# Chapter 6

# Lebesgue spaces

The Lebesgue spaces $L^p(\Omega)$ play a central role in many applications of functional analysis. This chapter focuses upon their basic properties, as well as certain attendant issues that will be important later. Notable among these are the semicontinuity of integral functionals, and the existence of measurable selections.

## 6.1 Uniform convexity and duality

We begin by identifying a geometric property of the norm which, when present, turns out to have a surprising consequence. Let $X$ be a normed space.

**6.1 Definition.** *$X$ is* **uniformly convex** *if it satisfies the following property*:

$$\forall\, \varepsilon > 0\,,\ \exists\, \delta > 0 \text{ such that } x \in B\,,\, y \in B\,,\ \|x-y\| > \varepsilon \implies \left\| \frac{x+y}{2} \right\| < 1-\delta\,.$$

In geometric terms, this is a way of saying that the unit ball is curved.[1] The property depends upon the choice of the norm on $X$, even among equivalent norms, as one can see even in $\mathbb{R}^2$.

**6.2 Exercise.** The following three norms on $\mathbb{R}^2$ are equivalent:

$$\|(x,y)\|_1 = |x|+|y|, \quad \|(x,y)\|_2 = |(x,y)| = \{|x|^2+|y|^2\}^{1/2},$$
$$\|(x,y)\|_\infty = \max\big(|x|,|y|\big)\,.$$

Which ones make $\mathbb{R}^2$ a uniformly convex normed space? □

[1] It turns out that the ball in $\mathbb{R}$ is curved in this sense, although it may seem rather straight to the reader.

F. Clarke, *Functional Analysis, Calculus of Variations and Optimal Control*, Graduate Texts in Mathematics 264, DOI 10.1007/978-1-4471-4820-3_6,

Despite the fact that uniform convexity is a norm-dependent property, the very existence of such a norm yields an intrinsic property of the underlying space, one that does not depend on the choice of equivalent norm.

**6.3 Theorem. (Milman)** *Any uniformly convex Banach space is reflexive.*

**Proof.** Let $\theta \in X^{**}$ satisfy $\|\,\theta\,\|_{**} = 1$, and fix any $\varepsilon > 0$. We shall prove the existence of $x \in B$ such that $\|\,Jx - \theta\,\|_{**} \leqslant \varepsilon$. Since $JB$ is closed in $X^{**}$ (see Prop. 5.3), this implies $JB = B_{**}$, and consequently that $JX = X^{**}$, so that $X$ is reflexive.

Let $\delta$ correspond to $\varepsilon$ as in the definition of uniform convexity. We choose $\zeta \in X^*$, $\|\,\zeta\,\|_* = 1$, such that $\langle\,\theta, \zeta\,\rangle > 1 - \delta/2$, and we set

$$V = \big\{\,\theta' \in X^{**} : \big|\langle\,\theta' - \theta, \zeta\,\rangle\big| < \delta/2\,\big\},$$

which is an open neighborhood of $\theta$ in the topology $\sigma\big(X^{**}, X^*\big)$. By Goldstine's lemma (see the proof of Theorem 5.47), $V$ intersects $JB$: there exists $x \in B$ such that

$$\big|\langle\,\zeta, x\rangle - \langle\,\theta, \zeta\,\rangle\big| = \big|\langle Jx - \theta, \zeta\,\rangle\big| < \delta/2.$$

We claim that $\|\,Jx - \theta\,\|_{**} \leqslant \varepsilon$. We reason from the absurd, by supposing that $\theta$ lies in $W$, where $W$ is the complement in $X^{**}$ of the set $Jx + \varepsilon B_{**}$.

Since $Jx + \varepsilon B_{**}$ is closed in $\sigma\big(X^{**}, X^*\big)$, $W$ is open in $\sigma\big(X^{**}, X^*\big)$. Thus $V \cap W$ is an open neighborhood of $\theta$ in this topology. By Goldstine's lemma, there exists $y \in B$ such that $Jy \in V \cap W$. Thus we have $|\langle \zeta, y\rangle - \langle \theta, \zeta\rangle| < \delta/2$ by definition of $V$. We calculate

$$\begin{aligned} 1 - \delta/2 < \langle \theta, \zeta\rangle &= \tfrac{1}{2}\big\{\langle\,\theta, \zeta\,\rangle - \langle\,\zeta, y\rangle\big\} + \tfrac{1}{2}\big\{\langle\,\theta, \zeta\,\rangle - \langle\,\zeta, x\rangle\big\} + \tfrac{1}{2}\langle\,\zeta, x + y\rangle \\ &< \delta/4 + \delta/4 + \|x + y\|/2. \end{aligned}$$

It follows that $\|x + y\|/2 > 1 - \delta$, whence $\|x - y\| \leqslant \varepsilon$ (from uniform convexity). However, $Jy \in W$ yields $\varepsilon < \|Jy - Jx\| = \|y - x\|$, a contradiction which completes the proof. □

There exist reflexive spaces which fail to admit an equivalent norm that is uniformly convex; thus, the existence of such a norm is not a necessary condition for reflexivity. But it is a useful sufficient condition, notably in the study of the Lebesgue spaces introduced in Example 1.9.

**6.4 Theorem.** *If* $1 < p < \infty$, *the Banach space* $L^p(\Omega)$ *is reflexive.*

**Proof.** We treat first[2] the case $2 \leqslant p < \infty$. Then, we claim, $L^p(\Omega)$ is uniformly convex, and therefore reflexive by Theorem 6.3.

[2] We follow Brézis [8, théorème IV.10].

When $p \geqslant 2$, it is easy to show (by examining its derivative) that the function

$$\theta(t) = \big(t^2+1\big)^{p/2} - t^p - 1$$

is increasing on $[0,\infty)$, which implies, by writing $\theta(0) \leqslant \theta(\alpha/\beta)$, the inequality

$$\alpha^p + \beta^p \leqslant \big(\alpha^2+\beta^2\big)^{p/2} \quad \forall\, \alpha, \beta \geqslant 0.$$

Now let $a, b \in \mathbb{R}$ and take $\alpha = |a+b|/2,\ \beta = |a-b|/2$; we find

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \leqslant \left(\left|\frac{a+b}{2}\right|^2 + \left|\frac{a-b}{2}\right|^2\right)^{p/2} = \left(\frac{a^2}{2}+\frac{b^2}{2}\right)^{p/2} \leqslant \frac{a^p}{2}+\frac{b^p}{2}$$

(the last estimate uses the convexity of the function $t \mapsto |t|^{p/2}$, which holds because $p \geqslant 2$). This yields *Clarkson's inequality*:

$$\left\|\frac{f+g}{2}\right\|_{L^p}^p + \left\|\frac{f-g}{2}\right\|_{L^p}^p \leqslant \frac{1}{2}\Big(\|f\|_{L^p}^p + \|g\|_{L^p}^p\Big) \quad \forall f, g \in L^p(\Omega).$$

Fix $\varepsilon > 0$, and suppose that $f, g$ in the unit ball of $L^p(\Omega)$ satisfy $\|f-g\|_{L^p} > \varepsilon$. From Clarkson's inequality we deduce

$$\left\|\frac{f+g}{2}\right\|_{L^p}^p < 1 - \Big(\frac{\varepsilon}{2}\Big)^p \implies \left\|\frac{f+g}{2}\right\|_{L^p} < 1-\delta,$$

where $\delta = 1 - [1-(\varepsilon/2)^p]^{1/p}$. This verifies the uniform convexity, and completes the proof of the case $p \geqslant 2$.

We now prove that $L^p(\Omega)$ is reflexive for $1 < p < 2$. Let $q = p_*$, and consider the operator $T : L^p(\Omega) \to L^q(\Omega)^*$ defined as follows: for $u \in L^p(\Omega)$, the effect of $Tu$ on $L^q(\Omega)$ is given by

$$\langle Tu, g\rangle = \int_\Omega u(x)\, g(x)\, dx \quad \forall g \in L^q(\Omega).$$

Then we have (see Exer. 1.31)

$$\|Tu\|_{L^q(\Omega)^*} = \|u\|_{L^p(\Omega)}.$$

Thus $T$ is an isometry between $L^p(\Omega)$ and a *closed* subspace of $L^q(\Omega)^*$ (since $L^p(\Omega)$ is complete, see Prop. 5.3). Now $q > 2$, so $L^q(\Omega)$ is reflexive (by the case of the theorem proved above); thus, its dual $L^q(\Omega)^*$ is reflexive (Prop. 5.43). Then $T(L^p(\Omega))$, as a closed subspace, is reflexive (Exer. 5.49), and therefore $L^p(\Omega)$ is reflexive as well (Prop. 5.42). □

**6.5 Corollary.** *The spaces $AC^p[a,b]$ are reflexive for $1 < p < \infty$.*

**6.6 Exercise.** Let $\Lambda : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous function having the property that, for almost every $t \in [0,1]$, the function $(x,v) \mapsto \Lambda(t,x,v)$ is convex (see Example 2.30). We suppose in addition that, for certain numbers $r > 1$, $\alpha > 0$ and $\beta$, we have

$$\Lambda(t,x,v) \geqslant \alpha|v|^r + \beta \;\; \forall (t,x,v) \in [0,1]\times\mathbb{R}\times\mathbb{R}.$$

Fix $x_0, x_1 \in \mathbb{R}$, and consider the following minimization problem (P):

$$\min \;\; f(x) = \int_0^1 \Lambda\big(t, x(t), x'(t)\big)\, dt \;\; : x \in \mathrm{AC}[0,1],\, x(0) = x_0,\, x(1) = x_1.$$

Prove that (P) admits a solution. □

**6.7 Exercise.** Let $1 < r < \infty$, and let $x_i$ be a bounded sequence of functions in $\mathrm{AC}^r[a,b]$. Prove the existence of $x_* \in \mathrm{AC}^r[a,b]$ and a subsequence $x_{i_j}$ such that

$$x_{i_j} \to x_* \text{ uniformly on } [a,b], \quad {x_{i_j}}' \to x_*' \text{ weakly in } L^r(a,b). \qquad \square$$

**6.8 Theorem. (Riesz)** *For $1 < p < \infty$, the dual space of $L^p(\Omega)$ is isometric to $L^q(\Omega)$, where $q$ is the conjugate exponent of $p$. More precisely, each $\zeta$ of the dual admits a function $g \in L^q(\Omega)$ (necessarily unique) such that*

$$\langle \zeta, f \rangle = \int_\Omega f(x)\, g(x)\, dx \;\; \forall f \in L^p(\Omega).$$

*We then have $\|\zeta\|_{L^p(\Omega)^*} = \|g\|_{L^q(\Omega)}$.*

**Proof.** Let the linear mapping $T : L^q(\Omega) \to L^p(\Omega)^*$ be defined by

$$\langle Tg, f \rangle = \int_\Omega f(x)\, g(x)\, dx \;\; \forall f \in L^p(\Omega).$$

Then, as we know, $\|Tg\|_{L^p(\Omega)^*} = \|g\|_{L^q(\Omega)}$ (see Exer. 1.31), so that $T$ is injective. We proceed to prove that $T$ is surjective, which implies the theorem.

