Chapter 2
Modeling the Quad-Rotor Mini-Rotorcraft

The complete dynamics of an aircraft, taking into account aero-elastic effects, flexibility of the wings, internal dynamics of the engine and the whole set of changing variables are quite complex and somewhat unmanageable for the purposes of control. Therefore, it is interesting to consider a simplified model of an aircraft formed by a minimum number of states and inputs, but retaining the main features that must be considered when designing control laws for a real aircraft.

This chapter deals with the modeling of a quad-rotor rotorcraft, and is organized as follows. Section 2.1 gives a general overview of the quad-rotor aerial vehicle and its operation principle. Next, Sect. 2.2 deals with the quad-rotor modeling, presenting two different approaches: Euler–Lagrange in Sect. 2.2.1 and Newton–Euler in Sect. 2.2.2. Subsequently, it is shown in Sect. 2.2.3 how to derive Lagrange's equations from Newton's equations. Section 2.2.4 presents a Newton–Euler modeling for an "X-Flyer" quad-rotor configuration. Finally, some concluding remarks are presented in Sect. 2.3.

2.1 The Quad-Rotor Mini-Rotorcraft

The quad-rotor mini-rotorcraft is controlled by the angular speeds of four electric motors as shown in Fig. 2.1. Each motor produces a thrust and a torque, whose combination generates the main thrust, the yaw torque, the pitch torque, and the roll torque acting on the quad-rotor. Conventional helicopters modify the lift force by varying the collective pitch. Such aerial vehicles use a mechanical device known as swashplate. This system interconnects servomechanisms and blade pitch links in order to change the rotor blades pitch angle in a cyclic manner, so as to obtain the pitch and roll control torques of the vehicle. In contrast, the quad-rotor does not have a swashplate and has constant pitch blades. Therefore, in a quad-rotor we can only vary the angular speed of each one of the four rotors to obtain the pitch and roll control torques.

From Fig. 2.1 it can be observed that the motor M_i (for $i = 1, \dots, 4$) produces the force f_i, which is proportional to the square of the angular speed, that is, $f_i = kw_i^2$.

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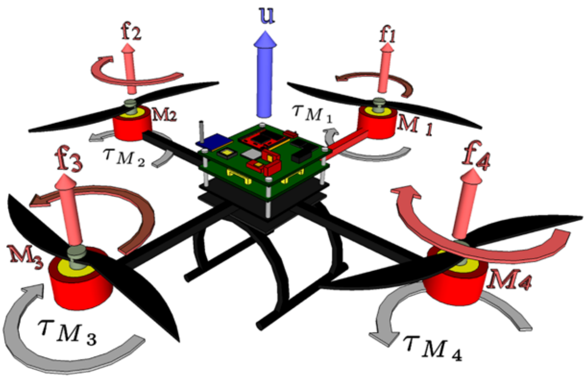


Fig. 2.1 The quad-rotor control input

Given that the quad-rotor's motors can only turn in a fixed direction, the produced force f_i is always positive. The front (M_1) and the rear (M_3) motors rotate counter-clockwise, while the left (M_2) and right (M_4) motors rotate clockwise. With this arrangement, gyroscopic effects and aerodynamic torques tend to cancel in trimmed flight. The main thrust u is the sum of individual thrusts of each motor. The pitch torque is a function of the difference $f_1 - f_3$, the roll torque is a function of $f_2 - f_4$, and the yaw torque is the sum $\tau_{M_1} + \tau_{M_2} + \tau_{M_3} + \tau_{M_4}$, where τ_{M_i} is the reaction torque of motor i due to shaft acceleration and blades drag. The motor torque is opposed by an aerodynamic drag τ_{drag}, such that

$$I_{\text{rot}}\dot{\omega} = \tau_{M_i} - \tau_{\text{drag}} \tag{2.1}$$

where I_{rot} is the moment of inertia of a rotor around its axis. The aerodynamic drag is defined as

$$\tau_{\text{drag}} = \frac{1}{2}\rho A v^2 \tag{2.2}$$

where ρ is the air density, the frontal area of the moving shape is defined by A, and v is its velocity relative to the air. In magnitude, the angular velocity ω is equal to the linear velocity v divided by the radius of rotation r

