Chapter 8
First-Order Logic: Deductive Systems

We extend the deductive systems $\mathscr{G}$ and $\mathscr{H}$ from propositional logic to first-order logic by adding axioms and rules of inference for the universal quantifier. (The existential quantifier is defined as the dual of the universal quantifier.) The construction of semantic tableaux for first-order logic included restrictions on the use of constants and similar restrictions will be needed here.

8.1 Gentzen System $\mathscr{G}$

Figure 8.1 is a closed semantic tableau for the negation of the valid formula

$$\forall x p(x) \vee \forall x q(x) \rightarrow \forall x (p(x) \vee q(x)).$$

The formulas to which rules are applied are underlined, while the sets of constants $C(n)$ in the labels of each node are implicit.

Let us turn the tree upside down and in every node n replace $U(n)$, the set of formulas labeling the node n, by $\bar{U}(n)$, the set of complements of the formulas in $U(n)$. The result (Fig. 8.2) is a Gentzen proof for the formula.

Here is the classification of quantified formulas into γ- and δ-formulas:

γ	$\gamma(a)$
$\exists x A(x)$	$A(a)$
$\neg \forall x A(x)$	$\neg A(a)$

δ	$\delta(a)$
$\forall x A(x)$	$A(a)$
$\neg \exists x A(x)$	$\neg A(a)$

Definition 8.1 The *Gentzen system* $\mathscr{G}$ is a deductive system. Its *axioms* are sets of formulas U containing a complementary pair of literals. The rules of inference are the rules given for α- and β-formulas in Sect. 3.2, together with the following rules for γ- and δ-formulas:

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$$
\begin{array}{c}
\underline{\neg\,(\forall x p(x) \vee \forall x q(x) \;\rightarrow\; \forall x (p(x) \vee q(x)))} \\
\downarrow \\
\underline{\forall x p(x) \vee \forall x q(x)},\ \neg \forall x (p(x) \vee q(x)) \\
\swarrow \qquad\qquad\qquad \searrow
\end{array}
$$

$$
\begin{array}{rl}
\forall x p(x),\ \underline{\neg \forall x (p(x) \vee q(x))} & \forall x q(x),\ \underline{\neg \forall x (p(x) \vee q(x))} \\
\downarrow & \downarrow \\
\forall x p(x),\ \underline{\neg\,(p(a) \vee q(a))} & \forall x q(x),\ \underline{\neg\,(p(a) \vee q(a))} \\
\downarrow & \downarrow \\
\underline{\forall x p(x)},\ \neg\, p(a),\ \neg q(a) & \underline{\forall x q(x)},\ \neg\, p(a),\ \neg q(a) \\
\downarrow & \downarrow \\
\forall x p(x),\ p(a),\ \neg\, p(a),\ \neg q(a) & \forall x q(x),\ q(a),\ \neg\, p(a),\ \neg q(a) \\
\times & \times
\end{array}
$$

Fig. 8.1 Semantic tableau in first-order logic

$$
\frac{U \cup \{\gamma, \gamma(a)\}}{U \cup \{\gamma\}}, \qquad \frac{U \cup \{\delta(a)\}}{U \cup \{\delta\}}.
$$

The rule for δ-formulas can be applied only if the constant a does not occur in any formula of U. ■

The γ-rule can be read: if an existential formula and some instantiation of it are true, then the instantiation is redundant.

The δ-rules formalizes the following frequently used method of mathematical reasoning: Let a be an *arbitrary* constant. Suppose that $A(a)$ can be proved. Since a was arbitrary, the proof holds for $\forall x A(x)$. In order to generalize from a specific constant to *for all*, it is essential that a be an arbitrary constant and not one of the constants that is constrained by another subformula.

$$
\begin{array}{rl}
\underline{\neg \forall x p(x),\ \neg\, p(a)},\ p(a),\ q(a) & \underline{\neg \forall x q(x),\ \neg q(a)},\ p(a),\ q(a) \\
\downarrow & \downarrow \\
\neg \forall x p(x),\ \underline{p(a),\ q(a)} & \neg \forall x q(x),\ \underline{p(a),\ q(a)} \\
\downarrow & \downarrow \\
\neg \forall x p(x),\ \underline{p(a) \vee q(a)} & \neg \forall x q(x),\ \underline{p(a) \vee q(a)} \\
\downarrow & \downarrow \\
\underline{\neg \forall x p(x)},\ \forall x (p(x) \vee q(x)) & \underline{\neg \forall x q(x)},\ \forall x (p(x) \vee q(x)) \\
\searrow & \swarrow
\end{array}
$$

$$
\begin{array}{c}
\underline{\neg\,(\forall x p(x) \vee \forall x q(x)),\ \forall x (p(x) \vee q(x))} \\
\downarrow \\
\forall x p(x) \vee \forall x q(x) \;\rightarrow\; \forall x (p(x) \vee q(x))
\end{array}
$$