Since $T(L^q(\Omega))$ is closed (as the image of a Banach space under an isometry, see Prop. 5.3), it suffices to prove that $T(L^q(\Omega))$ is dense in $L^p(\Omega)^*$. To prove this, it suffices in turn to prove that (see Theorem 2.39)

$$\theta \in L^p(\Omega)^{**},\; \langle \theta, Tg \rangle = 0 \;\; \forall g \in L^q(\Omega) \implies \theta = 0.$$

We proceed to establish this now. Since $L^p(\Omega)$ is reflexive, there exists $f \in L^p(\Omega)$ such that $\theta = Jf$. Then

$$\langle \theta, Tg \rangle = 0 = \langle Jf, Tg \rangle = \langle Tg, f \rangle = \int_\Omega f(x)\, g(x)\, dx \;\; \forall g \in L^q(\Omega).$$

We discover $f = 0$, by taking $g = |f|^{p-2} f$ (which lies in $L^q(\Omega)$) in the preceding relation, whence $\theta = 0$. □

**6.9 Exercise.** Characterize the dual of $\mathrm{AC}^p[a,b]$ for $1<p<\infty$. □

We now proceed to characterize the dual of $L^1(\Omega)$; the proof can no longer rely on reflexivity, however.

**6.10 Theorem.** *The dual of $L^1(\Omega)$ is isometric to $L^\infty(\Omega)$. More precisely, $\zeta$ belongs to $L^1(\Omega)^*$ if and only if there exists $z \in L^\infty(\Omega)$ (necessarily unique) such that*

$$\langle \zeta, f \rangle = \int_\Omega z(x) f(x)\, dx \quad \forall f \in L^1(\Omega).$$

*When this holds we have $\|\zeta\|_{L^1(\Omega)^*} = \|z\|_{L^\infty(\Omega)}$.*

**Proof.** That any $z \in L^\infty(\Omega)$ can be used as indicated to engender an element $\zeta$ in the dual of $L^1(\Omega)$ is clear, since

$$\langle \zeta, f \rangle \leqslant \|z\|_{L^\infty(\Omega)} \|f\|_{L^1(\Omega)}.$$

Thus any $\zeta$ defined in this way satisfies $\|\zeta\|_{L^1(\Omega)^*} \leqslant \|z\|_{L^\infty(\Omega)}$. Let us prove the opposite inequality, for which we may limit attention to the case $\|z\|_{L^\infty(\Omega)} > 0$. For any $\varepsilon > 0$, there exists a measurable subset $S \subset \Omega$ of positive finite measure such that

$$|z(x)| \geqslant \|z\|_{L^\infty(\Omega)} - \varepsilon, \ \ x \in S \text{ a.e.}$$

Set $f(x) = z(x)/|z(x)|$ for $x \in S$, and $f = 0$ elsewhere. Then $f \in L^1(\Omega)$, and we find

$$\langle \zeta, f \rangle = \int_\Omega z(x) f(x)\, dx \geqslant \big(\|z\|_{L^\infty(\Omega)} - \varepsilon\big) \operatorname{meas} S = \big(\|z\|_{L^\infty(\Omega)} - \varepsilon\big) \|f\|_{L^1(\Omega)}.$$

It follows that

$$\|\zeta\|_{L^1(\Omega)^*} \geqslant \|z\|_{L^\infty(\Omega)} - \varepsilon.$$

Since $\varepsilon > 0$ is otherwise arbitrary, the assertion concerning $\|\zeta\|_{L^1(\Omega)^*}$ is proved.

There remains to show that every $\zeta \in L^1(\Omega)^*$ is generated by some $z$ as above. We prove this first under the additional hypothesis that $\Omega$ is bounded.

Let $\zeta \in L^1(\Omega)^*$. Any $f \in L^2(\Omega)$ belongs to $L^1(\Omega)$, since $\Omega$ is bounded; by Hölder's inequality we have:

$$\langle \zeta, f \rangle \leqslant \|\zeta\|_{L^1(\Omega)^*} \|f\|_{L^1(\Omega)} \leqslant \|\zeta\|_{L^1(\Omega)^*} \big[\operatorname{meas}(\Omega)\big]^{1/2} \|f\|_{L^2(\Omega)}.$$

It follows that $\zeta$ can be viewed as an element of $L^2(\Omega)^*$. According to Theorem 6.8, there is a unique $z$ in $L^2(\Omega)$ such that (by the preceding inequality)

$$\langle \zeta, f \rangle = \int_\Omega z(x) f(x)\, dx \leqslant \|\zeta\|_{L^1(\Omega)^*} \|f\|_{L^1(\Omega)} \quad \forall f \in L^2(\Omega).$$

Thus, for any $f \in L^\infty(\Omega) \subset L^2(\Omega)$, we have (by rewriting):

$$\int_\Omega \{ \| \zeta \|_{L^1(\Omega)^*} |f(x)| - z(x) f(x) \} dx \geqslant 0.$$

This implies (here we must beg the reader's pardon for a regrettable forward reference: see Theorem 6.32)

$$\| \zeta \|_{L^1(\Omega)^*} |f| - z(x) f \geqslant 0 \;\; \forall f \in \mathbb{R}, \; x \in \Omega \text{ a.e.},$$

which yields $|z(x)| \leqslant \| \zeta \|_{L^1(\Omega)^*}$ a.e. Thus $z$ belongs to $L^\infty(\Omega)$, and satisfies

$$\langle \zeta, f \rangle = \int_\Omega z(x) f(x) dx \;\; \forall f \in L^\infty(\Omega).$$

Given any $f \in L^1(\Omega)$, there is a sequence $f_i \in L^\infty(\Omega)$ such that

$$\| f - f_i \|_{L^1(\Omega)} \to 0.$$

For instance, let $f_i(x) = f(x)$ if $|f(x)| \leqslant i$, and 0 otherwise. We have, by the above

$$\langle \zeta, f_i \rangle = \int_\Omega z(x) f_i(x) dx \;\; \forall i \geqslant 1.$$

Recalling that $\zeta$ is continuous, and passing to the limit, we obtain the same conclusion for $f$; it follows that $z$ represents $\zeta$ on $L^1(\Omega)$, as we wished to show. That $z$ is the *unique* function doing this is left as an exercise.

There remains to treat the case in which $\Omega$ is unbounded. Let $\zeta \in L^1(\Omega)^*$. For any sufficiently large positive integer $k$, the set $\Omega_k := \Omega \cap B^\circ(0,k)$ is nonempty. Then $\zeta$ induces an element of $L^1(\Omega_k)^*$: we simply extend to $\Omega$ any function $f \in L^1(\Omega_k)$ by setting it equal to 0 on $\Omega \setminus \Omega_k$, then apply $\zeta$ to the extension. By the above, there is a function $z_k \in L^\infty(\Omega_k)$ such that

$$\langle \zeta, f \rangle = \int_{\Omega_k} z_k(x) f(x) dx \;\; \forall f \in L^1(\Omega_k), \; \| z_k \|_{L^\infty(\Omega_k)} = \| \zeta \|_{L^1(\Omega_k)^*} \leqslant \| \zeta \|_{L^1(\Omega)^*}.$$

It is clear that each of the functions $z_k$ is necessarily an extension of the preceding ones (by uniqueness), so they define a function $z \in L^\infty(\Omega)$. We claim that this $z$ represents $\zeta$ as required. Let $f$ be any element of $L^1(\Omega)$, and let $f_k$ be the function which agrees with $f$ on $\Omega_k$ and which is zero on $\Omega \setminus \Omega_k$. Then

$$\langle \zeta, f_k \rangle = \int_{\Omega_k} z(x) f_k(x) dx = \int_\Omega z(x) f_k(x) dx.$$

But $f_k \to f$ in $L^1(\Omega)$, so in the limit we obtain

$$\langle \zeta, f \rangle = \int_\Omega z(x) f(x) dx.$$

□

**6.11 Exercise.** Let $f_i$ be a sequence in $L^\infty(0,1)$ such that, for each $g \in L^1(0,1)$, we have

$$\inf_{i \geqslant 1} \int_0^1 f_i(t) g(t)\, dt > -\infty .$$

Prove the existence of $M$ such that $\| f_i \|_{L^\infty(0,1)} \leqslant M \ \forall i$. □

**6.12 Exercise.** Let $\theta$ belong to $L^\infty(0,1)$. Prove the existence of a solution to the following minimization problem:

$$\min_{v \in L^1(0,1)} \int_0^1 e^{[(v(t)-1)^2]}\, dt \quad \text{subject to} \quad \int_0^1 \theta(t) v(t)\, dt = 0 .$$

□

**6.13 Proposition.** *The spaces $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive.*

**Proof.** For ease of exposition, as they say, let us suppose that $\Omega$ contains a ball $B(0,r)$. We define a function $z$ in $L^\infty(\Omega)$ as follows:

$$z(x) = \begin{cases} 1 - 2^{-n} & \text{if } 2^{-n-1} r \leqslant |x| < 2^{-n} r, \ n = 0,1,2\ldots \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\| z \|_{L^\infty(\Omega)} = 1$. If $f \neq 0$ is any nonnegative function in $L^1(\Omega)$, then

$$\int_\Omega f(x) z(x)\, dx < \int_\Omega f(x)\, dx = \| f \|_{L^1(\Omega)} .$$

If we denote by $\zeta$ the element of $L^1(\Omega)^*$ corresponding to $z$ as in Theorem 6.10, it follows that the supremum

$$\sup \big\{ \langle \zeta, f \rangle : \| f \|_{L^1(\Omega)} \leqslant 1 \big\} = \| \zeta \|_{L^1(\Omega)^*}$$

is *not* attained. But if the unit ball in $L^1(\Omega)$ were weakly compact, this supremum would be attained. We deduce from Theorem 5.47 that $L^1(\Omega)$ is not reflexive. It then follows from Theorem 6.10 and Prop. 5.42 that $L^\infty(\Omega)$ is not reflexive. □

We examine next the separability or otherwise of the Lebesgue spaces.