$$\omega = \frac{v}{r} \tag{2.3}$$

The aerodynamic drag can be rewritten as

$$\tau_{\text{drag}} = k_{\text{drag}}\omega^2 \tag{2.4}$$

where $k_{\text{drag}} > 0$ is a constant depending on the air density, the radius, the shape of the blade and other factors. For quasi-stationary maneuvers, ω is constant, then

$$\tau_{M_i} = \tau_{\text{drag}} \tag{2.5}$$

Forward pitch motion is obtained by increasing the speed of the rear motor M_3 while reducing the speed of the front motor M_1. Similarly, roll motion is obtained using the left and right motors. Yaw motion is obtained by increasing the torque of the

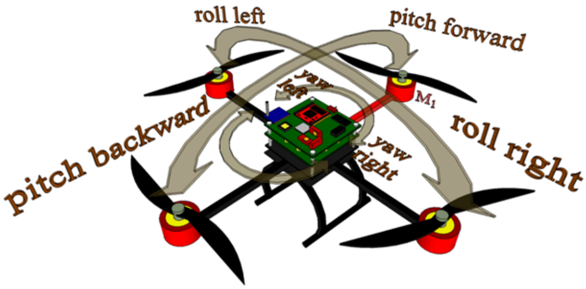


Fig. 2.2 Pitch, roll and yaw torques of the quad-rotor

front and rear motors (τ_{M1} and τ_{M3}, respectively) while decreasing the torque of the lateral motors (τ_{M2} and τ_{M4}, respectively). Such motions can be accomplished while maintaining the total thrust constant, see Fig. 2.2.

2.2 Quad-Rotor Dynamical Model

The quad-rotor model is obtained by representing the aircraft as a solid body evolving in a three dimensional space and subject to the main thrust and three torques: pitch, roll and yaw.

2.2.1 Euler–Lagrange Approach

Let the generalized coordinates of the rotorcraft be expressed by

$$\mathbf{q} = (x, y, z, \psi, \theta, \phi) \in \mathbb{R}^6 \tag{2.6}$$

where $\boldsymbol{\xi} = (x, y, z) \in \mathbb{R}^3$ denotes the position vector of the center of mass of the quad-rotor relative to a fixed inertial frame $\mathscr{I}$. The rotorcraft's Euler angles (the orientation of the rotorcraft) are expressed by $\boldsymbol{\eta} = (\psi, \theta, \phi) \in \mathbb{R}^3$, ψ is the yaw angle around the z-axis, θ is the pitch angle around the y-axis and ϕ is the roll angle around the x-axis (see [33] and [5]). An illustration of the generalized coordinates of the rotorcraft is shown in Fig. 2.3. Define the Lagrangian

$$L(\mathbf{q}, \dot{\mathbf{q}}) = T_{\text{trans}} + T_{\text{rot}} - U \tag{2.7}$$

where $T_{\text{trans}} = \frac{m}{2}\dot{\boldsymbol{\xi}}^{\mathrm{T}}\dot{\boldsymbol{\xi}}$ is the translational kinetic energy, $T_{\text{rot}} = \frac{1}{2}\boldsymbol{\Omega}^{\mathrm{T}} I \boldsymbol{\Omega}$ is the rotational kinetic energy, $U = mgz$ is the potential energy of the rotorcraft, z is the rotorcraft altitude, m denotes the mass of the quad-rotor, $\boldsymbol{\Omega}$ is the vector of the angular velocity, I is the inertia matrix and g is the acceleration due to gravity. The angular velocity vector $\boldsymbol{\omega}$ resolved in the body-fixed frame is related to the generalized velocities $\dot{\boldsymbol{\eta}}$ (in the region where the Euler angles are valid) by means of the standard kinematic relationship [38]

