Chapter 2

Foundations of Functional Analysis

We start with the Riemannian integral - and their Riemann integrable functions - and construct a considerably larger class of integrable functions via an extension procedure. Then we obtain Lebesgue's integral, which is distinguished by general convergence theorems for *pointwise* convergent sequences of functions. This extension procedure - from the Riemannian integral to Lebesgue's integral - will be provided by the Daniell integral. The measure theory for Lebesgue measurable sets will appear in this context as the theory of integration for characteristic functions. We shall present classical results from the theory of measure and integration in this chapter, e.g. the theorems of Egorov and Lusin.

Then we treat the Lebesgue spaces L^p with the exponents $1 \le p \le +\infty$ as classical Banach spaces. We investigate orthogonal systems of functions in the Hilbert space L^2. With ideas of J. von Neumann we determine the dual spaces $(L^p)^* = L^q$ and show the weak compactness of the Lebesgue spaces.

1 Daniell's Integral with Examples

Our point of departure is the following

Definition 1.1. *We consider an arbitrary set X, and by $M = M(X)$ we denote a space of functions $f : X \to \mathbb{R}$ which have the following properties:*

– *M is a linear space, which means*

$$\text{for all } f, g \in M \text{ and all } \alpha, \beta \in \mathbb{R} \text{ we have} \quad \alpha f + \beta g \in M. \tag{1}$$

– *M is closed with respect to the modulus operation, which means*

$$\text{for all } f \in M \text{ we have} \quad |f| \in M. \tag{2}$$

Furthermore, the symbol $I : M \to \mathbb{R}$ denotes a functional on M satisfying the following conditions:

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– *I is linear, which means*

$$\text{for all } f, g \in M \text{ and all } \alpha, \beta \in \mathbb{R} \text{ we have} \quad I(\alpha f + \beta g) = \alpha I(f) + \beta I(g). \tag{3}$$

– *I is nonnegative, which says*

$$\text{for all } f \in M \text{ with } f \geq 0 \text{ we have} \quad I(f) \geq 0. \tag{4}$$

Here the relation $f \geq 0$ indicates that $f(x) \geq 0$ for all $x \in X$ is correct.

– *I is continuous with respect to monotone convergence in M, which means*

$$\begin{aligned} &\text{for each sequence} \{f_n\}_{n=1,2,\ldots} \subset M \quad \text{with} \quad f_n \downarrow 0 \\ &\text{we have} \quad \lim_{n\to\infty} I(f_n) = I(0) = 0. \end{aligned} \tag{5}$$

Here we comprehend by $f_n \downarrow 0$ that the sequence $\{f_n(x)\}_{n=1,2,\ldots} \subset \mathbb{R}$ is weakly monotonically decreasing for all $x \in X$ and $\lim\limits_{n\to\infty} f_n(x) = 0$ holds true.

Then this functional I is named Daniell's integral *defined on M.*

Remarks:

1. From the linearity (1) and the lattice property (2) we infer

$$\max(f, g) = \frac{1}{2}\Big(f + g + |f - g|\Big) \in M$$

as well as

$$\min(f, g) = \frac{1}{2}\Big(f + g - |f - g|\Big) \in M$$

for two elements $f, g \in M$. In particular, with each element $f \in M$ we have

$$f^+(x) := \max\Big(f(x), 0\Big) = \frac{1}{2}\Big(f(x) + |f(x)|\Big) \in M$$

as well as

$$f^-(x) := \max\Big(-f(x), 0\Big) = (-f)^+(x) \in M.$$

We address f^+ as the *positive part of f* and f^- as the *negative part of f*. The definitions of f^+ and f^- imply the identities

$$f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^- = f^+ + (-f)^+.$$

Consequently, the lattice condition (2) is equivalent to

$$f \in M \quad \Longrightarrow \quad f^+ \in M. \tag{2'}$$

More generally, we see that finitely many functions $f_1, \ldots, f_m \in M$ with $m \in \mathbb{N}$ imply the inclusion

$$\max(f_1, \ldots, f_m) \in M \quad \text{and} \quad \min(f_1, \ldots, f_m) \in M.$$

2. The condition (4) is equivalent to the monotonicity of the integral, namely

$$I(f) \geq I(g) \quad \text{for all} \quad f, g \in M \text{ with } f \geq g. \tag{4'}$$

3. The condition (5) is equivalent to the following property:

All sequences $\{f_n\}_{n=1,2,\ldots} \subset M$ with $f_n \uparrow f$ and $f, g \in M$ with $g \leq f$ fulfill
$$I(g) \leq \lim_{n\to\infty} I(f_n). \tag{5'}$$

Proof: At first, we show the direction '(5′) ⇒ (5)'. Let the sequence of functions $\{f_n\}_{n=1,2,\ldots} \subset M$ with $f_n \downarrow 0$ be given. Then we infer $(-f_n) \uparrow 0$. We set $f(x) \equiv 0 \equiv g(x)$. The linearity of I implies $I(g) = 0$ immediately. The combination of (5′) and (4) reveals the relation

$$0 = I(g) \leq \lim_{n\to\infty} I(-f_n) = -\lim_{n\to\infty} \underbrace{I(f_n)}_{\geq 0} \leq 0.$$

This yields $\lim\limits_{n\to\infty} I(f_n) = I(0) = 0$.

Now we show the implication '(5) ⇒ (5′)'.
The sequence $\{f_n\}_{n=1,2,\ldots}$ may satisfy $f_n \uparrow f$ with an element $f \in M$, which immediately implies $(f - f_n) \downarrow 0$. From (5) we infer $0 = \lim_{n\to\infty} I(f - f_n)$, and the linearity of I yields

$$0 = I(f) - \lim_{n\to\infty} I(f_n).$$

With $g \leq f$ and (4′) we obtain

$$\lim_{n\to\infty} I(f_n) = I(f) \geq I(g),$$

and the proof is complete. q.e.d.

Now we provide examples of Daniell integrals, where we need the following

Theorem 1.2. (U. Dini)
Let the continuous functions $f_1, f_2, \ldots$ and $f \in C^0(K, \mathbb{R})$ be defined on the compact set $K \subset \mathbb{R}^n$. We have the relation $f_l \uparrow f$, which means that the sequence $\{f_l(x)\} \subset \mathbb{R}$ is weakly monotonically increasing for all $x \in K$ and furthermore
$$\lim_{l\to\infty} f_l(x) = f(x).$$
Then the sequence $\{f_l\}_{l=1,2,\ldots}$ converges uniformly on the set K towards the function f.

Remark: The transition to functions $g_l := f - f_l$ implies that the statement above is equivalent to the following:

A sequence of functions $\{g_l\}_{l=1,2,\ldots} \subset C^0(K, \mathbb{R})$ with $g_l \downarrow 0$ has necessarily the *property that* $\{g_l\}_{l=1,2,\ldots}$ *converges uniformly on* K *towards* 0.

Proof of Theorem 1.2: Let $\{g_l\}_{l=1,2,\ldots} \subset C^0(K,\mathbb{R})$ denote a sequence satisfying $g_l \downarrow 0$. We have to show that

$$\sup_{x\in K} |g_l(x)| \longrightarrow 0$$

is correct. If this property was not valid, then we could find indices $\{l_i\}$ with $l_i < l_{i+1}$ and points $\xi_i \in K$ such that

$$g_{l_i}(\xi_i) \geq \varepsilon > 0 \qquad \text{for all} \quad i \in \mathbb{N}$$

hold true with a fixed quantity $\varepsilon > 0$. According to the Weierstraß compactness theorem, we can assume - without loss of generality - that the relation $\xi_i \to \xi$ for $i \to \infty$ is valid, with the limit point $\xi \in K$. For the fixed index l_*, we now choose an index $i_* = i(l_*) \in \mathbb{N}$ such that $l_i \geq l_*$holds true for all $i \geq i_*$. Now the monotonicity of the sequence of functions $\{g_l\}$ implies

$$g_{l_*}(\xi_i) \geq g_{l_i}(\xi_i) \geq \varepsilon \qquad \text{for all} \quad i \geq i_*.$$

Since the function g_{l_*} is assumed to be continuous, we infer

$$g_{l_*}(\xi) = \lim_{i\to\infty} g_{l_*}(\xi_i) \geq \varepsilon \qquad \text{for all} \quad l_* \in \mathbb{N}.$$

Therefore, $\{g_l(\xi)\}$ does not constitute a null-sequence, which gives an obvious contradiction to the assumption.

q.e.d.

Main example 1: Let us consider $X = \Omega$ with the open set $\Omega \subset \mathbb{R}^n$ and the linear space

$$M_1 = M_1(X) := \left\{ f(x) \in C^0(\Omega,\mathbb{R}) \,:\, \int_\Omega |f(x)|\,dx < +\infty \right\}.$$

Here the symbol

$$\int_\Omega |f(x)|\,dx$$

means the improper Riemannian integral over the open set Ω. Then our space M_1 satisfies the conditions (1) and (2). Now we choose the functional

$$I_1(f) := \int_\Omega f(x)\,dx, \qquad f \in M_1,$$

where the improper Riemannian integral over Ω appears again on the right-hand side. Because the Riemannian integral is linear and nonnegative, the conditions (3) and (4) are evident. We still have to establish the continuity of our functional with respect to monotone convergence, namely (5). Let us consider with $\{f_n\}_{n=1,2,\ldots} \subset M_1$ a sequence of functions satisfying $f_n \downarrow 0$. If $K \subset \Omega$ denotes a compact subset, Dini's theorem tells us that $\{f_n\}$ converges uniformly on K towards 0. When we observe the properties $0 \le f_n(x) \le f_1(x)$ for all $n \in \mathbb{N}$ and $x \in \Omega$ as well as $\int\limits_\Omega |f_1(x)|\,dx < +\infty$, the fundamental convergence theorem for improper Riemannian integrals implies

$$\lim_{n\to\infty} I_1(f_n) = \lim_{n\to\infty} \int\limits_\Omega f_n(x)\,dx = \int\limits_\Omega \underbrace{\Big(\lim_{n\to\infty} f_n(x)\Big)}_{=0}\,dx = 0.$$

Therefore, I_1 represents a Daniell integral on the space M_1.

Remark: The set M_1 does not contain *all* functions whose improper Riemannian integral exists. The concept of Daniell's integral additionally necessitates the function space being closed with respect to the modulus operation, namely the lattice property (2). For instance, the integral

$$\int\limits_1^\infty \frac{\sin x}{x^\alpha}\,dx \qquad \text{for all powers} \quad \alpha \in (0,1)$$

does not converge absolutely, although it exists as an improper Riemannian integral.

Main example 2: As we described in Section 4 of Chapter 1, let $\mathcal{M} \subset \mathbb{R}^n$ denote a bounded m-dimensional manifold of the class C^1 with the regular boundary $\partial\mathcal{M}$. Then we can cover $\overline{\mathcal{M}}$ by finitely many charts, and we define the Riemannian integral over $\mathcal{M}$ via *partition of unity*, namely

$$I_2(f) := \int\limits_{\overline{\mathcal{M}}} f(x)\,d^m\sigma(x), \qquad f \in M_2$$

for all functions of the class

$$M_2 := \Big\{ f(x) : \overline{\mathcal{M}} \to \mathbb{R} \ : \ f \text{ is continuous on } \overline{\mathcal{M}} \Big\}.$$

Here the symbol $d^m\sigma$ means the m-dimensional surface element on $\mathcal{M}$. This integral I_2 gives us a further interesting Daniell integral: The linear space M_2 is closed with respect to the modulus operation. The properties (1) and (2) are consequently fulfilled. The existence of the integral above follows from the continuity - and therefore the boundedness - of f on the compact manifold $\overline{\mathcal{M}}$. The linearity and the positive-semidefinite character of I_2 are evident. The continuity of I_2 with respect to monotone convergence follows from Dini's theorem again.

2 Extension of Daniell's Integral to Lebesgue's Integral

In our main examples from Section 1, we already have an integral which allows, at least, to integrate the continuous functions with compact support. Now we consider an arbitrary Daniell integral $I : M \to \mathbb{R}$ due to Definition 1.1 in Section 1. We intend to extend this integral onto the larger linear space

$$L(X) \supset M(X),$$

in order to study convergence properties of the created integral on the space $L(X)$. This extension procedure is essentially based on the monotonicity property (4) and the associate continuity property (5) of this integral.

Developing our theory of integration simultaneously for *characteristic functions*

$$\chi_A(x) := \begin{cases} 1, \ x \in A \\ 0, \ x \in X \setminus A \end{cases}$$

of the subsets $A \subset X$, we obtain a measure theory which depends on our Daniell integral I for the subsets of X.

The extension procedure presented here was initiated by Carathéodory, later Daniell considered these particular functionals I, and Stone established the connection to measure theory. The consideration of minimal surfaces gave H. Lebesgue the impetus to study thoroughly the concept of surface area.

We prepare our considerations and introduce the function

$$\Phi(t) := \begin{cases} 0, \ t \le 0 \\ t, \ t \ge 0 \end{cases}$$

which is continuous and weakly monotonically increasing. Furthermore, we define

$$f^+(x) := \Phi(f(x)) = \max\,(f(x), 0), \quad x \in X$$

and study the following properties of the prescription $f \mapsto f^+$:

$$i.)\ f(x) \le f^+(x) \text{ for all } x \in X;$$

$$ii.)\ f_1(x) \le f_2(x) \quad \Longrightarrow \quad f_1^+(x) \le f_2^+(x) \text{ for all } x \in X;$$

$$iii.)\ f_n(x) \to f(x) \quad \Longrightarrow \quad f_n^+(x) \to f^+(x) \text{ for all } x \in X;$$

$$iv.)\ f_n(x) \downarrow f(x) \quad \Longrightarrow \quad f_n^+(x) \downarrow f^+(x) \text{ for all } x \in X;$$

$$v.)\ f_n(x) \uparrow f(x) \quad \Longrightarrow \quad f_n^+(x) \uparrow f^+(x) \text{ for all } x \in X.$$

Proposition 2.1. *Let* $\{g_n\} \subset M$ *and* $\{g'_n\} \subset M$, $n = 1, 2, \ldots$ *denote two sequences satisfying* $g_n(x) \uparrow g(x)$ *and* $g'_n(x) \uparrow g'(x)$ *defined on* X. *Here* $g, g' : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *represent two functions with the property* $g'(x) \geq g(x)$. *Then we infer the inequality*

$$\lim_{n\to\infty} I(g'_n) \geq \lim_{n\to\infty} I(g_n).$$

Proof: Since $\{I(g_n)\}_{n=1,2,\ldots}$ and $\{I(g'_n)\}_{n=1,2,\ldots}$ represent monotonically non-decreasing sequences, their limits exist for $n \to \infty$ in $\mathbb{R} \cup \{+\infty\}$. In the case $\lim\limits_{n\to\infty} I(g'_n) = +\infty$, the inequality above evidently holds true. Therefore, we can assume $\lim\limits_{n\to\infty} I(g'_n) < +\infty$ without loss of generality. With the index m being fixed, we observe

$$(g_m - g'_n)^+ \downarrow (g_m - g')^+ = 0 \qquad \text{for} \quad n \to \infty.$$

Then we invoke the properties of Daniell's integral I as follows:

$$\begin{aligned} I(g_m) - \lim_{n\to\infty} I(g'_n) &= \lim_{n\to\infty} \Big(I(g_m) - I(g'_n)\Big) = \lim_{n\to\infty} I(g_m - g'_n) \\ &\leq \lim_{n\to\infty} I\Big((g_m - g'_n)^+\Big) = 0. \end{aligned}$$

Now we see

$$I(g_m) \leq \lim_{n\to\infty} I(g'_n) \qquad \text{for all} \quad m \in \mathbb{N},$$

and we arrive at the relation

$$\lim_{m\to\infty} I(g_m) \leq \lim_{n\to\infty} I(g'_n).$$

q.e.d.

When we assume $g = g'$ on X in Proposition 2.1, we obtain equality for the two limits above. This justifies the following

Definition 2.2. *Let the symbol* $V(X)$ *denote the set of all functions* $f : X \to \mathbb{R} \cup \{+\infty\}$, *which can be approximated weakly monotonically increasing from* $M(X)$ *as follows: Each such element* f *possesses a sequence* $\{f_n\}_{n=1,2,\ldots}$ *in* $M(X)$ *with the property*

$$f_n(x) \uparrow f(x) \qquad \textit{for} \quad n \to \infty \quad \textit{and for all} \quad x \in X.$$

For the element $f \in V$, *we then define*

$$I(f) := \lim_{n\to\infty} I(f_n),$$

and we observe $I(f) \in \mathbb{R} \cup \{+\infty\}$.

Definition 2.3. *We set*

$$-V := \Big\{ f : X \to \mathbb{R} \cup \{-\infty\} \; : \; -f \in V \Big\}$$

and define

$$I(f) := -I(-f) \in \mathbb{R} \cup \{-\infty\} \qquad \text{for all} \quad f \in -V.$$

Remarks:

1. The set $-V$ represents the set of all functions f which can be approximated weakly monotonically decreasing from M as follows: There exists a sequence $\{f_n\}_{n=1,2,\ldots} \subset M$ satisfying $f_n \downarrow f$. Then we obtain
$$I(f) = \lim_{n\to\infty} I(f_n).$$
2. If $f \in V \cap (-V)$ holds true, we find sequences $\{f_n'\}_{n=1,2,\ldots}$ and $\{f_n''\}_{n=1,2,\ldots}$ in M which fulfill the approximative relations $f_n' \uparrow f$ and $f_n'' \downarrow f$, respectively. Now we see $f_n'' - f_n' \downarrow 0$, and the property (5) implies
$$0 = \lim_{n\to\infty} I(f_n'' - f_n') = \lim_{n\to\infty} I(f_n'') - \lim_{n\to\infty} I(f_n')$$
as well as
$$\lim_{n\to\infty} I(f_n'') = \lim_{n\to\infty} I(f_n').$$
Consequently, the functional I is uniquely defined on the set $V \cup (-V) \supset V \cap (-V) \supset M$.
3. The set V contains the element $f(x) \equiv +\infty$ as the monotonically increasing limit of $f_n(x) = n$; however, it does not contain the element $g(x) \equiv -\infty$. Therefore, the set V does not represent a linear space.

   According to Proposition 2.1, the functional I is monotonic on V as follows: Each two elements $f, g \in V$ with $f \leq g$ fulfill $I(f) \leq I(g)$. Furthermore, the linear combination $\alpha f + \beta g$ of two elements $f, g \in V$ with nonnegative scalars $\alpha \geq 0$ and $\beta \geq 0$ belongs to V as well, and we have
$$I(\alpha f + \beta g) = \alpha I(f) + \beta I(g).$$

Proposition 2.4. *The function $f : X \to [0, +\infty]$ satisfies the equivalence*

$$f \in V \quad \Longleftrightarrow \quad f(x) = \sum_{n=1}^{\infty} \varphi_n(x),$$

where $\varphi_n \in M(X)$ and $\varphi_n \geq 0$ for all $n \in \mathbb{N}$ hold true.

Proof: The direction '$\Longleftarrow$' is evident from the definition of the space V: The element f is constructed monotonically by the functions $\varphi_n \in M$, and this implies the conclusion.

Now we show the opposite direction '$\Longrightarrow$' as follows: Taking $f \in V$, we find a sequence $\{f_n\}_{n=1,2,\ldots} \subset M$ such that $f_n \uparrow f$, and we infer $f_n^+ \uparrow f^+ = f$. When we define

$$f_0(x) \equiv 0 \quad \text{and} \quad \varphi_n(x) := f_n^+(x) - f_{n-1}^+(x),$$

we observe

$$f_k^+(x) = \sum_{n=1}^{k} \varphi_n(x) \uparrow f(x)$$

and consequently

$$\sum_{n=1}^{\infty} \varphi_n(x) = f(x).$$

Obviously, the functions fulfill $\varphi_n(x) \in M$ and $\varphi_n(x) \geq 0$ for all $n \in \mathbb{N}$.

q.e.d.

Proposition 2.5. *Let the elements $f_i \in V$ with $f_i \geq 0$ for $i = 1, 2, \ldots$ be given. Then the function*

$$f(x) := \sum_{i=1}^{\infty} f_i(x)$$

belongs to the set V, and we have

$$I(f) = \sum_{i=1}^{\infty} I(f_i).$$

Proof: The double sequence $c_{ij} \in \mathbb{R}$ with $c_{ij} \geq 0$ satisfies the following equation:

$$\sum_{i,j=1}^{\infty} c_{ij} = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} c_{ij} \right) = \lim_{n \to \infty} \sum_{i,j=1}^{n} c_{ij}. \tag{1}$$

This equation holds true for convergent as well as for definitely divergent double series. On account of $f_i \in V$, we have functions $\varphi_{ij} \in M$ satisfying $\varphi_{ij} \geq 0$ such that

$$f_i(x) = \sum_{j=1}^{\infty} \varphi_{ij}(x) \quad \text{for all} \quad x \in X \quad \text{and all} \quad i \in \mathbb{N}$$

is correct. From Definition 2.2 we infer

$$I(f_i) = \lim_{n\to\infty} I\left(\sum_{j=1}^{n} \varphi_{ij}\right) = \lim_{n\to\infty} \left\{\sum_{j=1}^{n} I(\varphi_{ij})\right\} = \sum_{j=1}^{\infty} I(\varphi_{ij}).$$

Furthermore, we have the following representation for all $x \in X$:

$$f(x) = \sum_{i=1}^{\infty} f_i(x) = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \varphi_{ij}(x)\right) = \sum_{i,j=1}^{\infty} \varphi_{ij}(x) = \lim_{n\to\infty} \left(\sum_{i,j=1}^{n} \varphi_{ij}(x)\right).$$

Consequently, $f \in V$ holds true and Definition 2.2 yields

$$I(f) = \lim_{n\to\infty} I\left(\sum_{i,j=1}^{n} \varphi_{ij}\right) = \lim_{n\to\infty} \sum_{i,j=1}^{n} I(\varphi_{ij})$$

$$= \sum_{i,j=1}^{\infty} I(\varphi_{ij}) = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} I(\varphi_{ij})\right) = \sum_{i=1}^{\infty} I(f_i).$$

q.e.d.

Definition 2.6. *We consider an arbitrary function $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and define*

$$I^+(f) := \inf\Big\{I(h) \,:\, h \in V,\, h \geq f\Big\}, \quad I^-(f) := \sup\Big\{I(g) \,:\, g \in -V,\, g \leq f\Big\}.$$

We name $I^+(f)$ the upper *and $I^-(f)$ the* lower Daniell integral *of f.*

Proposition 2.7. *Let $f : X \to \overline{\mathbb{R}}$ denote an arbitrary function and (g, h) a pair of functions satisfying $g \in -V$ and $h \in V$ as well as $g(x) \leq f(x) \leq h(x)$ for all $x \in X$. Then we infer*

$$I(g) \leq I^-(f) \leq I^+(f) \leq I(h).$$

Proof: Definition 2.6 implies $I(h) \geq I^+(f)$ and $I(g) \leq I^-(f)$. Furthermore, we find sequences $\{g_n\}_{n=1,2,\ldots} \subset -V$ and $\{h_n\}_{n=1,2,\ldots} \subset V$ satisfying

$$g_n \leq f \leq h_n \quad \text{for all} \quad n \in \mathbb{N},$$

such that

$$\lim_{n\to\infty} I(g_n) = I^-(f) \quad \text{and} \quad \lim_{n\to\infty} I(h_n) = I^+(f)$$

holds true. On account of $0 \leq h_n + (-g_n) \in V$ for arbitrary $n \in \mathbb{N}$, we see

$$0 \leq I\Big(h_n + (-g_n)\Big) = I(h_n) + I(-g_n)$$

and consequently

$$I(g_n) \leq I(h_n)$$

and finally

$$I^-(f) = \lim_{n\to\infty} I(g_n) \leq \lim_{n\to\infty} I(h_n) = I^+(f).$$

q.e.d.

In the sequel, we consider functions with values in the extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. Within the set $\overline{\mathbb{R}}$ we need the following calculus rules:

– *Addition:*

$$\begin{aligned} a + (+\infty) = \quad & (+\infty) + a \quad = +\infty \text{ for all } a \in \mathbb{R} \cup \{+\infty\} \\ a + (-\infty) = \quad & (-\infty) + a \quad = -\infty \text{ for all } a \in \mathbb{R} \cup \{-\infty\} \\ (-\infty) + (+\infty) = (+\infty) + (-\infty) = \quad & 0 \end{aligned}$$

– *Multiplication:*

$$\left.\begin{array}{rcccl} a\,(+\infty) & = & (+\infty)\,a & = & +\infty \\ a\,(-\infty) & = & (-\infty)\,a & = & -\infty \end{array}\right\} \quad \text{for all } 0 < a \leq +\infty$$

$$\begin{array}{rcccl} 0\,(+\infty) & = & (+\infty)\,0 & = & +\infty \\ 0\,(-\infty) & = & (-\infty)\,0 & = & -\infty \end{array}$$

$$\left.\begin{array}{rcccl} a\,(+\infty) & = & (+\infty)\,a & = & -\infty \\ a\,(-\infty) & = & (-\infty)\,a & = & +\infty \end{array}\right\} \quad \text{for all } -\infty \leq a < 0$$

– *Subtraction:* For $a, b \in \overline{\mathbb{R}}$ we define

$$a - b := a + (-b),$$

where we set

$$-(+\infty) = -\infty \quad \text{and} \quad -(-\infty) = +\infty.$$

– *Ordering:* We have

$$-\infty \leq a \leq +\infty \qquad \text{for all} \quad a \in \overline{\mathbb{R}}.$$

Remark: Algebraically the set $\overline{\mathbb{R}}$ does not constitute a field, because the addition is not associative; consider for instance:

$$(-\infty) + \Big((+\infty) + (+\infty)\Big) = (-\infty) + (+\infty) \;=\; 0,$$

$$\Big((-\infty) + (+\infty)\Big) + (+\infty) = 0 + (+\infty) \;=\; +\infty.$$

With these calculus operations in $\overline{\mathbb{R}}$, we can uniquely define the functions $f+g$, $f-g$, cf for two functions $f : X \to \overline{\mathbb{R}}$ and $g : X \to \overline{\mathbb{R}}$ and arbitrary scalars $c \in \mathbb{R}$. Furthermore, we have the inequality $f \le g$ if and only if $g - f \ge 0$ is correct.

