Chapter 5

Potential Theory and Spherical Harmonics

In this chapter we investigate solutions of the potential equation due to Laplace in the homogeneous case and due to Poisson in the inhomogeneous case. Parallel to the theory of holomorphic functions we develop the theory of harmonic functions annihilating the Laplace equation. By the ingenious Perron method we shall solve Dirichlet's problem for harmonic functions. Then we present the theory of spherical harmonics initiated by Legendre and elaborated by Herglotz to the present form. This system of functions constitutes an explicit basis for the standard Hilbert space and simultaneously provides a model for the ground states of atoms.

1 Poisson's Differential Equation in $\mathbb{R}^n$

The solutions of 2-dimensional differential equations can often be obtained via integral representations over the circle S^1. As an example we remind the reader of Cauchy's integral formula. For n-dimensional differential equations will appear integrals over the $(n-1)$-dimensional sphere

$$S^{n-1} := \Big\{\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \ : \ \xi_1^2 + \ldots + \xi_n^2 = 1\Big\}, \qquad n \geq 2. \tag{1}$$

At first, we shall determine the area of this sphere S^{n-1}. Given the function $f = f(\xi) : S^{n-1} \to \mathbb{R} \in C^0(S^{n-1}, \mathbb{R})$ we set

$$\int\limits_{S^{n-1}} f(\xi)\, d\omega_\xi = \int\limits_{|\xi|=1} f(\xi)\, d\omega_\xi := \sum_{i=1}^{N} \int\limits_{\Sigma_i} f(\xi)\, d\omega_\xi. \tag{2}$$

By the symbols $\Sigma_1, \ldots, \Sigma_N$ we denote the $N \in \mathbb{N}$ regular surface parts with their surface elements $d\omega_\xi$ satisfying

$$S^{n-1} = \bigcup_{i=1}^{N} \overline{\Sigma}_i, \qquad \overline{\Sigma}_i \cap \overline{\Sigma}_j = \partial\Sigma_i \cap \partial\Sigma_j, \quad i \neq j.$$

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We now consider a continuous function

$$f : \Big\{ x = r\xi \in \mathbb{R}^n \ : \ a < r < b, \ \xi \in S^{n-1} \Big\} \to \mathbb{R}$$

with $0 \le a < b \le +\infty$, and we define the open sets

$$\mathcal{O}_i := \Big\{ x = r\xi \ : \ \xi \in \Sigma_i, \ r \in (a,b) \Big\}, \qquad i = 1, \ldots, N.$$

We require the integrability $\int\limits_{a<|x|<b} |f(x)|\, dx < +\infty$ and set

$$\int\limits_{a<|x|<b} f(x)\, dx = \sum_{i=1}^{N} \int\limits_{\mathcal{O}_i} f(x)\, dx. \tag{3}$$

The surface parts Σ_i are parametrized as follows

$$\Sigma_i : \quad \xi = \xi(t) = \xi(t_1, \ldots, t_{n-1}) : T_i \to \Sigma_i \in C^1(T_i, \Sigma_i), \qquad i = 1, \ldots, N$$

with the parameter domains $T_i \subset \mathbb{R}^{n-1}$. By the representation

$$x = x(t,r) = x(t_1, \ldots, t_{n-1}, r) = r\xi(t_1, \ldots, t_{n-1}), \qquad t \in T_i, \quad r \in (a,b) \tag{4}$$

we obtain a parametrization of the sets $\mathcal{O}_i$ for $i = 1, \ldots, N$. The Jacobian of this mapping is evaluated as follows:

$$J_x(t,r) = \begin{vmatrix} r\xi_{t_1}(t) \\ \vdots \\ r\xi_{t_{n-1}}(t) \\ \xi(t) \end{vmatrix} = r^{n-1} \begin{vmatrix} \xi_{t_1}(t) \\ \vdots \\ \xi_{t_{n-1}}(t) \\ \xi(t) \end{vmatrix} = r^{n-1} \Big(\xi(t) \cdot \xi_{t_1} \wedge \ldots \wedge \xi_{t_{n-1}} \Big).$$

Here the symbol $\wedge$ denotes the exterior vector product in $\mathbb{R}^n$. We have

$$\xi_{t_1} \wedge \ldots \wedge \xi_{t_{n-1}} = (D_1(t), \ldots, D_n(t))$$

with

$$D_j(t) := (-1)^{n+j} \frac{\partial(\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_n)}{\partial(t_1, \ldots, t_{n-1})}, \quad j = 1, \ldots, n.$$

We note $|\xi(t)| = 1$ and infer $\xi(t) \cdot \xi_{t_i}(t) = 0$ for all $i = 1, \ldots, n-1$. Therefore, the vectors $\xi(t)$ and $\xi_{t_1} \wedge \ldots \wedge \xi_{t_{n-1}}$ are parallel to each other and we deduce

$$J_x(t,r) = r^{n-1} \sqrt{\sum_{j=1}^{n} D_j(t)^2}. \tag{5}$$

Setting $d\omega_\xi = \sqrt{\sum\limits_{j=1}^{n} D_j(t)^2}\, dt_1 \dots dt_{n-1}$, $t \in T_i$ we obtain

$$\int\limits_{\mathcal{O}_i} f(x)\, dx = \int\limits_{T_i \times (a,b)} f(r\xi(t)) r^{n-1} \sqrt{\sum_{j=1}^{n} D_j(t)^2}\, dt_1 \dots dt_{n-1}\, dr$$

$$= \int\limits_a^b r^{n-1}\, dr \int\limits_{\Sigma_i} f(r\xi)\, d\omega_\xi, \qquad i = 1, \dots, N.$$

Summation over $i = 1, \dots, N$ finally yields

$$\int\limits_{a<|x|<b} f(x)\, dx = \int\limits_a^b r^{n-1}\, dr \int\limits_{S^{n-1}} f(r\xi)\, d\omega_\xi. \tag{6}$$

Especially the functions $f \in C^0(\mathbb{R}^n, \mathbb{R})$ with $\int\limits_{\mathbb{R}^n} |f(x)| dx < +\infty$ fulfill the identity

$$\int\limits_{\mathbb{R}^n} f(x)\, dx = \int\limits_0^{+\infty} r^{n-1}\, dr \int\limits_{S^{n-1}} f(r\xi)\, d\omega_\xi. \tag{7}$$

Before we continue to evaluate the area of the sphere S^{n-1}, we shall explicitly provide a calculus rule for the integral defined in (2). In this context we consider the following special parametrization of S^{n-1}:

$$\Sigma_\pm : \ \xi_i = t_i, \quad i = 1, \dots, n-1, \qquad \xi_n = \pm\sqrt{1 - t_1^2 - \dots - t_{n-1}^2},$$

$$t = (t_1, \dots, t_{n-1}) \in T := \Big\{ t \in \mathbb{R}^{n-1} \ : \ |t| < 1 \Big\}.$$

We calculate

$$\begin{vmatrix} \frac{\partial \xi_1}{\partial t_1} & \cdots & \frac{\partial \xi_{n-1}}{\partial t_1} & \frac{\partial \xi_n}{\partial t_1} \\ \vdots & & \vdots & \vdots \\ \frac{\partial \xi_1}{\partial t_{n-1}} & \cdots & \frac{\partial \xi_{n-1}}{\partial t_{n-1}} & \frac{\partial \xi_n}{\partial t_{n-1}} \\ \lambda_1 & \cdots & \lambda_{n-1} & \lambda_n \end{vmatrix} = \begin{vmatrix} 1 & \cdots & 0 & -\frac{\xi_1}{\xi_n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -\frac{\xi_{n-1}}{\xi_n} \\ \lambda_1 & \cdots & \lambda_{n-1} & \lambda_n \end{vmatrix} = \sum_{j=1}^{n-1} \lambda_j \frac{\xi_j}{\xi_n} + \lambda_n.$$

The surface element of $\Sigma_\pm$ consequently fulfills

$$d\omega_\xi = \sqrt{\sum_{j=1}^{n} D_j(t)^2}\,dt_1 \ldots dt_{n-1} = \sqrt{\frac{\sum\limits_{j=1}^{n} \xi_j(t)^2}{\xi_n(t)^2}}\,dt_1 \ldots dt_{n-1}$$
$$= \frac{dt_1 \ldots dt_{n-1}}{\sqrt{1 - t_1^2 - \ldots - t_{n-1}^2}}, \qquad t \in T.$$

Therefore, the relation (2) implies

$$\begin{aligned}&\int\limits_{|\xi|=1} f(\xi)\,d\omega_\xi \\ &= \int\limits_{|t|<1} \frac{f(t_1,\ldots,t_{n-1},+\sqrt{\ldots}) + f(t_1,\ldots,t_{n-1},-\sqrt{\ldots})}{\sqrt{1-t_1^2-\ldots-t_{n-1}^2}}\,dt_1 \ldots dt_{n-1}\end{aligned} \tag{8}$$

setting $\sqrt{\ldots} = \sqrt{1 - t_1^2 - \ldots - t_{n-1}^2}$.

We now return to evaluate the *area for the* $(n-1)$*-dimensional sphere* S^{n-1}

$$\omega_n := \int\limits_{S^{n-1}} d\omega_\xi.$$

We take a continuous function $g = g(r) : (0, +\infty) \to \mathbb{R}$, and require the function $f(x) = g(|x|)$ to fulfill

$$\int\limits_{\mathbb{R}^n} |f(x)|\,dx < +\infty.$$

Then the relation (7) yields

$$\begin{aligned}\int\limits_{\mathbb{R}^n} g(|x|)\,dx &= \left(\int\limits_0^{+\infty} r^{n-1} g(r)\,dr\right)\left(\int\limits_{S^{n-1}} d\omega_\xi\right) \\ &= \omega_n \int\limits_0^{+\infty} r^{n-1} g(r)\,dr.\end{aligned} \tag{9}$$

We insert the function $g(r) = e^{-r^2}$, $r \in (0, +\infty)$ and obtain

$$\begin{aligned}\omega_n \int\limits_0^{+\infty} r^{n-1} e^{-r^2}\,dr &= \int\limits_{\mathbb{R}^n} e^{-|x|^2}\,dx = \int\limits_{\mathbb{R}^n} e^{-x_1^2-\ldots-x_n^2}\,dx_1 \ldots dx_n \\ &= \left(\int\limits_{-\infty}^{+\infty} e^{-t^2}\,dt\right)^n = \sqrt{\pi}^n.\end{aligned} \tag{10}$$

Here we observe

$$\int_{-\infty}^{+\infty} e^{-t^2}\,dt = \sqrt{\iint_{\mathbb{R}^2} e^{-|x|^2}\,dx\,dy} = \sqrt{2\pi \int_0^{+\infty} e^{-r^2} r\,dr}$$

$$= \sqrt{\pi}\sqrt{\Big[-e^{-r^2}\Big]_0^{+\infty}} = \sqrt{\pi}.$$

Definition 1.1. *By the symbol*

$$\Gamma(z) := \int_0^{+\infty} t^{z-1} e^{-t}\,dt, \qquad z \in \mathbb{C} \quad \textit{with} \quad \mathit{Re}\, z > 0$$

we denote the Gamma-function*.*

Remark: We have

$$\Gamma(z+1) = z\Gamma(z) \qquad \text{for all} \quad z \in \mathbb{C} \quad \text{with} \quad \mathrm{Re}\, z > 0.$$

Therefore, we inductively obtain

$$\Gamma(n) = (n-1)! \qquad \text{for} \quad n = 1, 2, \ldots$$

With the aid of the substitution $t = \varrho^2$ and $dt = 2\varrho\,d\varrho$ we calculate

$$\Gamma\Big(\frac{1}{2}\Big) = \int_0^{+\infty} t^{-\frac{1}{2}} e^{-t}\,dt = \int_0^{+\infty} \frac{1}{\varrho} e^{-\varrho^2} 2\varrho\,d\varrho$$

$$= 2\int_0^{+\infty} e^{-\varrho^2}\,d\varrho = \int_{-\infty}^{+\infty} e^{-\varrho^2}\,d\varrho = \sqrt{\pi}.$$

Substituting $t = r^2$ and $dt = 2r\,dr$, we finally deduce

$$\Gamma\Big(\frac{n}{2}\Big) = \int_0^{+\infty} t^{\frac{n-2}{2}} e^{-t}\,dt = \int_0^{+\infty} r^{n-2} e^{-r^2} 2r\,dr = 2\int_0^{+\infty} r^{n-1} e^{-r^2}\,dr.$$

From the relation (10) we get the following identity for the area of the sphere S^{n-1}, namely

$$\omega_n = \frac{2\Big(\Gamma(\frac{1}{2})\Big)^n}{\Gamma(\frac{n}{2})}. \tag{11}$$

We now become acquainted with a class of functions which have similar properties as the class of holomorphic functions.

Definition 1.2. *On the open set* $\Omega \subset \mathbb{R}^n$ *with* $n \geq 2$ *we name the function* $\varphi = \varphi(x) \in C^2(\Omega, \mathbb{R})$ harmonic in Ω, *if* φ *satisfies the* Laplacian differential equation

$$\Delta\varphi(x) = \varphi_{x_1 x_1}(x) + \ldots + \varphi_{x_n x_n}(x) = 0 \qquad \textit{for all} \quad x \in \Omega. \tag{12}$$

At first, we shall find the radially symmetric harmonic functions in $\mathbb{R}^n \setminus \{0\}$. Here we begin with the ansatz

$$\varphi(x) = f(|x|), \qquad x \in \mathbb{R}^n \setminus \{0\}, \tag{13}$$

using the function $f = f(r) : (0, +\infty) \to \mathbb{R} \in C^2((0, +\infty), \mathbb{R})$. According to Chapter 1, Section 8 we decompose the Laplace operator with respect to n-dimensional polar coordinates $(\xi, r) \in S^{n-1} \times (0, +\infty)$ as follows:

$$\boldsymbol{\Delta} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\boldsymbol{\Lambda}. \tag{14}$$

Here the operator $\boldsymbol{\Lambda}$ is independent of the radius r . Therefore, the function φ is harmonic in $\mathbb{R}^n \setminus \{0\}$ if and only if the function f satisfies the following ordinary differential equation

$$\frac{\partial^2 f}{\partial r^2}(r) + \frac{n-1}{r}\frac{\partial f}{\partial r}(r) = 0, \qquad r \in (0, +\infty). \tag{15}$$

The linear solution space of this ordinary differential equation is 2-dimensional, and we easily verify: The general solution of (15) is given by

$$f(r) = a + b \log r, \quad r \in (0, +\infty), \quad a, b \in \mathbb{R}, \qquad \text{if} \quad n = 2,$$

$$f(r) = a + b r^{2-n}, \quad r \in (0, +\infty), \quad a, b \in \mathbb{R}, \qquad \text{if} \quad n \geq 3.$$

We observe that the solutions $f \not\equiv \text{const}$ of (15) behave at the origin like

$$\lim_{r \to 0+} |f(r)| = +\infty.$$

Therefore, the radially symmetric solutions $\varphi(x) = f(|x|)$, $x \in \mathbb{R}^n \setminus \{0\}$ of the Laplacian differential equation possess a *singularity* at the point $x = 0$. This phenomenon enables us to derive an integral representation for the solutions of Poisson's differential equation. We meet with a comparable situation in Cauchy's integral.

Definition 1.3. *A domain* $G \subset \mathbb{R}^n$ *satisfying the assumptions of the Gaussian integral theorem from Chapter 1, Section 5 is named a* normal domain in $\mathbb{R}^n$.

Definition 1.4. *On the normal domain* $G \subset \mathbb{R}^n$ *we define the function*

$$\varphi(y; x) := \frac{1}{2\pi} \log|y - x| + \psi(y; x), \quad x, y \in G \quad \textit{with} \quad x \neq y, \qquad n = 2, \tag{16}$$

and alternatively

$$\varphi(y;x) := \frac{1}{(2-n)\omega_n}|y-x|^{2-n} + \psi(y;x), \quad x, y \in G \quad \textit{with} \quad x \neq y, \qquad n \geq 3. \tag{17}$$

Here the function $\psi(\cdot;x)$ - defined by $y \mapsto \psi(y;x)$ - is harmonic in G and belongs to the class $C^1(\overline{G})$ for each fixed $x \in G$. Furthermore, we observe the regularity property $\psi \in C^0(\overline{G} \times \overline{G})$. Then we name $\varphi(y;x)$ a fundamental solution of the Laplace equation in G.

Of central significance for the potential theory is the following

Theorem 1.5. *On the normal domain $G \subset \mathbb{R}^n$ with $n \geq 2$, we consider a solution $u = u(x) \in C^2(G) \cap C^1(\overline{G})$ of* Poisson's differential equation

$$\Delta u(x) = f(x), \qquad x \in G \tag{18}$$

prescribing the function $f = f(x) \in C^0(\overline{G})$ as its right-hand side. *Then we have the* integral representation

$$u(x) = \int\limits_{\partial G} \Big(u(y)\frac{\partial \varphi}{\partial \nu}(y;x) - \varphi(y;x)\frac{\partial u}{\partial \nu}(y) \Big)\, d\sigma(y) + \int\limits_{G} \varphi(y;x) f(y)\, dy \tag{19}$$

for all $x \in G$. Here the symbol $\nu : \partial G \to \mathbb{R}^n$ denotes the exterior unit normal for the domain ∂G, $d\sigma(y)$ means the surface element on the boundary ∂G, and $\varphi(y;x)$ indicates a fundamental solution.

Proof:

1. We present our proof only for the case $n \geq 3$. Take a fixed point $x \in G$ and choose $\varepsilon_0 > 0$ so small that the condition

$$B_\varepsilon(x) := \Big\{ y \in \mathbb{R}^n \, : |y-x| < \varepsilon \Big\} \subset\subset G$$

is satisfied for all $0 < \varepsilon < \varepsilon_0$. We introduce the polar coordinates

$$y = x + r\xi, \qquad \xi \in \mathbb{R}^n \quad \text{with} \quad |\xi| = 1$$

about the point x, and denote the radial derivative by $\frac{\partial}{\partial r}$. On the domain $G_\varepsilon := G \setminus \overline{B_\varepsilon(x)}$ we apply Green's formula and obtain

$$\begin{aligned}
\int\limits_{G_\varepsilon} f(y)\varphi(y;x)\,dy
&= \int\limits_{G_\varepsilon} \Big(\Delta u(y)\varphi(y;x) - u(y)\Delta_y\varphi(y;x)\Big)\,dy \\
&= \int\limits_{\partial G_\varepsilon} \Big(\varphi(y;x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \varphi}{\partial \nu}(y;x)\Big)\,d\sigma(y) \\
&= \int\limits_{\partial G} \Big(\varphi(y;x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \varphi}{\partial \nu}(y;x)\Big)\,d\sigma(y) \\
&\quad - \int\limits_{\partial B_\varepsilon(x)} \Big(\varphi(y;x)\frac{\partial u}{\partial r}(y) - u(y)\frac{\partial \varphi}{\partial r}(y;x)\Big)\,d\sigma(y)
\end{aligned} \tag{20}$$

for all $\varepsilon \in (0, \varepsilon_0)$.