Fig. 8.2 Gentzen proof tree in first-order logic

$$\underline{\neg\forall yp(a,y),\ \neg p(a,b)},\ \exists xp(x,b),\ p(a,b)$$
$$\downarrow$$
$$\neg\forall yp(a,y),\ \underline{\exists xp(x,b),\ p(a,b)}$$
$$\downarrow$$
$$\neg\forall yp(a,y),\ \underline{\exists xp(x,b)}$$
$$\downarrow$$
$$\underline{\neg\forall yp(a,y)},\ \forall y\exists xp(x,y)$$
$$\downarrow$$
$$\underline{\neg\exists x\forall yp(x,y),\ \forall y\exists xp(x,y)}$$
$$\downarrow$$
$$\exists x\forall yp(x,y)\rightarrow\forall y\exists xp(x,y)$$

Fig. 8.3 Gentzen proof: use rules for γ-formulas followed by rules for δ-formulas

Example 8.2 The proof of $\exists x\forall yp(x,y)\rightarrow\forall y\exists xp(x,y)$ in Fig. 8.3 begins with the axiom obtained from the complementary literals $\neg p(a,b)$ and $p(a,b)$. Then the rule for the γ-formulas is used twice:

$$\frac{U,\neg\forall yp(a,y),\neg p(a,b)}{U,\neg\forall yp(a,y)},\qquad\frac{U,\exists xp(x,b),p(a,b)}{U,\exists xp(x,b)}.$$

Once this is done, it is easy to apply rules for the δ-formulas because the constants a and b appear only once so that the condition in the rule is satisfied:

$$\frac{U,\exists xp(x,b)}{U,\forall y\exists xp(x,y)},\qquad\frac{U,\neg\forall yp(a,y)}{U,\neg\exists x\forall y\exists xp(x,y)}.$$

A final application of the rule for the α-formula completes the proof. ■

We leave the proof of the soundness and completeness of $\mathscr{G}$ as an exercise.

Theorem 8.3 (Soundness and completeness) *Let U be a set of formulas in first-order logic. There is a Gentzen proof for U if and only if there is a closed semantic tableau for $\bar{U}$.*

8.2 Hilbert System $\mathscr{H}$

The Hilbert system $\mathscr{H}$ for propositional logic is extended to first-order logic by adding two axioms and a rule of inference.

Definition 8.4 The axioms of the Hilbert system $\mathscr{H}$ for first-order logic are:

Axiom 1 $\vdash (A \rightarrow (B \rightarrow A))$,

Axiom 2 $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$,

Axiom 3 $\vdash (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$,

Axiom 4 $\vdash \forall x A(x) \rightarrow A(a)$,

Axiom 5 $\vdash \forall x (A \rightarrow B(x)) \rightarrow (A \rightarrow \forall x B(x))$.

- In Axioms 1, 2 and 3, A, B and C are any formulas of first-order logic.
- In Axiom 4, $A(x)$ is a formula with a free variable x.
- In Axiom 5, $B(x)$ is a formula with a free variable x, while x is *not* a free variable of the formula A.

The rules of inference are ***modus ponens*** and ***generalization***:

$$\frac{\vdash A \rightarrow B \quad \vdash A}{\vdash B}, \qquad \frac{\vdash A(a)}{\vdash \forall x A(x)}.$$

■

Propositional Reasoning in First-Order Logic

Axioms 1, 2, 3 and the rule of inference *MP* are generalized to any formulas in first-order logic so all of the theorems and derived rules of inference that we proved in Chap. 3 can be used in first-order logic.

Example 8.5

$$\vdash \forall x p(x) \rightarrow (\exists y \forall x q(x, y) \rightarrow \forall x p(x))$$

is an instance of Axiom 1 in first-order logic and:

$$\frac{\vdash \forall x p(x) \rightarrow (\exists y \forall x q(x, y) \rightarrow \forall x p(x)) \qquad \vdash \forall x p(x)}{\vdash \exists y \forall x q(x, y) \rightarrow \forall x p(x)}$$

uses the rule of inference *modus ponens*. ■

In the proofs in this chapter, we will not bother to give the details of deductions that use propositional reasoning because these are easy to understand. The notation *PC* will be used for propositional deductions.