Definition 2.8. *The function $f : X \to \overline{\mathbb{R}}$ belongs to the class $L = L(X) = L(X, I)$ if and only if*

$$-\infty < I^-(f) = I^+(f) < +\infty$$

holds true. Then we define

$$I(f) := I^-(f) = I^+(f),$$

and we say that f is Lebesgue integrable *with respect to I.*

Remark: In our main example 1 from Section 1, we consider the open subset $\Omega \subset \mathbb{R}^n$ and obtain the class $L(X) =: L(\Omega)$ of *Lebesgue integrable functions in Ω*. In our main example 2, we get the class of *Lebesgue integrable functions on the manifold $\mathcal{M}$* with $L(X) =: L(\mathcal{M})$.

Proposition 2.9. *The function $f : X \to \overline{\mathbb{R}}$ belongs to the class $L(X)$ if and only if each quantity $\varepsilon > 0$ admits two functions $g \in -V$ and $h \in V$ satisfying*

$$g(x) \le f(x) \le h(x), \quad x \in X \quad \textit{and} \quad I(h) - I(g) < \varepsilon.$$

In particular, $I(g)$ and $I(h)$ are finite.

Proof:

'$\Longrightarrow$' We consider $f \in L(X)$ and note that $I^-(f) = I^+(f) \in \mathbb{R}$. According to Definition 2.6, we find functions $g \in -V$ and $h \in V$ with $g \le f \le h$ and $I(h) - I(g) < \varepsilon$.

'$\Longleftarrow$' For each quantity $\varepsilon > 0$, we have functions $g \in -V$ and $h \in V$ with $g \le f \le h$ and $I(h) - I(g) < \varepsilon$. On account of $I(h) \in (-\infty, +\infty]$ and $I(g) \in [-\infty, +\infty)$, we infer $I(h), I(g) \in \mathbb{R}$. Now Proposition 2.7 implies the estimate

$$0 \le I^+(f) - I^-(f) \le I(h) - I(g) < \varepsilon$$

for arbitrary $\varepsilon > 0$. Consequently, $I^+(f) = I^-(f) \in \mathbb{R}$ holds true and finally $f \in L(X)$. q.e.d.

Theorem 2.10. (Calculus rules for Lebesgue integrable functions) *The set $L(X)$ of Lebesgue integrable functions has the following properties:*

a) The statement

$$f \in L(X) \qquad \text{for each} \quad f \in V(X) \quad \text{with} \quad I(f) < +\infty$$

is correct, and the integrals from Definition 2.2 and Definition 2.8 coincide. Consequently, the functional $I : M(X) \to \mathbb{R}$ *has been extended onto* $L(X) \supset M(X)$*. Furthermore, we have*

$$I(f) \geq 0 \qquad \text{for all} \quad f \in L(X) \quad \text{with} \quad f \geq 0.$$

b) The space $L(X)$ *is linear, which means*

$$c_1 f_1 + c_2 f_2 \in L(X) \qquad \text{for all} \quad f_1, f_2 \in L(X) \quad \text{and} \quad c_1, c_2 \in \mathbb{R}.$$

Furthermore, $I : L(X) \to \mathbb{R}$ *represents a linear functional. Therefore, we have the calculus rule*

$$I(c_1 f_1 + c_2 f_2) = c_1 I(f_1) + c_2 I(f_2) \quad \text{for all} \quad f_1, f_2 \in L(X), \quad c_1, c_2 \in \mathbb{R}.$$

c) When $f \in L(X)$ *is given, then* $|f| \in L(X)$ *holds true and the estimate* $\big|I(f)\big| \leq I\big(|f|\big)$ *is valid.*

Proof:

a) Consider $f \in V(X)$ with $I(f) < +\infty$. Then we find a sequence

$$\{f_n\}_{n=1,2,\ldots} \subset M(X)$$

such that $f_n \uparrow f$ holds true. When we define $g_n := f_n$ and $h_n := f$ for all $n \in \mathbb{N}$, we infer $g_n \leq f \leq h_n$ with $g_n \in -V$ and $h_n \in V$, and we observe $I(h_n) - I(g_n) = I(f) - I(f_n) \to 0$. Proposition 2.9 tells us that $f \in L(X)$, and Definition 2.8 implies

$$-\infty < I(f) := I^+(f) = I^-(f) = \lim_{n\to\infty} I(f_n) < +\infty.$$

We consider $0 \leq f \in L(X)$, and we infer from $0 \in -V$ the statement $0 \leq I^-(f) = I(f)$.

b) At first, we show: If $f \in L(X)$ is chosen, we have $-f \in L(X)$ as well as $I(-f) = -I(f)$.

With $f \in L(X)$ given, each quantity $\varepsilon > 0$ admits a pair of functions $g \in -V$ and $h \in V$ satisfying $g \leq f \leq h$ as well as $I(h) - I(g) < \varepsilon$. This implies $-h \leq -f \leq -g$ with $-h \in -V$ and $-g \in V$. We note that $I(-g) = -I(g)$ and $I(-h) = -I(h)$ hold true, and we obtain

$$I(-g) - I(-h) = -I(g) + I(h) < \varepsilon \quad \text{for all} \quad \varepsilon > 0.$$

Finally, we arrive at $-f \in L(X)$ and $I(-f) = -I(f)$.

Now we show: With $f \in L(X)$ and $c > 0$, we have $cf \in L(X)$ and $I(cf) = cI(f)$.

Therefore, we consider $f \in L(X), c > 0$, and each $\varepsilon > 0$ admits functions $g \in -V$ and $h \in V$ with $g \le f \le h$ as well as $I(h) - I(g) < \varepsilon$. This implies $cg \le cf \le ch$, $cg \in -V$, $ch \in V$ and finally

$$I(ch) - I(cg) = c\Big(I(h) - I(g)\Big) < c\varepsilon.$$

We have thus proved $cf \in L(X)$ and $I(cf) = cI(f)$.

Finally, we deduce the calculus rule: From $f_1, f_2 \in L(X)$ we infer $f_1 + f_2 \in L(X)$ and $I(f_1 + f_2) = I(f_1) + I(f_2)$.

The elements $f_1, f_2 \in L(X)$ being given, we find to each $\varepsilon > 0$ the functions $g_1, g_2 \in -V$ and $h_1, h_2 \in V$ satisfying $g_i \le f_i \le h_i$ and $I(h_i) - I(g_i) < \varepsilon$ for $i = 1, 2$. This immediately implies $h_1 + h_2 \in V$, $g_1 + g_2 \in -V$, $g_1 + g_2 \le f_1 + f_2 \le h_1 + h_2$ and $I(h_1 + h_2) - I(g_1 + g_2) < 2\varepsilon$. We conclude $f_1 + f_2 \in L(X)$ and obtain the calculus rule $I(f_1 + f_2) = I(f_1) + I(f_2)$.

Therefore, $I : L(X) \to \mathbb{R}$ represents a linear functional on the linear space $L(X)$ of Lebesgue integrable functions.

c) With $f \in L(X)$, we find functions $g \in -V$ and $h \in V$ satisfying $g \le f \le h$ and $I(h) - I(g) < \varepsilon$ to each $\varepsilon > 0$, and we see $g^+ \le f^+ \le h^+$. Furthermore, we have sequences $g_n \downarrow g$ and $h_n \uparrow h$ in $M(X)$, which give us the approximations $g_n^+ \downarrow g^+$ and $h_n^+ \uparrow h^+$, respectively. Therefore, $h^+ \in V$ and $g^+ \in -V$ holds true as well as $h^+ - g^+ \in V$. From $h \ge g$ we infer $h^+ - g^+ \le h - g$ and see

$$\begin{aligned} I(h^+) - I(g^+) = I(h^+) + I(-g^+) \;&=\; I(h^+ - g^+) \\ \le I(h - g) \;&=\; I(h) - I(g) \;<\; \varepsilon. \end{aligned}$$

Consequently, the statements $f^+ \in L(X)$ and $|f| = f^+ + (-f)^+ \in L(X)$ are established. With $f \in L(X)$, the elements $-f$ and $|f|$ belong to $L(X)$ as well, and the inequalities $f \le |f|$, $-f \le |f|$ imply $I(f) \le I(|f|)$, $-I(f) = I(-f) \le I(|f|)$ and finally $|I(f)| \le I(|f|)$.

q.e.d.

Now we deduce convergence theorems for Lebesgue's integral: Fundamental is the following

Proposition 2.11. *Let the sequence* $\{f_k\}_{k=1,2,\ldots} \subset L(X)$ *with* $f_k \ge 0,\ k \in \mathbb{N}$ *and* $\sum\limits_{k=1}^{\infty} I(f_k) < +\infty$ *be given. Then the property*

$$f(x) := \sum_{k=1}^{\infty} f_k(x) \in L(X)$$

is fulfilled, and we have

$$I(f) = \sum_{k=1}^{\infty} I(f_k).$$

Proof: Given the quantity $\varepsilon > 0$, we find functions $g_k \in -V$ and $h_k \in V$ with $0 \le g_k \le f_k \le h_k$ and $I(h_k) - I(g_k) < \varepsilon\, 2^{-k}$ for all $k \in \mathbb{N}$, on account of $f_k \in L(X)$. Therefore, we have the inequalities

$$I(g_k) > I(h_k) - \frac{\varepsilon}{2^k} \ge I(f_k) - \frac{\varepsilon}{2^k} \quad \text{and} \quad I(h_k) < I(g_k) + \frac{\varepsilon}{2^k} \le I(f_k) + \frac{\varepsilon}{2^k}.$$

Now we choose n so large that $\sum\limits_{k=n+1}^{\infty} I(f_k) \le \varepsilon$ is correct. When we set

$$g := \sum_{k=1}^{n} g_k, \qquad h := \sum_{k=1}^{\infty} h_k,$$

we observe $g \in -V$ and $h \in V$, due to Proposition 2.5, as well as $g \le f \le h$. Furthermore, we see

$$I(g) = \sum_{k=1}^{n} I(g_k) > \sum_{k=1}^{n} \Big(I(f_k) - \frac{\varepsilon}{2^k} \Big) \ge \sum_{k=1}^{\infty} I(f_k) - 2\varepsilon$$

and

$$I(h) = \sum_{k=1}^{\infty} I(h_k) < \sum_{k=1}^{\infty} \Big(I(f_k) + \frac{\varepsilon}{2^k} \Big) = \sum_{k=1}^{\infty} I(f_k) + \varepsilon.$$

Consequently, we obtain $I(h) - I(g) < 3\varepsilon$ and additionally $f \in L(X)$. Finally, our estimates yield the identity

$$I(f) = \sum_{k=1}^{\infty} I(f_k).$$

q.e.d.

Theorem 2.12. (B.Levi's theorem on monotone convergence)
Let $\{f_n\}_{n=1,2,\ldots} \subset L(X)$ denote a sequence satisfying

$$f_n(x) \ne \pm\infty \qquad \text{for all} \quad x \in X \quad \text{and all} \quad n \in \mathbb{N}.$$

Furthermore, let the conditions

$$f_n(x) \uparrow f(x), \quad x \in X, \quad \text{and} \quad I(f_n) \le C, \quad n \in \mathbb{N}$$

be valid, with a constant $C \in \mathbb{R}$. Then we have $f \in L(X)$ and

$$\lim_{n \to \infty} I(f_n) = I(f).$$

Proof: On account of $f_k(x) \in \mathbb{R}$, the addition is associative there. Setting

$$\varphi_k(x) := (f_k(x) - f_{k-1}(x)) \in L(X), \qquad k = 2, 3, \dots,$$

we infer $\varphi_k \geq 0$ as well as

$$\sum_{k=2}^{n} \varphi_k(x) = f_n(x) - f_1(x), \qquad x \in X.$$

Now we observe

$$C - I(f_1) \geq I(f_n) - I(f_1) = \sum_{k=2}^{n} I(\varphi_k) \qquad \text{for all} \quad n \geq 2.$$

Proposition 2.11 implies

$$f - f_1 = \sum_{k=2}^{\infty} \varphi_k \in L(X)$$

and furthermore

$$\lim_{n\to\infty} I(f_n) - I(f_1) = \sum_{k=2}^{\infty} I(\varphi_k) = I\left(\sum_{k=2}^{\infty} \varphi_k\right) = I(f - f_1) = I(f) - I(f_1).$$

Therefore, we obtain $f \in L(X)$ and the following limit relation:

$$\lim_{n\to\infty} I(f_n) = I(f).$$

q.e.d.

Remark: The restrictive assumption $f_n(x) \neq \pm\infty$ will be eliminated in the next section.

Theorem 2.13. (Fatou's convergence theorem)
Let $\{f_n\}_{n=1,2,\dots} \subset L(X)$ denote a sequence of functions such that

$$0 \leq f_n(x) < +\infty \qquad \textit{for all} \quad x \in X \quad \textit{and all} \quad n \in \mathbb{N}$$

holds true. Furthermore, we assume

$$\liminf_{n\to\infty} I(f_n) < +\infty.$$

Then the function $g(x) := \liminf\limits_{n\to\infty} f_n(x)$ belongs to the space $L(X)$, and we observe the lower semicontinuity

$$I(g) \leq \liminf_{n\to\infty} I(f_n).$$

Proof: We note that

$$g(x) = \liminf_{n\to\infty} f_n(x) = \lim_{n\to\infty}\Big(\inf_{m\geq n} f_m(x)\Big) = \lim_{n\to\infty}\Big(\lim_{k\to\infty} g_{n,k}(x)\Big)$$

holds true with

$$g_{n,k}(x) := \min\Big(f_n(x), f_{n+1}(x), \dots, f_{n+k}(x)\Big) \in L(X).$$

When we define

$$g_n(x) := \inf_{m\geq n} f_m(x),$$

we infer the relations $g_{n,k} \downarrow g_n$ and $-g_{n,k} \uparrow -g_n$ for $k \to \infty$. Furthermore, we obtain $I(-g_{n,k}) \leq 0$ due to $f_n(x) \geq 0$. From Theorem 2.12 we infer $-g_n \in L(X)$ and consequently $g_n \in L(X)$ for all $n \in \mathbb{N}$.

Furthermore, we see $g_n(x) \leq f_m(x),\ x \in X$ for all $m \geq n$. Therefore, the inequality

$$I(g_n) \leq \inf_{m\geq n} I(f_m) \leq \lim_{n\to\infty}\Big(\inf_{m\geq n} I(f_m)\Big) = \liminf_{n\to\infty} I(f_n) < +\infty$$

is correct for all $n \in \mathbb{N}$. We utilize $g_n \uparrow g$ as well as Theorem 2.12, and we obtain $g \in L(X)$ and, moreover,

$$I(g) = \lim_{n\to\infty} I(g_n) \leq \liminf_{n\to\infty} I(f_n).$$

q.e.d.

Theorem 2.14. *Let* $\{f_n\}_{n=1,2,\dots} \subset L(X)$ *denote a sequence with*

$$|f_n(x)| \leq F(x) < +\infty, \qquad n \in \mathbb{N}, \quad x \in X,$$

where $F(x) \in L(X)$ *is correct. Furthermore, let us define*

$$g(x) := \liminf_{n\to\infty} f_n(x) \quad \textit{and} \quad h(x) := \limsup_{n\to\infty} f_n(x).$$

Then the elements g *and* h *belong to* $L(X)$*, and we have the inequalities*

$$I(g) \leq \liminf_{n\to\infty} I(f_n), \quad I(h) \geq \limsup_{n\to\infty} I(f_n).$$

Proof: We apply Theorem 2.13 on both sequences $\{F + f_n\}$ and $\{F - f_n\}$ of nonnegative finite-valued functions from $L(X)$. We observe the inequality

$$I(F \pm f_n) \leq I(F + F) \leq 2I(F) < +\infty \qquad \text{for all} \quad n \in \mathbb{N}.$$

Thus we obtain

$$L(X) \ni \liminf_{n\to\infty}(F + f_n) = F + \liminf_{n\to\infty} f_n = F + g$$

as well as $g \in L(X)$. Now Theorem 2.13 yields

$$I(F) + I(g) = I(F+g) \le \liminf_{n\to\infty} I(F+f_n) = I(F) + \liminf_{n\to\infty} I(f_n)$$

and

$$I(g) \le \liminf_{n\to\infty} I(f_n).$$

In the same way we deduce

$$L(X) \ni \liminf_{n\to\infty}(F - f_n) = F - \limsup_{n\to\infty} f_n = F - h$$

and consequently $h \in L(X)$. This implies

$$I(F) - I(h) = I(F-h) \le \liminf_{n\to\infty} I(F-f_n) = I(F) - \limsup_{n\to\infty} I(f_n)$$

and finally

$$I(h) \ge \limsup_{n\to\infty} I(f_n).$$

q.e.d.

Theorem 2.15. (H.Lebesgue's theorem on dominated convergence) *Let* $\{f_n\}_{n=1,2,\ldots} \subset L(X)$ *denote a sequence with*

$$f_n(x) \to f(x) \qquad \textit{for} \quad n \to \infty, \quad x \in X.$$

Furthermore, we assume

$$|f_n(x)| \le F(x) < +\infty, \qquad n \in \mathbb{N}, \quad x \in X$$

where $F \in L(X)$ *is valid. Then we infer* $f \in L(X)$ *as well as*

$$\lim_{n\to\infty} I(f_n) = I(f).$$

Proof: The limit relation

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad x \in X$$

implies

$$\liminf_{n\to\infty} f_n(x) = f(x) = \limsup_{n\to\infty} f_n(x).$$

According to Theorem 2.14, we have $f \in L(X)$ and

$$\limsup_{n\to\infty} I(f_n) \le I(f) \le \liminf_{n\to\infty} I(f_n).$$

Therefore, the subsequent limit exists

$$\lim_{n\to\infty} I(f_n),$$

and we deduce

$$I(f) = \lim_{n\to\infty} I(f_n).$$

q.e.d.

3 Measurable Sets

Beginning with this section, we have to require the following

Additional assumptions for the sets X and $M(X)$:

- We assume $X \subset \mathbb{R}^n$ with the dimension $n \in \mathbb{N}$. Then X becomes a topological space as follows: A subset $A \subset X$ is open (closed) if and only if we have an open (closed) subset $\widehat{A} \subset \mathbb{R}^n$ such that $A = X \cap \widehat{A}$ holds true.
- Furthermore, we assume that the inclusion $C_b^0(X,\mathbb{R}) \subset M(X) \subset C^0(X,\mathbb{R})$ is fulfilled. Here $C_b^0(X,\mathbb{R})$ describes the set of bounded continuous functions. This is valid for our main example 2. In our main example 1, this is fulfilled as well if the open set $\Omega \subset \mathbb{R}^n$ is subject to the following condition:

$$\int\limits_{\Omega} 1\,dx < +\infty.$$

  We see immediately that the function $f_0 \equiv 1$, $x \in X$ then belongs to the class $M(X)$.

Now we specialize our theory of integration from Section 2 to characteristic functions and obtain a measure theory. For an arbitrary set $A \subset X$ we define its *characteristic function* by

$$\chi_A(x) := \begin{cases} 1, & x \in A \\ 0, & x \in X \setminus A \end{cases}.$$

Definition 3.1. *A subset $A \subset X$ is called* finitely measurable *(or alternatively* integrable*) if its characteristic function satisfies $\chi_A \in L(X)$. We name*

$$\mu(A) := I(\chi_A)$$

the measure of the set A with respect to the integral I. *The* set of all finitely measurable sets in X *is denoted by $\mathcal{S}(X)$.*

From the additional assumptions above, namely $f_0 \equiv 1 \in M(X)$, we infer $\chi_X \in M(X) \subset L(X)$ and consequently $X \in \mathcal{S}(X)$. Therefore, we speak equivalently of *finitely measurable* and *measurable* sets.

Proposition 3.2. (σ-Additivity)
Let $\{A_i\}_{i=1,2,\ldots} \subset \mathcal{S}(X)$ denote a sequence of mutually disjoint sets. Then the set

$$A := \bigcup_{i=1}^{\infty} A_i$$

belongs to $\mathcal{S}(X)$ as well, and we have

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i).$$

Proof: We consider the sequence of functions

$$f_k := \sum_{l=1}^{k} \chi_{A_l} \uparrow \chi_A \le \chi_X \in L(X)$$

and note that $f_k \in L(X)$ for all $k \in \mathbb{N}$ holds true. Now Lebesgue's convergence theorem yields $\chi_A \in L(X)$ and consequently $A \in \mathcal{S}(X)$. Finally, we evaluate

$$\begin{aligned}\mu(A) = I(\chi_A) &= \lim_{k\to\infty} I(f_k) = \lim_{k\to\infty} I(\chi_{A_1} + \ldots + \chi_{A_k}) \\ &= \lim_{k\to\infty} \Big(\mu(A_1) + \ldots + \mu(A_k)\Big) = \sum_{l=1}^{\infty} \mu(A_l).\end{aligned}$$

q.e.d.

We show that with $A, B \in \mathcal{S}(X)$ their intersection $A \cap B$ belongs to $\mathcal{S}(X)$ as well. On account of $\chi_{A\cap B} = \chi_A \chi_B$, we have to verify that with $\chi_A, \chi_B \in L(X)$ their product satisfies $\chi_A \chi_B \in L(X)$ as well. In general, the product of two functions in $L(X)$ need not lie in $L(X)$ as demonstrated by the following

Example 3.3. With $X = (0, 1)$, we define the space

$$M(X) = \left\{ f : (0,1) \to \mathbb{R} \in C^0\Big((0,1), \mathbb{R}\Big) \; : \; \int_0^1 |f(x)|\, dx < +\infty \right\}$$

and the improper Riemannian integral $I(f) := \int_0^1 f(x)\, dx$. Then we observe

$$f(x) := \frac{1}{\sqrt{x}} \in L(X); \quad \text{however,} \quad f^2(x) := \frac{1}{x} \notin L(X).$$

Now we establish the following

Theorem 3.4. (Continuous combination of bounded L-functions) *Let $f_k(x) \in L(X)$ for $k = 1, \ldots, \kappa$ denote finitely many bounded functions, such that the estimate*

$$|f_k(x)| \le c \qquad \textit{for all points} \quad x \in X \quad \textit{and all indices} \quad k \in \{1, \ldots, \kappa\}$$

is valid, with a constant $c \in (0, +\infty)$. Furthermore, let the function $\Phi = \Phi(y_1, \ldots, y_\kappa) : \mathbb{R}^\kappa \to \mathbb{R} \in C^0(\mathbb{R}^\kappa, \mathbb{R})$ be given. Then the composition

$$g(x) := \Phi\Big(f_1(x), \ldots, f_\kappa(x)\Big), \qquad x \in X$$

belongs to the class $L(X)$ and is bounded.

Proof:

1. With $f : X \to \mathbb{R} \in L(X)$ let us consider a bounded function. At first, we show that its square satisfies $f^2 \in L(X)$. We observe

$$f^2(x) = \{f(x) - \lambda\}^2 + 2\lambda f(x) - \lambda^2$$

and infer

$$f^2(x) \geq 2\lambda f(x) - \lambda^2 \qquad \text{for all} \quad \lambda \in \mathbb{R},$$

where equality is attained only for $\lambda = f(x)$. Therefore, we can rewrite the square-function as follows:

$$f^2(x) = \sup_{\lambda \in \mathbb{R}} \Big(2\lambda f(x) - \lambda^2\Big).$$

Since the function $\lambda \mapsto (2\lambda f(x) - \lambda^2)$ is continuous with respect to λ for each fixed $x \in X$, it is sufficient to evaluate this supremum only over the set of rational numbers. Furthermore, we have $\mathbb{Q} = \{\lambda_l\}_{l=1,2,\ldots}$ and see

$$f^2(x) = \sup_{l \in \mathbb{N}} \Big(2\lambda_l f(x) - \lambda_l^2\Big) = \lim_{m \to \infty} \left(\max_{1 \leq l \leq m} \Big(2\lambda_l f(x) - \lambda_l^2\Big)\right).$$

With the aid of

$$\varphi_m(x) := \max_{1 \leq l \leq m} \Big(2\lambda_l f(x) - \lambda_l^2\Big)$$

we obtain

$$f^2(x) = \lim_{m \to \infty} \varphi_m(x) = \lim_{m \to \infty} \varphi_m^+(x),$$

where the last equality is inferred from the positivity of $f^2(x)$. Since $f \in L(X)$ holds true, the linearity and the closedness with respect to the maximum operation of $L(X)$ imply: The elements φ_m and consequently φ_m^+ belong to the space $L(X)$. Furthermore, for all points $x \in X$ and all $m \in \mathbb{N}$ we have the estimate

$$0 \leq \varphi_m^+(x) \leq f^2(x) \leq c$$

with a constant $c \in (0, +\infty)$. From the property $f_0(x) \equiv 1 \in L(X)$ we infer $f_c(x) \equiv c \in L(X)$, and the functions φ_m^+ have an integrable dominating function. Now Lebesgue's convergence theorem yields

$$f^2(x) = \lim_{m \to \infty} \varphi_m^+(x) \in L(X).$$

2. When $f, g \in L(X)$ represent bounded functions, its product $f \cdot g$ is bounded as well. On account of part 1 of our proof and the identity

$$fg = \frac{1}{4}(f+g)^2 - \frac{1}{4}(f-g)^2,$$

we deduce $fg \in L(X)$.

3. On the rectangle

$$Q := \Big\{y = (y_1, \ldots, y_\kappa) \in \mathbb{R}^\kappa \,:\, |y_k| \le c,\; k = 1, \ldots, \kappa\Big\}$$

we can approximate the continuous function Φ uniformly by polynomials

$$\Phi_l = \Phi_l(y_1, \ldots, y_\kappa), \qquad l = 1, 2, \ldots .$$

From part 2 we infer that the functions

$$g_l(x) := \Phi_l\Big(f_1(x), \ldots, f_\kappa(x)\Big), \qquad x \in X$$

are bounded and belong to the class $L(X)$. We have the estimate

$$|g_l(x)| \le C \qquad \text{for all} \quad x \in X \quad \text{and all} \quad l \in \mathbb{N}$$

with a fixed constant $C \in (0, +\infty)$. Since the function satisfies $\varphi(x) \equiv C \in L(X)$, Lebesgue's convergence theorem yields

$$g(x) = \Phi\Big(f_1(x), \ldots, f_\kappa(x)\Big) = \lim_{l \to \infty} g_l(x) \in L(X).$$

q.e.d.

