

Chapter 1

Financial Transmission Rights: Point-to Point Formulations

William W. Hogan

1.1 Introduction

Transmission rights stand at the center of market design in a restructured electricity industry. Beginning with the intuition that electricity markets require some rights to use the transmission system, simple models of transmission rights soon founder after confronting the limited capacity and complex interactions of a transmission grid. The industry searched for many years without success looking for a workable system of physical rights that would support decentralized decisions controlling use of the grid.

The physical interpretation of transmission rights was the principal complaint that buried the Federal Energy Regulatory Commission's (FERC) original Capacity Reservation Tariff (FERC 1996). Any attempt to match a large number of scheduled transactions to a set of transmission rights creates a burden that threatens the flexibility of trade needed to support a market or the flexibility of operations needed to maintain reliability. And in a design built on the centerpiece of a coordinated spot market (FERC 2002a), physical transmission rights or any associated scheduling priority would create perverse incentives and conflicts with priorities defined by the bids used in a security-constrained economic dispatch. The idea that a simple physical right can be made to work soon mutates into a complex system of rules intended to force market participants to act against market incentives. In the end, the right becomes not so physical and not much of a right. The idea dies hard, but the physical rights model deserves a decent burial.

If physical rights will not work, then something different is needed to achieve the same objective in providing a compatible definition of transmission rights for a competitive electricity market. As electricity market design developed, the focus

W.W. Hogan (✉)

Mossavar-Rahmani Center for Business and Government, John F. Kennedy School of Government, Harvard University, Cambridge, MA 02138, USA
e-mail: william_hogan@harvard.edu

turned from so-called physical transmission rights to a redefinition of transmission rights as financial instruments defined with a close connection to both the transmission grid and a spot market organized through a bid-based, security-constrained, economic dispatch (Hogan 1992). The financial approach separates actual use of the grid from ownership of the transmission rights and provides many simplifications that avoid the principal obstacles encountered in the search for physical rights. A coordinated spot market with locational prices complemented by financial transmission rights is a hallmark of market design that works.

There are many possible definitions of financial transmission rights, each with its advantages and disadvantages. Further, the basic building blocks of financial transmission rights could support a secondary market with a wide variety of other trading instruments, just as a forward contract can be decomposed into a variety of elements with different risk properties.

The basic building blocks under different definitions have different properties. The purpose here is to organize a common analysis covering different types of point-to-point financial transmission rights and compare them in regards to four critical aspects of the transmission rights model. The common notation is an eclectic synthesis designed to bridge the electrical engineering and economic market formulations.¹

The four aspects of the design cover modeling approximations, revenue adequacy, auction formulation, and computational requirements. These do not include important related subjects such as investment incentives. However, an understanding of at least these four aspects of the formulations would be important in choosing among the types of rights to include in a market design. The same would be true of a decision to include all types of rights, where the market participants could ask for any combination (O'Neill et al. 2002).

Approximation refers to the simplifications inherent in the transmission rights model in comparison to the complexity of the real transmission system. To illustrate the point, the simplification that there are no loop flows makes the contract-path transmission model workable in theory. But the simplification deviates from the reality and the contract-path model became recognized as inefficient and unworkable in practice. The different transmission right definitions depend to different degrees on approximations of the reality of the network. The discussion here begins with a simplified but explicit characterization of an alternating current load flow to then specialize it in the market context for an examination of different transmission rights.

Revenue adequacy refers to a financial counterpart of physical “available transmission capacity.” A financial transmission right as defined here is a contract for a financial payment that depends on the outcome of the spot market. By definition, the system is revenue adequate whenever the net revenue collected by the system operator for any period of the spot market is at least equal to the payment

¹This paper is an abridged version of the working paper, Hogan (2002). The working paper includes an elaboration of flowgate financial transmission rights and hybrid models.

obligations under the transmission rights. The analogous physical problem would be to define the available capacity for transmission usage rights such that the transmission schedules could be guaranteed to flow in any given period. A common requirement of both is to maintain the capability of the grid, but the complex interactions make it impossible to guarantee that physical rights could flow no matter what the dispatch conditions. By contrast, we examine here conditions that do ensure revenue adequacy for the financial transmission rights.

A natural approach to allocating some or all transmission rights is through an auction. The auction design also extends to regular and continuing coordinated auctions that could be employed to reconfigure the pattern of transmission rights, supplemented by secondary market trading. The auction formulation interacts with the conditions for revenue adequacy, with different implications for different definitions of financial transmission rights.

The computational requirements for execution of a transmission rights auction differ for the different models. The inherent scale of the security-constrained economic dispatch model takes the discussion into a realm where the ability to solve the problem cannot be taken for granted. In some cases, the auction model is no more complicated than a conventional security-constrained economic dispatch, and commercial software could be and has been adapted successfully for this purpose. In other cases, the ability to solve the formal model is not assured, and new approaches or various restrictions might be required. Hence, proposals for more ambitious financial transmission right formulations have been offered with the caveat that the expanded service beyond point-to-point rights should be offered “as soon as it is technically feasible” (FERC 2002b).²

The purpose here is to identify some of the issues raised in the evaluation of technical feasibility. The comparison of transmission rights models involves tradeoffs. Some versions may be impossible to implement. At a minimum, ease of both implementation and use for alternative transmission rights models should not be taken for granted.

1.2 Transmission Line Load Flow Model

Every alternating current (AC) electrical network has both real and reactive power flows. The sinusoidal pattern of instantaneous power flow produces a complex power representation with real and imaginary parts that correspond to real and reactive power. The real power flows are measured in Mega-Watts (MWs), and the reactive power flows are measured in Mega-Volt-Amperes-Reactive (MVARs). The VAR is the product of voltage and current, which is the same unit as the watt; the notational difference is maintained to distinguish between real and reactive power. Real power is defined as the average value of the instantaneous

² Similar qualifications appear in discussions of an introduction of options or flowgate rights in PJM, New York, New England, the Midwest, and so on.

power and is the “active” or “useful” power. Reactive power is the peak value of the power that “travels back and forth” over the line and has average value of zero and is “capable of no useful work . . . [and] represents a ‘nonactive,’ or ‘reactive,’ power (Elgerd 1982).”³ The combination of real and reactive power flow is the apparent power in Mega-Volt-Amperes (MVA), which is a measure of the magnitude of the total power flow.

The basic model characterizing electricity markets and financial transmission rights (FTR) centers on the description of a network of lines and buses operating in an electrical steady-state. A critical element is the representation of a transmission line. There is a developed literature on this subject. The choices here do not exhaust all that is relevant, but illustrate the basic issues in the treatment of AC networks for purposes of modeling economic dispatch, locational pricing and the related definition of financial transmission rights. In particular, although the focus is on real power flow, the model includes non-linear features of real and reactive power and control devices to illustrate the implications of various simplifications and approximations often suggested for economic dispatch, pricing and definition of financial transmission rights. Further extensions to include other elements of flexible AC transmission systems (FACTS) could be added, with the associated non-linear characterizations of even the effects on real power flows (Ge and Chung 1999).

A generic transmission line as represented here is illustrated in Fig. 1.1. The data include the resistance (r), reactance (x), and line charging capacitance ($2B_{cap}$). Variable controls include a transformer with winding tap ratio (t) and a phase shift angle (α). The voltage magnitude at bus i is V_i and the voltage angle is δ_i . The flow of real and reactive power bus from i towards j is the complex variable Z_{ij} . Assuming a steady-state flow can be achieved, the conditions relate the flow of complex power on a line to the control parameters including the voltage magnitudes and angles. Due to losses, the flow out of one bus is not the same as the flow into the other. With these sign conventions, positive flow away from a bus adds to net load at the bus.

The sign conventions support an interpretation of an increase in net load as typically adding to economic benefit and associated with a positive price. Correspondingly, an increase in generation reduces net load and typically adds to cost.⁴

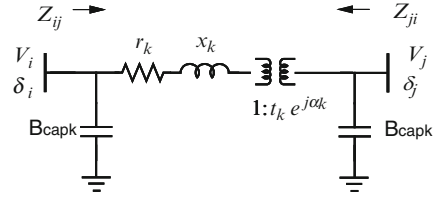
The flow of power in an AC electric network can be described by a system of equations known as the AC load flow model.⁵

³ For an excellent summary of the basics for those other than electrical engineers, see Elgerd (1982), pp. 19–32.

⁴ Atypical negative prices are allowed, and in the presence of system congestion may not be so atypical.

⁵ In anticipation of later simplifications, the notation here follows the development of the “DC” Load Flow model in Schweppe et al. (1988), Appendices A and D. The DC Load flow refers to the real power half of the nonlinear AC load flow model. Under the maintained assumptions, there is a weak link between the reactive power and real power halves of the full problem. And the real power flow equations have the same general form as the direct current flow equations in a purely resistive network; hence the name “DC Load Flow.” Similar linear approximations are available for reactive power flow, but the approximation is poor in a heavily loaded system. Hence, if in addition to real power flow, voltage constraints and the associated reactive power are important, then we require the full AC model and spot pricing theory as in Caramanis (1982).

Fig. 1.1 Generic transmission line representation



Let:

n_B = Number of buses,

n_L = Number of transmission lines, with each line having per unit resistance r_k , reactance x_k , and shunt capacitance $Bcap_{ij}$ for the Π -equivalent representation of line k ,⁶

$\tilde{y}_P = d_P - g_P = n_{B-1}$ vector of net real power bus loads, i.e. demand minus generation, $y_P^t = (y_{P_s}, \tilde{y}_P^t)$ where y_{P_s} is at the swing bus,

$\tilde{y}_Q = d_Q - g_Q = n_{B-1}$ vector of reactive power bus loads, i.e. demand minus generation, $y_Q^t = (y_{Q_s}, \tilde{y}_Q^t)$ where y_{Q_s} is at the swing bus,

$\delta = n_B$ Vector of voltage angles relative to the swing bus, where by definition $\delta_s = 0$,

$V = n_B$ Vector of voltage magnitudes, where by assumption the voltage at the swing bus, V_s , is exogenous,

t_k = ideal transformer tap ratio on line k ,

α_k = ideal transformer phase angle shift on line k ,

A = the oriented line-node incidence matrix, the network incidence matrix with elements of 0, 1, -1 corresponding to the network interconnections. If link k originates at bus i and terminates at bus j , then $a_{ki} = 1 = -a_{kj}$.

⁶ For a development of the Π -equivalent representation of a transmission line, see Bergen (1986), Chap. 4. Here we follow Wood and Wollenberg (1984) in representing $Bcap$ as one-half the total line capacitance in the Π -equivalent representation; (Wood and Wollenberg 1984), p.75. A. See also Skilling (1951), pp. 126–133.

Define⁷

$$G_k = r_k / (r_k^2 + x_k^2),$$

$$\Omega_k = x_k / (r_k^2 + x_k^2),$$

z_{Pijk} = real power (MWs) flowing out of bus i towards bus j along line k , and

z_{Qijk} = reactive power (MVARs) flowing out of bus i towards bus j along line k .

Then the complex power flow Z_{ij} includes the real and reactive components⁸:

$$\begin{aligned} z_{Pijk} &= G_k [V_i^2 - (V_i V_j / t_k) \cos(\delta_i - \delta_j + \alpha_k)] + \Omega_k (V_i V_j / t_k) \sin(\delta_i - \delta_j + \alpha_k), \\ z_{Pjik} &= G_k [(V_j / t_k)^2 - (V_j V_i / t_k) \cos(\delta_j - \delta_i - \alpha_k)] + \Omega_k (V_j V_i / t_k) \sin(\delta_j - \delta_i - \alpha_k). \end{aligned} \quad (1.1)$$

and

$$\begin{aligned} z_{Qijk} &= \Omega_k [V_i^2 - (V_i V_j / t_k) \cos(\delta_i - \delta_j + \alpha_k)] - G_k (V_i V_j / t_k) \sin(\delta_i - \delta_j + \alpha_k) \\ &\quad - V_i^2 B_{capk}, \\ z_{Qjik} &= \Omega_k [(V_j / t_k)^2 - (V_j V_i / t_k) \cos(\delta_j - \delta_i - \alpha_k)] \\ &\quad - G_k (V_j V_i / t_k) \sin(\delta_j - \delta_i - \alpha_k) - V_j^2 B_{capk}. \end{aligned}$$

Real losses on line k are given by

$$l_{Pk} = z_{Pijk} + z_{Pjik}.$$

Hence, in terms of the angles and voltages we have

$$l_{Pk}(\delta, V, t, \alpha) = G_k \left[V_i^2 + (V_j / t_k)^2 - 2(V_i V_j / t_k) \cos(\delta_i - \delta_j + \alpha_k) \right].$$

Similarly, reactive power losses are

$$l_{Qk} = z_{Qijk} + z_{Qjik},$$

⁷ Here the notation follows Schweppe et al. (1988). The purpose is to connect to the discussion of the economics of spot markets and the definition of FTRs. However, the electrical engineering literature follows different notational conventions. For example, Wood and Wollenberg (1984) and others use a different sign convention for Ω . Also note that here V_i is the magnitude of the complex voltage at bus i , not the complex voltage itself as in the appendix. Finally, we use y to denote the net loads at the buses. This should not be confused with the complex admittance matrix, often denoted as Y , which is composed of the elements of G and Ω . See the appendix for further discussion.

