Chapter 1
Doubly Periodic Functions

In this chapter we present meromorphic functions on the complex plane which are periodic in two different directions, hence the wording 'doubly periodic'. These are constructed using infinite sums. The dependence of these sums on the periods leads us to the notion of a modular form.

1.1 Definition and First Properties

We recall the notion of a meromorphic function. Let D be an open subset of the complex plane $\mathbb{C}$. A *meromorphic function* f on D is a holomorphic function $f : D \smallsetminus P \to \mathbb{C}$, where $P \subset D$ is a countable subset and the function f has poles at the points of P.

The set of poles P can be empty, so every holomorphic function is an example of a meromorphic function. An accumulation point of poles is always an essential singularity. As we do not allow essential singularities, this means that the set P has no accumulation points inside D, so poles can accumulate only on the boundary of D.

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the one-point compactification of the complex plane, also called the *Riemann sphere* (see Exercise 1.1). Let f be meromorphic on D and let P be its set of poles. We extend f to a map $f : D \to \widehat{\mathbb{C}}$, by setting $f(p) = \infty$ for every $p \in P$.

A meromorphic function can thus be viewed as an everywhere defined, $\widehat{\mathbb{C}}$-valued map.

For a point $p \in D$ and a meromorphic function f on D, there exists exactly one integer $r \in \mathbb{Z}$ such that $f(z) = h(z)(z-p)^r$, where h is a function that is holomorphic and non-vanishing at p. This integer r is called the *order* of f at p. For this we write

$$r = \operatorname{ord}_p f.$$

Note: the order of f at p is positive if p is a zero of f, and negative if p is a pole of f.

A. Deitmar, *Automorphic Forms*, Universitext,
DOI 10.1007/978-1-4471-4435-9_1, © Springer-Verlag London 2013

Definition 1.1.1 A *lattice* in $\mathbb{C}$ is a subgroup Λ of the additive group $(\mathbb{C}, +)$ of the form

$$\Lambda = \Lambda(a, b) = \mathbb{Z}a \oplus \mathbb{Z}b = \{ka + lb : k, l \in \mathbb{Z}\},$$

where $a, b \in \mathbb{C}$ are supposed to be linearly independent over $\mathbb{R}$. In this case one says that the lattice is generated by a and b, or that a, b is a $\mathbb{Z}$-*basis* of the lattice.

A lattice has many sublattices; for example, $\Lambda(na, mb)$ is a sublattice of $\Lambda(a, b)$ for any $n, m \in \mathbb{N}$. A subgroup $\Sigma \subset \Lambda$ is a sublattice if and only if the quotient group Λ/Σ is finite (see Exercise 1.3). For instance, one has

$$\Lambda(a, b)/\Lambda(ma, nb) \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$

Definition 1.1.2 Let Λ be a lattice in $\mathbb{C}$. A meromorphic function f on $\mathbb{C}$ is said to be *periodic* with respect to the lattice Λ if

$$f(z + \lambda) = f(z)$$

for every $z \in \mathbb{C}$ and every $\lambda \in \Lambda$. If f is periodic with respect to Λ, then it is so with respect to every sublattice. A function f is called *doubly periodic* if there exists a lattice Λ with respect to which f is periodic (see Exercise 1.2).

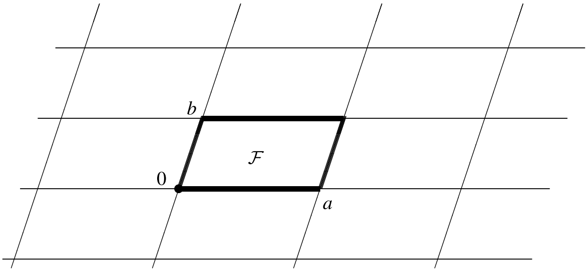
Proposition 1.1.3 *A doubly periodic function which is holomorphic is necessarily constant.*

Proof Let f be holomorphic and doubly periodic. Then there is a lattice $\Lambda = \Lambda(a, b)$ with $f(z + \lambda) = f(z)$ for every $\lambda \in \Lambda$. Let

$$\mathcal{F} = \mathcal{F}(a, b) = \{ta + sb : 0 \le s, t < 1\}.$$

The set $\mathcal{F}$ is a bounded subset of $\mathbb{C}$, so its closure $\overline{\mathcal{F}}$ is compact. The set $\mathcal{F}$ is called a *fundamental mesh* for the lattice Λ.

Two points $z, w \in \mathbb{C}$ are said to be *congruent modulo* Λ if $z - w \in \Lambda$.



