Chapter 1

Differentiation and Integration on Manifolds

In this chapter we lay the foundations for our treatise on partial differential equations. A detailed description for the contents of Chapter 1 is given in the Introduction to Volume 1 above. At first, we fix some familiar notations used throughout the two volumes of our textbook.

By the symbol $\mathbb{R}^n$ we denote the n-dimensional Euclidean space with the points $x = (x_1, \ldots, x_n)$ where $x_i \in \mathbb{R}$, and we define their modulus

$$|x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}.$$

In general, we denote open subsets in $\mathbb{R}^n$ by the symbol Ω. By the symbol $\overline{M}$ we indicate the topological closure and by $\overset{\circ}{M}$ the open kernel of a set $M \subset \mathbb{R}^n$. In the sequel, we shall use the following linear spaces of functions:

$C^0(\Omega)$	continuous functions on Ω
$C^k(\Omega)$	k-times continuously differentiable functions on Ω
$C_0^k(\Omega)$	k-times continuously differentiable functions f on Ω with the compact support $\operatorname{supp} f = \overline{\{x \in \Omega : f(x) \neq 0\}} \subset \Omega$
$C^k(\overline{\Omega})$	k-times continuously differentiable functions on Ω, whose derivatives up to the order k can be continuously extended onto the closure $\overline{\Omega}$
$C_0^k(\Omega \cup \Theta)$ ..	k-times continuously differentiable functions f on Ω, whose derivatives up to the order k can be extended onto the closure $\overline{\Omega}$ continuously with the property $\operatorname{supp} f \subset \Omega \cup \Theta$
$C_*^*(*, K)$ ...	space of functions as above with values in $K = \mathbb{R}^n$ or $K = \mathbb{C}$.

Finally, we utilize the notations

∇u	gradient $(u_{x_1}, \ldots, u_{x_n})$ of a function $u = u(x_1, \ldots, x_n) \in C^1(\mathbb{R}^n)$

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Δu .... Laplace operator $\sum\limits_{i=1}^{n} u_{x_i x_i}$ of a function $u \in C^2(\mathbb{R}^n)$

J_f .... functional determinant or *Jacobian* of a function $f : \mathbb{R}^n \to \mathbb{R}^n \in C^1(\mathbb{R}^n, \mathbb{R}^n)$.

1 The Weierstraß Approximation Theorem

Let $\Omega \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ denote an open set and $f(x) \in C^k(\Omega)$ with $k \in \mathbb{N} \cup \{0\} =: \mathbb{N}_0$ a k-times continuously differentiable function. We intend to prove the following statement:
There exists a sequence of polynomials $p_m(x)$, $x \in \mathbb{R}^n$ for $m = 1, 2, \ldots$ which converges on each compact subset $C \subset \Omega$ uniformly towards the function $f(x)$. Furthermore, all partial derivatives up to the order k of the polynomials p_m converge uniformly on C towards the corresponding derivatives of the function f. The coefficients of the polynomials p_m depend on the approximation, in general. If this were not the case, the function

$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{x^2}\right), & x > 0 \\ 0, & x \le 0 \end{cases}$$

could be expanded into a power series. However, this leads to the evident contradiction:

$$0 \equiv \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, x^k.$$

In the following Proposition, we introduce a *mollifier* which enables us to smooth systematically integrable functions.

Proposition 1.1. *We consider the following function to each $\varepsilon > 0$, namely*

$$\begin{aligned} K_\varepsilon(z) &:= \frac{1}{\sqrt{\pi\varepsilon}^{\,n}} \exp\left(-\frac{|z|^2}{\varepsilon}\right) \\ &= \frac{1}{\sqrt{\pi\varepsilon}^{\,n}} \exp\left(-\frac{1}{\varepsilon}\left(z_1^2 + \ldots + z_n^2\right)\right), \qquad z \in \mathbb{R}^n. \end{aligned}$$

Then this function $K_\varepsilon = K_\varepsilon(z)$ possesses the following properties:

1. *We have $K_\varepsilon(z) > 0$ for all $z \in \mathbb{R}^n$;*
2. *The condition $\int\limits_{\mathbb{R}^n} K_\varepsilon(z)\, dz = 1$ holds true;*
3. *For each $\delta > 0$ we observe: $\lim\limits_{\varepsilon \to 0+} \int\limits_{|z| \ge \delta} K_\varepsilon(z)\, dz = 0$.*