⁸ For details, see the [appendix](#).

or

$$l_{Qk}(\delta, V, t, \alpha) = \Omega_k \left[V_i^2 + (V_j/t_k)^2 - 2(V_i V_j/t_k) \cos(\delta_i - \delta_j + \alpha_k) \right] - (V_i^2 + V_j^2) Bcap_k.$$

Given these flows on the lines, conservation of power at each bus requires that the net power loads balance the summation of the flows in and out of each bus. Under our sign conventions and summing over every link connected to bus i , we have

$$d_{P_i} + \sum_{k(i,j)} z_{P_{ijk}} = g_{P_i} - \sum_{k(j,i)} z_{P_{jik}}, \quad \text{and}$$

$$d_{Q_i} + \sum_{k(i,j)} z_{Q_{ijk}} = g_{Q_i} - \sum_{k(j,i)} z_{Q_{jik}}.$$

Here the summation includes each directed line that terminates at i ($k(j,i)$) or originates at i ($k(i,j)$) Hence, the net loads satisfy:

$$y_{P_i} \equiv d_{P_i} - g_{P_i} = - \sum_{k(j,i)} z_{P_{jik}} - \sum_{k(i,j)} z_{P_{ijk}}, \quad \text{and}$$

$$y_{Q_i} \equiv d_{Q_i} - g_{Q_i} = - \sum_{k(j,i)} z_{Q_{jik}} - \sum_{k(i,j)} z_{Q_{ijk}}.$$

Recognizing that the individual flows can be expressed in terms of the several variables, we obtain the relation between net loads, bus angles, voltage magnitudes, transformer ratios, and phase angle changes:

$$\begin{bmatrix} \tilde{y}_P \\ \tilde{y}_Q \end{bmatrix} = \begin{bmatrix} \tilde{y}_P(\delta, V, t, \alpha) \\ \tilde{y}_Q(\delta, V, t, \alpha) \end{bmatrix} = \tilde{Y}(\delta, V, t, \alpha).$$

Assuming that there is convergence to a non-singular solution for the steady-state load flow, this system can be inverted to obtain the relation between the bus angles, voltage magnitudes and the net power loads given the transformer ratios and phase angle changes⁹:

⁹The convention here is that gradients are row vectors. Hence, with

$$f(u, v) = \begin{bmatrix} f_1(u, v) \\ f_2(u, v) \end{bmatrix}, \quad \nabla f = \begin{bmatrix} \partial f_1(u, v)/\partial u & \partial f_1(u, v)/\partial v \\ \partial f_2(u, v)/\partial u & \partial f_2(u, v)/\partial v \end{bmatrix}.$$

$$\begin{bmatrix} \delta \\ V \end{bmatrix} = \begin{bmatrix} J_\delta(\tilde{y}_P, \tilde{y}_Q, t, \alpha) \\ J_V(\tilde{y}_P, \tilde{y}_Q, t, \alpha) \end{bmatrix} = J(\tilde{y}_P, \tilde{y}_Q, t, \alpha), \text{ and}$$

$$\nabla J = \begin{bmatrix} \nabla J_{\delta P} & \nabla J_{\delta Q} \\ \nabla J_{VP} & \nabla J_{VQ} \end{bmatrix} = \begin{bmatrix} \nabla y_{P\delta} & \nabla y_{PV} \\ \nabla y_{Q\delta} & \nabla y_{QV} \end{bmatrix}^{-1} = \nabla \tilde{Y}^{-1}.$$

This formulation treats all buses, other than the swing bus, as load buses, with given real and reactive power loads. These are sometimes referred to as PQ buses.¹⁰ In practice, many generator buses are operated as PV buses, where \tilde{y}_P and V are given and the required reactive power is determined in order to maintain the voltage (Bergen 1986). There are $4(n_{B-1})$ variables (i.e., \tilde{y}_P , \tilde{y}_Q , δ , V) and $2(n_{B-1})$ independent node balance equations. Hence, half of the variables must be specified and then the solution obtained for the remainder. The corresponding change on the representation of the equations for different treatment of the buses is straightforward. For example, in the DC-Load model discussed below, all buses are treated as PV where the first step is to fix \tilde{y}_P and V to solve for δ and implicitly \tilde{y}_Q .

The power flow entering a line differs from the power leaving the line by the amount of the losses on the line. Typically, but not always, real power losses will be a small fraction of the total flow and it is common to speak of the power flow on the line. In the DC-Load case discussed below, losses are ignored and the real power flow is defined as the same at the source and destination. In the case of an AC line, we could select either or both ends of the line as metered and focus on the flow at that location for purposes of defining transmission constraints.

We can use these relations to define the link between the power flows on the lines and the net loads at the buses:

$$z = \begin{bmatrix} z_P(\delta, V, t, \alpha) \\ z_Q(\delta, V, t, \alpha) \end{bmatrix} = \begin{bmatrix} z_P(J(\tilde{y}_P, \tilde{y}_Q, t, \alpha), t, \alpha) \\ z_Q(J(\tilde{y}_P, \tilde{y}_Q, t, \alpha), t, \alpha) \end{bmatrix} = \begin{bmatrix} \tilde{K}_P(\tilde{y}_P, \tilde{y}_Q, t, \alpha) \\ \tilde{K}_Q(\tilde{y}_P, \tilde{y}_Q, t, \alpha) \end{bmatrix} = \tilde{K}(\tilde{y}_P, \tilde{y}_Q, t, \alpha),$$

and

$$\nabla \tilde{K}_y(\tilde{y}_P, \tilde{y}_Q, t, \alpha) = \begin{bmatrix} \nabla \tilde{K}_P \\ \nabla \tilde{K}_Q \end{bmatrix} = \begin{bmatrix} \nabla z_P \nabla J \\ \nabla z_Q \nabla J \end{bmatrix} = \begin{bmatrix} \nabla z_P \nabla \tilde{Y}^{-1} \\ \nabla z_Q \nabla \tilde{Y}^{-1} \end{bmatrix}. \quad (1.2)$$

¹⁰The swing bus is a δV bus for which the angle and the voltage are exogenous.

Summing over all lines gives total losses as:

$$\begin{bmatrix} L_P \\ L_Q \end{bmatrix} = \begin{bmatrix} \sum_k l_{Pk}(\delta, V, t, \alpha) \\ \sum_k l_{Qk}(\delta, V, t, \alpha) \end{bmatrix} = \begin{bmatrix} l_P(J(\tilde{y}_P, \tilde{y}_Q, t, \alpha), t, \alpha) \\ l_Q(J(\tilde{y}_P, \tilde{y}_Q, t, \alpha), t, \alpha) \end{bmatrix} = \begin{bmatrix} L_P(\tilde{y}_P, \tilde{y}_Q, t, \alpha) \\ L_Q(\tilde{y}_P, \tilde{y}_Q, t, \alpha) \end{bmatrix},$$

and

$$\nabla L = \nabla l \nabla J = \nabla l \nabla \tilde{Y}^{-1}.$$

Finally, conservation of power determines the required generation at the swing bus, g_{Ps} and g_{Qs} , as:

$$\begin{aligned} g_{Ps} &= -y_{Ps} = L_P(\tilde{y}_P, \tilde{y}_Q, t, \alpha) + t^t \tilde{y}_P, \text{ and} \\ g_{Qs} &= -y_{Qs} = L_Q(\tilde{y}_P, \tilde{y}_Q, t, \alpha) + t^t \tilde{y}_Q. \end{aligned}$$

where t is a unity column vector, $t^t = (1 \quad 1 \quad \dots \quad 1)$. Equivalently,

$$\begin{aligned} L_P(\tilde{y}_P, \tilde{y}_Q, t, \alpha) + t^t y_P &= 0, \text{ and} \\ L_Q(\tilde{y}_P, \tilde{y}_Q, t, \alpha) + t^t y_Q &= 0. \end{aligned}$$

These relationships summarize Kirchoff's Laws that define the AC load flow model in terms convenient for our subsequent characterization of the optimal dispatch problem. Given the configuration of the network consisting of the buses, lines, transformer settings, resistances and reactances, the load flow equations define the relationships among (1) the net inputs at each bus, (2) the voltage magnitudes and angles, and (3) the flows on the individual lines.

1.3 Optimal Power Flow

The optimal power flow or economic dispatch problem is to choose the net loads, typically by controlling the dispatch of power plants, in order to achieve maximum net benefits within the limits of the transmission grid. Under its economic interpretation, the solution of the power flow problem produces locational prices in the usual way. For our present purposes we define abstract benefit and cost functions. The model developed here includes three simplifications. First, strictly for notational convenience, we assume that all transmission constraints are defined in terms of the effects of net loads at buses. In reality, transmission constraints may treat loads and generation differently. Incorporating different buses for generation and load connected by a zero impedance line would accommodate different effects of load and generation. This would allow for different prices for load and generation by treating them as at different locations.

The second simplification is to focus on the real power part of the problem, even in the AC case. Here we anticipate a market in which we have FTRs for real power

but none are required or available for reactive power and there is no reactive power market. This is not a trivial simplification. It would be appropriate as a model under the assumption that there are no direct costs of producing reactive power and the dispatch of reactive power sources is fully under the control of the system operator. Finally, we abstract from explicit consideration of generation operating reserves.¹¹

With these assumptions, we formulate the economic dispatch problem and then extend it to the case of security-constrained economic dispatch.

1.3.1 Economic Dispatch

We first specialize the notation to represent the transmission constraints, and then the simplified aggregate benefit function.

The constraints for the economic dispatch problem derive from the characterization of the power flow in transmission lines. Under the simplifying assumptions, we treat the real and reactive power elements differently. Henceforth, we drop the subscript and treat the variable $y = y_P = d_P - g_P$ as the real power bus loads, including for the swing bus ($y^t = (y_s, \tilde{y}^t)$). We further subsume all other parameters above in the generic control vector u , with its own constraints as in:

$$u = \begin{pmatrix} y_Q \\ t \\ \alpha \end{pmatrix}$$

$$u \in U.$$

In addition to these control variables, we recognize that system operators may change to topology of the network as summarized in A . For simplicity, we limit attention to differentiable elements of u . However, in the applications discussed below, the incidence matrix could change. The principal impact of changes in A is to introduce discrete choices with complications for the optimization problem but not for the main results for FTRs.

With this notational adjustment, we restate the transmission flows as the function $\tilde{K}(y, u)$ and the losses as $L(y, u)$. We assume that the flows are constrained. In addition, we incorporate the constraint limits as part of the function and append any other constraints on the real power flows. For example, a constraint on MVA of apparent power flow at a metered end of the line would be:

$$z_{Pijk}^2 + z_{Qijk}^2 - b_{MVA_MAXk} \leq 0. \quad (1.3)$$

¹¹ Cadwalader et al. (1998) provides an outline of transmission rights and revenue adequacy in the context of explicit reserve markets. The analysis is limited to point-to-point obligations, as discussed below, but could be extended to include other types of financial transmission rights.

We treat this as simply another element of $\tilde{K}(y, u)$. All joint constraints on real power flows and the various control parameters, including interface and other operating limits, appear under $\tilde{K}(y, u)$. The separate limits on the control variables appear in the set U . Hence, the summary of the constraints is:

$$\begin{aligned} L(y, u) + t'y &= 0, \\ \tilde{K}(y, u) &\leq 0, \\ u &\in U. \end{aligned}$$

The objective function for the net loads derives from the benefits of load less the costs of generation. Anticipating a bid-based economic dispatch from a coordinated spot market, we formulate the benefit function for net loads as:

$$\begin{aligned} B(y) &= \underset{d \in D, g \in G}{\text{Max}} \text{Benefits}(d) - \text{Costs}(g) \\ &\text{s.t.} \\ &d - g = y. \end{aligned}$$

Under the usual convexity assumptions, the constraint multipliers for this optimization problem define a sub-gradient for this optimal value problem. For simplicity in the discussion here, we treat the sub-gradient as unique so that B is differentiable with gradient ∇B . This gives the right intuition for the resulting prices, with the locational prices of net loads at $p^t = \nabla B$. The more general case would require little more than recognizing that market-clearing prices might not be unique, as for example at a step in a supply function.

Then the economic dispatch problem is:¹²

$$\begin{aligned} &\underset{y, u \in U}{\text{Max}} B(y) \\ &\text{s.t.} \\ &L(y, u) + t'y = 0, \\ &\tilde{K}(y, u) \leq 0. \end{aligned} \tag{1.4}$$

In general, this can be a complicated non-linear and typically non-convex problem. In many cases, but not all, the economic dispatch problem is well-behaved in the sense that there is a solution with a corresponding set of Lagrange multipliers and no duality gap. The problem may still be hard to solve, but that is the challenge for software implementation.

¹²This is similar to the formulation in Caramanis et al. (1982); the principal difference is in imposing the thermal limit not just on the real power flow, but on the total MVA flow to account for the total thermal impact. The constraints could also include generator capability tradeoffs. See Feinstein et al. (1988), pp. 22–26, for a discussion of the generator capability curve tradeoffs between real and reactive power.

Cases where there may be no solution present a real challenge to electrical systems, as when there is no convergence to a stable load flow, or for markets, when there may be no price incentives that can support a feasible equilibrium solution. Both pathological circumstances would present difficulties for electricity markets that go beyond the discussion of FTR formulations. Hence, while not claiming that all such economic dispatch problems are well-behaved, we will restrict attention to the case when (1.4) is well-behaved.

There are many conditions that could be imposed to guarantee that the economic dispatch problem in (1.4) meets this condition. For our purposes, it is simple to restrict attention to problems that satisfy the optimality conditions¹³:

$$\begin{aligned}
 & \text{There exists } (y^*, u^*, \lambda, \eta), \text{ such that} \\
 & L(y^*, u^*) + t'y^* = 0, \\
 & \tilde{K}(y^*, u^*) \leq 0, \quad \eta^t \tilde{K}(y^*, u^*) = 0, \\
 & \eta \geq 0, \quad u^* \in U, \\
 & (y^*, u^*) \in \arg \max_{y, u \in U} [B(y) - \lambda(L(y, u) + t'y) - \eta^t \tilde{K}(y, u)].
 \end{aligned}$$

Hence, there is no duality gap (Bertsekas 1995). The Lagrange multipliers provide the “shadow prices” for the constraints. The solution for the economic dispatch problem is also a solution for the corresponding dual function for this economic dispatch problem:

$$\text{Max}_{y, u \in U} [B(y) - \lambda(L(y, u) + t'y) - \eta^t \tilde{K}(y, u)].$$

Assuming differentiability, the first order conditions for an optimum (y^*, u^*) include:

$$\nabla B(y^*) - \lambda(\nabla L_y(y^*, u^*) + t') - \eta^t \nabla \tilde{K}_y(y^*, u^*) = 0.$$

Hence, we have the locational prices as

$$p^t = \nabla B(y^*) = \lambda t' + \lambda \nabla L_y(y^*, u^*) + \eta^t \nabla \tilde{K}_y(y^*, u^*).$$

The locational prices have the usual interpretation as the price of power at the swing bus ($p_G = \lambda$), the marginal cost of losses ($p_L = \lambda \nabla L_y(y^*, u^*)$) and the

¹³ As an historical note, apparently the early work on optimality conditions by Kuhn and Tucker was motivated by an inquiry into the theory of electrical networks. Kuhn (2002), p. 132.

marginal cost of congestion ($p_C = \eta^t \nabla \tilde{K}_y(y^*, u^*)$).¹⁴ These locational prices play an important role in a coordinated spot market and in the definition of FTRs.