2. Observing (17), we now see

$$\lim_{\varepsilon\to 0+} \int\limits_{\partial B_\varepsilon(x)} \varphi(y;x)\frac{\partial u}{\partial r}(y)\,d\sigma(y) = 0. \tag{21}$$

Furthermore, we calculate

$$\begin{aligned}
\lim_{\varepsilon\to 0+} \int\limits_{\partial B_\varepsilon(x)} u(y)\frac{\partial \varphi}{\partial r}(y;x)\,d\sigma(y)
&= \lim_{\varepsilon\to 0+} \int\limits_{\partial B_\varepsilon(x)} u(y)\frac{1}{\omega_n}|y-x|^{1-n}\,d\sigma(y) \\
&\quad + \lim_{\varepsilon\to 0+} \int\limits_{\partial B_\varepsilon(x)} u(y)\frac{\partial}{\partial r}\psi(y;x)\,d\sigma(y) \\
&= \lim_{\varepsilon\to 0+} \int\limits_{\partial B_\varepsilon(x)} \Big(u(y)-u(x)\Big)\frac{1}{\omega_n}|y-x|^{1-n}\,d\sigma(y) \\
&\quad + u(x)\lim_{\varepsilon\to 0+} \int\limits_{\partial B_\varepsilon(x)} \frac{1}{\omega_n}\varepsilon^{1-n}\,d\sigma(y) \\
&= u(x).
\end{aligned} \tag{22}$$

3. From (20), (21), and (22) together with the passage to the limit $\varepsilon \to 0+$ we now infer the stated identity

$$\int\limits_G f(y)\varphi(y;x)\,dy + \int\limits_{\partial G} \Big(u(y)\frac{\partial\varphi}{\partial\nu}(y;x) - \varphi(y;x)\frac{\partial u}{\partial\nu}(y)\Big)\,d\sigma(y) = u(x)$$

for arbitrary points $x \in G$. q.e.d.

Theorem 1.6. *Given the point* $\overset{\circ}{x} = (\overset{\circ}{x}_1, \ldots, \overset{\circ}{x}_n) \in \mathbb{R}^n$ *and the radius* $R \in (0, +\infty)$, *we consider the ball* $B_R(\overset{\circ}{x}) := \{x \in \mathbb{R}^n : |x - \overset{\circ}{x}| < R\}$. *Let the function*

$$u = u(x_1, \ldots, x_n) \in C^2(B_R(\overset{\circ}{x})) \cap C^1(\overline{B_R(\overset{\circ}{x})})$$

solve the Laplace equation $\Delta u(x_1, \ldots, x_n) = 0$ *in* $B_R(\overset{\circ}{x})$. *Then we have a power series*

$$\mathcal{P}(x_1, \ldots, x_n) = \sum_{k_1,\ldots,k_n=0}^{\infty} a_{k_1\ldots k_n} x_1^{k_1} \cdot \ldots \cdot x_n^{k_n}$$

$$\text{for} \quad x_j \in \mathbb{C} \quad \text{with} \quad |x_j| \le \frac{R}{4n}, \quad j = 1, \ldots, n$$

with the real coefficients $a_{k_1\ldots k_n} \in \mathbb{R}$ *for* $k_1, \ldots, k_n = 0, 1, 2, \ldots$, *converging absolutely in the designated complex polycylinder such that*

$$u(x) = \mathcal{P}(x_1 - \overset{\circ}{x}_1, \ldots, x_n - \overset{\circ}{x}_n) \qquad \text{for} \quad x \in \mathbb{R}^n \quad \text{with} \quad |x_j - \overset{\circ}{x}_j| \le \frac{R}{4n}. \tag{23}$$

Proof:

1. It suffices only to prove the statement above in the case $\overset{\circ}{x} = 0$ and $R = 1$, which can easily be verified with the aid of the transformation

$$Ty := \overset{\circ}{x} + Ry, y \in B_1(0) \quad \text{satisfying} \quad T : B_1(0) \to B_R(\overset{\circ}{x}).$$

Furthermore, we only consider the situation $n \ge 3$. With the function

$$\varphi(y;x) := \frac{1}{(2-n)\omega_n}|y-x|^{2-n}, \qquad y \in B := B_1(0)$$

we obtain a fundamental solution of the Laplace equation in B for each fixed $x \in B$. Theorem 1.5 yields the representation formula

$$u(x) = \int\limits_{\partial B} \Big(u(y)\frac{\partial\varphi}{\partial\nu}(y;x) - \varphi(y;x)\frac{\partial u}{\partial\nu}(y)\Big)\,d\sigma(y), \qquad x \in B. \tag{24}$$

The points $x \in B$ being fixed and $y \in \partial B$ arbitrary, we comprehend

$$\begin{aligned} \tfrac{\partial}{\partial\nu}\varphi(y;x) = y \cdot \nabla_y\varphi(y;x) &= \frac{1}{\omega_n} y \cdot \Big(|y-x|^{1-n}\nabla_y|y-x|\Big) \\ = \frac{1}{\omega_n} y \cdot \Big(|y-x|^{-n}(y-x)\Big) &= \frac{1}{\omega_n|y-x|^n}\, y \cdot (y-x). \end{aligned} \tag{25}$$

2. We take arbitrary $\lambda \in \mathbb{R}$, $y \in \partial B$ and $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ satisfying $|x_j| \leq \frac{1}{4n}$ for $j = 1, \dots, n$ and consider the composite quantity

$$|y - x|^\lambda := \Big(\sum_{j=1}^{n} (y_j - x_j)^2 \Big)^{\frac{\lambda}{2}} = \Big(1 - 2 \sum_{j=1}^{n} y_j x_j + \sum_{j=1}^{n} x_j^2 \Big)^{\frac{\lambda}{2}}.$$

Abbreviating

$$\varrho := -2 \sum_{j=1}^{n} y_j x_j + \sum_{j=1}^{n} x_j^2 \quad \in \mathbb{C}$$

we see

$$|y - x|^\lambda = (1 + \varrho)^{\frac{\lambda}{2}} = \sum_{l=0}^{\infty} \binom{\frac{\lambda}{2}}{l} \varrho^l = \sum_{l=0}^{\infty} \binom{\frac{\lambda}{2}}{l} \Big(-2 \sum_{j=1}^{n} y_j x_j + \sum_{j=1}^{n} x_j^2 \Big)^l.$$

Here we observe

$$|\varrho| = \Big| -2 \sum_{j=1}^{n} y_j x_j + \sum_{j=1}^{n} x_j^2 \Big| \leq 2 \sum_{j=1}^{n} |y_j|\,|x_j| + \sum_{j=1}^{n} |x_j|^2$$
$$\leq 2 \frac{1}{4n} n + \frac{1}{16n^2} n \leq \frac{3}{4} < 1.$$

3. The function

$$\psi(x) := |y - x|^\lambda, \qquad x_k \in \mathbb{C} \quad \text{with} \quad |x_j| \leq \frac{1}{4n}, \quad j = 1, \dots, n$$

is consequently holomorphic for each fixed point $y \in \partial B$. On account of the relation (25), the function

$$F(x, y) := u(y) \frac{\partial \varphi}{\partial \nu}(y; x) - \varphi(y; x) \frac{\partial u}{\partial \nu}(y), \qquad |x_j| \leq \frac{1}{4n}$$

is holomorphic on the given polycylinder for each fixed $y \in \partial B$ and bounded. Now Theorem 2.12 from Chapter 4, Section 2 about holomorphic parameter integrals, together with (24), now yields that the function $u(x)$ is holomorphic on the given polycylinder. Therefore, the function u can be expanded into the power series specified above. Since the function $u(x)$ is real-valued, the coefficients $a_{k_1 \dots k_n}$ are real as well. They are namely the coefficients of the associate Taylor series.

q.e.d.

Of central interest is the following

Theorem 1.7. *Let us take the point* $\overset{\circ}{x} \in \mathbb{R}^n$*, the radius* $R \in (0, +\infty)$*, and the number* $\lambda \in \mathbb{R}$ *with* $\lambda < n$*. Furthermore, let the function* $f = f(y_1, \dots, y_n)$ *be* *holomorphic in an open neighborhood* $\mathcal{U} \subset \mathbb{C}^n$ *satisfying* $\mathcal{U} \supset\supset B_R(\overset{\circ}{x})$*. Then the function*

$$F(x_1,\ldots,x_n) := \int\limits_{B_R(\overset{\circ}{x})} \frac{f(y)}{|y-x|^\lambda}\,dy, \qquad x \in B_R(\overset{\circ}{x}) \tag{26}$$

can be locally expanded into a convergent power series about the point $\overset{\circ}{x}$*.*

Proof: Applying the transformation $Ty := \overset{\circ}{x} + Ry$, $y \in B_1(0)$ we can concentrate our considerations on the case $\overset{\circ}{x} = 0$ and $R = 1$. We therefore investigate the singular integral

$$F(x_1,\ldots,x_n) := \int\limits_{|y|<1} \frac{f(y)}{|y-x|^\lambda}\,dy, \qquad x \in B := B_1(0).$$

The point $x \in B$ being fixed, we consider the transformation of variables due to E. E. Levi, namely

$$y = x + \varrho(\xi - x) = (1-\varrho)x + \varrho\xi, \qquad 0 < \varrho \le 1, \quad |\xi| = 1;$$

$$\xi_n = \xi_n(\xi_1,\ldots,\xi_{n-1}) = \pm\sqrt{1 - \sum_{i=1}^{n-1} \xi_i^2}.$$

The so-defined mapping $(\xi_1,\ldots,\xi_{n-1},\varrho) \mapsto y$ is bijective, and we have

$$\frac{\partial(y_1,\ldots,y_n)}{\partial(\xi_1,\ldots,\xi_{n-1},\varrho)} = \begin{vmatrix} \dfrac{\partial y_1}{\partial \xi_1} & \cdots & \dfrac{\partial y_n}{\partial \xi_1} \\ \vdots & & \vdots \\ \dfrac{\partial y_1}{\partial \xi_{n-1}} & \cdots & \dfrac{\partial y_n}{\partial \xi_{n-1}} \\ \dfrac{\partial y_1}{\partial \varrho} & \cdots & \dfrac{\partial y_n}{\partial \varrho} \end{vmatrix}$$

$$= \begin{vmatrix} \varrho & \cdots & 0 & -\varrho\dfrac{\xi_1}{\xi_n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \varrho & -\varrho\dfrac{\xi_{n-1}}{\xi_n} \\ \xi_1 - x_1 & \cdots & \xi_{n-1} - x_{n-1} & \xi_n - x_n \end{vmatrix}$$

$$= \varrho^{n-1} \begin{vmatrix} 1 & \cdots & 0 & -\dfrac{\xi_1}{\xi_n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -\dfrac{\xi_{n-1}}{\xi_n} \\ \xi_1 - x_1 & \cdots & \xi_{n-1} - x_{n-1} & \xi_n - x_n \end{vmatrix}$$

$$= \frac{\varrho^{n-1}}{\xi_n} \left(\sum_{i=1}^{n-1} \xi_i(\xi_i - x_i) + \xi_n(\xi_n - x_n) \right)$$

$$= \frac{\varrho^{n-1}}{\xi_n} \left(1 - \sum_{i=1}^{n} \xi_i x_i \right) \neq 0 \qquad \text{for} \quad |\xi| = 1, \quad |x| < 1.$$

The transformation formula for multiple integrals now yields

$$F(x) = \int\limits_{|y|<1} \frac{f(y)}{|y-x|^\lambda}\, dy$$

$$= \int\limits_0^1 \int\limits_{\substack{\xi_1^2+\ldots+\xi_{n-1}^2<1 \\ \xi_n(\xi_1,\ldots,\xi_{n-1})>0}} \frac{f(x+\varrho(\xi-x))}{\varrho^\lambda |\xi-x|^\lambda} \frac{\varrho^{n-1}}{|\xi_n|} \left(1 - \sum_{k=1}^n \xi_k x_k\right) d\xi_1 \ldots d\xi_{n-1}\, d\varrho$$

$$+ \int\limits_0^1 \int\limits_{\substack{\xi_1^2+\ldots+\xi_{n-1}^2<1 \\ \xi_n(\xi_1,\ldots,\xi_{n-1})<0}} \frac{f(x+\varrho(\xi-x))}{\varrho^\lambda |\xi-x|^\lambda} \frac{\varrho^{n-1}}{|\xi_n|} \left(1 - \sum_{k=1}^n \xi_k x_k\right) d\xi_1 \ldots d\xi_{n-1}\, d\varrho$$

$$= \int\limits_0^1 \varrho^{n-1-\lambda} \left(\int\limits_{|\xi|=1} \frac{f(x+\varrho(\xi-x))}{|\xi-x|^\lambda} (1 - \xi\cdot x)\, d\omega_\xi \right) d\varrho.$$

As in the proof of Theorem 1.6 we expand the function $|\xi - x|^\lambda$ into a convergent power series. With the aid of Theorem 2.12 from Chapter 4, Section 2 we infer that the function $F(x)$ can be expanded into a convergent power series in a neighborhood of the point $x = 0$.

q.e.d.

Definition 1.8. *A function* $\varphi = \varphi(x_1, \ldots, x_n) : \Omega \to \mathbb{R}$ *defined on the open set* $\Omega \subset \mathbb{R}^n$ *is named* real-analytic in Ω *if the following condition holds true: For each point* $\overset{\circ}{x} = (\overset{\circ}{x}_1, \ldots, \overset{\circ}{x}_n) \in \Omega$ *there exists a sufficiently small number* $\varepsilon = \varepsilon(\overset{\circ}{x}) > 0$ *and a convergent power series*

$$\mathcal{P}(z_1, \ldots, z_n) = \sum_{k_1,\ldots,k_n=0}^{\infty} a_{k_1\ldots k_n} z_1^{k_1} \cdot \ldots \cdot z_n^{k_n}$$

$$\textit{for} \quad z_j \in \mathbb{C} \quad \textit{with} \quad |z_j| \le \varepsilon, \quad j = 1, \ldots, n$$

with the real coefficients

$$a_{k_1\ldots k_n} \in \mathbb{R} \quad \textit{for} \quad k_1, \ldots, k_n = 0, 1, 2, \ldots$$

such that the identity

$$\varphi(x_1, \ldots, x_n) = \mathcal{P}(x_1 - \overset{\circ}{x}_1, \ldots, x_n - \overset{\circ}{x}_n), \qquad |x_j - \overset{\circ}{x}_j| \le \varepsilon, \quad j = 1, \ldots, n$$

is satisfied.

Theorem 1.9. (Analyticity theorem for Poisson's equation)
The real-analytic function $f = f(x_1, \ldots, x_n) : \Omega \to \mathbb{R}$ is defined on the open set $\Omega \subset \mathbb{R}^n$ with $n \ge 2$. Furthermore, let the function $u = u(x_1, \ldots, x_n) \in C^2(\Omega)$ represent a solution of Poisson's differential equation

$$\Delta u(x_1, \ldots, x_n) = f(x_1, \ldots, x_n), \qquad (x_1, \ldots, x_n) \in \Omega.$$

Then this function $u(x)$ is real-analytic in the set Ω.

Proof: Taking $\overset{\circ}{x} \in \Omega$ and $B_R(\overset{\circ}{x}) \subset\subset \Omega$, Theorem 1.5 allows us to represent the solution $u(x)$ by the fundamental solution φ in the following form

$$u(x) = \int\limits_{\partial B_R(\overset{\circ}{x})} \Big(u(y) \frac{\partial \varphi}{\partial \nu}(y; x) - \varphi(y; x) \frac{\partial u}{\partial \nu}(y) \Big) \, d\sigma(y) + \int\limits_{B_R(\overset{\circ}{x})} \varphi(y; x) f(y) \, dy$$

with $x \in B_R(\overset{\circ}{x})$. According to Theorem 1.6, the first integral on the right-hand side represents a real-analytic function about the point $\overset{\circ}{x}$. From Theorem 1.7 we infer that the second integral yields a real-analytic function about the point $\overset{\circ}{x}$ as well.

q.e.d.

2 Poisson's Integral Formula with Applications

In Theorem 1.5 from Section 1 we have constructed an integral representation for the solutions of Poisson's equation in normal domains G with the aid of the fundamental solution $\varphi(y; x)$. The representation formula becomes particularly simple if the function $\varphi(.; x)$ vanishes on the boundary ∂G. This motivates the following

Definition 2.1. *On a normal domain $G \subset \mathbb{R}^n$ we have the fundamental solution $\varphi = \varphi(y;x)$ given. We call this function a* Green's function of the domain *G, if the boundary condition*

$$\varphi(y;x) = 0 \qquad \text{for all} \quad y \in \partial G \tag{1}$$

is satisfied for all $x \in G$.

Theorem 2.2. *Given the ball $B_R := \{y \in \mathbb{R}^n : |y| < R\}$ with $R \in (0, +\infty)$ and $n \geq 2$, we have the following Green's function:*

$$\varphi(y;x) = \frac{1}{2\pi} \log \left| \frac{R(y-x)}{R^2 - \overline{x}y} \right|, \qquad y \in \overline{B}_R, \quad x \in B_R, \tag{2}$$

in the case $n = 2$ and

$$\begin{aligned} \varphi(y;x) &= \frac{1}{(2-n)\omega_n} \left(\frac{1}{|y-x|^{n-2}} - \frac{\left(\frac{R}{|x|}\right)^{n-2}}{\left|y - \frac{R^2}{|x|^2}x\right|^{n-2}} \right) \\ &= \frac{1}{(2-n)\omega_n} \left(\frac{1}{|y-x|^{n-2}} - \frac{R^{n-2}}{(R^4 - 2R^2(x \cdot y) + |x|^2|y|^2)^{\frac{n-2}{2}}} \right) \end{aligned} \tag{3}$$

for $y \in \overline{B}_R$, $x \in B_R$ in the case $n \geq 3$.

Proof:

1. At first, we consider the case $n = 2$. Taking the point $x \in B_R$ as fixed, the expression

$$f(y) := \frac{R(y-x)}{R^2 - \overline{x}y} = \frac{Ry - Rx}{-\overline{x}y + R^2}, \qquad y \in \mathbb{C}$$

is a Möbius transformation with the nonsingular coefficient matrix

$$\begin{pmatrix} R & -Rx \\ -\overline{x} & R^2 \end{pmatrix}, \qquad \det \begin{pmatrix} R & -Rx \\ -\overline{x} & R^2 \end{pmatrix} = R(R^2 - |x|^2) > 0.$$

Furthermore, we have

$$\begin{aligned} |f(R)| &= \left| \frac{R^2 - Rx}{-\overline{x}R + R^2} \right| = \left| \frac{R^2 - Rx}{R^2 - Rx} \right| = 1, \\ |f(-R)| &= \left| \frac{-R^2 - Rx}{R\overline{x} + R^2} \right| = \left| \frac{R^2 + Rx}{R^2 + Rx} \right| = 1, \\ |f(iR)| &= \left| \frac{iR^2 - Rx}{-iR\overline{x} + R^2} \right| = \left| \frac{iR^2 - Rx}{R^2 + iRx} \right| = \left| \frac{R^2 + iRx}{R^2 + iRx} \right| = 1, \\ f(0) &= -\frac{x}{R} \quad \in B_1. \end{aligned}$$

This implies

$$|f(y)| = 1 \qquad \text{for all} \quad y \in \partial B_R$$

and then

$$\varphi(y;x) = \frac{1}{2\pi} \log \left| \frac{R(y-x)}{R^2 - \overline{x}y} \right| = 0$$

for all $y \in \partial B_R$ and all $x \in B_R$. Finally, we note that

$$\begin{aligned}
\varphi(y;x) &= \frac{1}{2\pi} \log \left| \frac{y-x}{R - \frac{\overline{x}}{R} y} \right| = \frac{1}{2\pi} \log |y-x| - \frac{1}{2\pi} \log \left| R - \frac{\overline{x}}{R} y \right| \\
&= \frac{1}{2\pi} \log |y-x| - \frac{1}{2\pi} \log \left| -\frac{\overline{x}}{R} \left(y - \frac{R^2}{\overline{x}} \right) \right| \\
&= \frac{1}{2\pi} \log |y-x| - \frac{1}{2\pi} \log \left| y - \frac{R^2}{|x|^2} x \right| - \frac{1}{2\pi} \log \left| \frac{\overline{x}}{R} \right| \\
&=: \frac{1}{2\pi} \log |y-x| + \psi(y;x), \qquad y \in B_R, \quad x \in B_R \setminus \{0\}.
\end{aligned}$$

The function $\psi(\cdot;x)$ is harmonic in $\overline{B}_R$ as the real part of a holomorphic function.