**6.14 Proposition.** *$L^p(\Omega)$ is separable for $1 \leqslant p < \infty$.*

**Proof.** We sketch the proof in the case $p = 1$, the other cases being much the same. We also take $\Omega$ bounded, a reduction that is easy to justify. Let $f \in L^1(\Omega)$, and let $f_i$ be the function which coincides with $f$ when $|f| \leqslant i$, and which equals 0 otherwise. Then $f_i$ is measurable, and it follows that $f_i \to f$ in $L^1(\Omega)$. By Lusin's theorem[3] there exists a continuous function $g_i$ on $\Omega$ having compact support, which is bounded in absolute value by $i$, and which agrees with $f_i$ except on a set $S_i$ of measure less than $1/i^2$. Then

$$\int_\Omega |f_i - g_i|\,dx = \int_{\Omega\setminus S_i} |f_i - g_i|\,dx + \int_{S_i} |f_i - g_i|\,dx \leqslant (2i)/i^2 \to 0.$$

Since $f_i \to f$, we deduce that $C(\overline{\Omega})$ is dense in $L^1(\Omega)$. However, the set of polynomials with rational coefficients is dense in $C(\overline{\Omega})$, by the Weierstrass approximation theorem, whence the separability of $L^1(\Omega)$. □

The proof shows that $C_c(\Omega)$, the continuous functions on $\Omega$ having compact support in $\Omega$, is dense in $L^1(\Omega)$. It can be shown that $C_c^\infty(\Omega)$ has the same property.

**6.15 Exercise.** Prove that $L^\infty(\Omega)$ is not separable. □

**Weak compactness without reflexivity.** Certain useful compactness properties do hold in $L^1(\Omega)$ and $L^\infty(\Omega)$, despite the fact that these spaces fail to be reflexive. We identify two such cases below, in each of which the separability of $L^1(\Omega)$ plays a role.

**6.16 Exercise.** Let $f_i$ be a bounded sequence in $L^\infty(\Omega)$. Prove the existence of a subsequence $f_{i_j}$ and $f \in L^\infty(\Omega)$ such that

$$g \in L^1(\Omega) \implies \int_\Omega g(x)\,f_{i_j}(x)\,dx \to \int_\Omega g(x)\,f(x)\,dx.$$

□

In applications to come, the reader will find that it is common to deal with a sequence of functions $f_i$ in $L^1(0,T)$ satisfying a uniform bound of the type $|f_i(t)| \leqslant k(t)$ a.e., where $k$ is summable. The following establishes a sequential compactness result that applies to such a situation.

**6.17 Proposition.** *Let $k(\cdot) \in L^1(\Omega)$, where $\Omega$ is an open subset of $\mathbb{R}^n$. Then the set*

$$K = \{ f \in L^1(\Omega) : |f(x)| \leqslant k(x),\ x \in \Omega \text{ a.e.} \}$$

*is weakly compact and sequentially weakly compact in $L^1(\Omega)$.*

**Proof.** Let us set

$$X = L^\infty(\Omega) \text{ equipped with the weak}^* \text{ topology } \sigma\big(L^\infty(\Omega), L^1(\Omega)\big),$$
$$Y = L^1(\Omega) \text{ equipped with the weak topology } \sigma\big(L^1(\Omega), L^\infty(\Omega)\big).$$

[3] Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, and let $\varphi : \Omega \to \mathbb{R}$ be measurable, $|\varphi(x)| \leqslant M$ a.e. For every $\varepsilon > 0$ there exists $g : \Omega \to \mathbb{R}$, continuous with compact support, having $\sup_\Omega |g| \leqslant M$, such that meas$\{x \in \Omega : \varphi(x) \neq g(x)\} < \varepsilon$. See Rudin [37, p. 53].

We define a linear functional $\Lambda : X \to Y$ by $\Lambda g = kg$. We claim that $\Lambda$ is continuous. By Theorem 3.1, we need only show that, for any $h \in L^\infty(\Omega)$, the map

$$f \mapsto \int_\Omega h(x)\,k(x)\,f(x)\,dx$$

is continuous on $X$. This follows from the fact that $hk \in L^1(\Omega)$, so that the map in question is an evaluation of the type that is rendered continuous by the topology $\sigma(L^\infty(\Omega), L^1(\Omega))$.

Then $K$ is the image under the continuous map $\Lambda$ of the unit ball in $L^\infty(\Omega)$, which is compact in $X$ by Theorem 3.15. Thus $K$ is compact in $Y$.

Now let $f_i$ be a sequence in $K$; then $f_i = kg_i$, where $g_i$ lies in the unit ball of $L^\infty(\Omega)$. (One may take $g_i(x) = f_i(x)/k(x)$ when $k(x) \neq 0$, and $g_i(x) = 0$ otherwise.) Because $L^1(\Omega)$ is separable, the weak* topology on the ball is metrizable (Theorem 3.21). Thus, a subsequence $g_{i_j}$ converges weak* to a limit $g$ in $L^\infty(\Omega)$. This means that

$$\int_\Omega g_{i_j}(x)\,h(x)\,dx \to \int_\Omega g(x)\,h(x)\,dx \quad \forall\, h \in L^1(\Omega).$$

It follows that

$$\int_\Omega g_{i_j}(x)\,k(x)\,u(x)\,dx \to \int_\Omega g(x)\,k(x)\,u(x)\,dx \quad \forall\, u \in L^\infty(\Omega),$$

which implies that $f_{i_j} := kg_{i_j}$ converges weakly in $L^1(\Omega)$ to $gk$. □

**6.18 Exercise.** For each $x \in \Omega$, let $F(x)$ be a closed convex subset of $\mathbb{R}$ satisfying $|F(x)| \leqslant k(x)$. Prove that the set

$$\Phi = \big\{ f \in L^1(\Omega) : f(x) \in F(x),\ x \in \Omega \text{ a.e.} \big\}$$

is sequentially weakly compact in $L^1(\Omega)$. □

**6.19 Exercise.** A *sawtooth function* $x$ on $[0,1]$ is a Lipschitz, piecewise affine function $x : [0,1] \to \mathbb{R}$ with $x(0) = x(1) = 0$ such that $|x'(t)| = 1$ a.e. Let $x_i$ be a sequence of such functions satisfying

$$\|x_i\|_{C[0,1]} \leqslant 1/i,$$

and set $v_i = x_i'$. Prove that $v_i$ converges weakly in $L^1(0,1)$ to $0$. Deduce that the set

$$\big\{ f \in L^1(0,1) : f(x) \in \{-1, 1\} \text{ a.e.} \big\}$$

is not weakly compact. □

## 6.2 Measurable multifunctions

Let $\Omega$ be a subset of $\mathbb{R}^m$. A *multifunction* $\Gamma$ from $\Omega$ to $\mathbb{R}^n$ is a mapping from $\Omega$ to the subsets of $\mathbb{R}^n$; thus, we associate with each $x \in \Omega$ a set $\Gamma(x)$ in $\mathbb{R}^n$, possibly the empty set. Such mappings arise rather frequently later on, and a recurrent issue will be that of finding a *measurable selection* of $\Gamma$. This means a measurable function $\gamma : \Omega \to \mathbb{R}^n$ such that $\gamma(x)$ belongs to $\Gamma(x)$ for almost all $x \in \Omega$.

Consider the following simple example, in which $n = m$. Let $U$ be an open convex subset of $\mathbb{R}^n$, and $f : U \to \mathbb{R}$ a convex function. We have learned that the subdifferential $\Gamma(x) := \partial f(x)$ is nonempty for each $x \in U$. It follows from the axiom of choice that there is a function $\zeta : U \to \mathbb{R}^n$ such that $\zeta(x) \in \partial f(x) \;\; \forall x \in U$. Is there, however, a *measurable* function having this property?

Answering a question such as this requires a theory. We develop one in this section, in the context of Euclidean spaces.

**Notation.** We write $\Gamma : \Omega \rightsquigarrow \mathbb{R}^n$ to denote a multifunction $\Gamma$ that maps a subset $\Omega$ of $\mathbb{R}^m$ to the subsets of $\mathbb{R}^n$.

One of the major ingredients in the theory is the following extension to multifunctions of the concept of measurable function.

**Measurable multifunctions.** The multifunction $\Gamma : \Omega \rightsquigarrow \mathbb{R}^n$ is *measurable* provided that $\Omega$ is measurable, and provided that the set

$$\Gamma^{-1}(V) = \big\{ x \in \Omega : \Gamma(x) \cap V \neq \emptyset \big\}$$

is (Lebesgue) measurable for every closed subset $V$ of $\mathbb{R}^n$.

We obtain an equivalent definition by taking compact sets $V$ in the definition. To see this, observe that any closed set $V$ is the union of countably many compact sets $V_i$. Then we have

$$\Gamma^{-1}(V) = \bigcup\nolimits_{i \geqslant 1} \Gamma^{-1}(V_i)\,.$$

If each $\Gamma^{-1}(V_i)$ is measurable, then so is $\Gamma^{-1}(V)$, as the countable union of measurable sets. The reader may show by a somewhat similar argument that when $\Gamma$ is measurable, then the set $\Gamma^{-1}(V)$ is measurable for every *open* set $V$ (this property, however, does not characterize measurability).

**6.20 Exercise.** Suppose that $\Gamma$ is a singleton $\{\gamma(x)\}$ for each $x$. Prove that the multifunction $\Gamma$ is measurable if and only if the function $\gamma$ is measurable. □

The *effective domain* dom $\Gamma$ of the multifunction $\Gamma : \Omega \rightsquigarrow \mathbb{R}^n$ is defined as follows:

$$\operatorname{dom} \Gamma = \big\{ x \in \Omega : \Gamma(x) \neq \emptyset \big\}.$$

By taking $V = \mathbb{R}^n$ in the definition of measurability, it follows that the effective domain of a measurable multifunction is measurable. We remark that as in the case of a function, redefining $\Gamma$ on a set of measure zero does not affect its measurability, so in discussing measurable multifunctions we deal implicitly with equivalence classes, as we do with Lebesgue spaces.