Proof:

1. The exponential function is positive, and the statement is obvious.
2. We substitute $z = \sqrt{\varepsilon}x$ with $dz = \sqrt{\varepsilon}^{\,n}\, dx$ and calculate

$$\int\limits_{\mathbb{R}^n} K_\varepsilon(z)\, dz = \frac{1}{\sqrt{\pi\varepsilon}^{\,n}} \int\limits_{\mathbb{R}^n} \exp\Big(-\frac{|z|^2}{\varepsilon}\Big)\, dz$$
$$= \frac{1}{\sqrt{\pi}^{\,n}} \int\limits_{\mathbb{R}^n} \exp\Big(-|x|^2\Big)\, dx = \Bigg(\frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} \exp\Big(-t^2\Big)\, dt\Bigg)^n = 1.$$

3. We utilize the substitution from part 2 of our proof and obtain

$$\int\limits_{|z|\geq\delta} K_\varepsilon(z)\, dz = \frac{1}{\sqrt{\pi}^{\,n}} \int\limits_{|x|\geq\delta/\sqrt{\varepsilon}} \exp\Big(-|x|^2\Big)\, dx \longrightarrow 0 \qquad \text{for} \quad \varepsilon \to 0+\,.$$

q.e.d.

Proposition 1.2. *Let us consider $f(x) \in C_0^0(\mathbb{R}^n)$ and additionally the function*

$$f_\varepsilon(x) := \int\limits_{\mathbb{R}^n} K_\varepsilon(y-x) f(y)\, dy, \qquad x \in \mathbb{R}^n$$

for $\varepsilon > 0$. Then we infer

$$\sup_{x\in\mathbb{R}^n} |f_\varepsilon(x) - f(x)| \longrightarrow 0 \quad \textit{for} \quad \varepsilon \to 0+,$$

and consequently the functions $f_\varepsilon(x)$ converge uniformly on the space $\mathbb{R}^n$ towards the function $f(x)$.

Proof: On account of its compact support, the function $f(x)$ is uniformly continuous on the space $\mathbb{R}^n$. The number $\eta > 0$ being given, we find a number $\delta = \delta(\eta) > 0$ such that

$$x, y \in \mathbb{R}^n,\ |x-y| \leq \delta \quad \Longrightarrow \quad |f(x) - f(y)| \leq \eta.$$

Since f is bounded, we find a quantity $\varepsilon_0 = \varepsilon_0(\eta) > 0$ satisfying

$$2 \sup_{y\in\mathbb{R}^n} |f(y)| \int\limits_{|y-x|\geq\delta} K_\varepsilon(y-x)\, dy \leq \eta \quad \text{for all} \quad 0 < \varepsilon < \varepsilon_0.$$

We note that

$$|f_\varepsilon(x) - f(x)| = \Bigg| \int\limits_{\mathbb{R}^n} K_\varepsilon(y-x)\, f(y)\, dy - f(x) \int\limits_{\mathbb{R}^n} K_\varepsilon(y-x)\, dy \Bigg|$$
$$\leq \Bigg| \int\limits_{|y-x|\leq\delta} K_\varepsilon(y-x)\, \{f(y) - f(x)\}\, dy \Bigg|$$
$$+ \Bigg| \int\limits_{|y-x|\geq\delta} K_\varepsilon(y-x)\, \{f(y) - f(x)\}\, dy \Bigg|,$$

and we arrive at the following estimate for all points $x \in \mathbb{R}^n$ and all numbers $0 < \varepsilon < \varepsilon_0$, namely

$$\begin{aligned} |f_\varepsilon(x) - f(x)| &\le \int\limits_{|y-x|\le\delta} K_\varepsilon(y-x)\,|f(y)-f(x)|\,dy \\ &\quad + \int\limits_{|y-x|\ge\delta} K_\varepsilon(y-x)\,\{|f(y)|+|f(x)|\}\,dy \\ &\le \eta + 2\sup_{y\in\mathbb{R}^n}|f(y)| \int\limits_{|y-x|\ge\delta} K_\varepsilon(y-x)\,dy \le 2\eta. \end{aligned}$$

We summarize our considerations to

$$\sup_{x\in\mathbb{R}^n} |f_\varepsilon(x) - f(x)| \longrightarrow 0 \qquad \text{for} \quad \varepsilon \to 0+\,.$$

q.e.d.