To conclude the proof of the proposition, we need a lemma.

Lemma 1.1.4 *Let $\mathcal{F}$ be a fundamental mesh for the lattice $\Lambda \subset \mathbb{C}$. Then $\mathbb{C} = \mathcal{F} + \Lambda$, or more precisely, for every $z \in \mathbb{C}$ there is exactly one $\lambda \in \Lambda$ such that $z + \lambda \in \mathcal{F}$. Equivalently, we can say that for every $z \in \mathbb{C}$ there is exactly one $w \in \mathcal{F}$ such that $z - w \in \Lambda$.*

Proof of the Lemma Let a, b be the $\mathbb{Z}$-basis of Λ, which defines the fundamental mesh $\mathcal{F}$, so $\mathcal{F} = \mathcal{F}(a, b)$. Since a and b are linearly independent over $\mathbb{R}$, they form a basis of $\mathbb{C}$ as an $\mathbb{R}$-vector space. Thus, for a given $z \in \mathbb{C}$ there are uniquely determined $r, v \in \mathbb{R}$ with $z = ra + vb$. There are uniquely determined $m, n \in \mathbb{Z}$ and $t, s \in [0, 1)$ such that

$$r = m + t \quad \text{and} \quad v = n + s.$$

This implies

$$z = ra + vb = \underbrace{ma + nb}_{\in \Lambda} + \underbrace{ta + sb}_{\in \mathcal{F}}$$

and this representation is unique. □

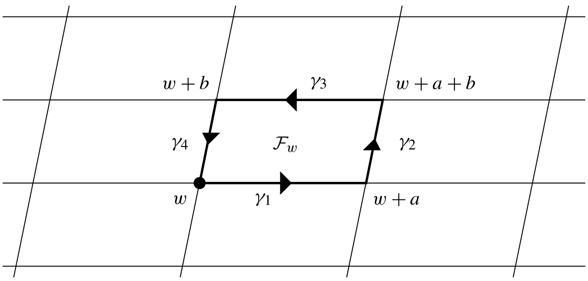
We now show the proposition. As the function f is holomorphic, it is continuous, so $f(\overline{\mathcal{F}})$ is compact, hence bounded. For an arbitrary $z \in \mathbb{C}$ there is, by the lemma, a $\lambda \in \Lambda$ with $z + \lambda \in \mathcal{F}$, so $f(z) = f(z + \lambda) \in f(\mathcal{F})$, which means that the function f is bounded, hence constant by Liouville's theorem. □

Proposition 1.1.5 *Let $\mathcal{F}$ be a fundamental mesh of a lattice $\Lambda \subset \mathbb{C}$ and let f be an Λ-periodic meromorphic function. Then there is $w \in \mathbb{C}$ such that f has no pole on the boundary of the translated mesh $\mathcal{F}_w = \mathcal{F} + w$. For every such w one has*

$$\int_{\partial \mathcal{F}_w} f(z)\, dz = 0,$$

where $\partial \mathcal{F}_w$ is the positively oriented boundary of $\mathcal{F}_w$.

Proof If f had poles on the boundary of $\mathcal{F}_w$ for every w, then f would have uncountably many poles, contradicting the meromorphicity of the function f. We can therefore choose w in such a way that no poles of f are located on the boundary of $\mathcal{F}_w$.



The path of integration $\partial \mathcal{F}_w$ is composed of the paths $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ as in the picture. The path γ_3 is the same as γ_1, only translated by $b \in \Lambda$ and running in the reverse direction. The function f does not change when one translates the argument by b and the change of direction amounts to a change of sign in the integral. Therefore we get

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz = 0 \quad \text{and similarly} \quad \int_{\gamma_2} f(z)\,dz + \int_{\gamma_4} f(z)\,dz = 0,$$

which together give $\int_{\partial \mathcal{F}_w} f(z)\,dz = 0$ as claimed. □

Proposition 1.1.6 *Let $f \neq 0$ be a meromorphic function, periodic with respect to the lattice $\Lambda \subset \mathbb{C}$ and let $\mathcal{F}$ be a fundamental mesh for the lattice Λ. For every $w \in \mathbb{C}$ we have*

$$\sum_{z \in \mathcal{F}_w} \operatorname{res}_z(f) = 0.$$

Proof In case there is no pole on the boundary of $\mathcal{F}_w$, the assertion follows from the last proposition together with the residue theorem. It follows in general, since the sum does not depend on w, as congruent points have equal residues. Hence we have

$$\sum_{z \in \mathcal{F}_w} \operatorname{res}_z(f) = \sum_{z \in \mathbb{C} \bmod \Lambda} \operatorname{res}_z(f).$$