1.3.2 Security-Constrained Economic Dispatch

The optimal power flow formulation in (1.4) ignores the standard procedure of imposing security constraints to protect against contingent events. Although the formulation could be interpreted as including security constraints, it is helpful here to be explicit about the separate security constraints in anticipation of the later discussion of FTR formulations and auctions that include the many contingency limits.

The basic idea of security-constrained dispatch is to identify a set of possible contingencies, such as loss of a line or major facility, and to limit the normal dispatch so that the system would still remain within security limits if the contingency occurs. The modeled loss of the facility leaves the remaining elements in place, suggesting the name of $n-1$ contingency analysis.¹⁵

Hence, a single line may have a normal limit of 100 MW and an emergency limit of 115 MW.¹⁶ The actual flow on the line at a particular moment might be only 90 MW, and the corresponding dispatch might appear to be unconstrained. However, this dispatch may actually be constrained because of the need to protect against a contingency. For example, the binding contingency might be the loss of some other line. In the event of the contingency, the flows for the current pattern of generation and load would redistribute instantly to cause 115 MW to flow on the line in question, hitting the emergency limit. No more power could be dispatched than for the 90 MW flow without potentially violating this emergency limit. The net loads that produced the 90 MW flow, therefore, would be constrained by the dispatch rules in anticipation of the contingency. It would be the contingency constraint and not the 90 MW flow that would set the limit. The corresponding prices would reflect these contingency constraints (Boucher et al 1998).

Depending on conditions, any one of many possible contingencies could determine the current limits on the transmission system. During any given hour, therefore, the actual flow may be, and often is, limited by the impacts that would occur in the event that the contingency came to pass. Hence, the

¹⁴ The dispatch and prices are not changed by the arbitrary designation of the swing bus. However, the choice of the reference bus for pricing, which need not be the same as the swing bus, does affect the decomposition of the prices.

¹⁵ A simultaneous loss of multiple facilities would be defined as a single contingency.

¹⁶ Expressing the limits in terms of MW and real power is shorthand for ease of explanation. Line limits in AC models appear in terms of MVA for real and reactive power.

contingencies do not just limit the system when they occur; they are anticipated and can limit the system all the time. In other words, analysis of the power flows during contingencies is not just an exception to the rule; it is the rule. The binding constraints on transmission generally are on the level of flows or voltage in post-contingency conditions, and flows in the actual dispatch are limited to ensure that the system could sustain a contingency.

For instance, suppose that the contingency ω is the loss of a line. For sake of simplicity in the illustration, assume that the only adjustment in the case of the contingency is to change the net load at the swing bus to rebalance the system. Then there would be a different network, different flows, and different losses, leading to a new set of power flow constraints described as:

$$\begin{aligned} L^\omega(y_s^\omega, \tilde{y}, u) + y_s^\omega + t\tilde{y} &= 0, \\ \tilde{K}^\omega(y_s^\omega, \tilde{y}, u) &\leq 0, \\ u &\in U. \end{aligned} \tag{1.5}$$

The values of the constraint limits could be different in different contingencies, including changes in monitored elements. Extension of this model to allow other changes in dispatch or control parameters present no problem in principle, but would add to the complexity of the notation. The set of constraints and balancing equations would be different for each contingency.

If we treat normal operations as the contingency $\omega = 0$, then the combined set of constraints on the dispatch would be:

$$\begin{aligned} L^\omega(y_s^\omega, \tilde{y}, u) + y_s^\omega + t\tilde{y} &= 0, \quad \omega = 0, 1, 2, \dots, N, \\ \tilde{K}^\omega(y_s^\omega, \tilde{y}, u) &\leq 0, \quad \omega = 0, 1, 2, \dots, N, \\ u &\in U. \end{aligned}$$

The security-constrained economic dispatch imposes all these constraints on the net loads in advance of the realization of any of the contingencies. However, since the swing bus net load is different in every contingency, we subsume the load balance impacts for $\omega > 0$ in the definition of the constraints, and keep explicit only the loss balance in normal conditions. Then with the appropriate change in notation with $(y^t = (y_s^0, \tilde{y}^t))$, we arrive at a compact representation of the constraints as:

$$\begin{aligned}
& L(y, u) + t'y = 0, \\
& K(y, u) \equiv \begin{pmatrix} \tilde{K}^0(y_s^0, \tilde{y}, u) \\ \tilde{K}^1(y_s^1, \tilde{y}, u) \\ \vdots \\ \tilde{K}^\omega(y_s^\omega, \tilde{y}, u) \\ \vdots \\ \tilde{K}^N(y_s^N, \tilde{y}, u) \end{pmatrix} \leq 0, \\
& u \in U.
\end{aligned}$$

With this notational convention, we can then restate the security-constrained economic dispatch problem as:

$$\begin{aligned}
& \underset{y, u \in U}{\text{Max}} B(y) \\
& \text{s.t.} \\
& L(y, u) + t'y = 0, \\
& K(y, u) \leq 0.
\end{aligned} \tag{1.6}$$

However, we now recognize that the single loss balance equation that affects the benefit function is appended by many contingency constraints that limit normal operations. If there are thousands of monitored elements for possible overloads of lines, transformers, or voltage constraints, and there are hundreds of contingencies that enter the protection set, the total number of constraints in K would be on the order of hundreds of thousands. This large scale is inherent in the problem, and a challenge for FTR models.

It is a remarkable fact that system operators solve just such contingency-constrained economic dispatch problems on a regular basis. Below we summarize a basic outline of a solution procedure to capture the elements relevant to the FTR formulations. This method exploits a relaxation strategy and the feature that as we get closer to the actual dispatch, the pattern if loads are better known and the list of plausible contingencies and monitored elements reduces accordingly. Anticipating the discussion of FTRs, however, the larger potential set of constraints would be relevant.

Under the assumed optimality conditions, the corresponding prices obtained from the solution appear as:

$$p^t = \nabla B(y^*) = \lambda^t + \lambda \nabla L_y(y^*, u^*) + \eta^t \nabla K_y(y^*, u^*).$$

Hence, the congestion cost could arise from any of the (many) contingency constraints.

1.3.3 Market Equilibrium

The security-constrained economic dispatch problem has the familiar close connection to the competitive partial equilibrium model where market participants act as profit maximizing or welfare maximizing price takers.

Assume that each market participant has an associated benefit function for electricity defined as $B_i(y_i)$, which is concave and continuously differentiable.¹⁷ In FERC terminology, the market participants are the transmission service customers. The customers' benefit functions can arise from a mixture of load or demand benefits and generation or supply costs. In this framework, the producing sector is the electricity transmission provider, with customers injecting power into the grid at some points and drawing power out of the grid at other points. The system operator receives and delivers power, coordinates a spot market, and provides transmission service across locations.

The competitive market equilibrium applied here is based on the conventional partial equilibrium framework that stands behind the typical supply and demand curve analysis.¹⁸ The market consists of the supply and demand of electric energy and transmission service plus an aggregate or numeraire "good" that represents the rest of the economy. Each customer is assumed to have an initial endowment \tilde{w}_i of the numeraire good. In addition, each customer has an ownership share s_i in the profits " π " of the electricity transmission provider, with $\sum_i s_i = 1$.

An assumption of the competitive model is that all customers are price takers. Hence, given market prices, p , customers choose the level of consumption of the aggregate good, c_i , and electric energy including the use of the transmission system

¹⁷ A sufficient condition for these to obtain would be that the demand and supply functions at each node are continuous, additively separable and aggregate into a downward sloping net demand curve. The benefit function would be the area under the demand curves minus the area under the supply curves in the usual consumer plus producer surplus interpretation at equilibrium. To avoid notational complexity, the assumption here is that each participant has a continuously differentiable concave benefit function defined across the net loads at every location. Concavity is important for the analysis below of the equivalence of economic dispatch and market equilibrium, if there is a market equilibrium. This would eliminate from this competitive market analysis the related unit commitment problem which includes non-convex start-up conditions. As is well known, in the presence of non-concave benefit functions there may be no competitive market equilibrium. Differentiability can be relaxed, with no more than the possibility of multiple equilibrium prices. Restricting the benefit function to definition at a subset of the locations would be more realistic, but different only in the need to account for the corresponding variable definitions. It would not affect the results presented here. In practice, as is often assumed, the benefits functions may be separable across locations.

¹⁸ The partial equilibrium assumptions are that electricity is a small part of the overall economy with consequent small wealth effects, and prices of other goods and services are approximately unaffected by changes in the electricity market. See Mas-Colell et al. (1995), pp. 311–343. Importantly, we adopt here a relaxed set of assumptions that do not include convexity of the set of feasible net loads.

according to the individual optimization problem maximizing benefits subject to an income constraint:

$$\begin{aligned}
 & \underset{y_i, c_i}{\text{Max}} B_i(y_i) + c_i \\
 & \text{s.t.} \\
 & p^t y_i + c_i \leq \tilde{w}_i + s_i \pi.
 \end{aligned} \tag{1.7}$$

In this simple partial equilibrium model of the economy, there is only one producing entity, which is the system operator providing transmission service. Under the competitive market assumption, the producer is constrained to operate as a price taker who chooses inputs and outputs (y_i) that are feasible and that maximize profits. The profits amount to $\pi = p^t \sum_i y_i$. Hence, the transmission system operator's problem is seen as:

$$\begin{aligned}
 & \underset{y_i, u \in U}{\text{Max}} p^t y \\
 & \text{s.t.} \\
 & y = \sum_i y_i, \\
 & L(y, u) + t^t y = 0, \\
 & K(y, u) \leq 0.
 \end{aligned} \tag{1.8}$$

Of course, the transmission service provider is a monopoly and would not be expected to follow the competitive assumption in the absence of regulatory oversight. However, the conventional competitive market definition provides the standard for the service that should be required of the system operator.¹⁹

Given the initial endowment of goods \tilde{w}_i , and the ownership shares s_i , a competitive market equilibrium is defined as a vector of prices, p , profits, π , controls, u , and a set of net loads, y_i , for all i that simultaneously solve (1.7) and (1.8).

A competitive equilibrium will have a number of important properties that we can exploit. First, note that $\sum_i c_i = \sum_i \tilde{w}_i$, which is implied and necessary for feasibility. Furthermore, every customer's income constraint is binding and the derivative of each benefit function will equal the common market prices, $p = \nabla B_i^t$. Hence, the equilibrium price at each location is equal to the market clearing

¹⁹ It is the standard formulation to include both the consumption (1.7) and production (1.8) sectors as part of the definition of competitive market equilibrium. Failure to follow this well established convention leads to confusion when the term "market equilibrium" is applied excluding the producing sector in (1.8), as in Wu et al. (1996), pp. 5–24. For a further discussion of equivalence results, see Boucher and Smeers (2001), pp. 821–838.

marginal benefit of net load and the marginal cost of generation and redispatch to meet incremental load.

Finally, a motivation for the connection with economic dispatch is that a market equilibrium $(\{y_i^*\}, u^*)$ must also be a solution to the economic dispatch problem with $B(y) = \sum_i B_i(y_i)$. If not, there would be a set of feasible net loads $\{y_i^1\}$ with $\sum_i B_i(y_i^1) > \sum_i B_i(y_i^*)$. Therefore, by concavity of B we would have:

$$p^t \left(\sum_i (y_i^1 - y_i^*) \right) = \sum_i \nabla B_i(y_i^1 - y_i^*) \geq \sum_i (B_i(y_i^1) - B_i(y_i^*)) > 0.$$

But this would violate the optimality of $(\{y_i^*\}, u^*)$. Hence, a market equilibrium is also a solution to the economic dispatch problem.

Therefore, under the optimality conditions assumed, the market equilibrium would satisfy the same local first-order necessary conditions as an optimal solution to the economic dispatch. In particular, for a market equilibrium we have the pricing condition that:

$$p^t = \nabla B(y^*) = \lambda^t + \lambda \nabla L_y(y^*, u^*) + \eta^t \nabla K_y(y^*, u^*).$$

Another way to look at this problem is to interpret the equilibrium as satisfying the “no arbitrage” condition. At equilibrium, there are no feasible trades of electric loads in (1.8) that would be profitable at the prices p . Hence, let y^1 be any other feasible set of net loads, such that there is a u^1 with:

$$\begin{aligned} L(y^1, u^1) + t^t y^1 &= 0, \\ K(y^1, u^1) &\leq 0, \\ u^1 &\in U. \end{aligned}$$

Then by (1.8), we have,

$$p^t (y^* - y^1) \geq 0. \tag{1.9}$$

This no arbitrage condition will be important as part of the analysis of revenue adequacy in the FTR formulations. Importantly, the condition allows for the controls to change from u^* . This implies a great degree of flexibility in changing the dispatch while maintaining the no-arbitrage condition for a market equilibrium.

1.3.4 Linear Approximation of Constraints

The full AC security-constrained economic dispatch problem is a large optimization problem with very many constraints. Solution procedures for solving this problem often rely on local linearizations of at least the constraints and exploit the condition that in any particular dispatch only relatively few (tens to hundreds) of the many potential constraints might be binding.