2. We now consider the case $n \geq 3$, and begin with the following ansatz:

$$\varphi(y;x) = \frac{1}{(2-n)\omega_n} \left(\frac{1}{|y-x|^{n-2}} - \frac{K}{|y-\lambda x|^{n-2}} \right), \qquad y \in \overline{B}_R.$$

Here the point $x \in B_R$ is fixed; the constants K and λ have still to be chosen adequately. At first, we see that the function

$$\psi(y;x) := -\frac{1}{(2-n)\omega_n} \frac{K}{|y-\lambda x|^{n-2}}$$

is harmonic in $y \in \overline{B}_R$ if $\lambda x \notin \overline{B}_R$ holds true. The condition $\varphi(y;x) = 0$ for all $y \in \partial B_R$ is satisfied if and only if

$$\frac{1}{|y-x|^{n-2}} = \frac{K}{|y-\lambda x|^{n-2}}$$

or equivalently

$$K^{\frac{2}{n-2}} |y-x|^2 = |y-\lambda x|^2 \qquad \text{for all} \quad y \in \partial B_R$$

is correct. On account of $|y| = R$ we can transform this identity into

$$K^{\frac{2}{n-2}} (R^2 - 2(y \cdot x) + |x|^2) = R^2 - 2\lambda(y \cdot x) + \lambda^2 |x|^2$$

and finally into

$$R^2\Big(K^{\frac{2}{n-2}}-1\Big)-2(x\cdot y)\Big(K^{\frac{2}{n-2}}-\lambda\Big)+|x|^2\Big(K^{\frac{2}{n-2}}-\lambda^2\Big)=0.$$

Setting $\lambda := K^{\frac{2}{n-2}}$ we obtain

$$0=R^2(\lambda-1)+|x|^2(\lambda-\lambda^2)=(\lambda-1)\{R^2-\lambda|x|^2\}.$$

Since the case $\lambda = 1$, $K = 1$ and consequently $\varphi \equiv 0$ has to be excluded as the trivial one, we choose $\lambda := \left(\frac{R}{|x|}\right)^2$ and $K = \lambda^{\frac{n-2}{2}} = \left(\frac{R}{|x|}\right)^{n-2}$. Now we obtain Green's function of the domain B_R with the following expression

$$\varphi(y;x)=\frac{1}{(2-n)\omega_n}\left(\frac{1}{|y-x|^{n-2}}-\frac{\left(\frac{R}{|x|}\right)^{n-2}}{\left|y-\left(\frac{R}{|x|}\right)^2 x\right|^{n-2}}\right),\qquad y\in\overline{B}_R,$$

for $x \in B_R \setminus \{0\}$. We note

$$\frac{\frac{R}{|x|}}{\left|y-\frac{R^2}{|x|^2}x\right|}=\frac{R}{\left||x|y-R^2\frac{x}{|x|}\right|}=\left(\frac{R^2}{|x|^2|y|^2-2R^2(x\cdot y)+R^4}\right)^{\frac{1}{2}},$$

and Green's function satisfies

$$\varphi(y;x)=\frac{1}{(2-n)\omega_n}\left(\frac{1}{|y-x|^{n-2}}-\frac{R^{n-2}}{(|x|^2|y|^2-2R^2(x\cdot y)+R^4)^{\frac{n-2}{2}}}\right)$$

for all $y \in \overline{B}_R$ and $x \in B_R$. q.e.d.

Theorem 2.3. (Poisson's integral formula)
In the ball $B_R := \{y \in \mathbb{R}^n : |y| < R\}$ of radius $R \in (0,+\infty)$ in the Euclidean space $\mathbb{R}^n$ with $n \geq 2$, let the function $u = u(x) = u(x_1,\ldots,x_n) \in C^2(B_R) \cap C^0(\overline{B}_R)$ solve Poisson's differential equation

$$\Delta u(x)=f(x),\qquad x\in B_R$$

for the right-hand side $f = f(x) \in C^0(\overline{B_R})$. Then we have the Poisson integral representation

$$u(x)=\frac{1}{R\omega_n}\int\limits_{|y|=R}\frac{|y|^2-|x|^2}{|y-x|^n}u(y)\,d\sigma(y)+\int\limits_{|y|\le R}\varphi(y;x)f(y)\,dy \tag{4}$$

for all $x \in B_R$. Here the symbol $\varphi = \varphi(y;x)$ denotes Green's function given in Theorem 2.2.

Proof:

1. At first, we assume the regularity $u \in C^2(\overline{B}_R)$. Theorem 1.5 from Section 1 yields the identity
$$u(x) = \int\limits_{|y|=R} u(y)\frac{\partial \varphi}{\partial \nu}(y;x)\,d\sigma(y) + \int\limits_{|y|\le R} \varphi(y;x)f(y)\,dy, \qquad x \in B_R.$$
We confine ourselves to the case $n \ge 3$. According to Theorem 2.2 we have Green's function
$$\varphi(y;x) = \frac{1}{(2-n)\omega_n}\Big(|y-x|^{2-n} - K|y-\lambda x|^{2-n}\Big), \qquad y \in \overline{B}_R,\ x \in B_R,$$
$$\text{with} \quad \lambda := \left(\frac{R}{|x|}\right)^2 \quad \text{and} \quad K = \left(\frac{R}{|x|}\right)^{n-2} = \lambda^{\frac{n-2}{2}}.$$
Taking $x \in B_R$ as fixed and $y \in \partial B_R$ arbitrarily, we calculate
$$\begin{aligned}\frac{\partial}{\partial \nu}\varphi(y;x) &= \frac{y}{R}\cdot \nabla_y \varphi(y;x)\\ &= \frac{1}{R\omega_n} y \cdot \left(|y-x|^{1-n}\frac{y-x}{|y-x|} - K|y-\lambda x|^{1-n}\frac{y-\lambda x}{|y-\lambda x|}\right)\\ &= \frac{1}{R\omega_n} y \cdot \left(\frac{y-x}{|y-x|^n} - K\frac{y-\lambda x}{|y-\lambda x|^n}\right).\end{aligned}$$
This formula remains true for $n = 2$ as well, where $K = 1$ is fulfilled in this case. We additionally note that
$$\begin{aligned}|y-\lambda x|^2 &= R^2 - 2\lambda(x\cdot y) + \lambda^2|x|^2\\ &= R^2 - 2\frac{R^2}{|x|^2}(x\cdot y) + \frac{R^4}{|x|^2}\\ &= \frac{R^2}{|x|^2}\Big(|x|^2 - 2(x\cdot y) + R^2\Big) = \lambda|y-x|^2\end{aligned}$$
and consequently
$$|y-\lambda x|^n = \lambda^{\frac{n}{2}}|y-x|^n.$$
Finally, we obtain
$$\begin{aligned}\frac{\partial}{\partial \nu}\varphi(y;x) &= \frac{1}{R\omega_n|y-x|^n} y\cdot\Big(y - x - K\lambda^{-\frac{n}{2}}(y-\lambda x)\Big)\\ &= \frac{1}{R\omega_n|y-x|^n} y\cdot\Big((1-\lambda^{-\frac{n}{2}}K)y - (1-K\lambda^{\frac{-n+2}{2}})x\Big)\\ &= \frac{|y|^2}{R\omega_n|y-x|^n}\Big(1-\frac{1}{\lambda}\Big) \;=\; \frac{|y|^2}{R\omega_n|y-x|^n}\Big(1-\frac{|x|^2}{R^2}\Big)\\ &= \frac{|y|^2-|x|^2}{R\omega_n|y-x|^n} \qquad \text{for all} \quad y \in \partial B_R \quad \text{and} \quad x \in B_R.\end{aligned}$$

Therefore, we get the Poisson integral representation

$$u(x) = \frac{1}{R\omega_n} \int\limits_{|y|=R} \frac{|y|^2 - |x|^2}{|y-x|^n} u(y)\, d\sigma(y) + \int\limits_{|y|\le R} \varphi(y;x) f(y)\, dy, \qquad x \in B_R.$$

2. Now assuming $u \in C^2(B_R) \cap C^0(\overline{B}_R)$, part 1 of our proof yields the following identity for all $\varrho \in (0,R)$:

$$u(x) = \frac{1}{\varrho\omega_n} \int\limits_{|y|=\varrho} \frac{|y|^2 - |x|^2}{|y-x|^n} u(y)\, d\sigma(y) + \int\limits_{|y|\le \varrho} \varphi(y;x,\varrho) f(y)\, dy.$$

   Here $\varphi(y;x,\varrho)$ denotes Green's function for the ball B_ϱ. We observe the transition to the limit $\varrho \to R-$ and obtain

$$u(x) = \frac{1}{R\omega_n} \int\limits_{|y|=R} \frac{|y|^2 - |x|^2}{|y-x|^n} u(y)\, d\sigma(y) + \int\limits_{|y|\le R} \varphi(y;x,R) f(y)\, dy$$

   for all $x \in B_R$. q.e.d.

Remarks:

1. In the special case $n = 2$ and $f = 0$ we obtain for $0 \le \varrho < R$ and $0 \le \vartheta < 2\pi$:

$$u(\varrho\cos\vartheta, \varrho\sin\vartheta) = \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{R^2 - \varrho^2}{R^2 - 2\varrho R\cos(\lambda - \vartheta) + \varrho^2}\, u(R\cos\lambda, R\sin\lambda)\, d\lambda.$$

2. We name

$$P(x,y,R) := \frac{1}{R\omega_n} \frac{|y|^2 - |x|^2}{|y-x|^n}, \qquad y \in \overline{B}_R, \quad x \in B_R$$

   the *Poisson kernel.*
3. Later in Chapter 9 we shall investigate the boundary behavior of Poisson's integral.

Theorem 2.4. *We consider a solution $u = u(x) \in C^2(G)$ of Poisson's differential equation $\Delta u(x) = f(x)$, $x \in G$ in the domain $G \subset \mathbb{R}^n$. For each ball $B_R(a) \subset\subset G$ we then have the identity*

$$u(a) = \frac{1}{2\pi R} \int\limits_{|x-a|=R} u(x)\, d\sigma(x) - \frac{1}{2\pi} \iint\limits_{|x-a|\le R} \log\Big(\frac{R}{|x-a|}\Big) f(x)\, dx \qquad (5)$$

in the case $n = 2$, and alternatively

$$\begin{aligned} u(a) = &\frac{1}{R^{n-1}\omega_n} \int\limits_{|x-a|=R} u(x)\,d\sigma(x) \\ &-\frac{1}{(n-2)\omega_n} \int\limits_{|x-a|\le R} \Big(|x-a|^{2-n} - R^{2-n}\Big) f(x)\,dx \end{aligned} \tag{6}$$

in the case $n \ge 3$.

Proof: Via an adequate translation we can achieve $a = 0$. We then consider Green's function

$$\varphi(y;0) = \frac{1}{2\pi}\log\left|\frac{y}{R}\right| = -\frac{1}{2\pi}\log\frac{R}{|y|}, \qquad y \in \overline{B}_R, \qquad n = 2,$$

and alternatively

$$\varphi(y;0) = -\frac{1}{(n-2)\omega_n}\left(\frac{1}{|y|^{n-2}} - \frac{1}{R^{n-2}}\right), \qquad y \in \overline{B}_R, \qquad n \ge 3.$$

Poisson's integral formula now yields

$$u(0) = \frac{1}{2\pi R}\int\limits_{|y|=R} u(y)\,d\sigma(y) - \frac{1}{2\pi}\iint\limits_{|y|\le R} \log\Big(\frac{R}{|y|}\Big) f(y)\,dy$$

in the case $n = 2$ and

$$u(0) = \frac{1}{R^{n-1}\omega_n}\int\limits_{|y|=R} u(y)\,d\sigma(y) - \frac{1}{(n-2)\omega_n}\int\limits_{|y|\le R}\left(\frac{1}{|y|^{n-2}} - \frac{1}{R^{n-2}}\right) f(y)\,dy$$

in the case $n \ge 3$. q.e.d.

Corollary: Harmonic functions u have the *mean value property*

$$u(a) = \frac{1}{R^{n-1}\omega_n}\int\limits_{|y-a|=R} u(y)\,d\sigma(y), \tag{7}$$

if $B_R(a) \subset\subset G$ is satisfied.

Theorem 2.5. (Harnack's inequality)
Let the function $u(x) \in C^2(B_R)$ *be harmonic in the ball* $B_R = \{y \in \mathbb{R}^n : |y| < R\}$ *of radius* $R \in (0, +\infty)$, *and we assume* $u(x) \ge 0$ *for all* $x \in B_R$. *Then we have the estimate*

$$\frac{1 - \frac{|x|}{R}}{\Big(1 + \frac{|x|}{R}\Big)^{n-1}}\, u(0) \le u(x) \le \frac{1 + \frac{|x|}{R}}{\Big(1 - \frac{|x|}{R}\Big)^{n-1}}\, u(0) \qquad \textit{for all} \quad x \in B_R. \tag{8}$$

Proof: At first we assume $u \in C^2(\overline{B}_R)$, and later we establish the inequality above for functions $u \in C^2(B_R)$ by a passage to the limit. From Theorem 2.3 we infer

$$u(x) = \int\limits_{|y|=R} P(x,y,R)u(y)\,d\sigma(y), \qquad x \in B_R.$$

For arbitrary points $y \in \mathbb{R}^n$ with $|y| = R$ and $x \in B_R$ we have the following inequality:

$$\frac{|y|^2 - |x|^2}{(R+|x|)^n} \le \frac{|y|^2 - |x|^2}{|y-x|^n} \le \frac{|y|^2 - |x|^2}{(R-|x|)^n}.$$

We multiply this inequality by $\frac{1}{R\omega_n}u(y)$ and then integrate over the boundary ∂B_R:

$$\frac{1}{R\omega_n}\frac{R^2-|x|^2}{(R+|x|)^n}\int\limits_{|y|=R} u(y)\,d\sigma(y) \le u(x) \le \frac{1}{R\omega_n}\frac{R^2-|x|^2}{(R-|x|)^n}\int\limits_{|y|=R} u(y)\,d\sigma(y).$$

Using the mean value property of harmonic functions we obtain

$$R^{n-2}\frac{R^2-|x|^2}{(R+|x|)^n}\,u(0) \le u(x) \le R^{n-2}\frac{R^2-|x|^2}{(R-|x|)^n}\,u(0)$$

and consequently

$$\frac{1-\frac{|x|^2}{R^2}}{\left(1+\frac{|x|}{R}\right)^n}\,u(0) \le u(x) \le \frac{1-\frac{|x|^2}{R^2}}{\left(1-\frac{|x|}{R}\right)^n}\,u(0), \qquad x \in B_R.$$

Finally, this implies

$$\frac{1-\frac{|x|}{R}}{\left(1+\frac{|x|}{R}\right)^{n-1}}\,u(0) \le u(x) \le \frac{1+\frac{|x|}{R}}{\left(1-\frac{|x|}{R}\right)^{n-1}}\,u(0), \qquad x \in B_R.$$

q.e.d.

Theorem 2.6. (Liouville's theorem for harmonic functions)
Let $u(x) : \mathbb{R}^n \to \mathbb{R}$ denote a harmonic function satisfying $u(x) \le M$ for all $x \in \mathbb{R}^n$, with a constant $M \in \mathbb{R}$. Then we have $u(x) \equiv const$, $x \in \mathbb{R}^n$.

Proof: We consider the harmonic function $v(x) := M - u(x)$, $x \in \mathbb{R}^n$ and note that $v(x) \ge 0$ for all $x \in \mathbb{R}^n$. Harnack's inequality now yields

$$\frac{1-\frac{|x|}{R}}{\left(1+\frac{|x|}{R}\right)^{n-1}}v(0) \le v(x) \le \frac{1+\frac{|x|}{R}}{\left(1-\frac{|x|}{R}\right)^{n-1}}v(0), \qquad x \in B_R, \quad R > 0.$$

We observe $R \to +\infty$ and obtain $v(x) = v(0)$ for all $x \in \mathbb{R}^n$ and finally $u(x) \equiv \text{const}$, $x \in \mathbb{R}^n$.

q.e.d.

Fundamentally important in the sequel is

Definition 2.7. *Let $G \subset \mathbb{R}^n$ denote a domain and $u = u(x) = u(x_1, \dots, x_n) : G \to \mathbb{R} \in C^0(G)$ a continuous function. We name u* weakharmonic (superharmonic, subharmonic)*, if*

$$u(a) \; = \; (\; \geq, \; \leq \;) \; \frac{1}{r^{n-1}\omega_n} \int\limits_{|x-a|=r} u(x)\, d\sigma(x) \; = \; \frac{1}{\omega_n} \int\limits_{|\xi|=1} u(a + r\xi)\, d\sigma(\xi)$$

for all $a \in G$ and $r \in (0, \vartheta(a))$ with a certain $\vartheta(a) \in (0, dist(a, \mathbb{R}^n \setminus G)]$ is correct.

Remarks:

1. The function $u : G \to \mathbb{R} \in C^0(G)$ is superharmonic if and only if the function $-u$ is subharmonic.
2. A function is weakharmonic if and only if this function is simultaneously superharmonic and subharmonic.
3. A weakharmonic function is characterized by the mean value property - and should be carefully distinguished from certain weak solutions of the Laplace equation in Sobolev spaces, which are not necessarily continuous functions in general.
4. If the functions $u, v : G \to \mathbb{R}$ are superharmonic and the constant $\alpha \in [0, +\infty)$ is given, then the following continuous functions

$$\begin{aligned} w_1(x) &:= \alpha u(x), \\ w_2(x) &:= u(x) + v(x), \\ w_3(x) &:= \min\{u(x), v(x)\}, \qquad x \in G, \end{aligned}$$

are superharmonic as well. For w_1 and w_2 this statement is evident, and we investigate the function w_3. Taking the point $a \in G$ and the radius $r \in (0, \vartheta(a))$ we infer

$$\begin{aligned} \frac{1}{\omega_n} \int\limits_{|\xi|=1} w_3(a + r\xi)\, d\sigma(\xi) &= \frac{1}{\omega_n} \int\limits_{|\xi|=1} \min\{u(a + r\xi), v(a + r\xi)\}\, d\sigma(\xi) \\ &\leq \min\left\{ \frac{1}{\omega_n} \int\limits_{|\xi|=1} u(a + r\xi)\, d\sigma(\xi), \frac{1}{\omega_n} \int\limits_{|\xi|=1} v(a + r\xi)\, d\sigma(\xi) \right\} \\ &\leq \min\{u(a), v(a)\} \; = \; w_3(a). \end{aligned}$$

5. If the functions $u, v : G \to \mathbb{R}$ are subharmonic and the constant $\alpha \in [0, +\infty)$ is given, then the following functions

$$\begin{aligned} w_1(x) &:= \alpha u(x), \\ w_2(x) &:= u(x) + v(x), \\ w_3(x) &:= \max\{u(x), v(x)\}, \qquad x \in G, \end{aligned}$$

are subharmonic functions in G as well.

Theorem 2.8. *Let the function $u = u(x) \in C^2(G)$ be defined on the domain $G \subset \mathbb{R}^n$. Then this twice continuously differentiable function u is weakharmonic (superharmonic, subharmonic) in G if and only if the relation*

$$\Delta u(x) = 0 \ (\leq 0,\ \geq 0) \qquad \text{for all} \quad x \in G$$

is correct.

