Ricci-flat Kähler metrics on X. After fixing M, i.e., the topological type of the involution, the invariant $\tau_M(X,\iota)$ gives rise to a function on the moduli space of $K3$ surfaces with involution, which is again a certain arithmetic quotient of a symmetric bounded domain of type IV, with the discriminant divisor removed. As before in Theorem 7, the resulting function τ_M is the Petersson norm of an automorphic form on the moduli space of $K3$ surfaces with involution. It is remarkable that the corresponding automorphic form on the moduli space of $K3$ surfaces with involution thus obtained, is very often expressed as the product of a certain Borcherds lift and Igusa's Siegel modular form. We refer the reader to [34, 36] for more details about the analytic torsion invariant τ_M of $K3$ surfaces with involution.

5 Resultants and Borcherds $\varPhi$-Function: An Algebraic Counter Part

In this section, we explain an algebro-geometric counterpart of the Borcherds $\varPhi$-function.

5.1 (2, 2, 2)-Model of an Enriques Surface

Let

$$f_1(x), g_1(x), h_1(x) \in \mathbf{C}[x_1, x_2, x_3], \qquad f_2(x), g_2(x), h_2(x) \in \mathbf{C}[x_4, x_5, x_6]$$

be homogeneous polynomials of degree 2. We define $f, g, h \in \mathbf{C}[x_1, x_2, x_3, x_4, x_5, x_6]$ by

$$f(x) := f_1(x) + f_2(x), \qquad g(x) := g_1(x) + g_2(x), \qquad h(x) := h_1(x) + h_2(x)$$

and the corresponding surface $X_{(f,g,h)}$ by

$$X_{(f,g,h)} := \left\{[x] \in \mathbf{P}^5;\ f(x) = g(x) = h(x) = 0\right\}.$$

If the quadratic forms f_1, g_1, h_1, f_2, g_2, h_2 are generic enough, then $X_{(f,g,h)}$ equipped with the line bundle $\mathcal{O}_{\mathbf{P}^5}(1)$ is a $K3$ surface of degree 8 by the adjunction formula. Let ι be the involution on $\mathbf{C}^6$ defined as

$$\iota(x_1, x_2, x_3, x_4, x_5, x_6) := (x_1, x_2, x_3, -x_4, -x_5, -x_6).$$

The involution on $\mathbf{P}^5$ induced by ι is again denoted by the same symbol ι. Since the set of fixed points of the ι-action on $\mathbf{P}^5$ is the disjoint union of two projective planes $P_1 := \{x_1 = x_2 = x_3 = 0\}$ and $P_2 := \{x_4 = x_5 = x_6 = 0\}$, we see that $X^{\iota}_{(f,g,h)}$, the set of fixed points of the ι-action on $X_{(f,g,h)}$, is given by

$$X^{\iota}_{(f,g,h)} = (X_{(f,g,h)} \cap P_1) \amalg (X_{(f,g,h)} \cap P_2).$$

For three quadratic forms in three variables $q_1(x, y, z)$, $q_2(x, y, z)$, $q_3(x, y, z)$, let $R(q_1, q_2, q_3)$ be the resultant of q_1, q_2, q_3. Then $R(q_1, q_2, q_3)$ is the polynomial of degree 12 of the coefficients of q_1, q_2, q_3 characterizing the existence of common intersection points of the three conics of $\mathbf{P}^2$ defined by $q_1 = 0$, $q_2 = 0$ and $q_3 = 0$. Namely,

$$R(q_1, q_2, q_3) = 0 \quad \Longleftrightarrow \quad \left\{(x : y : z) \in \mathbf{P}^2;\ q_1 = q_2 = q_3 = 0\right\} \neq \emptyset.$$

If $q_i(x, y, z) = a_{i1}x^2 + a_{i2}y^2 + a_{i3}z^2 + a_{i4}xy + a_{i5}xz + a_{i6}yz$, then $R(q_1, q_2, q_3)$ is expressed as an explicit integral linear combination of the polynomials of the form

$$[j_1, j_2, j_3][k_1, k_2, k_3][l_1, l_2, l_3][m_1, m_2, m_3],$$

where

$$[j_1, j_2, j_3] := \begin{vmatrix} a_{1,j_1} & a_{1,j_2} & a_{1,j_3} \\ a_{2,j_1} & a_{2,j_2} & a_{2,j_3} \\ a_{3,j_1} & a_{3,j_2} & a_{3,j_3} \end{vmatrix}.$$

See [15, p. 215 Table 1] for an explicit formula for $R(q_1, q_2, q_3)$.

If the quadrics $f_1, g_1, h_1, f_2, g_2, h_2$ are generic enough, then we may assume that $R(f_1, g_1, h_1)R(f_2, g_2, h_2) \neq 0$, so that ι has no fixed points on $X_{(f,g,h)}$ in that case. Hence, if $R(f_1, g_1, h_1)R(f_2, g_2, h_2) \neq 0$ and $X_{(f,g,h)}$ is smooth, then

$$Y_{(f,g,h)} := X_{(f,g,h)}/\iota$$

is an Enriques surface. Let us see that a generic Enriques surface is constructed in this manner.

Assume that $R(f_1, g_1, h_1)R(f_2, g_2, h_2) \neq 0$ and that $X_{(f,g,h)}$ is smooth. For simplicity, set $X_0 := X_{(f,g,h)}$. Let $S := \mathrm{Gr}_3(\mathrm{Sym}^2\mathbf{C}^6) \cong \mathrm{Gr}_3(\mathbf{C}^{\binom{7}{2}})$ be the Grassmann variety of 3-dimensional subspaces in the vector space of quadratic forms in the variables $x_1, \ldots, x_6$. Then S is equipped with the ι-action induced from the one on $\mathbf{C}^6$ and with the $PGL(\mathbf{C}^6)$-action induced from the standard $GL(\mathbf{C}^6)$-action on $\mathbf{C}^6$. By choosing $f_1, g_1, h_1, f_2, g_2, h_2$ generic enough, we may assume that $\mathfrak{sl}(\mathbf{C}^6)$ is a subspace of the tangent space of S at the point $\mathrm{Span}\{f, g, h\} \in S$.

For $s \in S$, we define $X_s := \{[x] \in \mathbf{P}^5; q(x) = 0\,(\forall q \in s)\}$. Then we get a flat family $\pi\colon X \to S$ with $\pi^{-1}(s) = X_s$. Write $[X_0] \in S$ for $\mathrm{Span}\{f, g, h\} \in S$. We get a flat deformation $\pi\colon (X, X_0) \to (S, [X_0])$ of $K3$ surfaces of degree 8. Since ι preserves X_0 and hence $\iota([X_0]) = [X_0]$, we get a subfamily $\pi\colon (X|_{S^\iota}, \iota, X_0) \to (S^\iota, [X_0])$ of $K3$ surfaces with involution, where $S^\iota := \{s \in S; \iota(s) = s\}$ is the fixed-point-set of the ι-action on S. Since ι has no fixed points on X_0 by assumption and since the set of fixed points of the ι-action on X is a closed subset of X, we see that ι has no fixed points on X_s if $s \in S^\iota$ is sufficiently close to $[X_0]$. We define $Y := (X|_{S^\iota})/\iota$ and $Y_0 := X_0/\iota$. Let $p\colon Y \to S$ be the projection induced from $\pi\colon X \to S$. Since ι has no fixed points on X_s, Y_s is an Enriques surface for $s \in S$ sufficiently close to $[X_0]$. Hence $p\colon (Y, Y_0) \to (S^\iota, [X_0])$ is a flat deformation of Y_0.

Let $\rho_{X_0}\colon T_{[X_0]}S \to H^1(X_0, \Theta_{X_0})$ and $\rho_{Y_0}\colon T_{[X_0]}S^\iota \to H^1(Y_0, \Theta_{Y_0})$ be the Kodaira-Spencer maps of the deformations $\pi\colon (X, X_0) \to (S, [X_0])$ and $p\colon (Y, Y_0) \to (S^\iota, [X_0])$, respectively. Let $(T_{[X_0]}S)_+$ and $H^1(X_0, \Theta_{X_0})_+$ be the invariant subspaces of $T_{[X_0]}S$ and $H^1(X_0, \Theta_{X_0})$ with respect to the ι-action, respectively. Since ρ_{X_0} commutes with the ι-action, we set $(\rho_{X_0})_+ := \rho_{X_0}|_{(T_{[X_0]}S)_+}\colon (T_{[X_0]}S)_+ \to H^1(X_0, \Theta_{X_0})_+$. Since $(\rho_{X_0})_+$ can be identified with ρ_{Y_0} under the identifications $(T_{[X_0]}S)_+ = T_{[X_0]}S^\iota$ and $H^1(X_0, \Theta_{X_0})_+ = H^1(Y_0, \Theta_{Y_0})$, we get

$$\ker \rho_{Y_0} \cong \ker(\rho_{X_0})_+ = \mathfrak{sl}\big(\mathbf{C}^6\big) \cap \ker(\iota_* - 1) \cong \mathfrak{sl}\big(\mathbf{C}^3\big) \oplus \mathfrak{sl}\big(\mathbf{C}^3\big) \oplus \mathbf{C} \cong \mathbf{C}^{17}.$$

Here the second equality follows from the equality $\ker \rho_{X_0} = \mathfrak{sl}(\mathbf{C}^6)$, which is a consequence of the fact that $X_s \cong X_{s'}$ as polarized $K3$ surfaces of degree 8 if and only if s and s' lie on the same $PGL(\mathbf{C}^6)$-orbit. (We can also see the equality $\ker \rho_{X_0} = \mathfrak{sl}(\mathbf{C}^6)$ as follows. Set $\mathcal{L}_0 := \mathcal{O}_{\mathbf{P}^5}(1)|_{X_0}$. We consider the semiuniversal deformation $q\colon ((\mathfrak{X}, \mathcal{L}), (X_0, \mathcal{L}_0)) \to (\mathrm{Def}(X_0, \mathcal{L}_0), [X_0])$ of the polarized $K3$ surface (X_0, L_0) of degree 8. Since $\mathcal{L}_0$ is very ample on X_0, we may assume that $\mathcal{L}$ is very ample on $\mathfrak{X}_t$ for $t \in \mathrm{Def}(X_0, \mathcal{L}_0)$. Since $\deg \mathcal{L}|_{\mathfrak{X}_t} = 8$, the image of the projective embedding $\Phi_{|\mathcal{L}|_{\mathfrak{X}_t}|}\colon \mathfrak{X}_t \to \mathbf{P}^5$ must be a $(2, 2, 2)$-complete intersection. Namely, $(\mathfrak{X}_t, \mathcal{L}|_{\mathfrak{X}_t})$ is isomorphic to $(X_s, \mathcal{O}_{\mathbf{P}^5}(1))$ for some $s \in S$. Hence the deformation germ of polarized $K3$ surfaces $\pi: (X, X_0) \to (S, [X_0])$ is complete, which implies the equality $\dim\ker\rho_{X_0} = \dim S - \dim \mathrm{Def}(X_0, \mathcal{L}_0) = 35 = \dim \mathfrak{sl}(\mathbf{C}^6)$. This, together with the inclusion $\mathfrak{sl}(\mathbf{C}^6) \subset \ker\rho_{X_0}$, yields the equality $\ker\rho_{X_0} = \mathfrak{sl}(\mathbf{C}^6)$.)

Since $\dim S^\iota = 27$ and $\dim\ker\rho_{Y_0} = 17$, we get $\dim \mathrm{Im}\,\rho_{Y_0} = 27 - 17 = 10 = \dim H^1(Y_0, \Theta_{Y_0})$. Hence the Kodaira-Spencer map ρ_{Y_0} is surjective and the family $p\colon (Y, Y_0) \to (S^\iota, [X_0])$ is complete.

Set $U := \{s \in S^\iota;\ \mathrm{Sing}\,X_s = X_s^\iota = \emptyset\}$. Then U is a Zariski open subset of S^ι. For $s \in U$, $Y_s = X_s/\iota$ is an Enriques surface. Let $\varpi: U \ni s \to \varpi(X_s/\iota) \in \mathcal{M}$ be the period mapping for the family of Enriques surfaces $p\colon Y|_U \to U$. By the Borel-Kobayashi-Ochiai extension theorem, ϖ extends to a rational map from S^ι to the Baily-Borel compactification of $\Omega^+_{\boldsymbol{\Lambda}}/O^+(\boldsymbol{\Lambda})$. By the completeness of the deformation germ $p\colon (Y, Y_0) \to (S^\iota, [X_0])$, the image of ϖ contains a dense Zariski open subset of $\mathcal{M}$, say $\mathcal{U}$. If Y is an Enriques surface with $\varpi(Y) \in \mathcal{U}$, then $Y = Y_{(F,G,H)}$ for some quadratic forms F, G, H.

5.2 An Algebraic Expression of Borcherds Φ-Function

Since we have a nice projective model of Enriques surfaces of degree 4, it is natural to expect that the Borcherds Φ-function may admit an algebraic expression analogous to the one for the Dedekind η-function associated to the plane cubic model or the $(2, 2)$-complete intersection model. In fact, this is the case.

Theorem 8 ([16]) *Let $Y_{(f,g,h)}$ be the $(2,2,2)$-model of an Enriques surface defined by the quadric polynomials $f = f_1 + f_2$, $g = g_1 + g_2$, $h = h_1 + h_2 \in \mathbf{C}[x_1, x_2, x_3, x_4, x_5, x_6]$. Then the following equality holds*

$$\left\|\Phi(Y_{(f,g,h)})\right\|^2 = \left|R(f_1, g_1, h_1)R(f_2, g_2, h_2)\right|\left(\frac{2}{\pi^4}\int_{X_{(f,g,h)}} \alpha_{(f,g,h)} \wedge \overline{\alpha_{(f,g,h)}}\right)^4.$$

Here $\alpha_{(f,g,h)} \in H^0(X_{(f,g,h)}, \Omega^2_{X_{(f,g,h)}})$ is defined as the residue of f, g, h, i.e.,

$$\alpha_{(f,g,h)} := \Xi|_{X_{(f,g,h)}},$$

where Ξ is a meromorphic 2-form on $\mathbf{P}^5$ satisfying the equation

$$df \wedge dg \wedge dh \wedge \Xi = \sum_{i=1}^{6} (-1)^i x_i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_6.$$

We remark that a weaker version of this result was obtained by Maillot-Roessler [20] under a certain arithmeticity assumption on $X_{(f,g,h)}$. In their formula, the contribution from the resultants is understood as the contribution from the bad primes with respect to the reductions of $X_{(f,g,h)}$. When f, g, h are defined over the ring of integers of a number field K, Theorem 8 implies that the Borcherds Φ-function detects the degenerations of ι over $\mathrm{Spec}(\mathcal{O}_K)$, since $R(f_1, g_1, h_1)R(f_2, g_2, h_2) \in \mathfrak{p}$ for a prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K)$ if and only if ι has non-empty fixed points on the reduction $X_{(f,g,h)}(\mathcal{O}_K/\mathfrak{p})$. This picture of the Borcherds Φ-function is quite analogous to the corresponding picture of the Dedekind η-function: For an elliptic curve $E = \{y^2 = 4x^3 - g_2 x - g_3\}$ over K, $\|\eta\|^{24}$ is identified with the discriminant of E up to the L^2-norm of dx/y. Hence the algebraic part of $\|\eta\|$ detects the degenerations of E over $\mathrm{Spec}(\mathcal{O}_K)$. See [10] for more explanation of this view point.

The proof of Theorem 8 shall be given in [16]. The strategy is as follows. We compare the $\partial\bar{\partial}$ of the both hand sides as currents on S. Then it turns out that they satisfy the same $\partial\bar{\partial}$-equation of currents on S. For this, we use Theorem 7 and a formula for the asymptotic behavior of equivariant analytic torsion for degenerating family of algebraic manifolds [35]. In this way, we get the desired equality, up to an absolute constant. To fix the absolute constant, we compare the behavior of the both hand sides for certain explicit 2-parameter family of Enriques surfaces, whose universal coverings are Kummer surfaces of product type.

In fact, Theorem 8 holds even if $Y_{(f,g,h)}$ has at most rational double points by the continuity of the both hand sides at those points of S^ι corresponding to Enriques surfaces with rational double points. This continuity is a consequence of the existence of simultaneous resolution of 2-dimensional rational double points.

By Theorem 8, we get a Thomae type formula for the Borcherds Φ-function.

Corollary 1 ([16]) *Let* $\mathbf{v}, \mathbf{v}' \in H^2(X_{(f,g,h)}, \mathbf{Z})$ *be anti-ι-invariant, primitive, isotropic vectors with* $\langle \mathbf{v}, \mathbf{v}' \rangle = 1$ *and let* $\mathbf{v}^\vee \in H_2(X_{(f,g,h)}, \mathbf{Z})$ *be the Poincaré dual of* $\mathbf{v}$. *Under the identification of lattices* $(\mathbf{Z}\mathbf{v} + \mathbf{Z}\mathbf{v}')^\perp \cong \mathbb{U}(2) \oplus \mathbb{E}_8(-2) =: L$, *the vector*

$$z_{(f,g,h),\mathbf{v},\mathbf{v}'} := \frac{\alpha - \langle \alpha, \mathbf{v}' \rangle \mathbf{v} - \langle \alpha, \mathbf{v} \rangle \mathbf{v}'}{\langle \alpha, \mathbf{v} \rangle} \in L \otimes \mathbf{R} + i\, \mathcal{C}_L^+$$

is regarded as the period of $Y_{(f,g,h)}$. *Then, by a suitable choice of the* 2*-cocycles* $\{\mathbf{v}, \mathbf{v}'\}$, *one has*

$$\Phi(z_{(f,g,h),\mathbf{v},\mathbf{v}'})^2 = R(f_1, g_1, h_1)R(f_2, g_2, h_2)\left(\frac{2}{\pi^2}\int_{\mathbf{v}^\vee} \alpha_{(f,g,h)}\right)^8.$$

When $X_{(f,g,h)}$ is birational to a Kummer surface of product type, the 2-cycle $\mathbf{v}^\vee$ can be given explicitly. See [16] for the details.