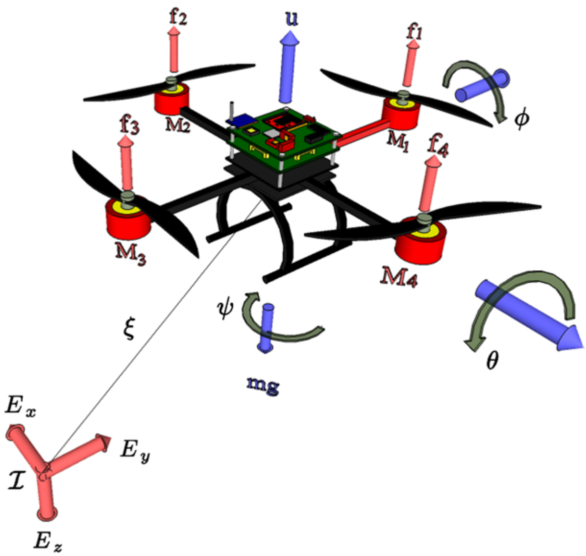


Fig. 2.3 The quad-rotor in an inertial frame. f_1, f_2, f_3, f_4 represent the thrust of each motor, ψ, θ and ϕ represent the Euler angles, and u is the main thrust

$$\boldsymbol{\Omega} = W_\eta \dot{\boldsymbol{\eta}} \tag{2.8}$$

where

$$W_\eta = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \cos\theta\sin\phi & \cos\phi & 0 \\ \cos\theta\cos\phi & -\sin\phi & 0 \end{bmatrix} \tag{2.9}$$

then

$$\boldsymbol{\Omega} = \begin{bmatrix} \dot{\phi} - \dot{\psi}\sin\theta \\ \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi \\ \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi \end{bmatrix} \tag{2.10}$$

Define

$$\mathbb{J} = \mathbb{J}(\boldsymbol{\eta}) = W_\eta^{\mathrm{T}} I W_\eta \tag{2.11}$$

where

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \tag{2.12}$$

so that

$$T_{\mathrm{rot}} = \frac{1}{2}\dot{\boldsymbol{\eta}}^{\mathrm{T}}\mathbb{J}\dot{\boldsymbol{\eta}} \tag{2.13}$$

Thus, the matrix $\mathbb{J} = \mathbb{J}(\boldsymbol{\eta})$ acts as the inertia matrix for the full rotational kinetic energy of the quad-rotor, expressed directly in terms of the generalized coordinates $\boldsymbol{\eta}$.

The model of the full rotorcraft dynamics is obtained from Euler–Lagrange equations with external generalized forces

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathscr{L}}{\partial \mathbf{q}} = \begin{bmatrix} \mathbf{F}_\xi \\ \boldsymbol{\tau} \end{bmatrix} \tag{2.14}$$

where $\mathbf{F}_\xi = R\hat{\mathbf{F}} \in \mathbb{R}^3$ is the translational force applied to the rotorcraft due to main thrust, $\boldsymbol{\tau} \in \mathbb{R}^3$ represents the yaw, pitch and roll moments and R denotes the rotational matrix. $R(\psi, \theta, \phi) \in SO(3)$ represents the orientation of the aircraft relative to a fixed inertial frame:

$$R = \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & s_\phi s_\psi + c_\phi c_\psi s_\theta \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} \tag{2.15}$$

where c_θ stands for $\cos\theta$ and s_θ for $\sin\theta$. From Fig. 2.1, it follows that

$$\hat{\mathbf{F}} = \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix} \tag{2.16}$$

where u is the main thrust directed out of the bottom of the aircraft and expressed as

$$u = \sum_{i=1}^{4} f_i \tag{2.17}$$

and, for $i = 1, \ldots, 4$, f_i is the force produced by motor M_i, as shown in Fig. 2.1. Typically $f_i = k\omega_i^2$, where k_i is a constant and ω_i is the angular speed of the ith motor. The generalized torques are thus

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_\psi \\ \tau_\theta \\ \tau_\phi \end{bmatrix} \triangleq \begin{bmatrix} \sum_{i=1}^{4} \tau_{M_i} \\ (f_2 - f_4)\ell \\ (f_3 - f_1)\ell \end{bmatrix} \tag{2.18}$$

where ℓ is the distance between the motors and the center of gravity, and τ_{M_i} is the moment produced by motor M_i, for $i = 1, \ldots, 4$, around the center of gravity of the aircraft.