Proof: We present our proof only in the case $n \geq 3$. We define $f(x) := \Delta u(x)$, $x \in G$ and see $f \in C^0(G)$. Theorem 2.4 yields the following identity for all points $a \in G$ and radii $r \in (0, \vartheta(a))$:

$$u(a) = \frac{1}{r^{n-1}\omega_n} \int\limits_{|x-a|=r} u(x)\, d\sigma(x) - \frac{1}{(n-2)\omega_n} \int\limits_{|x-a|\leq r} (|x-a|^{2-n} - r^{2-n}) f(x)\, dx.$$

Setting

$$\chi(a,r) := -\frac{1}{(n-2)\omega_n} \int\limits_{|x-a|\leq r} (|x-a|^{2-n} - r^{2-n}) f(x)\, dx$$

we easily see: The function u is weakharmonic (superharmonic, subharmonic) if and only if

$$\chi(a,r) = 0 \ (\geq 0,\ \leq 0) \qquad \text{for all} \quad a \in G, \quad r \in (0, \vartheta(a))$$

holds true. We finally note the inequality $|x-a|^{2-n} - r^{2-n} \geq 0$ for all $x \in G$ with $|x - a| \leq r$, and we obtain the statement above.

q.e.d.

Theorem 2.9. (Maximum and minimum principle)
The superharmonic (subharmonic) function $u = u(x) : G \to \mathbb{R}$ - defined on the domain $G \subset \mathbb{R}^n$ - may attain its global minimum (maximum) at a point $\overset{\circ}{x} \in G$; this means

$$u(x) \geq u(\overset{\circ}{x}) \quad \Big(u(x) \leq u(\overset{\circ}{x})\Big) \qquad \text{for all} \quad x \in G.$$

Then we have

$$u(x) \equiv const \qquad in \quad G.$$

Proof: Since the reflection $u \to -u$ transfers subharmonic functions into superharmonic ones, the statement has only to be shown for superharmonic functions. Now the superharmonic function $u : G \to \mathbb{R} \in C^0(G)$ may attain its global minimum at the point $\overset{\circ}{x} \in G$. We then consider the nonvoid set

$$G^* := \Big\{ x \in G : \ u(x) = \inf_{y \in G} u(y) = u(\overset{\circ}{x}) \Big\}$$

which is closed in the domain G. We now show that this set G^* is open as well. If namely $a \in G^*$ is an arbitrary point, we observe

$$\inf_{y \in G} u(y) = u(a) \geq \frac{1}{\omega_n} \int\limits_{|\xi|=1} u(a + r\xi)\, d\sigma(\xi) \qquad \text{for all} \quad r \in (0, \vartheta(a)). \tag{9}$$

This implies $u(x) = u(a)$ for all points $x \in \mathbb{R}^n$ with $|x - a| < \vartheta(a)$. Consequently, the set G^* is open. Since G is a domain and especially connected, we easily see by continuation along paths: $u(x) \equiv u(\overset{\circ}{x})$ for all $x \in G$. We finally obtain $u(x) \equiv const$, $x \in G$.

q.e.d.

Theorem 2.10. *Let the function $u : G \to \mathbb{R} \in C^0(G)$ be superharmonic (subharmonic) in the bounded domain $G \subset \mathbb{R}^n$. Furthermore, all sequences of points $\{x^{(k)}\}_{k=1,2,\ldots} \subset G$ satisfying $\lim_{k\to\infty} x^{(k)} = x \in \partial G$ have the property*

$$\liminf_{k\to\infty} u(x^{(k)}) \geq M \qquad \Big(\limsup_{k\to\infty} u(x^{(k)}) \leq M \Big)$$

with a constant $M \in \mathbb{R}$. Then we have the behavior

$$u(x) \geq M \quad \Big(u(x) \leq M \Big) \qquad \textit{for all} \quad x \in G.$$

Proof: It suffices to consider superharmonic functions $u : G \to \mathbb{R}$. If the statement $u(x) \geq M$ for all $x \in G$ were false, we have a point $\xi \in G$ with $\mu := u(\xi) < M$. We now construct a sequence of connected compact subsets of G exhausting the set G; this means $\Theta_j \uparrow G$ for $j \to \infty$ satisfying

$$\xi \in \Theta_1 \subset \Theta_2 \subset \ldots .$$

Due to Theorem 2.9, the superharmonic function u attains its minimum at a boundary point $y^{(j)} \in \partial\Theta_j$ of each compact set Θ_j. Therefore, we have the inequalities

$$u(y^{(j)}) \leq u(\xi) = \mu \qquad \text{for} \quad j = 1, 2, \ldots$$

From the sequence $\{y^{(j)}\}_{j=1,2,\ldots} \subset \overline{G}$ we now select a convergent subsequence $\{x^{(k)}\}_{k=1,2,\ldots} \subset \{y^{(j)}\}_{j=1,2,\ldots}$. We then obtain a sequence $\{x^{(k)}\}_{k=1,2,\ldots} \subset G$ satisfying

$$\lim_{k\to\infty} x^{(k)} = x \in \partial G \qquad \text{and} \qquad \liminf_{k\to\infty} u(x^{(k)}) \le \mu < M.$$

However, this contradicts the assumption

$$\liminf_{k\to\infty} u(x^{(k)}) \ge M \qquad \text{for all} \quad \{x^{(k)}\}_{k=1,2,\ldots} \subset G \quad \text{with} \quad \lim_{k\to\infty} x^{(k)} \in \partial G.$$

q.e.d.

Theorem 2.11. *Let $G \subset \mathbb{R}^n$ denote a bounded domain. Furthermore, we consider two functions $u = u(x)$, $v = v(x) : \overline{G} \to \mathbb{R} \in C^0(\overline{G})$, which are weakharmonic in G. Then we have the estimate*

$$\sup_{x\in\overline{G}} |u(x) - v(x)| \le \sup_{x\in\partial G} |u(x) - v(x)|.$$

Proof: The function $w(x) := u(x) - v(x)$, $x \in \overline{G}$ is continuous in $\overline{G}$ and weakharmonic in G. Setting $M := \sup\limits_{x\in\partial G} |u(x)-v(x)|$, Theorem 2.10 yields the inequality

$$-M \le w(x) \le M \qquad \text{for all} \quad x \in G.$$

This implies the stated estimate. q.e.d.

Theorem 2.12. *Let $G \subset \mathbb{R}^n$ denote a bounded domain. Then the Green function $\varphi_G(y;x)$ for this domain is uniquely determined, and we have*

$$\varphi_G(y;x) < 0 \qquad \textit{for all} \quad y \in G \quad \textit{and fixed} \quad x \in G. \tag{10}$$

Proof: (Only for $n \ge 3$.)

1. Let the two Green functions

$$\varphi_j(y;x) = \frac{1}{(2-n)\omega_n}|y-x|^{2-n} + \psi_j(y;x), \qquad y \in \overline{G}, \quad x \in G; \quad j = 1,2$$

be given. Then we infer $0 = \varphi_1(y;x) = \varphi_2(y;x)$ for $y \in \partial G$, $x \in G$ and therefore

$$\psi_1(y;x) = \psi_2(y;x), \qquad y \in \partial G, \quad x \in G.$$

Theorem 2.11 now implies $\psi_1(y;x) \equiv \psi_2(y;x)$, and finally

$$\varphi_1 \equiv \varphi_2, \qquad y \in G, \quad x \in G.$$

2. We take the point $x \in G$ as fixed and consider Green's function

$$\varphi_G(y;x) = \frac{1}{(2-n)\omega_n}|y-x|^{2-n} + \psi(y;x), \qquad y \in \overline{G}$$

for the domain G. Then the function $\chi(y) := \varphi(y;x) : G \setminus \{x\} \to \mathbb{R}$ is harmonic. Arbitrary sequences $\{y^{(k)}\}_{k=1,2,\ldots} \subset G' := G \setminus \{x\}$ with $\lim\limits_{k\to\infty} y^{(k)} \in \partial G' = \partial G \cup \{x\}$ now satisfy

$$\limsup_{k\to\infty} \chi(y^{(k)}) \le 0.$$

Therefore, Theorem 2.10 yields $\chi(y) \le 0$ for all $y \in G'$ and Theorem 2.9 implies the inequality (10).

q.e.d.

Remark: The existence question for Green's function on *Dirichlet domains G* will be answered affirmatively in the next section.

3 Dirichlet's Problem for the Laplace Equation in $\mathbb{R}^n$

In this paragraph the symbol $G \subset \mathbb{R}^n$ always means a bounded domain, and $f = f(x) : \partial G \to \mathbb{R} \in C^0(\partial G)$ denotes a continuous function on its boundary ∂G. Our interest is devoted to the following *Dirichlet's boundary value problem for the Laplace equation*

$$\begin{aligned} &u = u(x) \in C^2(G) \cap C^0(\overline{G}), \\ &\Delta u(x) = 0 \qquad \text{for all} \quad x \in G, \\ &u(x) = f(x) \qquad \text{for all} \quad x \in \partial G. \end{aligned} \tag{1}$$

Theorem 3.1. (Uniqueness theorem)
Consider two solutions $u(x)$, $v(x)$ of the Dirichlet problem (1) for the data G and f. Then we have

$$u(x) \equiv v(x) \qquad \textit{in} \quad \overline{G}.$$

Proof: The function $w(x) := v(x) - u(x)$, $x \in \overline{G}$ belonging to the class $C^2(G) \cap C^0(\overline{G})$ is especially weakharmonic in G and has the boundary values

$$\begin{aligned} w(x) &= v(x) - u(x) \\ &= f(x) - f(x) = 0 \qquad \text{for all} \quad x \in \partial G. \end{aligned}$$

Theorem 2.11 from Section 2 implies $w(x) \equiv 0$ in $\overline{G}$ and therefore

$$v(x) \equiv u(x), \qquad x \in \overline{G}.$$

q.e.d.

With the aid of Poisson's integral formula we can explicitly solve the Dirichlet problem on balls.

Theorem 3.2. *On the ball $B_R(a) := \{y \in \mathbb{R}^n : |y - a| < R\}$ with the center $a \in \mathbb{R}^n$ and the radius $R \in (0, +\infty)$ we consider Poisson's integral*

$$u(x) := \frac{1}{R\omega_n} \int\limits_{|y-a|=R} \frac{|y-a|^2 - |x-a|^2}{|y-x|^n} f(y)\, d\sigma(y), \qquad x \in B_R(a). \tag{2}$$

Then the function u belongs to the regularity class $C^2(B_R(a)) \cap C^0(\overline{B_R(a)})$ *and is harmonic in* $B_R(a)$*. Furthermore, we have the boundary behavior*

$$\lim_{\substack{x \to \overset{\circ}{x} \\ x \in B_R(a)}} u(x) = f(\overset{\circ}{x}) \qquad \text{for all} \quad \overset{\circ}{x} \in \partial B_R(a). \tag{3}$$

Consequently, the given function u solves Dirichlet's problem (1) on the ball $G = B_R(a)$ *for the continuous boundary function* $f : \partial B_R(a) \to \mathbb{R}$ *being prescribed.*

Proof:

1. At first, we consider the situation $a = 0$, $R = 1$ and set $B := B_1(0) \subset \mathbb{R}^n$. Then we obtain the function

$$u(x) = \frac{1}{\omega_n} \int\limits_{|y|=1} \frac{|y|^2 - |x|^2}{|y-x|^n} f(y)\, d\sigma(y) = \int\limits_{|y|=1} P(y;x) f(y)\, d\sigma(y), \quad x \in B \tag{4}$$

with Poisson's kernel

$$P(y;x) := \frac{1}{\omega_n} \frac{|y|^2 - |x|^2}{|y-x|^n}, \qquad y \in \partial B, \quad x \in B.$$

2. Formula (4) immediately implies the regularity $u \in C^2(B)$. According to part 1 in the proof of Theorem 2.3 from Section 2 the following identity is satisfied:

$$\begin{aligned} P(y;x) &= \frac{1}{\omega_n} \frac{|y|^2 - |x|^2}{|y-x|^n} = \frac{\partial}{\partial \nu} \varphi(y;x) \\ &= y \cdot \nabla_y \varphi(y;x), \qquad y \in \partial B, \quad x \in B. \end{aligned} \tag{5}$$

Here the symbol $\varphi(y;x)$ denotes Green's function for the unit ball B described in Section 2, Theorem 2.2. We note that φ is symmetric, more precisely

$$\varphi(x;y) = \varphi(y;x) \qquad \text{for all} \quad x, y \in B \quad \text{with} \quad x \neq y. \tag{6}$$

Furthermore, we have

$$\Delta_x P(y;x) = y \cdot \nabla_y \Big(\Delta_x \varphi(y;x) \Big) = 0, \qquad x \in B, \quad y \in \partial B. \tag{7}$$

Consequently, we obtain

$$\Delta u(x) = \int\limits_{|y|=1} \Delta_x P(y;x) f(y)\, d\sigma(y) = 0 \qquad \text{for all} \quad x \in B. \tag{8}$$

3. Applying Theorem 2.3 from Section 2 to the harmonic function $v(x) \equiv 1$, $x \in \overline{B}$ we deduce

$$1 = \frac{1}{\omega_n} \int\limits_{|y|=1} \frac{|y|^2 - |x|^2}{|y-x|^n} \, 1 \, d\sigma(y) = \int\limits_{|y|=1} P(y;x) \, d\sigma(y) \quad \text{for all}\, x \in B. \tag{9}$$

Furthermore, $P(y;x) > 0$ for all $y \in \partial B$ and all $x \in B$ is satisfied.

4. We now show that the relation

$$\lim_{\substack{x \to \mathring{x} \\ x \in B}} u(x) \;=\; f(\mathring{x})$$

is correct for all boundary points $\mathring{x} \in \partial B$. We take an arbitrary point $x \in B$ and see

$$\begin{aligned} u(x) - f(\mathring{x}) &= \frac{1}{\omega_n} \int\limits_{|y|=1} \frac{|y|^2 - |x|^2}{|y-x|^n} \Big(f(y) - f(\mathring{x}) \Big) \, d\sigma(y) \\ &= \frac{1}{\omega_n} \int\limits_{\substack{y \in \partial B \\ |y - \mathring{x}| \geq 2\delta}} \frac{|y|^2 - |x|^2}{|y-x|^n} \Big(f(y) - f(\mathring{x}) \Big) \, d\sigma(y) \\ &\quad + \frac{1}{\omega_n} \int\limits_{\substack{y \in \partial B \\ |y - \mathring{x}| \leq 2\delta}} \frac{|y|^2 - |x|^2}{|y-x|^n} \Big(f(y) - f(\mathring{x}) \Big) \, d\sigma(y). \end{aligned} \tag{10}$$

The function f is continuous at the point $\mathring{x}$. Given the quantity $\varepsilon > 0$ we therefore have a number $\delta = \delta(\varepsilon) > 0$ such that $|f(y) - f(\mathring{x})| \leq \varepsilon$ holds true for all points $y \in \partial B$ with $|y - \mathring{x}| \leq 2\delta$. This implies

$$\begin{aligned} &\left| \frac{1}{\omega_n} \int\limits_{\substack{y \in \partial B \\ |y - \mathring{x}| \leq 2\delta}} \frac{|y|^2 - |x|^2}{|y-x|^n} \Big(f(y) - f(\mathring{x}) \Big) \, d\sigma(y) \right| \\ &\qquad \leq \; \frac{1}{\omega_n} \int\limits_{\substack{y \in \partial B \\ |y - \mathring{x}| \leq 2\delta}} \frac{|y|^2 - |x|^2}{|y-x|^n} \Big| f(y) - f(\mathring{x}) \Big| \, d\sigma(y) \\ &\qquad \leq \; \varepsilon \qquad \text{for all} \quad x \in B. \end{aligned} \tag{11}$$

Choosing a point $x \in B$ with $|x - \mathring{x}| \leq \delta$ we infer the following estimate for all $y \in \partial B$ with $|y - \mathring{x}| \geq 2\delta$, namely

$$|y - x| \geq |y - \mathring{x}| - |\mathring{x} - x| \geq 2\delta - \delta = \delta.$$

Consequently, for all $y \in \partial B$ with $|y - \mathring{x}| \geq 2\delta$ and $x \in B$ with $|x - \mathring{x}| \leq \eta < \delta$ we have

$$\begin{aligned} \frac{|y|^2 - |x|^2}{|y-x|^n} &\leq \frac{(|y|+|x|)(|y|-|x|)}{\delta^n} \\ &\leq \frac{2}{\delta^n}(|\mathring{x}| - |x|) \leq \frac{2}{\delta^n}|\mathring{x} - x| \\ &\leq \frac{2\eta}{\delta^n}. \end{aligned}$$

Setting $M := \sup_{y \in \partial B} |f(y)|$ we now can estimate as follows:

$$\begin{aligned} &\left| \frac{1}{\omega_n} \int_{\substack{y \in \partial B \\ |y-\mathring{x}| \geq 2\delta}} \frac{|y|^2 - |x|^2}{|y-x|^n} \Big(f(y) - f(\mathring{x}) \Big) \, d\sigma(y) \right| \\ &\qquad \leq \frac{1}{\omega_n} \int_{\substack{y \in \partial B \\ |y-\mathring{x}| \geq 2\delta}} \frac{|y|^2 - |x|^2}{|y-x|^n} \Big| f(y) - f(\mathring{x}) \Big| \, d\sigma(y) \\ &\qquad \leq \frac{2M}{\omega_n} \int_{\substack{y \in \partial B \\ |y-\mathring{x}| \geq 2\delta}} \frac{|y|^2 - |x|^2}{|y-x|^n} \, d\sigma(y) \\ &\qquad \leq \frac{2M}{\omega_n \delta^n} 2\eta\omega_n \ \leq \ \varepsilon, \end{aligned} \tag{12}$$

if we choose $\eta \in (0, \delta)$ sufficiently small. With the aid of (10), (11), and (12) we deduce

$$|u(x) - f(\mathring{x})| \leq 2\varepsilon \qquad \text{for all} \quad x \in B \quad \text{with} \quad |x - \mathring{x}| \leq \eta. \tag{13}$$

This implies

$$\lim_{\substack{x \to \mathring{x} \\ x \in B}} u(x) \ = \ f(\mathring{x}) \qquad \text{for all} \quad \mathring{x} \in \partial B.$$

5. The function

$$u(x) := \frac{1}{\omega_n} \int_{|y|=1} \frac{|y|^2 - |x|^2}{|y-x|^n} f(y) \, d\sigma(y), \qquad x \in B$$

solves Dirichlet's problem on the unit ball B. We now utilize the transformation

$$x = T\xi = \frac{1}{R}(\xi - a), \qquad \xi \in \overline{B_R(a)}.$$

Then the function $v(\xi) := u(T\xi)$, $\xi \in \overline{B_R(a)}$ gives us a solution of Dirichlet's problem

$$\begin{aligned} &v = v(\xi) \in C^2(B_R(a)) \cap C^0(\overline{B_R(a)}),\\ &\Delta v(\xi) = 0 \qquad \text{for all} \quad \xi \in B_R(a), \\ &v(\xi) = g(\xi) \qquad \text{for all} \quad \xi \in \partial B_R(a), \end{aligned} \tag{14}$$

where we have set $g(\xi) := f(T\xi)$, $\xi \in \partial B_R(a)$. Taking

$$\eta := T^{-1}y = Ry + a, \qquad y \in \partial B$$

we see $\eta \in \partial B_R(a)$ and $d\sigma(\eta) = R^{n-1}\, d\sigma(y)$. On this basis we calculate

$$\begin{aligned} v(\xi) = u(T\xi) &= \frac{1}{\omega_n} \int\limits_{|y|=1} \frac{|y|^2 - |T\xi|^2}{|y - T\xi|^n} f(y)\, d\sigma(y) \\ &= \frac{1}{\omega_n} \int\limits_{|\eta-a|=R} \frac{|T\eta|^2 - |T\xi|^2}{|T\eta - T\xi|^n} f(T\eta) \frac{1}{R^{n-1}}\, d\sigma(\eta) \\ &= \frac{1}{R^{n-1}\omega_n} \int\limits_{|\eta-a|=R} \frac{\frac{1}{R^2}\Big(|\eta - a|^2 - |\xi - a|^2\Big)}{\frac{1}{R^n}|\eta - \xi|^n} g(\eta)\, d\sigma(\eta) \\ &= \frac{1}{R\omega_n} \int\limits_{|\eta-a|=R} \frac{|\eta - a|^2 - |\xi - a|^2}{|\eta - \xi|^n} g(\eta)\, d\sigma(\eta), \qquad \xi \in B_R(a). \end{aligned}$$

q.e.d.