One motivation for the linearization follows from the first order conditions for an optimum. Suppose we have a solution to the economic dispatch problem at (y^*, u^*) . The usual Taylor approximation gives:

$$L(y, u) \approx L(y^*, u^*) + \nabla L(y^*, u^*) \begin{pmatrix} y - y^* \\ u - u^* \end{pmatrix},$$

$$K(y, u) \approx K(y^*, u^*) + \nabla K(y^*, u^*) \begin{pmatrix} y - y^* \\ u - u^* \end{pmatrix}.$$

Then if we have a solution that satisfies the first order conditions for the security-constrained economic dispatch problem (1.6), this would also satisfy the first order conditions for the linearized constraints as in:

$$\begin{aligned} & \underset{y, u \in U}{\text{Max}} B(y) \\ & \text{s.t.} \\ & L(y^*, u^*) + \nabla L(y^*, u^*) \begin{pmatrix} y - y^* \\ u - u^* \end{pmatrix} + t'y = 0, \\ & K(y^*, u^*) + \nabla K(y^*, u^*) \begin{pmatrix} y - y^* \\ u - u^* \end{pmatrix} \leq 0. \end{aligned}$$

If the functions are well behaved, then finding a solution to this approximate problem might also provide a good estimate of the solution to the full problem. Although the functions are not so well behaved as to be everywhere convex, practical computational approaches for solving this problem search for a solution that satisfies the first order conditions. It is not fail safe, and when it fails other approaches would be necessary. However, given a starting point close to the optimum, and some judicious choices, this approximation can work well. Since the actual dispatch involves reoptimization starting with a good solution from the immediate previous period, as well as feedback from metering actual flows and a fair bit of operator judgment, this linearization of the model can be a reasonable approximation. However, as discussed below, the linearization changes with the dispatch.

The local linear approximation suggests an outline for solving this large problem through a familiar relaxation approach by ignoring non-binding constraints (Geoffrion 1970).

Relaxation Solution Procedure

- Step 1:** Select an initial candidate solution (y^0, u^0) , ignore most (or all) of the constraints in the economic dispatch using only the small subset $K^0(y, u)$, and set the iteration count to $m = 0$.
- Step 2:** Construct the relaxed master problem as:

$$\begin{aligned} & \underset{y, u \in U}{\text{Max}} B(y) \\ & \text{s.t.} \\ & L(y^m, u^m) + \nabla L(y^m, u^m) \begin{pmatrix} y - y^m \\ u - u^m \end{pmatrix} + t^t y = 0, \\ & K^m(y^m, u^m) + \nabla K^m(y^m, u^m) \begin{pmatrix} y - y^m \\ u - u^m \end{pmatrix} \leq 0. \end{aligned}$$

Let a solution be (y^{m+1}, u^{m+1}) and update $m = m + 1$.

- Step 3:** Check to see if the candidate solution (y^m, u^m) violates any of the constraints in (1.6). If so, create a new $K^m(y, u)$ including some or all of these constraints and repeat Step 2. Else done.

The central idea here is that the master problem is much smaller than the full problem and relatively easy to solve. With judicious choices of the initial solution and constraint set, the method works well in practice with relatively few iterations required. In the case that the objective function is represented by a piecewise linearization (as would be true naturally with step-wise representation of supply and demand), the master problem is a linear program for which there are efficient algorithms. Furthermore, in the case of this dispatch problem, evaluation of constraints in Step 3 requires only that a standard load flow be solved for each contingency. Although not trivial, this is well-understood albeit non-linear problem.

One difficulty with this computational approach is the need to calculate $\nabla K^m(y^m, u^m)$.²⁰ This gradient is the set of “shift factors” summarizing the marginal impact on constraints from changes in the loads and controls. Although it is possible to solve the load flow problem exploiting the sparsity of the network arising from the few links connected to each bus, this sparsity depends on explicit representation of the angles and voltage magnitudes. By contrast, the inverse presentation in $K(y, u)$ is dense. In a sufficiently meshed network, every net load affects every constraint. Hence, virtually every element of $\nabla K^m(y^m, u^m)$ could be non-zero. Part of the art of implementation of this computational outline is in the details of exploiting sparse representations to evaluate load

²⁰ For more detail on the construction of the gradients, see Weber (1997).

flows, and minimizing the need to calculate or represent $\nabla K^m(y^m, u^m)$. Such commercial dispatch software is well developed and in regular use.²¹

Further note that in general $\nabla K(y^{m_1}, u^{m_1}) \neq \nabla K(y^{m_2}, u^{m_2})$, and this may require frequent updates of the linearization. Finally, in general we have:

$$K(y^{m_1}, u^{m_1}) - \nabla K(y^{m_1}, u^{m_1}) \begin{pmatrix} y^{m_1} \\ u^{m_1} \end{pmatrix} \neq K(y^{m_2}, u^{m_2}) - \nabla K(y^{m_2}, u^{m_2}) \begin{pmatrix} y^{m_2} \\ u^{m_2} \end{pmatrix}.$$

Hence, the “right hand side” of the linearized constraint can be different for each candidate solution. These differences can be quite large, especially for interface constraints in DC-Load approximations.²²

This presents no difficulty in principle for the dispatch problem. However, these complications are relevant in the discussion of the DC-Load model and in the adaptation of the security-constrained economic dispatch formulation for FTR auctions.

1.3.5 DC-Load Approximations

A common simplification of the load flow model for real power is known as the DC-Load approximation (Schweppe et al. 1988). In terms of the present discussion, the DC-Load model adds further restrictive assumptions that allow us to ignore both real power losses and reactive power loads in determining the real power flows, further specializing the linearization of the constraints.

The key assumptions include:

- There is sufficient reactive power net load at each bus to maintain per unit voltages equal to 1.0 ($V_i \approx 1.0$);
- All phase angle settings are at zero angle change and a fixed tap ratio for transformers ($t = 1.0$, $\alpha = 0$)²³;
- The voltage angle differences across lines are small.

These assumptions imply a choice of controls ($u = u^0$) that yield full decoupling between real and reactive power flow and no transmission losses. The real power flow in (1.1) reduces to:

²¹ For example, firms providing such software include ALSTOM ESCA Corporation, Nexant, Inc., Open Access Technology International, Inc.

²² For examples, see Hogan (2000).

²³ For simplicity, we can assume that the ideal transformers with a fixed tap ratio have been incorporated in a per unit normalization, which results in a simplified Π -equivalent representation of a transmission line. See the appendix for further details

$$\begin{aligned} z_{Pijk} &= G_k[1 - \cos(\delta_i - \delta_j)] + \Omega_k \sin(\delta_i - \delta_j), \\ z_{Pjik} &= G_k[1 - \cos(\delta_j - \delta_i)] + \Omega_k \sin(\delta_j - \delta_i). \end{aligned}$$

Under the small angle difference assumption, we have:

$$\begin{aligned} \cos(\delta_i - \delta_j) &\approx 1, \\ \sin(\delta_i - \delta_j) &\approx \delta_i - \delta_j. \end{aligned}$$

Hence, the real power flow approximation becomes:

$$z_{Pk} = z_{Pijk} = \Omega_k(\delta_i - \delta_j) = -z_{Pjik} = -\Omega_k(\delta_j - \delta_i).$$

This linearity produces a substantial simplification. Let:

Ω = the diagonal matrix of line transfer factors,

z = the vector of line flows (z_{Pk}) in the DC-Load approximation.

Then, with our sign conventions we have:

$$\begin{aligned} y &= -A'z, \\ z &= \Omega A\delta. \end{aligned}$$

Furthermore, the inversion that eliminates the angles as in (1.2) reduces to another linear equation for the DC-Load formulation with

$$H = \nabla K_y(0, u^0).$$

This is the matrix of shift factors. Under the DC-Load assumptions, $H = (0 \quad \tilde{H})$, where $\tilde{H} = -\Omega\tilde{A}(\tilde{A}'\Omega\tilde{A})^{-1}$ with the swing bus dropped in defining \tilde{A} .²⁴ Although A is sparse, the matrix of shift factors is dense, meaning that nearly every net load affects nearly every line. Calculating an element of row of H , meaning the shift factors for a particular line in a particular contingency, is about the same amount of work as finding a DC-Load flow for that contingency.

For a given contingency the matrix that links the angles and the net loads, as in

$$y = -A'\Omega A\delta,$$

is quite sparse, with the only non-zero elements being for the nodes that are directly connected. Furthermore, solving for the angles given the vector of net injections, y , involves no more than finding a particular solution for a set of linear equations. In general, this is much less work than solving for the full matrix inverse, and in advanced optimization algorithms this is done quickly and cheaply using sparse

²⁴ Also the transfer admittance matrix as described in Scheweppe et al. (1988), p. 316.

matrix techniques. Once the vector of angles is known for a given set of net loads, it is an easy matter to complete the one matrix multiplication to obtain the complete load flow in z for each contingency. The import of all this is the simplicity of evaluating a particular load flow as compared to calculating the full transfer admittance matrix in H .

Note that calculating a particular row of H is about the same order of difficulty as evaluating the load flow for that particular contingency. Let ε^i be the elementary row vector with all zeros but a 1 in the i_{th} position. We can obtain any row of H , say h_i , as the solution to a set of sparse linear equations. By construction:

$$h_i = \varepsilon^i \tilde{H} = -\varepsilon^i \Omega \tilde{A} \left(\tilde{A}^t \Omega \tilde{A} \right)^{-1}.$$

Hence, we have the sparse system:

$$h_i \left(\tilde{A}^t \Omega \tilde{A} \right) = -\varepsilon^i \Omega \tilde{A}. \quad (1.10)$$

In other words, calculating a complete load flow for all the lines is about as much work as calculating the shift factors for one line. Both require solution of a sparse set of linear equations of the dimension equal to the number of nodes. There are specialized sparse matrix techniques for this computation as a part of commercial dispatch software.

With these approximations, the constraints could be restated as:

$$\begin{aligned} t'y &= 0, \\ K(0, u^0) + Hy &\leq 0. \end{aligned}$$

Letting $b = -K(0, u^0)$, the familiar DC-Load restatement of the security-constrained economic dispatch becomes:

$$\begin{aligned} & \underset{y}{\text{Max}} B(y) \\ & \text{s.t.} \\ & t'y = 0, \\ & Hy \leq b. \end{aligned} \quad (1.11)$$

It is an easy matter to extend the definition of H to include other linear constraints on y , including interface constraints expressed as limits on aggregations of flows on lines.

As above, the matrix H for the full security-constrained problem is very large and dense, and successful solution of the security-constrained economic dispatch exploits approaches such as the relaxation algorithm outlined above that avoid unnecessary computation of the elements of H and include only the binding

constraints. Furthermore, the DC-Load model is convex and the relaxation algorithm will assure convergence to a global solution.

As discussed below, many models for transmission rights exploit the specialized structure of (1.11) to simplify the problem and guarantee various equivalence conditions between and among different FTRs. In this context, it is important to remember that (1.11) is only a simplified approximation and that key elements of these assumptions are violated by regular operating conditions in the system. The different approximations have different effects on the alternative FTR models and the associated auction problems.

Here we consider the implications of various modifications of these assumptions. Suppose that the phase shifting transformers are set to shift the angles. If we hold the angle shifts fixed, then the approximation under the other DC-Load assumptions becomes:

$$z_{pk} = \Omega_k(\delta_i - \delta_j) + \Omega_k\alpha_k.$$

In principle, this changes the inversion in (1.2) to eliminate the bus angles such that even under zero net loads there would be real power flow on all the lines in order to maintain balance at every node. This preserves linearity and a constant H , but changes the residual limits for the constraints. Hence, we would have $b = b(\alpha)$, meaning that the limits on the power flow equations would be changing to reflect the phase angle settings. In principle, a phase shift on one line could affect the residual limit on every line.

If the ideal transformer tap ratio (t) were to change from 1.0, there would be a modified $\hat{\Omega}$ to reflect the changing impedance.²⁵ In addition, the inversion depends on the topology of the network as summarized in A . This may change from one dispatch to another. In each case, the inversion to eliminate the voltage angles and the associated linearization of the constraints actually depends on the values of (t, α, A) . To the extent that these are treated as variables in the economic dispatch, their constraints in U create additional non-linearities. For instance, if a phase-shifting transformer is controlling flow but reaches a limit on the ability to control a line, the representation of the phase angle regulator changes. Although the details depend on the particular case, if there is any possibility of actual changing the topology or settings of phase-shifting transformers, even for the simplified real power only DC-Load approximation we have $H(u^0) = \nabla K_y(0, u^0)$ and $b(u^0) = -K(0, u^0)$. In other words, the linear approximation is not the same across the dispatches.

Therefore, the security-constrained, economic dispatch of the DC-Load approximation could be written as:

²⁵ For example, see Oliveira et al. (1999), pp. 111–118.

$$\begin{aligned}
& \underset{y}{\text{Max}} B(y) \\
& \text{s.t.} \\
& t'y = 0, \\
& H(u^0)y \leq b(u^0).
\end{aligned} \tag{1.12}$$

When this problem is solved at any given hour, for fixed u^0 the resulting model takes on the form of the DC-Load approximation. Both constraint limits and shift factors adjust regularly. Hence, it is important below to be explicit about the fact that the linearizations, and therefore, the model itself, changes from dispatch to dispatch, especially for any changes in topology A .

Finally, in addition to these changes, other slight modifications of the DC-Load model retain most of the computational simplicity but make the approximation further sensitive to the non-linear properties of the system. For example, consider incorporating line losses:

$$l_{Pk}(\delta, 1, 1, 0) = G_k [2 - 2 \cos(\delta_i - \delta_j)].$$

Using the approximation that for small angle differences,

$$\cos(\delta_i - \delta_j) \approx 1 - \frac{(\delta_i - \delta_j)^2}{2},$$

the approximate line losses are:

$$l_{Pk}(\delta, 1, 1, 0) = G_k [2 - 2 \cos(\delta_i - \delta_j)] \approx G_k (\delta_i - \delta_j)^2 \approx r_k z_{Pk}^2.$$

Here we have used the condition that $r_k \ll x_k$.²⁶

Define R as the diagonal matrix of line resistances, $|A|$ as the matrix of the absolute values of the incidence matrix, and z^2 as the vector of squares of the individual line flows. Then we could include losses in the economic dispatch problem that is almost like the DC-Load model²⁷:

²⁶ This approximation applies to high voltage systems, but is less usable on lower voltage circuits.