Since the Lagrangian contains no cross terms in the kinematic energy combining $\dot{\boldsymbol{\xi}}$ with $\dot{\boldsymbol{\eta}}$, the Euler–Lagrange equation can be partitioned into dynamics for $\boldsymbol{\xi}$ coordinates and $\boldsymbol{\eta}$ coordinates. The Euler–Lagrange equation for the translational motion is

$$\frac{d}{dt}\left[\frac{\partial L_{\text{trans}}}{\partial \dot{\boldsymbol{\xi}}}\right] - \frac{\partial L_{\text{trans}}}{\partial \boldsymbol{\xi}} = \mathbf{F}_\xi \tag{2.19}$$

then

$$m\ddot{\boldsymbol{\xi}} + mg\mathbf{E}_z = \mathbf{F}_\xi \tag{2.20}$$

As for the η coordinates, it can be written

$$\frac{d}{dt}\left[\frac{\partial L_{\text{rot}}}{\partial \dot{\boldsymbol{\eta}}}\right] - \frac{\partial L_{\text{rot}}}{\partial \boldsymbol{\eta}} = \boldsymbol{\tau} \tag{2.21}$$

or

$$\frac{d}{dt}\left[\dot{\boldsymbol{\eta}}^{\mathrm{T}}\mathbb{J}\frac{\partial\dot{\boldsymbol{\eta}}}{\partial\boldsymbol{\eta}}\right]-\frac{1}{2}\frac{\partial}{\partial\boldsymbol{\eta}}\left(\dot{\boldsymbol{\eta}}^{\mathrm{T}}\mathbb{J}\dot{\boldsymbol{\eta}}\right)=\boldsymbol{\tau} \tag{2.22}$$

Thus one obtains

$$\mathbb{J}\ddot{\boldsymbol{\eta}}+\dot{\mathbb{J}}\dot{\boldsymbol{\eta}}-\frac{1}{2}\frac{\partial}{\partial\boldsymbol{\eta}}\left(\dot{\boldsymbol{\eta}}^{\mathrm{T}}\mathbb{J}\dot{\boldsymbol{\eta}}\right) \tag{2.23}$$

Defining the Coriolis-centripetal vector

$$\bar{V}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}})=\dot{\mathbb{J}}\dot{\boldsymbol{\eta}}-\frac{1}{2}\frac{\partial}{\partial\boldsymbol{\eta}}\left(\dot{\boldsymbol{\eta}}^{\mathrm{T}}\mathbb{J}\dot{\boldsymbol{\eta}}\right) \tag{2.24}$$

one writes

$$\mathbb{J}\ddot{\boldsymbol{\eta}}+\bar{V}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}})=\boldsymbol{\tau} \tag{2.25}$$

but $\bar{V}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}})$ can be expressed as

$$\begin{aligned}\bar{V}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}})&=\left(\dot{\mathbb{J}}-\frac{1}{2}\frac{\partial}{\partial\boldsymbol{\eta}}\left(\dot{\boldsymbol{\eta}}^{\mathrm{T}}\mathbb{J}\right)\right)\dot{\boldsymbol{\eta}}\\&=C(\boldsymbol{\eta},\dot{\boldsymbol{\eta}})\dot{\boldsymbol{\eta}}\end{aligned} \tag{2.26}$$

where $C(\boldsymbol{\eta},\dot{\boldsymbol{\eta}})$ is referred to as the Coriolis term and contains the gyroscopic and centrifugal terms associated with the $\boldsymbol{\eta}$ dependence of $\mathbb{J}$. This yields

$$m\ddot{\boldsymbol{\xi}}+mg\mathbf{E}_z=\mathbf{F}_\xi \tag{2.27}$$

$$\mathbb{J}\ddot{\boldsymbol{\eta}}=\boldsymbol{\tau}-C(\boldsymbol{\eta},\dot{\boldsymbol{\eta}})\dot{\boldsymbol{\eta}} \tag{2.28}$$

To simplify let us take

$$\tilde{\boldsymbol{\tau}}=\begin{pmatrix}\tilde{\tau}_\psi\\ \tilde{\tau}_\theta\\ \tilde{\tau}_\phi\end{pmatrix}=\mathbb{J}^{-1}\left(\boldsymbol{\tau}-C(\boldsymbol{\eta},\dot{\boldsymbol{\eta}})\dot{\boldsymbol{\eta}}\right) \tag{2.29}$$