□

Proposition 1.1.7 *Let $\mathcal{F}$ be a fundamental mesh for the lattice $\Lambda \subset \mathbb{C}$ and let $f \neq 0$ be a Λ-periodic meromorphic function. Then for every $w \in \mathbb{C}$ the number of zeros of f in $\mathcal{F}_w$ equals the number of poles of f in $\mathcal{F}_w$. Here zeros and poles are both counted with multiplicities, so a double pole, for instance, is counted twice.*

Proof A complex number z_0 is a zero or a pole of f of order $k \in \mathbb{Z}$ if the function $\frac{f'}{f}$ has a pole at z_0 of residue k. Hence the assertion follows from the last proposition, since the function $\frac{f'}{f}$ is doubly periodic with respect to the lattice Λ as well. □

1.2 The ℘-Function of Weierstrass

Except for constant functions, we have not yet seen any doubly periodic function. In this section we are going to construct some by giving Mittag-Leffler sums which have poles at the lattice points.

We first need a criterion for the convergence of the series that we consider. We prove this in a sharper form than is needed now, which will turn out useful later. Let $b \in \mathbb{C} \smallsetminus \{0\}$ be a fixed number. For every $a \in \mathbb{C} \smallsetminus \mathbb{R}b$ the set $\Lambda_a = \mathbb{Z}a \oplus \mathbb{Z}b$ is a lattice.

Lemma 1.2.1 *Let $\Lambda \subset \mathbb{C}$ be a lattice and let $s \in \mathbb{C}$. The series*

$$\sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \frac{1}{|\lambda|^s}$$

converges absolutely if $\mathrm{Re}(s) > 2$. *Furthermore, fix* $b \in \mathbb{C} \smallsetminus \{0\}$ *and consider the lattice* Λ_a *for* $a \in \mathbb{C} \smallsetminus \mathbb{R}b$. *The sum* $\sum_{\lambda\in\Lambda_a,\lambda\neq 0} \frac{1}{|\lambda|^s}$ *converges uniformly for all* $(a, s) \in C \times \{\mathrm{Re}(s) \geq \alpha\}$, *where* C *is a compact subset of* $\mathbb{C} \smallsetminus \mathbb{R}b$ *and* $\alpha > 2$.

Proof Let α and C be as in the lemma. We can assume $\mathrm{Re}(s) > 0$, because otherwise the series cannot converge as the sequence of its summands does not tend to zero. Further it suffices to consider the case $s \in \mathbb{R}$ since for $s \in \mathbb{C}$ the absolute value of $|\lambda|^{-s}$ equals $|\lambda|^{-\mathrm{Re}(s)}$. So, assuming $s > 0$, the function $x \mapsto x^s$ is monotonically increasing for $x > 0$. Let $\mathcal{F}(a)$ be a fundamental mesh for the lattice Λ_a and let

$$\psi_{a,s}(z) = \sum_{\substack{\lambda\in\Lambda_a\\ \lambda\neq 0}} \frac{1}{|\lambda|^s} \mathbf{1}_{\mathcal{F}(a)+\lambda}(z).$$

We then have

$$\big|\mathcal{F}(a)\big| \sum_{\substack{\lambda\in\Lambda_a\\ \lambda\neq 0}} \frac{1}{|\lambda|^s} = \int_{\mathbb{C}} \psi_{a,s}(x+iy)\,dx\,dy,$$

where $|\mathcal{F}(a)|$ is the area of the fundamental mesh $\mathcal{F}(a)$. The continuous map $a \mapsto |\mathcal{F}(a)|$ assumes its minimum and maximum values on the compact set C. One has $\psi_{a,s} \leq \psi_{a,\alpha}$ if $s \geq \alpha$, so it suffices to show the uniform convergence of $\int_{\mathbb{C}} \psi_{a,\alpha}(z)\,dx\,dy$ in a.