5.3 A 4-Parameter Family of Enriques Surfaces Associated to $M_{3,6}(\mathbf{C})$

For a non-zero 3×6-complex matrix $A \in M_{3,6}(\mathbf{C})$, we define

$$X_A := \left\{ [x] \in \mathbf{P}^5;\ \begin{matrix} f(x) = a_{11}x_1^2 + a_{12}x_2^2 + a_{13}x_3^2 + a_{14}x_4^2 + a_{15}x_5^2 + a_{16}x_6^2 = 0 \\ g(x) = a_{21}x_1^2 + a_{22}x_2^2 + a_{23}x_3^2 + a_{24}x_4^2 + a_{25}x_5^2 + a_{26}x_6^2 = 0 \\ h(x) = a_{31}x_1^2 + a_{32}x_2^2 + a_{33}x_3^2 + a_{34}x_4^2 + a_{35}x_5^2 + a_{36}x_6^2 = 0 \end{matrix} \right\}.$$

For $A = (\mathbf{a}_1, \ldots, \mathbf{a}_6) \in M(3, 6; \mathbf{C})$ and $i < j < k$, we define

$$\Delta_{ijk}(A) = \det(\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k).$$

A matrix $A \in M(3, 6; \mathbf{C})$ is said to be *non-degenerate* if $\prod_{i<j<k} \Delta_{ijk}(A) \neq 0$. Then, for a non-degenerate $A \in M_{3,6}(\mathbf{C})$, X_A is a $K3$ surface. We write α_A for $\alpha_{(f,g,h)}$. As an immediate consequence of Theorem 8, we get the following:

Corollary 2 ([16]) *Let $A \in M_{3,6}(\mathbf{C})$ be non-degenerate. For a partition of* 6 *letters* $\{1, 2, 3, 4, 5, 6\}$

$$\binom{ijk}{lmn} := \{i, j, k\} \cup \{l, m, n\} = \{1, 2, 3, 4, 5, 6\},$$

define an involution $\iota_{\binom{ijk}{lmn}}$ on $\mathbf{P}^5$ by

$$\iota_{\binom{ijk}{lmn}}(x_i, x_j, x_k, x_l, x_m, x_n) = (x_i, x_j, x_k, -x_l, -x_m, -x_n).$$

Then $\iota_{\binom{ijk}{lmn}}$ is a free involution on X_A called a switch *such that*

$$\left\| \Phi(X_A/\iota_{\binom{ijk}{lmn}}) \right\|^2 = \left|\Delta_{ijk}(A)\right|^4 \left|\Delta_{lmn}(A)\right|^4 \left(\frac{2}{\pi^4} \int_{X_A} \alpha_A \wedge \overline{\alpha}_A \right)^4.$$

By Corollary 2 , if $A \in M_{3,6}(K)$ with $K \subset \mathbf{C}$, then for any partitions $\binom{ijk}{lmn}$ and $\binom{i'j'k'}{l'm'n'}$, one has

$$\frac{\|\Phi(X_A/\iota_{\binom{ijk}{lmn}})\|^2}{\|\Phi(X_A/\iota_{\binom{i'j'k'}{l'm'n'}})\|^2} = \frac{|\Delta_{ijk}(A)|^4 |\Delta_{lmn}(A)|^4}{|\Delta_{i'j'k'}(A)|^4 |\Delta_{l'm'n'}(A)|^4} \in K.$$

Since $|\Delta_{ijk}(A)|^4 |\Delta_{lmn}(A)|^4 / |\Delta_{i'j'k'}(A)|^4 |\Delta_{l'm'n'}(A)|^4 \neq 1$ for all pairs of partitions $\binom{ijk}{lmn}$, $\binom{i'j'k'}{l'm'n'}$ for generic non-degenerate A, we conclude that all of the 10 Enriques surfaces $X_A/\iota_{\binom{ijk}{lmn}}$ are mutually distinct for a generic choice of A.

6 Theta Function and Borcherds Φ-Function

In this section, we explain a relation between the Borcherds Φ-function and Freitag's theta function.

6.1 The Matsumoto-Sasaki-Yoshida Model

Recall that, for $A \in M_{2,4}(\mathbf{C})$, we could associate two distinct models E_A and C_A of an elliptic curve. By a similar construction, we can associate another $K3$ surface to $A \in M_{3,6}(\mathbf{C})$ as follows. For $A \in M_{3,6}(\mathbf{C})$, define a $K3$ surface

$$Z_A := \left\{ \big((x_1 : x_2 : x_3), y\big) \in \mathcal{O}_{\mathbf{P}^2}(3);\ y^2 = \prod_{i=1}^{6}(a_{1i}x_1 + a_{2i}x_2 + a_{3i}x_3) \right\},$$

which is identified with its minimal resolution. Then Z_A is (the minimal resolution of) the double covering of $\mathbf{P}^2$, whose branch divisor is the union of 6 lines in general position $a_{1i}x_1 + a_{2i}x_2 + a_{3i}x_3 = 0$ $(i = 1, \ldots, 6)$. The period mapping and its inverse for the family of $K3$ surfaces Z_A over a certain open subset of $M_{3,6}(\mathbf{C})$ were worked out by Matsumoto-Sasaki-Yoshida [23] and Matsumoto [21].

We define a holomorphic 2-form η_A on Z_A by

$$\eta_A := \frac{x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2}{y}.$$

By Matsumoto-Sasaki-Yoshida [23], there are 6 independent transcendental 2-cycles $\{\gamma_{ij}\}_{1\le i<j\le 4}$ on Z_A and 16 independent algebraic 2-cycles on Z_A, which form a basis of $H_2(Z_A, \mathbf{Q})$.

Following Matsumoto-Sasaki-Yoshida [23], define the period of Z_A as the matrix

$$\Omega_A := \frac{1}{\eta_{34}(A)} \begin{pmatrix} \eta_{14}(A) & -\frac{\eta_{13}(A) - \sqrt{-1}\eta_{24}(A)}{1+\sqrt{-1}} \\ -\frac{\eta_{13}(A)+\sqrt{-1}\eta_{24}(A)}{1-\sqrt{-1}} & -\eta_{23}(A) \end{pmatrix},$$

where

$$\eta_{ij}(A) := \int_{\gamma_{ij}} \eta_A.$$

By a suitable choice of the cycles $\{\gamma_{ij}\}_{1\le i<j\le 4}$, one has

$$\Omega_A \in \mathbb{D} := \left\{ T \in M_{2,2}(\mathbf{C});\ \big(T - {}^t\overline{T}\big)/2i > 0 \right\},$$

where $\mathbb{D}$ is isomorphic to a symmetric bounded domain of type IV of dimension 4.

6.2 Theta Function on $\mathbb{D}$

Write $\mathbf{e}(x) := \exp(2\pi i x)$.

Definition 5 For $\Omega \in \mathbb{D}$ and $a, b \in \mathbf{Z}[i]^2$, define the Freitag theta function as

$$\begin{aligned}
&\Theta_{\frac{a}{1+i},\frac{b}{1+i}}(\Omega)\\
&\quad := \sum_{n\in\mathbb{Z}[i]^2} \mathbf{e}\left[\frac{1}{2}\left(n+\frac{a}{1+i}\right)\Omega^t\overline{\left(n+\frac{a}{1+i}\right)} + \Re\left(n+\frac{a}{1+i}\right)^t\overline{\left(\frac{b}{1+i}\right)}\right].
\end{aligned}$$

Following [32], we identify the characteristic $\binom{a}{b}$ with the partition $\binom{ijk}{lmn}$ by the rule:

$$\begin{array}{ccccccccccc}
\binom{a}{b} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} i & i \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} i & i \\ i & i \end{pmatrix} & \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ i & i \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
\binom{ijk}{lmn} & \binom{123}{456} & \binom{124}{356} & \binom{125}{346} & \binom{126}{345} & \binom{134}{256} & \binom{135}{246} & \binom{136}{245} & \binom{145}{236} & \binom{146}{235} & \binom{156}{234}
\end{array}$$

Under this identification, we define

$$\Theta_{\binom{ijk}{lmn}}(\Omega) := \Theta_{\frac{a}{1+i},\frac{b}{1+i}}(\Omega)$$

and its Petersson norm by

$$\left\|\Theta_{\binom{ijk}{lmn}}(\Omega)\right\|^2 := \det\left(\frac{\Omega - {}^t\overline{\Omega}}{2\sqrt{-1}}\right)\left|\Theta_{\binom{ijk}{lmn}}(\Omega)\right|^2.$$

Theorem 9 ([16]) *For a non-degenerate $A = (A_1, A_2) \in M_{3,6}(\mathbf{C})$ with $A_1, A_2 \in M_3(\mathbf{C})$, define*

$$A^\vee := \left({}^tA_1^{-1}, {}^tA_2^{-1}\right).$$

Then

$$\left\|\Phi(X_A/\iota_{\binom{ijk}{lmn}})\right\| = \left\|\Theta_{\binom{ijk}{lmn}}(Z_{A^\vee})\right\|^4.$$

The proof of Theorem 9 shall be given in [16]. We use Matsumoto-Terasoma's Thomae type formula [22] to rewrite the right hand side of Theorem 9. Comparing this with Theorem 8, we get the result. See [16] for the details. We remark that, after Freitag-Salvati-Manni [12, Th. 5.6], Theorem 9 is not very surprising, because they proved that the Borcherds Φ-function itself is expressed as a linear combination of certain additive Borcherds lifts.

6.3 The Case of Jacobian Kummer Surfaces

For $\lambda = (\lambda_1, \dots, \lambda_6) \in \mathbf{C}^6$ with $\lambda_i \neq \lambda_j$ $(i \neq j)$, define a genus 2 curve C_λ by the affine equation

$$C_\lambda := \left\{ (x, y) \in \mathbf{C}^2;\ y^2 = \prod_{i=1}^{6} (x - \lambda_i) \right\}.$$

Define holomorphic differentials ω_1 and ω_2 on C_λ by

$$\omega_1 := \frac{dx}{y}, \qquad \omega_2 := \frac{x\,dx}{y}.$$

Let $\{A_1, A_2, B_1, B_2\}$ be a certain symplectic basis of $H_1(C_\lambda, \mathbf{Z})$ and set

$$T_\lambda := \begin{pmatrix} \int_{B_1} \omega_1 & \int_{B_2} \omega_1 \\ \int_{B_1} \omega_2 & \int_{B_2} \omega_2 \end{pmatrix}^{-1} \begin{pmatrix} \int_{A_1} \omega_1 & \int_{A_2} \omega_1 \\ \int_{A_1} \omega_2 & \int_{A_2} \omega_2 \end{pmatrix} \in \mathfrak{S}_2.$$

Then the Kummer surface $K(C_\lambda)$ of the Jacobian variety $\mathrm{Jac}(C_\lambda)$ is expressed as follows:

$$K(C_\lambda) \cong X_A, \qquad A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \end{pmatrix} \in M_{3,6}(\mathbf{C}).$$

By Theorem 9, we get the following.

Corollary 3 ([16]) *If the partition* $\binom{pqr}{stu}$ *corresponds to the characteristic* (a, b), *then*

$$\left\| \Phi\left(K(C_\lambda)/\iota_{\binom{pqr}{stu}}\right) \right\| = (\det \Im T_\lambda)^2 \left| \theta_{\Re(\frac{a}{1+i}),\Re(\frac{b}{1+i})}(T_\lambda)\, \theta_{\Im(\frac{a}{1+i}),\Im(\frac{b}{1+i})}(T_\lambda) \right|^4.$$

Here $\theta_{\alpha,\beta}(T)$, $\alpha, \beta \in \{0, 1/2\}^2$, *is the Riemann theta constant*

$$\theta_{\alpha,\beta}(T) := \sum_{n \in \mathbf{Z}^2} \mathbf{e}\left[\frac{1}{2}(n+\alpha) T\, {}^t\overline{(n+\alpha)} + (n+\alpha)\, {}^t\overline{\beta} \right], \quad T \in \mathfrak{S}_2.$$

Recall that Igusa's Siegel modular form Δ_5 is defined as the product of all even theta constants

$$\Delta_5(T) := \prod_{(\alpha,\beta)\ \text{even}} \theta_{\alpha,\beta}(T), \quad T \in \mathfrak{S}_2.$$

For a genus 2 curve C with period $T \in \mathfrak{S}_2$, its Petersson norm

$$\left\| \Delta_5(C) \right\|^2 := (\det \Im T)^5 \left| \Delta_5(T) \right|^2$$

is independent of the choice of a symplectic basis of $H_1(C, \mathbf{Z})$. Hence $\|\Delta_5(C)\|$ is an invariant of C. Form Corollary 3, it follows the following:

Corollary 4 ([16]) *The Igusa cusp form Δ_5 is the average of Φ with respect to the* 10 *switches*, *i.e.*,

$$\prod_{\binom{ijk}{lmn}} \left\| \Phi\big(K(C)\big)/\iota_{\binom{ijk}{lmn}} \right\| = \left\| \Delta_5(C) \right\|^8.$$

7 Some Problems

Problem 1 For elliptic curves, two distinct models E_A and C_A yield distinct algebro-geometric expressions of $\|\eta\|$. For projective models of Enriques surfaces distinct from the $(2, 2, 2)$-complete intersection of $\mathbf{P}^5$, find the corresponding algebro-geometric expressions of $\|\Phi\|$.

Problem 2 On a generic Jacobian Kummer surface, there exists 31 conjugacy classes of free involutions ([25, 28]), which split into three families:

- 10 switches,
- 15 Hutchinson-Göpel involutions,
- 6 Hutchinson-Weber involutions.

Recall that, as the average of the Borcherds Φ-function by 10 switches, we get Igusa's Siegel modular form Δ_5. Determine the Siegel modular form constructed as the average of the Borcherds Φ-function by the 15 Hutchinson-Göpel involutions (resp. 6 Hutchinson-Weber involutions).

Problem 3 As mentioned in Sect. 4.2, there exists an analytic torsion invariant τ_M for $K3$ surfaces with involution [34], which is often expressed as the Petersson norm of the tensor product of an explicit Borcherds lift and Igusa's Siegel modular form [36]. After Theorem 8, it is an interesting problem to find an algebro-geometric expression of τ_M for general M.

Problem 4 (The inverse of the period mapping for Enriques surfaces) For elliptic curves, the inverse of the period mapping was constructed by Jacobi by using theta constants. We ask the same problem for the $(2, 2, 2)$-model of Enriques surfaces: For $1 \leq i < j \leq 3$ and $4 \leq k < l \leq 6$, find a system of automorphic forms

$$\alpha_{ij}^{(1)}(Z),\ \alpha_{kl}^{(2)}(Z),\ \beta_{ij}^{(1)}(Z),\ \beta_{kl}^{(2)}(Z),\ \gamma_{ij}^{(1)}(Z),\ \gamma_{kl}^{(2)}(Z)$$

on $\Omega_{\boldsymbol{\Lambda}}^+$ for (a finite index subgroup of) $O^+(\boldsymbol{\Lambda})$ such that

$$Y_Z := X_Z/\iota, \quad \iota(x) = (x_1, x_2, x_3, -x_4, -x_5, -x_6)$$

is the Enriques surface whose period is the given by $Z \in \Omega^+_\Lambda$. Here

$$X_Z = \left\{ [x] \in \mathbf{P}^5;\ \begin{array}{l} \sum_{1\leq i<j\leq 3} \alpha^{(1)}_{ij}(Z)x_i x_j + \sum_{4\leq k<l\leq 6} \alpha^{(2)}_{kl}(Z)x_k x_l = 0 \\ \sum_{1\leq i<j\leq 3} \beta^{(1)}_{ij}(Z)x_i x_j + \sum_{4\leq k<l\leq 6} \beta^{(2)}_{kl}(Z)x_k x_l = 0 \\ \sum_{1\leq i<j\leq 3} \gamma^{(1)}_{ij}(Z)x_i x_j + \sum_{4\leq k<l\leq 6} \gamma^{(2)}_{kl}(Z)x_k x_l = 0 \end{array} \right\}.$$

Kondō [18] and Freitag-Salvati-Manni [12] constructed certain (birational) projective embeddings of the moduli space of Enriques surfaces with some level structure. Are the system of automorphic forms appearing in their embeddings regarded as the set of coefficients of the defining equations of appropriately polarized Enriques surfaces?

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Sum Rule for the Eight-Vertex Model on Its Combinatorial Line

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Abstract We investigate the conjectured ground state eigenvector of the 8-vertex model inhomogeneous transfer matrix on its combinatorial line, i.e., at $\eta = \pi/3$, where it acquires a particularly simple form. We compute the partition function of the model on an infinite cylinder with certain restrictions on the inhomogeneities, and taking the homogeneous limit, we obtain an expression for the squared norm of the ground state of the XYZ spin chain as a solution of a differential recurrence relation.

1 Introduction

The purpose of this article is to investigate the *inhomogeneous eight-vertex model* on a particular one-dimensional family of the globally defined parameters of the model, namely, with the conventions of Baxter [2], when $\eta = \pi/3$. More precisely, we study a certain eigenvector (conjecturally, the ground state eigenvector in an appropriate range of parameters) of the transfer matrix of this model with periodic boundary conditions and an odd number of sites. Ultimately, the goal is to compare with some observations and conjectures [6, 28] made for the homogeneous eight-vertex model and the closely related XYZ spin chain, but the introduction of inhomogeneities (spectral parameters) turns out to be quite useful, as was previously found for the six-vertex model [12, 14, 29], a special case of the eight-vertex model.