Corollary from Theorem 3.4: If $f(x) \in L(X)$ represents a bounded function, its power $|f|^p$ belongs to the class $L(X)$ for all exponents $p > 0$.

Proposition 3.5. *With the sets $A, B \in \mathcal{S}(X)$ the following sets $A \cap B$, $A \cup B$, $A \setminus B$, $A^c := X \setminus A$ belong to $\mathcal{S}(X)$ as well.*

Proof: Let us take $A, B \in \mathcal{S}(X)$, and the associate characteristic functions χ_A, χ_B are bounded and belong to the class $L(X)$. Via Proposition 3.4, we deduce

$$\chi_{A\cap B} = \chi_A \chi_B \in L(X) \quad \text{and consequently} \quad A \cap B \in \mathcal{S}(X).$$

Now we see $A \cup B \in \mathcal{S}(X)$ due to $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B} \in L(X)$. Furthermore, we observe

$$\chi_{A\setminus B} = \chi_{A\setminus(A\cap B)} = \chi_A - \chi_{A\cap B} \in L(X) \quad \text{and consequently} \quad A \setminus B \in L(X).$$

On account of $X \in \mathcal{S}(X)$, we finally infer $A^c = (X \setminus A) \in \mathcal{S}(X)$. q.e.d.

Proposition 3.6. (σ-Subadditivity)
Let $\{A_i\}_{i=1,2,\ldots} \subset \mathcal{S}(X)$ denote a sequence of sets. Then their denumerable union

$$A := \bigcup_{i=1}^{\infty} A_i$$

belongs to $\mathcal{S}(X)$ as well, and we have the following estimate:

$$\mu(A) \le \sum_{i=1}^{\infty} \mu(A_i) \,\in [0, +\infty].$$

Proof: We make the transition from the sequence $\{A_i\}_{i=1,2,\ldots}$ to the sequence $\{B_i\}_{i=1,2,\ldots}$ of mutually disjoint sets:

$$B_1 := A_1,\ B_2 := A_2 \setminus B_1, \ldots,\ B_k := A_k \setminus (B_1 \cup \cdots \cup B_{k-1}), \ldots$$

Now Proposition 3.5 yields $\{B_i\}_{i=1,2,\ldots} \subset \mathcal{S}(X)$. Furthermore, we note that $B_i \subset A_i$ holds true for all $i \in \mathbb{N}$ and, moreover, $A = \bigcup\limits_{i=1}^{\infty} B_i$. Then Proposition 3.2 implies $A \in \mathcal{S}(X)$ as well as $\mu(A) = \sum_{i=1}^{\infty} \mu(B_i) \le \sum_{i=1}^{\infty} \mu(A_i)$.
q.e.d.

Definition 3.7. *A system $\mathcal{A}$ of subsets of a set X is called* σ-algebra *if we have the following properties :*

1. *$X \in \mathcal{A}$.*
2. *With $B \in \mathcal{A}$, its complement satisfies $B^c = (X \setminus B) \in \mathcal{A}$ as well.*
3. *For each sequence of sets $\{B_i\}_{i=1,2,\ldots}$ in $\mathcal{A}$, their denumerable union $\bigcup\limits_{i=1}^{\infty} B_i$ belongs to $\mathcal{A}$ as well.*

Remark: We infer $\emptyset \in \mathcal{A}$ immediately from these conditions. Furthermore, with the sets $\{B_i\}_{i=1,2,\ldots} \subset \mathcal{A}$ their denumerable intersection satisfies $\bigcap\limits_{i=1}^{\infty} B_i \in \mathcal{A}$ as well.

Definition 3.8. *We name the function $\mu : \mathcal{A} \to [0, +\infty]$ on a σ-algebra $\mathcal{A}$ a* measure *if the following conditions are fulfilled:*

1. *$\mu(\emptyset) = 0$.*
2. *$\mu\Big(\bigcup\limits_{i=1}^{\infty} B_i\Big) = \sum\limits_{i=1}^{\infty} \mu(B_i)$ for all mutually disjoint sets $\{B_i\}_{i=1,2,\ldots} \subset \mathcal{A}$.*

We call this measure finite *if $\mu(X) < +\infty$ holds true.*

Remark: Property 2 is called the σ-additivity of the measure. If we only have finite additivity - that means $\mu\left(\bigcup_{i=1}^{N} B_i\right) = \sum_{i=1}^{N} \mu(B_i)$ for all mutually disjoint sets $\{B_i\}_{i=1,2,\ldots,N} \subset \mathcal{A}$ - we speak of a *content*.

From our Propositions 3.2 to 3.6, we immediately infer

Theorem 3.9. *The set $\mathcal{S}(X)$ of the finitely measurable subsets of X constitutes a σ-algebra. The prescription*

$$\mu(A) := I(\chi_A), \qquad A \in \mathcal{S}(X)$$

defines a finite measure on the σ-algebra $\mathcal{S}(X)$.

Remark: Carathéodory developed axiomatically the measure theory, on which the integration theory can be based. We have presented the inverse approach here. The axiomatic measure theory begins with Definitions 3.7 and 3.8 above.

Definition 3.10. *A set $A \subset X$ is named* null-set *if $A \in \mathcal{S}(X)$ and $\mu(A) = 0$ hold true.*

Remark: The measure μ from Definition 3.1 has the property that each subset of a null-set is a null-set again. For $B \subset A$ and $A \in \mathcal{S}(X)$ with $\mu(A) = 0$ we namely deduce

$$0 = I^+(\chi_A) \geq I^+(\chi_B) \geq I^-(\chi_B) \geq 0,$$

and consequently

$$I^+(\chi_B) = I^-(\chi_B) = 0.$$

Therefore, we obtain $\chi_B \in L(X)$ and finally $B \in \mathcal{S}(X)$ with $\mu(B) = 0$.

Proposition 3.6 immediately implies

Theorem 3.11. *The denumerable union of null-sets is a null-set again.*

Now we show the following

Theorem 3.12. *Each open and each closed set $A \subset X$ belongs to $\mathcal{S}(X)$.*

Proof:

1. At first, let the set A be closed in X and bounded in $\mathbb{R}^n \supset X$. Then we have a compact set $\widehat{A}$ in $\mathbb{R}^n$ satisfying $A = \widehat{A} \cap X$. For the set $\widehat{A}$ we construct - with the aid of Tietze's extension theorem - a sequence of functions $f_l : \mathbb{R}^n \to \mathbb{R} \in C_0^0(\mathbb{R}^n)$ such that

$$f_l(x) = \begin{cases} \quad 1\,, & x \in \widehat{A} \\ \quad 0\,, & x \in \mathbb{R}^n \text{ with dist}\,(x, \widehat{A}) \geq \dfrac{1}{l} \\ \in \big[0,1\big]\,, \text{ elsewhere} & \end{cases}$$

holds true for $l = 1, 2, \ldots$. We observe $f_l(x) \to \chi_{\widehat{A}}(x)$, set $g_l = f_l\big|_X$, and obtain

$$g_l \in C_b^0(X) \subset M(X) \subset L(X)$$

as well as

$$0 \leq g_l(x) \leq 1 \quad \text{and} \quad g_l(x) \to \chi_A(x), \qquad x \in X.$$

On account of $f_0(x) \equiv 1 \in M(X)$, we can apply Lebesgue's convergence theorem and see

$$\chi_A(x) = \lim_{l \to \infty} g_l(x) \; \in L(X).$$

Therefore, $A \in \mathcal{S}(X)$ is satisfied.

2. For an arbitrary closed set $A \subset X$ we consider the sequence

$$A_l := A \cap \Big\{x \in \mathbb{R}^n \,:\, |x| \le l\Big\}.$$

Due to part 1 of our proof, the sets A_l belong to the system $\mathcal{S}(X)$ and consequently $A = \bigcup\limits_{l=1}^{\infty} A_l$ as well. Finally, the open sets belong to $\mathcal{S}(X)$ as complements of closed sets. q.e.d.

Proposition 3.13. *Let us consider $f \in V(X)$. Then the level set*

$$\mathcal{O}(f,a) := \Big\{x \in X \,:\, f(x) > a\Big\} \subset X$$

is open for all $a \in \mathbb{R}$.

Proof: We note that $f \in V(X)$ holds true and find a sequence

$$\{f_n\}_{n=1,2,\dots} \subset M(X) \subset C^0(X,\mathbb{R})$$

satisfying $f_n \uparrow f$ on X. Let us consider a point $\xi \in \mathcal{O}(f,a)$ which means $f(\xi) > a$. Then we have an index $n_0 \in \mathbb{N}$ with $f_{n_0}(\xi) > a$. Since the function $f_{n_0} : X \to \mathbb{R}$ is continuous, there exists an open neighborhood $U \subset X$ of ξ such that $f_{n_0}(x) > a$ for all $x \in U$ holds true. Due to $f_{n_0} \le f$ on X, we infer $f(x) > a$ for all $x \in U$, which implies $U \subset \mathcal{O}(f,a)$. Consequently, the level set $\mathcal{O}(f,a)$ is open.
q.e.d.

The following criterion illustrates the connection between open and measurable sets.

Theorem 3.14. *A set $B \subset X$ belongs to the system $\mathcal{S}(X)$ if and only if the following condition is valid: For all $\delta > 0$ we can find a closed set $A \subset X$ and an open set $O \subset X$, such that the properties $A \subset B \subset O$ and $\mu(O \setminus A) < \delta$ hold true.*

Proof:

'$\Longrightarrow$' When we take $B \in \mathcal{S}(X)$, we infer $\chi_B \in L(X)$ and Proposition 2.9 in Section 2 gives us a function $f \in V(X)$ satisfying $0 \le \chi_B \le f$ and $I(f) - \mu(B) < \varepsilon$ for all $\varepsilon > 0$. According to Proposition 3.13, the level sets

$$\mathcal{O}_\varepsilon := \{x \in X \mid f(x) > 1 - \varepsilon\} \supset B$$

with $\varepsilon > 0$ are open in X. Now we deduce

$$\chi_B \le \chi_{\mathcal{O}_\varepsilon} = \frac{1}{1-\varepsilon}(1-\varepsilon)\chi_{\mathcal{O}_\varepsilon} \le \frac{1}{1-\varepsilon} f \qquad \text{in } X,$$

and we see

$$\mu(\mathcal{O}_\varepsilon) - \mu(B) = I(\chi_{\mathcal{O}_\varepsilon}) - \mu(B) \le \frac{1}{1-\varepsilon} I(f) - \mu(B)$$

$$= \frac{1}{1-\varepsilon}\Big(I(f) - \mu(B)\Big) + \frac{\varepsilon}{1-\varepsilon}\mu(B) < \frac{\varepsilon}{1-\varepsilon}\Big(1 + \mu(B)\Big)$$

for all $\varepsilon > 0$. For the quantity $\delta > 0$ being given, we now choose a sufficiently small $\varepsilon > 0$ such that the set $O := \mathcal{O}_\varepsilon \supset B$ satisfies the estimate

$$\mu(O) - \mu(B) < \frac{\delta}{2}.$$

Furthermore, we attribute to each measurable set $B^c = X \setminus B$ an open set $\widetilde{O} = A^c$ such that $A^c = \widetilde{O} \supset B^c$ and $\mu(\widetilde{O} \cap B) < \frac{\delta}{2}$ hold true. Therefore, the closed set $A \subset X$ fulfills the inclusion $A \subset B \subset O$ and the estimate

$$\mu(O \setminus A) = \mu(O) - \mu(A) = \Big(\mu(O) - \mu(B)\Big) + \Big(\mu(B) - \mu(A)\Big)$$

$$< \frac{\delta}{2} + \mu(B \setminus A) = \frac{\delta}{2} + \mu(B \cap \widetilde{O}) < \delta.$$

'$\Longleftarrow$' The quantity $\delta > 0$ being given, we find an open set $O \supset B$ and a closed set $A \subset B$ - they are measurable due to Proposition 3.13 - such that the estimate $I(\chi_O - \chi_A) < \delta$ is fulfilled. Since $\chi_A, \chi_O \in L(X)$ is fulfilled, Proposition 2.9 in Section 2 provides functions $g \in -V(X)$ and $h \in V(X)$ satisfying

$$g \le \chi_A \le \chi_B \le \chi_O \le h \qquad \text{in} \quad X \quad \text{and} \quad I(h-g) < 3\delta.$$

Using Proposition 2.9 in Section 2 again, we deduce $\chi_B \in L(X)$ and consequently $B \in \mathcal{S}(X)$. q.e.d.

In the sequel, we shall intensively study the null-sets. These appear as sets of exemption for Lebesgue integrable functions and can be neglected in the Lebesgue integration. We start our investigations with the following

Proposition 3.15. *A set $N \subset X$ is a null-set if and only if we have a function $h \in V(X)$ satisfying $h(x) \ge 0$ for all $x \in X$, $h(x) = +\infty$ for all $x \in N$, and $I(h) < +\infty$.*

Proof:

'$\Longrightarrow$' Let $N \subset X$ denote a null-set. Then $\chi_N \in L(X)$ and $I(\chi_N) = 0$ hold true. For each index $k \in \mathbb{N}$ we obtain a function $h_k \in V(X)$ satisfying $0 \le \chi_N \le h_k$ in X and $I(h_k) \le 2^{-k}$, due to Proposition 2.9 in Section 2. According to Proposition 2.5 in Section 2, the element

$$h(x) := \sum_{k=1}^{\infty} h_k(x)$$

belongs to the space $V(X)$ and fulfills

$$I(h) = \sum_{k=1}^{\infty} I(h_k) \leq 1.$$

On the other hand, the estimates $h_k(x) \geq 1$ in N for all $k \in \mathbb{N}$ imply that the relation $h(x) = +\infty$ for all $x \in N$ is correct. We note that $h_k(x) \geq 0$ in X holds true, and we deduce $h(x) \geq 0$ for all $x \in X$.

'$\Longleftarrow$' Let the conditions $h \in V(X)$, $h(x) \geq 0$ for all $x \in X$, $h(x) = +\infty$ for all $x \in N$, and $I(h) < +\infty$ be fulfilled. When we define

$$h_\varepsilon(x) := \frac{\varepsilon}{1 + I(h)}\, h(x),$$

we immediately deduce $h_\varepsilon \in V(X)$, $h_\varepsilon(x) \geq 0$ for all $x \in X$, and $I(h_\varepsilon) < \varepsilon$ for all $\varepsilon > 0$. On account of $h(x) = +\infty$ for all $x \in N$, we infer

$$0 \leq \chi_N(x) \leq h_\varepsilon(x) \qquad \text{in} \quad X \quad \text{for all} \quad \varepsilon > 0.$$

Proposition 2.9 in Section 2 yields $I(\chi_N) = 0$, which means that N is a null-set. q.e.d.

Definition 3.16. *A property holds true* almost everywhere *in X (symbolically: a.e.), if there exists a null-set $N \subset X$ such that this property is valid for all points $x \in X \setminus N$.*

Theorem 3.17. (a.e.-Finiteness of L-functions)
Let the function $f \in L(X)$ be given. Then the set

$$N := \Big\{ x \in X \,:\, |f(x)| = +\infty \Big\}$$

constitutes a null-set.

Proof: With $f \in L(X)$ being given, we obtain $|f| \in L(X)$ and find a function $h \in V(X)$ satisfying $0 \leq |f(x)| \leq h(x)$ in X as well as $I(h) < +\infty$. Furthermore, $h(x) = +\infty$ in N holds true and Proposition 3.15 tells us that N represents a null-set.

q.e.d.

Theorem 3.18. *Let the function $f \in L(X)$ be given such that $I(|f|) = 0$ is correct. Then the set*

$$N := \Big\{ x \in X \,:\, f(x) \neq 0 \Big\}$$

constitutes a null-set.

Proof: With $f \in L(X)$ being given, we infer $|f| \in L(X)$. Setting

$$f_k(x) := |f(x)|, \qquad k \in \mathbb{N},$$

we observe

$$\sum_{k=1}^{\infty} I(f_k) = 0.$$

According to Proposition 2.11 in Section 2, the function

$$g(x) := \sum_{k=1}^{\infty} f_k(x)$$

is Lebesgue integrable as well. Now we see $N = \{x \in X : g(x) = +\infty\}$, and Theorem 3.17 implies that N is a null-set. q.e.d.

Now we want to show that an L-function can be arbitrarily modified on a null-set, without the value of the integral being changed! In this way we can confine ourselves to consider *finite-valued functions* $f \in L(X)$, which are functions f with $f(x) \in \mathbb{R}$ for all $x \in X$, more precisely. A bounded function is finite-valued; however, a finite-valued function is not necessarily bounded. In this context, we mention the function $f(x) = \frac{1}{x}$, $x \in (0,1)$.

Proposition 3.19. *Let $N \subset X$ denote a null-set. Furthermore, the function $f : X \to \overline{\mathbb{R}}$ may satisfy $f(x) = 0$ for all $x \in X \setminus N$. Then we infer $f \in L(X)$ as well as $I(f) = 0$.*

Proof: Due to Proposition 3.15, we find a function $h \in V(X)$ satisfying $h(x) \geq 0$ for all $x \in X$, $h(x) = +\infty$ for all $x \in N$, and $I(h) < +\infty$. For all numbers $\varepsilon > 0$, we see $\varepsilon h \in V$ and $-\varepsilon h \in -V$ as well as

$$-\varepsilon h(x) \leq f(x) \leq \varepsilon h(x) \qquad \text{for all} \quad x \in X.$$

Furthermore, the identity

$$I(\varepsilon h) - I(-\varepsilon h) = 2\varepsilon I(h) \qquad \text{for all} \quad \varepsilon > 0$$

is correct. We infer $f \in L(X)$ and, moreover, $I(f) = 0$ from Proposition 2.9 in Section 2.

q.e.d.

Theorem 3.20. *Consider the function $f \in L(X)$ and the null-set $N \subset X$. Furthermore, let the function $\widetilde{f} : X \to \overline{\mathbb{R}}$ with the property $\widetilde{f}(x) = f(x)$ for all $x \in X \setminus N$ be given. Then we infer $\widetilde{f} \in L(X)$ as well as $I(|f - \widetilde{f}|) = 0$, and consequently $I(f) = I(\widetilde{f})$.*

Proof: Since $f \in L(X)$ holds true, the following set

$$N_1 := \Big\{x \in X \,:\, |f(x)| = +\infty\Big\}$$

constitutes a null-set, due to Theorem 3.17. Now we find a function $\varphi(x) : X \to \overline{\mathbb{R}}$ such that

$$\widetilde{f}(x) = f(x) + \varphi(x) \qquad \text{for all} \quad x \in X.$$

Evidently, we have the identity $\varphi(x) = 0$ outside the null-set $N \cup N_1$. Proposition 3.19 yields $\varphi \in L(X)$ and $I(\varphi) = 0$. Consequently, $\widetilde{f} \in L(X)$ is correct and we see

$$I(\widetilde{f}) = I(f + \varphi) = I(f) + I(\varphi) = I(f).$$

When we apply these arguments on the function

$$\psi(x) := |f(x) - \widetilde{f}(x)|, \qquad x \in X,$$

Proposition 3.19 shows us $\psi \in L(X)$ and finally

$$0 \;=\; I(\psi) \;=\; I(|f - \widetilde{f}|).$$

q.e.d.

Remark: When a function $\widetilde{f}$ coincides a.e. with an L-function f, then $\widetilde{f} \in L(X)$ holds true and their integrals are identical.

We are now prepared to provide general convergence theorems of the Lebesgue integration theory.

Theorem 3.21. (General convergence theorem of B.Levi)
Let $\{f_k\}_{k=1,2,\ldots} \subset L(X)$ denote a sequence of functions satisfying $f_k \uparrow f$ a.e. in X. Furthermore, let $I(f_k) \le c$ for all $k \in \mathbb{N}$ be valid - with the constant $c \in \mathbb{R}$. Then we infer $f \in L(X)$ and

$$\lim_{k\to\infty} I(f_k) = I(f).$$

Proof: We consider the null-sets

$$N_k := \Big\{x \in X \,:\, |f_k(x)| = +\infty\Big\} \qquad \text{for} \quad k \in \mathbb{N}$$

as well as

$$N_0 := \Big\{x \in X \,:\, f_k(x) \uparrow f(x) \text{ is not valid}\Big\}.$$

We define the null-set

$$N := \bigcup_{k=0}^{\infty} N_k,$$

and modify f, f_k on N to 0. Then we obtain the functions $\widetilde{f}_k \in L(X)$ with

$$I(\widetilde{f}_k) = I(f_k) \le c \qquad \text{for all} \quad k \in \mathbb{N}$$

and $\widetilde{f}$ with $\widetilde{f}_k \uparrow \widetilde{f}$. According to Theorem 2.12 from Section 2, we deduce $\widetilde{f} \in L(X)$ as well as

$$\lim_{k\to\infty} I(\widetilde{f}_k) = I(\widetilde{f}).$$

Now Theorem 3.20 yields $f \in L(X)$ and

$$I(f) = I(\widetilde{f}) = \lim_{k\to\infty} I(\widetilde{f}_k) = \lim_{k\to\infty} I(f_k).$$

q.e.d.

Modifying the functions to 0 on the relevant null-sets as above, we easily prove the following Theorems 3.22 and 3.23 with the aid of Theorem 2.13 and 2.15 from Section 2, respectively.

Theorem 3.22. (General convergence theorem of Fatou)
Let $\{f_k\}_{k=1,2,\dots} \subset L(X)$ *denote a sequence of functions with* $f_k(x) \ge 0$ *a.e. in* X *for all* $k \in \mathbb{N}$*, and we assume*

$$\liminf_{k\to\infty} I(f_k) < +\infty.$$

Then the function

$$g(x) := \liminf_{k\to\infty} f_k(x)$$

belongs to the class $L(X)$ *as well, and we have lower semicontinuity as follows:*

$$I(g) \le \liminf_{k\to\infty} I(f_k).$$

Theorem 3.23. (General convergence theorem of Lebesgue)
Let $\{f_k\}_{k=1,2,\dots} \subset L(X)$ *denote a sequence with* $f_k \to f$ *a.e. on* X *and* $|f_k(x)| \le F(x)$ *a.e. in* X *for all* $k \in \mathbb{N}$*, where* $F \in L(X)$ *holds true. Then we infer* $f \in L(X)$ *and the identity*

$$\lim_{k\to\infty} I(f_k) = I(f).$$

We conclude this section with the following

Theorem 3.24. *Lebesgue's integral* $I : L(X) \to \mathbb{R}$ *constitutes a Daniell integral.*

Proof: We invoke Theorem 2.10 in Section 2 and obtain the following: The space $L(X)$ is linear and closed with respect to the modulus operation. Furthermore, $L(X)$ satisfies the properties (1) and (2) in Section 1. The Lebesgue integral I is nonnegative, linear, and closed with respect to monotone convergence - due to Theorem 3.21. Therefore, the functional I fulfills the conditions (3)–(5) in Section 1. Consequently, Lebesgue's integral $I : L(X) \to \mathbb{R}$ represents a Daniell integral as described in Definition 1.1 from Section 1.

q.e.d.

4 Measurable Functions

Fundamental is the following

Definition 4.1. *The function* $f : X \to \overline{\mathbb{R}}$ *is named* measurable *if the* level set - above the level a -

$$\mathcal{O}(f,a) := \Big\{x \in X \,:\, f(x) > a\Big\}$$

is measurable for all $a \in \mathbb{R}$*.*

Remark: Each continuous function $f : X \to \mathbb{R} \in C^0(X,\mathbb{R})$ is measurable. Then $\mathcal{O}(f,a) \subset X$ is an open set for all $a \in \mathbb{R}$, which is measurable due to Section 3, Theorem 3.12. Furthermore, Proposition 3.13 in Section 3 shows us that each function $f \in V(X)$ is measurable as well.

Proposition 4.2. *Let* $f : X \to \overline{\mathbb{R}}$ *denote a measurable function. Furthermore, let us consider the numbers* $a, b \in \overline{\mathbb{R}}$ *with* $a \le b$ *and the interval* $I = [\,a,b\,]$*; for* $a < b$ *we consider the intervals* $I = (a,b\,]$*,* $I = [\,a,b)$*,* $I = (a,b)$ *as well. Then the following sets*

$$A := \Big\{x \in X \,:\, f(x) \in I\Big\}$$

are measurable.

Proof: Definition 4.1 implies that the level sets

$$\mathcal{O}_1(f,c) := \mathcal{O}(f,c) = \Big\{x \in X \,:\, f(x) > c\Big\}$$

are measurable for all $c \in \mathbb{R}$. For a given $c \in \mathbb{R}$, we now choose a sequence $\{c_n\}_{n=1,2,\ldots}$ satisfying $c_n \uparrow c$, and we obtain again a measurable set via

$$\mathcal{O}_2(f,c) := \Big\{x \in X \,:\, f(x) \ge c\Big\} = \bigcap_{n=1}^{\infty} \Big\{x \in X \,:\, f(x) > c_n\Big\}.$$

The measurable sets $\mathcal{S}(X)$ namely constitute a σ-algebra due to Section 3, Definition 3.7 and Theorem 3.9. Furthermore, we have the relations

$$\mathcal{O}_2(f,+\infty) = \bigcap_{n=1}^{\infty} \mathcal{O}_2(f,n), \quad \mathcal{O}_1(f,-\infty) = \bigcup_{n=1}^{\infty} \mathcal{O}_1(f,-n),$$

and these sets are measurable as well. The transition to their complements shows that

$$\mathcal{O}_3(f,c) := \Big\{x \in X \,:\, f(x) \le c\Big\} \quad \text{and} \quad \mathcal{O}_4(f,c) := \Big\{x \in X \,:\, f(x) < c\Big\}$$

are measurable for all $c \in \overline{\mathbb{R}}$. Here

$$A := \Big\{ x \in X \ : \ f(x) \in I \Big\}$$

can be generated by an intersection of the sets $\mathcal{O}_1$–$\mathcal{O}_4$, when we replace c by a or b, respectively. This proves the measurability of the sets A. q.e.d.