Let $r > 0$ be so large that for every $a \in C$ the *diameter* of the fundamental mesh $\mathcal{F}(a)$,

$$\mathrm{diam}\big(\mathcal{F}(a)\big) = \sup\big\{|z - w| : z, w \in \mathcal{F}(a)\big\}$$

is less than r. For every $z \in \mathbb{C}$ we have $\psi_{a,\alpha}(z) = \frac{1}{|\lambda_{a,z}|^s}$ for some $\lambda_{a,z} \in \Lambda_a$ with $|z - \lambda_{a,z}| < r$. For every $a \in C$ and $z \in \mathbb{C}$ with $|z| \geq r$ one has the inequality

$$|\lambda_{a,z}| = |\lambda_{a,z} - z + z| \leq |\lambda_{a,z} - z| + |z| < r + |z| \leq 2|z|.$$

On the other hand, for $|z| \geq 2r$ we have

$$|\lambda_{a,z}| = \big|\lambda_{a,z} - z - (-z)\big| \geq \big||\lambda_{a,z} - z| - |z|\big| \geq \frac{1}{2}|z|.$$

Let $R = 2r$. For $|z| \geq R$ we have $\frac{1}{2^s}|z|^{-s} \leq \psi_{a,\alpha}(z) \leq 2^s|z|^{-s}$ for every $a \in C$. The continuous map $a \mapsto \int_{|z|\leq R} \psi_{a,\alpha}(z)\,dx\,dy$ is bounded on the compact set C. Therefore the series converges uniformly for $a \in C$, if $\int_{|z|>R} \frac{1}{|z|^\alpha}\,dx\,dy < \infty$. We now use *polar coordinates* on $\mathbb{C}$. Recall that the map $P : (0, \infty) \times (-\pi, \pi] \to \mathbb{C}$, given by

$$P(r, \theta) = re^{i\theta} = r\cos\theta + ir\sin\theta$$

is a bijection onto the image $\mathbb{C} \smallsetminus \{0\}$. The Jacobian determinant of this map is r, so we get, by the change of variables formula,

$$\int_{\mathbb{C}^\times} f(x+iy)\,dx\,dy = \int_{-\pi}^{\pi}\int_0^\infty f\big(re^{i\theta}\big)r\,dr\,d\theta$$

for every integrable function f. Therefore

$$\int_{|z|>R} \frac{1}{|z|^\alpha}\, dx\, dy = 2\pi \int_R^\infty r^{1-\alpha}\, dr,$$

which gives the claim. □

The following theorem contains the definition of the Weierstrass $\wp$-function.

Theorem 1.2.2 *Let Λ be a lattice in $\mathbb{C}$. The series*

$$\wp(z) \stackrel{\text{def}}{=} \frac{1}{z^2} + \sum_{\lambda \in \Lambda \smallsetminus \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$$

converges locally uniformly absolutely in $\mathbb{C} \smallsetminus \Lambda$. It defines a meromorphic Λ-periodic function, called the Weierstrass $\wp$-function.

Proof For $|z| < \frac{1}{2}|\lambda|$ we have $|\lambda - z| \geq \frac{1}{2}|\lambda|$. Further it holds that $|2\lambda - z| \leq \frac{5}{2}|\lambda|$. So that

$$\left|\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right| = \left|\frac{\lambda^2 - (z-\lambda)^2}{\lambda^2(z-\lambda)^2}\right| = \left|\frac{z(2\lambda - z)}{\lambda^2(z-\lambda)^2}\right| \leq \frac{|z|\frac{5}{2}|\lambda|}{|\lambda|^2\frac{1}{4}|\lambda|^2} = \frac{10|z|}{|\lambda|^3}.$$

Using Lemma 1.2.1, we get locally uniform convergence.

The way the sum is formed, it is not immediate that the $\wp$-function is actually periodic. For this we first show that it is an even function:

$$\wp(-z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \smallsetminus \{0\}} \frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \smallsetminus \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} = \wp(z),$$

by replacing λ in the sum with $-\lambda$. Since the series converges locally uniformly, and the summands are holomorphic, we are allowed to differentiate the series term-wise. Its derivative,

$$\wp'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3},$$

is Λ-periodic. Hence for $\lambda \in \Lambda \smallsetminus 2\Lambda$ the function $\wp(z+\lambda) - \wp(z)$ is constant. We compute this constant by setting $z = -\frac{\lambda}{2}$ to get $\wp(\frac{\lambda}{2}) - \wp(-\frac{\lambda}{2}) = 0$, as $\wp$ is even. □

Theorem 1.2.3 (Laurent-expansion of $\wp$) *Let $r = \min\{|\lambda| : \lambda \in \Lambda \smallsetminus \{0\}\}$. For $0 < |z| < r$ one has*

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^\infty (2n+1) G_{2n+2} z^{2n},$$

where the sum $G_k = G_k(\Lambda) = \sum_{\lambda \in \Lambda \smallsetminus \{0\}} \frac{1}{\lambda^k}$ converges absolutely for $k \geq 4$.