Specialization and Generalization

Axiom 4 can also be used as a rule of inference:

Rule 8.6 (Axiom 4)

$$\frac{U \vdash \forall x A(x)}{U \vdash A(a)}.$$

Any occurrence of $\forall x A(x)$ can be replaced by $A(a)$ for *any* a. If $A(x)$ is true whatever the assignment of a domain element of an interpretation $\mathscr{I}$ to x, then $A(a)$ is true for the domain element that $\mathscr{I}$ assigns to a.

The generalization rule of inference states that if a occurs in a formula, we may bind *all* occurrences of a with the quantifier. Since a is arbitrary, that is the same as saying that $A(x)$ is true for *all* assignments to x.

There is a reason that the generalization rule was given only for formulas that can be proved *without* a set of assumptions U:

$$\frac{\vdash A(a)}{\vdash \forall x A(x)}.$$

Example 8.7 Suppose that we were allowed to apply generalization to $A(a) \vdash A(a)$ to obtain $A(a) \vdash \forall x A(x)$ and consider the interpretation:

$$(\mathscr{Z}, \{even(x)\}, \{2\}).$$

The assumption $A(a)$ is true but $\forall x A(x)$ is not, which means that generalization is not sound as it transforms $A(a) \models A(a)$ into $A(a) \not\models \forall x A(x)$. ■

Since proofs invariably have assumptions, a constraint must be placed on the generalization rule to make it useful:

Rule 8.8 (Generalization)

$$\frac{U \vdash A(a)}{U \vdash \forall x A(x)},$$

provided that a does not appear in U.

The Deduction Rule

The Deduction rule is essential for proving theorems from assumptions.

Rule 8.9 (Deduction rule)

$$\frac{U \cup \{A\} \vdash B}{U \vdash A \rightarrow B}.$$

Theorem 8.10 (Deduction Theorem) *The deduction rule is sound.*

Proof The proof is by induction on the length of the proof of $U \cup \{A\} \vdash B$. We must show how to obtain a proof of $U \vdash A \rightarrow B$ that does not use the deduction rule. The proof for propositional logic (Theorem 3.14) is modified to take into account the new axioms and generalization.

The modification for the additional axioms is trivial.

Consider now an application of the generalization rule, where, without loss of generality, we assume that the generalization rule is applied to the immediately preceding formula in the proof:

i	$U \cup \{A\} \vdash B(a)$	
$i+1$	$U \cup \{A\} \vdash \forall x B(x)$	Generalization

By the condition on the generalization rule in the presence of assumptions, a does not appear in either U or A.

The proof that the deduction rule is sound is as follows:

i	$U \cup \{A\} \vdash B(a)$	
i'	$U \vdash A \rightarrow B(a)$	Inductive hypothesis, i
$i'+1$	$U \vdash \forall x (A \rightarrow B)$	Generalization, i'
$i'+2$	$U \vdash \forall x (A \rightarrow B) \rightarrow (A \rightarrow \forall x B)$	Axiom 5
$i'+3$	$U \vdash A \rightarrow \forall x B$	MP, $i'+1$, $i'+2$

The fact that a does not appear in U is used in line $i'+1$ and the fact that a does not appear in A is used in line $i'+2$. ∎

8.3 Equivalence of $\mathscr{H}$ and $\mathscr{G}$

We prove that any theorem that can be proved in $\mathscr{G}$ can also be proved in $\mathscr{H}$. We already know how to transform propositional proofs in $\mathscr{G}$ to proofs in $\mathscr{H}$; what remains is to show that any application of the γ- and δ-rules in $\mathscr{G}$ can be transformed into a proof in $\mathscr{H}$.

Theorem 8.11 *The rule for a γ-formula can be simulated in $\mathscr{H}$.*

Proof Suppose that the rule:

$$\frac{U \vee \neg \forall x A(x) \vee \neg A(a)}{U \vee \neg \forall x A(x)}$$

was used. This can be simulated in $\mathscr{H}$ as follows:

1.	$\vdash \forall x A(x) \rightarrow A(a)$	Axiom 4
2.	$\vdash \neg \forall x A(x) \vee A(a)$	PC 1
3.	$\vdash U \vee \neg \forall x A(x) \vee A(a)$	PC 2
4.	$\vdash U \vee \neg \forall x A(x) \vee \neg A(a)$	Assumption
5.	$\vdash U \vee \neg \forall x A(x)$	PC 3, 4

∎

Theorem 8.12 *The rule for a* δ*-formula can be simulated in* $\mathscr{H}$.