In this section we briefly describe the model and some conjectured properties at $\eta = \pi/3$. The rest of this paper is devoted to showing how some of these properties arise from specializing formulae for the inhomogeneous model. The main object of study will be the "partition function" of the model on an infinite cylinder (equivalently, a quadratic functional of the ground state eigenvector), for which we derive an inhomogeneous sum rule (with a certain restriction on the inhomogeneities, which we call "half-specialization") and a detailed discussion of its homogeneous limit. Note that this paper is not meant to be fully mathematically rigorous; firstly, it is

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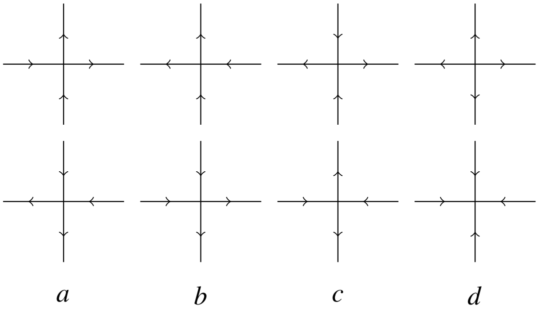
based on a conjecture (Conj. 1) which we hope to prove in future work [35]. Secondly, some calculations involving theta and elliptic functions are skipped; though they are in principle elementary, they can be quite tedious.

It should be noted that a special case of the eight-vertex model on its combinatorial line, namely the six-vertex model at $\Delta = -1/2$, is much better understood [1, 14, 26, 29, 32], and in this case many formulae of this work are already known and proved; we provide in Appendix A the connection to earlier work by taking the limit to the six-vertex point.

1.1 Inhomogeneous Eight-Vertex Transfer Matrix

The eight-vertex model is a two-dimensional statistical lattice model defined on the square lattice by the assignment of arrows to each edge of the lattice, according to eight possible local configurations around a vertex:



They are given Boltzmann weights denoted by a, b, c, d which are parameterized as follows:

$$
\begin{aligned}
a(x) &= \vartheta_4\big(2\eta, p^2\big)\vartheta_4\big(x, p^2\big)\vartheta_1\big(x+2\eta, p^2\big)\\
b(x) &= \vartheta_4\big(2\eta, p^2\big)\vartheta_1\big(x, p^2\big)\vartheta_4\big(x+2\eta, p^2\big)\\
c(x) &= \vartheta_1\big(2\eta, p^2\big)\vartheta_4\big(x, p^2\big)\vartheta_4\big(x+2\eta, p^2\big)\\
d(x) &= \vartheta_1\big(2\eta, p^2\big)\vartheta_1\big(x, p^2\big)\vartheta_1\big(x+2\eta, p^2\big)
\end{aligned} \tag{1}
$$

where x is the spectral parameter and $p = e^{i\pi\tau}$, $\operatorname{Im}\tau > 0$, is the elliptic nome. The weights have period 2π and pseudo-period $2\pi\tau$, i.e., they are multiplied by a *common* factor when x is replaced with $x + 2\pi\tau$.

Ordering the edge states as $(\uparrow, \downarrow)$ and $(\rightarrow, \leftarrow)$, these weights can be encoded into the R-matrix

$$
R(x) = \begin{pmatrix} a(x) & 0 & 0 & d(x) \\ 0 & b(x) & c(x) & 0 \\ 0 & c(x) & b(x) & 0 \\ d(x) & 0 & 0 & a(x) \end{pmatrix}
$$

We shall also need in what follows the $\check{R}$-matrix defined as

$$\check{R}(x) = \mathcal{P}R(x) = \begin{pmatrix} a(x) & 0 & 0 & d(x) \\ 0 & c(x) & b(x) & 0 \\ 0 & b(x) & c(x) & 0 \\ d(x) & 0 & 0 & a(x) \end{pmatrix}$$

where $\mathcal{P}$ permutes factors of the tensor product.

The Boltzmann weights satisfy the Yang–Baxter equation and unitarity equation; in terms of $\check{R}$, these are expressed as

$$\check{R}_{i,i+1}(x)\check{R}_{i+1,i+2}(x+y)\check{R}_{i,i+1}(y) = \check{R}_{i+1,i+2}(y)\check{R}_{i,i+1}(x+y)\check{R}_{i+1,i+2}(x) \quad (2)$$

and

$$\check{R}(x)\check{R}(-x) = r(x)r(-x)1 \quad (3)$$

where $r(x) = \vartheta_4(0; p^2)\vartheta_1(x - 2\eta; p^2)\vartheta_4(x - 2\eta; p^2)$.

We consider here the model in size L with periodic boundary conditions in the horizontal direction, i.e., with the geometry of a cylinder of width L. The state of the L vertical edges at same height on the cylinder are encoded by a sequence in $\{\uparrow, \downarrow\}^L$. The *transfer matrix* is a $2^L \times 2^L$ matrix, or equivalently an operator on $(\mathbb{C}^2)^{\otimes L}$ with its standard basis indexed by $\{\uparrow, \downarrow\}^L$, describing the transition from one row of vertical edges to the next; the fully inhomogeneous transfer matrix has the formal expression

$$T_L(u|x_1, \ldots, x_L) = \mathrm{Tr}_0\, R_{01}(x_1 - u)R_{02}(x_2 - u)\ldots R_{0L}(x_L - u)$$

where we use the following convention: the indices of operators R (and all other local operators) are the spaces on which they act in the tensor product $(\mathbb{C}^2)^{\otimes L}$. $u, x_1, \ldots, x_L$ are spectral parameters of the model. The system has rotational invariance in the sense that shifting cyclically sites in the tensor product *and* spectral parameters leaves T_L invariant. In what follows, all indices in $\{1, \ldots, L\}$ must be understood modulo L.

Finally we need Pauli matrices $\sigma^{x,y,z}$, which are local operators acting on one site; we give alternate names to two of them. The flip operator (σ^x Pauli matrix) is $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the spin operator (σ^z Pauli matrix) is $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Finally $\sigma^y = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$. Denote $F_* = \prod_{i=1}^{L} F_i$.

The transfer matrix T_L is invariant by reversal of all spins, i.e., $[T_L, F_*] = 0$.

1.2 Combinatorial Line

In all the rest of this paper, we assume that L is an odd number, $L = 2n + 1$, and that $\eta = \pi/3$. This second condition is what we call "combinatorial line", because of the occurrence of integer numbers in the ground state, as we shall see below. The value $\eta = \pi/3$ was first noticed to have special significance by Baxter [3]; the importance of odd L was emphasized by Stroganov [31]. More recently, Razumov and Stroganov [28] and Bazhanov and Mangazeev [4–6] studied the model with such conditions. It is also known [4, 17] that $\eta = \pi/3$ corresponds to a supersymmetric point for the XYZ spin chain.

Although the work [28] is mostly concerned with the homogeneous limit (see below), the following conjecture is made there (translated into our present conventions): the transfer matrix $T_L(u|x_1,\dots,x_L)$ possesses the eigenvalue

$$t_L(u|x_1,\dots,x_L) = \prod_{i=1}^{L}\big(a(x_i-u)+b(x_i-u)\big)$$

In fact, this eigenvalue is found to be doubly degenerate; in [5, 6, 28], this degeneracy is lifted by fixing the parity of the number of $\uparrow$ in the eigenvector. Here we find it more convenient to choose a different convention, which is to diagonalize simultaneously F_*.

Note the identities at $\eta = \pi/3$:

$$r(x) = \vartheta_4\big(0,p^2\big)\vartheta_1\big(x+\eta,p^2\big)\vartheta_4\big(x+\eta,p^2\big) = a(x)+b(x)$$

1.3 Homogeneous Limit

If we assume that all x_i are equal (homogeneous situation), then the transfer matrix T_L commutes with the *XYZ Hamiltonian*, which can be written as

$$H_L = -\frac{1}{2}\sum_{i=1}^{L}\big(J_4\sigma_i^x\sigma_{i+1}^x + J_3\sigma_i^y\sigma_{i+1}^y + J_2\sigma_i^z\sigma_{i+1}^z\big)$$

The numbering of the coupling constants will be explained later. The value $\eta = \pi/3$ implies that up to normalization, the three coupling constants can be expressed in terms of a single quantity, which we choose to be

$$\zeta = \left(\frac{\vartheta_1(\eta;p^2)}{\vartheta_4(\eta;p^2)}\right)^2$$

If we choose τ purely imaginary, then as p goes from 0 to 1, ζ goes from 0 to 1.

The coupling constants are given up to overall normalization by

$$J_2 = -\frac{1}{2} \qquad J_3 = \frac{1}{1+\zeta} \qquad J_4 = \frac{1}{1-\zeta}$$

The XXZ Hamiltonian (corresponding to the six-vertex transfer matrix), is the case $\zeta = 0$ (or $p = 0$). This case was already studied in detail, as mentioned in the introduction.

Another special case is $\zeta \to 1$ (or $p \to 1$), for which after rescaling the weights, $J_2 = J_3 = 0$, so the model becomes the Ising model, but with a $\sigma^x\sigma^x$ interaction. The ground state becomes of course trivial; some details are provided in Appendix B.

The simple eigenvalue of the eight-vertex transfer matrix translates into a simple eigenvalue of the Hamiltonian H_L, namely

$$E_L = -\frac{L}{2}(J_2 + J_3 + J_4)$$

It is conjectured to be the ground state eigenvalue of H_L.

Many remarkable observations were made on the corresponding eigenvector, Ψ_L in [6, 28]. Its entries can be chosen to be *polynomials* in ζ, and the form of some of these polynomials was conjectured. We shall not discuss these conjectures here. The values at $\zeta = 0$ (XXZ model) of these polynomials were calculated in [29].

In both [6, 28], the *squared norm* of Ψ_L was introduced:

$$|\Psi_L|^2 = \sum_{\alpha \in \{\uparrow,\downarrow\}^L} \Psi_{L;\alpha}^2 \tag{4}$$

where the normalization of the components is chosen so that they are coprime polynomials in ζ, and $\Psi_{L;\underbrace{\uparrow\dots\uparrow}_{n}\underbrace{\downarrow\dots\downarrow}_{n+1}}(\zeta = 0) = 1$. An expression for $|\Psi_L|^2$ was conjectured in [6] in terms of certain polynomials, themselves defined by differential recurrence relations which are special cases of certain Bäcklund transformations for Painlevé VI. Since the formulae are rather complicated, we shall not write them out here and derive our own (similar) formulae by specializing inhomogeneous expressions.

The main result of this paper is the factorization of this squared norm into four factors, as summarized in Sect. 4.1, which are all determined by differential bilinear recurrence relations which are given explicitly in Appendix D.

2 Properties of the Ground State Eigenvector

We consider once again the eigenvector equation in size $L = 2n + 1$

$$T_L(u|x_1, \dots, x_L)\Psi_L(x_1, \dots, x_L) = t_L(u|x_1, \dots, x_L)\Psi_L(x_1, \dots, x_L) \tag{5a}$$

$$F_*\Psi_L(x_1, \dots, x_L) = (-1)^n \Psi_L(x_1, \dots, x_L) \tag{5b}$$

for the inhomogeneous eight-vertex transfer matrix, where we recall that $t_L(u|x_1, \dots, x_L) = \prod_{i=1}^L r(x_i - u)$, $r(x) = a(x) + b(x) = \vartheta_4(0, p^2)\vartheta_1(x + \eta, p^2)\vartheta_4(x + \eta, p^2)$. The choice of eigenvalue of F_* will turn out convenient in what follows.

2.1 Pseudo-periodicity

Based on extensive study of the ground state entries by computer for small sizes $L = 3, 5, 7$, the following conjecture seems valid:

Conjecture 1 *The eigenvector equations* (5a), (5b) *possess a solution* $\Psi_L(x_1, \ldots, x_L)$ *whose entries are theta functions of degree* $L - 1 = 2n$ *and nome* p^2 *in each variable* x_i (*generically non zero and without common factor*); *i.e., they are holomorphic functions with pseudo-periodicity property*:

$$\Psi_L(\ldots, x_i + 2\pi\tau, \ldots) = p^{-4n} z_i^{2n} \prod_{j(\neq i)} z_j^{-1} \Psi_L(\ldots, x_i, \ldots) \tag{6a}$$

$$\Psi_L(\ldots, x_i + \pi, \ldots) = \prod_{j(\neq i)} \sigma_j \Psi_L(\ldots, x_i, \ldots) \tag{6b}$$

where the $\ldots$ *mean unspecified variables* x_1, x_2, *etc,* $p = e^{\mathrm{i}\pi\tau}$ *and* $z_i = e^{-2\mathrm{i}x_i}$, $i = 1, \ldots, L$.

(Note that the factor $\prod_{j(\neq i)} z_j^{-1}$ is to be expected since Ψ_L only depends on differences of spectral parameters. The factor p^{-4n} can be absorbed in a redefinition of Ψ_L, but is convenient. The factor $\prod_{j(\neq i)} \sigma_j$ is again expected from the properties of the R-matrix by shift of π; it could be absorbed in a simultaneous redefinition of the R-matrix and of Ψ_L.)

Similar properties have been observed and (in some cases) proved for models based on trigonometric or rational solutions of the Yang–Baxter equation at special points of their parameter space [7, 9–13, 29, 37, 38], except the entries are ordinary polynomials of prescribed degree (the main difficulty being to prove this degree). In particular, in the limit $\zeta \to 0$, Ψ_L reduces to the eigenvector of the inhomogeneous six-vertex transfer matrix whose existence and uniqueness was proved rigorously in [29] and references therein. Therefore, if such a solution of (5a), (5b) exists, it is necessarily unique (for generic p) up to normalization. In principle this normalization might contain a non-trivial function of the x_i, which is why we added to the conjecture the fact that the entries have no common factor. So there remains only an arbitrary constant in the normalization of Ψ_L, which will be fixed later.

2.2 Exchange Relation

As a direct application of the Yang–Baxter equation, we have the following intertwining relation:

$$T_L(u|\ldots, x_{i+1}, x_i, \ldots)\check{R}_{i,i+1}(x_{i+1} - x_i) = \check{R}_{i,i+1}(x_{i+1} - x_i) T_L(u|\ldots, x_i, x_{i+1}, \ldots) \tag{7}$$

(see Lemma 1 of [12] for the same formula in a similar setting, and its graphical proof).

Now apply $\Psi_L(x_1, \dots, x_L)$ to Eq. (7) and use the eigenvalue equation (5a):

$$T_L(u|x_1, \dots, x_{i+1}, x_i, \dots, x_L)\check{R}_{i,i+1}(x_{i+1} - x_i)\Psi_L(\dots, x_i, x_{i+1}, \dots)$$
$$= t_L(u|x_1, \dots, x_L)\check{R}_{i,i+1}(x_{i+1} - x_i)\Psi_L(\dots, x_i, x_{i+1}, \dots)$$

$t_L(u|x_1, \dots, x_L)$ being invariant by permutation of x_i, x_{i+1}, and F_* commuting with $\check{R}_{i,i+1}$, we conclude by the uniqueness of the solution of (5a), (5b) that

$$\check{R}_{i,i+1}(x_{i+1} - x_i)\Psi_L(\dots, x_i, x_{i+1}, \dots) = r_i(x_1, \dots, x_L)\Psi_L(\dots, x_{i+1}, x_i, \dots)$$

where r_i is some scalar function which is a ratio of theta functions, but cannot have a non-trivial denominator because it would be a common factor of the Ψ_α, which would contradict Conj. 1; so it is a theta function of degree 2 in x_i, x_{i+1} (and zero in all others, hence a constant) with given pseudo-periodicity property; by applying the identity twice and using unitarity equation (3), we find $r_i(x_i, x_{i+1})r_i(x_{i+1}, x_i) = r(x_i - x_{i+1})r(x_{i+1} - x_i)$. The only theta function which divides the right hand side and has the same pseudo-periodicity properties as $r_i(x_i, x_{i+1})$ is $r(x_{i+1} - x_i)$; so $r_i(x_i, x_{i+1}) = \pm r(x_{i+1} - x_i)$. The simplest way to fix the sign is to use the $\zeta \to 0$ limit where it is known [29] that the correct sign is $+$. By continuity in ζ, we have in the end $r_i(x_i, x_{i+1}) = r(x_{i+1} - x_i)$, so that

$$\check{R}_{i,i+1}(x_{i+1} - x_i)\Psi_L(\dots, x_i, x_{i+1}, \dots) = r(x_{i+1} - x_i)\Psi_L(\dots, x_{i+1}, x_i, \dots) \quad (8)$$

2.3 Spin Flip

Next, note that weights $a(x)$ and $b(x)$ (resp. $c(x)$ and $d(x)$) are exchanged by shift of x by $\pi\tau$. More precisely, we have the following identity:

$$R_{01}(x + \pi\tau) = -p^{-1} z F_1 R_{01}(x) F_1$$

where $z = e^{-2ix}$ and F_1 is the operator that flips the second spin (of course the same would be true with F_1 replaced with F_0, since the R matrix commutes with $F_0 F_1$).

Applying this to the transfer matrix, we find:

$$T_L(\dots, x_i + \pi\tau, \dots)F_i = -p^{-1} z_i F_i T_L(\dots, x_i, \dots) \quad (9)$$

where we recall that F_i flips spin i.