**6.21 Exercise.** Let $u : \mathbb{R}^m \to \mathbb{R}^n$ and $r : \mathbb{R}^m \to \mathbb{R}_+$ be measurable functions, and let $W$ be a measurable subset of $\mathbb{R}^n$. Prove that the multifunction $\Gamma$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ defined by $\Gamma(x) = W + B(u(x), r(x))$ is measurable. □

It is not hard to show that if $\gamma_i$ is a sequence of measurable functions, then the multifunction $\Gamma(x) = \{\, \gamma_i(x) : i \geqslant 1 \,\}$ is measurable. The following shows that all *closed-valued* measurable multifunctions are in fact generated this way. ($\Gamma$ is said to be closed-valued, of course, when $\Gamma(x)$ is a closed set for each $x \in \Omega$.)

**6.22 Theorem.** *Let $\Gamma : \Omega \rightsquigarrow \mathbb{R}^n$ be closed-valued and measurable. Then there exists a countable family $\{\, \gamma_i : \operatorname{dom} \Gamma \to \mathbb{R}^n \,\}$ of measurable functions such that*

$$\Gamma(x) = \operatorname{cl}\{\, \gamma_i(x) : i \geqslant 1 \,\}, \;\; x \in \operatorname{dom} \Gamma \text{ a.e.}$$

**Proof.** Let $\Delta = \operatorname{dom} \Gamma$. We begin by noting that, for any $u$ in $\mathbb{R}^n$, the function $s \to d_{\Gamma(s)}(u)$ restricted to $\Delta$ is measurable (where $d_{\Gamma(s)}$ is as usual the Euclidean distance function). This follows from the identity (for $0 \leqslant r < R$)

$$d_{\Gamma(\cdot)}(u)^{-1}(r, R) = \{ s \in \Delta : \Gamma(s) \cap B(u, r) = \emptyset \} \cap \Gamma^{-1}(B^{\circ}(u, R)).$$

Now let $\{\, u_j \,\}_{j \geqslant 1}$ be a dense sequence in $\mathbb{R}^n$, and define a function $f_0 \colon \Delta \to \mathbb{R}^n$ as follows:

$$f_0(s) = \text{ the first } u_j \text{ such that } d_{\Gamma(s)}(u_j) \leqslant 1.$$

**Lemma.** *The functions $s \to f_0(s)$ and $s \to d_{\Gamma(s)}(f_0(s))$ are measurable on $\Delta$.*

To see this, observe that $f_0$ assumes countably many values, and that we have, for each $i \geqslant 1$:

$$\{\, s \colon f_0(s) = u_i \,\} = \bigcap_{j=1}^{i-1} \{\, s \colon d_{\Gamma(s)}(u_j) > 1 \,\} \bigcap \{\, s \colon d_{\Gamma(s)}(u_i) \leqslant 1 \,\}.$$

This implies that $f_0$ is measurable. Since the function $(s, u) \mapsto d_{\Gamma(s)}(u)$ is measurable in $s$ and continuous in $u$, it is known (and actually proved in the next section, in the midst of more general goings on) that the function $s \to d_{\Gamma(s)}(f_0(s))$ is measurable. The lemma is proved.

We pursue the process begun above by defining for each integer $i \geqslant 0$ a function $f_{i+1}$ such that $f_{i+1}(s)$ is the first $u_j$ for which *both* the following hold:

$$|u_j - f_i(s)| \leqslant \tfrac{2}{3}\, d_{\Gamma(s)}\big(f_i(s)\big), \quad d_{\Gamma(s)}(u_j) \leqslant \tfrac{2}{3}\, d_{\Gamma(s)}\big(f_i(s)\big).$$

It follows as above that each $f_i$ is measurable. Moreover, we deduce

$$d_{\Gamma(s)}\big(f_{i+1}(s)\big) \leqslant \big(\tfrac{2}{3}\big)^{i+1} d_{\Gamma(s)}\big(f_0(s)\big) \leqslant \big(\tfrac{2}{3}\big)^{i+1},$$

together with $|f_{i+1}(s) - f_i(s)| \leqslant (2/3)^{i+1}$. This implies that $\{f_i(s)\}$ is a Cauchy sequence converging for each $s \in \Delta$ to a value which we denote by $\gamma_0(s)$, and that $\gamma_0(x) \in \Gamma(x)$ a.e. in $\Delta$. As a limit of measurable functions, $\gamma_0$ is measurable.

For every pair of positive integers $i, j$, we define a multifunction $\Gamma_{i,j} : \Omega \rightsquigarrow \mathbb{R}^n$ as follows:

$$\Gamma_{i,j}(x) = \begin{cases} \emptyset & \text{if } x \notin \Delta \\ \Gamma(x) \cap B(u_i, 1/j) & \text{if } x \in \Delta \text{ and } \Gamma(x) \cap B(u_i, 1/j) \neq \emptyset \\ \{\gamma_0(x)\} & \text{otherwise.} \end{cases}$$

For any closed subset $V$ of $\mathbb{R}^n$, the set $\Gamma_{i,j}^{-1}(V)$ is given by

$$\big\{x : \Gamma(x) \cap V \cap B(u_i, 1/j) \neq \emptyset\big\} \bigcup \Big[\big\{x \in \Delta : \Gamma(x) \cap B(u_i, 1/j) = \emptyset\big\} \cap \gamma_0^{-1}(V)\Big].$$

It follows that $\Gamma_{i,j}$ is measurable and closed-valued; its effective domain is $\Delta$. By the argument above (applied to $\Gamma_{i,j}$ rather than $\Gamma$), there exists a measurable function $\gamma_{i,j}$ such that $\gamma_{i,j}(x) \in \Gamma_{i,j}(x)$, $x \in \Delta$ a.e.

We claim that the countable collection $\gamma_{i,j}$, together with $\gamma_0$, satisfies the conclusion of the theorem.

To see this, let $S_{i,j}$ be the null set of $x \in \Delta$ for which the inclusion $\gamma_{i,j}(x) \in \Gamma(x)$ fails. Now let $x \in \Delta \backslash [\cup_{i,j} S_{i,j}]$, and fix any $\gamma \in \Gamma(x)$, $\gamma \neq \gamma_0(x)$. There exists a sequence $u_{i_k}$ in $\{u_i\}$ and an increasing sequence of integers $j_k \to \infty$ such that $|u_{i_k} - \gamma| < 1/j_k$. Then we have

$$\gamma_{i_k, j_k}(x) \in B(u_{i_k}, 1/j_k) \implies |\gamma_{i_k, j_k}(x) - u_{i_k}| < 1/j_k \implies |\gamma_{i_k, j_k}(x) - \gamma| < 2/j_k.$$

Thus, $\Gamma(x) = \operatorname{cl}\{\gamma_{i,j}(x)\}$, $x \in \Delta$ a.e. □

**6.23 Corollary. (Measurable selections)** *Let $\Gamma : \Omega \rightsquigarrow \mathbb{R}^n$ be closed-valued and measurable. Then there exists a measurable function $\gamma : \operatorname{dom} \Gamma \to \mathbb{R}^n$ such that*

$$\gamma(x) \in \Gamma(x), \quad x \in \operatorname{dom} \Gamma \text{ a.e.}$$

**6.24 Exercise.** Let $\Gamma : \Omega \rightsquigarrow \mathbb{R}^n$ and $G : \Omega \rightsquigarrow \mathbb{R}^n$ be two measurable closed-valued multifunctions. Prove that $\Gamma + G$ is measurable. □

The measurable multifunctions that the reader is likely to encounter will most often have the following structure.

**6.25 Proposition.** *Let $\Omega$ be a measurable subset of $\mathbb{R}^m$, and $\varphi : \Omega \times \mathbb{R}^n \times \mathbb{R}^\ell \to \mathbb{R}$ a function with the following properties:*

- *The mapping $x \mapsto \varphi(x,p,q)$ is measurable on $\Omega$ for each $(p,q) \in \mathbb{R}^n \times \mathbb{R}^\ell$, and*
- *The mapping $(p,q) \mapsto \varphi(x,p,q)$ is continuous for each $x \in \Omega$.*

*Let $P, Q : \Omega \rightsquigarrow \mathbb{R}^n$ be measurable closed-valued multifunctions, and $c,d : \Omega \to \mathbb{R}$ measurable functions. Then $\Gamma : \Omega \rightsquigarrow \mathbb{R}^n$ defined by*

$$\Gamma(x) = \big\{ p \in P(x) : c(x) \leqslant \varphi(x,p,q) \leqslant d(x) \text{ for some } q \in Q(x) \big\}$$

*is measurable.*

**Proof.** Let $p_i$ be a countable family of measurable selections of $P$ that generate the multifunction $P$ as described in Theorem 6.22, and similarly, let $q_i$ generate $Q$. Let $\Delta_P$ and $\Delta_Q$ be the effective domains of $P$ and $Q$.