Finally one obtains

$$m\ddot{x}=u(\sin\phi\sin\psi+\cos\phi\cos\psi\sin\theta) \tag{2.30}$$

$$m\ddot{y}=u(\cos\phi\sin\theta\sin\psi-\cos\psi\sin\phi) \tag{2.31}$$

$$m\ddot{z}=u\cos\theta\cos\phi-mg \tag{2.32}$$

$$\ddot{\psi}=\tilde{\tau}_\psi \tag{2.33}$$

$$\ddot{\theta}=\tilde{\tau}_\theta \tag{2.34}$$

$$\ddot{\phi}=\tilde{\tau}_\phi \tag{2.35}$$

where x and y are coordinates in the horizontal plane, z is the vertical position, and $\tilde{\tau}_\psi$, $\tilde{\tau}_\theta$ and $\tilde{\tau}_\phi$ are the yawing moment, pitching moment and rolling moment, respectively, which are related to the generalized torques τ_ψ, τ_θ, τ_ϕ.

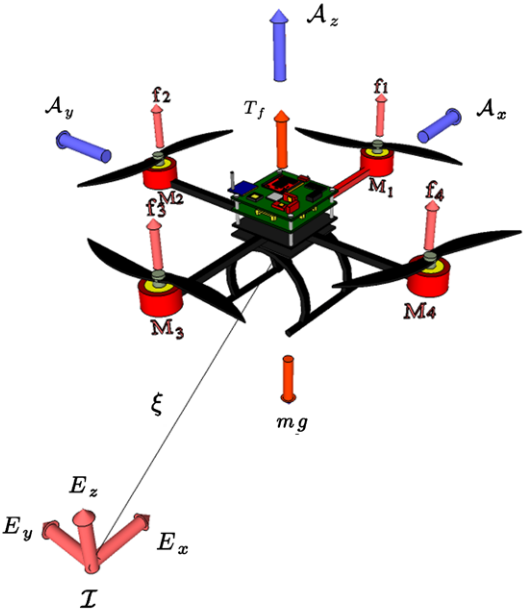


Fig. 2.4 The quad-rotor in an inertial frame. f_i represent the thrust of motor i and T_f is the main thrust

2.2.2 Newton–Euler Approach

The general motion of a rigid body in space is a combination of translational and rotational motions. Consider a rigid body moving in inertial space, undergoing both rotations and translations. Let us define now an earth fixed frame $\mathscr{I}$ and a body-fixed frame $\mathscr{A}$, as seen in Fig. 2.4. The center of mass and the body-fixed frame are assumed to coincide. Using Euler angles parametrization, the airframe orientation in space is given by a rotation R from $\mathscr{A}$ to $\mathscr{I}$, where $R \in SO(3)$ is the rotation matrix. Using the Newton–Euler formalism, the dynamics of a rigid body under external forces applied to the center of mass and expressed on earth fixed frame is

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \mathbf{v} \\ m\dot{\mathbf{v}} &= \mathbf{f} \\ \dot{R} &= R\hat{\boldsymbol{\Omega}} \\ I\dot{\boldsymbol{\Omega}} &= -\boldsymbol{\Omega} \times I\boldsymbol{\Omega} + \boldsymbol{\tau} \end{aligned} \tag{2.36}$$

where $\boldsymbol{\xi} = (x, y, z)^{\mathrm{T}}$ denotes the position of the center of mass of the airframe with respect to the frame $\mathscr{I}$ relative to a fixed origin, $\mathbf{v} \in \mathscr{I}$ denotes the linear velocity expressed in the inertial frame, and $\boldsymbol{\Omega} \in \mathscr{A}$ denotes the angular velocity of the airframe expressed in the body-fixed frame. The mass of the rigid body is denoted by m, and $I \in \mathbb{R}^{3\times3}$ denotes the constant inertia matrix around the center of mass (expressed in the body-fixed frame $\mathscr{A}$). $\hat{\boldsymbol{\omega}}$ denotes the skew-symmetric matrix of the vector $\boldsymbol{\omega}$. $\mathbf{f} \in \mathscr{I}$ represents the vector of the principal non-conservative forces applied to the object; including thrusts T_f and drag terms associated with the rotors.

$\boldsymbol{\tau} \in \mathscr{A}$ is derived from differential thrust associated with pairs of rotors along with aerodynamics effects and gyroscopic effects.