In the sequel, we need

Proposition 1.3. (Partial integration in $\mathbb{R}^n$)
When the functions $f(x) \in C_0^1(\mathbb{R}^n)$ and $g(x) \in C^1(\mathbb{R}^n)$ are given, we infer

$$\int\limits_{\mathbb{R}^n} g(x)\frac{\partial}{\partial x_i} f(x)\,dx = -\int\limits_{\mathbb{R}^n} f(x)\frac{\partial}{\partial x_i} g(x)\,dx \quad \textit{for} \quad i = 1,\dots,n.$$

Proof: On account of the property $f(x) \in C_0^1(\mathbb{R}^n)$, we find a radius $r > 0$ such that $f(x) = 0$ and $f(x)g(x) = 0$ is correct for all points $x \in \mathbb{R}^n$ with $|x_j| \ge r$ for one index $j \in \{1,\dots,n\}$ at least. The fundamental theorem of differential- and integral-calculus yields

$$\begin{aligned} &\int\limits_{\mathbb{R}^n} \frac{\partial}{\partial x_i}\Big\{f(x)g(x)\Big\}\,dx \\ &= \int\limits_{-r}^{+r}\dots\int\limits_{-r}^{+r}\left(\int\limits_{-r}^{+r} \frac{\partial}{\partial x_i}\Big\{f(x)g(x)\Big\}\,dx_i\right) dx_1\dots dx_{i-1}dx_{i+1}\dots dx_n = 0. \end{aligned}$$

This implies

$$0 = \int\limits_{\mathbb{R}^n} \frac{\partial}{\partial x_i}\Big\{f(x)g(x)\Big\}\,dx = \int\limits_{\mathbb{R}^n} g(x)\frac{\partial}{\partial x_i} f(x)\,dx + \int\limits_{\mathbb{R}^n} f(x)\frac{\partial}{\partial x_i} g(x)\,dx.$$

q.e.d.

Proposition 1.4. *Let the function $f(x) \in C_0^k(\mathbb{R}^n, \mathbb{C})$ with $k \in \mathbb{N}_0$ be given. Then we have a sequence of polynomials with complex coefficients*

$$p_m(x) = \sum_{j_1,\ldots,j_n=0}^{N(m)} c^{(m)}_{j_1\ldots j_n} x_1^{j_1}\ldots x_n^{j_n} \quad \textit{for} \quad m=1,2,\ldots$$

such that the limit relations

$$D^\alpha p_m(x) \longrightarrow D^\alpha f(x) \qquad \textit{for} \quad m\to\infty, \quad |\alpha|\le k$$

are satisfied uniformly in each ball $B_R := \{x\in\mathbb{R}^n : |x|\le R\}$ *with the radius* $0<R<+\infty$. *Here we define the differential operator* D^α *with* $\alpha = (\alpha_1,\ldots,\alpha_n)$ *by*

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}\,, \qquad |\alpha| := \alpha_1+\ldots+\alpha_n,$$

where $\alpha_1,\ldots,\alpha_n\ge 0$ *represent nonnegative integers.*

Proof: We differentiate the function $f_\varepsilon(x)$ with respect to the variables x_i, and together with Proposition 1.3 we see

$$\begin{aligned}\frac{\partial}{\partial x_i} f_\varepsilon(x) &= \int\limits_{\mathbb{R}^n}\left\{\frac{\partial}{\partial x_i}K_\varepsilon(y-x)\right\} f(y)\,dy\\ &= -\int\limits_{\mathbb{R}^n}\left\{\frac{\partial}{\partial y_i}K_\varepsilon(y-x)\right\} f(y)\,dy\\ &= \int\limits_{\mathbb{R}^n} K_\varepsilon(y-x)\frac{\partial}{\partial y_i} f(y)\,dy\end{aligned}$$

for $i=1,\ldots,n$. By repeated application of this device, we arrive at

$$D^\alpha f_\varepsilon(x) = \int\limits_{\mathbb{R}^n} K_\varepsilon(y-x) D^\alpha f(y)\,dy, \quad |\alpha|\le k.$$