Proof Suppose that the rule:

$$\frac{U \vee A(a)}{U \vee \forall x A(x)}$$

was used. This can be simulated in $\mathscr{H}$ as follows:

1.	$\vdash U \vee A(a)$	Assumption
2.	$\vdash \neg U \rightarrow A(a)$	PC 1
3.	$\vdash \forall x(\neg U \rightarrow A(x))$	Gen. 2
4.	$\vdash \forall x(\neg U \rightarrow A(x)) \rightarrow (\neg U \rightarrow \forall x A(x))$	Axiom 5
5.	$\vdash \neg U \rightarrow \forall x A(x)$	MP 3, 4
6.	$\vdash U \vee \forall x A(x)$	PC 5

The use of Axiom 5 requires that a not occur in U, but we know that this holds by the corresponding condition on the rule for the δ-formula. ■

Simulations in $\mathscr{G}$ of proofs in $\mathscr{H}$ are left as an exercise. From this follows:

Theorem 8.13 (Soundness and completeness) *The Hilbert system* $\mathscr{H}$ *is sound and complete.*

8.4 Proofs of Theorems in $\mathscr{H}$

We now give a series of theorems and proofs in $\mathscr{H}$.

The first two are elementary theorems using existential quantifiers.

Theorem 8.14 $\vdash A(a) \rightarrow \exists x A(x)$.

Proof

1.	$\vdash \forall x \neg A(x) \rightarrow \neg A(a)$	Axiom 4
2.	$\vdash A(a) \rightarrow \neg \forall x \neg A(x)$	PC 1
3.	$\vdash A(a) \rightarrow \exists x A(x)$	Definition $\exists$

■

Theorem 8.15 $\vdash \forall x A(x) \rightarrow \exists x A(x)$.

Proof

1.	$\forall x A(x) \vdash \forall x A(x)$	Assumption
2.	$\forall x A(x) \vdash A(a)$	Axiom 4
3.	$\forall x A(x) \vdash A(a) \rightarrow \exists x A(x)$	Theorem 8.14
4.	$\forall x A(x) \vdash \exists x A(x)$	MP 2, 3
5.	$\vdash \forall x A(x) \rightarrow \exists x A(x)$	Deduction

■

Theorem 8.16 $\vdash \forall x(A(x) \rightarrow B(x)) \rightarrow (\forall x A(x) \rightarrow \forall x B(x))$.

Proof

1.	$\forall x(A(x) \rightarrow B(x)),\ \forall x A(x) \vdash \forall x A(x)$	Assumption
2.	$\forall x(A(x) \rightarrow B(x)),\ \forall x A(x) \vdash A(a)$	Axiom 4
3.	$\forall x(A(x) \rightarrow B(x)),\ \forall x A(x) \vdash \forall x(A(x) \rightarrow B(x))$	Assumption
4.	$\forall x(A(x) \rightarrow B(x)),\ \forall x A(x) \vdash A(a) \rightarrow B(a)$	Axiom 4
5.	$\forall x(A(x) \rightarrow B(x)),\ \forall x A(x) \vdash B(a)$	PC 2, 4
6.	$\forall x(A(x) \rightarrow B(x)),\ \forall x A(x) \vdash \forall x B(x)$	Gen. 5
7.	$\forall x(A(x) \rightarrow B(x)) \vdash \forall x A(x) \rightarrow \forall x B(x)$	Deduction
8.	$\vdash \forall x(A(x) \rightarrow B(x)) \rightarrow (\forall x A(x) \rightarrow \forall x B(x))$	Deduction

■

Rule 8.17 (Generalization)

$$\frac{\vdash A(a) \rightarrow B(a)}{\vdash \forall x A(x) \rightarrow \forall x B(x)}.$$

The next theorem was previously proved in the Gentzen system. Make sure that you understand why Axiom 5 can be used.

Theorem 8.18 $\vdash \exists x \forall y A(x, y) \rightarrow \forall y \exists x A(x, y)$.