Theorem 3.3. (Regularity theorem for weakharmonic functions) *Let the weakharmonic function $u = u(x) : G \to \mathbb{R} \in C^0(G)$ be given on the domain $G \subset \mathbb{R}^n$. Then the function u is real-analytic in G and satisfies the Laplace equation $\Delta u(x) = 0$ for all $x \in G$.*

Proof: Let the point $a \in G$ be chosen arbitrarily. For a suitable radius $R \in (0, +\infty)$ we then consider the ball $B_R(a) \subset\subset G$, where we solve Dirichlet's problem with the aid of Theorem 3.2, namely

$$\begin{aligned} &v = v(x) \in C^2(B_R(a)) \cap C^0(\overline{B_R(a)}),\\ &\Delta v(x) = 0 \qquad \text{for all} \quad x \in B_R(a), \\ &v(x) = u(x) \qquad \text{for all} \quad x \in \partial B_R(a). \end{aligned} \tag{15}$$

Theorem 2.11 from Section 2 now yields $u(x) \equiv v(x)$ in $\overline{B_R(a)}$. Consequently, we have $u \in C^2(G)$ and $\Delta u(x) = 0$ for all $x \in G$. According to Theorem 1.9 in Section 1, the function u is real-analytic in G.

q.e.d.

We now intend to solve Dirichlet's problem (1) for a large class of domains G. In this context we use an ingenious *method* proposed *by O. Perron.*

Definition 3.4. *Let $G \subset \mathbb{R}^n$ denote a bounded domain on which the continuous function $u = u(x) : G \to \mathbb{R} \in C^0(G)$ is given. Then we define the* harmonically modified function

$$v(x) := [u]_{a,R}(x)$$
$$:= \begin{cases} u(x), & x \in G \text{ with } |x-a| \geq R \\ \dfrac{1}{R\omega_n} \displaystyle\int\limits_{|y-a|=R} \frac{|y-a|^2 - |x-a|^2}{|y-x|^n} u(y)\, d\sigma(y), & x \in G \text{ with } |x-a| < R \end{cases}$$

for all $a \in G$ and $R \in (0, dist(a, \mathbb{R}^n \setminus G))$.

Remark: The function $v = v(x) : G \to \mathbb{R} \in C^0(G)$ is harmonic in $B_R(a)$ and coincides with the original function on the complement of this ball $G \setminus B_R(a)$.

In the sequel we need the important

Proposition 3.5. *Let the point $a \in G$ and the radius $R \in (0, dist(a, \mathbb{R}^n \setminus G))$ be chosen as fixed, whereas $u = u(x)$ denotes a superharmonic function in G. Then the harmonically modified function*

$$v(x) := [u]_{a,R}(x), \qquad x \in G$$

is superharmonic in G as well, and we have

$$v(x) \leq u(x) \qquad \textit{for all} \quad x \in G.$$

Proof:

1. At first, we show the inequality $v(x) \leq u(x)$ for all $x \in G$. In this context we only have to verify $v(x) \leq u(x)$ for all $x \in \overline{B_R(a)}$. The function

$$w(x) := u(x) - v(x), \qquad x \in \overline{B_R(a)}$$

is superharmonic in the ball $B_R(a)$. Each sequence of points

$$\{x^{(k)}\}_{k=1,2,\cdots} \subset B_R(a)$$

with $\lim\limits_{k\to\infty} x^{(k)} = \overset{\circ}{x} \in \partial B_R(a)$ satisfies

$$\liminf_{k\to\infty} w(x^{(k)}) = w(\overset{\circ}{x}) = 0.$$

From Section 2, Theorem 2.10 we infer $w(x) \geq 0$, $x \in B_R(a)$ and consequently

$$v(x) \leq u(x) \qquad \text{for all} \quad x \in B_R(a).$$

2. We now show that v is superharmonic in G. Choose an arbitrary point $\xi \in \partial B_R(a)$ and a quantity $\vartheta(\xi) \in (0, \operatorname{dist}(\xi, \mathbb{R}^n \setminus G)]$. Using part 1 of our proof, we then obtain

$$\frac{1}{\varrho^{n-1}\omega_n} \int\limits_{|x-\xi|=\varrho} v(x)\, d\sigma(x) \leq \frac{1}{\varrho^{n-1}\omega_n} \int\limits_{|x-\xi|=\varrho} u(x)\, d\sigma(x) \leq u(\xi) = v(\xi)$$

for all $\varrho \in (0, \vartheta(\xi))$. Consequently, the function v is superharmonic in G: In the ball $B_R(a)$ the function v is harmonic anyway, and in $G \setminus \overline{B_R(a)}$ this function v is superharmonic.

q.e.d.

We additionally need the following

Proposition 3.6. (Harnack's lemma)
We consider a sequence $w_k(x) : G \to \mathbb{R}$, $k = 1, 2, \dots$ of harmonic functions in G, which are descending in the following way:

$$w_1(x) \geq w_2(x) \geq w_3(x) \geq \dots \qquad \textit{for all} \quad x \in G.$$

Furthermore, let the sequence converge at one point $\overset{\circ}{x} \in G$ which means

$$\lim_{k\to\infty} w_k(\overset{\circ}{x}) > -\infty.$$

Then the sequence of functions $\{w_k(x)\}_{k=1,2,}$ uniformly converges in each compact set $\Theta \subset G$ towards a function harmonic in G, namely

$$w(x) := \lim_{k\to\infty} w_k(x), \qquad x \in G.$$

Proof: Without loss of generality we assume $\overset{\circ}{x}= 0$ and for the ball the inclusion $B_R \subset G$ with a radius $R \in (0, +\infty)$. For the indices $k, l \in \mathbb{N}$ with $k \leq l$ we define the nonnegative functions $v_{kl}(x) := w_k(x) - w_l(x) \geq 0$, $x \in B_R$. We apply Harnack's inequality and obtain

$$0 \leq v_{kl}(x) \leq \frac{1 + \frac{|x|}{R}}{\left(1 - \frac{|x|}{R}\right)^{n-1}} v_{kl}(0) \leq \frac{1 + \frac{1}{2}}{\left(1 - \frac{1}{2}\right)^{n-1}} v_{kl}(0), \qquad x \in \overline{B_{\frac{R}{2}}}.$$

Setting $K := \frac{3}{2} \cdot (\frac{1}{2})^{1-n} = 3 \cdot 2^{n-2}$ we infer

$$|w_k(x) - w_l(x)| \leq K |w_k(0) - w_l(0)| \qquad \text{for all} \quad x \in \overline{B_{\frac{R}{2}}} \quad \text{and all} \quad k, l \in \mathbb{N}. \tag{16}$$

Since the limit $\lim\limits_{k\to\infty} w_k(0)$ exists, the sequence $\{w_k(x)\}_{k=1,2,\dots}$ converges uniformly in $\overline{B_{\frac{R}{2}}}$ towards the function $w(x)$. When we cover a compact set

$\Theta \subset G$ by finitely many balls we comprehend that the sequence of functions $\{w_k(x)\}_{k=1,2,\ldots}$ converges uniformly in Θ towards the function $w(x)$. The transition to the limit in Poisson's integral formula shows that the limit function $w(x)$ is harmonic in G.

q.e.d.

In order to solve Dirichlet's problem we utilize the following set of admissible functions

$$\mathcal{M} := \Big\{ v : G \to \mathbb{R} \in C^0(G) \ : \ v \text{ is in } G \text{ superharmonic, and for all sequences } \{x^{(k)}\}_{k=1,2,\ldots} \subset G \text{ with } \lim_{k\to\infty} x^{(k)} = x^* \in \partial G \text{ we have } \liminf_{k\to\infty} v(x^{(k)}) \geq f(x^*) \Big\}.$$

Here the symbol $f : \partial G \to \mathbb{R}$ denotes a continuous boundary function. Since

$$v(x) := M := \max_{x \in \partial G} f(x) \quad \in \ \mathcal{M}$$

holds true, we have $\mathcal{M} \neq \emptyset$.

Proposition 3.7. *Let us define the function*

$$u(x) := \inf_{v \in \mathcal{M}} v(x), \qquad x \in G.$$

Then u is harmonic in G and we have

$$m \leq u(x) \leq M \qquad \text{for all} \quad x \in G.$$

Here we abbreviate $m := \inf_{x \in \partial G} f(x)$ *and* $M := \sup_{x \in \partial G} f(x)$.

Proof:

1. We take a sequence of points $\{x^i\}_{i=1,2,3,\ldots} \subset G$ which are dense in G. For each index $i \in \mathbb{N}$, there exists a sequence of functions $\{v_{ij}\}_{j=1,2,\ldots} \subset \mathcal{M}$ satisfying

$$\lim_{j\to\infty} v_{ij}(x^i) = u(x^i).$$

The minimum principle implies the estimate $v_{ij}(x) \geq m$ for all $x \in G$ and all $i, j \in \mathbb{N}$. We now define the functions

$$v_k(x) := \min_{1 \leq i,j \leq k} v_{ij}(x), \qquad x \in G$$

for each index $k \in \mathbb{N}$. Evidently, we have $v_k(x) \geq v_{k+1}(x)$, $x \in G$ for all $k \in \mathbb{N}$. The minimum of finitely many superharmonic functions is superharmonic again according to a previous remark, and we infer

$$v_k \in \mathcal{M}, \qquad k = 1, 2, \ldots$$

We observe $u(x^i) \leq v_k(x^i) \leq v_{ik}(x^i)$ for $1 \leq i \leq k$, and we obtain

$$\lim_{k\to\infty} v_k(x^i) = u(x^i) \qquad \text{for all} \quad i = 1, 2, \ldots$$

2. In the disc $B_R(a) \subset\subset G$ we harmonically modify the function v_k to the following function
$$w_k(x) := [v_k]_{a,R}(x), \qquad x \in G.$$
With the aid of Proposition 3.5 we see $\{w_k\}_{k=1,2,\ldots} \subset \mathcal{M}$. Furthermore, we have $w_k(x) \geq w_{k+1}(x)$ in $B_R(a)$ for all $k \in \mathbb{N}$ and
$$u(x^i) \leq w_k(x^i) \leq v_k(x^i) \qquad \text{for all} \quad i,k \in \mathbb{N}.$$
Therefore, we obtain
$$\lim_{k\to\infty} w_k(x^i) = u(x^i) \qquad \text{for all} \quad i \in \mathbb{N}.$$
According to Harnack's lemma the sequence $\{w_k(x)\}_{k=1,2,\ldots}$ converges uniformly in $B_R(a)$ towards a harmonic function $w(x)$, and we comprehend
$$w(x^i) = u(x^i) \qquad \text{for all} \quad x^i \in B_R(a), \quad i = 1,2,\ldots$$
Since w and u are continuous functions, we infer the identity $u(x) = w(x)$, $x \in \overline{B_R(a)}$. Consequently, the function u has to be harmonic in G, because the ball $B_R(a) \subset\subset G$ has been chosen arbitrarily.
3. The inclusion $M \in \mathcal{M}$ implies the estimate $u(x) \leq M$ for all $x \in G$. Since the inequality $v_{ij}(x) \geq m$ for all $x \in G$ and all $i,j \in \mathbb{N}$ holds true and consequently $v_k(x) \geq m$ in G for all $k \in \mathbb{N}$ is valid, we finally obtain
$$u(x) = \lim_{k\to\infty} v_k(x) \geq m \qquad \text{for all} \quad x \in G.$$
q.e.d.

Definition 3.8. *Let us consider the bounded domain $G \subset \mathbb{R}^n$. We name a* boundary point $x \in \partial G$ regular *if we have a superharmonic function*
$$\Phi(y) = \Phi(y;x) : G \to \mathbb{R} \quad \text{with} \quad \lim_{\substack{y\to x\\ y\in G}} \Phi(y) = 0$$
and
$$\varrho(\varepsilon) := \inf_{\substack{y\in G\\ |y-x|\geq\varepsilon}} \Phi(y) > 0 \qquad \text{for all} \quad \varepsilon > 0.$$
If each boundary point of the domain G is regular, we speak of a Dirichlet domain.

Remark: A point $x \in \partial G$ is regular if and only if we have a number $r > 0$ and a superharmonic function $\Psi = \Psi(y) : G \cap B_r(x) \to \mathbb{R}$ satisfying
$$\lim_{\substack{y\to x\\ y\in G\cap B_r(x)}} \Psi(y) = 0 \qquad \text{and} \qquad \inf_{\substack{r>|y-x|\geq\varepsilon\\ y\in G}} \Psi(y) > 0, \quad 0 < \varepsilon < r.$$

Here we set $m := \inf_{\substack{r > |y-x| \geq \frac{1}{2}r \\ y \in G}} \Psi(y) > 0$ and consider the following function

$$\Phi(y) := \begin{cases} \min\left(1, \frac{2\Psi(y)}{m}\right), & y \in G \cap B_r(x) \\ 1, & y \in G \setminus B_r(x) \end{cases}$$

which is superharmonic in G.

Theorem 3.9. (Dirichlet problem for the Laplacian)
Let $G \subset \mathbb{R}^n$ denote a bounded domain with $n \geq 2$. Then the Dirichlet problem

$$\begin{aligned} &u = u(x) \in C^2(G) \cap C^0(\overline{G}), \\ &\Delta u(x) = 0 \qquad in \quad G, \\ &u(x) = f(x) \qquad on \quad \partial G \end{aligned} \tag{17}$$

can be solved for all continuous boundary functions $f : \partial G \to \mathbb{R}$ if and only if G is a Dirichlet domain in the sense of Definition 3.8.

Proof:

'$\Longrightarrow$' Let the Dirichlet problem be solvable for all continuous boundary functions $f : \partial G \to \mathbb{R}$. Taking an arbitrary point $\xi \in \partial G$ we define the function $f(y) := |y - \xi|$, $y \in \partial G$, and we solve Dirichlet's problem (17) for these boundary values. We apply the minimum principle to the harmonic function $u = u(x) : \overline{G} \to \mathbb{R}$ and obtain

$$u(x) > 0 \qquad \text{for all} \quad x \in \overline{G} \setminus \{\xi\}.$$

Therefore, the boundary point ξ is regular.

'$\Longleftarrow$' Let G be a Dirichlet domain and $x \in \partial G$ an arbitrary regular boundary point. Then we have an associate superharmonic function $\Phi(y) = \Phi(y; x) : G \to \mathbb{R}$ due to Definition 3.8. Since the function $f : \partial G \to \mathbb{R}$ is continuous, we can prescribe $\varepsilon > 0$ and obtain a quantity $\delta = \delta(\varepsilon) > 0$ satisfying

$$|f(y) - f(x)| \leq \varepsilon \quad \text{for all} \quad y \in \partial G \quad \text{with} \quad |y - x| \leq \delta.$$

We now define

$$\eta(\varepsilon) := \inf_{\substack{y \in G \\ |y-x| \geq \delta(\varepsilon)}} \Phi(y) > 0.$$

1. Let the *upper barrier function*

$$v^+(y) := f(x) + \varepsilon + (M - m)\frac{\Phi(y)}{\eta(\varepsilon)}, \qquad y \in G$$

be given. Evidently, the function v^+ is superharmonic in G. Furthermore, an arbitrary sequence $\{y^{(k)}\}_{k=1,2,\ldots} \subset G$ with $y^{(k)} \to y^+ \in \partial G$ for $k \to \infty$ satisfies

$$\liminf_{k\to\infty} v^+(y^{(k)}) \geq f(y^+).$$

Consequently, $v^+ \in \mathcal{M}$ holds true.

2. Now we consider the *lower barrier function*

$$v^-(y) := f(x) - \varepsilon - (M-m)\frac{\Phi(y)}{\eta(\varepsilon)}, \qquad y \in G.$$

We choose $v \in \mathcal{M}$ arbitrarily. Considering a sequence $\{y^{(k)}\}_{k=1,2,\ldots} \subset G$ with $y^{(k)} \to y^- \in \partial G$ for $k \to \infty$, we can estimate

$$\begin{aligned} &\liminf_{k\to\infty} \Big(v(y^{(k)}) - v^-(y^{(k)}) \Big) \\ &\qquad \geq \liminf_{k\to\infty} \Big(v(y^{(k)}) - f(y^-) \Big) + \liminf_{k\to\infty} \Big(f(y^-) - v^-(y^{(k)}) \Big) \\ &\qquad \geq 0. \end{aligned}$$

Furthermore, the function $v - v^-$ is superharmonic in G, and Theorem 2.10 from Section 2 yields $v - v^- \geq 0$ in G. This implies

$$v(y) \geq v^-(y), \qquad y \in G \quad \text{for all} \quad v \in \mathcal{M}.$$

3. The harmonic function

$$u(y) := \inf_{v \in \mathcal{M}} v(y), \qquad y \in G$$

constructed in Proposition 3.7 now attains the prescribed boundary values f continuously. On account of 1. and 2. the estimate

$$v^-(y) \leq u(y) \leq v^+(y) \qquad \text{for all} \quad y \in G$$

is fulfilled, which means

$$f(x) - \varepsilon - (M-m)\frac{\Phi(y)}{\eta(\varepsilon)} \leq u(y) \leq f(x) + \varepsilon + (M-m)\frac{\Phi(y)}{\eta(\varepsilon)}, \qquad y \in G.$$

Using the relation $\lim\limits_{\substack{y \in G \\ y \to x}} \Phi(y) = 0$ we obtain

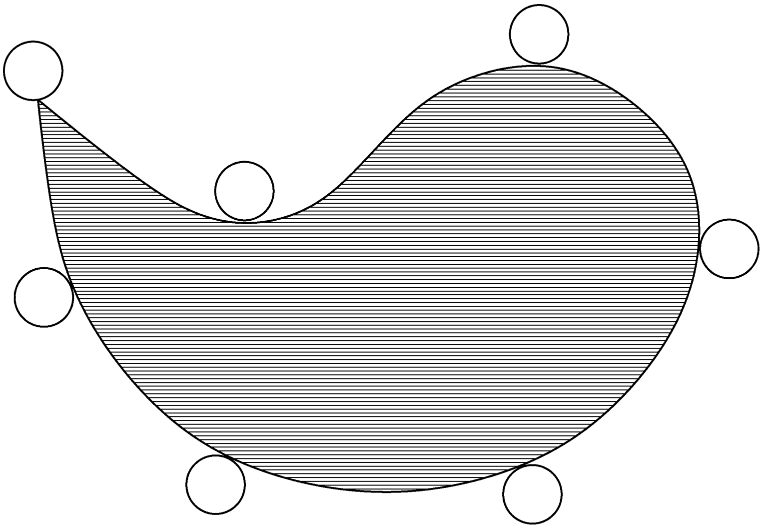
$$|f(x) - u(y)| \leq \varepsilon + (M-m)\frac{\Phi(y)}{\eta(\varepsilon)} \leq 2\varepsilon$$

for all $y \in G$ with $|y - x| \leq \delta^*(\varepsilon)$. This implies

$$\lim_{\substack{y \in G \\ y \to x}} u(y) = f(x).$$

Therefore, the function u solves Dirichlet's problem (17) for the boundary values f. q.e.d.

Figure 1.6 Poincaré's condition of exterior support balls



Theorem 3.10. (Poincaré's condition)
A boundary point $x \in \partial G$ is regular, if we have a ball $B_r(a)$ with the center $a \in \mathbb{R}^n$ and the radius $r \in (0, +\infty)$ satisfying $\overline{G} \cap \overline{B_r(a)} = \{x\}$. Especially, bounded domains with a regular C^2-boundary are Dirichlet domains.