²⁷ This approach is from Transpower in New Zealand.

$$\begin{aligned}
& \text{Max } B(y) \\
& y, z, \delta \\
& \text{s.t.} \\
& y = -A^t z - \frac{1}{2}|A|^t R z^2, \\
& z = \Omega A \delta, \\
& \delta_s = 0, \\
& z \leq b.
\end{aligned}$$

Note that this computational form of the problem does not need a separate overall balance equation, as this is accounted for in the individual node equations. Hence, we have net loads (generation) balancing losses as in:

$$t'(g - d) = -t'(d - g) = -t'y = t'A^t z + \frac{1}{2}t'|A|^t R z^2 = t'R z^2.$$

This is no longer a linear problem, but the addition of the few quadratic terms in the node balance equations is easier to deal with than a full AC model. However, this simplified formulation would capture some of the interaction between losses and congestion, with the additional power flows needed to account for losses adding to losses and congestion. The inverse linearization of the solution in terms of the net loads would now differ further from the pure DC-Load approximation.²⁸

1.4 Point-to-Point Financial Transmission Rights

Financial transmission rights are defined in terms of payments related to market prices. Although many years were spent in the search for well-defined and workable physical transmission rights, the complexity of the grid and rapidly changing conditions of the real market outcomes made it impossible to design physical rights that could be used to determine the use of the transmission system.²⁹ By contrast, financial transmission rights specify payments that are connected to the market outcomes but do not control use of the system. Rather, the actual dispatch or spot market produces a set of market-clearing prices, and these prices in turn define the payments under the FTRs.

The system operator accepts schedules and coordinates the spot market as a bid-based, security-constrained, economic dispatch. The resulting locational prices apply to purchases and sales through the spot market, or the difference in the locational prices defines the price for transmission usage for bilateral schedules.

²⁸ A version of this DC-Load-Flow implementation with losses appears in a GAMS model available at www.whogan.com.

²⁹ For further details, see Harvey et al. (1997).

The need for transmission rights to hedge the locational price differences leads to the interest in FTRs.³⁰

1.4.1 PTP Obligations

The definition of point-to-point (PTP) forward obligations as FTRs follows closely the notion of bilateral transmission schedules. A generic definition includes both balanced and unbalanced rights. Given a vector of inputs and outputs by location, the k th PTP forward obligation is defined by:

$$PTP_k^f = \begin{pmatrix} 0 \\ -g_i \\ 0 \\ d_j \\ 0 \end{pmatrix}.$$

With a corresponding vector of market clearing prices, this FTR is a contract to receive

$$p^t PTP_k^f = p^t \begin{pmatrix} 0 \\ -g_i \\ 0 \\ d_j \\ 0 \end{pmatrix} = p_j d_j - p_i g_i.$$

Although any such vector could be allowed, it is clear that any such FTR could be restated as a mix of balanced and unbalanced rights:

$$PTP_k^f = \begin{pmatrix} 0 \\ -d_j \\ 0 \\ d_j \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ g_i - d_j \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Motivated by the discussion of options below, it is convenient to define two types of forward obligations, balanced (τ_k^f) and unbalanced (\bar{g}_k^f), such as

³⁰ For further discussion of market structure, see Chandley and Hogan (2002).

$$\tau_k^f = \begin{pmatrix} 0 \\ -x \\ 0 \\ x \\ 0 \end{pmatrix}, \quad \bar{g}_k^f = \begin{pmatrix} 0 \\ 0 \\ g \\ 0 \\ 0 \end{pmatrix}.$$

We can think of the balanced PTP-FTRs providing for the same input and output at different locations. More generally, all that is required of a balanced PTP-FTR is that the inputs and outputs sum to zero, $t^f \tau_k^f = 0$. The unbalanced FTRs can be thought of as forward energy sales at any location and would be a contribution towards losses to balance the system. The notation suggests that individuals could hold either or both types of PTP-FTR forward obligations, and there is no need that the locations be the same.

The intended role of the PTP-FTR is to provide a hedge against variable transmission costs. If a market participant has a balanced FTR between two locations and schedules a corresponding bilateral transaction with the same inputs and outputs (x), then the charge for using the system would be $(p_j - p_i)x$, which is exactly the payment that would be received under the FTR. Hence, the balanced FTR provides a perfect hedge of the variable transmission charge for the bilateral transaction.

The holder of an unbalanced forward obligation FTR has an obligation to make the payment equal to the value of the energy at the relevant location. If the holder also sells an equal amount of energy at the same location in the actual dispatch, the payment received for the energy is $p_i g$, equal to the payment required under the FTR. Hence, we can think of the unbalanced FTR as a forward sale of energy. Although in principle there would be no difficulty in allowing negative unbalanced PTP-FTRs, equivalent to forward purchases of energy, it is convenient to interpret unbalanced PTP-FTR obligations as forward sales of energy.

In this case of obligations, the PTP-FTRs are easily decomposable. For example, an FTR from bus 1 to bus 2 can be decomposed into two PTP-FTR obligations from 1 to a Hub and the Hub to 2. The total payment is $(p_2 - p_{HUB}) + (p_{HUB} - p_1) = (p_2 - p_1)$. This provides support for trading at market hubs and the associated trading flexibility. Periodic FTR auctions provide other opportunities to obtain other reconfigurations of the pattern of FTRs

An attraction of the FTR is that the spot market can operate to set the actual use of the transmission system and the FTRs operate in parallel through the settlements system to administer financial hedges. Importantly, the system of payments will be consistent as long as the set of PTP-FTRs satisfies a simultaneous feasibility condition.

1.4.1.1 Revenue Adequacy

Suppose that we have a set of balanced ($\tau_k^f, k = 1, \dots, N$) and unbalanced ($\bar{g}_k^f, k = 1, \dots, N$) PTP-FTRs obligations for any possible locations. Consider the constraints from the security-constrained dispatch in (1.6) or equivalently in (1.8). We say that the set of FTRs is simultaneously feasible if there is a $u \in U$ such that:

$$\begin{aligned} y &= \sum_k \tau_k^f - \sum_k \bar{g}_k^f, \\ L(y, u) + t'y &= 0, \\ K(y, u) &\leq 0. \end{aligned} \tag{1.13}$$

Assume the set of PTP-FTR forward obligations is simultaneously feasible. If we have a market equilibrium (p, y^*, u^*) in the spot market, then from (1.9) it follows immediately that we meet the revenue adequacy condition,

$$p^t y^* - p^t \left(\sum_k \tau_k^f - \sum_k \bar{g}_k^f \right) = p^t \left(y^* - \sum_k \tau_k^f + \sum_k \bar{g}_k^f \right) = p^t (y^* - y) \geq 0.$$

In other words, at the market equilibrium prices the net payments collected by the system operator through the actual dispatch ($p^t y^*$) would be greater than or equal to the payments required under the PTP-FTR forward obligations $\left(p^t \left(\sum_k \tau_k^f - \sum_k \bar{g}_k^f \right) \right)$. This revenue adequacy condition is general enough to accommodate a great deal of flexibility.

Note that the simultaneous feasibility condition does not require that the set of PTP-FTRs be feasible at the current set of controls (u^*) associated with the market equilibrium. All that is required is that the system operator could choose a set of controls that would make the PTP-FTRs feasible. There could be a very different set of actual operating conditions, including changes in the configuration of the grid, but as long as the controls and configuration could be set to make the PTP-FTRs feasible, the simultaneous feasibility condition holds and revenue adequacy follows. This is true even though actual physical delivery to match the FTRs would be impossible at the current settings of the grid controls at u^* . This is an important simplification compared to physical rights and a primary attraction of using financial rights.

The intuition of revenue adequacy is clear. If the dispatch of PTP-FTRs were more valuable than the market equilibrium, in violation of the revenue adequacy condition, the system operator could have selected this dispatch outcome. Since we have by assumption a market equilibrium that differs from the PTP-FTRs, and the PTP-FTRs are simultaneously feasible, the market equilibrium

from (1.8) must be at least as valuable as the payment obligation under the PTP-FTRs.³¹

1.4.1.2 PTP-FTR Auction

Allocation rules for FTRs follow different procedures. For example, in PJM Load Serving Entities (LSE) are required to purchase network service and meet installed capacity requirements. As part of this process, LSEs acquire FTRs. Grandfathering rules under existing contracts might be another source of allocation, and so on.

A natural way to allocate PTP-FTR forward obligations would be to conduct an auction. Suppose that we represent bids for balanced forward-obligations by (t_k^f, τ_k^f) and for unbalanced forward obligations by (ρ_k^f, \bar{g}_k^f) . Here the first element is the scalar amount of the FTR and the second element is the vector pattern of inputs and outputs. For simplicity, we subsume any upper bounds on the awards are part of the concave and differentiable bid function $\beta_k(t_k^f, \rho_k^f)$. With these notational conventions, a formulation of the PTP-FTR forward obligation auction would be:

$$\begin{aligned}
 & \text{Max}_{y, u \in U, t_k^f \geq 0, \rho_k^f \geq 0} \sum_k \beta_k(t_k^f, \rho_k^f) \\
 & \quad \text{s.t.} \\
 & y = \sum_k t_k^f \tau_k^f - \sum_k \rho_k^f \bar{g}_k^f, \\
 & L(y, u) + t^f y = 0, \\
 & K(y, u) \leq 0.
 \end{aligned} \tag{1.14}$$

A solution of this problem would determine the award of FTRs and the associated market clearing prices for the awards. The locational price \hat{p} would be of the same form as in the market equilibrium model, with

$$\hat{p}^t = \hat{\lambda} t^t + \hat{\lambda} \nabla L_y(y^*, u^*) + \hat{\eta}^t \nabla K_y(y^*, u^*).$$

However, the prices here would be based on the expected value of the hedge over the many dispatches to which it applies. The corresponding market clearing prices

³¹ The definition of FTRs could be extended to include the sharing rule for allocation of any difference between the collections and payments. This is formalized in the market equilibrium model as s_j . In practice, the FTR implementations for existing system redistribute any excess collection to reduce access charges or some similar purpose. Although this is a more important issue for defining incentives for system expansion, it does not affect the analysis here.

for the auction awards would be the difference in the locational prices for the balanced obligations and the locational price for loss contributions. Hence,

$$\widehat{p}_{t_k^f} = \frac{\partial \beta_k(t_k^f, \rho_k^f)}{\partial t_k^f} = \widehat{p}^t \tau_k^f, \quad \widehat{p}_{\bar{g}_k^f} = -\frac{\partial \beta_k(t_k^f, \rho_k^f)}{\partial \rho_k^f} = \widehat{p}^t \bar{g}_k^f.$$

By construction, the FTRs would be simultaneously feasible. In addition to an initial sale to allocate FTRs for the existing grid, this same format accommodates offers to sell existing FTRs. By this means, regular auctions of this form also provide opportunities to reconfigure the pattern of FTRs.

It is obvious that the PTP-FTR auction problem in (1.14) is essentially of the same form as the security-constrained economic dispatch problem in (1.6) or the market equilibrium problem in (1.8), with the addition of a set of simple linear constraints on the net loads as dictated by the bids. Furthermore, the addition of the linear constraints on the awards could be included in the master problem of the relaxation solution procedure described above, allowing for a direct adaptation of familiar optimal dispatch software to solve the auction problem. This is the essence of the AC-formulation of the PTP-FTR obligation auction conducted by the New York Independent System Operator (NYISO), where the computational feasibility of the solution procedure has been verified in practice.³²

In the case of a dispatch that prices losses and includes losses in the PTP-FTRs, the consistent model anticipates that market participants will take on the forward commitment to meet the financial requirements for losses. Various approximations might be considered where this is a requirement is modified.³³ In the early implementations, the focus of PTP-FTRs was on congestion costs.

1.4.1.3 PTP-FTR for Congestion

The initial PJM implementation employed a DC-Load dispatch model similar to (1.12).³⁴ The dispatch and the resulting market prices do not explicitly treat marginal losses. Hence, the prices differ across locations only due to the effects of congestion. The PTP-FTRs are defined for payments on congestion cost, and in this case are the full hedge for the difference in locational prices. Under this system, the payments for losses are treated as part of an uplift charge, and not covered by the FTRs. Since the congestion costs define the only locational price differences

³² For results of New York auctions, see: http://www.nyiso.com/markets/tcc_auctions/2001_2002_winter.html.

³³ For a further discussion see Harvey and Hogan (2002).

³⁴ In PJM, financial transmission rights are called fixed transmission rights (FTR). <http://www.pjm.com/energy/fttr/ftrauc.html>.

charged or hedged, revenue adequacy follows from the simultaneous feasibility condition for the PTP-FTRs.

The implementation in New York differs in its treatment of losses. Losses are included in the dispatch model and the associated market prices. However, the PTP-FTRs are defined as balanced rights only and provide for payment of congestion costs but not the cost of losses. The auction for FTRs uses an AC formulation as in (1.14). Market participants obtain balanced FTRs and the NYISO includes provisions for losses in the auction, in order to obtain a feasible solution in the auction. However, the NYISO does not assume financial responsibility for loss hedges. In New York, the FTRs provide a hedge only for congestion costs.³⁵ This New York type implementation leads to a different version of the revenue adequacy condition.