As in the previous section, apply $\Psi_L(x_1, \dots, x_L)$ to Eq. (9) and use the eigenvalue equation (5a):

$$T_L(\dots, x_i + \pi\tau, \dots)F_i\Psi_L(\dots, x_i, \dots) = -p^{-1} z_i t_L(u|\dots, x_i, \dots)F_i\Psi_L(\dots, x_i, \dots)$$

We have $t_L(u|\ldots,x_i+\pi\tau,\ldots) = -p^{-1}z_i t_L(u|\ldots,x_i,\ldots)$, as should be, and F_* and F_i commute, so we conclude as before that

$$F_i\Psi_L(\ldots,x_i,\ldots) = f_i(x_1,\ldots,x_L)\Psi_L(\ldots,x_i+\pi\tau,\ldots) \tag{10}$$

where $f_i(x_1,\ldots,x_L)$ is a scalar function with the following properties: it is a ratio of theta functions, but cannot have a non-trivial denominator because it would be a common factor of the Ψ_α, which would contradict Conj. 1; so it is a holomorphic function, with pseudo-periodicity properties determined by shifting one of the x_j by $\pi, \pi\tau$ in Eq. (10) and comparing with (6a), (6b); we find

$$\begin{aligned} f_i(\ldots,x_i+\pi,\ldots) &= f_i(\ldots,x_i,\ldots) \\ f_i(\ldots,x_j+\pi,\ldots) &= -f_i(\ldots,x_j,\ldots) \quad j\neq i \\ f_i(\ldots,x_i+2\pi\tau,\ldots) &= p^{4n} f_i(\ldots,x_i,\ldots) \\ f_i(\ldots,x_j+2\pi\tau,\ldots) &= p^{-2} f_i(\ldots,x_j,\ldots) \quad j\neq i \end{aligned}$$

This fixes it to be $f_i(x_1,\ldots,x_L) = ce^{2n\mathrm{i}x_i - \mathrm{i}\sum_{j(\neq i)} x_j}$. By rotational invariance, the constant c is independent of i. Iterating Eq. (10) results in $f_i(\ldots,x_i,\ldots)f_i(\ldots,x_i+\pi\tau,\ldots) = p^{4n}e^{4n\mathrm{i}x_i - 2\mathrm{i}\sum_{j(\neq i)} x_j}$, which imposes that $c=\pm p^n$. In order to fix the sign, we use the invariance by shift of all the spectral parameters and the fact that $F_*\Psi_L = (-1)^n\Psi_L$ with $F_* = \prod_{i=1}^L F_i$ to conclude that $c^L p^{-L(L-1)/2} = (-1)^n$ and therefore $c = (-p)^n$.

We finally obtain:

$$F_i\Psi_L(\ldots,x_i,\ldots) = (-p)^n e^{2n\mathrm{i}x_i - \mathrm{i}\sum_{j(\neq i)} x_j}\Psi_L(\ldots,x_i+\pi\tau,\ldots) \tag{11}$$

2.4 Wheel Condition and Recurrence Relations

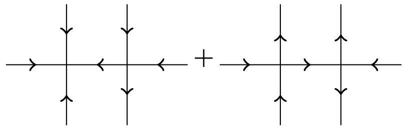
We are now interested in the situation where two successive spectral parameters have difference 2η. In this paragraph, we denote to simplify $T_L^+ = T_L(u|\ldots,x,x+2\eta,\ldots)$ and $T_L^- = T_L(u|\ldots,x+2\eta,x,\ldots)$ where the two specialized spectral parameters are at sites $i, i+1$. Applying the intertwining relation (7) with $x_i = x+2\eta$, $x_{i+1} = x$, we find:

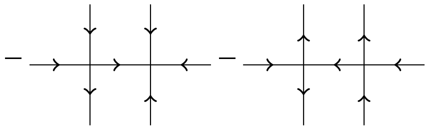
$$T_L^+\check{R}_{i,i+1}(-2\eta) = \check{R}_{i,i+1}(-2\eta)T_L^-$$

A direct calculation shows that $\check{R}(-2\eta) = 2\vartheta_4(2\eta,p^2)\vartheta_1(2\eta,p^2)\vartheta_4(0,p^2)P$ where P is the projector $P = \frac{1}{2}(1-\mathcal{P}) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Therefore the equality above says that T_L^+ leaves $\operatorname{Im} P_{i,i+1}$ stable (and that restricted to that subspace it is equal to the projection of T_L^-: $T_L^+|_{\operatorname{Im} P_{i,i+1}} = P_{i,i+1}T_L^-|_{\operatorname{Im} P_{i,i+1}}$).

We shall need to check this explicitly. Set $s = |{\uparrow\downarrow}\rangle - |{\downarrow\uparrow}\rangle = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ to be a generator of the image of projector P, and compute $R_{0,i}(x)R_{0,i+1}(x+2\eta)s_{i,i+1}$, e.g.,

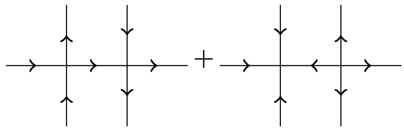
$\langle{\rightarrow}|R_{0,i}(x)R_{0,i+1}(x+2\eta)s_{i,i+1}|{\leftarrow}\rangle =$

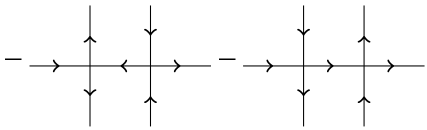




$$= \big(d(x)a(x+2\eta) - b(x)d(x+2\eta)\big)|{\downarrow\downarrow}\rangle$$
$$+ \big(a(x)c(x+2\eta) - c(x)b(x+2\eta)\big)|{\uparrow\uparrow}\rangle$$
$$= 0$$

$\langle{\rightarrow}|R_{0,i}(x)R_{0,i+1}(x+2\eta)s_{i,i+1}|{\rightarrow}\rangle =$





$$= \big(a(x)b(x+2\eta) - c(x)c(x+2\eta)\big)|{\uparrow\downarrow}\rangle$$
$$+ \big(d(x)d(x+2\eta) - b(x)a(x+2\eta)\big)|{\downarrow\uparrow}\rangle$$
$$= r(x+6\eta)r(x+2\eta)s_{i,i+1}$$

and similarly with all arrows reversed. Thus, $R_{0,i}(x)R_{0,i+1}(x+2\eta)s_{i,i+1} = r(x+6\eta)r(x+2\eta)s_{i,i+1} \otimes 1_0$, and therefore, after shift of $x \to x-u$, and use of $\eta = \pi/3$ to get rid of the 6η, we find

$$T_L^+|_{\mathrm{Im}\,P_{i,i+1}} = r(x-u)r(x+2\eta-u)T_{L-2} \tag{12}$$

where it is understood that T_{L-2} acts only on sites distinct from $i, i+1$.

Now apply Ψ_{L-2} (with parameters x_j except x_i, x_{i+1}) tensor $s_{i,i+1}$ and use eigenvector equation (5a):

$$T_L^+\Psi_{L-2}(\ldots) \otimes s_{i,i+1} = r(x-u)r(x+2\eta-u)t_{L-2}(u|\ldots)\Psi_{L-2}(\ldots) \otimes s_{i,i+1}$$

By definition, $t_L(u|\ldots,x,x+2\eta,\ldots) = r(x-u)r(x+2\eta-u)t_{L-2}(u|\ldots)$. Also, from Eq. (5b), $F_*\Psi_{L-2} \otimes s_{i,i+1} = ((-1)^{n-1}\Psi_{L-2}) \otimes (-s_{i,i+1}) = (-1)^n\Psi_{L-2} \otimes$

$s_{i,i+1}$. By uniqueness of the solution of (5a), (5b), we conclude that

$$\Psi_L(\ldots, \underset{i}{x}, \underset{i+1}{x+2\eta}, \ldots) = \psi_i(x;\ldots)\Psi_{L-2}(\ldots) \otimes s_{i,i+1} \tag{13}$$

where, by the same kind of argument as in previous sections, ψ_i is a theta function of its arguments of degree 1 in the x_j, $j \neq i, i+1$ and of degree $4n$ in x.

In order to fix the function ψ_i, we shall need the so-called wheel condition vanishing relation. Let us first consider a special case of it: suppose three successive spectral parameters x_i, x_{i+1}, x_{i+2} are of the form $x, x+2\eta, x+4\eta$. Then according to Eq. (13) applied at $(i, i+1)$ and $(i+1, i+2)$,

$$\begin{aligned}\mathcal{P}_{i,i+1}\Psi_L(\ldots, x, x+2\eta, x+4\eta, \ldots) &= \mathcal{P}_{i+1,i+2}\Psi_L(\ldots, x, x+2\eta, x+4\eta, \ldots)\\ &= -\Psi_L(\ldots, x, x+2\eta, x+4\eta, \ldots)\end{aligned}$$

But the action of the symmetric group $\mathcal{S}_3$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ does not possess the sign representation as a sub-representation; therefore

$$\Psi_L(\ldots, x, x+2\eta, x+4\eta, \ldots) = 0$$

Now assume all other parameters x_j, $j \neq i, i+1, i+2$, are generic; then according to Eq. (3), $\check{R}(x_j - x_k)$ $(j \neq i, i+1, i+2, k = i, i+1, i+2)$ is an invertible operator. Applying repeatedly the exchange relation (8) to the equality above, we conclude that

$$\Psi_L(\ldots, x, \ldots, x+2\eta, \ldots, x+4\eta, \ldots) = 0 \tag{14}$$

where the location of the three arguments is now arbitrary, as long as the cyclic order is respected. This is the general wheel condition (the equality is true for generic x_j, therefore for all x_j).

Finally, using pseudo-periodicity relations (6a), (6b), as well as flip relation (11), we conclude that the wheel condition vanishing relation (14) is valid provided the triplet of spectral parameters forms a wheel $x, x+2\eta, x+4\eta$ modulo $\pi, \pi\tau$ (not just $2\pi\tau$! a crucial technical point which will be used repeatedly below).

We can now come back to our recurrence relation (13). On the left hand side, we notice that as soon as one of the x_j, $j \neq i, i+1$, is equal to $x - 2\eta$ (mod $\pi, \pi\tau$), a wheel is formed and Ψ_L vanishes. Therefore $\psi_i(x;\ldots)$ contains factors $\prod_{j(\neq i,i+1)} \vartheta_1(x - 2\eta - x_j; p^2)\vartheta_4(x - 2\eta - x_j; p^2)$; moreover these exhaust its degree, and noting that these factors can also be written up to a multiplicative constant as $\vartheta_1(x - 2\eta - x_j; p)$, we can rewrite Eq. (13)

$$\Psi_L(\ldots, \underset{i}{x}, \underset{i+1}{x+2\eta}, \ldots) = cst \prod_{j(\neq i,i+1)} \vartheta_1(x - 2\eta - x_j; p)\Psi_{L-2}(\ldots) \otimes s_{i,i+1} \tag{15}$$

More explicitly, it means that

$$\Psi_{L;\alpha_1,\dots,\alpha_L}|_{x_{i+1}=x_i+2\eta} = \begin{cases} 0 & \alpha_i = \alpha_{i+1} \\ cst\alpha_i \prod_{j(\neq i,i+1)} \vartheta_1(x_i - 2\eta - x_j; p)\Psi_{L-2;\alpha_1,\dots,\alpha_{i-1},\alpha_{i+2},\dots,\alpha_L} & \alpha_i \neq \alpha_{i+1} \end{cases}$$

The constant remains undetermined at this stage, since we have not fixed the normalization of Ψ_L yet.

A similar recurrence relation can be written for $x_{i+1} = x_i + 2\eta + \pi\tau$ (the non-zero result occurring when $\alpha_i = \alpha_{i+1}$), but we shall not need it.

3 Partition Function

In the rest of this paper, we denote $\vartheta(x) := \vartheta_1(x; p)$ and $\vartheta_k(x) := \vartheta_k(x; p)$, $k = 2, 3, 4$. Since the contents of this section are not expected to generalize outside $\eta = \pi/3$, we shall use $3\eta = 0 \pmod{\pi}$ to replace 2η with $-\eta$ whenever possible.

3.1 Definition

We now introduce a quantity that naturally generalizes the squared norm of the XYZ ground state (Eq. (4)) to the inhomogeneous case:

$$Z_L(x_1, \dots, x_L) = \big\langle \Psi_L(-x_1, \dots, -x_L) | \Psi_L(x_1, \dots, x_L) \big\rangle$$

where we have used the (real) scalar product: $\langle \Phi | \Phi' \rangle = \sum_{\alpha \in \{\uparrow,\downarrow\}^L} \Phi_\alpha \Phi'_\alpha$.

$Z_L(x_1, \dots, x_L)$ has the following interpretation: it is the "partition function" of the inhomogeneous eight-vertex model on an infinite cylinder. Indeed, assuming that we are in a regime of parameters where Ψ_L is associated to the largest eigenvalue of the transfer matrix, $\Psi_L(x_1, \dots, x_L)$ corresponds to the partition function on a half-infinite cylinder (pointing upwards) with given arrows at the boundary at the bottom. A vertical mirror symmetry of the eight vertices correspond in the weights (1) to $x \to -2\eta - x$ and a change of sign of the weights a and b, the latter being irrelevant with periodic boundary conditions. So the partition function of the other half-infinite cylinder (pointing downwards) is $\Psi_L(-x_1, \dots, -x_L)$ (Ψ_L only depends on the differences of its arguments so the -2η term is irrelevant). We mean partition function in the following sense: a one-point correlation function will be expressed as $\langle \mathcal{O} \rangle = \frac{1}{Z} \sum_{\alpha,\beta \in \{\uparrow,\downarrow\}^L} \Psi_{L;\alpha}(-x_1, \dots, -x_L) \mathcal{O}_{\alpha,\beta} \Psi_{L;\beta}(x_1, \dots, x_L)$.

3.2 Pseudo-periodicity

According to its definition and (6b), Z_L is invariant by $x_i \to x_i + \pi$ for any given i. Furthermore,

$$\begin{aligned}
\mathrm{Z}_L(\dots, x_i + \pi\tau, \dots) &= \sum_{\alpha\in\{\uparrow,\downarrow\}^L} \big\langle \Psi_L(\dots, -x_i - \pi\tau, \dots) | \Psi_L(\dots, x_i + \pi\tau, \dots)\big\rangle \\
&= (-p)^n e^{2n\mathrm{i}(-x_i - \pi\tau) + \mathrm{i}\sum_{j(\neq i)} x_j} \big\langle \Psi_L(\dots, -x_i, \dots) | F_i \\
&\quad (-p)^{-n} e^{-2n\mathrm{i}x_i + \mathrm{i}\sum_{j(\neq i)} x_j} F_i | \Psi_L(\dots, x_i, \dots)\big\rangle \quad \text{by Eq. (11)} \\
&= p^{-2n} z_i^{2n} \prod_{j(\neq i)} z_j^{-1} \mathrm{Z}_L(\dots, x_i, \dots)
\end{aligned}$$

where it is reminded that $z_j = e^{-2\mathrm{i}x_j}$.

We reach the conclusion that Z_L is a theta function of degree $2n$ and nome p (as opposed to p^2 for Ψ_L) in each variable x_i.

3.3 Symmetry

Given $i = 1, \dots, L-1$, we can use the exchange relation (8) and unitarity relation (3) to write

$$\begin{aligned}
&\mathrm{Z}_L(x_1, \dots, x_{i+1}, x_i, \dots, x_L) \\
&\quad = \big\langle \Psi_L(-x_1, \dots, -x_{i+1}, -x_i, \dots, -x_L) \big| \Psi_L(x_1, \dots, x_{i+1}, x_i, \dots, x_L)\big\rangle \\
&\quad = \left\langle \frac{\check{R}_{i,i+1}(x_i - x_{i+1})}{r(x_i - x_{i+1})} \Psi_L(-x_1, \dots, -x_i, -x_{i+1}, \dots, -x_L) \right| \\
&\qquad \left| \frac{\check{R}_{i,i+1}(x_{i+1} - x_i)}{r(x_{i+1} - x_i)} \Psi_L(x_1, \dots, x_i, x_{i+1}, \dots, x_L) \right\rangle \\
&\quad = \big\langle \Psi_L(-x_1, \dots, -x_i, -x_{i+1}, \dots, -x_L) \big| \frac{\check{R}_{i,i+1}(x_i - x_{i+1})}{r(x_i - x_{i+1})} \\
&\qquad \frac{\check{R}_{i,i+1}(x_{i+1} - x_i)}{r(x_{i+1} - x_i)} \big| \Psi_L(x_1, \dots, x_i, x_{i+1}, \dots, x_L)\big\rangle \\
&\quad = \mathrm{Z}_L(x_1, \dots, x_L)
\end{aligned}$$

where in the intermediate step we also used the fact that the $\check{R}$ matrix is self-adjoint. We conclude from this calculation that Z_L is a *symmetric* function of its arguments.

3.4 Recurrence Relation

The recurrence relation (15) for Ψ_L implies one for Z_L:

$$
\begin{aligned}
& Z_L(\dots, x, x+2\eta) \\
&\quad = \big\langle \Psi_L(\dots, x, x+2\eta) | \Psi_L(\dots, -x, -x-2\eta) \big\rangle \\
&\quad \propto \prod_{i=1}^{L-2} \vartheta(x-2\eta-x_i) \big\langle \Psi_{L-2}(\dots) \otimes s_{L-1,L} | \Psi_L(\dots, -x, -x-2\eta) \big\rangle \\
&\quad \propto \prod_{i=1}^{L-2} \vartheta(x-2\eta-x_i) \big\langle \Psi_{L-2}(\dots) \otimes s_{L-1,L} | P_{L-1,L} | \Psi_L(\dots, -x, -x-2\eta) \big\rangle \\
&\quad \propto \prod_{i=1}^{L-2} \vartheta(x-2\eta-x_i) \big\langle \Psi_{L-2}(\dots) \otimes s_{L-1,L} | \Psi_L(\dots, -x-2\eta, -x) \big\rangle \\
&\qquad \text{by Eq. (8) with } x_{L-1} = -x, x_L = -x-2\eta \\
&\quad \propto \prod_{i=1}^{L-2} \vartheta(x-2\eta-x_i) \vartheta\big((-x-2\eta) - 2\eta - (-x_i)\big) \\
&\qquad\qquad \big\langle \Psi_{L-2}(x_1, \dots, x_{L-2}) \otimes s_{L-1,L} | \Psi_{L-2}(-x_1, \dots, -x_{L-2}) \otimes s_{L-1,L} \big\rangle \\
&\quad \propto \prod_{i=1}^{L-2} \vartheta^2(x-2\eta-x_i) Z_{L-2}(\dots)
\end{aligned}
$$

where $\propto$ means equal up to a multiplicative constant. At this stage, we fix the normalization of Ψ_L in such a way that this constant disappears in the recurrence formula for Z_L, which becomes:

$$
Z_L(\dots, x, x+\eta) = \prod_{i=1}^{L-2} \vartheta^2(x-\eta-x_i) Z_{L-2}(\dots) \tag{16}
$$

where we have also shifted $x \to x+\eta$ and used $3\eta = 0 \pmod{\pi}$.