Proof For $0 < |z| < r$ and $\lambda \in \Lambda \setminus \{0\}$ we have $|z/\lambda| < 1$, so

$$\frac{1}{(z-\lambda)^2} = \frac{1}{\lambda^2(1-\frac{z}{\lambda})^2} = \frac{1}{\lambda^2}\left(1 + \sum_{k=1}^{\infty}(k+1)\left(\frac{z}{\lambda}\right)^k\right),$$

and so

$$\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} = \sum_{k=1}^{\infty} \frac{k+1}{\lambda^{k+2}} z^k.$$

We sum over all λ and find

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(k+1)\sum_{\lambda \neq 0}\frac{1}{\lambda^{k+2}} z^k = \frac{1}{z^2} + \sum_{k=1}^{\infty}(k+1)G_{k+2}z^k,$$

where we have changed the order of summation, as we may by absolute convergence. This absolute convergence follows from

$$\sum_{k=1}^{\infty} \frac{k+1}{|\lambda|^{k+2}}|z|^k \leq \frac{1}{|\lambda|^3}\sum_{k=1}^{\infty}\frac{k+1}{|\lambda|^{k-1}}|z|^k$$

and Lemma 1.2.1. Since $\wp$ is even, the G_{k+2} vanish for odd k. □

1.3 The Differential Equation of the ℘-Function

The differential equation of the $\wp$-function connects doubly periodic functions to elliptic curves, as explained in the notes at the end of this chapter.

Theorem 1.3.1 *The $\wp$-function satisfies the differential equation*

$$\left(\wp'(z)\right)^2 = 4\wp^3(z) - 60G_4\wp(z) - 140G_6.$$

Proof We show that the difference of the two sides has no pole, i.e. is a holomorphic Λ-periodic function, hence constant.

If $z \neq 0$ is small we have

$$\wp'(z) = -\frac{2}{z^3} + 6G_4 z + 20G_6 z^3 + \cdots,$$

so

$$\left(\wp'(z)\right)^2 = \frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 + \cdots.$$

On the other hand,

$$4\wp^3(z) = \frac{4}{z^6} + \frac{36G_4}{z^2} + 60G_6 + \cdots,$$

so that

$$\left(\wp'(z)\right)^2 - 4\wp^3(z) = -\frac{60G_4}{z^2} - 140G_6 + \cdots.$$

We finally get

$$\left(\wp'(z)\right)^2 - 4\wp^3(z) + 60G_4\wp(z) = -140G_6 + \cdots$$

where the left-hand side is a holomorphic Λ-periodic function, hence constant. If on this right-hand side we put $z = 0$, we see that this constant is $-140G_6$. □

1.4 Eisenstein Series

For a ring R we denote by $M_2(R)$ the set of all 2×2 matrices with entries from R. In a linear algebra course you prove that a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(R)$ is invertible if and only if its determinant is invertible in R, i.e. if $ad - bd \in R^\times$. You may have done this for R being a field only, but for a ring it is just the same proof. Let $GL_2(R)$ be the group of all invertible matrices in $M_2(R)$. It contains the subgroup $SL_2(R)$ of all matrices of determinant 1. Consider the example $R = \mathbb{Z}$. We have $\mathbb{Z}^\times = \{1, -1\}$. So $GL_2(\mathbb{Z})$ is the group of all integer matrices with determinant ± 1. The subgroup $SL_2(\mathbb{Z})$ is therefore a subgroup of index 2.

For $k \in \mathbb{N}$, $k \geq 4$ the series $G_k(\Lambda) = \sum_{\lambda \in \Lambda \smallsetminus \{0\}} \lambda^{-k}$ converges. The set $w\Lambda$ is again a lattice if $w \in \mathbb{C}^\times$ and we have

$$G_k(w\Lambda) = w^{-k} G_k(\Lambda).$$

Recall for $\alpha, \beta \in \mathbb{C}$, linearly independent over $\mathbb{R}$ we have the lattice

$$\Lambda(\alpha, \beta) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta.$$