Here we note that $D^\alpha f(y)\in C^0_0(\mathbb{R}^n)$ holds true. Due to Proposition 1.2, the family of functions $D^\alpha f_\varepsilon(x)$ converges uniformly on the space $\mathbb{R}^n$ towards $D^\alpha f(x)$ - for all $|\alpha|\le k$ - when $\varepsilon\to 0+$ holds true. Now we choose the radius $R>0$ such that $\operatorname{supp} f\subset B_R$ is valid. Taking the number $\varepsilon>0$ as fixed, we consider the power series

$$K_\varepsilon(z) = \frac{1}{\sqrt{\pi\varepsilon}^{\,n}}\exp\left(-\frac{|z|^2}{\varepsilon}\right) = \frac{1}{\sqrt{\pi\varepsilon}^{\,n}}\sum_{j=0}^{\infty}\frac{1}{j!}\left(-\frac{|z|^2}{\varepsilon}\right)^j,$$

which converges uniformly in B_{2R}. Therefore, each number $\varepsilon>0$ possesses an index $N_0 = N_0(\varepsilon,R)$ such that the polynomial

$$P_{\varepsilon,R}(z) := \frac{1}{\sqrt{\pi\varepsilon}^{\,n}}\sum_{j=0}^{N_0(\varepsilon,R)}\frac{1}{j!}\left(-\frac{z_1^2+\ldots+z_n^2}{\varepsilon}\right)^j$$

is subject to the following estimate:

$$\sup_{|z|\le 2R} |K_\varepsilon(z) - P_{\varepsilon,R}(z)| \le \varepsilon.$$

With the expression

$$\widetilde{f}_{\varepsilon,R}(x) := \int\limits_{\mathbb{R}^n} P_{\varepsilon,R}(y-x) f(y)\, dy$$

we obtain a polynomial in the variables $x_1, \ldots, x_n$ - for each $\varepsilon > 0$. Furthermore, we deduce

$$D^\alpha \widetilde{f}_{\varepsilon,R}(x) = \int\limits_{\mathbb{R}^n} P_{\varepsilon,R}(y-x) D^\alpha f(y)\, dy \quad \text{for all} \quad x \in \mathbb{R}^n, \quad |\alpha| \le k.$$

Now we arrive at the subsequent estimate for all $|\alpha| \le k$ and $|x| \le R$, namely

$$\begin{aligned} |D^\alpha f_\varepsilon(x) - D^\alpha \widetilde{f}_{\varepsilon,R}(x)| &= \Big| \int\limits_{|y|\le R} \Big\{K_\varepsilon(y-x) - P_{\varepsilon,R}(y-x)\Big\} D^\alpha f(y)\, dy \Big| \\ &\le \int\limits_{|y|\le R} |K_\varepsilon(y-x) - P_{\varepsilon,R}(y-x)| |D^\alpha f(y)|\, dy \\ &\le \varepsilon \int\limits_{|y|\le R} |D^\alpha f(y)|\, dy. \end{aligned}$$

Therefore, the polynomials $D^\alpha \widetilde{f}_{\varepsilon,R}(x)$ converge uniformly on B_R towards the derivatives $D^\alpha f(x)$. Choosing the null-sequence $\varepsilon = \frac{1}{m}$ with $m = 1, 2, \ldots$, we obtain an approximating sequence of polynomials $p_{m,R}(x) := \widetilde{f}_{\frac{1}{m},R}(x)$ in B_R, which is still dependent on the radius R. We take $r = 1, 2, \ldots$ and find polynomials $p_r = p_{m_r,r}$ satisfying

$$\sup_{x\in B_r} |D^\alpha p_r(x) - D^\alpha f(x)| \le \frac{1}{r} \quad \text{for all} \quad |\alpha| \le k.$$

The sequence p_r satisfies all the properties stated above. q.e.d.

We are now prepared to prove the fundamental

Theorem 1.5. (The Weierstraß approximation theorem)
Let $\Omega \subset \mathbb{R}^n$ denote an open set and $f(x) \in C^k(\Omega, \mathbb{C})$ a function with the degree of regularity $k \in \mathbb{N}_0$. Then we have a sequence of polynomials with complex coefficients of the degree $N(m) \in \mathbb{N}_0$, namely

$$f_m(x) = \sum_{j_1,\ldots,j_n=0}^{N(m)} c^{(m)}_{j_1\ldots j_n} x_1^{j_1} \cdot \ldots \cdot x_n^{j_n}, \qquad x \in \mathbb{R}^n, \quad m = 1, 2, \ldots,$$

such that the limit relations

$$D^\alpha f_m(x) \longrightarrow D^\alpha f(x) \qquad \textit{for} \quad m \to \infty, \quad |\alpha| \le k$$

are satisfied uniformly on each compact set $C \subset \Omega$.