Combined with the symmetry in its arguments, the recurrence relation (16) satisfied by Z_L means that we can express its specialization at $x_1 = x_2 \pm \eta, \dots, x_L \pm \eta$ in terms of Z_{L-2}. So we possess $4n$ values of Z_L as a function of x_1; since it is a theta function of degree $2n$, these relations are more than enough to determine Z_L inductively (say, by Lagrange interpolation).

3.5 Half-specialization

At the moment, we do not know how to solve in a closed form the recurrence relation above. However, note that we have twice as many recurrence relations as needed to determine Z_L. This suggests to “half-specialize” Z_L in such a way that the number of recurrence relations now matches the degree.

Explicitly, assume $x_{i+n} = -x_i$, $i = 1, \ldots, n$, and $x_L = 0$. After such a specialization, Z_L is an even function of $x_1, \ldots, x_n$, and it has a *double zero* at $x_i = \pm\eta$, $i = 1, \ldots, n$. Let us check the latter statement carefully. Since Z_L is a symmetric function of its arguments, let us assume that we order them as $x, 0, -x \ldots$ (where x is one of the x_i) and that we send x to -2η (which is equal to η modulo π). Then it is clear that $\Psi_L(x, 0, -x, \ldots)$ forms a wheel and therefore $Z_L(x, 0, -x, \ldots)$ vanishes. However, $\Psi_L(-x, 0, x, \ldots)$ does not vanish, so that to show that the zero is double, we need to go further. Write

$$\begin{aligned}\frac{\partial}{\partial x}\mathrm{Z}_L(x, 0, -x, \ldots)|_{x=-2\eta} &= \left\langle \frac{\partial}{\partial x}\Psi_L(x, 0, 2\eta, \ldots)\Big|_{x=-2\eta} - \frac{\partial}{\partial x}\Psi_L(-2\eta, 0, x, \ldots)\Big|_{x=2\eta}\right. \\ &\quad \left.\Big| \Psi_L(2\eta, 0, -2\eta, \ldots)\right\rangle \end{aligned} \tag{17}$$

Now apply recurrence relation (15) to $\Psi_L(x, 0, 2\eta, \ldots)$; we find that it is proportional to some vector at sites $j \neq 2, 3$ tensor $s_{2,3}$, and therefore the same its true of its derivative w.r.t. x (one can be more explicit using $x = -2\eta$ but we shall not need it). Similarly, $\Psi_L(-2\eta, 0, x, \ldots)$ and its derivative w.r.t. x are equal to $s_{1,2}$ tensor some vector at other sites.

On the other hand, applying the exchange relation (8) to $\Psi_L(2\eta, 0, -2\eta)$ at $i = 1$ implies that $P_{1,2}\Psi_L(2\eta, 0, -2\eta, \ldots) \propto \Psi_L(0, 2\eta, -2\eta, \ldots) = 0$ since a wheel is formed. Similarly, the exchange relation at $i = 2$ implies that $P_{2,3}\Psi_L(2\eta, 0, -2\eta, \ldots) = 0$.

We conclude that the expression (17) is zero by inserting $P_{2,3}$ (resp. $P_{1,2}$) in the first (resp. second) term. Therefore, taking into account evenness, we can write

$$\mathrm{Z}_L(x_1, \ldots, x_n, -x_1, \ldots, -x_n, 0) = \prod_{i=1}^{n} \vartheta^2(x_i - \eta)\vartheta^2(x_i + \eta)\mathrm{X}_n(x_1, \ldots, x_n) \tag{18}$$

where $\mathrm{X}_n(x_1, \ldots, x_n)$ has the following properties, as a direct consequence of the corresponding properties for Z_L:

- X_n is a symmetric function of its arguments, and an even theta function of degree $2(2n-1)$ in each.
- It satisfies the recurrence relations:

$$X_n(\ldots, x, x+\eta) = \varphi^2(x)\varphi^2(x+\eta) \times \prod_{i=1}^{n-2} \vartheta^4(x-\eta-x_i)\vartheta^4(x-\eta+x_i)X_{n-2}(\ldots) \quad (19)$$

$$X_n(\ldots, \beta_k) = \frac{\kappa_k}{\vartheta_k^2(\eta)} \prod_{i=1}^{n-1} \vartheta_k^4(x_i)X_{n-1}(\ldots) \quad k = 2, 3, 4 \quad (20)$$

where $\varphi(x) = \frac{\vartheta(2x)}{\vartheta(x)} = \frac{1}{\kappa}\vartheta_2(x)\vartheta_3(x)\vartheta_4(x)$, $\kappa = \frac{\vartheta_2\vartheta_3\vartheta_4}{2} = \vartheta_2(\eta)\vartheta_3(\eta)\vartheta_4(\eta)$ and $\kappa_2 = 1$, $\kappa_3 = \kappa_4 = -\nu^{2n-1}$ ($\nu = e^{-2\pi i/3}/\sqrt{p}$) are pseudo-periodicity constants where $\beta_{2,3,4}$ are representatives of the three solutions of $2\beta_k + \eta = 0$ (mod $\pi, \pi\tau$) excluding η, namely, $\beta_2 = \pi/2 + \eta$, $\beta_3 = \pi/2 + \pi\tau/2 + \eta$, $\beta_4 = \pi\tau/2 + \eta$.

We now have at our disposal the specializations $x_1 = \pm x_i \pm \eta, \pm\beta_k, i = 2, \ldots, n$, $k = 2, 3, 4$, that is $4(n-1) + 6 = 2(2n+1)$. An even theta function of degree $2(2n-1)$ being determined by $2 \times 2n$ values, we have enough recurrence relations to determine X_n.

3.6 Solution as Pfaffians

We first introduce the function:

$$A_2(x, y) = -\nu^2 \frac{\vartheta_2(\eta)}{\vartheta_2(0)} \big(\vartheta_3(x+\eta)\vartheta_3(x-\eta)\vartheta_4^2(y) + \vartheta_4(x+\eta)\vartheta_4(x-\eta)\vartheta_3^2(y)\big)$$

which has the following properties:

- It is symmetric function of x, y, and is an even theta function of degree 2 in each.
- It satisfies the following recurrence relations:

$$A_2(x, x+\eta) = -\nu^2\vartheta_3(x)\vartheta_3(x+\eta)\vartheta_4(x)\vartheta_4(x+\eta) \quad (21)$$

$$A_2(x, \beta_k) = \nu^3\vartheta_2^2(\eta)\vartheta_k^2(x) \quad k = 3, 4 \quad (22)$$

Next we claim the following: define

$$A_n(x_1, \ldots, x_n) = \prod_{1 \le i < j \le n} \frac{h(x_i, x_j)}{\vartheta(x_i - x_j)\vartheta(x_i + x_j)} \operatorname{Pf} M_n \quad (23)$$

where

$$h(x, y) = \vartheta(\eta + x - y)\vartheta(\eta + x + y)\vartheta(\eta - x - y)\vartheta(\eta - x + y)$$

and M_n is a skew-symmetric $2m \times 2m$ matrix, $m = \lceil n/2 \rceil$, given by

$$(\mathrm{M}_n)_{ij} = \begin{cases} \mathrm{f}(x_i, x_j) & n \text{ even or } i, j < 2m \\ -1 & n \text{ odd}, i = 2m, j < 2m \\ 1 & n \text{ odd}, j = 2m, i < 2m \\ 0 & n \text{ odd}, i = j = 2m \end{cases} \tag{24}$$

and

$$\mathrm{f}(x, y) = \frac{\vartheta(x-y)\vartheta(x+y)\mathrm{A}_2(x, y)}{\mathrm{h}(x, y)} \tag{25}$$

Also define

$$\mathrm{B}_n(x_1, \ldots, x_n) = \mathrm{A}_{n+1}(x_1, \ldots, x_n, \beta_2) \tag{26}$$

Then:

- A_n (resp. B_n) is a symmetric function of its arguments, and an even theta function of degree $2(n-1)$ (resp. $2n$) in each.
- They satisfy the recurrence relations:

$$\mathrm{A}_n(\ldots, x, x+\eta) = -\nu^2 \vartheta_3(x)\vartheta_3(x+\eta)\vartheta_4(x)\vartheta_4(x+\eta) \times \prod_{i=1}^{n-2} \vartheta^2(x-\eta-x_i)\vartheta^2(x-\eta+x_i)\mathrm{A}_{n-2}(\ldots) \tag{27}$$

$$\mathrm{B}_n(\ldots, x, x+\eta) = -\nu^2 \vartheta_2^2(x)\vartheta_2^2(x+2\eta)\vartheta_3(x)\vartheta_3(x+\eta)\vartheta_4(x)\vartheta_4(x+\eta) \times \prod_{i=1}^{n-2} \vartheta^2(x-\eta-x_i)\vartheta^2(x-\eta+x_i)\mathrm{B}_{n-2}(\ldots) \tag{28}$$

$$\mathrm{A}_n(\ldots, \beta_2) = \mathrm{B}_{n-1}(\ldots) \tag{29}$$

$$\mathrm{B}_n(\ldots, \beta_2) = -\nu^2 \vartheta_3^2(\eta)\vartheta_4^2(\eta) \prod_{i=1}^{n-1} \vartheta_2^4(x_i)\mathrm{A}_{n-1}(\ldots) \tag{30}$$

$$\mathrm{A}_n(\ldots, \beta_k) = \nu^n \vartheta_2(\eta)\big(\nu\vartheta_2(\eta)\big)^{(-1)^n} \times \prod_{i=1}^{n-1} \vartheta_k^2(x_i)\mathrm{A}_{n-1}(\ldots) \quad k = 3, 4 \tag{31}$$

$$\mathrm{B}_n(\ldots, \beta_k) = \nu^{n+1} \vartheta_2(\eta)\big(\nu\vartheta_2(\eta)\big)^{-(-1)^n} \frac{\vartheta_3^2(\eta)\vartheta_4^2(\eta)}{\vartheta_k^2(\eta)} \times \prod_{i=1}^{n-1} \vartheta_k^2(x_i)\mathrm{B}_{n-1}(\ldots) \quad k = 3, 4 \tag{32}$$

where conventionally $\mathrm{A}_0 = \mathrm{B}_0 = 1$.

Let us show for example (27) for $n=2m$ even. Assume that $x_{2m-1}=x$ approaches $x_{2m}+\eta=x'$. Then the matrix element $(\mathrm{M}_n)_{2m-1,2m}$ develops a pole: $(\mathrm{M}_n)_{2m-1,2m}\propto 1/(x-x')$; and the other entries $(\mathrm{M}_n)_{ij}$ $(i<j)$ remaining finite, the only relevant contributions to the Pfaffian are those pairing $2m-1$ and $2m$, so we immediately have

$$\mathrm{A}_n(\ldots,x,x+\eta)=\prod_{i=1}^{n-2}\frac{\mathrm{h}(x_i,x)\mathrm{h}(x_i,x+\eta)}{\vartheta(x_i-x)\vartheta(x_i-x-\eta)\vartheta(x_i+x)\vartheta(x_i+x+\eta)}\\ \times\mathrm{A}_2(x,x+\eta)\mathrm{A}_{n-2}(\ldots)$$

where we have cancelled all factors in common to A_n and A_{n-2}.

Now the remarkable phenomenon (using in a crucial way $\eta=\pi/3$) is that there are compensations in the product, which simplifies to $\prod_{i=1}^{n-2}\vartheta^2(x-\eta-x_i)\vartheta^2(x-\eta+x_i)$. Finally, we use Eq. (21) for A_2 to reproduce the remaining prefactors on the r.h.s. of Eq. (27).

The other equations follow from similar reasonings.

Finally, we find that $\mathrm{A}_n\mathrm{B}_n$ satisfies all the recurrence relations of X_n, or more precisely,

$$\mathrm{X}_n(x_1,\ldots,x_n)=\left(-\nu^2\kappa^2\right)^{-n}\mathrm{A}_n(x_1,\ldots,x_n)\mathrm{B}_n(x_1,\ldots,x_n)$$

3.7 *Further Factorization as Determinants*

Consider the following elliptic version of Tsuchiya's determinant [23, 34]: (see a similar determinant in [18])

$$\mathrm{H}_{2m}(x_1,\ldots,x_m;x_{m+1},\ldots,x_{2m})\\ =\frac{\prod_{i=1}^{m}\prod_{j=m+1}^{2m}\mathrm{h}(x_i,x_j)}{\prod_{\substack{1\le i<j\le m\\ \text{or}\\ m+1\le i<j\le 2m}}\vartheta(x_i-x_j)\vartheta(x_i+x_j)}\det_{\substack{1\le i\le m\\ m+1\le j\le 2m}}\frac{1}{\mathrm{h}(x_i,x_j)}\tag{33}$$

Conventionally, $\mathrm{H}_0=1$. Note that $\mathrm{H}_2=1$ as well.

The expression of H_{2m} has the disadvantage that it is only (apparently) symmetric in the variables $\{x_1,\ldots,x_m\}$ and $\{x_{m+1},\ldots,x_{2m}\}$; in fact we show in Appendix C that thanks to $\eta=\pi/3$, it is indeed symmetric in all variables. In terms of each, it is an even theta function of degree $2(m-1)$.

It is not hard to see that H_{2m} satisfies the following recurrence relation:

$$\mathrm{H}_{2m}(\ldots,x;\ldots,x+\eta)=\prod_{\substack{i=1,\ldots,m-1,\\ m+1,\ldots,2m}}\vartheta(x-\eta-x_i)\vartheta(x-\eta+x_i)\mathrm{H}_{2m-2}(\ldots;\ldots)\tag{34}$$

and the same if one exchanges x and $x+\eta$. These are the usual recurrence relations satisfied by such determinants, as in the classical case of the Izergin–Korepin determinant [19–21], and similarly to the Pfaffians of Sect. 3.6; the reasoning to derive Eq. (34) is identical—a pole develops in one of the entries of the determinant, reducing it to a determinant one size smaller.

Now consider the function

$$\mathrm{A}'_{2m}(x_1,\dots,x_{2m})$$
$$= \mathrm{H}_{2m}(x_1,\dots,x_m;x_{m+1},\dots,x_{2m})\mathrm{H}_{2m+2}(x_1,\dots,x_m,\beta_3;x_{m+1},\dots,x_{2m},\beta_4)$$

A'_{2m} is an even theta function of its arguments, of degree $2(2m-1)$, and using Eq. (34), it satisfies the same recurrence relation (27) as A_{2m}, for say $x_1=\pm x_j\pm\eta$, $j=m+1,\dots,2m$. Thus the function is known at $2\times 2m$ values of x_1, which determines it uniquely. Combined with $\mathrm{A}'_2=\mathrm{A}_2=1$, we conclude by induction that $\mathrm{A}'_{2m}=\mathrm{A}_{2m}$.

Similar arguments can be made for A_{2m-1} and B_n. Together, we find

$$\mathrm{A}_{2m}(x_1,\dots,x_{2m}) = \mathrm{H}_{2m}(x_1,\dots,x_{2m})\mathrm{H}_{2m+2}(x_1,\dots,x_{2m},\beta_3,\beta_4) \tag{35}$$

$$\mathrm{A}_{2m-1}(x_1,\dots,x_{2m-1}) = \mathrm{H}_{2m}(x_1,\dots,x_{2m-1},\beta_3)\mathrm{H}_{2m}(x_1,\dots,x_{2m-1},\beta_4) \tag{36}$$

$$\mathrm{B}_{2m}(x_1,\dots,x_{2m}) = \mathrm{H}_{2m+2}(x_1,\dots,x_{2m},\beta_2,\beta_3) \times \mathrm{H}_{2m+2}(x_1,\dots,x_{2m},\beta_2,\beta_4) \tag{37}$$

$$\mathrm{B}_{2m-1}(x_1,\dots,x_{2m-1}) = \mathrm{H}_{2m}(x_1,\dots,x_{2m-1},\beta_2) \times \mathrm{H}_{2m+2}(x_1,\dots,x_{2m-1},\beta_2,\beta_3,\beta_4) \tag{38}$$

These are the only 8 possible specializations at $\beta_{2,3,4}$, corresponding to subsets of $\{\beta_2,\beta_3,\beta_4\}$, since applying any such specialization twice amounts to the shift $n\to n-2$.