Proof

1.	$\vdash A(a, b) \rightarrow \exists x A(x, b)$	Theorem 8.14
2.	$\vdash \forall y A(a, y) \rightarrow \forall y \exists x A(x, y)$	Gen 1
3.	$\vdash \neg \forall y \exists x A(x, y) \rightarrow \neg \forall y A(a, y)$	PC 2
4.	$\vdash \forall x(\neg \forall y \exists x A(x, y) \rightarrow \neg \forall y A(x, y))$	Gen. 3
5.	$\vdash (\forall x(\neg \forall y \exists x A(x, y) \rightarrow \neg \forall y A(x, y)))\rightarrow$ $(\neg \forall y \exists x A(x, y) \rightarrow \forall x \neg \forall y A(x, y))$	Axiom 5
6.	$\vdash \neg \forall y \exists x A(x, y) \rightarrow \forall x \neg \forall y A(x, y)$	MP 4, 5
7.	$\vdash \neg \forall x \neg \forall y A(x, y) \rightarrow \forall y \exists x A(x, y)$	PC 6
8.	$\vdash \exists x \forall y A(x, y) \rightarrow \forall y \exists x A(x, y)$	Definition of $\exists$

■

The proof of the following theorem is left as an exercise:

Theorem 8.19 *Let A be a formula that does not have x as a free variable.*

$$\vdash \forall x(A \rightarrow B(x)) \leftrightarrow (A \rightarrow \forall x B(x)),$$
$$\vdash \exists x(A \rightarrow B(x)) \leftrightarrow (A \rightarrow \exists x B(x)).$$

The name of a bound variable can be changed if necessary:

Theorem 8.20 $\vdash \forall x A(x) \leftrightarrow \forall y A(y)$.

Proof

1.	$\vdash \forall x A(x) \rightarrow A(a)$	Axiom 4
2.	$\vdash \forall y(\forall x A(x) \rightarrow A(y))$	Gen. 1
3.	$\vdash \forall x A(x) \rightarrow \forall y A(y)$	Axiom 5
4.	$\vdash \forall y A(y) \rightarrow \forall x A(x)$	Similarly
5.	$\vdash \forall x A(x) \leftrightarrow \forall y A(y)$	PC 3, 4

■

The next theorem shows a non-obvious relation between the quantifiers.

Theorem 8.21 *Let B be a formula that does not have x as a free variable.*

$$\vdash \forall x (A(x) \rightarrow B) \leftrightarrow (\exists x A(x) \rightarrow B).$$

Proof

1.	$\forall x (A(x) \rightarrow B) \vdash \forall x (A(x) \rightarrow B)$	Assumption
2.	$\forall x (A(x) \rightarrow B) \vdash \forall x (\neg B \rightarrow \neg A(x))$	Exercise
3.	$\forall x (A(x) \rightarrow B) \vdash \neg B \rightarrow \forall x \neg A(x)$	Axiom 5
4.	$\forall x (A(x) \rightarrow B) \vdash \neg \forall x \neg A(x) \rightarrow B$	PC 3
5.	$\forall x (A(x) \rightarrow B) \vdash \exists x A(x) \rightarrow B$	Definition of $\exists$
6.	$\vdash \forall x (A(x) \rightarrow B) \rightarrow (\exists x A(x) \rightarrow B)$	Deduction
7.	$\exists x A(x) \rightarrow B \vdash \exists x A(x) \rightarrow B$	Assumption
8.	$\exists x A(x) \rightarrow B \vdash \neg \forall x \neg A(x) \rightarrow B$	Definition of $\exists$
9.	$\exists x A(x) \rightarrow B \vdash \neg B \rightarrow \forall x \neg A(x)$	PC 8
10.	$\exists x A(x) \rightarrow B \vdash \forall x (\neg B \rightarrow \neg A(x))$	Theorem 8.19
11.	$\exists x A(x) \rightarrow B \vdash \forall x (A(x) \rightarrow B)$	Exercise
12.	$\vdash \forall x (A(x) \rightarrow B) \leftrightarrow (\exists x A(x) \rightarrow B)$	PC 6, 11

■

8.5 The C-Rule *

The C-rule is a rule of inference that is useful in proofs of existentially quantified formulas. The rule is the formalization of the argument: if there exists an object satisfying a certain property, let a be that object.