If z is a complex number with $\mathrm{Im}(z) > 0$, then z and 1 are linearly independent over $\mathbb{R}$. We define the *Eisenstein series* as a function on the *upper half plane*

$$\mathbb{H} = \left\{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \right\}$$

by

$$G_k(z) = G_k\left(\Lambda(z, 1)\right) = \sum_{(m,n) \neq (0,0)} \frac{1}{(mz+n)^k},$$

where the sum runs over all $m, n \in \mathbb{Z}$ which are not both zero. Using matrix multiplication, we can write $mz + n = (z\ 1)\binom{m}{n}$. The group $\Gamma_0 = \mathrm{SL}_2(\mathbb{Z})$ acts on the set of pairs $\binom{m}{n}$ by multiplication from the left. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0$ we have

$$\begin{aligned} G_k(z) &= \sum_{m,n} \left((z \quad 1) \begin{pmatrix} m \\ n \end{pmatrix} \right)^{-k} && = \sum_{m,n} \left((z \quad 1) \gamma^t \begin{pmatrix} m \\ n \end{pmatrix} \right)^{-k} \\ &= \sum_{m,n} \left((z \quad 1) \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} \right)^{-k} && = \sum_{m,n} \left((az+b, cz+d) \begin{pmatrix} m \\ n \end{pmatrix} \right)^{-k} \\ &= (cz+d)^{-k} \sum_{m,n} \left(\left(\frac{az+b}{cz+d}, 1 \right) \begin{pmatrix} m \\ n \end{pmatrix} \right)^{-k} && = (cz+d)^{-k} G_k\left(\frac{az+b}{cz+d} \right), \end{aligned}$$

or

$$G_k\left(\frac{az+b}{cz+d}\right) = (cz+d)^k G_k(z).$$

Proposition 1.4.1 *If $k \geq 4$ is even, then*

$$\lim_{y \to \infty} G_k(iy) = 2\zeta(k),$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1,$$

is the Riemann zeta function. (*See Exercise* 1.4.)

Proof One has

$$G_k(iy) = 2\zeta(k) + \sum_{\substack{(m,n) \\ m \neq 0}} \frac{1}{(miy+n)^k}.$$

We show that the second summand tends to zero for $y \to \infty$. Consider the estimate

$$\left| \sum_{\substack{(m,n) \\ m \neq 0}} \frac{1}{(miy+n)^k} \right| \leq \sum_{\substack{(m,n) \\ m \neq 0}} \frac{1}{n^k + m^k y^k}.$$

Here, every summand on the right-hand side is monotonically decreasing in as $y \to \infty$ and tends to zero. Further, the right-hand side converges for every $y > 0$, so we conclude by dominated convergence, that the entire sum tends to zero as $y \to \infty$. □

1.5 Bernoulli Numbers and Values of the Zeta Function

We have seen that Eisenstein series assume zeta-values 'at infinity'. We shall need the following exact expressions for these zeta-values later. We now define the Bernoulli numbers B_k.

Lemma 1.5.1 *For $k = 1, 2, 3, \dots$ there are uniquely determined rational numbers B_k such that for $|z| < 2\pi$ one has*

$$\frac{z}{e^z - 1} + \frac{z}{2} = \frac{z}{2}\frac{e^z + 1}{e^z - 1} = 1 - \sum_{k=1}^{\infty} (-1)^k B_k \frac{z^{2k}}{(2k)!}.$$

The first of these numbers are $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$.

Proof Let $f(z) = \frac{z}{e^z-1} + \frac{z}{2} = \frac{z}{2}\frac{e^z+1}{e^z-1}$. Then f is holomorphic in $\{|z| < 2\pi\}$, so its power series expansion converges in this circle. We show that f is an even function:

$$f(-z) = -\frac{z}{2}\frac{e^{-z} + 1}{e^{-z} - 1} = -\frac{z}{2}\frac{1 + e^z}{1 - e^z} = f(z).$$

Therefore there is such an expression with $B_k \in \mathbb{C}$.

Let $g(z) = \frac{z}{e^z-1} = \sum_{k=0}^{\infty} c_k z^k$. We show that the c_k are all rational numbers. The equation $z = g(z)(e^z - 1)$ gives

$$z = \sum_{n=0}^{\infty} z^n \left(\sum_{j=0}^{n-1} \frac{c_j}{(n-j)!} \right).$$

So $c_0 = 1$ and for every $n \geq 2$ the number c_{n-1} is a rational linear combination of the c_j with $j < n - 1$. Inductively we conclude $c_j \in \mathbb{Q}$. □

Proposition 1.5.2 *For every natural number k one has*

$$\zeta(2k) = \frac{2^{2k-1}}{(2k)!} B_k \pi^{2k}.$$

The first values are $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{\pi^6}{945}$.