3.8 Alternative Determinant Formula

Here we follow the same general method as in [33] (see also Appendix B of [12]). Define

$$\mathrm{g}(x,y) = \frac{\vartheta(2x)\vartheta(2y)}{\mathrm{h}(x,y)}$$

$\mathrm{g}(x,y)$ is an odd elliptic function of x and y. One further observes that for all x, y,

$$\mathrm{g}(x,y)+\mathrm{g}(x+\eta,y)+\mathrm{g}(x+2\eta,y)=0$$

and similarly for y. Therefore, $\det_{i,j}\mathrm{g}(x_i,x_j)$, as a function of any of its arguments, satisfies the same three-term relation. Now define

$$\begin{aligned}
&\mathrm{S}_{2m}(x_1,\ldots,x_{2m})\\
&\quad = \prod_{1\le i<j\le 2m} \vartheta(x_i - x_j) \prod_{1\le i\le j\le 2m} \vartheta(x_i + x_j)\mathrm{H}_{2m}(x_1,\ldots,x_{2m})\\
&\quad = \prod_{\substack{1\le i<j\le m\\ \text{or}\\ m+1\le i<j\le 2m}} \prod_{k=0}^{2} \vartheta(x_i - x_j + k\eta)\vartheta(x_i + x_j + k\eta) \det_{\substack{1\le i\le m\\ m+1\le j\le 2m}} \mathrm{g}(x_i, x_j)
\end{aligned}$$

Since the prefactor is invariant by $x_i \to x_i + \eta$ for any i, we have the same three-term relation for S_{2m}. In summary:

- S_{2m} is a skew-symmetric function of its arguments x_i, and an odd theta function of degree $6m$ in each.
- It satisfies

$$\mathrm{S}_{2m}(\ldots,x,\ldots) + \mathrm{S}_{2m}(\ldots,x+\eta,\ldots) + \mathrm{S}_{2m}(\ldots,x+2\eta,\ldots) = 0$$

The space of odd theta functions of degree $6m$ is of dimension $3m$, a possible basis being

$$s_k(x) = e^{2\mathrm{i}kx}\vartheta_3\big(k\pi\tau + 6mx, p^{6m}\big) - e^{-2\mathrm{i}kx}\vartheta_3\big(k\pi\tau - 6mx, p^{6m}\big) \quad 0 \le k \le 3m-1$$

$s_k(x)$ satisfies the relation $s_k(x) + s_k(x+\eta) + s_k(x+2\eta) = 0$ iff $k \ne 0 \pmod 3$. The sequence $(1, 2, 4, 5, \ldots, 3m-2, 3m-1) = (k_1, \ldots, k_{2m})$ is of cardinality $2m$, which is the number of variables of S_{2m}, so we conclude that S_{2m} is proportional to the “Slater determinant”

$$\mathrm{S}_{2m}(x_1,\ldots,x_{2m}) \propto \det_{i,j=1,\ldots,2m} s_{k_j}(x_i) \tag{39}$$

We shall not need the proportionality constant, only that it is nonzero (for generic p).

3.9 *Uniformization*

Although the formulae above are simple to derive, they are a bit too cumbersome to be used, especially in the homogeneous limit. Since all functions we consider are theta functions of definite parity, there is a rational uniformization, and we use from now on the following parameterization:

$$w(x) = \big(1-\zeta^2\big)^{-1/3} \frac{\vartheta^2(x)}{\vartheta(x-\eta)\vartheta(x+\eta)}$$

In terms of the original Boltzmann weights (1), we have $(1-\zeta^2)w(x) = \frac{(a(x-\eta)+b(x-\eta))^2}{a(x-\eta)b(x-\eta)}$.

Note the special values

$$w(\beta_2) = -\frac{1}{2} = J_2 \qquad w(\beta_3) = \frac{1}{1+\zeta} = J_3 \qquad w(\beta_4) = \frac{1}{1-\zeta} = J_4$$

which explains the labelling we have chosen for the coupling constants $J_{2,3,4}$.

This parameterization has the advantage that the wheel condition becomes simple to express: three spectral parameters form a "wheel" $\pm x, \pm(x+\eta), \pm(x+2\eta)$ iff the corresponding variables w, w', w'' satisfy

$$\begin{cases} w + w' + w'' = \dfrac{3+\zeta^2}{1-\zeta^2} \\ ww'w'' = \dfrac{1}{1-\zeta^2} \end{cases} \tag{40}$$

and therefore, two parameters form a "2-string" $\pm x, \pm(x+\eta)$ iff the corresponding variables w, w' satisfy $h(w, w') = 0$, where

$$h(w, w') = 1 - (3+\zeta^2)ww' + (1-\zeta^2)ww'(w+w') \tag{41}$$

This formula allows to rewrite the recurrence formulae in this new parameterization, but due to the fact that it is quadratic in w and w', the result is somewhat cumbersome and we shall not write it explicitly.

We also redefine the functions by dividing them by a "reference" even theta function of degree 2 to the appropriate power, here $\vartheta(x-\eta)\vartheta(x+\eta)$, and absorbing some constants in the normalization. That is, we define

$$A_n\big(w(x_1), \ldots, w(x_n)\big) = a_n \frac{\mathrm{A}_n(x_1, \ldots, x_n)}{\prod_{i=1}^n (\vartheta(x_i-\eta)\vartheta(x_i+\eta))^{n-1}}$$

$$B_n\big(w(x_1), \ldots, w(x_n)\big) = b_n \frac{\mathrm{B}_n(x_1, \ldots, x_n)}{\prod_{i=1}^n (\vartheta(x_i-\eta)\vartheta(x_i+\eta))^{n-1}}$$

where a_n and b_n are constants which are implicitly defined by the expressions below, and whose explicit expression we shall not need.

In particular,

$$A_2(w, w') = ww' - (w+w') + \frac{1+\zeta^2}{1-\zeta^2}$$

and if we define

$$f(w, w') = \frac{(w-w')A_2(w, w')}{h(w, w')}$$

which is such that $f(w(x), w(y)) = -\zeta q/\sqrt{p}\mathrm{f}(x, y)$, then we have:

$$A_n(w_1, \ldots, w_n) = \prod_{1 \le i < j \le n} \frac{h(w_i, w_j)}{w_i - w_j} \operatorname{Pf} M_n \tag{42}$$

where M_n is identical to M_n, except entries $\mathrm{f}(x_i, x_j)$ are replaced with entries $f(w_i, w_j)$; and

$$B_n(w_1, \ldots, w_n) = A_{n+1}(w_1, \ldots, w_n, J_2)$$

as well as

$$X_n(w_1, \ldots, w_n) = 2^{n+1} A_n(w_1, \ldots, w_n) B_n(w_1, \ldots, w_n)$$

where the numerical coefficient has been adjusted so that in the rational limit, the normalization of X_n coincides with the one discussed in Sect. 1.3.

A further advantage of this new normalization is that A_n and B_n are polynomials in $w_1, \ldots, w_n$ and also of ζ, up to a conventional denominator in powers of $1 - \zeta^2$ which we have added for convenience.

Similarly, we can define

$$H_{2m}(w_1, \ldots, w_{2m}) = \frac{\prod_{i=1}^{m} \prod_{j=m+1}^{2m} h(w_i, w_j)}{\prod_{\substack{1 \le i < j \le m \\ \text{or} \\ m+1 \le i < j \le 2m}} (w_i - w_j)} \det_{\substack{1 \le i \le m \\ m+1 \le j \le 2m}} \frac{1}{h(w_i, w_j)}$$

and then the relations (35)–(38) expressing A, B in terms of H remain the same; more compactly, one can write:

$$X_n(\ldots) = 2^{n+1} \prod_{\substack{S \subset \{J_2, J_3, J_4\} \\ |S| = n \pmod 2}} H_{n+|S|}(\ldots, S)$$

There are various alternative formulae, for example

$$H_{2m+2}(w_1, \ldots, w_{2m}, J_3, J_4) = \frac{\prod_{i=1}^{m} \prod_{j=m+1}^{2m} h(w_i, w_j)}{\prod_{\substack{1 \le i < j \le m \\ \text{or} \\ m+1 \le i < j \le 2m}} (w_i - w_j)} \det_{\substack{1 \le i \le m \\ m+1 \le j \le 2m}} \frac{A_2(w_i, w_j)}{h(w_i, w_j)}$$

Finally, the transformations

$$\zeta \to -\zeta \qquad \zeta \to \frac{\zeta + 3}{\zeta - 1}$$

generate the group of permutations of the three coupling constants J_2, J_3, J_4. Via the uniformization $w(x) = w_\zeta(x)$, this translates into the symmetry of permutations of non-trivial solutions of $2x + \eta = 0$. The function $h(w, w') = h_\zeta(w, w')$ itself possesses this symmetry, in the sense that

$$h_\zeta(w,w') = h_{-\zeta}(w,w') \qquad h_\zeta(w,w') = h_{(\zeta+3)/(\zeta-1)}\left(\frac{\zeta-1}{2}w, \frac{\zeta-1}{2}w'\right)$$

which is consistent with $w_\zeta(\beta_{3,4}) = w_{-\zeta}(\beta_{4,3})$ and

$$\begin{aligned} w_{(\zeta+3)/(\zeta-1)}(\beta_2) &= -1/2 = \frac{\zeta-1}{2} J_4 \\ w_{(\zeta+3)/(\zeta-1)}(\beta_3) &= \frac{\zeta-1}{2(\zeta+1)} = \frac{\zeta-1}{2} J_3 \\ w_{(\zeta+3)/(\zeta-1)}(\beta_4) &= \frac{1-\zeta}{4} = \frac{\zeta-1}{2} J_2 \end{aligned}$$

4 Homogeneous Limit of the Partition Function

4.1 Summary

The homogeneous limit is obtained by setting all spectral parameters equal; in the half-specialized partition function X_n, this is achieved by sending all x_i to zero.

In this section, we use the following notation: we omit parameters that are set to zero, e.g., $H_{2m} = H_{2m}(\underbrace{0,\ldots,0}_{2m})$. This is unambiguous because the total number of variables is given in subscript. Here are some values of H_{2m} for $m = 0, 1, 2, 3$:

$$\begin{aligned}
H_{2m} &= 1, && 1, && 3+\zeta^2, && 26+29\zeta^2+8\zeta^4+\zeta^6 \\
2^{m-1}H_{2m}(J_2) &= && 1, && 7+\zeta^2, && 143+99\zeta^2+13\zeta^4+\zeta^6 \\
H_{2m}(J_3) &= && 1, && 2+\zeta+\zeta^2, && 11+12\zeta+21\zeta^2+10\zeta^3 \\
&&&&&&& +7\zeta^4+2\zeta^5+\zeta^6 \\
H_{2m}(J_4) &= && 1, && 2-\zeta+\zeta^2, && 11-12\zeta+21\zeta^2-10\zeta^3 \\
&&&&&&& +7\zeta^4-2\zeta^5+\zeta^6 \\
2^{m-1}H_{2m}(J_2,J_3) &= && 1, && 5+2\zeta+\zeta^2, && 66+63\zeta+81\zeta^2+30\zeta^3 \\
&&&&&&& +12\zeta^4+3\zeta^5+\zeta^6 \\
2^{m-1}H_{2m}(J_2,J_4) &= && 1, && 5-2\zeta+\zeta^2, && 66-63\zeta+81\zeta^2-30\zeta^3 \\
&&&&&&& +12\zeta^4-3\zeta^5+\zeta^6 \\
H_{2m}(J_3,J_4) &= && 1, && 1+\zeta^2, && 3+9\zeta^2+3\zeta^4+\zeta^6 \\
2^{m-1}H_{2m}(J_2,J_3,J_4) &= && && 3+\zeta^2, && 21+39\zeta^2+3\zeta^4+\zeta^6
\end{aligned}$$

We recognize the reciprocal polynomials of those occurring in Conjecture E of [6]: the correspondence of notations is that for $m \geq 1$, $H_{2m} = \zeta^{m(m-1)} q_{m-1}(1/\zeta)$, $H_{2m}(J_3) = \zeta^{m(m-1)} p_{m-1}(1/\zeta)$, $2^{m-1} H_{2m}(J_2, J_4) = \zeta^{m(m-1)} p_{-m}(1/\zeta)$, $2^{m-1} H_{2m}(J_2, J_3, J_4) = \zeta^{m(m-1)} q_{-m}(1/\zeta)$. All other sequences can be obtained by permutations of the $\{J_2, J_3, J_4\}$, and can therefore be obtained by iterating the transformations $\zeta \to -\zeta$ and $\zeta \to \frac{\zeta+3}{\zeta-1}$, as explained at the end of last section. All the properties listed in Conjecture E of [6] can thus be checked on the H_{2m}.

If we recombine the H_{2m} in pairs to form A_n and B_n, we recognize the reciprocal polynomials of the s_n of [6]: (see their Appendix A)

$$A_n = \zeta^{2\lfloor n^2/4 \rfloor} s_n\big(1/\zeta^2\big) \qquad n \geq 0 \tag{43}$$

$$B_n = (2/3)^n \zeta^{2\lfloor (n+1)^2/4 \rfloor} s_{-n-1}\big(1/\zeta^2\big) \quad n \geq 0 \tag{44}$$

from which we conclude

$$X_n = 2^{n+1} A_n B_n = 2(4/3)^n \zeta^{n(n+1)} s_n\big(1/\zeta^2\big) s_{-n-1}\big(1/\zeta^2\big)$$

which coincides with the expression given in Conjecture 1 of [6] up to the factor of two (which is due to our slightly different way of lifting the two-fold degeneracy: $F_* \Psi_L = (-1)^n \Psi_L$ effectively duplicates every entry of Ψ_L compared to [6]).

See also Appendix A for an explanation of the constant terms of the various polynomials above.

In the rest of this section, it is convenient to denote $\alpha = 1 - \zeta^2$. We shall show that the various polynomials above satisfy (differential) recurrence relations.

4.2 Linear Relations

We first derive certain linear relations satisfied by $H_{2m}(w_1, \ldots, w_{2m})$. We shall need them to relate the various derivatives of H_{2m} at $w_i = 0$.

Define

$$D_{2m}(w_1, \ldots, w_m; w_{m+1}, \ldots, w_{2m}) = \det_{\substack{1 \leq i \leq m \\ m+1 \leq j \leq 2m}} g(w_i, w_j) \tag{45}$$

with $g(u, v) = \frac{1}{h(u,v)}$; we recall that $h(u, v) = 1 + uv(\alpha(u + v + 1) - 4)$. In other words,

$$H_{2m}(w_1, \ldots, w_{2m}) = \frac{\prod_{i=1}^{m} \prod_{j=m+1}^{2m} h(w_i, w_j)}{\prod_{\substack{1 \leq i < j \leq m \\ \text{or} \\ m+1 \leq i < j \leq 2m}} (w_i - w_j)} D_{2m}(w_1, \ldots, w_m; w_{m+1}, \ldots, w_{2m}) \tag{46}$$

Also define

$$S_{2m}(w_1,\dots,w_{2m}) = \prod_{1\le i<j\le 2m} (w_i - w_j) H_{2m}(w_1,\dots,w_{2m}) \tag{47}$$

4.2.1 A First Order Differential/Divided Difference Equation

We start from the following identity, which can be checked directly:

$$\begin{aligned}&\rho(u)\partial_u g(u,v) + \rho(v)\partial_v g(u,v)\\ &\quad + 2(1-\alpha)(8+\alpha)\partial_\alpha g(u,v) + \big(\sigma(u)+\sigma(v)\big)g(u,v) + (\delta_u+\delta_v)g(u,v) = 0\end{aligned}$$

where $\rho(u) = (1+2u)(4-6u+u\alpha+u^2\alpha)$, $\sigma(u) = 5u(\alpha-4+4\alpha u)$, ∂_u is the usual partial derivative $\frac{\partial}{\partial u}$, and δ_u is the *divided difference* operator: $\delta_u\phi(u) = \frac{\phi(u)-\phi(0)}{u}$ for any function $\phi(u)$.

Then, one can easily prove starting from (45) (for example by writing D_{2m} as a sum over permutations and grouping together the summands for values of the index connected by the permutation)

$$\begin{aligned}&\left(\sum_{i=1}^{2m}\big(\rho(w_i)\partial_{w_i} + \sigma(w_i) + \delta_{w_i}\big) + 2(1-\alpha)(8+\alpha)\partial_\alpha\right)\\ &\qquad \times D_{2m}(w_1,\dots,w_m;w_{m+1},\dots,w_{2m}) = 0\end{aligned} \tag{48}$$

In principle, by using relation (46), one can reformulate this identity in terms of H_{2m}, but the result is not particularly illuminating and we shall not need it.

4.2.2 A Second Order Differential Equation

Starting from the differential equation satisfied by ϑ_3, namely, $(\frac{\partial^2}{\partial x^2} + 4p\frac{\partial}{\partial p})\vartheta_3(x;p) = 0$, we find

$$\left(\frac{\partial^2}{\partial x^2} + 24mp\frac{\partial}{\partial p}\right)s_k = k^2 s_k$$

According to Eq. (39), this implies that

$$\left(\sum_{i=1}^{2m}\frac{\partial^2}{\partial x_i^2} + 24mp\frac{\partial}{\partial p}\right)\mathrm{S}_{2m}(x_1,\dots,x_{2m}) = c_m \mathrm{S}_{2m}(x_1,\dots,x_{2m}) \tag{49}$$

where c_m is $m(6m^2-1)$ plus some p-dependent constant related to the normalization of S_{2m}.

After switching to our rational parameterization and from p to α, we find the following equation for S_{2m}:

$$\left(\sum_{i=1}^{2m}\big(\gamma_2(w_i)\partial_i^2 + \gamma_1(w_i)\partial_i + \gamma_0(w_i)\big) + 24m\alpha(1-\alpha)(8+\alpha)\partial_\alpha\right) \times S_{2m}(w_1,\dots,w_{2m}) = 0 \tag{50}$$

where the coefficients are entirely determined except the constant term of $\gamma_0(w)$. The latter is determined by the large $w = (w_1,\dots,w_{2m})$ expansion: from (45)–(47) one easily derives

$$S_{2m}(w_1,\dots,w_{2m}) = \alpha^{m(m-1)}\prod_{i=1}^{2m} w_i^{m-1}\prod_{1\le i<j\le 2m}(w_i - w_j)\,\big(1 + O\big(w^{-3}\big)\big)$$

and expanding (50) up to second subleading order fixes the constant. We find the rather unpleasant expressions:

$$\begin{aligned}
\gamma_0(w) &= 18\alpha^2(m-1)(3m-2)w^2 + 6\alpha(\alpha-4)(3m-2)(4m-3)w \\
&\quad + 64m^2 - 12\alpha^2 m + 96\alpha m - 192m + 80 + 5\alpha^2 - 40\alpha \\
&\quad + 10\alpha^2 m^2 - 20\alpha m^2 \\
\gamma_1(w) &= -36\alpha^2(m-1)w^3 - 6\alpha(\alpha-4)(10m-9)w^2 \\
&\quad + 6\big(3\alpha^2 - 24\alpha - 4\alpha^2 m + 12\alpha m - 32m + 48\big)w - 36\alpha \\
\gamma_2(w) &= 6w(\alpha w - 4)\big(\alpha + \alpha w^2 + 2\alpha w - 4w\big)
\end{aligned}$$

4.2.3 Homogeneous Limit

We now take the homogeneous limit in two steps: we first send $w_1,\dots,w_m$ to u and $w_{m+1},\dots,w_{2m}$ to v and then expand around $u, v = 0$. H_{2m}, being a symmetric function of $w_1,\dots,w_{2m}$, only has one independent first derivative (resp. two independent second derivatives), which with our specialization correspond to $\frac{\partial}{\partial u}H_{2m} = \frac{\partial}{\partial v}H_{2m}$ (resp. $\frac{\partial^2}{\partial u^2}H_{2m} = \frac{\partial^2}{\partial v^2}H_{2m}$ and $\frac{\partial^2}{\partial u\partial v}H_{2m}$).