Definition 8.22 (C-Rule) The following rule may be used in a proof:

i	$U \vdash \exists x A(x)$	(an existentially quantified formula)
$i+1$	$U \vdash A(a)$	C-rule

provided that

- The constant a is new and does not appear in steps $1, \ldots, i$ of the proof.
- Generalization is never applied to a free variable or constant in the formula to which the C-rule is applied:

i	$U \vdash \exists x A(x, y)$	(an existentially quantified formula)
$i+1$	$U \vdash A(a, y)$	C-rule
	$\ldots$	
j	$U \vdash \forall y A(a, y)$	Illegal!

■

For a proof that the rule is sound, see Mendelson (2009, Proposition 2.10).

We use the C-Rule to give a more intuitive proof of Theorem 8.18.

Theorem 8.23 $\vdash \exists x \forall y A(x, y) \rightarrow \forall y \exists x A(x, y)$

Proof

1.	$\exists x \forall y A(x, y) \vdash \exists x \forall y A(x, y)$	Assumption
2.	$\exists x \forall y A(x, y) \vdash \forall y A(a, y)$	C-Rule
3.	$\exists x \forall y A(x, y) \vdash A(a, b)$	Axiom 4
4.	$\exists x \forall y A(x, y) \vdash \exists x A(x, b)$	Theorem 8.14
5.	$\exists x \forall y A(x, y) \vdash \forall y \exists x A(x, y)$	Gen. 4
6.	$\vdash \exists x \forall y A(x, y) \rightarrow \forall y \exists x A(x, y)$	Deduction

■

The conditions in the C-rule are necessary. The first condition is similar to the condition on the deduction rule. The second condition is needed so that a formula that is true for one specific constant is not generalized for all values of a variable. Without the condition, we could prove the converse of Theorem 8.18, which is not a valid formula:

1.	$\forall x \exists y A(x, y) \vdash \forall x \exists y A(x, y)$	Assumption
2.	$\forall x \exists y A(x, y) \vdash \exists y A(a, y)$	Axiom 4
3.	$\forall x \exists y A(x, y) \vdash A(a, b)$	C-rule
4.	$\forall x \exists y A(x, y) \vdash \forall x A(x, b)$	Generalization (illegal!)
5.	$\forall x \exists y A(x, y) \vdash \exists y \forall x A(x, y)$	Theorem 8.14
6.	$\vdash \forall x \exists y A(x, y) \rightarrow \exists y \forall x A(x, y)$	Deduction

8.6 Summary

Gentzen and Hilbert deductive systems were defined for first-order logic. They are sound and complete. Be careful to distinguish between completeness and decidability. Completeness means that every valid formula has a proof. We can discover the proof by constructing a semantic tableau for its negation. However, we cannot decide if an arbitrary formula is valid and provable.

8.7 Further Reading

Our presentation is adapted from Smullyan (1968) and Mendelson (2009). Chapter X of (Smullyan, 1968) compares various proofs of completeness.

8.8 Exercises

8.1 Prove in $\mathscr{G}$:

$$\vdash \forall x(p(x) \rightarrow q(x)) \rightarrow (\exists x p(x) \rightarrow \exists x q(x)),$$
$$\vdash \exists x(p(x) \rightarrow q(x)) \leftrightarrow (\forall x p(x) \rightarrow \exists x q(x)).$$

8.2 Prove the soundness and completeness of $\mathscr{G}$ (Theorem 8.3).

8.3 Prove that Axioms 4 and 5 are valid.

8.4 Show that a proof in $\mathscr{H}$ can be simulated in $\mathscr{G}$.

8.5 Prove in $\mathscr{H}$: $\vdash \forall x(p(x) \rightarrow q) \leftrightarrow \forall x(\neg q \rightarrow \neg p(x))$.

8.6 Prove in $\mathscr{H}$: $\vdash \forall x(p(x) \leftrightarrow q(x)) \rightarrow (\forall x p(x) \leftrightarrow \forall x q(x))$.

8.7 Prove the theorems of Exercise 8.1 in $\mathscr{H}$.

8.8 Prove Theorem 8.19 in $\mathscr{H}$. Let A be a formula that does not have x as a free variable.

$$\vdash \forall x(A \rightarrow B(x)) \leftrightarrow (A \rightarrow \forall x B(x)),$$
$$\vdash \exists x(A \rightarrow B(x)) \leftrightarrow (A \rightarrow \exists x B(x)).$$

8.9 Let A be a formula built from the quantifiers and the Boolean operators $\neg$, $\vee$, $\wedge$ only. A', the dual of A is obtained by exchanging $\forall$ and $\exists$ and exchanging $\vee$ and $\wedge$. Prove that $\vdash A$ iff $\vdash \neg A'$.

References

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