Proof By definition, the cotangent function satisfies

$$z \cot z = zi \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}.$$

Replacing z by $z/2i$, this becomes

$$\frac{z}{2i} \cot\left(\frac{z}{2i}\right) = \frac{z}{2}\frac{e^z + 1}{e^z - 1} = f(z),$$

so that

$$z \cot z = 1 - \sum_{k=1}^{\infty} B_k \frac{2^{2k} z^{2k}}{(2k)!}.$$

The partial fraction expansion of the cotangent function (Exercise 1.8) is

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{m=1}^{\infty} \left(\frac{1}{z+m} + \frac{1}{z-m} \right).$$

Therefore

$$z\cot z = 1 + 2\sum_{n=1}^{\infty}\frac{z^2}{z^2 - n^2\pi^2} = 1 - 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{z^{2k}}{n^{2k}\pi^{2k}},$$

from which we get

$$\sum_{k=1}^{\infty} B_k \frac{2^{2k}z^{2k}}{(2k)!} = 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{z^{2k}}{n^{2k}\pi^{2k}}.$$

Comparing coefficients gives the claim. □

1.6 Exercises and Remarks

Exercise 1.1 The *one-point compactification* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$ has the following topology. Open sets are

- open subsets of $\mathbb{C}$, or
- sets V which contain ∞ and have the property that $\mathbb{C} \smallsetminus V$ is a compact subset of $\mathbb{C}$.

Show that $\widehat{\mathbb{C}}$ is compact. Consider the three-dimensional space $\mathbb{C} \times \mathbb{R}$ and let

$$S = \left\{(z,t) \in \mathbb{C} \times \mathbb{R} : |z|^2 + t^2 = 1\right\}.$$

Then S is the two-dimensional sphere. Consider the point $N = (0,1) \in S$, called the *north pole*. Show that for every $z \in \mathbb{C}$ the line through z and N meets the sphere S in exactly one other point $\phi(z)$. Show that the resulting map $\phi : \mathbb{C} \to S \smallsetminus \{N\}$ is a homeomorphism which extends to a homeomorphism $\widehat{\mathbb{C}} \to S$ by sending ∞ to N. This homeomorphism is the reason why $\widehat{\mathbb{C}}$ is also called the *Riemann sphere*.

Exercise 1.2 Let $a \in \mathbb{C} \smallsetminus \{0\}$. A function f on $\mathbb{C}$ is called *simply periodic* of period a, or a-periodic, if $f(z+a) = f(z)$ for every $z \in \mathbb{C}$. Show that if $a, b \in \mathbb{C}$ are linearly independent over $\mathbb{R}$, then a function f is $\Lambda(a,b)$-periodic if and only if it is a-periodic and b-periodic simultaneously. This explains the notion *doubly periodic*.

Exercise 1.3 A subgroup $\Lambda \subset \mathbb{C}$ of the additive group $(\mathbb{C},+)$ is called a *discrete subgroup* if Λ is discrete in the subset topology, i.e. if for every $\lambda \in \Lambda$ there exists an open set $U_\lambda \subset \mathbb{C}$ such that $\Lambda \cap U_\lambda = \{\lambda\}$. Show

1. A subgroup $\Lambda \subset \mathbb{C}$ is discrete if and only if there is an open set $U_0 \subset \mathbb{C}$ with $U_0 \cap \Lambda = \{0\}$.
2. If $\Lambda \subset \mathbb{C}$ is a discrete subgroup, then there are three possibilities: either $\Lambda = \{0\}$, or there is a $\lambda_0 \in \Lambda$ with $\Lambda = \mathbb{Z}\lambda_0$, or Λ is a lattice.
3. A discrete subgroup $\Lambda \subset \mathbb{C}$ is a lattice if and only if the quotient group $\mathbb{C}/\Lambda$ is compact in the quotient topology.
4. If $\Lambda \subset \mathbb{C}$ is a lattice, then a subgroup $\Sigma \subset \Lambda$ is a lattice if and only if it has finite index, i.e. if the group Λ/Σ is finite.

Exercise 1.4 Show that the sum defining the Riemann zeta function, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, converges absolutely for $\mathrm{Re}(s) > 1$. One can adapt the proof of Lemma 1.2.1.

Exercise 1.5 Show that the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ has the Euler product

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}, \quad \mathrm{Re}(s) > 1,$$

where the product runs over all prime numbers p. (Hint: consider the sequence $s_N(s) = \prod_{p \le N} \frac{1}{1-p^{-s}}$. Using the geometric series, write $\frac{1}{1-p^{-s}} = \sum_{k=0}^{\infty} p^{-ks}$ and use absolute convergence of the Dirichlet series defining $\zeta(s)$.)