Proof: For $n = 2$ we consider in G the harmonic function

$$\Phi(y) := \log\left(\frac{|y-a|}{r}\right), \qquad y \in G,$$

and for $n \geq 3$ we consider the harmonic function

$$\Phi(y) := r^{2-n} - |y-a|^{2-n}, \qquad y \in G.$$

Then we immediately obtain the statements above. q.e.d.

Theorem 3.11. *Let $B_R := \{x \in \mathbb{R}^n : |x| < R\}$ denote the ball about the origin of radius $R > 0$ and consider the pointed ball $\dot{B}_R := B_R \setminus \{0\}$. The function $u = u(x) \in C^2(\dot{B}_R) \cap C^0(\overline{B_R})$ is assumed to be harmonic in $\dot{B}_R$. Then the function u is harmonic in B_R.*

Proof: We restrict our considerations to the case $n \geq 3$ and set

$$v(x) := \frac{1}{R\omega_n} \int\limits_{|y|=R} \frac{R^2 - |x|^2}{|y-x|^n} u(y)\, d\sigma(y), \qquad x \in B_R.$$

This function v is harmonic in B_R and continuous in $\overline{B_R}$ with the boundary values

$$v(x) = u(x), \qquad x \in \partial B_R.$$

Since the functions u and v are continuous in $\overline{B_R}$, we have a constant $M > 0$ such that

$$\sup_{x \in B_R} |u(x) - v(x)| \le M$$

holds true. Given the quantity $\varepsilon > 0$, we now can choose a sufficiently small number $\delta = \delta(\varepsilon) \in (0, R)$ such that

$$M \le \varepsilon\Big(|x|^{2-n} - R^{2-n}\Big) \qquad \text{for all} \quad x \in \mathbb{R}^n \quad \text{with} \quad |x| = \delta(\varepsilon).$$

We consider the spherical shell $K_\varepsilon := \{x \in \mathbb{R}^n \ : \ \delta(\varepsilon) \le |x| \le R\}$ and see

$$|u(x) - v(x)| \le \varepsilon\Big(|x|^{2-n} - R^{2-n}\Big) \qquad \text{for all} \quad x \in \partial K_\varepsilon.$$

The maximum principle for harmonic functions now yields

$$|u(x) - v(x)| \le \varepsilon\Big(|x|^{2-n} - R^{2-n}\Big) \qquad \text{for all} \quad x \in K_\varepsilon.$$

Since the number $\varepsilon > 0$ has been chosen arbitrarily and the behavior $\delta(\varepsilon) \downarrow 0$ for $\varepsilon \downarrow 0$ can be achieved, we obtain

$$u(x) \equiv v(x), \qquad x \in \dot{B}_R.$$

Now the functions u and v are continuous in $\overline{B_R}$, and we infer

$$u(x) \equiv v(x), \qquad x \in \overline{B_R}.$$

Therefore, the function u is harmonic in B_R. q.e.d.

Remarks:

1. When we consider the Riemannian theorem on removable singularities for holomorphic functions, it suffices to assume the boundedness of the functions in the neighborhood of a singular point in order to continue them holomorphically into this point.
2. There are bounded domains, where the Dirichlet problem cannot be solved for arbitrary boundary values. For example, we consider the domain

$$G := \dot{B}_R, \qquad \partial G = \partial B_R \cup \{0\}.$$

On account of Theorem 3.11, there does not exist a harmonic function for the boundary values $f(x) = 1$, $|x| = R$ and $f(0) = 0$.

4 Theory of Spherical Harmonics in 2 Variables: Fourier Series

The theory of spherical harmonics has been founded by Laplace and Legendre and is applied in quantum mechanics to the investigation of the spectrum for the hydrogen atom. We owe the theory in arbitrary spatial dimensions $n \geq 2$ to G.Herglotz. In the next two paragraphs we utilize Banach and Hilbert spaces introduced in Chapter 2, Section 6. At first, we consider the case $n = 2$.

On the unit circle line $S^1 := \{x \in \mathbb{R}^2 \ : \ |x| = 1\}$ we consider the functions $u = u(x) \in C^0(S^1, \mathbb{R})$. They are identified with the 2π-periodic continuous functions

$$C^0_{2\pi}(\mathbb{R}, \mathbb{R}) := \left\{ v : \mathbb{R} \to \mathbb{R} \in C^0(\mathbb{R}, \mathbb{R}) \ : \ \begin{array}{l} v(\varphi + 2\pi k) = v(\varphi) \\ \text{for all } \varphi \in \mathbb{R}, \ k \in \mathbb{Z} \end{array} \right\}$$

via $\hat{u}(\varphi) := u(e^{i\varphi})$, $0 \leq \varphi \leq 2\pi$. We endow the space $C^0(S^1, \mathbb{R})$ with the norm

$$\|u\|_0 := \max_{x \in S^1} |u(x)|, \qquad u \in C^0(S^1, \mathbb{R}) \tag{1}$$

and get a Banach space with the topology of uniform convergence. By the inner product

$$(u, v) := \int_0^{2\pi} u(e^{i\varphi}) v(e^{i\varphi}) \, d\varphi, \qquad u, v \in C^0(S^1, \mathbb{R}) \tag{2}$$

the set $C^0(S^1, \mathbb{R})$ becomes a pre-Hilbert-space. We complete this space with respect to the L^2-norm induced by the inner product (2), namely

$$\|u\| := \sqrt{(u, u)}, \qquad u \in C^0(S^1, \mathbb{R}), \tag{3}$$

and obtain the Lebesgue space $L^2(S^1, \mathbb{R})$ of the square integrable, measurable functions on S^1. Furthermore, we note the inequality

$$\|u\| \leq \sqrt{2\pi} \|u\|_0 \qquad \text{for all} \quad u \in C^0(S^1, \mathbb{R}). \tag{4}$$

If a sequence converges with respect to the Banach-space-norm $\| \cdot \|_0$, this is as well the case with respect to the Hilbert-space-norm $\| \cdot \|$. However, the opposite direction is not true, since the Hilbert space $L^2(S^1, \mathbb{R})$ also contains discontinuous functions.

Theorem 4.1. (Fourier series)
The system of functions

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{1}{\sqrt{\pi}} \cos k\varphi, \quad \frac{1}{\sqrt{\pi}} \sin k\varphi, \qquad \varphi \in [0, 2\pi], \quad k = 1, 2, \ldots$$

represents a complete orthonormal system - briefly c.o.n.s. - in the pre-Hilbert-space $\mathcal{H} := C^0(S^1, \mathbb{R})$ *endowed with the inner product from (2).*

Proof:

1. We easily verify that the system of functions $\mathcal{S}$ given is orthonormal, which means $\|u\| = 1$ for all $u \in \mathcal{S}$ and $(u, v) = 0$ for all $u, v \in \mathcal{S}$ with $u \neq v$. It remains for us to comprehend that this orthonormal system of functions is complete in the pre-Hilbert-space $\mathcal{H}$. According to Theorem 6.19 from Chapter 2, Section 6 we have to show that the Fourier series for each element $u \in \mathcal{H}$ approximates this element with respect to the Hilbert-space-norm $\|\cdot\|$ from (3).
2. Let the function
$$u = u(x) \in \mathcal{H} = C^0(S^1, \mathbb{R})$$
be given arbitrarily. We then continue u harmonically onto the disc
$$B = \{x \in \mathbb{R}^2 \ : \ |x| < 1\}$$
via
$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{|e^{i\varphi} - z|^2} u(e^{i\varphi})\, d\varphi, \qquad |z| < 1; \tag{5}$$
here we have set $z = re^{i\vartheta}$. We now expand Poisson's kernel as follows:
$$\begin{aligned} \frac{1 - r^2}{|e^{i\varphi} - z|^2} &= \frac{1 - r^2}{|e^{i\varphi} - re^{i\vartheta}|^2} \\ &= \frac{1 - r^2}{|1 - re^{i(\vartheta - \varphi)}|^2} \\ &= \frac{1 - r^2}{(1 - re^{i(\vartheta - \varphi)})(1 - re^{i(\varphi - \vartheta)})} \\ &= -1 + \frac{1}{1 - re^{i(\varphi - \vartheta)}} + \frac{1}{1 - re^{-i(\varphi - \vartheta)}} \\ &= -1 + \sum_{k=0}^{\infty} r^k e^{ik(\varphi - \vartheta)} + \sum_{k=0}^{\infty} r^k e^{-ik(\varphi - \vartheta)} \\ &= 1 + 2 \sum_{k=1}^{\infty} r^k \cos k(\varphi - \vartheta). \end{aligned} \tag{6}$$
Here the series converges locally uniformly for $0 \leq r < 1$ and $\varphi, \vartheta \in \mathbb{R}$. Now we have
$$\cos k(\varphi - \vartheta) = \cos k\varphi \cos k\vartheta + \sin k\varphi \sin k\vartheta,$$
and we obtain the following identity with $g(\varphi) := u(e^{i\varphi})$, $\varphi \in [0, 2\pi)$:

$$u(re^{i\vartheta}) = \frac{1}{2\pi}\int_0^{2\pi}\Big\{1 + 2\sum_{k=1}^{\infty} r^k\Big(\cos k\varphi\cos k\vartheta + \sin k\varphi\sin k\vartheta\Big)\Big\}g(\varphi)\,d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} g(\varphi)\,d\varphi + \sum_{k=1}^{\infty}\Big\{\Big(\frac{1}{\pi}\int_0^{2\pi} g(\varphi)\cos k\varphi\,d\varphi\Big)r^k\cos k\vartheta$$

$$+\Big(\frac{1}{\pi}\int_0^{2\pi} g(\varphi)\sin k\varphi\,d\varphi\Big)r^k\sin k\vartheta\Big\}.$$

Finally, we set

$$a_k := \frac{1}{\pi}\int_0^{2\pi} g(\varphi)\cos k\varphi\,d\varphi, \qquad k = 0, 1, 2, \ldots \tag{7}$$

and

$$b_k := \frac{1}{\pi}\int_0^{2\pi} g(\varphi)\sin k\varphi\,d\varphi, \qquad k = 1, 2, \ldots. \tag{8}$$

With the representation

$$u(re^{i\vartheta}) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}\Big(a_k\cos k\vartheta + b_k\sin k\vartheta\Big)r^k,\ 0 \le r < 1,\ 0 \le \vartheta < 2\pi \tag{9}$$

we obtain the *Fourier expansion of a harmonic function within the unit disc.*

3. Since the function $u(z)$ is continuous in $\overline{B}$, we find a radius $r \in (0,1)$ to each given $\varepsilon > 0$, such that

$$|u(re^{i\vartheta}) - g(\vartheta)| \le \varepsilon \qquad \text{for all} \quad \vartheta \in [0, 2\pi). \tag{10}$$

Furthermore, we can choose an integer $N = N(\varepsilon) \in \mathbb{N}$ so large that

$$\Big|\frac{a_0}{2} + \sum_{k=1}^{N} r^k\Big(a_k\cos k\vartheta + b_k\sin k\vartheta\Big) - u(re^{i\vartheta})\Big| \le \varepsilon \quad \text{for all} \quad \vartheta \in [0, 2\pi) \tag{11}$$

is satisfied. For the quantity $\varepsilon > 0$ given, we therefore find real coefficients $A_0, \ldots, A_N$ and $B_1, \ldots, B_N$, such that the *trigonometric polynomial*

$$F_\varepsilon(\vartheta) := A_0 + \sum_{k=1}^{N}\Big(A_k\sin k\vartheta + B_k\cos k\vartheta\Big), \qquad 0 \le \vartheta < 2\pi$$

fulfills the following inequality

$$|F_\varepsilon(\vartheta) - g(\vartheta)| \le 2\varepsilon \qquad \text{for all} \quad \vartheta \in [0, 2\pi). \tag{12}$$

From the relation (4) we infer

$$\|F_\varepsilon - g\| \le 2\sqrt{2\pi}\,\varepsilon. \tag{13}$$

On account of the minimal property for the Fourier coefficients due to Chapter 2, Section 6, Proposition 6.17, the Fourier series belonging to the system of functions above approximates the given function with respect to the Hilbert-space-norm. From Theorem 6.19 in Chapter 2, Section 6 we infer that this system of functions represents a complete orthonormal system in $\mathcal{H}$.

q.e.d.

Remark: We leave the following question unanswered: Which functions $g = g(\vartheta)$ satisfy the identity (9) pointwise even for the radius $r = 1$, which concerns the validity of the pointwise equation

$$u(e^{i\vartheta}) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}\Big(a_k \cos k\vartheta + b_k \sin k\vartheta\Big), \qquad 0 \le \vartheta < 2\pi.$$

We have shown only the convergence in the square mean. For continuous functions the identity above is *not* satisfied, in general. The investigations on the convergence of Fourier series gave an important motivation for the development of the analysis.

We now present the relationship of trigonometric functions to the Laplace operator. At first, we remind the reader of the decomposition for the Laplacian in polar coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}. \tag{14}$$

For an arbitrary C^2-function $f = f(r)$ we therefore have the identity

$$\Delta\left(f(r)\,\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}\right) = \left(f''(r) + \frac{1}{r}f'(r) - \frac{k^2}{r^2}f(r)\right)\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix} = \Big(L_k f(r)\Big)\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}.$$

Here we abbreviate

$$L_k f(r) := f''(r) + \frac{1}{r}f'(r) - \frac{k^2}{r^2}f(r), \qquad r > 0.$$

We note that

$$L_k(r^k) = k(k-1)r^{k-2} + kr^{k-2} - k^2 r^{k-2} = 0, \qquad k = 0, 1, 2, \ldots$$

and obtain

$$\Delta(r^k \cos k\varphi) = 0 = \Delta(r^k \sin k\varphi), \qquad k = 0, 1, 2, \ldots \tag{15}$$

Proposition 4.2. *Let the function $u = u(x_1, x_2) \in C^2(B_R)$ be given on the disc $B_R := \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1^2 + x_2^2 < R^2\}$. By the symbols*

$$a_k(r) = \frac{1}{\pi}\int_0^{2\pi} u(re^{i\varphi})\cos k\varphi\,d\varphi, \qquad b_k(r) = \frac{1}{\pi}\int_0^{2\pi} u(re^{i\varphi})\sin k\varphi\,d\varphi \tag{16}$$

we denote the Fourier coefficients of the function u and by

$$\tilde{a}_k(r) = \frac{1}{\pi}\int_0^{2\pi} \Delta u(re^{i\varphi})\cos k\varphi\,d\varphi, \qquad \tilde{b}_k(r) = \frac{1}{\pi}\int_0^{2\pi} \Delta u(re^{i\varphi})\sin k\varphi\,d\varphi \tag{17}$$

we mean the Fourier coefficients of the function Δu for $0 < r < R$. Now we have the equation

$$\tilde{a}_k(r) = L_k a_k(r), \qquad \tilde{b}_k(r) = L_k b_k(r), \qquad 0 < r < R. \tag{18}$$

Remark: The Fourier coefficients of Δu are consequently obtained by formal differentiation of the Fourier series

$$u(re^{i\vartheta}) = \frac{1}{2}a_0(r) + \sum_{k=1}^{\infty}\Big(a_k(r)\cos k\vartheta + b_k(r)\sin k\vartheta\Big).$$

Proof of Proposition 4.2: We evaluate as follows:

$$\begin{aligned}
\tilde{a}_k(r) &= \frac{1}{\pi}\int_0^{2\pi} \Delta u(re^{i\varphi})\cos k\varphi\,d\varphi,\\
&= \frac{1}{\pi}\int_0^{2\pi}\left\{\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)u(re^{i\varphi})\right\}\cos k\varphi\,d\varphi\\
&= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left\{\frac{1}{\pi}\int_0^{2\pi} u(re^{i\varphi})\cos k\varphi\,d\varphi\right\} - \frac{k^2}{\pi r^2}\int_0^{2\pi} u(re^{i\varphi})\cos k\varphi\,d\varphi\\
&= L_k a_k(r), \qquad 0 < r < R, \quad k = 0, 1, 2, \ldots
\end{aligned}$$

Similarly we show the relation (18) for the functions $b_k(r)$.

q.e.d.

Theorem 4.3. *We choose $k \in \mathbb{R}$ and define $\dot{\mathbb{R}}^2 := \mathbb{R}^2 \setminus \{0\}$. Furthermore, the symbol $H_k = H_k(\xi) : S^1 \to \mathbb{R}$ denotes a function defined on the unit circle S^1 with the properties*

$$|x|^k H_k\left(\frac{x}{|x|}\right) \in C^2(\dot{\mathbb{R}}^2) \quad \textit{and} \quad \Delta\left\{|x|^k H_k\left(\frac{x}{|x|}\right)\right\} = 0,\ x \in \dot{\mathbb{R}}^2.$$

Then we infer $k \in \mathbb{Z}$, and we have the identity

$$H_k(e^{i\vartheta}) = A_k \cos k\vartheta + B_k \sin k\vartheta$$

with the real constants A_k, B_k.

Proof: At first, we calculate

$$\begin{aligned}
0 &= \Delta\left\{|x|^k H_k\left(\frac{x}{|x|}\right)\right\} \\
&= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right)\left[r^k H_k(e^{i\varphi})\right] \\
&= \left[k(k-1)r^{k-2} + kr^{k-2}\right] H_k(e^{i\varphi}) + r^{k-2}\frac{\partial^2}{\partial \varphi^2} H_k(e^{i\varphi}).
\end{aligned}$$

Therefore, the functions $H_k(e^{i\varphi})$ satisfy the linear ordinary differential equation

$$\frac{d^2}{d\varphi^2} H_k(e^{i\varphi}) + k^2 H_k(e^{i\varphi}) = 0, \qquad 0 \le \varphi \le 2\pi.$$

This means that

$$H_k(e^{i\varphi}) = A_k \cos k\varphi + B_k \sin k\varphi, \qquad A_k, B_k \in \mathbb{R}$$

holds true if $k \neq 0$ is correct. Since the function H_k is periodic in $[0, 2\pi]$, we infer $k \in \mathbb{Z}$. In the case $k = 0$ we obtain the solution

$$H_0(e^{i\varphi}) = A_0 + B_0\varphi, \qquad A_0, B_0 \in \mathbb{R}.$$

Therefore, $B_0 = 0$ holds true, and the theorem is proved. q.e.d.

5 Theory of Spherical Harmonics in n Variables

Theorem 4.3 from Section 4 suggests the following definition of the spherical harmonics in $\mathbb{R}^n$:

Definition 5.1. *Let $H_k = H_k(x_1, \ldots, x_n) \in C^2(\dot{\mathbb{R}}^n)$ denote a harmonic function on the set $\dot{\mathbb{R}}^n := \mathbb{R}^n \setminus \{0\}$ which is homogeneous of degree k, more precisely*

$$H_k(tx_1, \ldots, tx_n) = t^k H(x_1, \ldots, x_n) \qquad \text{for all} \quad x \in \dot{\mathbb{R}}^n, \quad t \in (0, +\infty).$$

Then we name

$$H_k = H_k(\xi_1, \ldots, \xi_n) : S^{n-1} \to \mathbb{R}$$

an n-dimensional spherical harmonic (or spherically harmonic function) of degree *k; here the symbol*

$$S^{n-1} := \{\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \;:\; \xi_1^2 + \ldots + \xi_n^2 = 1\}$$

denotes the $(n-1)$-dimensional unit sphere in the Euclidean space $\mathbb{R}^n$.

In this paragraph we answer the following questions for $n \geq 2$:

1. Are there spherical harmonics in all spatial dimensions, and for which degrees of homogeneity k do they exist?
2. Is the system of spherically harmonic functions complete?
3. In which relationship do the spherical harmonics appear with respect to the Laplace operator?