Translational Force and Gravitational Force The only forces acting on the body are given by the translational force T_f and the gravitational force g. From Fig. 2.4, the thrust applied to the vehicle is

$$T_f = \sum_{i=1}^{4} f_i \tag{2.37}$$

where the lift f_i generated by a rotor in free air can be modeled as $f_i k\omega_i^2$ in the z-direction, where $k > 0$ is a constant and ω_i is the angular speed of the ith motor. Equation (2.37) can be rewritten as

$$T_f = k\left(\sum_{i=1}^{4} \omega_i^2\right) \tag{2.38}$$

Then

$$\mathbf{F} = \begin{bmatrix} 0 \\ 0 \\ T_f \end{bmatrix} \tag{2.39}$$

The gravitational force applied to the vehicle is

$$\mathbf{f}_g = -mg\mathbf{E}_z \tag{2.40}$$

This yields

$$\mathbf{f} = \mathbf{R}_{E_z} T_f + \mathbf{f}_g \tag{2.41}$$

Torques Due to the rigid rotor constraint, the dynamics of each rotor disc around its axis of rotation can be treated as a decoupled system in the generalized variable ω_i, denoting angular velocity or a rotor around its axis. The torque exerted by each electrical motor is denoted by τ_{M_i}. The motor's torque is opposed by an aerodynamic drag $\tau_{\text{drag}} = k_\tau \omega_i^2$. Using Newton's second law one has

$$I_M \dot{\omega}_i = -\tau_{\text{drag}} + \tau_{M_i} \tag{2.42}$$

where I_M is the angular moment of the ith motor and $k_\tau > 0$ is a constant for quasi-stationary maneuvers in free flight. In steady state, i.e., when $\dot{\omega}_i = 0$, the yaw torque is

$$\tau_{M_i} = k_\tau \omega_i^2 \tag{2.43}$$

The generalized torques are thus

$$\boldsymbol{\tau}_{\mathscr{A}} = \begin{bmatrix} \sum_{i=1}^{4} \tau_{M_i} \\ (f_2 - f_4)\ell \\ (f_3 - f_1)\ell \end{bmatrix} = \begin{bmatrix} \tau_\psi \\ \tau_\theta \\ \tau_\phi \end{bmatrix} \tag{2.44}$$

where ℓ represents the distance between the motors and the center of gravity. Rewriting (2.44) one has

$$\tau_\psi = k_\tau\left(\omega_1^2 + \omega_3^2 - \omega_2^2 - \omega_4^2\right) \tag{2.45}$$

$$\tau_\theta = \ell k\left(\omega_2^2 - \omega_4^2\right) \tag{2.46}$$

$$\tau_\phi = \ell k\left(\omega_3^2 - \omega_1^2\right) \tag{2.47}$$

where τ_ψ, τ_θ and τ_ϕ are the generalized torques (yawing moment, pitching moment and rolling moment, respectively). Each rotor may be thought of as a rigid disc rotating around the axis E_z in the body-fixed frame, with angular velocity ω_i. The rotor's axis of rotation is itself moving with the angular velocity of the frame. This leads to the following gyroscopic torques applied to the airframe:

$$\begin{aligned}\boldsymbol{\tau}_{G_\mathscr{A}} &= -\sum_{i=1}^{4} I_M(\boldsymbol{\omega} \times \mathbf{E}_z)\omega_i \\ &= -(\boldsymbol{\omega} \times \mathbf{E}_z)\sum_{i=1}^{4} I_M\omega_i \end{aligned} \tag{2.48}$$

This yields

$$\boldsymbol{\tau} = \boldsymbol{\tau}_\mathscr{A} + \boldsymbol{\tau}_{G_\mathscr{A}} \tag{2.49}$$

Rewriting (2.36), one has

$$\begin{aligned}\dot{\boldsymbol{\xi}} &= \mathbf{v} \\ m\dot{\mathbf{v}} &= \mathbf{R}_{E_z}T_f - mg\mathbf{E}_z \\ \dot{R} &= R\hat{\boldsymbol{\Omega}} \\ I\dot{\boldsymbol{\Omega}} &= -\boldsymbol{\Omega} \times I\boldsymbol{\Omega} + \boldsymbol{\tau}_\mathscr{A} + \boldsymbol{\tau}_{G_\mathscr{A}}\end{aligned} \tag{2.50}$$