Let the allocation of balanced FTRs in the auction be $\tau^f = \sum_k t_k^f \tau_k^f$. Choose an arbitrary unbalanced vector of loss contributions \bar{g}^f such that (τ^f, \bar{g}^f) is simultaneously feasible. Let there be a market equilibrium (p, y^*, u^*) from the actual dispatch. The prices decompose into the price of generation ($p_G = \lambda$), the marginal contribution to losses ($p_L^t = \lambda \nabla L_y(y^*, u^*)$), and the cost of congestion ($p_C^t = \eta^t \nabla K_y(y^*, u^*)$). By the simultaneous feasibility of the PTP-FTRs, we have

$$p^t y^* \geq p^t (\tau^f - \bar{g}^f) = p_C^t \tau^f + (p_G^t + p_L^t) \tau^f - p^t \bar{g}^f.$$

Define the loss rentals on the FTRs as the difference between the payment for losses at the marginal cost and the average cost of the losses. Hence,

$$\pi_L \equiv p_L^t \tau^f - p^t \bar{g}^f = (p_G^t + p_L^t) \tau^f - p^t \bar{g}^f.$$

If we have these loss rentals as non-negative, $\pi_L \geq 0$, then the simultaneous feasibility test coupled with this condition is enough to ensure that the total net payments from the dispatch are at least as large as the congestion payments under the PTP-FTRs, as in:

$$p^t y^* \geq p_C^t \tau^f. \quad (1.15)$$

Since \bar{g}^f is arbitrary but feasible, we could have chosen \bar{g}^f to maximize the loss rentals for the FTRs given the prices for this hour. In other words, if we have sufficiently inexpensive locations at which to deem the unbalanced FTR loss contribution, the loss rentals would be non-negative and along with the

³⁵ In New York, financial transmission rights are called Transmission Congestion Contracts (TCC).

simultaneous feasibility condition would be sufficient to ensure revenue adequacy in the sense of (1.15) for congestion hedges only.³⁶

In the case of New York, the loss prices and loss rentals may be small, and the typical situation would be that losses would be costly with the maximum loss rentals implied for the FTRs being positive. Under typical conditions, therefore, simultaneous feasibility would guarantee revenue adequacy for the congestion payments under the FTRs.

1.4.2 PTP Options

A PTP-FTR obligation is a financial contract for the payment of the locational price difference. When matched with a corresponding delivery of power, the charge for transmission usage just balances the FTR payment, and there is a perfect hedge. This is true whether or not the price difference is positive or negative. If the price difference is negative, the schedule provides valuable counterflow for which the provider is paid, and the payment from the spot market dispatch just balances the obligation under the FTR. There is a perfect match either way.

A natural complement to the PTP-FTR obligation would be a PTP-FTR option that did not require payment when the price difference was negative. Hence for the balanced PTP-FTR option τ_k^o the payment would be $\max(0, p^t \tau_k^o)$. This financial contract might be more attractive as a tool for hedging purposes, and it is typically the first suggestion from market participants because of the perception that there is a closer analogy to the presumed option not to schedule under a physical right. The option might also be more valuable for speculators who want to trade rights but don't plan to match the FTR with a schedule.

Unlike obligations, PTP-FTRs are not decomposable in the sense of to and from a hub. The difficulty is inherent in the option. For example, an FTR option from bus 1 to bus 2 cannot be decomposed into two PTP-FTR options from 1 to a Hub and the Hub to 2. The total payment under the two options would be $\max(0, p_2 - p_{HUB}) + \max(0, p_{HUB} - p_1) \neq \max(0, p_2 - p_1)$. Hence, reconfiguration of options would require coordination in a formal auction.

Whatever the merits of the PTP-FTR option, it presents complications that do not arise in the case of obligations. The difficulty flows from the simple fact that the dispatch formulation (1.6) does not include options; in the real dispatch everything is an obligation. Hence the auction model for options does not follow directly from the formulation for economic dispatch. Further, the associated settlement rules for options do not follow immediately from the analysis for obligations.

³⁶ It is a conjecture, but not proven, that this "optimized" FTR-loss rental is always non-negative, and that simultaneous feasibility alone is sufficient for revenue adequacy in this congestion-only case.

The analytical problem for options is similar to the problem for physical rights. Without knowing all the other flows on the system, it is not possible in general to know if any particular transaction will be feasible. Hence, to guarantee feasibility it is necessary to consider all possible combinations of the exercise of options. For example, if too few of the other options are exercised, there may be insufficient counterflow to support a particular transaction; or if all the options are exercised, some other constraint might be limiting. This ambiguity does not arise with obligations, which by definition are always exercised.

1.4.2.1 Revenue Adequacy for Options

As with PTP-FTR obligations, simultaneous feasibility of the exercised options is a necessary condition to guarantee revenue adequacy.³⁷ To demonstrate that simultaneous feasibility is also sufficient requires an expansion of the definition and test for simultaneous feasibility. Once we know which options are exercised, we can treat them like obligations for settlement purposes, so if the exercised rights are simultaneously feasible, we will have revenue adequacy. But the test of feasibility of all possible combinations of exercise of options requires an expansion of the model.

Here we consider only the possibility of balanced PTP-FTR options, combined with both balanced and unbalanced forward obligations. As above, we have a set of balanced $(\tau_k^f, k = 1, \dots, N)$ and unbalanced $(\bar{g}_k^f, k = 1, \dots, N)$ PTP-FTR obligations for any possible locations. In addition, define the balanced options as $(\tau_k^0, k = 1, \dots, N)$. Let x_k be the fraction of each option exercised. Since different exercise patterns produce different losses, we need some flexibility in the total loss provision. As with contingency constraints, we impose this balancing adjustment at the swing bus. For the moment, assume the unbalanced obligations are large enough to ensure that this adjustment is non-negative. Then for feasibility we require by analogy to (1.13) that there is a $u \in U$ and a scalar balancing adjustment at the swing bus with $\varepsilon_s^0 \geq 0$ such that:

$$y = \sum_k \tau_k^f + \sum_k x_k \tau_k^0 - \sum_k \bar{g}_k^f + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varepsilon_s^0,$$

$$L(y, u) + t^s y = 0,$$

$$K(y, u) \leq 0.$$

Since this must be true for an arbitrary exercise of options and applies to all constraints collectively, it must be true for each contingency and constraint

³⁷ The FTRs may be revenue adequate under some dispatch cases without simultaneous feasibility, but not under all dispatch cases. For instance, if the FTRs follow the same pattern as the dispatch, but imply even more of the valuable flows than is feasible, the FTRs would not be revenue adequate.

combination. A formulation that allowed for a different $u \in U$ for each exercise of the options would be the weakest condition. A somewhat simpler test that provides a sufficient condition for simultaneous feasibility is to require that any exercise of the options be feasible for the same $u \in U$.³⁸

Consider first the constraints in $K(y, u)$. The constraints do not depend on the value of y at the swing bus that is merely a balancing adjustment. Hence, the constraints would be satisfied if there is a $u \in U$ such that

$$y = \sum_k \tau_k^f + \sum_{\substack{k \\ 0 \leq x_k \leq 1}} x_k \tau_k^o - \sum_k \bar{g}_k^f, \quad \text{Max}_{i, \omega} K_i^\omega(y, u) = \text{Max}_{i, \omega} \sum_k \tau_k^f + \sum_{\substack{k \\ 0 \leq x_k \leq 1}} x_k \tau_k^o - \sum_k \bar{g}_k^f, \quad K_i^\omega(y, u) \leq 0. \quad (1.16)$$

Recall from (1.5) that there is a loss function for each contingency, and many constraints. Here we represent these loss functions and constraints explicitly to make clear the nature of the constraints induced by the options. Hence, define a new function w_i^ω , meaning constraint i in contingency ω :

$$\begin{aligned} w_i^\omega(\tau^f, \{t_k^o\}, \bar{g}^f, u) &= \text{Max}_{\epsilon_s, y} K_i^\omega(y, u) \\ &\quad 0 \leq x_k \leq 1 \\ & \text{s.t.} \\ y &= \tau^f + \sum_k x_k t_k^o - \bar{g}^f + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \epsilon_s, \\ L^\omega(y, u) + t^l y &= 0. \end{aligned} \quad (1.17)$$

The notation $\{t_k^o\}$ refers to the vector of award levels of the options. Here ϵ_s is the load adjustment at the swing bus to achieve balanced loads in the contingency. This notation allows and anticipates a different solution y for every constraint and contingency combination. Apparently the condition that the constraint K_i^ω is satisfied for all possible exercise of options is equivalent to:

$$w_i^\omega(\tau^f, \{t_k^o\}, \bar{g}^f, u) \leq 0.$$

This w_i^ω is an optimal-value function, the result itself of an optimization problem (Shimuzu et al. 1997). However, it is a well-defined function that would allow restatement of the auction problem in terms of the variables defining the auction awards.

For the contingency we define:

³⁸ These two definitions would be the same if there is a saddle point for the function $f(y, u) = \text{Max}_{i, \omega} K_i^\omega(y, u)$. However, the usual convexity arguments would not apply to guarantee a saddle point as it seems unlikely that f would be concave in y , Ponstein (1965), pp. 181–188. In any event, the former computational problem appears more difficult.

$$w^\omega(\tau^f, \{t_k^\omega\}, \bar{g}^f, u) \equiv \begin{pmatrix} w_1^\omega(\tau^f, \{t_k^\omega\}, \bar{g}^f, u) \\ w_2^\omega(\tau^f, \{t_k^\omega\}, \bar{g}^f, u) \\ \vdots \\ w_n^\omega(\tau^f, \{t_k^\omega\}, \bar{g}^f, u) \end{pmatrix}.$$

Hence, the sufficient condition in (1.16) for simultaneous feasibility of PTP-FTRs with options requires:

$$w(\tau^f, \{t_k^\omega\}, \bar{g}^f, u) \equiv \begin{pmatrix} w^0(\tau^f, \{t_k^\omega\}, \bar{g}^f, u) \\ w^1(\tau^f, \{t_k^\omega\}, \bar{g}^f, u) \\ \vdots \\ w^\omega(\tau^f, \{t_k^\omega\}, \bar{g}^f, u) \\ \vdots \\ w^m(\tau^f, \{t_k^\omega\}, \bar{g}^f, u) \end{pmatrix} \leq 0.$$

Finally, to treat losses and ensure that $\varepsilon_s^0 \geq 0$, define the worst case for the contribution of losses and the unbalanced obligations:

$$L_O^0(\tau^f, \{t_k^\omega\}, \bar{g}^f, u) = \underset{\substack{y \\ 0 \leq x_k \leq 1}}{\text{Max}} L^0(y, u)$$

s.t.

$$y = \tau^f + \sum_k x_k t_k^\omega \tau_k^\omega - \bar{g}^f,$$

$$L^0(y, u) + t^i y = 0.$$

If we have enough loss obligations to meet this maximized exercise of FTR losses, then we have enough total forward unbalanced obligations to meet or exceed the exercised FTR losses and ensure that we meet the assumption above that $\varepsilon_s^0 \geq 0$. Therefore, we set the simultaneous feasibility condition with PTP-FTR obligations and options as:

$$\begin{aligned} L_O^0(\tau^f, \{t_k^\omega\}, \bar{g}^f, u) - t^i \bar{g}^f &= 0, \\ w(\tau^f, \{t_k^\omega\}, \bar{g}^f, u) &\leq 0, \\ u &\in U. \end{aligned}$$

Consider a market equilibrium (p, y^*, u^*) . Let $\tau^{\omega*}$ be the corresponding aggregate of exercised options from the simultaneously feasible combination, $(\tau^f, \{t_k^\omega\}, \bar{g}^f, u)$.

In other words, τ^{o^*} is the aggregate of all the options with $p^t \tau_k^o \geq 0$. Then let ε_s^* be the difference in the net load at the swing bus required to achieve balance of the FTR in the pre-contingency case $\omega = 0$, i.e.,

$$y = \tau^f + \tau^{o^*} - \bar{g}^f + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varepsilon_s^*,$$

$$L^0(y, u) + t'y = 0.$$

By construction it must be that $\varepsilon_s^* \geq 0$. Further, let

$$\bar{g}^{f*} = \bar{g}^f - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varepsilon_s^*.$$

Then since \bar{g}^{f*} differs from \bar{g}^f only for the swing bus, which is allowed to adjust freely for each contingency in the definition of w , we have a $u \in U$ with

$$w(\tau^f, \{t_k^o\}, \bar{g}^{f*}, u) \leq 0.$$

Therefore, the exercise of the options must be feasible. Hence, we have a balanced load that satisfies every constraint, or

$$y = \tau^f + \tau^{o^*} - \bar{g}^{f*},$$

$$L(y, u) + t'y = 0,$$

$$K(y, u) \leq 0,$$

$$u \in U.$$

Following (1.9) we must have:

$$p^t(y^* - y) \geq 0.$$

The payments under the PTP-FTRs equal $p^t(\tau^f + \tau^{o^*} - \bar{g}^f) = p^t y - p_s \varepsilon_s^*$. By construction, $\varepsilon_s^* \geq 0$. Hence, if the swing bus price $p_s \geq 0$, the net revenue from the dispatch will be adequate to pay out the obligations and exercise of options for the PTP-FTRs. Typically, ε_s^* should be small so that even with a negative price at the swing bus, any revenue inadequacy would be bounded by the small value of the difference in losses.

1.4.2.2 PTP-FTR Auction with Options

With this background, the natural extension of the auction for PTP-FTRs in (1.14) becomes:

$$\begin{aligned}
& \underset{u \in U, t_k^f \geq 0, t_k^o \geq 0, \rho_k^f \geq 0}{\text{Max}} \quad \sum_k \beta_k \left(t_k^f, t_k^o, \rho_k^f \right) \\
& \text{s.t.} \\
& L^0 \left(\sum_k t_k^f \tau_k^f, \{t_k^o\}, \sum_k \rho_k^f \bar{g}_k^f, u \right) - t^t \sum_k \rho_k^f \bar{g}_k^f = 0, \\
& w \left(\sum_k t_k^f \tau_k^f, \{t_k^o\}, \sum_k \rho_k^f \bar{g}_k^f, u \right) \leq 0.
\end{aligned} \tag{1.18}$$

This is a well-defined model and the objective function is well-behaved.³⁹ The major change from the AC auction model with obligations only is that the conventional constraint functions K have been replaced with the more complicated constraint functions w . Evaluating any element of the function K requires solving an AC load flow problem, one for each contingency. Evaluating any element of w requires solution of an AC optimal power flow problem, one for each contingency and constraint combination. This is a significant increase in computational burden.