For $a, b \in \overline{\mathbb{R}}$ with $a < b$, we define the function

$$\phi_{a,b}(t) := \begin{cases} a\,, & -\infty \le t \le a \\ t\,, & a \le t \le b \\ b\,, & b \le t \le +\infty \end{cases}$$

as a *cut-off function*. Given the function $f : X \to \overline{\mathbb{R}}$, we set

$$f_{a,b}(x) := \phi_{a,b}(f(x)) := \begin{cases} a\,, & -\infty \le f(x) \le a \\ f(x)\,, & a \le f(x) \le b \\ b\,, & b \le f(x) \le +\infty \end{cases} .$$

Evidently, we have the estimate

$$|\, f_{a,b}(x)| \le \max\,(|\,a|, |\,b|) < +\infty \qquad \text{for all} \quad x \in X, \quad a, b \in \mathbb{R}.$$

Furthermore, we note that

$$f^{+}(x) = f_{0,+\infty}(x) \quad \text{and} \quad f^{-}(x) = f_{-\infty,0}(x), \qquad x \in X.$$

Theorem 4.3. *A function $f : X \to \overline{\mathbb{R}}$ is measurable if and only if the function $f_{a,b}$ belongs to $L(X)$ for all $a, b \in \mathbb{R}$ with $a < b$.*

Proof:

'$\Longrightarrow$' Let $f : X \to \overline{\mathbb{R}}$ be measurable and $-\infty < a < b < +\infty$ hold true. We define the intervals

$$I_0 := [-\infty, a); \ I_k := \Big[\, a + (k-1)\frac{b-a}{m}\,,\ a + k\frac{b-a}{m} \Big); \ I_{m+1} := [\,b, +\infty]$$

with $k = 1, \ldots, m$ for arbitrary $m \in \mathbb{N}$. Furthermore, we choose the intermediate values

$$\eta_l = a + (l-1)\frac{b-a}{m}\,, \qquad l = 0, \ldots, m+1.$$

We infer from Proposition 4.2 that the sets

$$A_l := \Big\{ x \in X \ : \ f(x) \in I_l \Big\}$$

are measurable. The function

$$f_m := \sum_{l=0}^{m+1} \eta_l \, \chi_{A_l}$$

is Lebesgue integrable, and we observe

$$|\, f_m(x)| \le \max\left(|\, 2a - b|, |\, b|\right) \quad \text{for all} \quad x \in X \quad \text{and all} \quad m \in \mathbb{N}.$$

Since constant functions are integrable, Lebesgue's convergence theorem yields

$$f_{a,b}(x) = \lim_{m \to \infty} f_m(x) \, \in L(X).$$

'$\Longleftarrow$' We have to show that the set $\mathcal{O}(f, \tilde{a})$ is measurable for all $\tilde{a} \in \mathbb{R}$. Here we prove: The set $\{x \in X \, : \, f(x) \ge b\}$ is measurable for all $b \in \mathbb{R}$. Then we obtain the measurability of

$$\mathcal{O}(f, \tilde{a}) = \bigcup_{l=1}^{\infty} \left\{ x \in X \mid f(x) \ge \tilde{a} + \frac{1}{l} \right\}$$

via Proposition 3.6 from Section 3. Choosing $b \in \mathbb{R}$ arbitrarily, we take $a = b - 1$ and consider the function

$$g(x) := f_{a,b}(x) - a \, \in L(X).$$

Evidently, $g : X \to [\,0, 1\,]$ holds true and, moreover,

$$g(x) = 1 \quad \Longleftrightarrow \quad f(x) \ge b.$$

The corollary from Theorem 3.4 in Section 3 yields $g^l(x) \in L(X)$ for all $l \in \mathbb{N}$. Now Lebesgue's convergence theorem implies

$$\chi(x) := \lim_{l \to \infty} g^l(x) = \begin{cases} 1\,, & x \in X \text{ with } f(x) \ge b \\ 0\,, & x \in X \text{ with } f(x) < b \end{cases} \quad \in L(X),$$

and consequently $\{x \in X \, : \, f(x) \ge b\}$ is measurable for all $b \in \mathbb{R}$. q.e.d.

Corollary: Each function $f \in L(X)$ is measurable.

Proof: We take $f \in L(X)$, and see that $N := \{x \in X \, : \, |f(x)| = +\infty\}$ is a null-set. Then we define

$$\widetilde{f}(x) := \begin{cases} f(x)\,, & x \in X \setminus N \\ 0\,, & x \in N \end{cases} \quad \in L(X).$$

According to Definition 4.1, the function f is measurable if and only if $\widetilde{f}$ is measurable. We now apply the criterion of Theorem 4.3 on $\widetilde{f}$. When $-\infty < a < b < +\infty$ is arbitrary, we immediately infer

$$\widetilde{f}_{-\infty,b}(x) = \min\left(\widetilde{f}(x), b\right) = \frac{1}{2}\left(\widetilde{f}(x) + b\right) - \frac{1}{2}\,|\, \widetilde{f}(x) - b| \quad \in L(X),$$

because $\widetilde{f} \in L(X)$. Analogously, we deduce $g_{a,+\infty} \in L(X)$ for $g \in L(X)$. Taking the following relation

$$\widetilde{f}_{a,b} = \big(\widetilde{f}_{-\infty,b}\big)_{a,+\infty}$$

into account, we infer $\widetilde{f}_{a,b} \in L(X)$. q.e.d.

In the next theorem there will appear an adequate notion of convergence for measurable functions.

Theorem 4.4. (a.e.-Convergence)
Let $\{f_k\}_{k=1,2,\ldots}$ denote a sequence of measurable functions with the property $f_k(x) \to f(x)$ a.e. in X. Then f is measurable.

Proof: Let us take $a, b \in \mathbb{R}$ with $a < b$. Then the functions $(f_k)_{a,b}$ belong to $L(X)$ for all $k \in \mathbb{N}$, and we have

$$|(f_k)_{a,b}(x)| \le \max(|\,a|, |\,b|) \quad \text{and} \quad (f_k)_{a,b} \to f_{a,b} \quad \text{a.e. in} \quad X.$$

The general convergence theorem of Lebesgue yields $f_{a,b} \in L(X)$. Due to Theorem 4.3, the function f is measurable.

q.e.d.

Theorem 4.5. (Combination of measurable functions)
We have the following statements:

a) Linear Combination: *When f, g are measurable and $\alpha, \beta \in \mathbb{R}$ are chosen, the four functions $\alpha f + \beta g$, $\max(f, g)$, $\min(f, g)$, $|\,f|$ are measurable as well.*
b) Nonlinear Combination: *Let the $\kappa \in \mathbb{N}$ finite-valued measurable functions $f_1, \ldots, f_\kappa$ be given, and furthermore the continuous function $\phi = \phi(y_1, \ldots, y_\kappa) \in C^0(\mathbb{R}^\kappa, \mathbb{R})$. Then the composed function*

$$g(x) := \phi\Big(f_1(x), \ldots, f_\kappa(x)\Big), \quad x \in X$$

is measurable.

Proof:

a) According to Theorem 4.3, we have $f_{-p,p}\,, g_{-p,p} \in L(X)$ for all $p \in \mathbb{R}$. When we note that $f = \lim\limits_{p\to\infty} f_{-p,p}$ holds true, Theorem 4.4 combined with the linearity of the space $L(X)$ imply that the function

$$\alpha f + \beta g = \lim_{p\to+\infty} (\alpha f_{-p,p} + \beta g_{-p,p})$$

is measurable for all $\alpha, \beta \in \mathbb{R}$. In the same way, we see the measurability of the functions

$$\max(f,g) = \lim_{p\to+\infty} \max(f_{-p,p}, g_{-p,p})$$

and

$$\min(f,g) = \lim_{p\to+\infty} \min(f_{-p,p}, g_{-p,p}),$$

as well as $|f|$ - due to $|f| = \max(f,-f)$.

b) The functions $(f_k)_{-p,p} \in L(X)$ are bounded for all $p > 0$ and $k = 1,\ldots,\kappa$. According to Theorem 3.4 in Section 3 and Theorem 4.3 in Section 4, the function $\phi\Big((f_1)_{-p,p}(x),\ldots,(f_\kappa)_{-p,p}(x)\Big)$ belongs to the class $L(X)$. Furthermore, we have the limit relation

$$g(x) = \lim_{p\to+\infty} \phi\Big((f_1)_{-p,p}(x),\ldots,(f_\kappa)_{-p,p}(x)\Big)$$

for all $x \in X$, and Theorem 4.4 finally yields the measurablity of g.q.e.d.

Now we define improper Lebesgue integrals.

Definition 4.6. *We set for a nonnegative measurable function f the integral*

$$I(f) := \lim_{N\to+\infty} I(f_{0,N}) \ \in [0,+\infty].$$

Theorem 4.7. *A measurable function f belongs to the class $L(X)$ if and only if the following limit*

$$\lim_{\substack{a\to-\infty \\ b\to+\infty}} I(f_{a,b}) \ \in \mathbb{R}$$

exists. In this case we have the identity

$$I(f) \ = \lim_{\substack{a\to-\infty \\ b\to+\infty}} I(f_{a,b}) \ = \ I(f^+) - I(f^-).$$

Therefore, a measurable function f belongs to $L(X)$ if and only if $I(f^+) < +\infty$ as well as $I(f^-) < +\infty$ are valid.

Proof: On account of $f_{a,b} = (f^+)_{0,b} - (f^-)_{0,-a}$ for all $-\infty < a < 0 < b < +\infty$ we see

$$\lim_{\substack{a\to-\infty \\ b\to+\infty}} I(f_{a,b}) \ \text{exists in } \mathbb{R} \quad \Longleftrightarrow \quad \lim_{N\to+\infty} I\Big((f^\pm)_{0,N}\Big) \ \text{exist in } \mathbb{R}.$$

Consequently, it suffices to show:

$$f \in L(X) \quad \Longleftrightarrow \quad \lim_{N\to+\infty} I\Big((f^\pm)_{0,N}\Big) \quad \text{exist in} \quad \mathbb{R}.$$

'$\Longrightarrow$' : Let us take $f \in L(X)$. Then we infer $f^\pm \in L(X)$, and B.Levi's theorem on monotone convergence yields

$$\lim_{N\to+\infty} I\Big((f^{\pm})_{0,N}\Big) = I(f^{\pm}) \in \mathbb{R}.$$

'$\Longleftarrow$' : If

$$\lim_{N\to+\infty} I\Big((f^{\pm})_{0,N}\Big)$$

in $\mathbb{R}$ exist, the theorem of B.Levi implies $f^{\pm} \in L(X)$, and together with the identity $f = f^{+} - f^{-}$ the property $f \in L(X)$ is deduced.

q.e.d.

Theorem 4.8. *Let $f : X \to \overline{\mathbb{R}}$ denote a measurable function satisfying*

$$|f(x)| \le F(x), \quad x \in X,$$

with a dominating function $F \in L(X)$. Then we have

$$f \in L(X) \quad \textit{and} \quad I(|f|) \le I(F).$$

Proof: According to Theorem 4.5, the functions f^{+} and f^{-} are measurable, and we see $0 \le f^{\pm} \le F$. Consequently, the estimates $0 \le (f^{\pm})_{0,N} \le F$ and $(f^{\pm})_{0,N} \in L(X)$ are correct. Furthermore, we have

$$I\Big((f^{\pm})_{0,N}\Big) \le I(F) < +\infty \qquad \text{for all} \quad N > 0.$$

B.Levi's theorem now yields $I(f^{\pm}) < +\infty$ and $f^{\pm} \in L(X)$, which implies $f \in L(X)$. On account of the monotonicity of Lebesgue's integral, the estimate $I(|f|) \le I(F)$ follows from the inequality $|f(x)| \le F(x)$.

q.e.d.

Theorem 4.9. *Let $\{f_l\}_{l=1,2,\dots}$ denote a sequence of nonnegative measurable functions satisfying $f_l(x) \uparrow f(x)$, $x \in X$. Then the function f is measurable, and we have*

$$I(f) = \lim_{l\to\infty} I(f_l).$$

Proof: From Theorem 4.4 we infer the measurability of f. According to Definition 4.6, two measurable functions $0 \le g \le h$ satisfy the inequality $I(g) \le I(h)$. Therefore, $\{I(f_l)\}_{l=1,2,\dots} \in [0,+\infty]$ represents a monotonically nondecreasing sequence, such that $I(f) \ge I(f_l)$ for all $l \in \mathbb{N}$ holds true. We distinguish between the following two cases:

a) Let us consider

$$\lim_{l\to\infty} I(f_l) \le c < +\infty.$$

Then we have $I(f_l) \le c$, which implies $f_l \in L(X)$ due to Theorem 4.7. B.Levi's theorem now yields $f \in L(X)$ and

$$I(f) = \lim_{l\to\infty} I(f_l).$$

b) Let us consider
$$\lim_{l\to\infty} I(f_l) = +\infty.$$
Then we note that $I(f) \geq I(f_l)$ for all $l \in \mathbb{N}$ holds true, and we obtain immediately
$$I(f) = +\infty = \lim_{l\to\infty} I(f_l).$$
q.e.d.

Definition 4.10. *We name a function $g : X \to \overline{\mathbb{R}}$* simple *if there exist finitely many mutually disjoint sets $A_1, \ldots, A_{n^*} \in \mathcal{S}(X)$ and numbers $\eta_1, \ldots, \eta_{n^*} \in \mathbb{R}$ with $n^* \in \mathbb{N}$, such that the following representation holds true in X:*
$$g = \sum_{k=1}^{n^*} \eta_k \, \chi_{A_k}.$$

Remark: Evidently, we then have $g \in L(X)$ and
$$I(g) = \sum_{k=1}^{n^*} \eta_k \, \mu(A_k).$$

Let us take an arbitrary decomposition $\mathcal{Z} : -\infty < y_0 < y_1 < \ldots < y_{n^*} < +\infty$ in the real line $\mathbb{R}$, with the intervals $I_k := [\, y_{k-1}, y_k)$ for $k = 1, \ldots, n^*$. Furthermore, we consider an arbitrary measurable function $f : X \to \overline{\mathbb{R}}$ and select arbitrary intermediate values $\eta_k \in I_k$ for $k = 1, \ldots, n^*$. Now we attribute the following simple function to the data $f, \mathcal{Z}$ and η, namely
$$f^{(\mathcal{Z},\eta)} := \sum_{k=1}^{n^*} \eta_k \, \chi_{A_k}$$
with $A_k := \{x \in X \,:\, f(x) \in I_k\}$ for $k = 1, \ldots, n^*$. Then we observe
$$I\Big(f^{(\mathcal{Z},\eta)}\Big) = \sum_{k=1}^{n^*} \eta_k \, \mu(A_k).$$

We denote by a *canonical sequence of decompositions* such a sequence of decompositions, whose start- and end-points tend towards $-\infty$ and $+\infty$, respectively, and whose maximal interval-lengths tend to 0.

Theorem 4.11. *When we consider $f : X \to \overline{\mathbb{R}} \in L(X)$, each canonical sequence of decompositions $\{\mathcal{Z}^{(p)}\}_{p=1,2,\ldots}$ in $\mathbb{R}$ and each choice of intermediate values $\{\eta^{(p)}\}_{p=1,2,\ldots}$ gives us the asymptotic identity*
$$I(f) = \lim_{p\to\infty} I\Big(f^{(\mathcal{Z}^{(p)},\eta^{(p)})}\Big) = \lim_{p\to\infty} \sum_{k=1}^{n^{(p)}} \eta_k^{(p)} \mu(A_k^{(p)}).$$

Remark: Therefore, Lebesgue's integral can be approximated by the Lebesgue sums as above, and the notation

$$I(f) = \int\limits_X f(x)\,d\mu(x)$$

is justified. However, the Riemannian intermediate sums can be evaluated numerically much better than the Lebesgue sums.

Proof of Theorem 4.11: Let us consider the function $f \in L(X)$, a decomposition $\mathcal{Z}$ with its *fineness* $\delta(\mathcal{Z}) = \max\{(y_k - y_{k-1}) : k = 1, \ldots, n^*\}$, and arbitrary intermediate values $\{\eta_k\}_{k=1,\ldots,n^*}$. Then we infer the estimate

$$|f^{(\mathcal{Z},\eta)}(x)| \le \delta(\mathcal{Z}) + |f(x)| \quad \text{for all} \quad x \in X.$$

When $\{\mathcal{Z}^{(p)}\}_{p=1,2,\ldots}$ describes a canonical sequence of decompositions and $\{\eta^{(p)}\}_{p=1,2,\ldots}$ denote arbitrary intermediate values, we observe the limit relation

$$f^{(\mathcal{Z}^{(p)},\eta^{(p)})}(x) \to f(x) \quad \text{a.e. for} \quad p \to \infty,$$

which is valid for all $x \in X$ with $|f(x)| \ne +\infty$. Now Lebesgue's convergence theorem yields

$$I(f) = \lim_{p\to\infty} I\Big(f^{(\mathcal{Z}^{(p)},\eta^{(p)})}\Big) = \lim_{p\to\infty} \sum_{k=1}^{n^{(p)}} \eta_k^{(p)} \mu(A_k^{(p)}).$$

q.e.d.

Now we shall present a selection theorem related to a.e.-convergence.

Theorem 4.12. (Lebesgue's selection theorem)
Let $\{f_k\}_{k=1,2,\ldots}$ denote a sequence in $L(X)$ satisfying

$$\lim_{k,l\to\infty} I(|\,f_k - f_l|) = 0.$$

Then a null-set $N \subset X$ as well as a monotonically increasing subsequence $\{k_m\}_{m=1,2,\ldots}$ exist, such that the sequence of functions $\{f_{k_m}(x)\}_{m=1,2,\ldots}$ converges for all points $x \in X \setminus N$ and their limit fulfills

$$\lim_{m\to\infty} f_{k_m}(x) =: f(x) \in L(X).$$

Therefore, we can select an a.e. convergent subsequence from a Cauchy sequence with respect to the integral I.

Proof: On the null-set

$$N_1 := \bigcup_{k=1}^{\infty} \Big\{x \in X \,:\, |f_k(x)| = +\infty\Big\}$$

we modify the functions f_k and obtain

$$\widetilde{f}_k(x) := \begin{cases} f_k(x)\,, & x \in X \setminus N_1 \\ 0\,, & x \in N_1 \end{cases}.$$

Without loss of generality, we can assume the functions $\{f_k\}_{k=1,2,\ldots}$ to be finite-valued. On account of

$$\lim_{p,l\to\infty} I(|\,f_p - f_l|) = 0,$$

we find a subsequence $k_1 < k_2 < \cdots$ with the property

$$I(|\,f_p - f_l|) \le \frac{1}{2^m} \quad \text{for all} \quad p, l \ge k_m, \quad m = 1, 2, \ldots.$$

In particular, we infer the following estimates:

$$I(|\,f_{k_{m+1}} - f_{k_m}|) \le \frac{1}{2^m}, \qquad m = 1, 2, \ldots$$

and

$$\sum_{m=1}^{\infty} I(|\,f_{k_{m+1}} - f_{k_m}|) \le 1.$$

B.Levi's theorem tells us that the function

$$g(x) := \sum_{m=1}^{\infty} |\,f_{k_{m+1}}(x) - f_{k_m}(x)|, \qquad x \in X$$

belongs to $L(X)$, and $N_2 := \{x \in X \setminus N_1 \;:\; |\,g(x)| = +\infty\}$ represents a null-set. Therefore, the series

$$\sum_{m=1}^{\infty} |\,f_{k_{m+1}}(x) - f_{k_m}(x)| \quad \text{for all} \quad x \in X \setminus N \quad \text{with} \quad N := N_1 \cup N_2$$

converges, as well as the series

$$\sum_{m=1}^{\infty} \Big(f_{k_{m+1}}(x) - f_{k_m}(x)\Big).$$

Consequently, the limit

$$\lim_{m\to\infty} \Big(f_{k_m}(x) - f_{k_1}(x)\Big) =: f(x) - f_{k_1}(x)$$

exists for all points $x \in X \setminus N$, and the sequence $\{f_{k_m}\}_{m=1,2,\ldots}$ converges on $X \setminus N$ towards f. We note that $g \in L(X)$ and $|\,f_{k_m}(x) - f_{k_1}(x)| \le |\,g(x)|$ are valid, and Lebesgue's convergence theorem is applicable. Finally, we infer $f \in L(X)$ and the relation

$$I(f) = \lim_{m\to\infty} I(f_{k_m}).$$

q.e.d.

Proposition 4.13. (Approximation in the integral)
Let the function $f \in L(X)$ be given. To each quantity $\varepsilon > 0$, we then find a function $f_\varepsilon \in M(X)$ satisfying

$$I(|\,f - f_\varepsilon|) < \varepsilon.$$

Proof: Since $f \in L(X)$ holds true, Proposition 2.9 from Section 2 provides two functions $g \in -V$ and $h \in V$ such that

$$g(x) \le f(x) \le h(x), \quad x \in X, \quad \text{and} \quad I(h) - I(g) < \frac{\varepsilon}{2}.$$

Recalling the definition of the space $V(X)$, we find a function $h'(x) \in M(X)$ satisfying

$$h'(x) \le h(x), \quad x \in X, \quad \text{and} \quad I(h) - I(h') < \frac{\varepsilon}{2}.$$

This implies

$$|\,f - h'| \le |\,f - h| + |\,h - h'| \le (h - g) + (h - h'),$$

and the monotonicity and linearity of the integral yield

$$I(|\,f - h'|) \le (I(h) - I(g)) + (I(h) - I(h')) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

With $f_\varepsilon := h'$ we obtain the desired function. q.e.d.

Theorem 4.14. (a.e.-Approximation)
Let f denote a measurable function satisfying $|\,f(x)| \le c$, $x \in X$ with the constant $c \in (0, +\infty)$. Then we have a sequence $\{f_k\}_{k=1,2,\ldots} \subset M(X)$ satisfying $|\,f_k(x)| \le c,\ x \in X$ for all $k \in \mathbb{N}$, such that $f_k(x) \to f(x)$ a.e. in X holds true.

Proof: Since f is measurable and dominated by the constant function $c \in L(X)$, we infer $f \in L(X)$ from Theorem 4.8. Now Proposition 4.13 allows us to find a sequence $\{g_k(x)\}_{k=1,2,\ldots} \subset M(X)$ satisfying $I(|\,f - g_k|) \to 0$ for $k \to \infty$. We set

$$h_k(x) := (g_k)_{-c,c}(x)$$

and observe $h_k \in M(X)$ as well as $|\,h_k(x)| \le c$ for all $x \in X$ and all $k \in \mathbb{N}$. We note that

$$|\,h_k - f| = |\,(g_k)_{-c,c} - f_{-c,c}| = |\,(g_k - f)_{-c,c}| \le |\,g_k - f|$$

is correct and see

$$\lim_{k\to\infty} I(|\,h_k - f|) \le \lim_{k\to\infty} I(|\,g_k - f|) = 0.$$

On account of the relation

$$I(|\,h_k - h_l|) \le I(|\,h_k - f|) + I(|\,f - h_l|) \longrightarrow 0 \qquad \text{for} \quad k,l \to \infty,$$

Lebesgue's selection theorem yields a null-set $N_1 \subset X$ and a monotonically increasing subsequence $\{k_m\}_{m=1,2,\ldots}$ such that the following limit exists:

$$h(x) := \lim_{m\to\infty} h_{k_m}(x) \quad \text{for all} \quad x \in X \setminus N_1.$$

We extend h onto the null-set by the prescription $h(x) := 0$ for all $x \in N_1$. Now we conclude

$$\lim_{m\to\infty} |\,h_{k_m}(x) - f(x)| = |\,h(x) - f(x)| \quad \text{in} \quad X \setminus N_1.$$

The theorem of Fatou yields

$$I(|\,h - f|) \le \lim_{m\to\infty} I(|\,h_{k_m} - f|) = 0.$$

Consequently, we find a null-set $N_2 \subset X$ such that

$$f(x) = h(x) \qquad \text{for all} \quad x \in X \setminus N_2$$

holds true. When we define $N := N_1 \cup N_2$ and $f_m(x) := h_{k_m}(x)$, we obviously infer $f_m(x) \in M(X)$, $|f_m(x)| \le c$ for all $x \in X$ and all $m \in \mathbb{N}$, and the following limit relation:

$$\lim_{m\to\infty} f_m(x) = \lim_{m\to\infty} h_{k_m} \overset{x\notin N_1}{=} h(x) \overset{x\notin N_2}{=} f(x) \qquad \text{for all} \quad x \in X \setminus N.$$

Consequently, we obtain $f_m(x) \to f(x)$ for all $x \in X \setminus N$. q.e.d.

Uniform convergence and a.e.-convergence are connected by the following result.

Theorem 4.15. (Egorov)
Let the measurable set $B \subset X$ as well as the measurable a.e.-finite-valued functions $f : B \to \overline{\mathbb{R}}$ and $f_k : B \to \overline{\mathbb{R}}$ for all $k \in \mathbb{N}$ be given, with the convergence property $f_k(x) \to f(x)$ a.e. in B. To each quantity $\delta > 0$, we then find a closed set $A \subset B$ satisfying $\mu(B \setminus A) < \delta$ such that the limit relation, $f_k(x) \to f(x)$ uniformly on A, holds true.