Taking this limit in Eqs. (48) and (50) is a rather tedious procedure which we shall not describe in detail. Expanding to first non-trivial order these equations produces the same result, namely the first equation below. This equation is a first order differential equation, and so we can differentiate it once w.r.t. α, resulting in a second order equation (second equation below). Expanding to the next order Eqs. (48)

and (50) produces two *distinct* second order differential equations. Finally, we find:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 2(4m+1) & 2m(1-\alpha)(\alpha+8) & m^2(m-1)\times(2+\alpha) \\ 0 & 0 & 2m\times(1-\alpha)(\alpha+8) & 2(4m+1) & 0 & (\alpha+2)m^2(m-1)-m(4\alpha+14) & m^2(m-1) \\ 2(2m+1) & 4m & 0 & 2m\times(1-\alpha)(\alpha+8) & m(m^2-m+1)\times(2+\alpha) & 0 & m^2(m-1)\times(4-\alpha) \\ (4m^2+m-2)\times 2\alpha & -2\alpha\times m(4m+1) & 0 & 0 & m(2m+1)\times(\alpha-4)^2 & 0 & -(m-1)m^2\times(2m+1)\times\alpha(\alpha-4) \end{pmatrix} \begin{pmatrix} \frac{\partial^2}{\partial u\partial v}H_{2m} \\ \frac{\partial^2}{\partial u^2}H_{2m} \\ \frac{\partial^2}{\partial \alpha^2}H_{2m} \\ \frac{\partial^2}{\partial u\partial \alpha}H_{2m} \\ \frac{\partial}{\partial u}H_{2m} \\ \frac{\partial}{\partial \alpha}H_{2m} \\ H_{2m} \end{pmatrix} = 0 \qquad (51)$$

There are 4 relations for seven derivatives, so they can all be expressed in terms of derivatives w.r.t. α only.

A similar reasoning can be made when all variables are specialized to 0 except one, or two, or three, are specialized to a subset of $\{J_2, J_3, J_4\}$. The result is given in Appendix D.

4.3 Bilinear Recurrence Relations

We now show how to derive differential bilinear recurrence relations for H_{2m} and its variants. In fact these relations were mentioned, but not written explicitly, in paragraph 3 of [6].

Similarly to the previous paragraph, we first consider the quantity

$$H_{2m}(\underbrace{u,\ldots,u}_{m},\underbrace{v,\ldots,v}_{m}) = g(u,v)^{-m^2} \det_{0\le i,j\le m-1}\left(\frac{1}{i!j!}\frac{\partial^{i+j}}{\partial u^i\partial v^j}g(u,v)\right)$$

A standard application of the Jacobi–Desnanot identity (see [22] for the simpler case of the Izergin–Korepin determinant) to the determinant in the right hand side produces the Toda lattice equation:

$$\frac{1}{m^2}\frac{\partial^2}{\partial u\partial v}\log H_{2m}(u,\ldots,v,\ldots) = -\frac{\partial^2}{\partial u\partial v}\log g(u,v)$$

$$+\frac{H_{2(m+1)}(u,\ldots,v,\ldots)H_{2(m-1)}(u,\ldots,v,\ldots)}{g(u,v)^2 H_{2m}(u,\ldots,v,\ldots)^2}$$

The left hand side involves first and second derivative of $H_{2m}(u,\ldots,v,\ldots)$, which at $u=v=0$ can be reexpressed in terms of derivatives w.r.t. α thanks to Eq. (51). The result is:

$$C_0 H_{2(m+1)}H_{2(m-1)} = C_1 H_{2m}H''_{2m} - C_2\big(H'_{2m}\big)^2 + C_3 H_{2m}H'_{2m} + C_4 H^2_{2m} \qquad (52)$$

where all derivatives are w.r.t. α, and

$$\begin{aligned}
C_0 &= 4\alpha(4m-1)(4m+1)^2(4m+3)\\
C_1 &= 4(\alpha-1)^2\alpha(\alpha+8)^2(4m+1)^2\\
C_2 &= 4(\alpha-1)^2\alpha(\alpha+8)^2(4m-1)(4m+3)\\
C_3 &= 2(\alpha-1)(\alpha+8)\big(\alpha^2+28\alpha+24\alpha^2m^2+304\alpha m^2-256m^2+20\alpha^2 m\\
&\qquad +208\alpha m-192m-32\big)\\
C_4 &= 6\alpha^2-24\alpha+4\alpha^3m^4-1008\alpha^2m^4+3408\alpha m^4+512m^4-4\alpha^3m^3\\
&\qquad -984\alpha^2m^3+4032\alpha m^3-128m^3+\alpha^3m^2-142\alpha^2m^2+1028\alpha m^2\\
&\qquad -320m^2-\alpha^3 m+28\alpha^2 m-44\alpha m-64m
\end{aligned}$$

Note that contrary to Eq. (51), Eq. (52) is a closed relation allowing to compute inductively the H_{2m} as polynomials of $\alpha=1-\zeta^2$.

A similar computation produces differential recurrence relations of the same form for the other factors of X_n. The coefficients are given in Appendix D. Together, they allow to compute the full squared norm X_n inductively.

5 Conclusion and Prospects

In this paper, we have considered the inhomogeneous eight-vertex model with periodic boundary conditions in odd size and crossing parameter $\eta=\pi/3$. We have provided a basic setup for the computation of the fully inhomogeneous generalization of the ground state eigenvector of the XYZ spin chain, and then went on to compute the partition function on an infinite cylinder, which generalizes the squared norm of the ground state eigenvector, when the spectral parameters are "half-specialized", i.e., form pairs $x,-x$. We have provided a variety of explicit expressions for this partition function in terms of Pfaffians and determinants. Interestingly, one can then obtain self-contained expressions in the homogeneous limit for the squared norm, without any more reference to the inhomogeneous case, by allowing differentiation w.r.t. the variable parameterizing the line $\eta=\pi/3$ (elliptic nome, or ζ). These expressions take the form of bilinear differential recurrence relations (cf. Eq. (52)).

In order to derive such differential relations, we have used certain differential (and divided difference) relations satisfied by the inhomogeneous partition function. In fact, we have strictly limited ourselves to the relations that were needed for our purposes, but it seems that this is only the tip of the iceberg: one should investigate in more detail the structure of the set of such equations. It would be interesting to understand the role of the full symmetry of arguments of the Izergin–Korepin type determinant (33).

Note that we have not been able to obtain an expression for the fully inhomogeneous partition function, but if we compare to the work of Rosengren for the 8VSOS model [30] there is also no simple expression for the fully inhomogeneous partition function. Inversely, it would be interesting to see if the "half-specialization" trick helps in this context. More generally, as noted in [6], there are many ressemblances between the work [30] and our present setup, which should be clarified.

Another connection which should be more thoroughly explored is with the supersymmetric models of lattice fermions of [16, 17].

It is clear that the present methods should allow to compute more quantities such as individual entries of the ground state, or certain correlation functions (see the recent work [8] in the XXZ setting).

Some more directions which should be explored are: the relation to the quantum Knizhnik–Zamolodchikov–Bernard (qKZB) equation and to the qKZB heat equation [15], which should be the right framework for part of Sect. 2, especially in view of a generalization to arbitrary η; the connection to nonsymmetric elliptic Macdonald polynomials; the use of matrix model techniques to analyze the determinants of Izergin–Korepin type found here, as in [36]; and the meaning of the connection to the Painlevé VI equation, which is emphasized in [5, 6].

Finally, it would be interesting to find a combinatorial interpretation for the (positive integer) entries of the polynomials of Sect. 4.1, beyond their value at $\zeta = 0$.

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Appendix A: The $\zeta \to 0$ Trigonometric Limit

The trigonometric limit is obtained by sending ζ to 0. The Boltzmann weights (1) of the eight-vertex model turn into those of the six-vertex (the weight d go to zero). In this limit the results of this paper should be closely related to the computations of [14]. Note that the "quadratic" sum rule considered here was actually not computed in [14]—instead the quantity $\sum_\alpha \Psi_\alpha(z_1,\dots,z_L)^2$ was used there. However, the same argument of degeneracy of the scalar product allows to conclude that

$$Z_L(x_1,\dots,x_L) = 3^{-n}\left(\sum_\alpha \Psi_\alpha(x_1,\dots,x_L)\right)\left(\sum_\alpha \Psi_\alpha(-x_1,\dots,-x_L)\right)$$

$$= 3^{-n^2} s_{Y_L}(z_1,\ldots,z_L) s_{Y_L}\big(z_1^{-1},\ldots,z_L^{-1}\big) \tag{53}$$

where s_λ is the Schur function associated to partition λ, and $Y_L = (\lfloor (L-i)/2\rfloor)_{i=1,\ldots,L}$. In the homogeneous limit,

$$s_{Y_n}(1,\ldots,1) = 3^{n(n-1)/2}\prod_{j=1}^n \frac{(3j)!(j-1)!}{(2j)!(2j-1)!}$$

and together we have $Z_L = A_{HT}(L)$, where $A_{HT}(L) = 1, 3, 25, 588\ldots$ is the number of Half-Turn Symmetric Alternating Sign Matrices [23, 27].

The half-specialization of Sect. 3.5 produces the following factorization:

$$s_{Y_L}\big(1, z_1, z_1^{-1},\ldots,z_n,z_n^{-1}\big) = \prod_{i=1}^n \big(1+z_i+z_i^{-1}\big)\chi_{Y_n}(z_1,\ldots,z_n)\chi_{Y_{n+1}}(z_1,\ldots,z_n,\omega) \tag{54}$$

where χ_λ is the symplectic character, defined by:

$$\chi_\lambda(z_1,\ldots,z_n) = \frac{\det(z_i^{\lambda_j+n-j+1} - z_i^{-\lambda_j-n+j-1})}{\det(z_i^{n-j+1} - z_i^{-n+j-1})},$$

and $\omega = e^{i\pi/3}$; this formula can be proved by induction, or can be seen as a byproduct of this paper, as we now show.

In the limit $\zeta\to 0$, the parameterization w is related to the multiplicative spectral parameter z by $w = (z-1)^2/(1+z+z^2)$; this way we find

$$h\big(z,z'\big) = \frac{9(z^2+zz'+z'^2)(1+zz'+z^2z'^2)}{(1+z+z^2)^2(1+z'+z'^2)^2}$$

The denominator factors out of Pfaffians and determinants.

A.1 Pfaffians

We now recognize the Pfaffian A_n (Eq. (42)) in even size:

$$\begin{aligned}
&A_{2m}(w_1,\ldots,w_{2m})\\
&\quad = 3^m \prod_{i=1}^n z_i \prod_{1\le i<j\le 2m} \frac{3(z_i^2+z_iz_j+z_j^2)(1+z_iz_j+z_i^2z_j^2)}{(1+z_i+z_i^2)(1+z_j+z_j^2)(z_i-z_j)(1-z_iz_j)}\\
&\qquad \times \operatorname{Pf}\frac{(z_i-z_j)(1-z_iz_j)}{(z_i^2+z_iz_j+z_j^2)(1+z_iz_j+z_i^2z_j^2)}
\end{aligned}$$

which up to some prefactors is exactly the Pfaffian given in [11] (Eq. (3.27)) for the *square* of the partition function Z_{UASM} of U-turn symmetric ASMs of [23]. The latter is known to coincide with $\chi_{Y_{2m}}(z_1,\dots,z_{2m})$ [25] and so we reproduce the first factor of the l.h.s. of Eq. (54). More precisely, we find $A_{2m}(w_1,\dots,w_{2m}) = 3^{2m^2}\prod_{i=1}^{2m}(1+z_i+z_i^{-1})^{-2m+1}\chi_{Y_{2m}}(z_1,\dots,z_{2m})^2$. The odd case can be reduced to the even case by sending one of the z_i to zero (something which did not make sense in the elliptic setting), so that for both parities we have

$$A_n(w_1,\dots,w_n) = 3^{2\lfloor n/2\rfloor\lfloor (n+1)/2\rfloor}\prod_{i=1}^{n}\left(1+z_i+z_i^{-1}\right)^{-n+1}\chi_{Y_n}(z_1,\dots,z_n)^2$$

or in terms of the original quantities,

$$\mathrm{A}_n(x_1,\dots,x_n) = 3^{-2\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor}\chi_{Y_n}(z_1,\dots,z_n)^2$$

The second factor is simply obtained by noting that $w = J_2 = -1/2$ corresponds to $z = \omega = e^{i\pi/3}$, so

$$B_n(w_1,\dots,w_n) = 2^{-n}3^{2\lfloor n/2+1\rfloor\lfloor (n+1)/2\rfloor}\prod_{i=1}^{n}\left(1+z_i+z_i^{-1}\right)^{-n}\chi_{Y_{n+1}}(z_1,\dots,z_n,\omega)^2$$

or $\mathrm{B}_n(x_1,\dots,x_n) = 3^{-2\lfloor n/2\rfloor\lfloor (n+1)/2\rfloor}\chi_{Y_{n+1}}(z_1,\dots,z_n,\omega)^2$. Finally,

$$\begin{aligned}\mathrm{Z}_n &= \prod_{i=1}^{n}\left(\frac{1+z_i+z_i^{-1}}{3}\right)^2\mathrm{X}_n\\ &= 3^{-n^2}\prod_{i=1}^{n}\left(1+z_i+z_i^{-1}\right)^2\chi_{Y_n}(z_1,\dots,z_n)^2\chi_{Y_{n+1}}(z_1,\dots,z_n,\omega)^2\end{aligned}$$

which is consistent with Eqs. (53) and (54).

A.2 Determinants

Similarly, the determinants simplify as $\zeta\to 0$. Noting that $w = J_3$ and $w = J_4$ both correspond to $z = 0$, we conclude that there are only two distinct determinants for each parity; Tsuchiya's determinant [23, 34] is known to be equal at a cubic root of unity to the symplectic character introduced above [25]

$$\begin{aligned}&\frac{\prod_{\substack{1\le i\le m,\\ m+1\le j\le 2m}}(z_i^2+z_iz_j+z_j^2)(1+z_iz_j+z_i^2z_j^2)}{\prod_{\substack{1\le i<j\le m,\\ m+1\le i<j\le 2m}}(z_j-z_i)(1-z_iz_j)}\\ &\times\det_{\substack{1\le i\le m,\\ m+1\le j\le 2m}}\frac{1}{(z_i^2+z_iz_j+z_j^2)(1+z_iz_j+z_i^2z_j^2)}\end{aligned}$$

$$= \chi_{Y_{2m}}(z_1, \dots, z_{2m})$$

and then we have:

$$H_{2m}(w_1, \dots, w_{2m}) = 3^{m(m-1)} \prod_{i=1}^{2m} \left(1 + z_i + z_i^{-1}\right)^{-m+1} \times \chi_{Y_{2m}}(z_1, \dots, z_{2m})$$

$$H_{2m}(w_1, \dots, w_{2m-1}, J_2) = 3^{m(m-1)} \prod_{i=1}^{2m-1} \left(1 + z_i + z_i^{-1}\right)^{-m+1} \times \chi_{Y_{2m}}(z_1, \dots, z_{2m-1}, \omega)$$

$$H_{2m}(w_1, \dots, w_{2m-1}, J_{3/4}) = 3^{m(m-1)} \prod_{i=1}^{2m-1} \left(1 + z_i + z_i^{-1}\right)^{-m+1} \times \chi_{Y_{2m-1}}(z_1, \dots, z_{2m-1})$$

$$H_{2m+2}(w_1, \dots, w_{2m}, J_2, J_{3/4}) = 3^{m(m+1)} 2^{-m} \prod_{i=1}^{2m} \left(1 + z_i + z_i^{-1}\right)^{-m} \times \chi_{Y_{2m+1}}(z_1, \dots, z_{2m}, \omega)$$

$$H_{2m+2}(w_1, \dots, w_{2m}, J_3, J_4) = 3^{m(m+1)} \prod_{i=1}^{2m} \left(1 + z_i + z_i^{-1}\right)^{-m} \times \chi_{Y_{2m}}(z_1, \dots, z_{2m})$$

$$H_{2m+2}(w_1, \dots, w_{2m-1}, J_2, J_3, J_4) = 3^{m(m+1)} \prod_{i=1}^{2m-1} \left(1 + z_i + z_i^{-1}\right)^{-m} \times \chi_{Y_{2m}}(z_1, \dots, z_{2m-1}, \omega)$$

A.3 More Determinants

The expression (39) of S_{2m} as a Slater determinant reduces to the numerator of our definition of the symplectic character $\chi_{Y_{2m}}$ (since $k_j = Y_{2m;m+1-j} + 2m - j + 1$, $j = 1, \dots, 2m$)

$$S_{2m}(z_1, \dots, z_{2m}) = \det_{i,j=1,\dots,2m} \left(z_i^{k_j} - z_i^{-k_j}\right)$$

The differential equation (49) reduces to

$$\sum_{i=1}^{2m} \left(z_i \frac{\partial}{\partial z_i}\right)^2 S_{2m}(z_1, \dots, z_{2m}) = m\left(6m^2 - 1\right) S_{2m}(z_1, \dots, z_{2m})$$

A.4 Homogeneous Limit

Finally, $A_{2m}^{1/2} = H_{2m} = 3^{-m(m-1)} \chi_{Y_{2m}}(1, \ldots, 1) = 1, 1, 3, 26, 646\ldots$ is the number of Vertically Symmetric Alternating Sign Matrices of size $2m+1$ (also, the number of Off-diagonally Symmetric Alternating Sign Matrices of size $2m$, and the number of Descending Plane Partitions of size m which are symmetric w.r.t. all reflections, i.e., Cyclically Symmetric Transpose Complement Plane Partitions of a hexagon of size $(m+1) \times (m-1)$ with a triangular hole cut out), while $A_{2m-1}^{1/2} = H_{2m}(J_{3/4}) = 3^{-(m-1)^2} \chi_{Y_{2m-1}}(1, \ldots, 1) = 1, 2, 11, 170\ldots$ is the number of Cyclically Symmetric Transpose Complement Plane Partitions of size m (also, the number of VSASMs of size $(2m-1) \times (2m+1)$ with a defect on the m^{th} row, the symmetry line). Note that the square of the number of VSASMs also appears in the observations of [28].