In Chapter 1, Section 8 we have represented the Laplace operator in $\mathbb{R}^n$ with respect to spherical coordinates. We utilize $r \in (0, +\infty)$ and $\xi = (\xi_1, \ldots, \xi_n) \in S^{n-1}$, and the function $u = u(r\xi)$ satisfies the identity

$$\boldsymbol{\Delta} u(r\xi) = \frac{\partial^2}{\partial r^2} u(r\xi) + \frac{n-1}{r} \frac{\partial}{\partial r} u(r\xi) + \frac{1}{r^2} \boldsymbol{\Lambda} u(r\xi); \tag{1}$$

here the symbol $\boldsymbol{\Lambda}$ denotes the invariant Laplace-Beltrami operator on the sphere S^{n-1}. We now endow the space of functions $C^0(S^{n-1}, \mathbb{R})$ with the inner product

$$(u, v) := \int\limits_{S^{n-1}} u(\xi) v(\xi)\, d\sigma(\xi), \qquad u, v \in C^0(S^{n-1}, \mathbb{R}) \tag{2}$$

and we obtain a pre-Hilbert-space $\mathcal{H} = C^0(S^{n-1}, \mathbb{R})$. Setting

$$\|u\| := \sqrt{(u, u)}$$

the set $\mathcal{H}$ becomes a normed space.

Theorem 5.2. *The function*

$$H_k = H_k(\xi_1, \ldots, \xi_n) : S^{n-1} \to \mathbb{R}$$

is an n-dimensional spherical harmonic of the degree $k \in \mathbb{R}$ if and only if the following differential equation

$$\boldsymbol{\Lambda} H_k(\xi) + k\Big\{k + (n-2)\Big\} H_k(\xi) = 0, \qquad \xi \in S^{n-1} \tag{3}$$

is satisfied. If H_k and H_l are two spherical harmonics with different degrees $k \neq l$ satisfying $k + l \neq 2 - n$, we then have the orthogonality relation

$$(H_k, H_l) = 0. \tag{4}$$

Proof:

1. On account of (1) we have the identity

$$\begin{aligned} 0 = \boldsymbol{\Delta} H_k(r\xi) &= \boldsymbol{\Delta}\Big\{r^k H_k(\xi)\Big\} \\ &= \Big\{k(k-1)r^{k-2} + k(n-1)r^{k-2}\Big\} H_k(\xi) + r^{k-2} \boldsymbol{\Lambda} H_k(\xi) \end{aligned}$$

and equivalently

$$\boldsymbol{\Lambda} H_k(\xi) + \Big\{k^2 + (n-2)k\Big\} H_k(\xi) = 0, \qquad \xi \in S^{n-1}.$$

2. The symmetry of the operator $\boldsymbol{\Lambda}$ from Theorem 8.7 in Chapter 1, Section 8 yields

$$\begin{aligned}\Big\{k^2 + (n-2)k\Big\} \int\limits_{S^{n-1}} H_k(\xi) H_l(\xi)\, d\sigma(\xi) \\ &= -\int\limits_{S^{n-1}} \Big(\boldsymbol{\Lambda} H_k(\xi)\Big) H_l(\xi)\, d\sigma(\xi) \\ &= -\int\limits_{S^{n-1}} H_k(\xi) \Big(\boldsymbol{\Lambda} H_l(\xi)\Big)\, d\sigma(\xi) \\ &= \Big\{l^2 + (n-2)l\Big\} \int\limits_{S^{n-1}} H_k(\xi) H_l(\xi)\, d\sigma(\xi).\end{aligned}$$

This implies that

$$0 = \Big\{k^2 - l^2 + (n-2)(k-l)\Big\}(H_k, H_l) = \{k-l\}\{k+l+n-2\}(H_k, H_l)$$

and therefore $(H_k, H_l) = 0$ if $k \neq l$ and $k + l \neq 2 - n$ is fulfilled.

q.e.d.

Remarks: The spherical harmonics of the degree k are consequently eigenfunctions of the Laplace-Beltrami operator $\boldsymbol{\Lambda}$ on the sphere S^{n-1} to the eigenvalue $-k\{k + (n-2)\}$. The orthogonality condition (4) is especially satisfied in the case $k \geq 0$, $l \geq 0$ and $k \neq l$.

At this moment we do not yet know for which degrees $k \in \mathbb{R}$ (nonvanishing) spherical harmonics of the degree k exist. This will be investigated now: Given the continuous boundary function, we shall construct a harmonic function with the aid of Poisson's integral and shall decompose this function into homogeneous harmonic functions of the degrees $k = 0, 1, 2, \ldots$. Here we have to expand Poisson's kernel suitably with the aid of power series.

We take $\nu > 0$ as fixed and choose $h = \cos\vartheta \in [-1, +1]$ with $\vartheta \in [0, \pi]$; then we consider the following expression in $t \in (-1, +1)$:

$$
\begin{aligned}
(1-2ht+t^2)^{-\nu} &= (1-2(\cos\vartheta)t+t^2)^{-\nu} \\
&= (1-e^{i\vartheta}t)^{-\nu}(1-e^{-i\vartheta}t)^{-\nu} \\
&= \left\{\sum_{m=0}^{\infty}\binom{-\nu}{m}(-e^{i\vartheta}t)^m\right\}\left\{\sum_{m=0}^{\infty}\binom{-\nu}{m}(-e^{-i\vartheta}t)^m\right\} \\
&= \left\{\sum_{m=0}^{\infty}\begin{bmatrix}\nu\\ m\end{bmatrix}e^{im\vartheta}t^m\right\}\left\{\sum_{m=0}^{\infty}\begin{bmatrix}\nu\\ m\end{bmatrix}e^{-im\vartheta}t^m\right\}.
\end{aligned}
$$

Here we set

$$
\begin{aligned}
\begin{bmatrix}\nu\\ m\end{bmatrix} &:= \binom{-\nu}{m}(-1)^m = \frac{-\nu(-\nu-1)(-\nu-2)\ldots(-\nu-m+1)}{m!}(-1)^m \\
&= \frac{\nu(\nu+1)(\nu+2)\ldots(\nu+m-1)}{m!}, \qquad m\in\mathbb{N}, \\
\begin{bmatrix}\nu\\ 0\end{bmatrix} &:= 1.
\end{aligned}
$$

Defining the real coefficients

$$
\begin{aligned}
c_m^{(\nu)}(h) &:= \sum_{k=0}^{m}\begin{bmatrix}\nu\\ k\end{bmatrix}\begin{bmatrix}\nu\\ m-k\end{bmatrix}e^{ik\vartheta}e^{-i(m-k)\vartheta} \\
&= \sum_{k=0}^{m}\begin{bmatrix}\nu\\ k\end{bmatrix}\begin{bmatrix}\nu\\ m-k\end{bmatrix}e^{-i(m-2k)\vartheta} \\
&= \frac{1}{2}\sum_{k=0}^{m}\begin{bmatrix}\nu\\ k\end{bmatrix}\begin{bmatrix}\nu\\ m-k\end{bmatrix}\left\{e^{i(m-2k)\vartheta}+e^{-i(m-2k)\vartheta}\right\} \\
&= \sum_{k=0}^{m}\begin{bmatrix}\nu\\ k\end{bmatrix}\begin{bmatrix}\nu\\ m-k\end{bmatrix}\cos(m-2k)\vartheta,
\end{aligned}
$$

we obtain the following identity for $t\in(-1,+1)$:

$$
(1-2ht+t^2)^{-\nu} = \sum_{m=0}^{\infty}c_m^{(\nu)}(h)t^m, \qquad t\in(-1,+1). \tag{5}
$$

On account of the Binomial Theorem, we have the following expansion for $p\in\mathbb{Z}$:

$$\cos p\vartheta = \frac{1}{2}\Big(e^{ip\vartheta} + e^{-ip\vartheta}\Big) \;=\; \frac{1}{2}\Big\{(e^{i\vartheta})^p + (e^{-i\vartheta})^p\Big\}$$
$$= \frac{1}{2}\Big\{(\cos\vartheta + i\sin\vartheta)^p + (\cos\vartheta - i\sin\vartheta)^p\Big\}$$
$$= (\cos\vartheta)^p - \binom{p}{2}(\cos\vartheta)^{p-2}(\sin\vartheta)^2 + \binom{p}{4}(\cos\vartheta)^{p-4}(\sin\vartheta)^4 - \ldots$$

Due to the formula $\sin^2\vartheta = 1 - \cos^2\vartheta$, *Gegenbaur's polynomials* $c_m^{(\nu)}(h)$ are polynomials in $h = \cos\vartheta$ of the degree m. Furthermore, we utilize the relation

$$\sum_{m=0}^{\infty} c_m^{(\nu)}(-h)(-t)^m = (1 - 2ht + t^2)^{-\nu} = \sum_{m=0}^{\infty} c_m^{(\nu)}(h)t^m,$$

and comparison of the coefficients yields

$$c_m^{(\nu)}(-h) = (-1)^m c_m^{(\nu)}(h), \qquad m = 0, 1, 2, \ldots \tag{6}$$

Therefore, Gegenbaur's polynomials can be represented in the form

$$c_m^{(\nu)}(h) = \gamma_m^{(\nu)} h^m + \gamma_{m-2}^{(\nu)} h^{m-2} + \ldots \tag{7}$$

with the real constants $\gamma_m^{(\nu)}, \gamma_{m-2}^{(\nu)}, \ldots$. Furthermore, we have the estimate

$$\left|c_m^{(\nu)}(h)\right| \le \sum_{k=0}^{m} \begin{bmatrix}\nu\\ k\end{bmatrix}\begin{bmatrix}\nu\\ m-k\end{bmatrix} = c_m^{(\nu)}(1) \qquad \text{for all} \quad h \in [-1,+1]. \tag{8}$$

With $\nu = \frac{1}{2}$ we obtain the *Legendre polynomials* by $c_m^{(\frac{1}{2})}(h)$. We now choose $n \in \mathbb{N} \setminus \{1\}$. With the aid of (5) we expand as follows for $t \in (-1,+1)$ and $h \in [-1,+1]$:

$$\frac{1-t^2}{(1-2ht+t^2)^{\frac{n}{2}}} = \sum_{m=0}^{\infty} c_m^{(\frac{n}{2})}(h)(1-t^2)t^m =: \sum_{m=0}^{\infty} P_m(h;n)t^m. \tag{9}$$

For the case $n = 2$ we have derived the following expansion in the proof of Theorem 4.1 from Section 4 (compare the formula (6)):

$$\frac{1-t^2}{1-2ht+t^2} = 1 + 2\sum_{m=1}^{\infty}(\cos m\vartheta)t^m, \qquad t \in (-1,+1). \tag{10}$$

Therefore, we have $P_0(h;2) = 1$ and $P_m(h;2) = 2\cos m\vartheta$, $m = 1, 2, \ldots$. For the case $n \ge 3$ we calculate

$$\left(1 + \frac{2t}{n-2}\frac{\partial}{\partial t}\right)\frac{1}{(1-2ht+t^2)^{\frac{n}{2}-1}} = \frac{1 - 2ht + t^2 + \frac{2-n}{2}\frac{2t}{n-2}(-2h+2t)}{(1-2ht+t^2)^{\frac{n}{2}}}$$
$$= \frac{1-t^2}{(1-2ht+t^2)^{\frac{n}{2}}}.$$

Therefore, we have the identity

$$\frac{1-t^2}{(1-2ht+t^2)^{\frac{n}{2}}} = \left(1+\frac{2t}{n-2}\frac{\partial}{\partial t}\right)\frac{1}{(1-2ht+t^2)^{\frac{n}{2}-1}}, \qquad t \in (-1,+1). \quad (11)$$

Together with (9) we infer

$$\sum_{m=0}^{\infty} P_m(h;n)t^m = \frac{1-t^2}{(1-2ht+t^2)^{\frac{n}{2}}} = \left(1+\frac{2t}{n-2}\frac{\partial}{\partial t}\right)\sum_{m=0}^{\infty} c_m^{(\frac{n}{2}-1)}(h)t^m,$$

and comparision of the coefficients yields the formula

$$P_m(h;n) = c_m^{(\frac{n}{2}-1)}(h)\left(\frac{2m}{n-2}+1\right), \qquad m = 0,1,2,\ldots \quad (12)$$

The relations (8) and (12) imply the estimate

$$|P_m(h;n)| \le P_m(1;n), \qquad h \in [-1,+1], \quad m \in \{0,1,2,\ldots\}. \quad (13)$$

This inequality holds true for $n = 2,3,\ldots$

We now can expand the Poisson kernel: We choose $\eta \in S^{n-1}$ as fixed and $x = r\xi$ with $r \in [0,1)$ and $\xi \in S^{n-1}$ to be variable. We utilize the parameter of homogeneity $\tau \in \mathbb{R}$ with $|\tau r| < 1$, and obtain the following relation with the aid of the expansion (9):

$$\begin{aligned}
\frac{|\eta|^2 - |\tau x|^2}{|\eta - \tau x|^n} &= \frac{1-(\tau r)^2}{\Big\{|\eta - (\tau r)\xi|^2\Big\}^{\frac{n}{2}}} \\
&= \frac{1-(\tau r)^2}{\Big\{1-2(\tau r)(\xi,\eta)+(\tau r)^2\Big\}^{\frac{n}{2}}} \qquad (14) \\
&= \sum_{m=0}^{\infty}\Big\{P_m\Big((\xi,\eta);n\Big)r^m\Big\}\tau^m.
\end{aligned}$$

For each $x \in \mathbb{R}^n$ with $|x| < 1$ and each $\tau \in \mathbb{R}$ with $|\tau x| < 1$ we have the identity

$$0 = \Delta_x\left\{\frac{|\eta|^2 - |\tau x|^2}{|\eta - \tau x|^n}\right\} = \sum_{m=0}^{\infty} \Delta_x\Big\{P_m\Big((\xi,\eta);n\Big)r^m\Big\}\tau^m.$$

Taking $\eta \in S^{n-1}$ fixed, the comparison of coefficients yields

$$\Delta_x\Big\{P_m\Big((\xi,\eta);n\Big)r^m\Big\} = 0, \qquad |x| < 1, \quad m = 0,1,2,\ldots. \quad (15)$$

On account of (7) and (12) we have the representation

$$P_m\Big((\xi,\eta);n\Big)r^m = \Big(\pi_m^{(m)}(\xi,\eta)^m + \pi_{m-2}^{(m)}(\xi,\eta)^{m-2} + \ldots\Big)r^m$$
$$= \pi_m^{(m)}(x,\eta)^m + \pi_{m-2}^{(m)}(x,\eta)^{m-2}|x|^2 + \ldots$$

with the real constants $\pi_m^{(m)}, \pi_{m-2}^{(m)}, \ldots$. Therefore, $P_m((\xi,\eta);n)r^m$ is a homogeneous polynomial of the degree m in the variables $x_1,\ldots,x_n$. On account of (15), we obtain an n-dimensional spherical harmonic of the degree $m \in \{0,1,2,\ldots\}$ with $P_m((\xi,\eta);n)$ for each fixed $\eta \in S^{n-1}$. Given the function $f = f(\eta) : S^{n-1} \to \mathbb{R} \in C^0(S^{n-1},\mathbb{R})$, then the integral

$$\tilde{f}(\xi) := \frac{1}{\omega_n}\int\limits_{|\eta|=1} P_m\Big((\xi,\eta);n\Big)f(\eta)\,d\sigma(\eta), \qquad \xi \in S^{n-1}$$

represents an n-dimensional spherical harmonic of the degree m. Here $\tilde{f}(\xi)r^m$ means a homogeneous polynomial in the variables $x_1,\ldots,x_n$.

Theorem 5.3. *Let the function $f = f(x) : S^{n-1} \to \mathbb{R} \in C^0(S^{n-1},\mathbb{R})$ be prescribed, and the function $u = u(x) : B := \{x \in \mathbb{R}^n \,:\, |x| < 1\} \to \mathbb{R}$ of the class $C^2(B) \cap C^0(\overline{B})$ solves the Dirichlet problem*

$$\Delta u(x) = 0 \qquad \text{for all} \quad x \in B,$$
$$u(x) = f(x) \qquad \text{for all} \quad x \in \partial B = S^{n-1}.$$

For each $R \in (0,1)$ we then have the representation

$$u(x) = \sum_{m=0}^{\infty}\left\{\frac{1}{\omega_n}\int\limits_{|\eta|=1} P_m\Big(\xi_1\eta_1 + \ldots + \xi_n\eta_n;n\Big)f(\eta)\,d\sigma(\eta)\right\}r^m \tag{16}$$

with $x = r\xi$, $\xi \in S^{n-1}$ and $0 \le r \le R$. The series on the right-hand side converges uniformly.

Proof: The unique solution of the Dirichlet problem above is given by Poisson's integral. With the aid of the expansion (14) for $\tau = 1$ we infer

$$u(x) = \frac{1}{\omega_n}\int\limits_{|\eta|=1} \frac{|\eta|^2 - |x|^2}{|\eta - x|^n} f(\eta)\,d\sigma(\eta)$$
$$= \frac{1}{\omega_n}\int\limits_{|\eta|=1} \left\{\sum_{m=0}^{\infty} P_m\Big((\xi,\eta);n\Big)r^m\right\}f(\eta)\,d\sigma(\eta), \qquad x \in B.$$

For all $\xi,\eta \in S^{n-1}$ and $0 \le r \le R < 1$ we obtain the inequality

$$\Big|\sum_{m=0}^{\infty} P_m\Big((\xi,\eta);n\Big)r^m\Big| \le \sum_{m=0}^{\infty}\Big|P_m\Big((\xi,\eta);n\Big)\Big|r^m \le \sum_{m=0}^{\infty} P_m(1;n)R^m$$

$$= \frac{1-R^2}{(1-2R+R^2)^{\frac{n}{2}}} = \frac{1+R}{(1-R)^{n-1}}$$

respecting (9) and (13). Due to the Weierstraß majorant test, the following series

$$\sum_{m=0}^{\infty} P_m\Big((\xi,\eta);n\Big)r^m$$

converges uniformly on $S^{n-1}\times S^{n-1}\times[0,R]$ for all $R\in(0,1)$. This implies

$$u(x) = \sum_{m=0}^{\infty}\left\{\frac{1}{\omega_n}\int\limits_{|\eta|=1} P_m\Big(\xi_1\eta_1+\ldots+\xi_n\eta_n;n\Big)f(\eta)\,d\sigma(\eta)\right\}r^m, \qquad |x|\le R,$$

where the given series converges uniformly for all $R\in(0,1)$.

q.e.d.

We choose $k=0,1,2,\ldots$ and denote by

$$\mathcal{M}_k := \Big\{f: S^{n-1}\to\mathbb{R} : f \text{ is } n\text{-dimensional spherical harmonic of degree } k\Big\}$$

the *linear space of the n-dimensional spherical harmonics of the order k*. We already know $\dim\mathcal{M}_k\ge 1$ for $k=0,1,2,\ldots$ and intend to show $\dim\mathcal{M}_k < +\infty$ in the sequel. For the function $f=f(\eta)\in\mathcal{H}=C^0(S^{n-1},\mathbb{R})$ we define the *projector on $\mathcal{M}_k$* by

$$\boldsymbol{P}_k f(\xi) = \hat{f}(\xi) := \frac{1}{\omega_n}\int\limits_{|\eta|=1} P_k\Big(\xi_1\eta_1+\ldots+\xi_n\eta_n;n\Big)f(\eta)\,d\sigma(\eta).$$

Theorem 5.4. *For each integer $k=0,1,2,\ldots$ the linear operator $\boldsymbol{P}_k:\mathcal{H}\to\mathcal{H}$ has the following properties:*

a) $(\boldsymbol{P}_k f,g)=(f,\boldsymbol{P}_k g)$ for all $f,g\in\mathcal{H}$;
b) $\boldsymbol{P}_k(\mathcal{H})=\mathcal{M}_k$;
c) $\boldsymbol{P}_k\circ\boldsymbol{P}_k=\boldsymbol{P}_k$.