2.2.3 Newton's Equations to Lagrange's Equations

Using the classical *yaw*, *pitch* and *roll* Euler angles (ψ, θ, ϕ) applied in aeronautical applications [5, 33], the rotation matrix can be expressed as

$$R = \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & s_\phi s_\psi + c_\phi c_\psi s_\theta \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} \tag{2.51}$$

The equations in (2.50) can be separated into the $\boldsymbol{\xi}$ coordinates dynamics and the $\boldsymbol{\eta}$ dynamics. Rewriting the $\boldsymbol{\xi}$ dynamics one has

$$\ddot{\boldsymbol{\xi}} = \frac{1}{m}(\mathbf{R}_{E_z}T_f - g\mathbf{E}_z) \tag{2.52}$$

where

$$\mathbf{R}_{E_z} = \begin{bmatrix} s_\phi s_\psi + c_\phi c_\psi s_\theta \\ c_\phi s_\theta s_\psi - c_\psi s_\phi \\ c_\theta c_\phi \end{bmatrix}$$

From Figs. 2.3 and 2.4 one has $u = T_f$, this yields

$$\ddot{x} = \frac{1}{m} u(\sin\phi \sin\psi + \cos\phi \cos\psi \sin\theta) \tag{2.53}$$

$$\ddot{y} = \frac{1}{m} u(\cos\phi \sin\theta \sin\psi - \cos\psi \sin\phi) \tag{2.54}$$

$$\ddot{z} = \frac{1}{m} u \cos\theta \cos\phi - g \tag{2.55}$$

From Newton–Euler formalism, one obtains in (2.53)–(2.55) the same equations as obtained in (2.30)–(2.32).

2.2.4 Newton–Euler Approach for an X-type Quad-Rotor

The quad-rotor model presented in Sects. 2.2.1 and 2.2.2 considers front and rear motors aligned with the longitudinal axis, and left and right motors aligned with the lateral axis. This section introduces an "X-type" quad-rotor flying configuration, considering two frontal motors and two rear motors. The quad-rotor dynamical model equations are based on Newton–Euler formalism, where the nonlinear dynamics is obtained in North-East-Down (NED) inertial and body-fixed coordinates, see Fig. 2.5. Let $\{N, E, D\}$ represent the inertial reference frame and $\{X, Y, Z\}$ represent the body-fixed frame. The position vector of the center of mass of the rotorcraft is denoted by $\boldsymbol{\xi} = (x, y, z)^{\mathrm{T}}$, representing the position coordinates of the vehicle relative to the NED inertial frame. The orientation vector of the aircraft with respect to the inertial frame is expressed by $\boldsymbol{\eta} = (\psi, \theta, \phi)^{\mathrm{T}}$, where ψ, θ and ϕ are the yaw, pitch and roll Euler angles, respectively. The full nonlinear dynamics of the quad-rotor can be expressed as

$$m\ddot{\boldsymbol{\xi}} = -mg\mathbf{D} + \mathbf{R}F \tag{2.56}$$

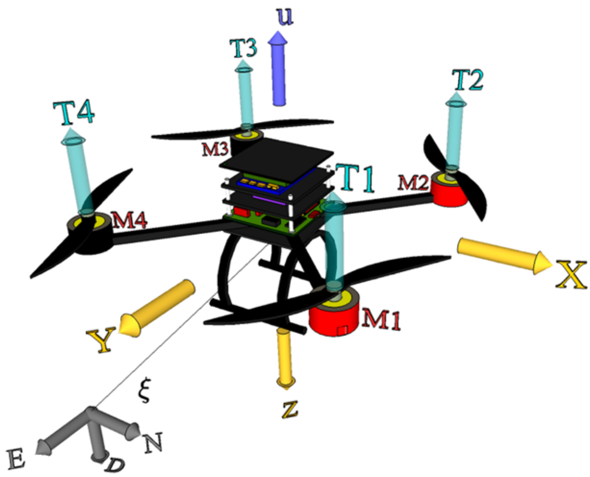
$$I\dot{\Omega} = -\boldsymbol{\Omega} \times I\boldsymbol{\Omega} + \boldsymbol{\tau} \tag{2.57}$$

where $R \in SO(3)$ is a rotation matrix that associates the inertial frame with the body-fixed frame, F denotes the total force applied to the vehicle, m is the total mass, g denotes the gravitational constant, $\boldsymbol{\Omega}$ represents the angular velocity of the vehicle expressed in the body-fixed frame, I describes the inertia matrix, and $\boldsymbol{\tau}$ is the total torque.