Then, if $V$ is a compact subset of $\mathbb{R}^n$, it follows (fairly easily, though we beg the reader's indulgence for the next expression) that

$$\Gamma^{-1}(V) = \bigcup\nolimits_{i \geqslant 1} \bigcap\nolimits_{j \geqslant 1} \bigcup\nolimits_{k \geqslant 1} \Big\{ x \in \Delta_P \cap \Delta_Q :$$
$$p_k(x) \in \big(V + j^{-1}B\big),\ |q_k(x)| \leqslant i,\ c(x) - j^{-1} < \varphi\big(x, p_k(x), q_k(x)\big) < d(x) + j^{-1} \Big\}.$$

This is recognized to be a measurable set, since the function

$$x \mapsto \varphi(x, p_k(x), q_k(x))$$

is measurable (a known result on measurable functions, see Props. 6.34 and 6.35 below). □

**6.26 Corollary.** *The intersection of two closed-valued measurable multifunctions $\Gamma_1, \Gamma_2 : \Omega \rightsquigarrow \mathbb{R}^n$ is measurable.*

**Proof.** Let $\Delta_1$ and $\Delta_2$ be the effective domains of $\Gamma_1$ and $\Gamma_2$. Define a function $\varphi$ on $\Omega \times \mathbb{R}^n$ by

$$\varphi(x,p) = \begin{cases} d_{\Gamma_1(x)}(p) + d_{\Gamma_2(x)}(p) & \text{if } x \in \Delta_1 \cap \Delta_2, \\ -1 & \text{otherwise.} \end{cases}$$

The proof of Theorem 6.22 showed that $\varphi$ is measurable in $x$; it is evidently continuous in $p$. Then the multifunction

$$\Gamma_1(x) \cap \Gamma_2(x) = \big\{ p : \varphi(x,p) = 0 \big\}$$

is measurable by Prop. 6.25. □

The **graph** of a multifunction $\Gamma : \Omega \rightsquigarrow \mathbb{R}^n$ is the set

$$\operatorname{gr}\Gamma = \big\{ (x,\gamma) \in \Omega \times \mathbb{R}^n : \gamma \in \Gamma(x) \big\}.$$

**6.27 Corollary.** *Let $\Omega \subset \mathbb{R}^m$ be measurable. If $\Gamma : \Omega \rightsquigarrow \mathbb{R}^n$ has the property that $\operatorname{gr}\Gamma$ is closed, then $\Gamma$ is measurable.*

**Proof.** We may assume that $\operatorname{gr}\Gamma \neq \emptyset$; then the function $(x,v) \mapsto d_{\operatorname{gr}\Gamma}(x,v)$ is continuous. For any $x \in \Omega$, the set $\Gamma(x)$ is given by $\{ v \in \mathbb{R}^n : d_{\operatorname{gr}\Gamma}(x,v) = 0 \}$, which leads to the required conclusion with the help of Prop. 6.25. □

**6.28 Corollary.** *Let $G : \Omega \rightsquigarrow \mathbb{R}^n$ be measurable and closed-valued. Then the multifunction $\Gamma$ defined by $\Gamma(x) = \operatorname{co} G(x)$ is measurable.*

**Proof.** Let $\Sigma$ denote the set of all nonnegative vectors $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_n) \in \mathbb{R}^{n+1}$ whose coordinates sum to 1. It is not hard to see that the multifunction

$$Q(x) = \Sigma \times G(x) \times G(x) \times \cdots \times G(x)$$

is measurable (where the Cartesian product contains $n+1$ factors equal to $G(x)$). Let $f$ be defined by

$$f(\lambda, g_0, g_1, \dots, g_n) = \textstyle\sum_{i=0}^{n} \lambda_i g_i,$$

where each $g_i$ lies in $\mathbb{R}^n$. Then, by Prop. 2.6, the set $\Gamma(x)$ is described by

$$\big\{ v \in \mathbb{R}^n : |v - f(\lambda, g_0, g_1, \dots, g_n)| = 0 \text{ for some } (\lambda, g_0, g_1, \dots, g_n) \in Q(x) \big\}.$$

The result now follows from Prop. 6.25. □

**6.29 Proposition.** *Let $\Omega \subset \mathbb{R}^m$ be measurable, and let $G : \Omega \rightsquigarrow \mathbb{R}^n$ be a multifunction whose values are nonempty compact convex sets. Let $H_{G(x)}(\cdot)$ be the support function of the set $G(x)$. Then $G$ is measurable if and only if, for any $p \in \mathbb{R}^n$, the function $x \mapsto H_{G(x)}(p)$ is measurable on $\Omega$.*

**Proof.** Suppose first that the support function has the stated measurability property. Let $V$ be a nonempty compact subset of $\mathbb{R}^n$, and let $\{ v_j \}$ be a countable dense set in $V$. Let $\{ p_k \}$ be a countable dense set in $\mathbb{R}^n$. Then (invoking the separation theorem for the last step) it follows that

$$\begin{aligned} \big\{ x \in \Omega : G(x) \cap V = \emptyset \big\} &= \textstyle\bigcup_{\varepsilon > 0} \big\{ x \in \Omega : G(x) \cap [V + \varepsilon B] = \emptyset \big\} \\ &= \textstyle\bigcup_{i \geqslant 1} \bigcap_{j \geqslant 1} \big\{ x \in \Omega : G(x) \cap B(v_j, i^{-1}) = \emptyset \big\} = \\ &\textstyle\bigcup_{i \geqslant 1} \bigcap_{j \geqslant 1} \bigcup_{k \geqslant 1} \big\{ x \in \Omega : H_{G(x)}(p_k) < \langle p_k, v_j \rangle + i^{-1} |p_k| \big\}. \end{aligned}$$

This implies that $\big\{ x \in \Omega : G(x) \cap V = \emptyset \big\}$ is measurable, so $G$ is measurable.

Conversely, let $G$ be measurable, and let the functions $\gamma_i$ generate $G$ as in Theorem 6.22. Then we have

$$H_{G(x)}(p) = \sup \{ \langle p, \gamma_i(x) \rangle : i \geqslant 1 \} \ \forall x \in \Omega,$$

which reveals the required measurability in $x$ of the function on the left. □

**6.30 Exercise.** Let $G : \Omega \rightsquigarrow \mathbb{R}^n$ be measurable and closed-valued. If $u : \mathbb{R}^m \to \mathbb{R}^n$ is measurable, prove that the function $x \mapsto d_{G(x)}(u(x))$ is measurable on dom $G$. □

The following fact, already invoked in proving Theorem 6.10, will be useful again later. It bears upon interchanging the integral and the supremum.

**6.31 Theorem.** *Let $\Omega$ be an open subset of $\mathbb{R}^m$, and let $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a function such that $\varphi(x,p)$ is measurable in the $x$ variable and continuous in the $p$ variable. Let $P : \Omega \rightsquigarrow \mathbb{R}^n$ be measurable and closed-valued. Let $\Sigma$ denote the set of all functions $p \in L^\infty(\Omega, \mathbb{R}^n)$ which satisfy*

$$p(x) \in P(x), \ x \in \Omega \text{ a.e.},$$

*and for which the integral*

$$\int_\Omega \varphi\big(x, p(x)\big)\, dx$$

*is well defined, either finitely or as $+\infty$. Then, if $\Sigma$ is nonempty, the integral*

$$\int_\Omega \sup_{p \in P(x)} \varphi(x,p)\, dx$$

*is well defined, either finitely or as $+\infty$, and we have*

$$\int_\Omega \sup_{p \in P(x)} \varphi(x,p)\, dx = \sup_{p(\cdot) \in \Sigma} \int_\Omega \varphi\big(x, p(x)\big)\, dx,$$

*where both sides may equal $+\infty$.*

**Proof.** The hypotheses imply that $P(x) \neq \emptyset$ for $x \in \Omega$ a.e. By Theorem 6.22, there exists a countable collection $\{ p_i \}$ of measurable selections of $P$ such that

$$P(x) = \mathrm{cl}\{ p_i(x) \}, \ x \in \Omega \text{ a.e.}$$

Since $\varphi(x, \cdot)$ is continuous, we have

$$\sigma(x) := \sup_{p \in P(x)} \varphi(x,p) = \sup_{i \geqslant 1} \varphi\big(x, p_i(x)\big) \text{ a.e.},$$

which shows that $\sigma : \Omega \to \mathbb{R}_\infty$ is measurable, as a countable supremum of measurable functions.

Let $\bar{p}$ be any element of $\Sigma$. Then $\sigma(x)$ is bounded below by the function $\varphi(x, \bar{p}(x))$, and it follows that the integral of $\sigma$ is well defined, possibly as $+\infty$; this is the first assertion of the theorem.[4]

If the integral over $\Omega$ of the function $\varphi(x, \bar{p}(x))$ is $+\infty$, then the remaining assertion is evident. We may proceed under the assumption, therefore, that the integral in question is finite. Fix a positive integer $N$, and define

$$\sigma_N(x) = \sup\left\{ \varphi(x,p) : p \in P(x) \cap B(\bar{p}(x), N) \right\}.$$

Using Cor. 6.26, and arguing as above, we find that $\sigma_N$ is measurable. Evidently we have $\varphi(x, \bar{p}(x)) \leqslant \sigma_N(x) \leqslant \sigma(x)$, $x \in \Omega$ a.e. The multifunction

$$\Gamma(x) = \left\{ p \in P(x) \cap B(\bar{p}(x), N) : \sigma_N(x) = \varphi(x,p) \right\}$$

is measurable by Prop. 6.25; since its values on $\Omega$ are closed and nonempty, it admits a measurable selection $p_N$. It follows that $p_N \in \Sigma$, whence

$$\sup_{p(\cdot) \in \Sigma} \int_\Omega \varphi\big(x, p(x)\big)\, dx \geqslant \int_\Omega \varphi\big(x, p_N(x)\big)\, dx \to \int_\Omega \sigma(x)\, dx,$$

by monotone convergence. But the supremum on the left in this expression is evidently bounded above by the integral on the right (and neither depend on $N$). Thus we obtain equality. □

The point of the next result is that a local minimum in $L^1(\Omega)$ translates into a global minimum (almost everywhere) at the pointwise level.

**6.32 Theorem.** *Let $\Omega$, $\varphi$, $P$, and $\Sigma$ be as described in* Theorem 6.31, *and let $\bar{p} \in \Sigma$ be such that the integral*

$$\int_\Omega \varphi(x, \bar{p}(x))\, dx$$

*is finite. Suppose that for some $\delta > 0$, we have:*

$$p(\cdot) \in \Sigma, \int_\Omega |p(x) - \bar{p}(x)|\, dx \leqslant \delta \implies \int_\Omega \varphi\big(x, p(x)\big)\, dx \geqslant \int_\Omega \varphi\big(x, \bar{p}(x)\big).$$

*Then, for almost every $x \in \Omega$, we have $\varphi(x,p) \geqslant \varphi(x, \bar{p}(x))\ \forall\, p \in P(x)$.*

**Proof.** We reason by the absurd. If the conclusion fails, there exist positive numbers $\varepsilon$ and $M$ such that the multifunction

$$\Gamma(x) = \left\{ p \in P(x) \cap B(\bar{p}(x), M) : \varphi\big(x, \bar{p}(x)\big) - M \leqslant \varphi(x,p) \leqslant \varphi\big(x, \bar{p}(x)\big) - \varepsilon \right\}$$

[4] We have used the following fact from integration: if two measurable functions $f$ and $g$ satisfy $f \geqslant g$, and if the integral of $g$ is well defined, either finitely or as $+\infty$, then the integral of $f$ is well defined, either finitely or as $+\infty$.

has an effective domain of positive measure. Then, for any $m > 0$ sufficiently small, we may use a measurable selection $\gamma$ of $\Gamma$ to define a function $p$ as follows: let $S_m$ be a measurable subset of dom $\Gamma$ satisfying meas $(S_m) = m$, and set $p(x) = \gamma(x)$ if $x \in S_m$, and $p(x) = \bar{p}(x)$ otherwise. It follows that $p \in \Sigma$. But for $m$ sufficiently small, $p$ satisfies

$$\int_\Omega |p(x) - \bar{p}(x)|\, dx \leqslant \delta, \quad \int_\Omega \varphi\big(x, p(x)\big)\, dx < \int_\Omega \varphi\big(x, \bar{p}(x)\big),$$

which is the desired contradiction. □

## 6.3 Integral functionals and semicontinuity

A technical issue of some importance to us later concerns the measurability of certain composite functions $f$ arising in the following way:

$$f(t) = \Lambda\big(t, x(t), x'(t)\big).$$

Here, $x$ is an element of AC$[0,1]$ (say), so that $x'(\cdot)$ is merely Lebesgue measurable. When the function $\Lambda(t,x,v)$ is continuous (in all its variables $(t,x,v)$), then, as the reader will recall, it is a basic result in measurability theory that $f$ is Lebesgue measurable (a continuous function of a measurable function is measurable). This is a minimal requirement for considering the integral of $f$, as we do later in the calculus of variations. When $\Lambda$ is less than continuous, the issue is more complex.