Proof:

a) Let the functions $f,g\in\mathcal{H}$ be chosen arbitrarily. Then we have

$$
\begin{aligned}
(\boldsymbol{P}_k f, g) &= \int\limits_{|\xi|=1} \boldsymbol{P}_k f(\xi) g(\xi)\, d\sigma(\xi) \\
&= \int\limits_{|\xi|=1} \int\limits_{|\eta|=1} P_k(\xi_1\eta_1 + \ldots + \xi_n\eta_n) f(\eta) g(\xi)\, d\sigma(\eta)\, d\sigma(\xi) \\
&= (f, \boldsymbol{P}_k g).
\end{aligned}
$$

b) and c) In our considerations preceding Theorem 5.3 we already have seen that

$$\hat{f}(\xi) = \boldsymbol{P}_k f(\xi) \quad \in \mathcal{M}_k \qquad \text{for all} \quad f \in \mathcal{H}.$$

Therefore, we have $\boldsymbol{P}_k(\mathcal{H}) \subset \mathcal{M}_k$. Choosing $f \in \mathcal{M}_k$ arbitrarily we infer $\Delta_x(f(\xi)r^k) = 0$ in $\dot{\mathbb{R}}^n$ with $x = r\xi$. Now our Theorem 5.3 yields the representation

$$f(\xi)r^k = \sum_{m=0}^{\infty} \Big(\boldsymbol{P}_m f(\xi)\Big) r^m, \quad \xi \in S^{n-1}, \quad r \in [0,1).$$

Comparison of the coefficients implies

$$f(\xi) = \boldsymbol{P}_k f(\xi), \qquad \xi \in S^{n-1}.$$

Consequently, we obtain $\mathcal{M}_k \subset \boldsymbol{P}_k(\mathcal{H})$ and $\boldsymbol{P}_k \circ \boldsymbol{P}_k = \boldsymbol{P}_k$. q.e.d.

We now show that $\dim \mathcal{M}_k \in \mathbb{N}$ for $k = 0,1,2,\ldots$ is correct. For a fixed index $k \in \{0,1,2,\ldots\}$ we choose an orthonormal system $\{\varphi_\alpha\}_{\alpha=1,\ldots,N}$ of dimension $N \in \mathbb{N}$ in the linear subspace $\mathcal{M}_k \subset \mathcal{H}$. Then we have

$$(\varphi_\alpha, \varphi_\beta) = \delta_{\alpha\beta} \qquad \text{for all} \quad \alpha, \beta \in \{1,\ldots,N\}$$

and

$$\boldsymbol{P}_k \varphi_\alpha(\xi) = \varphi_\alpha(\xi), \qquad \alpha = 1,\ldots,N.$$

For each $\xi \in S^{n-1}$ we infer

$$\int\limits_{|\eta|=1} \frac{1}{\omega_n} P_k\Big((\xi,\eta);n\Big) \varphi_\alpha(\eta)\, d\sigma(\eta) = \varphi_\alpha(\xi), \qquad \alpha = 1,\ldots,N.$$

Bessel's inequality now yields

$$
\begin{aligned}
\sum_{\alpha=1}^{N} \varphi_\alpha^2(\xi) &= \sum_{\alpha=1}^{N} \left\{ \int\limits_{|\eta|=1} \frac{1}{\omega_n} P_k\Big((\xi,\eta);n\Big) \varphi_\alpha(\eta)\, d\sigma(\eta) \right\}^2 \\
&\leq \int\limits_{|\eta|=1} \left\{ \frac{1}{\omega_n} P_k\Big((\xi,\eta);n\Big) \right\}^2 d\sigma(\eta) \qquad \text{for all} \quad \xi \in S^{n-1}.
\end{aligned}
$$

Therefore, we have

$$N = \int\limits_{|\xi|=1} \sum_{\alpha=1}^{N} \varphi_\alpha^2(\xi)\, d\sigma(\xi)$$
$$\leq \int\limits_{|\xi|=1} \int\limits_{|\eta|=1} \left\{ \frac{1}{\omega_n} P_k\Big((\xi,\eta);n\Big) \right\}^2 d\sigma(\eta)\, d\sigma(\xi).$$

Consequently, we get the following estimate for the dimension of $\mathcal{M}_k$, namely

$$\dim \mathcal{M}_k \leq \int\limits_{|\xi|=1} \int\limits_{|\eta|=1} \left\{ \frac{1}{\omega_n} P_k\Big((\xi,\eta);n\Big) \right\}^2 d\sigma(\eta)\, d\sigma(\xi) < +\infty, \quad k = 0,1,2,\ldots \tag{17}$$

We now set $N = N(k,n) := \dim \mathcal{M}_k$ and choose N orthonormal functions $H_{k1}(\xi), \ldots, H_{kN}(\xi)$ in $\mathcal{M}_k$ spanning the vector space $\mathcal{M}_k$. Each element $f \in \mathcal{M}_k$ can be represented in the form

$$f(\xi) = c_1 H_{k1}(\xi) + \ldots + c_N H_{kN}(\xi), \qquad \xi \in S^{n-1},$$

with the real coefficients $c_j = c_j[f]$ for $j = 1, \ldots, N$. More generally, taking $f = f(\xi) \in \mathcal{H}$ we have the identity

$$\frac{1}{\omega_n} \int\limits_{|\eta|=1} P_k\Big((\xi,\eta);n\Big) f(\eta)\, d\sigma(\eta) = c_1[f] H_{k1}(\xi) + \ldots + c_N[f] H_{kN}(\xi)$$

with the real constants $c_1[f], \ldots, c_N[f]$. This implies

$$c_l[f] = \int\limits_{|\xi|=1} H_{kl}(\xi) \left\{ \frac{1}{\omega_n} \int\limits_{|\eta|=1} P_k\Big((\xi,\eta);n\Big) f(\eta)\, d\sigma(\eta) \right\} d\sigma(\xi)$$
$$= \int\limits_{|\eta|=1} f(\eta) \left\{ \frac{1}{\omega_n} \int\limits_{|\xi|=1} P_k\Big((\xi,\eta);n\Big) H_{kl}(\xi)\, d\sigma(\xi) \right\} d\sigma(\eta)$$
$$= \int\limits_{|\eta|=1} f(\eta) H_{kl}(\eta)\, d\sigma(\eta).$$

Therefore, we obtain

$$\frac{1}{\omega_n} \int\limits_{|\eta|=1} P_k\Big((\xi,\eta);n\Big) f(\eta)\, d\sigma(\eta) = \int\limits_{|\eta|=1} \left\{ \sum_{l=1}^{N(k,n)} H_{kl}(\xi) H_{kl}(\eta) \right\} f(\eta)\, d\sigma(\eta)$$

and consequently

$$\int\limits_{|\eta|=1} \left\{ \frac{1}{\omega_n} P_k\Big((\xi,\eta);n\Big) - \sum_{l=1}^{N(k,n)} H_{kl}(\xi)H_{kl}(\eta) \right\} f(\eta)\, d\sigma(\eta) = 0$$

for all $\xi \in S^{n-1}$ and each $f = f(\eta) \in \mathcal{H}$. Since the functions $P_k((\xi,\eta);n)$ and $H_{kl}(\xi)$ are continuous, we get the *addition theorem for the n-dimensional spherical harmonics*

$$\sum_{l=1}^{N(k,n)} H_{kl}(\xi)H_{kl}(\eta) = \frac{1}{\omega_n} P_k\Big(\xi_1\eta_1 + \ldots + \xi_n\eta_n; n\Big), \qquad \xi,\eta \in S^{n-1} \tag{18}$$

for $k = 0,1,2,\ldots$ and $n = 2,3,\ldots$. We insert $\xi = \eta$ into (18) and integrate over the unit sphere S^{n-1}. Then we obtain

$$N(k,n) = \int\limits_{|\xi|=1} \sum_{l=1}^{N(k,n)} \Big(H_{kl}(\xi)\Big)^2 d\sigma(\xi) = P_k(1;n).$$

On account of (9), we finally deduce the expansion

$$\sum_{k=0}^{\infty} N(k,n)t^k = \sum_{k=0}^{\infty} P_k(1;n)t^k = \frac{1-t^2}{(1-t)^n} = \frac{1+t}{(1-t)^{n-1}}, \qquad |t| < 1.$$

We summarize our results as follows:

Theorem 5.5. *I. The cardinality $N(k,n)$ of all linear independent spherical harmonics in $\mathbb{R}^n$ of the order k is finite. The number $N(k,n) = \dim \mathcal{M}_k$ is determined by the equation*

$$\frac{1+t}{(1-t)^{n-1}} = \sum_{k=0}^{\infty} N(k,n)t^k, \quad |t| < 1. \tag{19}$$

II. Let $H_{k1}(\xi),\ldots,H_{kN}(\xi)$ represent the $N = N(k,n)$ orthonormal spherical harmonics of the order k, which means

$$\int\limits_{|\xi|=1} H_{kl}(\xi)H_{kl'}(\xi)\, d\sigma(\xi) = \delta_{ll'} \qquad \textit{for} \quad l,l' \in \{1,\ldots,N\} \tag{20}$$

is satisfied. Then we have the representation

$$\sum_{l=1}^{N(k,n)} H_{kl}(\xi)H_{kl}(\eta) = \frac{1}{\omega_n} P_k\Big(\xi_1\eta_1 + \ldots + \xi_n\eta_n; n\Big) \tag{21}$$

for all $\xi,\eta \in S^{n-1}$. Here the functions $P_k(h;n)$ are defined by the equation

$$\frac{1-t^2}{(1-2ht+t^2)^{\frac{n}{2}}} = \sum_{k=0}^{\infty} P_k(h;n)t^k, \qquad -1 < t < +1, \quad -1 \le h \le +1. \tag{22}$$

III. Each solution $u = u(x) \in C^2(B) \cap C^0(\overline{B})$ *of Dirichlet's problem*

$$\begin{aligned} \Delta u(x) &= 0 \qquad \text{in} \quad B, \\ u(x) &= f(x) \qquad \text{on} \quad \partial B = S^{n-1} \end{aligned}$$

possesses the representation as uniformly convergent series

$$u(x) = \sum_{k=0}^{\infty} \left\{ \sum_{l=1}^{N(k,n)} \left(\int_{|\eta|=1} f(\eta) H_{kl}(\eta) \, d\sigma(\eta) \right) H_{kl}(\xi) \right\} r^k \tag{23}$$

with $x = r\xi$, $\xi \in S^{n-1}$ *and* $0 \le r \le R$*; here* $R \in (0,1)$ *can be chosen arbitrarily.*

Proof: Statement III immediately follows from (18) together with Theorem 5.3. q.e.d.

Analogously to Theorem 4.1 from Section 4, we obtain the following result for arbitrary dimensions $n \ge 2$:

Theorem 5.6. (Completeness of spherical harmonics)
The n*-dimensional spherical harmonics* $\{H_{kl}(\xi)\}_{k=0,1,2,\ldots;\ l=1,\ldots,N(k,n)}$ *constitute a complete orthonormal system of functions in* $\mathcal{H}$*. More precisely,*

$$(H_{kl}, H_{k'l'}) = \delta_{kk'}\delta_{ll'}, \qquad k, k' = 0, 1, 2, \ldots, \quad l, l' = 1, \ldots, N(k,n)$$

holds true, and for each element $f \in \mathcal{H}$ *we have the relation*

$$\lim_{M \to \infty} \left\| f(\xi) - \sum_{k=0}^{M} \sum_{l=1}^{N(k,n)} f_{kl} H_{kl}(\xi) \right\| = 0$$

or equivalently

$$\|f\|^2 = \sum_{k=0}^{\infty} \sum_{l=1}^{N(k,n)} f_{kl}^2.$$

Here we have used the following abbreviations

$$f_{kl} := (f, H_{kl}), \qquad k = 0, 1, 2, \ldots, \quad l = 1, \ldots, N(k,n)$$

for the Fourier coefficients.

Proof: We have only to show the completeness for the system of the n-dimensional spherical harmonics. To each element $f \in \mathcal{H}$ we have a function $u = u(x)$ with the following properties:

1. the function u is harmonic for all $|x| < 1$;
2. the function u is continuous for $|x| \le 1$ and satisfies the boundary condition

$$u(x) = f(x) \qquad \text{for all} \quad |x| = 1.$$

According to Theorem 5.5, Statement III we see: For each $\varepsilon > 0$ there exists a radius $r \in (0,1)$ and an index $M = M(\varepsilon) \in \mathbb{N}$, such that

$$\left| f(\xi) - \sum_{k=0}^{M(\varepsilon)} r^k \sum_{l=1}^{N(k,n)} f_{kl} H_{kl}(\xi) \right| \leq \varepsilon \qquad \text{for all} \quad \xi \in S^{n-1}.$$

This implies

$$\left\| f(\xi) - \sum_{k=0}^{M(\varepsilon)} r^k \sum_{l=1}^{N(k,n)} f_{kl} H_{kl}(\xi) \right\| \leq \sqrt{\omega_n}\, \varepsilon,$$

and the minimal property of the Fourier coefficients yields

$$\left\| f(\xi) - \sum_{k=0}^{M(\varepsilon)} \sum_{l=1}^{N(k,n)} f_{kl} H_{kl}(\xi) \right\| \leq \sqrt{\omega_n}\, \varepsilon.$$

From this relation we immediately infer the statement. q.e.d.

Corollaries from Theorem 5.6:

1. With $f(\xi)$ and $g(\xi)$ we consider two real, continuous functions on S^{n-1}, and then *Parseval's equation*

$$\int\limits_{|\xi|=1} f(\xi) g(\xi)\, d\sigma(\xi) = \sum_{k=0}^{\infty} \sum_{l=1}^{N(k,n)} f_{kl} g_{kl}$$

   holds true with

$$f_{kl} = \int\limits_{|\xi|=1} f(\xi) H_{kl}(\xi)\, d\sigma(\xi), \qquad g_{kl} = \int\limits_{|\xi|=1} g(\xi) H_{kl}(\xi)\, d\sigma(\xi).$$

2. Nontrivial spherical harmonics H_j of the order $j \neq 0, \pm 1, \pm 2, \ldots$ do not exist. Due to Theorem 5.2 such a function would satisfy the orthogonality relations $(H_j, H_{kl}) = 0$. The system of functions $\{H_{kl}\}_{k=0,1,2,\ldots;\ l=1,\ldots,N(k,n)}$ being complete in $\mathcal{H}$, we infer $H_j = 0$ for all $j \neq 0, \pm 1, \pm 2, \ldots$

At the end of this paragraph we shall investigate the relationship of the spherical harmonics to the Laplace operator in $\mathbb{R}^n$. From (1) we infer the decomposition

$$\boldsymbol{\Delta} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \boldsymbol{\Lambda} \qquad \text{in} \quad \mathbb{R}^n.$$

We note (3) and obtain the following identity for arbitrary C^2-functions $f = f(r)$:

$$\begin{aligned} \Delta \Big\{ f(r) H_{kl}(\xi) \Big\} &= \left\{ f''(r) + \frac{n-1}{r} f'(r) - \frac{k(k+(n-2))}{r^2} f(r) \right\} H_{kl}(\xi) \\ &= \Big(L_{k,n} f(r) \Big) H_{kl}(\xi), \qquad l = 1, \ldots, N(k,n) \end{aligned} \tag{24}$$

with the operator

$$L_{k,n}f(r) := \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} - \frac{k(k+(n-2))}{r^2}\right)f(r).$$

Evidently, we have $L_{k,2} = L_k$ with the operator L_k from Section 4.

Let the function $u = u(x_1, \dots, x_n) \in C^2(B_R)$ with $B_R := \{x \in \mathbb{R}^n : |x| < R\}$ be chosen arbitrarily. We now expand u in $\mathcal{H}$ with respect to the spherical harmonics

$$u = u(r\xi) = \sum_{k=0}^{\infty} \sum_{l=1}^{N(k,n)} f_{kl}(r) H_{kl}(\xi), \qquad 0 \le r < R, \quad \xi \in S^{n-1}. \tag{25}$$

Here we utilize the n-dimensional Fourier coefficients

$$f_{kl}(r) := \int\limits_{|\eta|=1} u(r\eta) H_{kl}(\eta)\, d\sigma(\eta), \qquad k = 0, 1, 2, \dots, \quad l = 1, \dots, N(k,n). \tag{26}$$

We then expand the function $\tilde{u}(x) = \Delta u(x)$, $x \in B_R$ in $\mathcal{H}$ with respect to spherical harmonics as well, and we obtain the n-dimensional Fourier series

$$\Delta u(x) = \Delta u(r\xi) = \sum_{k=0}^{\infty} \sum_{l=1}^{N(k,n)} \tilde{f}_{kl}(r) H_{kl}(\xi), \qquad 0 \le r < R, \quad \xi \in S^{n-1}, \tag{27}$$

with the Fourier coefficients $\tilde{f}_{kl}(r) = L_{k,n} f_{kl}(r)$. We consequently obtain the series for Δu in $\mathcal{H}$ by formal differentiation of the series for u. This is the content of the following

Proposition 5.7. *Let the function $u = u(x) \in C^2(B_R)$ be given, and its Fourier coefficients $f_{kl}(r)$ are defined due to the formula (26). Then the Fourier coefficients $\tilde{f}_{kl}(r)$ of Δu, namely*

$$\tilde{f}_{kl}(r) := \int\limits_{|\eta|=1} \Delta u(r\eta) H_{kl}(\eta)\, d\sigma(\eta), \qquad k = 0, 1, 2, \dots, \quad l = 1, \dots, N(k,n),$$

satisfy the identity

$$\tilde{f}_{kl}(r) = L_{k,n} f_{kl}(r), \qquad k = 0, 1, 2, \dots, \quad l = 1, \dots, N(k,n), \tag{28}$$

with $0 \le r < R$.

Proof: We choose $0 \le r < R$, and calculate with the aid of (3) as follows:

$$
\begin{aligned}
\tilde{f}_{kl}(r) &= \int\limits_{|\xi|=1} \Delta u(r\xi) H_{kl}(\xi)\, d\sigma(\xi) \\
&= \int\limits_{|\xi|=1} \left\{ \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\boldsymbol{\Lambda} \right) u(r\xi) \right\} H_{kl}(\xi)\, d\sigma(\xi) \\
&= \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} \right) \int\limits_{|\xi|=1} u(r\xi) H_{kl}(\xi)\, d\sigma(\xi) \\
&\quad + \frac{1}{r^2} \int\limits_{|\xi|=1} u(r\xi) \boldsymbol{\Lambda} H_{kl}(\xi)\, d\sigma(\xi) \\
&= \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} - \frac{k(k+(n-2))}{r^2} \right) \int\limits_{|\xi|=1} u(r\xi) H_{kl}(\xi)\, d\sigma(\xi) \\
&= L_{k,n} f_{kl}(r) \qquad \text{for} \quad k = 0,1,2,\ldots, \quad l = 1,\ldots,N(k,n).
\end{aligned}
$$

q.e.d.

Remark: The most important partial differential equation of the second order in quantum mechanics, namely the Schrödinger equation, contains the Laplacian as its principal part. Therefore, the investigation of eigenvalues of this operator is of central interest. This will be presented in Chapter 8.

Figure 1.7 PORTRAIT OF JOSEPH A. F. PLATEAU (1801–1883) Universitätsbibliothek der Rheinischen Friedrich-Wilhelms-Univerität Bonn; taken from the book by *S. Hildebrandt and A. Tromba: Panoptimum – Mathematische Grundmuster des Vollkommenen*, Spektrum-Verlag Heidelberg (1986).