Proof: We consider the null-set

$$\begin{aligned} N &:= \Big\{x \in B \,:\, f_k(x) \to f(x) \text{ is not satisfied}\Big\} \\ &= \left\{x \in B \,:\, \begin{array}{l} \text{To } m \in \mathbb{N} \text{ and for all } l \in \mathbb{N} \text{ exists} \\ \text{an index } k \ge l \text{ with } |\,f_k(x) - f(x)| > \dfrac{1}{m} \end{array}\right\} \\ &= \bigcup_{m=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{k\ge l} \left\{x \in B \,:\, |f_k(x) - f(x)| > \frac{1}{m}\right\} = \bigcup_{m=1}^{\infty} B_m, \end{aligned}$$

where

$$B_m := \bigcap_{l=1}^{\infty} \bigcup_{k \geq l} \left\{ x \in B \,:\, |f_k(x) - f(x)| > \frac{1}{m} \right\}$$

has been defined. We observe $B_m \subset N$ and consequently $\mu(B_m) = 0$ for all $m \in \mathbb{N}$. We note that

$$B_{m,l} := \bigcup_{k \geq l} \left\{ x \in B \,:\, |f_k(x) - f(x)| > \frac{1}{m} \right\}$$

holds true and infer $B_{m,l} \supset B_{m,l+1}$ for all $m, l \in \mathbb{N}$. From the relation

$$B_m = \bigcap_{l=1}^{\infty} B_{m,l}$$

we then obtain

$$0 = \mu(B_m) = \lim_{l \to \infty} \mu(B_{m,l}).$$

Consequently, to each index $m \in \mathbb{N}$ we find an index $l_m \in \mathbb{N}$ with $l_m < l_{m+1}$ such that

$$\mu\left(\bigcup_{k \geq l_m} \left\{ x \in B \,:\, |f_k(x) - f(x)| > \frac{1}{m} \right\} \right) = \mu(B_{m,l_m}) < \frac{\delta}{2^{m+1}}$$

holds true. We define

$$\widehat{B}_m := B_{m,l_m} \quad \text{and} \quad \widehat{B} := \bigcup_{m=1}^{\infty} \widehat{B}_m.$$

Evidently, the set $\widehat{B}$ is measurable and the estimate

$$\mu(\widehat{B}) \leq \sum_{m=1}^{\infty} \mu(\widehat{B}_m) \leq \frac{\delta}{2}$$

is fulfilled. When we still define $\widehat{A} := B \setminus \widehat{B}$, we comprehend

$$\widehat{A} = B \cap \left(\bigcup_{m=1}^{\infty} \widehat{B}_m \right)^c = B \cap \left(\bigcap_{m=1}^{\infty} \widehat{B}_m^c \right)$$

$$= \bigcap_{m=1}^{\infty} \left\{ x \in B \,:\, |f_k(x) - f(x)| \leq \frac{1}{m} \text{ for all } k \geq l_m \right\}.$$

For all points $x \in \widehat{A}$, we find an index $l_m \in \mathbb{N}$ to a given $m \in \mathbb{N}$ such that

$$|\, f_k(x) - f(x)| \leq \frac{1}{m} \quad \text{for all} \quad k \geq l_m$$

holds true. Consequently, the sequence $\{f_k|_{\widehat{A}}\}_{k=1,2,\ldots}$ converges uniformly towards $f|_{\widehat{A}}$. According to Theorem 3.14 in Section 3, we now choose a closed set $A \subset \widehat{A}$ with

$$\mu(\widehat{A} \setminus A) < \frac{\delta}{2}.$$

We note that $A \subset \widehat{A}$ holds true, and the sequence of functions $\{f_k\,|_A\}_{k=1,2,\ldots}$ converges uniformly towards $f\,|_A$. When we additionally observe $B \setminus \widehat{A} = \widehat{B}$, we finally see

$$\mu(B \setminus A) = \mu(B \setminus \widehat{A}) + \mu(\widehat{A} \setminus A) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

q.e.d.

The interrelation between measurable and continuous functions is revealed by the following result.

Theorem 4.16. (Lusin)
Let $f : B \to \mathbb{R}$ denote a measurable function on the measurable set $B \subset X$. To each quantity $\delta > 0$, we then find a closed set $A \subset X$ with the property $\mu(B \setminus A) < \delta$ such that the restriction $f\,|_A : A \to \mathbb{R}$ is continuous.

Proof: For $j = 1, 2, \ldots$ we consider the truncated functions

$$f_j(x) := \begin{cases} -j\,, & f(x) \in [-\infty, -j] \\ f(x)\,, & f(x) \in [-j, +j] \\ +j\,, & f(x) \in [+j, +\infty] \end{cases}.$$

All functions $f_j : B \to \mathbb{R}$ are measurable, and we infer

$$|\,f_j(x)| \le j \qquad \text{for all} \quad x \in B.$$

We utilize Theorem 4.14 and the property $M(X) \subset C^0(X)$: For each index $j \in \mathbb{N}$, there exists a sequence of continuous functions $f_{j,k} : B \to \mathbb{R}$ satisfying

$$\lim_{k\to\infty} f_{j,k}(x) = f_j(x) \qquad \text{a.e. in} \quad B.$$

Via Egorov's theorem, we find a closed set $A_j \subset B$ to each $j = 1, 2, \ldots$ satisfying

$$\mu(B \setminus A_j) < \frac{\delta}{2^{j+1}},$$

such that the sequence of functions $\{f_{j,k}|_{A_j}\}_{k=1,2,\ldots}$ converges uniformly towards the function $f_j|_{A_j}$. The Weierstraß convergence theorem reveals continuity of the functions $f_j|_{A_j}$ for all $j \in \mathbb{N}$. The set

$$\widehat{A} := \bigcap_{j=1}^{\infty} A_j \subset B$$

is closed, and we arrive at the estimate

$$\mu(B \setminus \widehat{A}) \leq \sum_{j=1}^{\infty} \mu(B \setminus A_j) < \sum_{j=1}^{\infty} \frac{\delta}{2^{j+1}} = \frac{\delta}{2}.$$

Now the functions $f_j : \widehat{A} \to \mathbb{R}$ are continuous for all $j \in \mathbb{N}$, and we recall

$$f(x) = \lim_{j\to\infty} f_j(x) \qquad \text{in} \quad \widehat{A}.$$

Egorov's theorem supplies a closed set $A \subset \widehat{A}$ with

$$\mu(\widehat{A} \setminus A) < \frac{\delta}{2},$$

such that f_j converges uniformly on A towards f. Consequently, the function $f\,|_A$ is continuous, and we estimate as follows:

$$\mu(B \setminus A) = \mu(B \setminus \widehat{A}) + \mu(\widehat{A} \setminus A) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

q.e.d.

Remark: We have learned the *Three principles of Littlewood* in Lebesgue's theory of measure and integration. J.E.LITTLEWOOD: "There are three principles roughly expressible in the following terms: Every measurable set is nearly a finite union of intervals; every measurable function is nearly continuous; every a.e. convergent sequence of measurable functions is nearly uniformly convergent."

5 Riemann's and Lebesgue's Integral on Rectangles

With $d \in (0, +\infty)$ being given, we consider the rectangle

$$Q := \Big\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, |x_j| \leq d\,,\ j = 1, \ldots, n \Big\}, \quad \text{where} \quad n \in \mathbb{N}.$$

In our main example from Section 1, we choose $X = \Omega := \overset{\circ}{Q}$ and extend the improper Riemannian integral

$$I \,:\, M(X) \longrightarrow \mathbb{R}, \qquad \text{with} \quad f \mapsto I(f) := \int_{\Omega} f(x)\,dx$$

from the space

$$M(X) := \left\{ f \in C^0(\Omega) \,:\, \int_{\Omega} |\,f(x)|\,dx < +\infty \right\}$$

onto the space $L(X) \supset M(X)$ and obtain Lebesgue's integral $I : L(X) \to \mathbb{R}$.

Theorem 5.1. *For the set $E \subset \Omega$ being given, the following statements are equivalent:*

(1) E is a null-set.
(2) To each quantity $\varepsilon > 0$, we find with $\{Q_k\}_{k=1,2,\ldots} \subset \Omega$ denumerably many rectangles satisfying $E \subset \bigcup_{k=1}^{\infty} Q_k$ *and* $\sum_{k=1}^{\infty} |Q_k| < \varepsilon$.

Proof:

(1)$\Longrightarrow$(2): Since E represents a null-set, Proposition 3.15 from Section 3 provides a function $h \in V(X)$ with $h \geq 0$ on X, $h = +\infty$ on E, and $I(h) < +\infty$. With the constant $c \in [1, +\infty)$ chosen arbitrarily, we consider the open - and consequently measurable - set

$$E_c := \Big\{ x \in \Omega \,:\, h(x) > c \Big\} \supset E.$$

Then we observe

$$\mu(E_c) = I(\chi_{E_c}) = \frac{1}{c}\, I(c\,\chi_{E_c}) \leq \frac{1}{c}\, I(h) < \varepsilon$$

for $c > \frac{I(h)}{\varepsilon}$. The open set E_c can be represented as a denumerable union of closed rectangles Q_k which intersect, at most, in boundary points. Therefore, we deduce

$$E \subset E_c = \bigcup_{k=1}^{\infty} Q_k.$$

We note that the boundary points of a rectangle constitute a null-set and see

$$\sum_{k=1}^{\infty} |Q_k| = \mu(E_c) < \varepsilon.$$

(2)$\Longrightarrow$(1): For each index $k \in \mathbb{N}$ we find a function $h_k \in C_0^0(\Omega)$ satisfying

$$h_k(x) = \begin{cases} 1 & ,\ x \in Q_k \\ \in [0,1] & ,\ x \in \mathbb{R}^n \setminus Q_k \end{cases} \qquad \text{and} \qquad I(h_k) \leq 2|Q_k|.$$

The sequence $\{g_l(x)\}_{l=1,2,\ldots}$, defined by $g_l(x) := \sum_{k=1}^{l} h_k(x)$, converges monotonically and belongs to $M(X)$. This implies

$$h(x) := \sum_{k=1}^{\infty} h_k(x) \in V(X).$$

Furthermore, we have $\chi_E(x) \leq h(x),\ x \in \mathbb{R}^n$ and estimate

$$0 \leq I^-(\chi_E) \leq I^+(\chi_E) \leq I(h) = \sum_{k=1}^{\infty} I(h_k) \leq 2 \sum_{k=1}^{\infty} |Q_k| < 2\varepsilon$$

for all $\varepsilon > 0$. Therefore, E is a null-set. q.e.d.

Riemann's and Lebesgue's integral are compared as follows:

Theorem 5.2. *A bounded function $f : \Omega \to \mathbb{R}$ is Riemann integrable if and only if the set K, containing all points of discontinuities, constitutes a null-set. In this case the function f belongs to the class $L(\Omega)$, and we have the identity*

$$I(f) = \int\limits_{\Omega} f(x)\,dx = \int_Q f(x)\,dx;$$

this means that Riemann's integral of f coincides with Lebesgue's integral. Here we have to extend f to 0 onto the whole space $\mathbb{R}^n$.

Proof: We consider the functions

$$m^+(x) := \lim_{\varepsilon\to 0+} \sup_{|y-x|<\varepsilon} f(y) \quad \text{and} \quad m^-(x) := \lim_{\varepsilon\to 0+} \inf_{|y-x|<\varepsilon} f(y)\,,\ x \in \mathbb{R}^n.$$

We have the identity $m^+(x) = m^-(x)$ if and only if f is continuous at the point x. Let

$$\mathcal{Z}\ :\ Q = \bigcup_{k=1}^{N} Q_k$$

denote a canonical decomposition of Q into N closed rectangles Q_k. We define the simple functions

$$m_k^+ := \sup_{Q_k} f(y), \quad m_k^- := \inf_{Q_k} f(y) \quad \text{and} \quad f_{\mathcal{Z}}^{\pm}(x) := \sum_{k=1}^{N} m_k^{\pm}\chi_{Q_k}(x) \in L(X).$$

We observe the identity

$$I(f_{\mathcal{Z}}^{\pm}) = \sum_{k=1}^{N} m_k^{\pm}|\,Q_k|.$$

Therefore, Lebesgue's integral of the functions $f_{\mathcal{Z}}^{\pm}$ coincides with the Riemannian upper and lower sums, respectively, of the function f - associated with the decomposition $\mathcal{Z}$. When we denote by

$$\partial\mathcal{Z} := \bigcup_{k=1}^{N} \partial Q_k$$

the set of the boundary points for the decomposition $\mathcal{Z}$, then $\partial\mathcal{Z}$ constitutes a null-set in $\mathbb{R}^n$. Now we observe an arbitrary canonical sequence of decompositions $\{\mathcal{Z}_p\}_{p=1,2,\ldots}$ for the rectangle Q, such that its fineness tends to 0. We obtain the limit relation

$$\lim_{p\to\infty} f_{\mathcal{Z}_p}^{\pm}(x) = m^{\pm}(x) \quad \text{for all} \quad x \in \Omega \setminus N,$$

where

$$N = \bigcup_{p=1}^{\infty} \partial \mathcal{Z}_p \subset Q$$

is a null-set. Now we select an adequate canonical sequence of decompositions such that

$$\underline{\int_Q} f(x)\,dx = \lim_{p\to\infty} I(f^-_{\mathcal{Z}_p}) \quad \text{and} \quad \overline{\int_Q} f(x)\,dx = \lim_{p\to\infty} I(f^+_{\mathcal{Z}_p}).$$

Lebesgue's convergence theorem implies

$$\underline{\int_Q} f(x)\,dx = I(m^-) \quad \text{and} \quad \overline{\int_Q} f(x)\,dx = I(m^+).$$

Now we note that the function $f : \Omega \to \mathbb{R}$ is Riemann integrable if and only if

$$I(m^+) = \overline{\int_Q} f(x)\,dx = \underline{\int_Q} f(x)\,dx = I(m^-) \quad \text{or equivalently} \quad I(m^+ - m^-) = 0$$

holds true. Due to $m^+ \geq m^-$, this is exactly the case if $m^+ = m^-$ a.e. in Q holds true, or equivalently if f is continuous a.e. on Q.

q.e.d.

We intend to prove Fubini's theorem interchanging the order of integration for Lebesgue integrable functions. Here we consider two open bounded rectangles $Q \subset \mathbb{R}^p$ and $R \subset \mathbb{R}^q$ and begin with the following

Proposition 5.3. *Let* $f = f(x,y) : Q \times R \to \overline{\mathbb{R}} \in V(Q \times R)$ *be given. Then the function* $f(x,y)\,,\ y \in R$ *belongs to the class* $V(R)$ *for each* $x \in Q$*, and the function*

$$\varphi(x) := \int_R f(x,y)\,dy$$

belongs to the class $V(Q)$*. Furthermore, we have*

$$\iint_{Q\times R} f(x,y)\,dxdy = \int_Q \varphi(x)\,dx.$$

Proof: Since $f \in V(Q \times R)$ holds true, we find a sequence $\{f_n(x,y)\}_{n=1,2,\ldots} \subset C_0^0(Q \times R)$ satisfying $f_n(x,y) \uparrow f(x,y)$. For each $x \in Q$, the functions $f_n(x,y),\quad y \in R$ belong to the class $C_0^0(R)$ and consequently $f(x,y)$ to $V(R)$. When we define

$$\varphi_n(x) := \int_R f_n(x,y)\,dy, \qquad x \in Q,$$

we infer $\varphi_n \in C_0^0(Q)$ and $\varphi_n(x) \uparrow \varphi(x)$ in Q. This implies

$$\iint\limits_{Q\times R} f(x,y)\,dxdy := \lim_{n\to\infty} \iint\limits_{Q\times R} f_n(x,y)\,dxdy = \lim_{n\to\infty} \int\limits_Q \varphi_n(x)\,dx = \int\limits_Q \varphi(x)\,dx.$$

q.e.d.

Proposition 5.4. *Let N denote a null-set in $Q \times R$ and define*

$$N_x := \Big\{ y \in R \,:\, (x,y) \in N \Big\}.$$

Then we have a null-set $E \subset Q$, such that N_x constitutes a null-set in R for all points $x \in Q \setminus E$.

Proof: Since N is a null-set, we find a function $h(x,y) \in V(Q \times R)$ with $h \geq 0$ on $Q \times R$ and $h(x,y) = +\infty$ for all $(x,y) \in N$, such that the property

$$+\infty > \iint\limits_{Q\times R} h(x,y)\,dxdy = \int\limits_Q \varphi(x)\,dx \qquad \text{with} \quad \varphi(x) := \int\limits_R h(x,y)\,dy \geq 0$$

holds true - due to Proposition 5.3. We note that $\varphi \in V(Q)$ and

$$\int\limits_Q \varphi(x)\,dx < +\infty$$

is satisfied and deduce $\varphi \in L(Q)$. Furthermore, we find a null-set $E \subset Q$ with $\varphi(x) < +\infty$ for all $x \in Q \setminus E$. On account of $h = +\infty$ on N, the set N_x is a null-set for all $x \in Q \setminus E$.

q.e.d.

Theorem 5.5. (Fubini) *Let $f(x,y) : Q \times R \to [0,+\infty]$ represent a measurable function. Then we have a null-set $E \subset Q$, such that the function $f(x,y)$, $y \in R$ is measurable for all points $x \in Q \setminus E$. When we define*

$$\varphi(x) := \begin{cases} \displaystyle\int\limits_R f(x,y)\,dy\,, & x \in Q \setminus E \\ 0\,, & x \in E \end{cases},$$

the function φ is nonnegative and measurable. Furthermore, we have Fubini's identity

$$\iint\limits_{Q\times R} f(x,y)\,dxdy = \int\limits_Q \varphi(x)\,dx.$$

Proof: For $n = 1, 2, \ldots$ we consider the functions

$$f_n(x,y) := \begin{cases} f(x,y), & \text{if } f(x,y) \in [0,n] \\ n, & \text{otherwise} \end{cases}$$

with $f_n \in L(Q \times R)$. Applying Theorem 4.14 from Section 4, we find for each number $n \in \mathbb{N}$ a null-set $N_n \subset Q \times R$ and a sequence of functions

$$f_{n,m}(x,y) \in C_0^0(Q \times R) \quad \text{with} \quad |f_{n,m}| \leq n \quad \text{on} \quad Q \times R,$$

such that

$$\lim_{m\to\infty} f_{n,m}(x,y) = f_n(x,y) \qquad \text{for all} \quad (x,y) \in (Q \times R) \setminus N_n.$$

Each fixed number $n \in \mathbb{N}$ admits a null-set $E_n \subset Q$, such that

$$\{y \in R \, : \, (x,y) \in N_n\} \subset R$$

represents a null-set for all points $x \in Q \setminus E_n$. Now Lebesgue's convergence theorem yields

$$\iint_{Q\times R} f_n(x,y)\, dxdy$$

$$= \lim_{m\to\infty} \iint_{Q\times R} f_{n,m}(x,y)\, dxdy = \lim_{m\to\infty} \int_Q \left(\int_R f_{n,m}(x,y)\, dy \right) dx$$

$$= \lim_{m\to\infty} \int_{Q\setminus E_n} \left(\int_R f_{n,m}(x,y)\, dy \right) dx = \int_{Q\setminus E_n} \left(\int_R \underbrace{f_n(x,y)}_{\in L(R)}\, dy \right) dx.$$

In addition,

$$E := \bigcup_{n=1}^{\infty} E_n \subset Q$$

constitutes a null-set, and we see

$$\iint_{Q\times R} f_n(x,y)\, dxdy = \int_{Q\setminus E} \left(\int_R f_n(x,y)\, dy \right) dx.$$

Finally, Theorem 4.9 from Section 4 yields

$$\iint_{Q\times R} f(x,y)\, dxdy = \lim_{n\to\infty} \left(\iint_{Q\times R} f_n(x,y)\, dxdy \right)$$

$$= \lim_{n\to\infty} \int_{Q\setminus E} \left(\int_R f_n(x,y)\, dy \right) dx = \int_{Q\setminus E} \left(\int_R f(x,y)\, dy \right) dx = \int_Q \varphi(x)\, dx.$$

q.e.d.

6 Banach and Hilbert Spaces

We owe the basic concepts for linear spaces, which appear in the next sections, to the mathematicians D. Hilbert and S. Banach. Here we can equally consider real and complex vector spaces.

Definition 6.1. *Let $\mathcal{M}$ denote a real (or complex) linear space, which means*

$$f, g \in \mathcal{M},\ \alpha, \beta \in \mathbb{R}\ (or \mathbb{C}) \quad \Longrightarrow \quad \alpha f + \beta g \in \mathcal{M}.$$

Then we name $\mathcal{M}$ a normed real (or complex) linear space *and equivalently a* normed vector space *if we have a function*

$$\| \cdot \| : \mathcal{M} \longrightarrow [0, +\infty)$$

with the following properties:

(N1) $\|f\| = 0 \iff f = 0$;
(N2) Triangle inequality: $\|f + g\| \le \|f\| + \|g\|$ *for all* $f, g \in \mathcal{M}$;
(N3) Homogeneity: $\|\lambda f\| = |\lambda| \|f\|$ *for all* $f \in \mathcal{M},\ \lambda \in \mathbb{R}\ (or \mathbb{C})$.

The function $\| \cdot \|$ is called the norm on $\mathcal{M}$.

Remark: From the axioms (N1), (N2), and (N3) we immediately infer the inequality

$$\|f - g\| \ge \Big| \|f\| - \|g\| \Big| \quad \text{for all} \quad f, g \in \mathcal{M},$$

because we have

$$\|f\| - \|g\| = \|f - g + g\| - \|g\| \le \|f - g\| + \|g\| - \|g\| = \|f - g\|,$$

which yields our statement by interchanging f and g.

Definition 6.2. *The normed vector space $\mathcal{M}$ is named* complete*, if each Cauchy sequence in $\mathcal{M}$ converges. This means, to each sequence $\{f_n\} \subset \mathcal{M}$ satisfying $\lim_{k,l \to \infty} \|f_k - f_l\| = 0$ we find an element $f \in \mathcal{M}$ with* $\lim_{k \to \infty} \|f - f_k\| = 0$.

Definition 6.3. *A complete normed vector space is named a* Banach space.

Example 6.4. Choosing the compact set $K \subset \mathbb{R}^n$, we endow the space $\mathcal{B} := C^0(K, \mathbb{R})$ with the norm

$$\|f\| := \sup_{x \in K} |f(x)| = \max_{x \in K} |f(x)|, \qquad f \in \mathcal{B},$$

and thus obtain a Banach space. This norm generates the uniform convergence - a concept already introduced by Weierstraß.

Definition 6.5. *A complex linear space $\mathcal{H}'$ is named* pre-Hilbert-space *if an* inner product *is defined in $\mathcal{H}'$; more precisely, we have a function*

$$(\cdot,\cdot)\,:\,\mathcal{H}'\times\mathcal{H}'\longrightarrow\mathbb{C}$$

with the following properties:

(H1) $(f+g,h)=(f,h)+(g,h)$ *for all* $f,g,h\in\mathcal{H}'$;
(H2) $(f,\lambda g)=\lambda(f,g)$ *for all* $f,g\in\mathcal{H}',\ \lambda\in\mathbb{C}$;
(H3) Hermitian character: $(f,g)=\overline{(g,f)}$ *for all* $f,g\in\mathcal{H}'$;
(H4) Positive-definite character: $(f,f)>0,$ *if* $f\neq 0$.

Remarks:

1. We infer the following calculus rule from the axioms (H1) - (H4) immediately:
   (H5) For all $f,g,h\in\mathcal{H}'$ we have

$$(f,g+h)=\overline{(g+h,f)}=\overline{(g,f)}+\overline{(h,f)}=(f,g)+(f,h).$$

   (H6) Furthermore, the relation

$$(\lambda f,g)=\overline{\lambda}(f,g)\qquad\text{for all}\quad f,g\in\mathcal{H}',\quad\lambda\in\mathbb{C}$$

   is satisfied.
   Therefore, the inner product is antilinear in its first and linear in its second argument.
2. In a *real linear space* $\mathcal{H}'$, an inner product is characterized by the properties (H1) - (H4) as well, where (H3) then reduces to the symmetry condition

$$(f,g)=(g,f)\qquad\text{for all}\quad f,g\in\mathcal{H}'.$$

Example 6.6. Let us consider the numbers $-\infty<a<b<+\infty$ and the space $\mathcal{H}':=C^0([a,b],\mathbb{C})$ of continuous functions. Via the inner product

$$(f,g):=\int_a^b\overline{f(x)}g(x)\,dx,$$

the set $\mathcal{H}'$ becomes a pre-Hilbert-space.

Theorem 6.7. *Let $\mathcal{H}'$ represent a pre-Hilbert-space. With the aid of the norm*

$$\|f\|:=\sqrt{(f,f)},$$

the set $\mathcal{H}'$ becomes a normed vector space.

Proof:

1. At first, we show that the following inequality is valid in $\mathcal{H}'$, namely
$$|(g,f)| = |(f,g)| \leq \|f\|\|g\| \quad \text{for all} \quad f,g \in \mathcal{H}'.$$
With $f, g \in \mathcal{H}'$, we associate a quadratic form in $\lambda, \mu \in \mathbb{C}$ as follows:
$$\begin{aligned} 0 \ \leq \ Q(\lambda,\mu) &:= (\lambda f - \mu g, \lambda f - \mu g) \\ &= |\lambda|^2(f,f) - \lambda\overline{\mu}(g,f) - \overline{\lambda}\mu(f,g) + |\mu|^2(g,g). \end{aligned}$$
When $(g, f) = (f, g) = 0$ - in particular $f = 0$ or $g = 0$ - holds true, this inequality is evident. In the other case, we choose
$$\lambda = 1, \quad \overline{\mu} = \frac{\|f\|^2}{(g,f)}.$$
The nonnegative character of Q - easily seen from the property $(H4)$ - implies the inequality
$$0 \leq -\|f\|^2 + \frac{\|f\|^4\|g\|^2}{|(f,g)|^2}$$
and finally by rearrangement
$$|(f,g)| \leq \|f\|\,\|g\| \quad \text{for all} \quad f,g \in \mathcal{H}'.$$
2. Now we show that $\|f\| := \sqrt{(f,f)}$ satisfies the norm conditions (N1) - (N3). We infer for all elements $f, g \in \mathcal{H}'$ and $\lambda \in \mathbb{C}$ the following properties:
   i.) $\|f\| \geq 0$, and (H4) tells us that $\|f\| = 0$ is fulfilled if and only if $f = 0$ is correct;
   ii.) $\|\lambda f\| = \sqrt{(\lambda f, \lambda f)} = \sqrt{\lambda\overline{\lambda}(f,f)} = |\lambda|\,\|f\|$;
   iii.)
$$\begin{aligned} \|f+g\|^2 &= (f+g, f+g) = (f,f) + 2\mathrm{Re}(f,g) + (g,g) \\ &\leq \|f\|^2 + 2|(f,g)| + \|g\|^2 \\ &\leq \|f\|^2 + 2\|f\|\,\|g\| + \|g\|^2 \\ &= (\|f\| + \|g\|)^2, \end{aligned}$$
   and consequently
$$\|f+g\| \leq \|f\| + \|g\|.$$

Therefore, $\|\cdot\|$ gives us a norm on $\mathcal{H}'$. q.e.d.

Definition 6.8. *A pre-Hilbert-space* $\mathcal{H}$ *is named* Hilbert space*, if* $\mathcal{H}$ *endowed with the norm*
$$\|f\| := \sqrt{(f,f)}, \qquad f \in \mathcal{H}$$
is complete and consequently a Banach space.