In a relaxation and sequential approximation approach for solving the AC auction model with obligations only, the corresponding model from (1.14) is:

$$\begin{aligned}
& \underset{y, u \in U, 0 \leq t_k^f, 0 \leq \rho_k^f}{\text{Max}} \quad \sum_k \beta_k \left(t_k^f, \rho_k^f \right) \\
& \text{s.t.} \\
& y = \sum_k t_k^f \tau_k^f - \sum_k \rho_k^f \bar{g}_k^f, \\
& L(y, u) + t^t y = 0, \\
& K(y, u) \leq 0.
\end{aligned}$$

A computational approach to this problem would exploit the close similarity with security-constrained optimal dispatch problem. The sequential approximation approach begins with a simplified version of the problem that ignores many of the constraints and is solved via a sequential linearization. Then a candidate solution $(\hat{y}, \hat{u}) = (\hat{i}^f - \hat{g}^f, \hat{u})$ is tested for feasibility by solving a load flow to evaluate $K_i^\omega(\hat{y}, \hat{u})$. If the constraint is violated, determine the gradient of the function and impose the new constraint:

$$K_i^\omega(\hat{y}, \hat{u}) + \nabla K_i^\omega(\hat{y}, \hat{u})^t \begin{pmatrix} y - \hat{y} \\ u - \hat{u} \end{pmatrix} \leq 0.$$

³⁹This is a parametric satisfaction problem in the terminology of Shimuzu et al. (1997), p. 285.

This linearized constraint would be appended to the auction model, and there would be further iteration until a solution is found that optimizes the bid function and satisfies all the constraints. Typically we are limited to search algorithms that find solutions to the first-order Karush-Kuhn-Tucker (KKT) conditions and, therefore, to a guarantee only of local optimal solutions.

Applying this same idea to the AC auction with options would require a method for (1) evaluating w and (2) finding a linear approximation whenever the constraint is violated.

Consider first the question of evaluating a constraint. For each contingency constraint, a good guess as to the solution of the unconstrained optimal power flow in (1.17) would be to use the DC-Load approximation above to determine the value for x , the pattern of the exercise of the option. For each option, if $H_i^{\omega} \tau_k^{\omega} > 0$ set the corresponding k th element x to 1, otherwise set the element to zero. Let the result be the vector \tilde{x}_i^{ω} that achieves this value for the i th constraint in contingency ω . This is the same solution for x that would be obtained in the DC-Load case.

Then compute $\nabla K_i^{\omega} \left(\hat{t}^f + \sum_k \tilde{x}_{ik}^{\omega} \tau_k^{\omega} - \hat{g}^f, \hat{u} \right)$, the change in the constraint as we change the exercise of the options. If the solution satisfies the condition that the elements of this gradient vector have positive signs when and only when the corresponding elements of \tilde{x}_i^{ω} are at the upper bound, then we can show that \tilde{x}_i^{ω} satisfies the first-order conditions for achieving the maximum for the optimal value function. If so, then we would expect that this is the optimal solution for w_i^{ω} , at least for a well-behaved network. If the first order condition is satisfied at a local optimum that is not a global optimum, then an ordinary local search algorithm may not be able to find a global solution.

In practice, we accept approximate solutions of the first-order conditions as optimal solutions. If the problem is well-behaved, then the simple solution based on the DC-Load model should define the worst-case exercise of options for each constraint without the necessity to conduct a further search. (Note that this is not the same thing as saying that the DC-Load estimate of K is acceptable. We use the DC-Load guess for the solution x , but use a full AC load flow to evaluate the constraint).

If the first order condition is not satisfied, then this should be a good starting point for a search to find an acceptable solution to maximize $K_i^{\omega}(y, u)$. This case would require iterative solution of an optimal power flow problem for the applicable contingency. This is easier than finding the full security-constrained solution for the auction model.

In any event, let the end result of evaluating the optimal value function w_i^{ω} be \hat{x}_i^{ω} , with corresponding solution (\hat{y}, \hat{u}) where $w_i^{\omega} \left(\sum_k \hat{t}_k^f \tau_k^f, \{\hat{t}_k^{\omega}\}, \sum_k \hat{\rho}_k^f \bar{g}_k^f, \hat{u} \right) = K_i^{\omega}(\hat{y}, \hat{u})$. This gives us an evaluation of the constraint. If the value is greater than zero, the constraint is violated.

Recognize that there will be different value of \hat{x}_i^{ω} , the implied exercise of the options, for each constraint i and contingency ω . This is not an obstacle in principle because in using the optimal-value function we are interested only in the value of the

violated constraint and its linear approximation relative to the option awards, not to the exercised awards. Hence we need only use the exercised awards temporarily, at each constraint, to evaluate the function and calculate the linear approximation.⁴⁰

In the case of a violated constraint, the optimal-value function is not in general differentiable or even convex. However, it does have a generalized gradient $\partial^o w$ that serves a similar purpose (Clarke 1990).⁴¹ In the present application the generalized gradient of the optimal value function w_i^o has a simple form that limits the domain where it is nondifferentiable to those points where some of the elements of the options awards are zero. These are important points, since not all options will have positive awards. Hence, the lack of a regular gradient is relevant.

The following vector will always be an element of the generalized gradient:

$$\phi_i^o(\hat{\tau}^f, \{\hat{t}_k^o\}, \hat{g}^f, \hat{u})^t \equiv \left[\begin{array}{c} \nabla_y K_i^o(\hat{y}, \hat{u})^t \\ \{ \text{Max}(0, \nabla_y K_i^o(\hat{y}, \hat{u}) \tau_k^o) \}^t \\ -\nabla_y K_i^o(\hat{y}, \hat{u})^t \\ \nabla_u K_i^o(\hat{y}, \hat{u})^t \end{array} \right]^t \in \partial^o w_i^o(\hat{\tau}^f, \{\hat{t}_k^o\}, \hat{g}^f, \hat{u}). \quad (1.19)$$

To see this, note that the special nature of the problem in (1.17), where the swing bus net load is determined freely to meet the condition, could be restated as:

$$w_i^o(\tau^f, \{t_k^o\}, \bar{g}^f, u) = \underset{\substack{\varepsilon_s, y \\ 0 \leq x_k \leq 1}}{\text{Max}} \tilde{K}_i^o(\tilde{y}, u)$$

s.t.

$$\begin{pmatrix} y_s \\ \tilde{y} \end{pmatrix} = \tau^f + \sum_k x_k t_k^o \tau_k^o - \bar{g}^f + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varepsilon_s.$$

⁴⁰ Note: in the early stages of the computation, we might accept both the DC-Load solution and the associated DC-Load shift factors as the estimates of the linearized constraint. However, when close to the solution, the assumption that the DC-Load model is inadequate means that we need an exact evaluation of both the function and the linearized representation of any violated constraint.

⁴¹ Here we follow the applications Shimuzu et al. (1997), p. 28. A generalized gradient of a function $f(x)$ at the point \bar{x} is defined as $\partial^o f(\bar{x})$ in terms of the generalized directional derivative as the set of vectors

$$\partial^o f(\bar{x}) = \{ \gamma \in R^n \mid f^o(\bar{x} : s) \geq \gamma^t s, \forall s \in R^n \},$$

where

$$f^o(\bar{x} : s) = \limsup_{\substack{x \rightarrow \bar{x} \\ \tau \downarrow 0}} \frac{f(x + \tau s) - f(x)}{\tau}.$$

In other words, y_s does not enter the objective function and the resulting gradients depend only on the objective function derivatives. At most points, w is differentiable. But at points where it is not differentiable, the generalized gradient exists and equals the convex hull of the limit points of the gradients, including (1.19), (Shimuzu et al. 1997).

When the option award is zero, any element in the interval $[Max(0, \nabla_y K_i^\omega(\hat{y}, \hat{u})\tau_k^o), +\infty)$ would also give rise to a generalized gradient. Thus the vector $\phi_i^\omega(\hat{\tau}^f, \{\hat{t}_k^o\}, \hat{g}^f, \hat{u})$ is an extreme point of the generalized gradient. It should give an adequate linear representation of the constraint function in the range of interest over the non-negative allocations.

For a violated constraint, therefore, the idea is to introduce the linearized constraint:

$$w_i^\omega(\hat{\tau}^f, \{\hat{t}_k^o\}, \hat{g}^f, \hat{u}) + \phi_i^\omega(\hat{\tau}^f, \{\hat{t}_k^o\}, \hat{g}^f, \hat{u}) \begin{pmatrix} \sum_k t_k^f \tau_k^f - \hat{t}^f \\ \{t_k^o\} - \{\hat{t}_k^o\} \\ \sum_k \rho_k^f \bar{g}_k^f - \hat{g}^f \\ u - \hat{u} \end{pmatrix} \leq 0.$$

This would then serve as a constraint in the sequential approximation of the nonlinear AC auction problem in the corresponding way that the constraint would enter in the case of obligations only. For the linear approximation, the usual first order KKT conditions would generalize to finding zero as an element of the generalized gradient.

As a technical point, this application would depend on a slightly stronger set of assumptions to guarantee that w is Lipschitz near the solution. These conditions would apply for a slightly modified version of the problem where for a sufficiently large value of the penalty M we redefine the value function as:

$$w_i^\omega(\tau^f, \{t_k^o\}, \bar{g}^f, u) = \underset{\substack{\varepsilon_s, y, 0 \leq x_k, 0 \leq m_k, \\ x_k - m_k \leq 1}}{\text{Max}} K_i^\omega(y, u) - M \sum_k m_k$$

s.t.

$$y = \tau^f + \sum_k x_k t_k^o \tau_k^o - \bar{g}^f + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varepsilon_s,$$

$$L^\omega(y, u) + t^t y = 0.$$

This allows the function to be finite for all $(\tau^f, \{t_k^o\}, \bar{g}^f, u)$ and locally Lipschitz everywhere (Shimuzu et al. 1997). The generalized gradient at a non-differentiable point would be bounded by M , but the same lower extreme point should define the appropriate local linearization to use in the large optimization problem. The sequential linear approximations would use these function evaluations and

selections from the generalized gradient to search for the optimal solution that satisfied the generalized Karush-Kuhn-Tucker conditions for the master problem.

Relaxation Solution Procedure with PTP-FTR Options

Step 1: Select an initial candidate solution $(\hat{\tau}^f, \{\hat{t}_k^o\}, \hat{g}^f, \hat{u})^0$, ignore most (or all) of the constraints in the economic dispatch using only the small subset $w_i^\omega(\hat{\tau}^f, \{\hat{t}_k^o\}, \hat{g}^f, \hat{u})^0$, and set the iteration count to $m = 0$.

Step 2: Construct the relaxed master problem as:

$$\begin{aligned} & \underset{u \in U, t_k^f \geq 0, t_k^o \geq 0, \rho_k^f \geq 0}{Max} && \sum_k \beta_k (t_k^f, t_k^o, \rho_k^f) \\ & s.t. && \\ & L_O^0 \left(\sum_k t_k^f \tau_k^f, \{t_k^o\}, \sum_k \rho_k^f \bar{g}_k^f, u \right) - i^t \sum_k \rho_k^f \bar{g}_k^f = 0, \\ & w_i^\omega(\hat{\tau}^f, \{\hat{t}_k^o\}, \hat{g}^f, \hat{u})^m + \phi_i^\omega(\hat{\tau}^f, \{\hat{t}_k^o\}, \hat{g}^f, \hat{u})^{m^t} \begin{pmatrix} \sum_k t_k^f \tau_k^f - \hat{t}^{f^m} \\ \{t_k^o\} - \{\hat{t}_k^o\}^m \\ \sum_k \rho_k^f \bar{g}_k^f - \hat{g}^{f^m} \\ u - \hat{u}^m \end{pmatrix} \leq 0. \end{aligned}$$

Let a solution be $(\hat{\tau}^f, \{\hat{t}_k^o\}, \hat{g}^f, \hat{u})^{m+1}$ and update $m = m + 1$.

Step 3: Check to see if the candidate solution $(\hat{\tau}^f, \{\hat{t}_k^o\}, \hat{g}^f, \hat{u})^m$ violates any of the constraints. If so, create a new $w_i^\omega(\hat{\tau}^f, \{\hat{t}_k^o\}, \hat{g}^f, \hat{u})^m$ including some or all of these constraints and repeat Step 2. Else done.

Success with this proposed relaxation procedure for solving the auction problem with PTP-FTR options depends on the expectation that relatively few of the (very) many contingency constraints will be binding. This is a well-established condition in the dispatch model and the associated PTP-FTR obligation-only auction model that is of the same form as the dispatch. By contrast, if the introduction of options produces many more bids and many more binding constraints, then the scale of the problem may overwhelm current computational capabilities.