To illustrate this point, let $S$ be a non measurable subset of $[0,1]$, and define a subset $G$ of $\mathbb{R}^2$ as follows:

$$G = \big\{(s,s) : s \in S\big\}.$$

Since $G$ is a subset of the diagonal in $\mathbb{R}^2$, a set of measure zero, it follows that $G$ is a null set for two-dimensional Lebesgue measure, which is complete. Thus $G$ is a measurable set and its characteristic function $\chi_G$ is measurable.

Let us define $\Lambda(t,x,v) = \Lambda(t,x) = 1 - \chi_G(t,x)$, a measurable function. The reader may verify that $\Lambda(t,x)$ is lower semicontinuous separately in each variable; in particular, measurable as a function of $t$ for each $x$, and lower semicontinuous as a function of $x$ for each $t$. When we proceed to substitute the function $x(t) = t$ into $\Lambda$, we obtain

$$f(t) = \Lambda(t,t) = 1 - \chi_G(t,t).$$

This function $f$ *fails* to be measurable, however, since

$$\big\{t \in [0,1] : f(t) < 1/2\big\} = \big\{t \in [0,1] : \chi_G(t,t) = 1\big\} = S$$

is not a measurable set.

**LB measurability.** We shall prove below that when $\Lambda$ is measurable in $t$ and continuous in $(x,v)$, then the composition $f$ is measurable, as desired. The example above demonstrates, however, that when $\Lambda$ fails to be continuous in $(x,v)$, as it will later on occasion, mere measurability in $(x,v)$, or even lower semicontinuity (which is a natural hypothesis in the contexts to come), does not suffice to guarantee the measurability of $f$.

One could compensate for the lack of continuity by simple requiring that $\Lambda$, as a function of $(t,x,v)$, be Borel measurable. This is because of the fact that the composition of a Borel measurable function with a measurable one is measurable. Since lower semicontinuous functions are Borel measurable, it follows, as a special case, that our measurability concerns would disappear if we took $\Lambda$ to be lower semicontinuous in the entirety of its variables $(t,x,v)$. This is overly restrictive as a global hypothesis, however, and even Borel measurability is asking too much, since mere Lebesgue measurability in $t$ is desirable in certain applications.

A more common way to deal with the measurability issue is to employ a hybrid hypothesis of the following type:

**6.33 Definition.** *A function $F:\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}$ of two variables $(x,y)$, where $x\in\mathbb{R}^m$ and $y\in\mathbb{R}^n$, is said to be* LB *measurable in $x$ and $y$ when it is the case that $F$ is measurable with respect to the $\sigma$-algebra $L\times B$ generated by products of Lebesgue measurable subsets of $\mathbb{R}^m$ (for $x$) and Borel measurable subsets of $\mathbb{R}^n$ (for $y$).*

Can the reader can go so far back as to remember that a $\sigma$-algebra is a family of sets closed under taking complements, countable intersections, and countable unions? We remark that if $F:\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}$ is lower semicontinuous, then $F$ is Borel measurable, which implies that $F$ is LB measurable.

Returning to the context of the function $\Lambda$, we say that $\Lambda(t,x,v)$ is LB measurable if $\Lambda$ is LB measurable in the variables $t$ and $(x,v)$; that is, measurable with respect to the $\sigma$-algebra generated by products of Lebesgue measurable subsets of $[a,b]$ (for $t$) and Borel measurable subsets of $\mathbb{R}^2$ (for $(x,v)$). This property guarantees the measurability of the function $f$ above, and is satisfied in the important case in which $\Lambda(t,x,v)$ is measurable with respect to $t$ and continuous with respect to $(x,v)$, as we proceed to show in the next two results.

**6.34 Proposition.** *Let $F$ be* LB *measurable as in* Def. 6.33, *and let $g:\mathbb{R}^m\to\mathbb{R}^n$ be Lebesgue measurable. Then the mapping $x\mapsto F(x,g(x))$ is Lebesgue measurable.*

**Proof.** Let $U$ be a Lebesgue measurable set in $\mathbb{R}^m$ and $V$ a Borel set in $\mathbb{R}^n$. Then the set

$$\big\{x\in\mathbb{R}^m : \big(x,g(x)\big)\in U\times V\big\} = U\cap g^{-1}(V)$$

is clearly Lebesgue measurable. Let us denote by $A$ the collection of all subsets $S$ of $\mathbb{R}^m\times\mathbb{R}^n$ having the property that the set

$$\{x \in \mathbb{R}^m : (x, g(x)) \in S\}$$

is Lebesgue measurable. It is easy to verify that $A$ is a $\sigma$-algebra, and it contains the products $U \times V$. It follows that $A$ contains the $\sigma$-algebra $L \times B$ generated by products of Lebesgue measurable subsets of $\mathbb{R}^m$ and Borel measurable subsets of $\mathbb{R}^n$.

Now let $W$ be any open subset of $\mathbb{R}^n$. Since $F$ is LB measurable, the set $F^{-1}(W)$ is LB measurable, and hence lies in $A$. As a consequence of this fact, we deduce that the set

$$\{x : F(x, g(x)) \in W\} = \{x : (x, g(x)) \in F^{-1}(W)\}$$

is Lebesgue measurable. This confirms the required measurability. □

**6.35 Proposition.** *If a function $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ of two variables $(x, y)$ is measurable in $x$ and continuous in $y$, then $F$ is* LB *measurable in $x$ and $y$.*

**Proof.** Let $\{u_i\}$ be a dense sequence in $\mathbb{R}^n$, and for each positive integer $k$ define a function $f_k(x, y) = F(x, u_j)$, where $u_j$ is the first point of the dense set satisfying $|u_j - y| \leqslant k^{-1}$. (Thus, $j$ depends on $y$ and $k$.) Then $F(x, y) = \lim_{k \to \infty} f_k(x, y)$ for every $(x, y)$, by the continuity of $F$ in $y$.

It suffices therefore to prove that each $f_k$ is LB measurable. Let $W$ be any open subset of $\mathbb{R}$. Then the set $f_k^{-1}(W)$ is the union over $j \geqslant 1$ of the sets

$$\{x : F(x, u_j) \in W\} \times \{y : |u_j - y| \leqslant k^{-1} \text{ and } |u_i - y| > k^{-1}\ (i = 1, \dots, j-1)\}.$$

This reveals $f_k^{-1}(W)$ to be a countable union of products of the type which generate the $\sigma$-algebra $L \times B$, whence the required measurability. □

We remark that a function $F$ of two variables having the properties described in Prop. 6.35 is often referred to as a **Carathéodory function**.

The next result says that inserting a measurable function into a continuous slot preserves LB measurability.

**6.36 Proposition.** *Let $F : \mathbb{R}^m \times \mathbb{R}^\ell \times \mathbb{R}^n \to \mathbb{R}$ satisfy the following*:

(a) *The map $(x, z) \mapsto F(x, u, z)$ is* LB *measurable for each $u$*;

(b) *The function $u \mapsto F(x, u, z)$ is continuous for each $(x, z)$.*

*Then, for any Lebesgue measurable function $g : \mathbb{R}^m \to \mathbb{R}^\ell$, the function*

$$(x, z) \mapsto F(x, g(x), z)$$

*is* LB *measurable.*

**Proof.** Let $\{u_i\}$ be a dense sequence in $\mathbb{R}^\ell$, and for each positive integer $k$ define a function $f_k(x, z) = F(x, u_j, z)$, where $u_j$ is the first term of the sequence satisfying

$|u_j - g(x)| \leqslant k^{-1}$. (Thus, $j$ depends on $x$ and $k$.) Then $f_k(x,z)$ converges pointwise to $F(x,g(x),z)$, so it suffices to prove that each $f_k$ is LB measurable. This follows from the identity

$$f_k^{-1}(W) = \bigcup_{j \geqslant 1} \{ (x,z) : F(x,u_j,z)) \in W \} \cap \\ \{ (x,z) : |u_j - g(x)| \leqslant k^{-1} \text{ and } |u_i - g(x)| > k^{-1} \, (i = 1, \ldots, j-1) \}$$

(where $W$ is any open subset of $\mathbb{R}$), which expresses $f_k^{-1}(W)$ as a countable union of sets belonging to $L \times B$. □

**Semicontinuity of integral functionals.** Let $\Omega$ be an open subset of $\mathbb{R}^m$. We study the semicontinuity of the following integral functional:

$$J(u,z) = \int_\Omega F\big(x, u(x), z(x)\big)\, dx.$$

Here, $F : \Omega \times \mathbb{R}^\ell \times \mathbb{R}^n \to \mathbb{R}$ is a function whose three arguments are generically denoted by $x, u, z$. We are also given a subset $Q$ of $\Omega \times \mathbb{R}^\ell$ which defines a restriction on the functions $u : \Omega \to \mathbb{R}^\ell$ involved in the discussion: they must satisfy

$$\big(x, u(x)\big) \in Q, \ \ x \in \Omega \text{ a.e.}$$

We shall impose the following hypotheses on the data:

**6.37 Hypothesis.**

(a) *$F$ is lower semicontinuous in $(u,z)$, and convex with respect to $z$*;

(b) *For every measurable $u : \Omega \to \mathbb{R}^\ell$ having $(x,u(x)) \in Q$* a.e.*, and for every measurable $z : \Omega \to \mathbb{R}^n$, the function $x \mapsto F(x,u(x),z(x))$ is measurable*;

(c) *There exist $\alpha \in L^1(\Omega)$, $\beta \in L^\infty(\Omega, \mathbb{R}^n)$ such that*

$$F(x,u,z) \geqslant \alpha(x) + \langle \beta(x), z \rangle \ \ \forall (x,u) \in Q, \ z \in \mathbb{R}^n;$$

(d) *$Q$ is closed in $\Omega \times \mathbb{R}^\ell$.*

An immediate consequence of these hypotheses is that, for every measurable function $u : \Omega \to \mathbb{R}^\ell$ having $(x,u(x)) \in Q$ a.e., and for every summable function $z : \Omega \to \mathbb{R}^n$, the function

$$x \mapsto F(x,u(x),z(x))$$

is measurable, and it is bounded below as follows:

$$F(x,u(x),z(x)) \geqslant \alpha(x) + \langle \beta(x), z(x) \rangle, \ \ x \in \Omega \text{ a.e.}$$

Since the right side of this inequality is summable over $\Omega$, the integral $J(u,z)$ is well defined, either finitely or as $+\infty$ (a fact from integration theory). It is the lower semicontinuity of $J$ that is the essential point being considered.