Remarks:

1. We prove that the inner product (f, g) is continuous in $\mathcal{H}$. Here we note the following estimate for the elements $f, g, f_n, g_n \in \mathcal{H}$:
$$\begin{aligned} |(f_n, g_n) - (f, g)| &= |(f_n, g_n) - (f_n, g) + (f_n, g) - (f, g)| \\ &\leq |(f_n, g_n) - (f_n, g)| + |(f_n, g) - (f, g)| \\ &\leq |(f_n, g_n - g)| + |(f_n - f, g)| \\ &\leq \|f_n\| \, \|g_n - g\| + \|f_n - f\| \, \|g\|. \end{aligned}$$
Therefore, when the limit relations $f_n \to f$ and $g_n \to g$ for $n \to \infty$ in $\mathcal{H}$ hold true, we infer
$$\lim_{n\to\infty} (f_n, g_n) = (f, g).$$
We observe that the completeness of the space $\mathcal{H}$ is not needed for the proof of the continuity of the inner product.
2. The pre-Hilbert-space from Example 6.6 is not complete and consequently does not represent a Hilbert space.
3. In Section 3 from Chapter 8, we shall embed - parallel to the transition from rational numbers to real numbers - each pre-Hilbert-space $\mathcal{H}'$ into a Hilbert space $\mathcal{H}$. This means $\mathcal{H}' \subset \mathcal{H}$ and $\mathcal{H}'$ is dense in $\mathcal{H}$.
4. Hilbert spaces represent particular Banach spaces. The existence of an inner product in $\mathcal{H}$ allows us to introduce the notion of *orthogonality*: Two elements $f, g \in \mathcal{H}$ are named *orthogonal to each other* if $(f, g) = 0$ holds true.

Let $\mathcal{M} \subset \mathcal{H}$ denote an arbitrary linear subspace. We define the *orthogonal space* to $\mathcal{M}$ via
$$\mathcal{M}^\perp := \Big\{ g \in \mathcal{H} \,:\, (g, f) = 0 \text{ for all } f \in \mathcal{M} \Big\}.$$
We see immediately that $M^\perp$ is a linear subspace of $\mathcal{H}$, and the continuity of the inner product justifies the following

Remark: For an arbitrary linear subspace $\mathcal{M} \subset \mathcal{H}$, its associate orthogonal space $\mathcal{M}^\perp$ is closed. More precisely, each sequence
$$\{f_n\} \subset \mathcal{M}^\perp \quad \text{in} \quad \mathcal{M}^\perp \quad \text{satisfying} \quad f_n \to f \quad \text{for} \quad n \to \infty$$
fulfills $f \in \mathcal{M}^\perp$.

Proof: Since $\{f_n\} \subset \mathcal{M}^\perp$ holds true, we infer $(f_n, g) = 0$ for all $n \in \mathbb{N}$ and $g \in \mathcal{M}$. This implies
$$0 = \lim_{n\to\infty} (f_n, g) = (f, g) \quad \text{for all} \quad g \in \mathcal{M}.$$

q.e.d.

Fundamentally important is the following

Theorem 6.9. (Orthogonal projection)
Let $\mathcal{M} \subset \mathcal{H}$ denote a closed linear subspace of the Hilbert space $\mathcal{H}$. Then each element $f \in \mathcal{H}$ possesses the following representation:

$$f = g + h \qquad \text{with} \quad g \in \mathcal{M} \quad \text{and} \quad h \in \mathcal{M}^\perp.$$

Here the elements g and h are uniquely determined.

This theorem says that the Hilbert space $\mathcal{H}$ can be decomposed into two orthogonal subspaces $\mathcal{M}$ and $\mathcal{M}^\perp$ such that $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ holds true.
Proof:

1. At first, we show the uniqueness. Let us consider an element $f \in \mathcal{H}$ with

$$f = g_1 + h_1 = g_2 + h_2, \qquad g_j \in \mathcal{M}, \quad h_j \in \mathcal{M}^\perp.$$

Then we deduce

$$0 = f - f = (g_1 - g_2) + (h_1 - h_2).$$

The uniqueness follows from the identity

$$\begin{aligned} 0 &= \|(g_1 - g_2) + (h_1 - h_2)\|^2 \\ &= ((g_1 - g_2) + (h_1 - h_2), (g_1 - g_2) + (h_1 - h_2)) \\ &= \|g_1 - g_2\|^2 + \|h_1 - h_2\|^2. \end{aligned}$$

2. Now we have to establish the existence of the desired representation. The element $f \in \mathcal{H}$ being given, we solve the subsequent variational problem: Find an element $g \in \mathcal{M}$ such that

$$\|f - g\| = \inf_{\widetilde{g} \in \mathcal{M}} \|f - \widetilde{g}\| =: d$$

holds true. We choose a sequence $\{g_k\} \subset \mathcal{M}$ with the property

$$\lim_{k \to \infty} \|f - g_k\| = d.$$

Then we prove that this sequence converges towards an element $g \in \mathcal{M}$. Here we utilize the *parallelogram identity*

$$\left\| \frac{\varphi + \psi}{2} \right\|^2 + \left\| \frac{\varphi - \psi}{2} \right\|^2 = \frac{1}{2} \left(\|\varphi\|^2 + \|\psi\|^2 \right) \qquad \text{for all} \quad \varphi, \psi \in \mathcal{H},$$

which we easily check by evaluating the inner products on both sides. Now we apply this identity to the elements

$$\varphi = f - g_k, \quad \psi = f - g_l, \qquad k, l \in \mathbb{N}$$

and obtain

$$\left\| f - \frac{g_k + g_l}{2} \right\|^2 + \left\| \frac{g_k - g_l}{2} \right\|^2 = \frac{1}{2} \Big(\|f - g_k\|^2 + \|f - g_l\|^2 \Big).$$

Rearrangement of these equations implies

$$0 \leq \left\| \frac{g_k - g_l}{2} \right\|^2 = \frac{1}{2} \Big(\|f - g_k\|^2 + \|f - g_l\|^2 \Big) - \left\| f - \frac{g_k + g_l}{2} \right\|^2$$

$$\leq \frac{1}{2} \Big(\|f - g_k\|^2 + \|f - g_l\|^2 \Big) - d^2.$$

The passage to the limit $k, l \to \infty$ reveals that $\{g_k\}$ represents a Cauchy sequence. Since the linear subspace $\mathcal{M}$ is closed, we infer the existence of the limit $g \in \mathcal{M}$ for the sequence $\{g_k\}$.
Finally, we prove $h = (f - g) \in \mathcal{M}^\perp$ and obtain the desired representation $f = g + (f - g) = g + h$.
When $\varphi \in \mathcal{M}$ is chosen arbitrarily as well as the number $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we infer the inequality

$$\|(f - g) + \varepsilon\varphi\|^2 \geq d^2 = \|f - g\|^2.$$

We note that

$$\|f - g\|^2 + 2\varepsilon \operatorname{Re}(f - g, \varphi) + \varepsilon^2 \|\varphi\|^2 \geq \|f - g\|^2,$$

and deduce

$$2\varepsilon \operatorname{Re}(f - g, \varphi) + \varepsilon^2 \|\varphi\|^2 \geq 0$$

for all $\varphi \in \mathcal{M}$ and all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Therefore, the identity

$$\operatorname{Re}(f - g, \varphi) = 0 \qquad \text{for all} \quad \varphi \in \mathcal{M}$$

must be valid. When we replace φ by $i\varphi$, we obtain $(f - g, \varphi) = 0$. Since the element φ has been chosen arbitrarily within $\mathcal{M}$, the property

$$(f - g) \in \mathcal{M}^\perp$$

is shown. q.e.d.

The subsequent concepts on the continuity of linear operators in infinite-dimensional vector spaces have been created by S. Banach.

Definition 6.10. *Let* $\{\mathcal{M}_1, \|\cdot\|_1\}$ *and* $\{\mathcal{M}_2, \|\cdot\|_2\}$ *denote two normed linear spaces and* $A : \mathcal{M}_1 \to \mathcal{M}_2$ *a linear mapping. Then* A *is called* continuous *at the point* $f \in \mathcal{M}_1$*, if we can find a quantity* $\delta = \delta(\varepsilon, f) > 0$ *for all* $\varepsilon > 0$ *such that*

$$g \in \mathcal{M}_1, \ \|g - f\|_1 < \delta \quad \Longrightarrow \quad \|A(g) - A(f)\|_2 < \varepsilon.$$

Theorem 6.11. *Consider the linear functional* $A : \mathcal{M} \to \mathbb{C}$ *on the linear normed space* $\mathcal{M}$*, which means*

$$A(\alpha f + \beta g) = \alpha A(f) + \beta A(g) \quad \textit{for all} \quad f, g \in \mathcal{M}, \quad \alpha, \beta \in \mathbb{C}.$$

Then the following statements are equivalent:

(i) A *is continuous at all points* $f \in \mathcal{M}$*;*
(ii) A *is continuous at one point* $f \in \mathcal{M}$*;*
(iii) A *is bounded in the following sense: There exists a constant* $\alpha \in [0, +\infty)$ *such that*

$$|A(f)| \leq \alpha \|f\| \qquad \textit{for all} \quad f \in \mathcal{M}$$

holds true.

Proof:

$(i) \Rightarrow (iii):$ Let A be continuous in $\mathcal{M}$, then this holds true at the origin $0 \in \mathcal{M}$ in particular. For $\varepsilon = 1$ we find a quantity $\delta(\varepsilon) > 0$ such that $\|f\| \leq \delta$ implies $|A(f)| \leq 1$. We obtain

$$|A(f)| \leq \frac{1}{\delta}\|f\| \qquad \text{for all} \quad f \in \mathcal{M}.$$

$(iii) \Rightarrow (ii):$ We immediately infer the continuity of A at the origin 0 from the boundedness of A.

$(ii) \Rightarrow (i):$ Let A be be continuous at one point $f_0 \in \mathcal{M}$. For a number $\varepsilon > 0$ being given, we find a quantity $\delta > 0$ satisfying

$$\varphi \in \mathcal{M},\ \|\varphi\| \leq \delta \quad \Longrightarrow \quad |A(f_0 + \varphi) - A(f_0)| \leq \varepsilon.$$

The linearity of our functional A gives us the following estimate for all $f \in \mathcal{M}$:

$$\varphi \in \mathcal{M},\ \|\varphi\| \leq \delta \quad \Longrightarrow \quad |A(f + \varphi) - A(f)| \leq \varepsilon.$$

Therefore, A is continuous for all $f \in \mathcal{M}$. q.e.d.

Remark: This theorem remains true for linear mappings $A : \mathcal{M}_1 \to \mathcal{M}_2$ between the normed vector spaces $\{\mathcal{M}_1, \|\cdot\|_1\}$ and $\{\mathcal{M}_2, \|\cdot\|_2\}$. Here we mean by the notion *'A is bounded'* that we can find a number $\alpha \in [0, +\infty)$ such that

$$\|A(f)\|_2 \leq \alpha \|f\|_1 \quad \text{for all} \quad f \in \mathcal{M}_1$$

holds true.

Definition 6.12. *When we consider a bounded linear functional* $A : \mathcal{M} \to \mathbb{C}$ *on the normed linear space* $\mathcal{M}$*, we introduce the* norm of the functional A *as follows:*

$$\|A\| := \sup_{f \in \mathcal{M},\ \|f\| \leq 1} |A(f)|.$$

Definition 6.13. *By the symbol*

$$\mathcal{M}^* := \Big\{ A : \mathcal{M} \to \mathbb{C} \,:\, A \text{ is bounded on } \mathcal{M} \Big\},$$

we denote the dual space *of the normed linear space* $\mathcal{M}$*.*

Remarks:

1. We easily show that $\mathcal{M}^*$, endowed with the norm from Definition 6.12, constitutes a Banach space.
2. Let $\mathcal{H}$ denote a Hilbert space. Then its dual space $\mathcal{H}^*$ is isomorphic to $\mathcal{H}$, as we shall show now.

Theorem 6.14. (Representation theorem of Fréchet and Riesz) *Each bounded linear functional* $A : \mathcal{H} \to \mathbb{C}$*, defined on a Hilbert space* $\mathcal{H}$*, can be represented in the form*

$$A(f) = (g, f) \qquad \text{for all} \quad f \in \mathcal{H},$$

with a generating element $g \in \mathcal{H}$ *which is uniquely determined.*

Proof:

1. At first, we show the uniqueness. Let $f \in \mathcal{H}$ and $g_1, g_2 \in \mathcal{H}$ denote two generating elements. Then we see

$$A(f) = (g_1, f) = (g_2, f) \qquad \text{for all} \quad f \in \mathcal{H}.$$

We subtract these equations and obtain

$$(g_1, f) - (g_2, f) = (g_1 - g_2, f) = 0 \qquad \text{for all} \quad f \in \mathcal{H}.$$

When we choose $f = g_1 - g_2$, we infer $g_1 = g_2$ on account of

$$0 = (g_1 - g_2, g_1 - g_2) = \|g_1 - g_2\|^2.$$

2. In order to prove the existence of g, we consider

$$\mathcal{M} := \Big\{ f \in \mathcal{H} \,:\, A(f) = 0 \Big\} \subset \mathcal{H}$$

representing a closed linear subspace of $\mathcal{H}$.

i.) Let $\mathcal{M} = \mathcal{H}$ be satisfied. Then we set $g = 0 \in \mathcal{H}$ and obtain the identity

$$A(f) = (g, f) = 0 \qquad \text{for all} \quad f \in \mathcal{H}.$$

ii.) Let $\mathcal{M} \subsetneqq \mathcal{H}$ be satisfied. We invoke the theorem of the orthogonal projection and see $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ with $\{0\} \neq \mathcal{M}^\perp$. Consequently, there exists an element $h \in \mathcal{M}^\perp$ with $h \neq 0$. We now determine a number $\alpha \in \mathbb{C}$, such that the identity $A(h) = (g, h)$ for $g = \alpha h$ is correct. This is equivalent to

$$A(h) = (g, h) = (\alpha h, h) = \overline{\alpha}\,(h, h) = \overline{\alpha}\,\|h\|^2$$

and

$$g = \frac{\overline{A(h)}}{\|h\|^2}\,h.$$

Now the identity $A(f) = (g, f)$ is valid for all $f \in \mathcal{M}$ and for $f = h$. When $f \in \mathcal{H}$ is arbitrary, we define $c := \frac{A(f)}{A(h)}$. With $\widetilde{f} := f - ch$, we obtain

$$A(\widetilde{f}) = A(f) - cA(h) = A(f) - \frac{A(f)}{A(h)}\,A(h) = 0$$

and consequently $\widetilde{f} \in \mathcal{M}$. Therefore, we have the representation

$$f = \widetilde{f} + ch \quad \text{for} \quad f \in \mathcal{H}, \quad \text{where} \quad \widetilde{f} \in \mathcal{M} \quad \text{and} \quad ch \in \mathcal{M}^\perp.$$

This implies

$$A(f) = A(\widetilde{f}) + cA(h) = (g, \widetilde{f}) + c(g, h) = (g, \widetilde{f} + ch) = (g, f)$$

for all $f \in \mathcal{H}$. q.e.d.

Definition 6.15. *We name a Banach space* separable *if a sequence* $\{f_k\} \subset \mathcal{B}$ *exists, which lies densely in* $\mathcal{B}$. *More precisely, we find an index* $k \in \mathbb{N}$ *to each element* $f \in \mathcal{B}$ *and every* $\varepsilon > 0$ *such that* $\|f - f_k\| < \varepsilon$ *holds true.*

Definition 6.16. *In a pre-Hilbert-space* $\mathcal{H}'$, *we name the denumerably infinite many elements* $\{\varphi_1, \varphi_2, \ldots\} \subset \mathcal{H}'$ orthonormal *if*

$$(\varphi_i, \varphi_j) = \delta_{ij} \quad \textit{for all} \quad i, j \in \mathbb{N}$$

is valid.

Remark: When we have the system of denumerably many linearly independent elements in $\mathcal{H}'$, we can apply the *orthonormalization procedure of E. Schmidt* in order to transfer this into an orthonormal system.

Here we start with the linearly independent elements $\{f_1, \ldots, f_N\} \subset \mathcal{H}'$ of the pre-Hilbert-space $\mathcal{H}'$. Then we define inductively

$$\varphi_1 := \frac{1}{\|f_1\|}\,f_1, \quad \varphi_2 := \frac{f_2 - (\varphi_1, f_2)\varphi_1}{\|f_2 - (\varphi_1, f_2)\varphi_1\|}, \ldots \varphi_N := \frac{f_N - \sum\limits_{j=1}^{N-1} (\varphi_j, f_N)\varphi_j}{\left\| f_N - \sum\limits_{j=1}^{N-1} (\varphi_j, f_N)\varphi_j \right\|}.$$

The vector spaces spanned by $\{f_1, \dots, f_N\}$ and $\{\varphi_1, \dots, \varphi_N\}$ coincide, and we note that

$$(\varphi_i, \varphi_j) = \delta_{ij} \quad \text{for} \quad i, j = 1, \dots, N.$$

Proposition 6.17. *Let $\{\varphi_k\}$ with $k = 1, \dots, N$ represent a system of orthonormal elements in the pre-Hilbert-space $\mathcal{H}'$ and assume $f \in \mathcal{H}'$. Then we have the identity*

$$\Big\| f - \sum_{k=1}^{N} c_k \varphi_k \Big\|^2 = \Big\| f - \sum_{k=1}^{N} (\varphi_k, f)\varphi_k \Big\|^2 + \sum_{k=1}^{N} |c_k - (\varphi_k, f)|^2$$

for all numbers $c_1, \dots, c_N \in \mathbb{C}$.

Proof: At first, we define

$$g := f - \sum_{k=1}^{N} (\varphi_k, f)\varphi_k, \quad h := \sum_{k=1}^{N} \Big((\varphi_k, f) - c_k \Big) \varphi_k.$$

Then we deduce the equation

$$f - \sum_{k=1}^{N} c_k \varphi_k = f - \sum_{k=1}^{N} (\varphi_k, f)\varphi_k + \sum_{k=1}^{N} \Big((\varphi_k, f) - c_k \Big) \varphi_k = g + h.$$

Now we evaluate

$$\begin{aligned} (g, h) &= \left(f - \sum_{k=1}^{N} (\varphi_k, f)\varphi_k \, , \sum_{l=1}^{N} \Big((\varphi_l, f) - c_l \Big) \varphi_l \right) \\ &= \sum_{l=1}^{N} \Big((\varphi_l, f) - c_l \Big) \overline{(\varphi_l, f)} - \sum_{k,l=1}^{N} \overline{(\varphi_k, f)} \Big((\varphi_l, f) - c_l \Big) (\varphi_k, \varphi_l). \end{aligned}$$

We note that $(\varphi_k, \varphi_l) = \delta_{kl}$ and obtain $(g, h) = 0$. This implies

$$\begin{aligned} \Big\| f - \sum_{k=1}^{N} c_k \varphi_k \Big\|^2 &= (g + h, g + h) = \|g\|^2 + \|h\|^2 \\ &= \Big\| f - \sum_{k=1}^{N} (\varphi_k, f)\varphi_k \Big\|^2 + \sum_{k,l=1}^{N} \overline{\Big((\varphi_k, f) - c_k \Big)} \Big((\varphi_l, f) - c_l \Big) (\varphi_k, \varphi_l) \\ &= \Big\| f - \sum_{k=1}^{N} (\varphi_k, f)\varphi_k \Big\|^2 + \sum_{k=1}^{N} |(\varphi_k, f) - c_k|^2. \end{aligned}$$

q.e.d.

Corollary: For all numbers $c_1, \dots, c_N \in \mathbb{C}$ we have

$$\left\| f - \sum_{k=1}^{N} c_k \varphi_k \right\|^2 \geq \left\| f - \sum_{k=1}^{N} (\varphi_k, f) \varphi_k \right\|^2,$$

and equality is attained only if $c_k = (\varphi_k, f)$ for $k = 1, \ldots, N$ holds true. We name these numbers c_k the *Fourier coefficients* of f (with respect to the system (φ_k)).

When we set $c_1 = \ldots = c_N = 0$, we obtain

Proposition 6.18. *The following relation*

$$\left\| f - \sum_{k=1}^{N} (\varphi_k, f) \varphi_k \right\|^2 = \|f\|^2 - \sum_{k=1}^{N} |(\varphi_k, f)|^2 \geq 0$$

holds true.

From the last proposition we immediately infer

Theorem 6.19. *Let $\{\varphi_k\}$, $k = 1, 2, \ldots$ represent an orthonormal system in the pre-Hilbert-space $\mathcal{H}'$. For all elements $f \in \mathcal{H}'$,* Bessel's inequality

$$\sum_{k=1}^{\infty} |(\varphi_k, f)|^2 \leq \|f\|^2$$

holds true. An element $f \in \mathcal{H}'$ satisfies the equation

$$\sum_{k=1}^{\infty} |(\varphi_k, f)|^2 = \|f\|^2$$

if and only if the limit relation

$$\lim_{N \to \infty} \left\| f - \sum_{k=1}^{N} (\varphi_k, f) \varphi_k \right\| = 0$$

is valid.

Remark: The last statement means that $f \in \mathcal{H}'$ can be represented by its *Fourier series*

$$\sum_{k=1}^{\infty} (\varphi_k, f) \varphi_k$$

with respect to the Hilbert-space-norm $\| \cdot \|$.

Definition 6.20. *We say that an orthonormal system $\{\varphi_k\}$ is* complete *- we abbreviate this as c.o.n.s - if each element $f \in \mathcal{H}'$ of the pre-Hilbert-space $\mathcal{H}'$ satisfies the* completeness relation

$$\|f\|^2 = \sum_{k=1}^{\infty} |(\varphi_k, f)|^2.$$

Remarks:

1. In Section 4 and Section 5 of Chapter 5, we shall present explicit c.o.n.s. with the classical Fourier series and the spherical harmonic functions. More profound results are contained in Chapter 8 about Linear Operators in Hilbert Spaces.
2. With the aid of E. Schmidt's orthonormalization procedure, we can construct a complete orthonormal system in each separable Hilbert space.
3. When we have a complete orthonormal system $\{\varphi_k\} \subset \mathcal{H}'$ with $k = 1, 2, \ldots$ in the pre-Hilbert-space $\mathcal{H}'$, the representation via the Fourier series
$$f = \sum_{k=1}^{\infty} (\varphi_k, f)\varphi_k$$
holds true with respect to convergence in the Hilbert-space-norm. The interesting question remains open, whether a Fourier series converges pointwise or even uniformly (see e.g. H. Heuser: *Analysis II.* B. G. Teubner-Verlag, Stuttgart, 1992).

7 The Lebesgue Spaces $L^p(X)$

Now we continue our considerations from Section 1 to Section 4. We assume $n \in \mathbb{N}$ as usual, and we consider subsets $X \subset \mathbb{R}^n$ which we endow with the *relative topology* of the Euclidean space $\mathbb{R}^n$ as follows:

$$A \subset X \text{ is} \left\{ \begin{matrix} \text{open} \\ \text{closed} \end{matrix} \right\}$$
$$\Longleftrightarrow \quad \text{There exists } B \subset \mathbb{R}^n \left\{ \begin{matrix} \text{open} \\ \text{closed} \end{matrix} \right\} \text{with } A = B \cap X.$$

By the symbol $M(X)$ we denote a linear space of continuous functions $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ with the following properties:

(M1) *Linearity:* With $f, g \in M(X)$ and $\alpha, \beta \in \mathbb{R}$ we have $\alpha f + \beta g \in M(X)$.
(M2) *Lattice property:* From $f \in M(X)$ we infer $|f| \in M(X)$.
(M3) *Global property:* The function $f(x) \equiv 1$, $x \in X$ belongs to $M(X)$.

We name a linear functional $I : M \to \mathbb{R}$, which is defined on $M = M(X)$, *Daniell's integral* if the following properties are valid:

(D1) *Linearity:* $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ for all $f, g \in M$ and $\alpha, \beta \in \mathbb{R}$;
(D2) *Nonnegativity:* $I(f) \geq 0$ for all $f \in M$ with $f \geq 0$;
(D3) *Monotone continuity:* For all $\{f_k\} \subset M(X)$ with $f_k(x) \downarrow 0$ $(k \to \infty)$ on X we infer $I(f_k) \to 0$ $(k \to \infty)$.

Example 7.1. Let $X = \Omega \subset \mathbb{R}^n$ denote an open bounded set, and we define the linear space

$$M = M(X) := \left\{ f : X \to \mathbb{R} \in C^0(X) \; : \; \int\limits_{\Omega} |f(x)| \, dx < +\infty \right\}.$$

We utilize the improper Riemannian integral on the set X, namely

$$I(f) := \int\limits_{\Omega} f(x) \, dx, \qquad f \in M$$

as our linear functional.

Example 7.2. On the sphere $X = S^{n-1} := \left\{ x \in \mathbb{R}^n \; : \; |x| = 1 \right\}$, we consider the linear space of all continuous functions $M(X) = C^0(S^{n-1})$, and we introduce the Daniell integral

$$I(f) := \int\limits_{S^{n-1}} f(x) \, d\sigma^{n-1}(x), \qquad f \in M.$$

In Section 2 we have extended the functional I from $M(X)$ onto the space $L(X)$ of the *Lebesgue integrable functions.* In Section 3 we investigated sets which are Lebesgue measurable, more precisely those sets A whose characteristic functions χ_A are Lebesgue integrable.

Definition 7.3. *Let the exponent satisfy* $1 \le p < +\infty$. *We name a measurable function* $f : X \to \overline{\mathbb{R}}$ p-times integrable *if* $|f|^p \in L(X)$ *is correct. In this case we write* $f \in L^p(X)$. *With*

$$\|f\|_p := \|f\|_{L^p(X)} := \left(\int\limits_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} = \Big(I(|f|^p) \Big)^{\frac{1}{p}}$$

we obtain the L^p-norm *of the function* $f \in L^p(X)$*; here the symbol* μ *denotes the* Lebesgue measure *on* X.

Remark: Evidently, we have the identity $L^1(X) = L(X)$.

The central tool, when dealing with Lebesgue spaces, is provided by the subsequent result.