A concern with the potential number of binding constraints applies as well to the case of a DC-Load model for PTP-FTR obligations and options. However, the DC-Load formulation would have the computational advantage that evaluation of the constraints and the associated generalized gradient would be a relatively simple calculation that reduces to calculating the associated shift factors in $H(u^0)$ and evaluating the positive elements to construct the generalized gradient. In the DC-Load formulation ignoring losses, we would have:

$$\begin{aligned}
w_{DC_i}^\omega(\tau^f, \{t_k^\omega\}, u) &= \text{Max}_{0 \leq x_k \leq 1} H_i^\omega(u) \left(\tau^f + \sum_k x_k t_k^\omega \tau_k^\omega \right) - b(u), \\
&= H_i^\omega(u) \tau^f + \sum_k t_k^\omega \max(0, H_i^\omega(u) \tau_k^\omega) - b(u).
\end{aligned}$$

Combining all the constraints and contingencies, we have

$$w_{DC}(\tau^f, \{t_k^\omega\}, u) \leq 0.$$

The corresponding auction model with bids for balanced forward-obligations by (t_k^f, τ_k^f) and balanced forward-option by $(t_k^\omega, \tau_k^\omega)$ would be

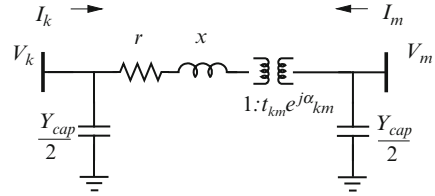
$$\begin{aligned}
&\text{Max}_{u \in U, t_k^f \geq 0, t_k^\omega \geq 0} \quad \sum_k \beta_k \left(t_k^f, t_k^\omega \right) \\
&\text{s.t.} \tag{1.20} \\
&w_{DC} \left(\sum_k t_k^f \tau_k^f, \{t_k^\omega\}, u \right) \leq 0.
\end{aligned}$$

Even in the DC-Load case, therefore, this computation is not trivial. For obligations we need to evaluate only the load flow for each contingency given τ^f , the aggregate of the obligations. Following the discussion of (1.10), this amounts to solving a system of linear equations for each contingency but evaluates all constraints in that contingency at once. But in order to evaluate the constraint in (1.20), we need to calculate the shift factors for every constraint in the contingency, each of which involves a similar system of linear equations. In other words, in the relaxation algorithm the need to calculate shift factors expands from the violated constraints only to every constraint when options are included.

Although this does require more computation, the evaluation of the constraints is separable and efficient means should be available to do the many evaluations, at least in the DC-Load case. Furthermore, not every constraint needs to be included in the relaxed master problem. As long as the number of binding constraints is small, meaning hundreds and not hundreds of thousands, this auction model might accommodate PTP-FTR options and obligations and be computationally feasible.

By construction of the constraints, exercise of the options along with the obligations would be simultaneously feasible under the condition that the system operator could select the set of controls needed to satisfy the constraints for the obligations and exercised options. Hence, the revenue collected in the final spot market dispatch would always be sufficient to pay the amounts required by the various PTP-FTR contracts.

Fig. 1.2 Transmission line and transformer



1.5 Conclusion

So-called physical transmission rights present so many complications for a restructured electricity market that some other approach is required to provide property rights for the grid. Under a standard market design built on a bid-based, security-constrained, economic dispatch with locational prices, the natural approach is to define financial transmission rights that offer payments based on prices in the actual dispatch. Different models have been proposed for point-to-point, including obligations and options. With consistent definitions, the rights can be shown to be simultaneously feasible and revenue adequate in various AC formulations or approximations. The conditions for simultaneous feasibility also define the form of auctions that would award or reconfigure the rights. In the case of point-to-point obligations, the practical feasibility of the approach has been demonstrated with adaptations of commercial dispatch software. In the case of point-to-point options, the computational strategies are more demanding but have been implemented in a limited way.

Appendix: Generic Transmission Line Representation

The generic transmission line analysis employs complex variables. To avoid confusion here, the indexes for the two terminals of the line are k and m . For a development of the model transmission line and transformer model, see Grainger and Stevenson (1994). By choice of parameters, this generic transmission line representation allows for a Π -equivalent representation of a line with no transformer, an ideal transformer, or a combination of both.

Here we follow Weber (1997)'s notation and conventions. This is useful in that Weber also provides an extensive detail on the characterization of the Jacobian of the power flow equations to provide further insight into the implications of the AC power flow model, including calculation of the derivatives with respect to the transformer parameters. As shown in Fig. 1.2 let V_k represent the complex voltage with magnitude $|V_k|$ and angle θ_k . The data include the line resistance (r), reactance (x). The transformer includes turns ratio (t_{km}) and angle change (α_{km}). The line charging capacitance is the complex Y_{cap} .

The line admittance (y) is the inverse of the line impedance (z) formed from the resistance and reactance.

$$y = \frac{1}{z} = \frac{1}{r + jx} = \frac{1}{r + jx} \frac{(r + jx)^*}{(r + jx)^*} = \frac{1}{r + jx} \frac{r - jx}{r - jx} = \frac{r - jx}{r^2 + x^2} = g + jb.$$

With P as the real power and Q as the reactive power, the general rules for complex power (S) have:

$$S = P + jQ = VI^* = zI^2 = z|I|^2 = (P - jQ)^* = (V^*I)^*.$$

The line capacitance is represented here as:

$$\frac{Y_{cap}}{2} = 0 + jB_{cap}.$$

Following Weber, for the generic representation in Fig. 1.2, complex current (I_k) from k towards m satisfies:

$$I_k = V_k \left(y + \frac{Y_{cap}}{2} \right) - V_m \frac{e^{-j\alpha_{km}}}{t_{km}} y.$$

Therefore, the complex power flow from k to m is:

$$\begin{aligned} S_k &= V_k I_k^* = V_k V_k^* \left(y + \frac{Y_{cap}}{2} \right)^* - V_k V_m^* \frac{e^{j\alpha_{km}}}{t_{km}} y^* \\ &= |V_k|^2 \left(y + \frac{Y_{cap}}{2} \right)^* - \frac{|V_k| |V_m| e^{j(\theta_k - \theta_m + \alpha_{km})}}{t_{km}} y^*, \\ &= |V_k|^2 (g - j(b + B_{cap})) - \frac{|V_k| |V_m|}{t_{km}} (\cos(\theta_k - \theta_m + \alpha_{km}) + j \sin(\theta_k - \theta_m + \alpha_{km})) (g - jb), \\ &= |V_k|^2 g - \frac{|V_k| |V_m|}{t_{km}} (g \cos(\theta_k - \theta_m + \alpha_{km}) + b \sin(\theta_k - \theta_m + \alpha_{km})) \\ &\quad + j \left(-\frac{|V_k| |V_m|}{t_{km}} (g \sin(\theta_k - \theta_m + \alpha_{km}) - b \cos(\theta_k - \theta_m + \alpha_{km})) - |V_k|^2 (b + B_{cap}) \right). \end{aligned}$$

The complex current (I_m) from m towards k is

$$I_m = -V_k \frac{e^{j\alpha_{km}}}{t_{km}} y + V_m \left(\frac{1}{t_{km}^2} y + \frac{Y_{cap}}{2} \right).$$

Hence,

$$\begin{aligned}
 S_m &= -V_m V_k^* \frac{e^{-j\alpha_{km}}}{t_{km}} y^* + V_m V_m^* \left(\frac{1}{t_{km}^2} y + \frac{Y_{cap}}{2} \right)^* , \\
 &= -\frac{|V_m||V_k| e^{j(\theta_m - \theta_k - \alpha_{km})}}{t_{km}} (g - jb) + |V_m|^2 \left(\frac{g}{t_{km}^2} - j \left(\frac{b}{t_{km}^2} + B_{cap} \right) \right) , \\
 &= |V_m|^2 \frac{g}{t_{km}^2} - \frac{|V_m||V_k|}{t_{km}} (g \cos(\theta_m - \theta_k - \alpha_{km}) + b \sin(\theta_m - \theta_k - \alpha_{km})) \\
 &\quad + j \left(-\frac{|V_m||V_k|}{t_{km}} (g \sin(\theta_m - \theta_k - \alpha_{km}) - b \cos(\theta_m - \theta_k - \alpha_{km})) - |V_m|^2 \left(\frac{b}{t_{km}^2} + B_{cap} \right) \right) .
 \end{aligned}$$

If the system is normal and the angle change is fixed, then the angle change can be included in the line admittance. Similarly for normal systems, if the transformer tap setting is fixed, the turns ratio can be included in the per unit normalization of the voltages, which would produce appropriately modified values of y but with the elimination of the separate transformer parameters (t , α).⁴² Ignoring the line capacitance, this simplified representation would be

$$\begin{aligned}
 S_k &= |V_k|^2 \hat{g} - |V_k||V_m| (\hat{g} \cos(\theta_k - \theta_m) + \hat{b} \sin(\theta_k - \theta_m)) \\
 &\quad + j \left(-|V_k||V_m| (\hat{g} \sin(\theta_k - \theta_m) - \hat{b} \cos(\theta_k - \theta_m)) - |V_k|^2 \hat{b} \right) .
 \end{aligned}$$

and

$$\begin{aligned}
 S_m &= |V_m|^2 \hat{g} - |V_m||V_k| (\hat{g} \cos(\theta_m - \theta_k) + \hat{b} \sin(\theta_m - \theta_k)) \\
 &\quad + j \left(-|V_m||V_k| (\hat{g} \sin(\theta_m - \theta_k) - \hat{b} \cos(\theta_m - \theta_k)) - |V_m|^2 \hat{b} \right) .
 \end{aligned}$$

This is a familiar simplification often seen in the electrical engineering literature. However, if the system is not normal, tap ratios are variable, or phase angle adjustments are variable, it will be necessary to use the more general representation as shown above.

The notation translation to the discussion in the main text has:

$$G_k = g, \quad \Omega_k = -b, \quad \delta_i = \theta_k, \quad Z_{ij} = S_k, \quad \alpha_k = \alpha_{km}, \quad t_k = t_{km}.$$

⁴²Normal is a term of art, not necessarily intended to mean "usual." A system is normal if for each parallel path the product of ideal transformer gain magnitudes is equal and the sum of ideal transformer phase shifts is the same. See Bergen and Vittal (2000), pp. 154–175.

References

- Bergen AR (1986) Power systems analysis. Prentice Hall, Englewood Cliffs, NJ
- Bergen AR, Vittal V (2000) Power systems analysis, 2nd edn. Prentice Hall, Upper Saddle River, pp 154–175
- Bertsekas DP (1995) Nonlinear programming. Athena Scientific, Belmont, MA, p 427
- Boucher J, Smeers Y (2001) Alternative models of restructured electricity systems, part 1: no market power. *Oper Res* 9(6):821–838
- Boucher J, Ghilain B, Smeers Y (1998) Security-constrained dispatch gives financially and economically significant nodal prices. *Elec J* 11:53–59
- Cadwalader MD, Harvey SM, Hogan WW, Pope SL (1998) Reliability, scheduling markets, and electricity pricing. Center for Business and Government, Harvard University, May 1998
- Caramanis MC, Bohn RE, Schweppe FC (1982) Investment decisions and long-term planning under electricity spot pricing. *IEEE Trans Power Ap Syst PAS-101(12)*:3234–3245
- Chandley JD, Hogan WW (2002) Independent transmission companies in a regional transmission organization. Center for Business and Government, Harvard University, Cambridge, MA, 8 Jan 2002
- Clarke FH (1990) Optimization and nonsmooth analysis. SIAM Reprints, Philadelphia, p 10
- Elgerd OI (1982) Electric energy systems and theory, 2nd edn. McGraw Hill, New York, p 23
- Federal Energy Regulatory Commission (FERC) (1996) Capacity reservation open access transmission tariffs. Notice of Proposed Rulemaking, RM96-11-000, Washington DC, 24 April 1996
- Federal Energy Regulatory Commission (FERC) (2002a) Working paper on standardized transmission service and wholesale electricity market design. Washington, DC, 15 March 2002
- Federal Energy Regulatory Commission (FERC) (2002b) Working paper on standardized transmission service and wholesale electricity market design. Washington, DC, 15 March 2002, p 11
- Feinstein J, Tscherne J, Koenig M (1988) Reactive load and reserve calculation in real-time computer control system. *IEEE Comput Appl Power* 1(3):22–26
- Ge SY, Chung TS (1999) Optimal active power flow incorporating power flow control needs in flexible AC transmission systems. *IEEE Trans Power Syst* 14(2):738–744
- Geoffrion AM (1970) Elements of large-scale mathematical programming, parts I and II. *Manag Sci* 16(11):652–691
- Grainger JD, Stevenson WD (1994) Power systems analysis. McGraw-Hill, New York, pp 361–367
- Harvey SM, Hogan WW (2002) Loss hedging financial transmission rights. Center for Business and Government, Harvard University, 15 Jan 2002
- Harvey SM, Hogan WW, Pope SL (1997) Transmission capacity reservations and transmission congestion contracts, Center for Business and Government, Harvard University, 6 June 1996, (Revised 8 March 8 1997)
- Hogan H (1992) Contract networks for electric power transmission. *J Regul Econ* 4:211–242
- Hogan WW (2000) Flowgate rights and wrongs. Center for Business and Government, Harvard University, Aug 2000
- Kuhn HW (2002) Being in the right place at the right time. *Oper Res* 50(1):132
- Mas-Colell A, Whinston MD, Green JR (1995) Microeconomic theory. Oxford University Press, New York, pp 311–343
- Oliveira EJ, Marangon JW, Lima JL, Pereira R (1999) Flexible AC transmission system devices: allocation and transmission pricing. *Elec Power Energy Syst* 21:111–118
- O'Neill RP, Helman U, Hobbs B, Stewart WR, Rothkopf MH (2002) A joint energy and transmission rights auction: proposal and properties. Federal Energy Regulatory Commission, Working Paper, Feb 2002
- Ponstein J (1965) An extension of the min-max theorem. *SIAM Rev* 7(2):181–188
- Schweppe FC, Caramanis MC, Tabors RD, Bohn RE (1988) Spot pricing of electricity. Kluwer Academic, Norwell, MA

- Shimuzu K, Ishizuka Y, Bard JF (1997) Nondifferentiable and two-level mathematical programming. Kluwer Academic, Boston, pp 188–228
- Skilling HH (1951) Electric transmission lines. McGraw Hill, New York, pp 126–133
- Weber JD (1997) Implementation of a Newton-based optimal power flow into a power system simulation environment. MS Thesis, University of Illinois at Urbana-Champaign, Urbana, Illinois
- Wood AJ, Wollenberg BF (1984) Power generation, control, and operation. Wiley, New York, p 75
- Wu F, Varaiya P, Spiller P, Oren S (1996) Folk theorems on transmission access. *J Regul Econ* 10(1):5–24