The sequence of numbers

$$2^m B_{2m}^{1/2} = 2^m H_{2m}(J_2, J_{3/4}) = 3^{-m(m-1)} \chi_{Y_{2m+1}}(1, \ldots, 1, \omega) = 1, 5, 66, 2431\ldots$$

appears as one of the factors of the enumeration of UUASMs in [23]. The last sequence,

$$2^m (B_{2m-1}/3)^{1/2} = H_{2m}(J_2) = 3^{-(m-1)^2} \chi_{Y_{2m}}(1, \ldots, 1, \omega) = 1, 7, 143, 8398, \ldots$$

is the number of ASMs of order $2m+1$ divided by the number of VSASMs of size $2m+1$.

As mentioned before, the last two cases, namely $H_{2m}(J_3, J_4)$, and $H_{2m}(J_2, J_3, J_4)$, are related to H_{2m} and $H_{2m}(J_2)$ by multiplication by powers of 3 and 2.

Appendix B: The $\zeta \to 1$ Limit

Besides the $\zeta \to 0$ limit, there is another trigonometric limit, namely $\zeta \to 1$ or $\alpha \to 0$. It is expected to be somewhat trivial since the corresponding Hamiltonian is the Ising Hamiltonian with interaction $\sigma^x \sigma^x$. Indeed, we find that the building block H_{2m} of the partition function becomes:

$$\begin{aligned} H_{2m}(w_1, \ldots, w_{2m})|_{\zeta=1} &= \frac{\prod_{\substack{1\le i\le m\\ m+1\le j\le 2m}} (1-4w_i w_j)}{\prod_{\substack{1\le i<j\le m \\ \text{or} \\ m+1\le i<j\le 2m}} (w_i - w_j)} \det_{\substack{1\le i\le m\\ m+1\le j\le 2m}} \frac{1}{1-4w_i w_j} \\ &= 2^{m(m-1)} \end{aligned}$$

This formula is valid as long as the w_i stay finite as $\zeta \to 1$. One special case is if one w_i is equal to $J_4 = 1/(1-\zeta)$. Then we find instead

$$H_{2m}(w_1, \ldots, w_{2m-1}, J_4)|_{\zeta=1} = 2^{(m-1)^2}$$

so that

$$X_n(\ldots) = 2^{n(n+1)+1}$$

This is compatible with a constant value of $\Psi_{n,\alpha} = 2^{n(n-1)/2}$ since $X_{2m} = 2^{2n+1}\Psi^2_{2m,\alpha}$.

Appendix C: Proof of Symmetry of H_{2m}

The symmetry of H_{2m}, defined by (33) can be seen as a particular case of a general result, which can be formulated as follows: (see also Thm. 4.2 in [24])

Proposition *Let ϕ_1, ϕ_2 be two functions (with values in $\mathbb{C}$) such that*

(i) $\phi(x, y) = -\phi(y, x)$,
(ii) $\phi(x_1, x_2)\phi(x_3, x_4) - \phi(x_1, x_3)\phi(x_2, x_4) + \phi(x_1, x_4)\phi(x_2, x_3) = 0$ *for* $\phi = \phi_1, \phi_2$.

Then, in the domain of the $(x_i)_{1\le i\le 2m}$ such that $\phi_2(x_i, x_j) \neq 0$ for all $1 \le i, j \le 2m$,

$$\Delta_{2m}(x_1, \ldots, x_{2m}) = \frac{\det\limits_{\substack{i=1,\ldots,m\\ j=m+1,\ldots,2m}} \left(\frac{\phi_1(x_i,x_j)}{\phi_2(x_i,x_j)}\right)}{\prod\limits_{\substack{1\le i<j\le m\\ \text{or}\\ m+1\le i<j\le 2m}} \phi_2(x_i, x_j)}$$

is symmetric in all arguments $\{x_1, \ldots, x_{2m}\}$.

Actually it is well-known that functions that satisfy (i) and (ii) are 2×2 determinants $\left|\begin{smallmatrix} a(x) & a(y)\\ b(x) & b(y)\end{smallmatrix}\right|$, so that, removing symmetric factors, one may without loss of generality write $\phi_i(x_1, x_2) = \phi_i(x_1) - \phi_i(x_2)$, $i = 1, 2$. The proposition then follows from the following representation (characteristic of Toda chain tau functions): starting from $\frac{\phi_1(x_i)-\phi_1(x_j)}{\phi_2(x_1)-\phi_2(x_j)} = \frac{1}{2\pi i}\oint_C \frac{dy}{(y-\phi_2(x_i))(y-\phi_2(x_j))}\phi_1(y)$ where C is any contour that surrounds once counterclockwise the $\phi_2(x_j)$, $j = 1, \ldots, 2m$, and expanding the determinant in Δ_{2m} we get

$$\Delta_{2m}(x_1, \ldots, x_{2m})$$
$$= \frac{1}{m!(2\pi i)^m} \oint_{C^m} \prod_{i=1}^{m} dy_i \phi_1(y_i)$$

$$\times \frac{\det_{\substack{i=1,\dots,m\\ j=1,\dots,m}}\left(\frac{1}{y_i-\phi_2(x_j)}\right)}{\prod_{1\le i<j\le m}(\phi_2(x_i)-\phi_2(x_j))}\frac{\det_{\substack{i=1,\dots,m\\ j=m+1,\dots,2m}}\left(\frac{1}{y_i-\phi_2(x_j)}\right)}{\prod_{m+1\le i<j\le 2m}(\phi_2(x_i)-\phi_2(x_j))}$$

$$= \frac{1}{m!(2\pi i)^m}\oint_{C^m}\prod_{i=1}^m dy_i\phi_1(y_i)\frac{\prod_{1\le i<j\le m}(y_i-y_j)^2}{\prod_{i=1}^m\prod_{j=1}^{2m}(y_i-\phi_2(x_j))}$$

which is explicitly symmetric in the x_i.

The application to H_{2m} consists in writing $\phi_2(x,y)=\mathrm{h}(x,y)\vartheta(x-y)\vartheta(x+y)$, $\phi_1(x,y)=\vartheta(x-y)\vartheta(x+y)$ and checking that they satisfy (i) and (ii), so that $\mathrm{H}_{2m}(x_1,\dots,x_{2m})=\prod_{1\le i<j\le 2m}\mathrm{h}(x_i,x_j)\Delta_{2m}(x_1,\dots,x_{2m})$. It is slightly easier to apply it to H_{2m}, i.e., after the change of variables from x to w, since we then have the more explicit expressions $\phi_1(w)=w$, $\phi_2(w)=w/(1+(3+\zeta^2)w^2-(1-\zeta^2)w^3)$.

Note that other identities following from integrability of the Toda chain, for example the Hankel determinant form

$$\Delta_{2m}(x_1,\dots,x_{2m})=\det(s_{i+j})_{i,j=0,\dots m-1},$$
$$s_k=\sum_{i=1}^{2m}\frac{\phi_2(x_i)^k}{\prod_{j(\ne i)}(\phi_2(x_i)-\phi_2(x_j))}\phi_1(x_i).$$

They also provide an alternative derivation of Eq. (39) ("first quantized" form of the tau function).

Appendix D: Differential Equations

We provide here analogues of Eqs. (51) and (52) when H_{2m} (that is, the function H_{2m} with all arguments set to zero) is replaced with $H_{2m}(S)$, $S\subset\{J_2,J_3,J_4\}$ (again, with all other arguments set to zero). Because of the permutation symmetry w.r.t. $\{J_2,J_3,J_4\}$, we only need to provide one formula for each possible cardinality of S. When taking derivatives w.r.t. u or v, the convention is that the arguments that are specialized to J_2,J_3,J_4 are among the u's.

After transposition (for display purposes), Eq. (51) is of the form

$$\left(\frac{\partial^2}{\partial u\partial v}H_{2m},\ \frac{\partial^2}{\partial u^2}H_{2m},\ \frac{\partial^2}{\partial\alpha^2}H_{2m},\ \frac{\partial^2}{\partial u\partial\alpha}H_{2m},\ \frac{\partial}{\partial u}H_{2m},\ \frac{\partial}{\partial\alpha}H_{2m},\ H_{2m}\right)P$$
$$=0$$

For H_{2m} itself, the matrix P is

$$
\begin{pmatrix}
0 & 0 & 2(2m+1) & 2(4m^2+m-2)\alpha \\
0 & 0 & 4m & -2m(4m+1)\alpha \\
0 & -2m(\alpha-1)(\alpha+8) & 0 & 0 \\
0 & -2(-4m-1) & -2m(\alpha-1)(\alpha+8) & 0 \\
2(4m+1) & 0 & m(m^2-m+1)(\alpha+2) & m(2m+1)(\alpha-4)^2 \\
-2m\times(\alpha-1)(\alpha+8) & m^2(m-1)(\alpha+2)-m(4\alpha+14) & 0 & 0 \\
(m-1)m^2\times(\alpha+2) & (m-1)m^2 & -(m-1)m^2(\alpha-4) & -(m-1)m^2(2m+1)(\alpha-4)\alpha
\end{pmatrix}
$$

For $H_{2m}(J_2)$:

$$
\begin{pmatrix}
0 & 0 & 2(2m+1) & 2(4m^2-5m-1)\alpha \\
0 & 0 & 4m & -2m(4m-1)\alpha \\
0 & -2(m-1)(\alpha-1)(\alpha+8) & 0 & 0 \\
0 & -2(1-4m) & -2m(\alpha-1)(\alpha+8) & 0 \\
2(4m-1) & 0 & m^3(\alpha+2)-m(m-1)(\alpha+4) & m(2m-1)\alpha^2-16m^2(\alpha-2) \\
-2(m-1)\times(\alpha-1)(\alpha+8) & (m-1)\times(\alpha m^2+2m^2-\alpha m-4m-4\alpha-12) & 0 & 0 \\
(m-1)^2\times(\alpha m+2m-2) & (m-1)^2 m & -(m-1)m\times(\alpha m-4m+4) & -(m-1)m^2\alpha\times(2\alpha m-8m-\alpha+8)
\end{pmatrix}
$$

For $H_{2m}(J_3, J_4)$:

$$\begin{pmatrix} 0 & 0 & 2(2m+1) & 2(4m^2-11m+4)\alpha \\ 0 & 0 & 4m & -2m(4m-3)\alpha \\ 0 & -2(m-2)(\alpha-1)(\alpha+8) & 0 & 0 \\ 0 & 2(4m-3) & -2m(\alpha-1)(\alpha+8) & 0 \\ 2(4m-3) & 0 & m(\alpha+2)(m^2+1) & m(2m-1)\alpha^2 \\ \begin{matrix}-2(m-2)\times \\ (\alpha-1)(\alpha+8)\end{matrix} & \begin{matrix}(m-2)\times \\ (\alpha m^2+2m^2-\alpha m-4\alpha-14)\end{matrix} & -\alpha m^2 & -16m(m-1)(\alpha-2) \\ & & 0 & 0 \\ \begin{matrix}(m-2)m\times \\ (\alpha m+2m-\alpha)\end{matrix} & (m-2)(m-1)m & \begin{matrix}-(m-2)m\times \\ (\alpha m-4m-\alpha)\end{matrix} & \begin{matrix}-(m-2)(m-1)m\alpha\times \\ (2\alpha m-8m-\alpha)\end{matrix} \end{pmatrix}$$

For $H_{2m}(J_2, J_3, J_4)$:

$$\begin{pmatrix} 0 & 0 & 2(2m+1) & 2(m-1)(4m-13)\alpha \\ 0 & 0 & 4m & -2m(4m-5)\alpha \\ 0 & -2(m-3)(\alpha-1)(\alpha+8) & 0 & 0 \\ 0 & 2(4m-5) & -2m(\alpha-1)(\alpha+8) & 0 \\ 2(4m-5) & 0 & m(m^2-m+1)(\alpha+2) & m(2m-3)(\alpha-4)^2 \\ \begin{matrix}-2(m-3)\times \\ (\alpha-1)(\alpha+8)\end{matrix} & \begin{matrix}m(m-1)(m-3)(\alpha+2) \\ -(m-3)(4\alpha+14)\end{matrix} & 0 & 0 \\ \begin{matrix}(m-3)(m-1)m \\ \times(\alpha+2)\end{matrix} & (m-3)(m-1)m & -(m-3)m^2(\alpha-4) & \begin{matrix}-(m-3)m^2(2m-3) \\ \times(\alpha-4)\alpha\end{matrix} \end{pmatrix}$$

As to Eq. (52):

$$C_0 H_{2(m+1)} H_{2(m-1)} = C_1 H_{2m} H''_{2m} - C_2 \left(H'_{2m}\right)^2 + C_3 H_{2m} H'_{2m} + C_4 H_{2m}^2$$

The coefficients for H_{2m} are:

$$C_0 = 4\alpha(4m-1)(4m+1)^2(4m+3)$$
$$C_1 = 2(\alpha-1)^2\alpha(\alpha+8)^2(4m+1)^2$$
$$C_2 = 4(\alpha-1)^2\alpha(\alpha+8)^2(4m-1)(4m+3)$$
$$C_3 = (\alpha-1)(\alpha+8)\big(\alpha^2+28\alpha+24\alpha^2m^2+304\alpha m^2-256m^2+20\alpha^2 m + 208\alpha m - 192m - 32\big)$$
$$C_4 = 6\alpha^2 - 24\alpha + 4\alpha^3m^4 - 1008\alpha^2m^4 + 3408\alpha m^4 + 512m^4 - 4\alpha^3m^3 - 984\alpha^2m^3 + 4032\alpha m^3 - 128m^3 + \alpha^3m^2 - 142\alpha^2m^2 + 1028\alpha m^2 - 320m^2 - \alpha^3 m + 28\alpha^2 m - 44\alpha m - 64m$$

For $H_{2m}(J_2)$:

$$C_0 = 4\alpha(4m-3)(4m-1)^2(4m+1)$$
$$C_1 = 2(4m-1)^2(-1+\alpha)^2\alpha(8+\alpha)^2$$
$$C_2 = 4(4m-3)(1+4m)(-1+\alpha)^2\alpha(8+\alpha)^2$$
$$C_3 = (-1+\alpha)(8+\alpha)\big(64m-256m^2-4\alpha-96m\alpha+304m^2\alpha+\alpha^2-4m\alpha^2 + 24m^2\alpha^2\big)$$
$$C_4 = -128m+768m^2-1152m^3+512m^4-36\alpha+336m\alpha-204m^2\alpha - 2784m^3\alpha+3408m^4\alpha+18\alpha^2-126m\alpha^2-6m^2\alpha^2+1032m^3\alpha^2 - 1008m^4\alpha^2-m\alpha^3+9m^2\alpha^3-12m^3\alpha^3+4m^4\alpha^3$$

For $H_{2m}(J_3, J_4)$:

$$C_0 = 4\alpha(4m-5)(4m-3)^2(4m-1)$$
$$C_1 = 2(4m-3)^2(-1+\alpha)^2\alpha(8+\alpha)^2$$
$$C_2 = 4(4m-5)(4m-1)(-1+\alpha)^2\alpha(8+\alpha)^2$$
$$C_3 = (-1+\alpha)(8+\alpha)\big(-192+448m-256m^2+204\alpha-512m\alpha+304m^2\alpha + 21\alpha^2-44m\alpha^2+24m^2\alpha^2\big)$$
$$C_4 = 384m^2-896m^3+512m^4+720\alpha-5208m\alpha+11892m^2\alpha-10848m^3\alpha + 3408m^4\alpha-90\alpha^2+1074m\alpha^2-2958m^2\alpha^2+3000m^3\alpha^2-1008m^4\alpha^2 + 3m\alpha^3-3m^2\alpha^3-4m^3\alpha^3+4m^4\alpha^3$$

For $H_{2m}(J_2, J_3, J_4)$:

$$C_0 = 4\alpha(4m-7)(4m-5)^2(4m-3)$$

$$C_1 = 2(4m-5)^2(-1+\alpha)^2\alpha(8+\alpha)^2$$

$$C_2 = -4(4m-7)(4m-3)(-1+\alpha)^2\alpha(8+\alpha)^2$$

$$C_3 = (-1+\alpha)(8+\alpha)\big(-480+704m-256m^2+540\alpha-816m\alpha+304m^2\alpha + 45\alpha^2-68m\alpha^2+24m^2\alpha^2\big)$$

$$C_4 = -960m+2368m^2-1920m^3+512m^4+8400\alpha-27740m\alpha+33572m^2\alpha - 17664m^3\alpha+3408m^4\alpha-2100\alpha^2+7240m\alpha^2-9142m^2\alpha^2+5016m^3\alpha^2 - 1008m^4\alpha^2-5m\alpha^3+13m^2\alpha^3-12m^3\alpha^3+4m^4\alpha^3$$

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