Theorem 7.4. (Hölder's inequality)
Let the exponents $p, q \in (1, +\infty)$ *be conjugate, which means* $p^{-1} + q^{-1} = 1$ *holds true. Furthermore, we assume* $f \in L^p(X)$ *and* $g \in L^q(X)$ *being given. Then we infer the property* $fg \in L^1(X)$ *and the inequality*

$$\|fg\|_{L^1(X)} \le \|f\|_{L^p(X)} \|g\|_{L^q(X)}.$$

Proof: We have to investigate only the case $\|f\|_p > 0$ and $\|g\|_q > 0$. Alternatively, we had $\|f\|_p = 0$, and consequently $f = 0$ a.e. as well as $f \cdot g = 0$ a.e. would hold. Analogously, we treat the case $\|g\|_q = 0$. Then we apply *Young's inequality*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

to the functions

$$\varphi(x) = \frac{1}{\|f\|_p}|f(x)|, \quad \psi(x) = \frac{1}{\|g\|_q}|g(x)|, \qquad x \in X,$$

and we obtain

$$\frac{1}{\|f\|_p\|g\|_q}\,|f(x)g(x)| = \varphi(x)\psi(x) \le \frac{1}{p}\,\frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q}\,\frac{|g(x)|^q}{\|g\|_q^q}$$

for all points $x \in X$. Theorem 4.8 from Section 4 implies $fg \in L(X) = L^1(X)$. Now integration yields the inequality

$$\frac{1}{\|f\|_p\|g\|_q}I(|fg|) \le \frac{1}{p}\,\frac{1}{\|f\|_p^p}\,I(|f|^p) + \frac{1}{q}\,\frac{1}{\|g\|_q^q}\,I(|g|^q) = 1,$$

and finally

$$I(|fg|) \le \|f\|_p\|g\|_q.$$

q.e.d.

Theorem 7.5. (Minkowski's inequality)
With the exponent $p \in [1, +\infty)$, let us consider the functions $f, g \in L^p(X)$. Then we infer $f + g \in L^p(X)$ and we have

$$\|f + g\|_{L^p(X)} \le \|f\|_{L^p(X)} + \|g\|_{L^p(X)}.$$

Proof: The case $p = 1$ can be easily derived by application of the triangle inequality on the integrand $|f + g|$. Therefore, we assume $p, q \in (1, +\infty)$ with $p^{-1} + q^{-1} = 1$. At first, convexity arguments yield

$$|f(x) + g(x)|^p \le 2^{p-1}\left(|f(x)|^p + |g(x)|^p\right)$$

and consequently $f + g \in L^p$ or equivalently $I(|f + g|^p) < +\infty$. Now we have

$$\begin{aligned}
|f(x) + g(x)|^p &= |f(x) + g(x)|^{p-1}|f(x) + g(x)| \\
&\le |f(x) + g(x)|^{p-1}|f(x)| + |f(x) + g(x)|^{p-1}|g(x)| \\
&= |f(x) + g(x)|^{\frac{p}{q}}|f(x)| + |f(x) + g(x)|^{\frac{p}{q}}|g(x)|.
\end{aligned}$$

The factors of the summands on the right-hand side are L^q- and L^p-functions, respectively. Therefore, we obtain

$$I(|f+g|^p) \le I(|f+g|^p)^{\frac{1}{q}}(\|f\|_p + \|g\|_p).$$

Finally, we see

$$(I(|f+g|^p)^{\frac{1}{p}} \le \|f\|_p + \|g\|_p$$

and the desired inequality

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

q.e.d.

Remark: Minkowski's inequality represents the triangle inequality for the $\|\cdot\|_p$-norm in the space L^p.

The following result guarantees the completeness of L^p-spaces, which means: Each Cauchy sequence converges towards a function in the respective space.

Theorem 7.6. (Fischer, Riesz)
Let us consider the exponent $p \in [1, +\infty)$ *and a sequence* $\{f_k\}_{k=1,2,\ldots} \subset L^p(X)$ *satisfying*

$$\lim_{k,l\to\infty} \|f_k - f_l\|_{L^p(X)} = 0.$$

Then we have a function $f \in L^p(X)$ *with the property*

$$\lim_{k\to\infty} \|f_k - f\|_{L^p(X)} = 0.$$

Proof: With the aid of Hölder's inequality we show the identity

$$\lim_{k,l\to\infty} I(|f_k - f_l|) = 0.$$

Here we estimate in the case $p > 1$ as follows:

$$I(|f_k - f_l|) = I(|f_k - f_l| \cdot 1) \le \|f_k - f_l\|_p \|1\|_q \longrightarrow 0.$$

The Lebesgue selection theorem gives us a subsequence $k_1 < k_2 < k_3 < \ldots$ and a null-set $N \subset X$, such that

$$\lim_{m\to\infty} f_{k_m}(x) = f(x), \qquad x \in X \setminus N$$

holds true. We observe that the function f is measurable. Now we choose $l \ge N(\varepsilon)$ and $k_m \ge N(\varepsilon)$, where $\|f_k - f_l\|_p \le \varepsilon$ for all $k, l \ge N(\varepsilon)$ is valid, and we infer

$$I(|f_{k_m} - f_l|^p) = \|f_{k_m} - f_l\|^p_{L^p(X)} \le \varepsilon^p.$$

For $m \to \infty$, Fatou's theorem implies the inequality

$$I(|f - f_l|^p) \le \varepsilon^p \qquad \text{for all} \quad l \ge N(\varepsilon)$$

and consequently

$$\|f - f_l\|_{L^p(X)} \leq \varepsilon \qquad \text{for all} \quad l \geq N(\varepsilon).$$

Since $L^p(X)$ is linear and f_l as well as $(f - f_l)$ belong to this space, we infer $f \in L^p(X)$. Furthermore, we observe

$$\lim_{l\to\infty} \|f - f_l\|_p = 0.$$

q.e.d.

Definition 7.7. *A measurable function $f : X \to \overline{\mathbb{R}}$ belongs to the class $L^\infty(X)$ if we have a null-set $N \subset X$ and a constant $c \in [0, +\infty)$ with the property*

$$|f(x)| \leq c \qquad \textit{for all} \quad x \in X \setminus N.$$

We name

$$\begin{aligned} \|f\|_\infty = \|f\|_{L^\infty(X)} &= \operatorname*{\mathit{ess}\,sup}_{x\in X} |f(x)| \\ &= \inf \left\{ c \geq 0 \ : \ \begin{array}{l} \textit{There exists a null-set } N \subset X \\ \textit{with } |f(x)| \leq c \textit{ for all } x \in X \setminus N \end{array} \right\} \end{aligned}$$

the L^∞-norm *or equivalently the* essential supremum *of the function f.*

Remark: Evidently, we have the inclusion

$$L^\infty(X) \subset \bigcap_{p\in[1,+\infty)} L^p(X).$$

Theorem 7.8. *A function $f \in \bigcap\limits_{p\geq 1} L^p(X)$ belongs to the class $L^\infty(X)$, if the condition*

$$\limsup_{p\to\infty} \|f\|_{L^p(X)} < +\infty$$

is correct. In this case we have

$$\|f\|_{L^\infty(X)} = \lim_{p\to\infty} \|f\|_{L^p(X)} < +\infty,$$

where the limit on the right-hand side exists.

Proof: Let $f \in \bigcap\limits_{p\geq 1} L^p(X)$ hold true. When we assume $f \in L^\infty(X)$, we infer $0 \leq \|f\|_\infty < +\infty$ as well as

$$|f|^p = |f|^q |f|^{p-q} \leq |f|^q \|f\|_\infty^{p-q} \quad \text{a.e. on} \quad X.$$

Therefore, we obtain

$$\|f\|_p \leq \|f\|_\infty^{1-\frac{q}{p}} \|f\|_q^{\frac{q}{p}}$$

and finally

$$\limsup_{p\to\infty} \|f\|_p \le \|f\|_\infty < +\infty. \tag{1}$$

In order to show the inverse direction, we consider the set

$$A_a := \Big\{x \in X \,:\, |f(x)| > a\Big\}$$

for an arbitrary number $a < \|f\|_\infty$. Therefore, A_a does not constitute a null-set. We obtain the estimate

$$\begin{aligned}+\infty > \limsup_{p\to\infty} \|f\|_p \;&\ge\; \liminf_{p\to\infty} \|f\|_p \\ = \liminf_{p\to\infty} \Big(I(|f|^p)\Big)^{\frac{1}{p}} \;&\ge\; a \liminf_{p\to\infty} \Big(\mu(A_a)\Big)^{\frac{1}{p}} \;=\; a.\end{aligned}$$

Now we infer

$$+\infty > \liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty \tag{2}$$

and consequently $f \in L^\infty(X)$. These inequalities immediately imply the existence of

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

q.e.d.

Corollary: Hölder's inequality remains valid for the case $p = 1$ and $q = \infty$. Furthermore, Minkowski's inequality holds true in the case $p = \infty$ as well.

Definition 7.9. *Let $1 \le p \le +\infty$ be satisfied. Then we introduce an equivalence relation on the space $L^p(X)$ as follows:*

$$f \sim g \quad \Longleftrightarrow \quad f(x) = g(x) \quad \text{a.e. in } X.$$

By the symbol $[f]$ we denote the equivalence class belonging to the element $f \in L^p(X)$. We name

$$\mathcal{L}^p(X) := \Big\{[f] \,:\, f \in L^p(X)\Big\}$$

the Lebesgue space of order $1 \le p \le +\infty$.

We summarize our considerations to the subsequent

Theorem 7.10. *For each fixed p with $1 \le p \le +\infty$, the Lebesgue space $\mathcal{L}^p(X)$ constitutes a real Banach space with the given L^p-norm. Furthermore, we have the inclusion*

$$\mathcal{L}^r(X) \supset \mathcal{L}^s(X)$$

for all $1 \le r < s \le +\infty$. Moreover, the estimate

$$\|f\|_{\mathcal{L}^r(X)} \le C(r,s)\|f\|_{\mathcal{L}^s(X)} \qquad \text{for all} \quad f \in \mathcal{L}^s(X)$$

holds true with a constant $C(r,s) \in [0,+\infty)$. *This means, the mapping for the embedding*

$$\Phi : \mathcal{L}^s(X) \longrightarrow \mathcal{L}^r(X), \quad f \mapsto \Phi(f) = f$$

is continuous. Therefore, a sequence converging in the space $\mathcal{L}^s(X)$ *is convergent in the space* $\mathcal{L}^r(X)$ *as well.*

Proof:

1. At first, we show that $\mathcal{L}^p(X)$ constitute normed spaces. Let us consider $[f] \in \mathcal{L}^p(X)$: We have $\|[f]\|_p = 0$ if and only if $\|f\|_p = 0$ and consequently $f = 0$ a.e. in X is fulfilled. This implies $[f] = 0$ and gives us the norm property (N1). Minkowski's inequality from Theorem 7.5 ascertains the norm property (N2), where Theorem 7.8 provides the triangle inequality in the space $L^\infty(X)$. The norm property (N3), namely the homogeneity, is obvious.
2. The Fischer-Riesz theorem implies completeness of the spaces $\mathcal{L}^p$ for $1 \leq p < +\infty$. Therefore, only completeness of the space $\mathcal{L}^\infty$ has to be shown. Here we consider a Cauchy sequence $\{f_k\} \subset L^\infty$ satisfying

$$\|f_k - f_l\|_\infty \to 0 \qquad \text{for} \quad k,l \to \infty.$$

We infer the inequality $\|f_k\|_\infty \leq c$ for all $k \in \mathbb{N}$, with a constant $c \in (0,+\infty)$. Then we find a null-set $N_0 \subset X$ with $|f_k(x)| \leq c$ for all points $x \in X \setminus N_0$ and all indices $k \in \mathbb{N}$. Furthermore, we have null-sets $N_{k,l}$ with

$$|f_k(x) - f_l(x)| \leq \|f_k - f_l\|_\infty \quad \text{for} \quad x \in X \setminus N_{k,l}.$$

We define

$$N := N_0 \cup \bigcup_{k,l} N_{k,l}$$

and observe

$$\lim_{k,l\to\infty} \sup_{x\in X\setminus N} |f_k(x) - f_l(x)| = 0.$$

When we introduce the function

$$f(x) := \begin{cases} \lim\limits_{k\to\infty} f_k(x) \ , & x \in X \setminus N \\ 0 \ , & x \in N \end{cases} \in L^\infty(X)$$

we infer

$$\lim_{k\to\infty} \sup_{x\in X\setminus N} |f_k(x) - f(x)| = 0$$

and finally

$$\lim_{k\to\infty} \|f_k - f\|_{L^\infty(X)} = 0.$$

3. Let us assume $1 \le r < s \le +\infty$. The function $f \in L^s(X)$ satisfies

$$\|f\|_r = \Big(I(|f|^r \cdot 1)\Big)^{\frac{1}{r}} \le \left\{\Big(I(|f|^s)\Big)^{\frac{r}{s}} \Big(\mu(X)\Big)^{\frac{s-r}{s}}\right\}^{\frac{1}{r}} = \Big(\mu(X)\Big)^{\frac{s-r}{rs}} \|f\|_s$$

for all elements $f \in L^s(X)$. q.e.d.

Definition 7.11. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ denote two Banach spaces with $\mathcal{B}_1 \subset \mathcal{B}_2$. Then we say $\mathcal{B}_1$* is continuously embedded into $\mathcal{B}_2$ *if the mapping*

$$I_1 : \mathcal{B}_1 \longrightarrow \mathcal{B}_2, \quad f \mapsto I_1(f) = f$$

is continuous. This means, the inequality

$$\|f\|_{\mathcal{B}_2} \le c\,\|f\|_{\mathcal{B}_1} \quad \textit{for all} \quad f \in \mathcal{B}_1$$

holds true with a constant $c \in [0, +\infty)$. Then we use the notation $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$.

Remarks:

1. The transition to equivalence classes will be made tacitly - such that we can identify $\mathcal{L}^p(X)$ and $L^p(X)$.
2. We have the embedding $\mathcal{L}^s(X) \hookrightarrow \mathcal{L}^r(X)$ for all $1 \le r \le s \le +\infty$.
3. On the space $C^0(X)$, we obtain with

$$\|f\|_0 := \sup_{x \in X} |f(x)|, \qquad f \in C^0(X)$$

the *supremum-norm* which induces uniform convergence. With the L^p-norms $\|\cdot\|_p$ for $1 \le p \le +\infty$, we have constructed a family of norms which constitute a continuum beginning with the weakest norm, namely the L^1-norm, and ending with the strongest norm, namely the L^∞-norm or the C^0-norm, respectively. Exactly in the centrum for $p = 2$, we find the Hilbert space $\mathcal{H} = L^2(X)$.

Example 7.12. Let the space

$$\mathcal{H} = L^2(X, \mathbb{C}) := \Big\{f = g + ih \,:\, g, h \in L^2(X, \mathbb{R})\Big\}$$

be endowed with the inner product

$$(f_1, f_2)_{\mathcal{H}} := I(\overline{f_1} f_2) \quad \text{for} \quad f_j = g_j + ih_j \ \in \mathcal{H} \quad \text{and} \quad j = 1, 2.$$

Here we define $I(f) = I(g + ih) := I(g) + i\,I(h)$. Then $\mathcal{H}$ represents a Hilbert space.

In the sequel, we use the space of functions

$$M^\infty(X) := \Big\{f \in M(X) \,:\, \sup_{x \in X} |f(x)| < +\infty\Big\} = M(X) \cap L^\infty(X).$$

Theorem 7.13. (Approximation of L^p-functions)
Given the exponent $p \in [1, +\infty)$, the space $M^\infty(X)$ lies densely in $L^p(X)$, which means: For each function $f \in L^p(X)$ and each $\varepsilon > 0$, we have a function $f_\varepsilon \in M^\infty(X)$ satisfying

$$\|f - f_\varepsilon\|_{L^p(X)} < \varepsilon.$$

Proof: Let $\varepsilon > 0$ be given. We choose $K > 0$ and consider the truncated function

$$f_{-K,+K}(x) := \begin{cases} f(x), \; x \in X \text{ with } |f(x)| \leq K \\ -K, \;\; x \in X \text{ with } f(x) \leq -K \\ +K, \;\; x \in X \text{ with } f(x) \geq +K \end{cases}$$

subject to the inequality

$$|f(x) - f_{-K,+K}(x)|^p \leq |f(x)|^p.$$

Furthermore, we have

$$\lim_{K \to \infty} |f(x) - f_{-K,+K}(x)|^p = 0$$

almost everywhere in X. Lebesgue's convergence theorem implies

$$\lim_{K \to \infty} I(|f - f_{-K,+K}|^p) = 0,$$

and we find a number $K = K(\varepsilon) > 0$ with

$$\|f(x) - f_{-K,+K}(x)\|_p \leq \frac{\varepsilon}{2}.$$

According to Theorem 4.14 in Section 4, the function $f_{-K,+K}$ possesses a sequence $\{\varphi_k\}_{k=1,2,\ldots} \subset M(X)$ with $|\varphi_k(x)| \leq K$ satisfying

$$\varphi_k(x) \longrightarrow f_{-K,+K}(x) \qquad \text{a.e. in } X.$$

The Lebesgue convergence theorem yields

$$\|f_{-K,+K} - \varphi_k\|_p^p = I(|f_{-K,+K} - \varphi_k|^p) \longrightarrow 0$$

for $k \to \infty$. Consequently, we find an index $k = k(\varepsilon)$ with

$$\|f_{-K,+K} - \varphi_k\|_p \leq \frac{\varepsilon}{2}.$$

The function $f_\varepsilon := \varphi_{k(\varepsilon)} \in M(X)$, which is uniformly bounded by $K(\varepsilon)$ on X, satisfies

$$\|f - f_\varepsilon\|_p \leq \|f - f_{-K,+K}\|_p + \|f_{-K,+K} - \varphi_{k(\varepsilon)}\|_p \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

q.e.d.

Theorem 7.14. (Separability of L^p-spaces)
Let $X \subset \mathbb{R}^n$ be an open bounded set and $p \in [1, +\infty)$ the exponent given. Then the Banach space $L^p(X)$ is separable: More precisely, there exists a sequence of functions $\{\varphi_k(x)\}_{k=1,2,\ldots} \subset C_0^\infty(X) \subset L^p(X)$ which lies densely in $L^p(X)$.

Proof: Let us consider the set

$$\mathcal{R} := \left\{ g(x) = \sum_{i_1,\ldots,i_n=0}^{N} a_{i_1\ldots i_n} x_1^{i_1} \ldots x_n^{i_n} \ : \ a_{i_1\ldots i_n} \in \mathbb{Q}, \ N \in \mathbb{N} \cup \{0\} \right\}$$

of polynomials in $\mathbb{R}^n$ with rational coefficients. Furthermore, let

$$\chi_j(x) : X \longrightarrow \mathbb{R} \in C_0^\infty(X), \qquad j = 1, 2, \ldots$$

denote an exhausting sequence for the set X, which means

$$\chi_j(x) \le \chi_{j+1}(x), \quad \lim_{j\to\infty} \chi_j(x) = 1 \quad \text{for all} \quad x \in X.$$

Now we show that the denumerable set

$$\mathcal{D}(X) := \Big\{ h(x) = \chi_j(x) g(x) \ : \ j \in \mathbb{N}, \ g \in \mathcal{R} \Big\}$$

lies densely in $L^p(X)$. Here we take the function $f \in L^p(X)$ and the quantity $\varepsilon > 0$ arbitrarily. Then we find a function $g \in M^\infty(X)$ with $\|f - g\|_p \le \varepsilon$. Now we infer

$$\begin{aligned} \|g - \chi_j g\|_p^p &= \int_X |g(x) - \chi_j(x) g(x)|^p \, d\mu(x) \\ &= \int_X \Big(1 - \chi_j(x)\Big)^p |g(x)|^p \, d\mu(x) \longrightarrow 0, \end{aligned}$$

and consequently we find an index $j \in \mathbb{N}$ satisfying $\|g - \chi_j g\|_p \le \varepsilon$. Now the function $\chi_j g$ has compact support in X. Via the Weierstraß approximation theorem, there exists a polynomial $h(x) \in \mathcal{R}$ such that $\sup_{x\in X} \chi_j |g - h| \le \delta(\varepsilon)$ is correct - with a quantity $\delta(\varepsilon) > 0$ given. Consequently, we find a polynomial $h(x) \in \mathcal{R}$ with the property

$$\|\chi_j g - \chi_j h\|_p \le \varepsilon.$$

This implies

$$\|f - \chi_j h\|_p \le \|f - g\|_p + \|g - \chi_j g\|_p + \|\chi_j g - \chi_j h\|_p \le 3\varepsilon.$$

Consequently, $\mathcal{D}(X)$ lies densely in $L^p(X)$. q.e.d.

8 Bounded Linear Functionals on $L^p(X)$ and Weak Convergence

We begin with

Theorem 8.1. (Extension of linear functionals)
Take $p \in [1,+\infty)$ and let $A : M^\infty(X) \to \mathbb{R}$ denote a linear functional with the following property: We have a constant $\alpha \in [0,+\infty)$ such that

$$|A(f)| \le \alpha \|f\|_{L^p(X)} \qquad \text{for all} \quad f \in M^\infty(X)$$

holds true. Then there exists exactly one bounded linear functional $\widehat{A} : L^p(X) \to \mathbb{R}$ satisfying

$$\|\widehat{A}\| \le \alpha \quad \text{and} \quad \widehat{A}(f) = A(f) \quad \text{for all} \quad f \in M^\infty(X).$$

Consequently, the functional $\widehat{A}$ can be uniquely continued from $M^\infty(X)$ onto $L^p(X)$.

Proof: The linear functional A is bounded on $\{M^\infty(X), \|\cdot\|_{L^p(X)}\}$ and therefore continuous. According to Theorem 7.13 from Section 7, each element $f \in L^p(X)$ possesses a sequence $\{f_k\}_{k=1,2,\ldots} \subset M^\infty(X)$ satisfying

$$\|f_k - f\|_{L^p(X)} \to 0 \quad \text{for} \quad k \to \infty.$$

Now we define

$$\widehat{A}(f) := \lim_{k\to\infty} A(f_k).$$

We immediately verify that $\widehat{A}$ has been defined independently of the sequence $\{f_k\}_{k=1,2,\ldots}$ chosen, and that the mapping $\widehat{A} : L^p(X) \to \mathbb{R}$ is linear. Furthermore, we have

$$\|\widehat{A}\| = \sup_{f\in L^p,\ \|f\|_p\le 1} |\widehat{A}(f)| = \sup_{f\in M^\infty,\ \|f\|_p\le 1} |A(f)| \le \alpha.$$

When we consider with $\widehat{A}$ and $\widehat{B}$ two extensions of A onto $L^p(X)$, we infer $\widehat{A} = \widehat{B}$ on $M^\infty(X)$. Since the functionals $\widehat{A}$ and $\widehat{B}$ are continuous, and $M^\infty(X)$ lies densely in $L^p(X)$, we obtain the identity $\widehat{A} = \widehat{B}$ on $L^p(X)$.

q.e.d.

Now we consider *multiplication functionals* A_g as follows:

Theorem 8.2. *Let us choose the exponent $1 \le p \le +\infty$ and with $q \in [1,+\infty]$ its conjugate exponent satisfying*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

For each function $g \in L^q(X)$ being given, the symbol $A_g : L^p(X) \to \mathbb{R}$ with

$$A_g(f) := I(fg), \qquad f \in L^p(X)$$

represents a bounded linear functional such that $\|A_g\| = \|g\|_q$ holds true.

Proof: Obviously, $A_g : L^p(X) \to \mathbb{R}$ constitutes a linear functional. Hölder's inequality yields the estimate

$$|A_g(f)| = |I(fg)| \le I(|f||g|) \le \|f\|_p \|g\|_q \quad \text{for all} \quad f \in L^p(X),$$

and we see

$$\|A_g\| \le \|g\|_q.$$

In the case $1 < p < +\infty$, we choose the function

$$f(x) = |g(x)|^{\frac{q}{p}} \operatorname{sign} g(x)$$

and calculate

$$A_g(f) = I(fg) = I\left(|g|^{\frac{q}{p}+1}\right) = I(|g|^q)$$

$$= \|g\|_q^q = \|g\|_q \|g\|_q^{\frac{q}{p}} = \|g\|_q \Big(I\big(|f|^p\big)\Big)^{\frac{1}{p}} = \|g\|_q \|f\|_p.$$

This implies

$$\frac{A_g(f)}{\|f\|_p} = \|g\|_q \qquad \text{and therefore} \quad \|A_g\| \ge \|g\|_q \tag{1}$$

and consequently $\|A_g\| = \|g\|_q$ for all $1 < p < +\infty$. In the case $p = +\infty$, we choose

$$f(x) = \operatorname{sign} g(x)$$

and we obtain

$$A_g(f) = I(g \operatorname{sign} g) = I(|g|) = \|g\|_1 \, \|f\|_\infty.$$

This implies

$$\frac{A_g(f)}{\|f\|_\infty} = \|g\|_1 \qquad \text{and therefore} \quad \|A_g\| = \|g\|_1.$$

In the case $p = 1$, we choose the following function to the element $g \in L^q(X) = L^\infty(X)$ and for all quantities $\varepsilon > 0$, namely

$$f_\varepsilon(x) := \begin{cases} 1, & x \in X \text{ with } \ g(x) \ge \|g\|_\infty - \varepsilon \\ 0, & x \in X \text{ with } |g(x)| < \|g\|_\infty - \varepsilon \\ -1, & x \in X \text{ with } g(x) \le -\|g\|_\infty + \varepsilon \end{cases}.$$

Therefore, we have

$$A_g(f_\varepsilon) = I(g f_\varepsilon) \ge (\|g\|_\infty - \varepsilon)\|f_\varepsilon\|_1 \qquad \text{for all} \quad \varepsilon > 0,$$

which reveals

$$\frac{A_g(f_\varepsilon)}{\|f_\varepsilon\|_1} \geq \|g\|_\infty - \varepsilon.$$

Consequently, $\|A_g\| \geq \|g\|_\infty - \varepsilon$ is correct and finally $\|A_g\| = \|g\|_\infty$. q.e.d.

We want to show that each bounded linear functional on $L^p(X)$ with $1 \leq p < \infty$ can be represented as a multiplication functional A_g via a generating element $g \in L^q(X)$, where $p^{-1} + q^{-1} = 1$ holds true.

Theorem 8.3. (Regularity in $L^p(X)$)
Let us consider $1 \leq p < +\infty$ and $g \in L^1(X)$. Furthermore, we have a constant $\alpha \in [0, +\infty)$ such that

$$|A_g(f)| = |I(fg)| \leq \alpha \|f\|_p \qquad \textit{for all} \quad f \in M^\infty(X) \tag{2}$$

holds true. Then we infer the property $g \in L^q(X)$ and the estimate $\|g\|_q \leq \alpha$.