Let $u = \sum_{i=1}^{4} T_i$ be the force applied to the vehicle, which is generated by the four rotors. Assuming that this force has only one component in the Z direction, the total force can be written as $\mathbf{F} = (0, 0, -u)^{\mathrm{T}}$. The rotation matrix R is defined as

$$R = \begin{bmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{bmatrix} \tag{2.58}$$

Fig. 2.5 NED diagram of the quad-rotor dynamical model



where $c_. = \cos(.)$ and $s_. = \sin(.)$. Let us define now an auxiliary vector $\tilde{\tau}$ related to the generalized torque τ and based on (2.57):

$$\tilde{\boldsymbol{\tau}} = \begin{bmatrix} \tilde{\tau}_\psi \\ \tilde{\tau}_\theta \\ \tilde{\tau}_\phi \end{bmatrix} = I^{-1} W^{-1} (-I \dot{W} \dot{\boldsymbol{\eta}} - W \dot{\boldsymbol{\eta}} \times I W \dot{\boldsymbol{\eta}} + \boldsymbol{\tau}) \tag{2.59}$$

where $\boldsymbol{\Omega} = W \dot{\boldsymbol{\eta}}$ and W is [38]:

$$W = \begin{bmatrix} -\sin(\theta) & 0 & 1 \\ \cos(\theta)\sin(\phi) & \cos(\phi) & 0 \\ \cos(\theta)\cos(\phi) & -\sin(\phi) & 0 \end{bmatrix} \tag{2.60}$$

Using (2.56)–(2.59), the quad-rotor dynamical model can be represented by

$$m\ddot{x} = -u\big(\cos(\psi)\sin(\theta)\cos(\phi) + \sin(\psi)\sin(\phi)\big) \tag{2.61}$$

$$m\ddot{y} = -u\big(\sin(\psi)\sin(\theta)\cos(\phi) - \cos(\psi)\sin(\phi)\big) \tag{2.62}$$

$$m\ddot{z} = -u\big(\cos(\theta)\cos(\phi)\big) + mg \tag{2.63}$$

$$\ddot{\psi} = \tilde{\tau}_\psi \tag{2.64}$$

$$\ddot{\theta} = \tilde{\tau}_\theta \tag{2.65}$$

$$\ddot{\phi} = \tilde{\tau}_\phi \tag{2.66}$$

In the "X-type" quad-rotor model, the motors M_1 and M_3 rotate clockwise, while motors M_2 and M_4 rotate counter-clockwise. Assuming that total thrust approximately counteracts gravity, i.e., the quad-rotor is in hover or near-hover flight conditions, we can consider that each thrust can be modeled as $\tau_i = C w_i^2$, where C is a constant value depending on the rotor characteristics and w_i denotes the speed of the rotor i [6]. For simplicity, it is also assumed that the torque τ_i generated by each rotor is proportional to its lift force, then $\tau_i = C_M T_i$. Taking into account the previous assumptions, we can obtain the generalized torques as

$$\begin{bmatrix} \tau_\psi \\ \tau_\theta \\ \tau_\phi \end{bmatrix} = \begin{bmatrix} -C_M & C_M & -C_M & C_M \\ -l & -l & l & l \\ -l & l & l & -l \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} \tag{2.67}$$

where l represents the distance between the center of mass and the center of the rotor.

2.3 Concluding Remarks

In this chapter Euler–Lagrange and Newton–Euler approaches have been applied for obtaining a simplified model of a quad-rotor rotorcraft. The model is formed by a minimum number of states and inputs, but retains the main features that must be considered when designing control laws. Two quad-rotor configurations were analyzed. The first configuration addressed a classical motor arrangement having one pair of motors aligned with the longitudinal axis while the other pair is aligned with the translational axis. The second configuration addressed an "X-Flyer" motor arrangement, having two frontal motors and two rear motors.

The models obtained here will be used in later sections for designing control laws devoted to attitude stabilization and autonomous positioning.