**Remark.** In view of our earlier results, we are able to identify certain cases in which hypothesis (b) above is guaranteed to hold:

- $F(x,u,z)$ is measurable in $x$ and continuous in $(u,z)$ (see Prop. 6.35);
- $F(x,u,z)$ is LB measurable in $x$ and $(u,z)$ (by Prop. 6.34);
- $F(x,u,z)$ is continuous in $u$ and LB measurable in $(x,z)$ (by Prop. 6.36).

The theorem below will provide one of the main ingredients in the recipe that we call the direct method.

**6.38 Theorem. (Integral semicontinuity)** *Let $u_i$ be a sequence of measurable functions on $\Omega$ having $(x, u_i(x)) \in Q$ a.e. which converges almost everywhere to a limit $u_*$. Let $z_i$ be a sequence of functions converging weakly in $L^r(\Omega, \mathbb{R}^n)$ to $z_*$, where $r > 1$. Then*

$$J(u_*, z_*) \leqslant \liminf_{i\to\infty} J(u_i, z_i).$$

**Proof.** Fix $\delta > 0$, and define, for $(x,u) \in Q,\ z \in \mathbb{R}^n$, the function

$$H(x,u,p) = \sup\left\{ \langle p,w\rangle - F(x,u,w) - \delta|w|^r/r : w \in \mathbb{R}^n \right\}.$$

The properties of $H$ play an essential role in the proof.

**Lemma 1.** *There is a positive constant $c$ such that*

$$H(x,u,p) \leqslant c|p-\beta(x)|^{r_*} - \alpha(x) \ \ \forall (x,u) \in Q,\ p \in \mathbb{R}^m,$$

*where $r_*$ is the conjugate exponent to $r$.*

**Proof.** Observe that, by the inequality in Hypothesis 6.37 (c), we have:

$$\begin{aligned} H(x,u,p) &= \sup_w \left\{ \langle p,w\rangle - F(x,u,w) - \delta|w|^r/r \right\} \\ &\leqslant \sup_w \left\{ \langle p,w\rangle - \alpha(x) - \langle \beta(x),w\rangle - \delta|w|^r/r \right\} \\ &= c|p-\beta(x)|^{r_*} - \alpha(x) \ \text{ (by explicit calculation)} \end{aligned}$$

where $c := (r_* \delta^{r_*-1})^{-1}$. □

**Lemma 2.** *Fix $(x,u) \in Q$ and $p \in \mathbb{R}^n$. Then:*

(a) *The function $H(x,u,\cdot)$ is continuous at $p$.*

(b) *The function $v \mapsto H(x,v,p)$ is upper semicontinuous at $u$ in the following sense:*

$$(x, v_i) \in Q \;\; \forall i, \;\; v_i \to u \implies H(x,u,p) \geqslant \limsup_{i \to \infty} H(x, v_i, p).$$

(c) *For all* $w \in \mathbb{R}^n$, *we have*

$$F(x,u,w) + \delta |w|^r / r = \sup_{p \in \mathbb{R}^n} \big\{ \langle w, p \rangle - H(x,u,p) \big\}.$$

**Proof.** Since the function $p \mapsto H(x,u,p)$ is convex and finite on $\mathbb{R}^n$ (by Lemma 1), we know it to be continuous, as affirmed in (a). We now fix $x$ and $p$ and turn to assertion (b).

Let $v_i$ be a sequence converging to $u$ for which $\lim_{i \to \infty} H(x, v_i, p) \geqslant \ell \in \mathbb{R}$. We establish (b) by proving that $H(x,u,p) \geqslant \ell$. Note that the supremum defining $H(x, v_i, p)$ may be restricted to those $w$ satisfying

$$\langle p, w \rangle - \alpha(x) - \langle \beta(x), w \rangle - \delta |w|^r / r \geqslant H(x, v_i, p) - 1,$$

and consequently, to the points $w$ in a compact set. It follows that the supremum is attained at a point $w_i$, and that the sequence $w_i$ is bounded. Taking a subsequence if necessary, and without relabeling, we may assume $w_i \to w$. Invoking the lower semicontinuity of $F$, we have

$$\begin{aligned} H(x,u,p) &\geqslant \langle p, w \rangle - F(x,u,w) - \delta |w|^r / r \\ &\geqslant \langle p, w \rangle - \liminf_{i \to \infty} F\big(x, v_i, w_i\big) - \delta |w|^r / r \\ &= \limsup_{i \to \infty} \big\{ \langle p, w_i \rangle - F\big(x, v_i, w_i\big) - \delta |w_i|^r / r \big\} = \lim_{i \to \infty} H(x, v_i, p) \geqslant \ell, \end{aligned}$$

as required.

For the final assertion, note that $H(x,u,\cdot)$ is defined as the conjugate of the convex lower semicontinuous function

$$w \mapsto F(x,u,w) + \delta |w|^r / r.$$

Thus the equality is a consequence of Theorem 4.21. □

**Lemma 3.** *Let* $u : \Omega \to \mathbb{R}^\ell$ *be a measurable function having* $(x, u(x)) \in Q$ a.e., *and let* $p : \Omega \to \mathbb{R}^n$ *be measurable. Then the function* $x \mapsto H(x, u(x), p(x))$ *is measurable.*

**Proof.** Note that the function $w \mapsto F(x,u,w)$ is continuous, since it is convex and finite. It follows that if $\{w_i\}$ is a countable dense set in $\mathbb{R}^n$, we have (almost everywhere)

$$H\big(x, u(x), p(x)\big) = \sup_{i \geqslant 1} \big\{ \langle p(x), w_i \rangle - F(x, u(x), w_i) - \delta |w_i|^r / r \big\}.$$

Thus the left side is the upper envelope of a countable family of measurable functions, and is therefore measurable. □

Note that the limit function $u_*$ satisfies

$$(x, u_*(x)) \in Q \text{ a.e.},$$

in view of Hypothesis 6.37 (d). We now write, without claiming that $J(u_*, z_*)$ is finite:

$$\begin{aligned}
J(u_*, z_*) &= \int_\Omega F\big(x, u_*(x), z_*(x)\big)\, dx \\
&\leqslant \int_\Omega \big\{ F\big(x, u_*(x), z_*(x)\big) + \delta |z_*(x)|^r \big\}\, dx \\
&= \int_\Omega \sup_{p \in \mathbb{R}^n} \big\{ \langle z_*(x), p \rangle - H(x, u_*(x), p) \big\}\, dx \text{ (by (c) of Lemma 2)} \\
&= \sup_{p(\cdot) \in L^\infty(\Omega)} \int_\Omega \big\{ \langle z_*(x), p(x) \rangle - H\big(x, u_*(x), p(x)\big) \big\}\, dx
\end{aligned}$$

(we use Theorem 6.31 and Lemma 3 to switch integral and supremum)

$$\leqslant \sup_{p(\cdot) \in L^\infty(\Omega)} \left[ \lim_{i \to \infty} \int_\Omega \langle p(x), z_i(x) \rangle\, dx - \int_\Omega \limsup_{i \to \infty} H\big(x, u_i(x), p(x)\big)\, dx \right]$$

(since $z_i$ converges weakly to $z_*$, and $u_i$ to $u$ a.e., and since $H$ is upper semicontinuous in $u$, by Lemma 2)

$$\leqslant \sup_{p(\cdot) \in L^\infty(\Omega)} \left[ \lim_{i \to \infty} \int_\Omega \langle p(x), z_i(x) \rangle\, dx - \limsup_{i \to \infty} \int_\Omega H\big(x, u_i(x), p(x)\big)\, dx \right]$$

(Fatou 's lemma applies, since $u_i$ has values in $Q$, $p \in L^\infty(\Omega)$, and the terms in $H$ are uniformly integrably bounded above, by Lemma 1; Lemma 3 is used to assert that the integrand is measurable)

$$\begin{aligned}
&= \sup_{p(\cdot) \in L^\infty(\Omega)} \left[ \liminf_{i \to \infty} \int_\Omega \big\{ \langle p(x), z_i(x) \rangle - H\big(x, u_i(x), p(x)\big) \big\}\, dx \right] \\
&\leqslant \liminf_{i \to \infty} \sup_{p(\cdot) \in L^\infty(\Omega)} \left[ \int_\Omega \big\{ \langle p(x), z_i(x) \rangle - H\big(x, u_i(x), p(x)\big) \big\}\, dx \right] \\
&= \liminf_{i \to \infty} \int_\Omega \sup_{p \in \mathbb{R}^n} \big\{ \langle p, z_i(x) \rangle - H\big(x, u_i(x), p\big) \big\}\, dx \text{ (by Theorem 6.31)} \\
&= \liminf_{i \to \infty} \int_\Omega \big\{ F\big(x, u_i(x), z_i(x)\big) + \delta |z_i(x)|^r \big\}\, dx \text{ (by (c) of Lemma 2)} \\
&\leqslant \liminf_{i \to \infty} J(u_i, z_i) + \delta \limsup_{i \to \infty} \| z_i \|_r^r .
\end{aligned}$$

Since $z_i$ is weakly convergent in $L^r(\Omega, \mathbb{R}^n)$, the sequence $z_i$ is norm bounded, so that $\limsup_{i\to\infty} \| z_i \|_r^r$ is finite. Since $\delta > 0$ is arbitrary, we obtain the required conclusion.