Proof:

1. At first, we deduce the following inequality from (2), namely
$$|I(fg)| \leq \alpha \|f\|_p \qquad \text{for all} \quad f \text{ measurable and bounded.} \tag{3}$$
According to Theorem 4.14 from Section 4, the bounded measurable function $f : X \to \mathbb{R}$ possesses a sequence of functions $\{f_k\}_{k=1,2,\ldots} \subset M^\infty(X)$ with
$$f_k(x) \to f(x) \qquad \text{a.e. in} \quad X$$
and
$$\sup_X |f_k(x)| \leq \sup_X |f(x)| =: c \in [0, +\infty).$$
Now Lebesgue's convergence theorem yields
$$|I(fg)| = \lim_{k\to\infty} |I(f_k g)| \leq \lim_{k\to\infty} \alpha \|f_k\|_p = \alpha \|f\|_p.$$
2. Let us assume $1 < p < +\infty$, at first. Then we consider the functions
$$g_k(x) := \begin{cases} g(x)\ , & x \in X \text{ with } |g(x)| \leq k \\ \ 0 \ , & x \in X \text{ with } |g(x)| > k \end{cases}.$$
Now the functions
$$f_k(x) = |g_k(x)|^{\frac{q}{p}} \operatorname{sign} g_k(x), \qquad x \in X,$$
are measurable and bounded. Consequently, we are allowed to insert $f_k(x)$ into (3) and obtain
$$I(f_k g) = I\Big(|g_k|^{\frac{q}{p}+1}\Big) = I(|g_k|^q) = \|g_k\|_q^q\ .$$
Then (3) implies

$$I(f_k g) \le \alpha \|f_k\|_p = \alpha (I(|g_k|^q))^{\frac{1}{p}} = \alpha \|g_k\|_q^{\frac{q}{p}}.$$

For $k = 1, 2, \ldots$ we have the estimate

$$\alpha \ge \|g_k\|_q^{q-\frac{q}{p}} = \|g_k\|_q\,, \quad \alpha^q \ge I(|g_k|^q).$$

We invoke Fatou's theorem and obtain

$$|g(x)|^q \overset{\text{a.e.}}{=} \liminf_{k\to\infty} |g_k(x)|^q \in L(X)$$

as well as

$$\alpha^q \ge I(|g|^q) \quad \text{and consequently} \quad \|g\|_q \le \alpha.$$

3. Now we assume $p = 1$. The quantity $\varepsilon > 0$ being given, we consider the set

$$E := \Big\{ x \in X \;:\; |g(x)| \ge \alpha + \varepsilon \Big\}.$$

We insert the function $f = \chi_E \operatorname{sign} g$ into (3) and obtain

$$\alpha \mu(E) = \alpha \|f\|_1 \ge |I(fg)| \ge (\alpha + \varepsilon)\mu(E).$$

This implies $\mu(E) = 0$ for all $\varepsilon > 0$ and finally $\|g\|_\infty \le \alpha$. q.e.d.

Until now, we considered only one Daniell integral $I : M^\infty(X) \to \mathbb{R}$ as fixed, which we could extend onto the Lebesgue space $L^1(X)$. When a statement refers to this functional, we do not mention this functional I explicitly: We simplify $L^p(X) = L^p(X, I)$, for instance, or $f(x) = 0$ almost everywhere in X if and only if we have an I-null-set $N \subset X$ such that $f(x) = 0$ for all $x \in X \setminus N$ holds true. We already know that

$$M^\infty(X) \subset L^\infty(X) \subset L^p(X), \qquad 1 \le p \le +\infty$$

is correct. Additionally, we consider the Daniell integral J.

Definition 8.4. *We name a Daniell integral*

$$J \,:\, M^\infty(X) \longrightarrow \mathbb{R},$$

which satisfies the conditions (M1) to (M3) as well as (D1) to (D3) from Section 7 and is extendable onto $L^1(X, J) \supset L^\infty(X)$, as absolutely continuous with respect to I *if the following property is valid:*

(D4) Each I-null-set is a J-null-set.

With the aid of ideas of John v. Neumann (see L.H. Loomis: Abstract harmonic analysis), we prove the profound

Theorem 8.5. (Radon, Nikodym)
Let the Daniell integral J be absolutely continuous with respect to I. Then a uniquely determined function $g \in L^1(X)$ exists such that

$$J(f) = I(fg) \quad \text{for all} \quad f \in M^\infty(X)$$

holds true.

Proof:

1. Let $f \in L^\infty(X)$ be given, then we have a null-set $N \subset X$ and a constant $c \in [0, +\infty)$ such that
$$|f(x)| \le c \qquad \text{for all} \quad x \in X \backslash N$$
is valid. We recall the property (D4), and see that N is a J-null-set as well, which implies $f \in L^\infty(X, J)$. A sequence $\{f_k\}_{k=1,2,\ldots} \subset L^\infty(X)$ with $f_k \downarrow 0$ $(k \to \infty)$ a.e. on X fulfills the limit relation
$$f_k \downarrow 0 \quad \text{J-a.e. on} \quad X \quad \text{for} \quad k \to \infty$$
due to (D4). Now B.Levi's theorem on the space $L^1(X, J)$ yields
$$\lim_{k \to \infty} J(f_k) = 0.$$
Consequently, $J : L^\infty(X) \to \mathbb{R}$ represents a Daniell integral. Then we introduce the Daniell integral
$$K(f) := I(f) + J(f), \quad f \in L^\infty(X). \tag{4}$$
As in Section 2 we extend this functional onto the space $L^1(X, K)$; here the a.e.-properties are sufficient. We consider the inclusion $L^1(X, K) \supset L^p(X, K)$ for all $p \in [1, +\infty]$.
2. We take the exponents $p, q \in [1, +\infty]$ with $p^{-1} + q^{-1} = 1$ and obtain the following estimate for all $f \in M^\infty(X)$, namely
$$\begin{aligned} |J(f)| &\le J(|f|) \le K(|f|) \\ &\le \|f\|_{L^p(X,K)} \, \|1\|_{L^q(X,K)} \\ &= \big(I(1) + J(1)\big)^{\frac{1}{q}} \, \|f\|_{L^p(X,K)}. \end{aligned}$$
Therefore, J represents a bounded linear functional on the space $L^p(X, K)$ for an arbitrary exponent $p \in [1, +\infty)$. In the Hilbert space $L^2(X, K)$ we can apply the representation theorem of Fréchet-Riesz and obtain
$$J(f) = K(fh) \qquad \text{for all} \quad f \in M^\infty(X) \tag{5}$$
with an element $h \in L^2(X, K)$. Now Theorem 8.3 - in the case $p = 1$ - is utilized and we see the regularity improvement $h \in L^\infty(X, K)$. Since J is nonnegative, we infer $h(x) \ge 0$ K-a.e. on X. Furthermore, the relation (4) together with the assumption (D4) tell us that the K-null-sets coincide with the I-null-sets, and we arrive at
$$h(x) \ge 0 \qquad \text{a.e. in} \quad X.$$

3. Taking $f \in M^\infty(X)$, we can iterate (5) and (4) as follows

$$\begin{aligned} J(f) = K(fh) &= I(fh) + J(fh) \\ &= I(fh) + K(fh^2) \\ &= I(fh) + I(fh^2) + J(fh^2) = \ldots, \end{aligned}$$

and we obtain

$$J(f) = I\left(f \sum_{k=1}^{l} h^k\right) + J(fh^l), \qquad l = 1, 2, \ldots \tag{6}$$

Let us define

$$A := \Big\{x \in X \ : \ h(x) \geq 1\Big\}$$

and $f = \chi_A$. Via approximation, we immediately see that this element f can be inserted into (6). Then we observe

$$+\infty > J(f) \geq I\left(f \sum_{k=1}^{l} h^k\right) \geq l\, I(\chi_A) \qquad \text{for all} \quad l \in \mathbb{N}$$

and consequently $I(\chi_A) = 0$. Therefore, the inequality $0 \leq h(x) < 1$ a.e. in X is satisfied and, moreover,

$$h^l(x) \downarrow 0 \qquad \text{a.e. in} \quad X \quad \text{for} \quad l \to \infty. \tag{7}$$

Via transition to the limit $l \to \infty$ in (6), then B.Levi's theorem implies

$$J(f) = I\left(f \sum_{k=1}^{\infty} h^k\right) \qquad \text{for all} \quad f \in M^\infty(X),$$

when we note that $f = f^+ - f^-$ holds true. Taking $f(x) \equiv 1$ on X in particular, we infer that

$$g(x) = \sum_{k=1}^{\infty} h^k(x) \overset{\text{a.e.}}{=} \frac{h(x)}{1 - h(x)} \in L^1(X)$$

is fulfilled. q.e.d.

Theorem 8.6. (Decomposition theorem of Jordan and Hahn)
Let the bounded linear functional $A : M^\infty(X) \to \mathbb{R}$ be given on the linear normed space $\{M^\infty(X), \|\cdot\|_p\}$, where $1 \leq p < +\infty$ is fixed. Then we have two nonnegative bounded linear functionals $A^\pm : M^\infty(X) \to \mathbb{R}$ with $A = A^+ - A^-$; this means, more precisely,

$$A(f) = A^+(f) - A^-(f) \qquad \textit{for all} \quad f \in M^\infty(X)$$

with

$$A^{\pm}(f) \geq 0 \qquad \textit{for all} \quad f \in M^{\infty}(X) \quad \textit{with} \quad f \geq 0.$$

Furthermore, we have the estimates

$$\|A^{\pm}\| \leq 2\|A\|, \quad \|A^{-}\| \leq 3\|A\|.$$

Here we define

$$\|A\| := \sup_{f \in M^{\infty},\ \|f\|_p \leq 1} |A(f)|, \quad \|A^{\pm}\| := \sup_{f \in M^{\infty},\ \|f\|_p \leq 1} |A^{\pm}(f)|.$$

Proof:

1. We take $f \in M^{\infty}(X)$ with $f \geq 0$ and set

$$A^{+}(f) := \sup\left\{A(g) \,:\, g \in M^{\infty}(X),\ 0 \leq g \leq f\right\}. \tag{8}$$

Evidently, we have $A^{+}(f) \geq 0$ for all $f \geq 0$. Moreover, the identity

$$\begin{aligned} A^{+}(cf) &= \sup\left\{A(g) \,:\, 0 \leq g \leq cf\right\} = \sup\left\{A(cg) \,:\, 0 \leq g \leq f\right\} \\ &= c \sup\left\{A(g) \,:\, 0 \leq g \leq f\right\} = cA^{+}(f) \end{aligned}$$

for all $f \geq 0$ and $c \geq 0$ holds true. When we take $f_j \in M^{\infty}(X)$ with $f_j \geq 0$ - for j=1,2 - we infer

$$\begin{aligned} &A^{+}(f_1) + A^{+}(f_2) \\ &\quad = \sup\left\{A(g_1) \,:\, 0 \leq g_1 \leq f_1\right\} + \sup\left\{A(g_2) \,:\, 0 \leq g_2 \leq f_2\right\} \\ &\quad = \sup\left\{A(g_1 + g_2) \,:\, 0 \leq g_1 \leq f_1,\ 0 \leq g_2 \leq f_2\right\} \\ &\quad \leq \sup\left\{A(g) \,:\, 0 \leq g \leq f_1 + f_2\right\} = A^{+}(f_1 + f_2). \end{aligned}$$

Given the function g with $0 \leq g \leq f_1 + f_2$, we introduce

$$g_1 := \min(g, f_1) \quad \text{and} \quad g_2 := (g - f_1)^{+}.$$

Then we observe $g_j \leq f_j$ for $j = 1, 2$ as well as $g_1 + g_2 = g$. Consequently, we obtain

$$A^{+}(f_1 + f_2) \leq A^{+}(f_1) + A^{+}(f_2)$$

and finally

$$A^{+}(f_1 + f_2) = A^{+}(f_1) + A^{+}(f_2).$$

Furthermore, the following inequality holds true for all $f \in M^{\infty}(X)$ with $f \geq 0$, namely

$$
\begin{aligned}
|A^+(f)| &= \Big| \sup \Big\{ A(g) \,:\, g \in M^\infty(X),\ 0 \le g \le f \Big\} \Big| \\
&\le \sup \Big\{ |A(g)| \,:\, g \in M^\infty(X),\ 0 \le g \le f \Big\} \\
&\le \sup \Big\{ \|A\|\, \|g\|_p \,:\, g \in M^\infty(X),\ 0 \le g \le f \Big\} \\
&\le \|A\|\, \|f\|_p.
\end{aligned}
$$

2. Now we extend $A^+ : M^\infty(X) \to \mathbb{R}$ via

$$M^\infty(X) \ni f(x) = f^+(x) - f^-(x) \qquad \text{with} \quad f^\pm(x) \ge 0$$

and define

$$A^+(f) := A^+(f^+) - A^+(f^-).$$

Consequently, we obtain with $A^+ : M^\infty(X) \to \mathbb{R}$ a bounded linear mapping. More precisely, we have the following estimate for all $f \in M^\infty(X)$:

$$
\begin{aligned}
|A^+(f)| &\le |A^+(f^+)| + |A^+(f^-)| \\
&\le \|A\| \big(\|f^+\|_p + \|f^-\|_p \big) \le 2\|A\|\, \|f\|_p.
\end{aligned}
$$

This implies $\|A^+\| \le 2\|A\|$.

3. Now we define

$$A^-(f) := A^+(f) - A(f) \qquad \text{for all} \quad f \in M^\infty(X).$$

Obviously, A^- represents a bounded linear functional. Here we observe

$$|A^-(f)| \le |A^+(f)| + |A(f)| \le 2\|A\| \cdot \|f\|_p + \|A\|\, \|f\|_p$$

and consequently $\|A^-\| \le 3\|A\|$. Finally, the inequality

$$A^-(f) = A^+(f) - A(f) = \sup \Big\{ A(g) \,:\, 0 \le g \le f \Big\} - A(f) \ge 0$$

for all $f \in M^\infty(X)$ with $f \ge 0$ is satisfied. q.e.d.

Theorem 8.7. (The Riesz representation theorem)
Let $1 \le p < +\infty$ be fixed. For each bounded linear functional $A \in (\mathcal{L}^p(X))^*$ being given, there exists exactly one generating element $g \in \mathcal{L}^q(X)$ with the property

$$A(f) = I(fg) \quad \text{for all } f \in \mathcal{L}^p(X).$$

Here the identity $p^{-1} + q^{-1} = 1$ holds true for the conjugate exponent $q \in (1, +\infty]$.

Proof: We perform our proof in two steps.

1. *Uniqueness:* Let the functions $g_1, g_2 \in \mathcal{L}^q(X)$ with
$$A(f) = I(fg_1) = I(fg_2) \qquad \text{for all} \quad f \in \mathcal{L}^p(X)$$
be given, and we deduce
$$0 = I\Big(f(g_1 - g_2)\Big) \qquad \text{for all} \quad f \in \mathcal{L}^p(X).$$
We recall Theorem 8.2 and obtain $0 = \|g_1 - g_2\|_{\mathcal{L}^q(X)}$, which implies $g_1 = g_2$ in $\mathcal{L}^q(X)$.
2. *Existence:* The functional $A : M^\infty(X) \to \mathbb{R}$ satisfies
$$|A(f)| \le \alpha \|f\|_p \qquad \text{for all} \quad f \in M^\infty(X) \tag{9}$$
with a bound $\alpha \in [0, +\infty)$. The decomposition theorem of Jordan-Hahn gives us nonnegative bounded linear functionals $A^\pm : M^\infty(X) \to \mathbb{R}$ satisfying
$$\|A^\pm\| \le 3\|A\| \le 3\alpha \quad \text{and} \quad A = A^+ - A^-.$$
Here the space $M^\infty(X)$ is endowed with the $\|\cdot\|_p$-norm. In particular, we observe $|A^\pm(f)| < +\infty$ for $f(x) = 1$, $x \in X$. A sequence $\{f_k\}_{k=1,2,\ldots} \subset M^\infty(X)$ with $f_k \downarrow 0$ in X converges uniformly on each compact set towards 0, due to Dini's theorem. Then we arrive at the estimate
$$|A^\pm(f_k)| \le 3\alpha \|f_k\|_p \longrightarrow 0 \qquad \text{for} \quad k \to \infty.$$
With $A^\pm$ we have two Daniell integrals, which are absolutely continuous with respect to I. When N namely is an I-null-set, we infer
$$|A^\pm(\chi_N)| \le 3\alpha \|\chi_N\|_p = 0.$$
Therefore, N is a null-set for the Daniell integrals $A^\pm$ as well. The Radon-Nikodym theorem provides elements $g^\pm \in \mathcal{L}^1(X)$ such that the representation
$$A^\pm(f) = I(fg^\pm) \qquad \text{for all} \quad f \in M^\infty(X)$$
holds true. This implies
$$\begin{aligned} A(f) &= A^+(f) - A^-(f) \\ &= I(fg^+) - I(fg^-) \\ &= I(fg) \qquad \text{for all} \quad f \in M^\infty(X), \end{aligned}$$
when we define $g := g^+ - g^- \in \mathcal{L}^1(X)$. On account of (9) our regularity theorem yields $g \in \mathcal{L}^q(X)$. When we extend the functional continuously onto $\mathcal{L}^p(X)$, we arrive at the representation
$$A(f) = I(fg) \qquad \text{for all} \quad f \in \mathcal{L}^p(X)$$
with a generating function $g \in \mathcal{L}^q(X)$. q.e.d.

Now we address the question of compactness in infinite-dimensional spaces of functions.

Definition 8.8. *A sequence* $\{x_k\}_{k=1,2,\ldots} \subset \mathcal{B}$ *in a Banach space* $\mathcal{B}$ *is called weakly convergent towards an element* $x \in \mathcal{B}$ *- symbolically* $x_k \rightharpoonup x$ *- if the limit relations*

$$\lim_{k\to\infty} A(x_k) = A(x)$$

hold true for each continuous linear functional $A \in \mathcal{B}^*$*.*

Theorem 8.9. (Weak compactness of $L^p(X)$)
Let us take the exponent $1 < p < +\infty$*. Furthermore, let* $\{f_k\}_{k=1,2,\ldots} \subset L^p(X)$ *denote a bounded sequence with the property*

$$\|f_k\|_p \le c \quad \textit{for a constant} \quad c \in [0,+\infty) \quad \textit{and all indices} \quad k \in \mathbb{N}.$$

Then we have a subsequence $\{f_{k_l}\}_{l=1,2,\ldots}$ *and a limit element* $f \in L^p(X)$ *such that* $f_{k_l} \rightharpoonup f$ *in* $L^p(X)$ *holds true.*

Proof:

1. We invoke the Riesz representation theorem and see the following: The relation $f_l \rightharpoonup f$ holds true if and only if $I(f_l g) \to I(fg)$ for all $g \in L^q(X)$ is correct; here we have $p^{-1}+q^{-1} = 1$ as usual. Theorem 7.14 from Section 7 tells us that the space $L^q(X)$ is separable. Therefore, we find a sequence $\{g_m\}_{m=1,2,\ldots} \subset L^q(X)$ which lies densely in $L^q(X)$. From the bounded sequence $\{f_k\}_{k=1,2,\ldots} \subset L^p(X)$ satisfying $\|f_k\|_p \le c$ for all $k \in \mathbb{N}$, we now extract successively the subsequences
$$\{f_k\}_{k=1,2,\ldots} \;\supset\; \{f_{k_l^{(1)}}\}_{l=1,2,\ldots} \;\supset\; \{f_{k_l^{(2)}}\}_{l=1,2,\ldots} \;\supset\; \cdots$$
such that
$$\lim_{l\to\infty} I(f_{k_l^{(m)}} g_m) =: \alpha_m \in \mathbb{R}, \qquad m = 1,2,\ldots.$$
Then we apply Cantor's diagonalization procedure, and we make the transition to the diagonal sequence $f_{k_l} := f_{k_l^{(l)}}$, $l = 1,2,\ldots$. Now we observe that
$$\lim_{l\to\infty} I(f_{k_l} g_m) = \alpha_m, \quad m = 1,2,\ldots$$
holds true.
2. By the symbol
$$\mathcal{D} := \left\{ g \in L^q(X) \;:\; \begin{array}{c} \text{There exist } N \in \mathbb{N} \text{ and } c_1,\ldots,c_N \in \mathbb{R} \\ \text{and } 1 \le i_1 < \ldots < i_N < +\infty \text{ with } g = \displaystyle\sum_{k=1}^N c_k g_{i_k} \end{array} \right\}$$

we denote the vector space of finite linear combinations of $\{g_m\}_{m=1,2,\ldots}$. Obviously, the limits

$$A(g) := \lim_{l\to\infty} I(f_{k_l} g) \qquad \text{for all} \quad g \in \mathcal{D}$$

exist. The linear functional $A : \mathcal{D} \to \mathbb{R}$ is bounded on the space $\mathcal{D}$ which lies densely in $L^q(X)$, and we have, more precisely,

$$|A(g)| \le c\|g\|_q \quad \text{for all} \quad g \in \mathcal{D}.$$

As described in Theorem 8.1, we continue our functional A from $\mathcal{D}$ onto the space $L^q(X)$, and the Riesz representation theorem provides an element $f \in L^p(X)$ such that

$$A(g) = I(fg) \;\; \text{for all } g \in L^q(X).$$

3. Now we show that $f_{k_l} \rightharpoonup f$ in $L^p(X)$ holds true. For each element $g \in L^q(X)$ we find a sequence $\{\widetilde{g}_j\}_{j=1,2,\ldots} \subset \mathcal{D}$ satisfying

$$g \overset{L^q}{=} \lim_{j\to\infty} \widetilde{g}_j \in L^q(X).$$

Then we obtain

$$\begin{aligned} |I(fg) - I(f_{k_l} g)| &\le |I(f(g-\widetilde{g}_j))| + |I((f-f_{k_l})\widetilde{g}_j)| + |I(f_{k_l}(\widetilde{g}_j - g))| \\ &\le 2C\|g-\widetilde{g}_j\|_q + |I((f-f_{k_l})\widetilde{g}_j)| \le \varepsilon \end{aligned}$$

for sufficiently large - but fixed - j and the indices $l \ge l_0$. q.e.d.

Remarks:

1. Similarly, we can introduce the notion of weak convergence in Hilbert spaces. Due to Hilbert's selection theorem, we can extract a weakly convergent subsequence from each bounded sequence in Hilbert spaces. However, it is not possible to extract a norm-convergent subsequence from an arbitrary bounded sequence in infinite-dimensional Hilbert spaces. Here we recommend the study of Section 6 in Chapter 8, in particular the first Definition and Example as well as Hilbert's selection theorem.
2. We assume $1 \le p_1 \le p_2 < +\infty$. Then the weak convergence $f_k \rightharpoonup f$ in $L^{p_2}(X)$ implies weak convergence $f_k \rightharpoonup f$ in $L^{p_1}(X)$, which is immediately inferred from the embedding relation $L^{p_2}(X) \hookrightarrow L^{p_1}(X)$.

Theorem 8.10. *The L^p-norm is* lower semicontinuous with respect to weak convergence*, which means:*

$$f_k \rightharpoonup f \;\; in \; L^p(X) \quad \Longrightarrow \quad \|f\|_p \le \liminf_{k\to\infty} \|f_k\|_p.$$

Here we assume $1 < p < +\infty$ for the Hölder exponent.

Proof: We start with $f_k \rightharpoonup f$ in $L^p(X)$ and deduce

$$I(f_k g) \to I(fg) \qquad \text{for all} \quad g \in L^q(X).$$

When we choose

$$g(x) := |f(x)|^{\frac{p}{q}} \operatorname{sign} f(x) \in L^q(X),$$

we infer

$$I\Big(f_k |f|^{\frac{p}{q}} \operatorname{sign} f(x)\Big) \to I(|f|^p) = \|f\|_p^p$$

with $p^{-1} + q^{-1} = 1$. For all quantities $\varepsilon > 0$, we find an index $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned} \|f\|_p^p - \varepsilon \le I\Big(f_k |f|^{\frac{p}{q}} \operatorname{sign} f(x)\Big) &\le I\Big(|f_k|\, |f|^{\frac{p}{q}}\Big) \\ &\le \|f_k\|_p \Big(I(|f|^p)\Big)^{\frac{1}{q}} = \|f_k\|_p (\|f\|_p)^{\frac{p}{q}} \end{aligned}$$

holds true for all indices $k \ge k_0(\varepsilon)$. When we assume $\|f\|_p > 0$ - without loss of generality - we find to each quantity $\varepsilon > 0$ an index $k_0(\varepsilon) \in \mathbb{N}$ such that

$$\|f_k\|_p \ge \|f\|_p - (\|f\|_p)^{-\frac{p}{q}} \varepsilon \quad \text{for all} \quad k \ge k_0(\varepsilon)$$

is correct. This implies

$$\liminf_{k \to \infty} \|f_k\|_p \ge \|f\|_p.$$

q.e.d.

9 Some Historical Notices to Chapter 2

The modern theory of partial differential equations requires to understand the class of Lebesgue integrable functions – extending the classical family of continuous functions. These more abstract concepts were only reluctantly accepted – even by some of the mathematical heroes of their time. A beautiful source of information, written within the *golden era for mathematics* in Poland between World War I and II, is the following textbook by
Stanisław Saks: Theory of the Integral; Warsaw 1933, Reprint by Hafner Publ. Co., New York (1937).

We would like to present a direct quotation from the preface of this monograph: “On several occasions attempts were made to generalize the old process of integration of Cauchy-Riemann, but it was Lebesgue who first made real progress in this matter. At the same time, Lebesgue’s merit is not only to have created a new and more general notion of integral, nor even to have established its intimate connection with the theory of measure: the value of his work consists primarily in his theory of derivation which is parallel to that of integration. This enabled his discovery to find many applications in the most widely different branches of analysis and, from the point of view of method, made it possible to reunite the two fundamental conceptions of integral, namely that of definite integral and that of primitive, which appeared to be forever separated as soon as integration went outside the domain of continuous functions."

The integral of Lebesgue (1875–1941) was wonderfully combined with the abstract spaces created by D. Hilbert (1862–1943) and S. Banach (1892–1945). When we develop the modern theory of partial differential equations in the next volume of our textook, we shall highly appreciate the great vision of the words above by Stanisław Saks – written already in 1933.

Figure 1.2 PORTRAIT OF STEFAN BANACH (1892–1945)
taken from the *Lexikon bedeutender Mathematiker* edited by S. Gottwald, H.-J. Ilgauds, and K.-H. Schlote in Bibliographisches Institut Leipzig (1988).