Exercise 1.6 Let $\alpha, \beta, \alpha'\beta' \in \mathbb{C}$ with $\mathbb{C} = \mathbb{R}\alpha + \mathbb{R}\beta = \mathbb{R}\alpha' + \mathbb{R}\beta'$. Show that the lattices $\Lambda(\alpha, \beta)$ and $\Lambda(\alpha', \beta')$ coincide if and only if there is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ with

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Exercise 1.7 Let f be meromorphic on $\mathbb{C}$ and Λ-periodic for a lattice Λ. Let $\mathcal{F}_w = \mathcal{F} + w$ a translated fundamental mesh for Λ, such that there are no poles or zeros of f on the boundary $\partial\mathcal{F}_w$. Let $S(0)$ be the sum of all zeros of f in $\mathcal{F}$, counted with multiplicities. Let $S(\infty)$ be the sum of all poles of f in $\mathcal{F}$, also with multiplicities. Show:

$$S(0) - S(\infty) \in \Lambda.$$

(Integrate the function $zf'(z)/f(z)$.)

Exercise 1.8 Prove the partial fraction expansion of the cotangent:

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{m=1}^{\infty} \left(\frac{1}{z+m} + \frac{1}{z-m} \right).$$

(The difference of the two sides is periodic and entire. Show that it is bounded and odd.)

Exercise 1.9 Let $z, w \in \mathbb{C}$ and let $\wp$ be the Weierstrass $\wp$-function for a lattice Λ. Show that $\wp(z) = \wp(w)$ if and only if $z + w$ or $z - w$ is in the lattice Λ.

Exercise 1.10 Let $\wp$ be the Weierstrass $\wp$-function for a lattice Λ.

(a) Let $a_1, \dots, a_n$ and $b_1, \dots, b_m$ be complex numbers. Show that the function

$$f(z) = \frac{\prod_{i=1}^{n} \wp(z) - \wp(a_i)}{\prod_{j=1}^{m} \wp(z) - \wp(b_j)}$$

is even and Λ-periodic.

(b) Show that every even Λ-periodic function is a rational function of $\wp$.
(c) Show that every Λ-periodic meromorphic function is of the form $R(\wp(z)) + \wp'(z)Q(\wp(z))$, where R and Q are rational functions.

Exercise 1.11 (Residue theorem for circle segments) Let $r_0 > 0$ and let f be a holomorphic function on the set $\{z \in \mathbb{C} : 0 < |z| < r_0\}$, which has a simple pole at $z = 0$. Let $a, b : [0, r_0) \to (-\pi, \pi)$ be continuous functions with $a(r) \le b(r)$ for every $0 \le r < r_0$ and for $0 < r < r_0$ let $\gamma_r : (a(r), b(r)) \to \mathbb{C}$ the circle segment $\gamma_r(t) = re^{it}$. Show:

$$\lim_{r\to 0} \frac{1}{2\pi i} \int_{\gamma_r} f(z)\,dz = \frac{b(0) - a(0)}{2\pi} \operatorname{res}_{z=0} f(z).$$

Exercise 1.12 Let (a_n) be a sequence in $\mathbb{C}$. Show that there exists an $r \in \mathbb{R} \cup \{\pm\infty\}$, such that for every $s \in \mathbb{C}$ with $\operatorname{Re}(s) > r$ and for no s with $\operatorname{Re}(s) < r$ the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges absolutely.

Remarks Putting $g_4 = 15G_4$ and $g_6 = 35G_6$ one sees that $(x, y) = (\wp, \wp'/2)$ satisfies the polynomial equation

$$y^2 = x^3 - g_4 x - g_6.$$

This means that the map $z \mapsto (\wp(z), \wp'(z)/2)$ maps the complex manifold $\mathbb{C}/\Lambda$ bijectively onto the *elliptic curve* given by this equation. Indeed, every elliptic curve is obtained in this way, so elliptic curves are parametrized by lattices. The book [Sil09] gives a good introduction to elliptic curves.

The Riemann zeta function featured in this section has a meromorphic extension to all of $\mathbb{C}$ and satisfies a functional equation, as shown in Theorem 6.1.3. The famous *Riemann hypothesis* says that every zero of the function $\zeta(s)$ in the strip $0 < \operatorname{Re}(s) < 1$ has real part $\frac{1}{2}$. This hypothesis is considered the hardest problem of all mathematics.