We remark that the crux of the proof is to find a way to exploit the (merely) weak convergence of the sequence $z_i$; this has been done by rewriting certain expressions so as to have $z_i$ appear only in *linear* terms. □

## 6.4 Weak sequential closures

In things to come, the reader will find that the closure properties of *differential inclusions* of the type

$$x'(t) \in \Gamma\big(t, x(t)\big),$$

where $\Gamma$ is a multifunction, will play an important role. The following abstract result is a basic tool in this connection. Note that weak convergence in $L^1$ is now involved.

**6.39 Theorem. (Weak closure)** *Let $[a,b]$ be an interval in $\mathbb{R}$ and $Q$ a closed subset of $[a,b]\times\mathbb{R}^\ell$. Let $\Gamma(t,u)$ be a multifunction mapping $Q$ to the closed convex subsets of $\mathbb{R}^n$. We assume that*

(a) *For each $t \in [a,b]$, the set*

$$G(t) = \big\{ (u,z) : (t,u,z) \in Q\times\mathbb{R}^n,\ z \in \Gamma(t,u) \big\}$$

*is closed and nonempty*;

(b) *For every measurable function $u$ on $[a,b]$ satisfying $(t,u(t)) \in Q$ a.e. and every $p \in \mathbb{R}^n$, the support function map*

$$t \mapsto H_{\Gamma(t,u(t))}(p) = \sup \big\{ \langle p, v \rangle : v \in \Gamma\big(t, u(t)\big) \big\}$$

*is measurable*;

(c) *For a summable function $k$, we have $\Gamma(t,u) \subset B(0,k(t))\ \forall\,(t,u) \in Q$.*

*Let $u_i$ be a sequence of measurable functions on $[a,b]$ having $(t,u_i(t)) \in Q$ a.e. and converging almost everywhere to $u_*$, and let $z_i : [a,b] \to \mathbb{R}^n$ be a sequence of functions satisfying $|z_i(t)| \leqslant k(t)$ a.e. whose components converge weakly in $L^1(a,b)$ to those of $z_*$. Suppose that, for certain measurable subsets $\Omega_i$ of $[a,b]$ satisfying $\lim_{i\to\infty} \text{meas}\, \Omega_i = b - a$, we have*

$$z_i(t) \in \Gamma\big(t, u_i(t)\big) + B\big(0, r_i(t)\big),\ \ t \in \Omega_i \text{ a.e.},$$

*where $r_i$ is a sequence of nonnegative functions converging in $L^1(a,b)$ to* $0$. *Then we have in the limit* $z_*(t) \in \Gamma(t,u_*(t))$, $t \in [a,b]$ a.e.

**Proof.** Let $H : Q \times \mathbb{R}^n \to \mathbb{R}$ be the support function associated with $\Gamma$:

$$H(t,u,p) = \sup \big\{ \langle p,v \rangle : v \in \Gamma(t,u) \big\}.$$

Note that $|H(t,u,p)| \leqslant |p|k(t)$, in view of hypothesis (c); it follows that for each $(t,u) \in Q$, the function $p \mapsto H(t,u,p)$ is continuous with respect to $p$, as the support function of a nonempty bounded set. Furthermore, for any $t \in [a,b]$, using the fact that $G(t)$ is closed, it is not hard to show that for fixed $p$, the map $u \mapsto H(t,u,p)$ is upper semicontinuous on the set $\{u : (t,u) \in Q\}$ (exercise).

In view of Prop. 2.42, and because $\Gamma$ is convex-valued, the conclusion that we seek may now be restated as follows: for some null set $N$, for all $t \in [a,b] \backslash N$, we have

$$H\big(t,u_*(t),p\big) \geqslant \langle p, z_*(t) \rangle \;\; \forall p \in \mathbb{R}^n. \tag{$*$}$$

By the continuity of $H$ in $p$, it is equivalent to obtain this conclusion for all $p$ having rational coordinates. Then, if $(*)$ holds for each such $p$ except on a null set (depending on $p$), we obtain the required conclusion, since the countable union of null sets is a null set.

We may summarize to this point as follows: it suffices to prove that for any fixed $p \in \mathbb{R}^n$, the inequality in $(*)$ holds almost everywhere.

This assertion in turn would result from knowing that the following inequality holds for any measurable subset $A$ of $[a,b]$:

$$\int_A \big\{ H\big(t,u_*(t),p\big) - \langle p, z_*(t) \rangle \big\}\, dt \;\geqslant 0.$$

(Note that the integrand in this expression is measurable by hypothesis (b), and summable because of the bound on $H$ noted above.) But we have

$$\begin{aligned}\int_A \big\{ H\big(t,u_*(t),p\big) - \langle p, z_*(t) \rangle \big\}\, dt &\geqslant \int_A \big\{ \limsup_{i\to\infty} H\big(t,u_i(t),p\big) - \langle p, z_*(t) \rangle \big\}\, dt \\ &\geqslant \limsup_{i\to\infty} \int_A \big\{ H\big(t,u_i(t),p\big) - \langle p, z_i(t) \rangle \big\}\, dt,\end{aligned}$$

as a result of the almost everywhere convergence of $u_i$ to $u_*$, the upper semicontinuity of $H$ in $u$, Fatou's lemma, and the weak convergence of $z_i$ to $z_*$. The last integral above may be written in the form

$$\int_{A \cap \Omega_i} \big\{ H(t,u_i,p) - \langle p, z_i \rangle \big\}\, dt + \int_{A \backslash \Omega_i} \big\{ H(t,u_i,p) - \langle p, z_i \rangle \big\}\, dt. \tag{$**$}$$

We have now reduced the proof to showing that the lower limit of this expression is nonnegative.

Using the bound on $|H|$ noted above, together with the given bound on $|z_i|$, we see that the *second* term in $(**)$ is bounded above in absolute value by

$$\int_{[a,b]\setminus\Omega_i} 2\,|p|\,k(t)\,dt$$

which tends to 0 as $i\to\infty$, since meas $\Omega_i \to b-a$.

As for the *first* term in $(**)$, the hypotheses imply

$$H\big(t,u_i(t),p\big) \geqslant \langle p, z_i(t)\rangle - r_i(t)|p|,\ \ t\in\Omega_i \text{ a.e.},$$

so that it is bounded below by

$$-\int_{A\cap\Omega_i} r_i(t)|p|\,dt \ \geqslant -\int_a^b r_i(t)|p|\,dt$$

(recall that the functions $r_i$ are nonnegative). But this last term also converges to 0, since $r_i$ converges to 0 in $L^1(a,b)$ by hypothesis. The proof is complete. □

**6.40 Exercise.** Let $A$ be a compact convex subset of $\mathbb{R}$, and $v_i$ a sequence converging weakly in $L^1(a,b)$ to a limit $v_*$, where, for each $i$, we have $v_i(t)\in A$ a.e. Prove that $v_*(t)\in A$ a.e. Show that this may fail when $A$ is not convex. □

In later chapters, Theorem 6.39 will be used when $u:[a,b]\to\mathbb{R}^n$ is absolutely continuous (that is, each component of $u$ belongs to AC$[a,b]$) and $z_i = u_i'$. Furthermore, the convergence hypotheses will most often be obtained with the help of the following well-known result, which we promote to the rank of a theorem:

**6.41 Theorem. (Gronwall's lemma)** *Let $x:[a,b]\to\mathbb{R}^n$ be absolutely continuous and satisfy*

$$|x'(t)| \leqslant \gamma(t)|x(t)| + \beta(t),\ \ t\in[a,b] \text{ a.e.},$$

*where $\gamma,\beta\in L^1(a,b)$, with $\gamma$ nonnegative. Then, for all $t\in[a,b]$, we have*

$$|x(t)-x(a)| \ \leqslant \int_a^t \exp\Big(\int_s^t \gamma(r)\,dr\Big)\big\{\gamma(s)|x(a)| + \beta(s)\big\}\,ds.$$

**Proof.** Let $r(t) = |x(t)-x(a)|$, a function which is absolutely continuous on $[a,b]$, as the composition of a Lipschitz function and an absolutely continuous one. Let $t$ be in that set of full measure in which both $x'(t)$ and $r'(t)$ exist. If $x(t)\neq x(a)$, we have

$$r'(t) = \Big\langle \frac{x(t)-x(a)}{|x(t)-x(a)|}, x'(t)\Big\rangle,$$

and otherwise $r'(t)=0$ (since $r$ attains a minimum at $t$). Thus we have

$$r'(t) \leqslant |x'(t)| \leqslant \gamma(t)|x(t)| + \beta(t) \leqslant \gamma(t)|x(t) - x(a)| + \gamma(t)|x(a)| + \beta(t)$$
$$= \gamma(t)r(t) + \gamma(t)|x(a)| + \beta(t).$$

We may rewrite this inequality in the form

$$\big[r'(t) - \gamma(t)r(t)\big] \exp\Big(-\int_a^t \gamma\Big) \leqslant \exp\Big(-\int_a^t \gamma\Big)\big\{\gamma(t)|x(a)| + \beta(t)\big\}.$$

Note that the left side is the derivative of the function

$$t \mapsto r(t) \exp\Big(-\int_a^t \gamma\Big).$$

With this in mind, integrating both sides of the preceding inequality from $a$ to $t$ yields the required estimate. □

**6.42 Exercise.** Let $p_i : [a,b] \to \mathbb{R}^n$ be a sequence of absolutely continuous functions with $|p_i(a)|$ uniformly bounded, and such that, for certain functions $\gamma, \beta$ in $L^1(a,b)$, we have, for each $i$:

$$|p_i'(t)| \leqslant \gamma(t)|p_i(t)| + \beta(t), \; t \in [a,b] \text{ a.e.}$$

Then there exist an absolutely continuous function $p : [a,b] \to \mathbb{R}^n$ and a subsequence $p_{i_j}$ such that (componentwise)

$$p_{i_j} \to p \text{ uniformly on } [a,b], \; p_{i_j}' \to p' \text{ weakly in } L^1(a,b).$$ □