Proof: We consider a sequence $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega$ of bounded open sets exhausting Ω. Here we have $\overline{\Omega_j} \subset \Omega_{j+1}$ for all indices j. Via the partition of unity (compare Theorem 1.8), we construct a sequence of functions $\phi_j(x) \in C_0^\infty(\Omega)$ satisfying $0 \le \phi_j(x) \le 1$, $x \in \Omega$ and $\phi_j(x) = 1$ on $\overline{\Omega_j}$ for $j = 1, 2, \ldots$. Then we observe the sequence of functions

$$f_j(x) := \begin{cases} f(x)\phi_j(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

with the following properties:

$$f_j(x) \in C_0^k(\mathbb{R}^n) \ \text{ and } \ D^\alpha f_j(x) = D^\alpha f(x), \qquad x \in \Omega_j, \quad |\alpha| \le k.$$

Due to Proposition 1.4, we find a polynomial $p_j(x)$ to each function $f_j(x)$ satisfying

$$\sup_{x \in \Omega_j} |D^\alpha p_j(x) - D^\alpha f_j(x)| = \sup_{x \in \Omega_j} |D^\alpha p_j(x) - D^\alpha f(x)| \le \frac{1}{j}\,, \qquad |\alpha| \le k,$$

since Ω_j is bounded. For a compact set $C \subset \Omega$ being given arbitrarily, we find an index $j_0 = j_0(C) \in \mathbb{N}$ such that the inclusion $C \subset \Omega_j$ for all $j \ge j_0(C)$ is correct. This implies

$$\sup_{x \in C} |D^\alpha p_j(x) - D^\alpha f(x)| \le \frac{1}{j}\,, \qquad j \ge j_0(C), \quad |\alpha| \le k.$$

When we consider the transition to the limit $j \to \infty$, we arrive at the statement

$$\sup_{x \in C} |D^\alpha p_j(x) - D^\alpha f(x)| \longrightarrow 0$$

for all $|\alpha| \le k$ and all compact subsets $C \subset \Omega$. q.e.d.

Theorem 1.5 above provides a uniform approximation by polynomials in the interior of the domain for the respective function. Continuous functions defined on compact sets can be uniformly approximated up to the boundary of the domain. Here we need the following

Theorem 1.6. (Tietze's extension theorem)
Let $C \subset \mathbb{R}^n$ denote a compact set and $f(x) \in C^0(C, \mathbb{C})$ a continuous function defined on C. Then we have a continuous extension of f onto the whole space $\mathbb{R}^n$ which means: There exists a function $g(x) \in C^0(\mathbb{R}^n, \mathbb{C})$ satisfying

$$f(x) = g(x) \qquad \textit{for all points} \quad x \in C.$$

Proof:

1. We take $x \in \mathbb{R}^n$ and define the function

$$d(x) := \min_{y \in C} |y - x|,$$

which measures the distance of the point x to the set C. Since C is compact, we find to each point $x \in \mathbb{R}^n$ a point $\overline{y} \in C$ satisfying $|\overline{y} - x| = d(x)$. When $x_1, x_2 \in \mathbb{R}^n$ are chosen, we infer the following inequality for $\overline{y}_2 \in C$ with $|\overline{y}_2 - x_2| = d(x_2)$, namely

$$\begin{aligned} d(x_1) - d(x_2) &= \inf_{y \in C} \Big(|x_1 - y| \Big) - |x_2 - \overline{y}_2| \\ &\leq |x_1 - \overline{y}_2| - |x_2 - \overline{y}_2| \\ &\leq |x_1 - x_2|. \end{aligned}$$

Interchanging the points x_1 and x_2, we obtain an analogous inequality and infer

$$|d(x_1) - d(x_2)| \leq |x_1 - x_2| \qquad \text{for all points} \quad x_1, x_2 \in \mathbb{R}^n.$$

In particular, the distance $d : \mathbb{R}^n \to \mathbb{R}$ represents a continuous function.

2. For $x \notin C$ and $a \in \mathbb{R}^n$, we consider the function

$$\varrho(x, a) := \max \left\{ 2 - \frac{|x - a|}{d(x)}, 0 \right\}.$$

The point a being fixed, the arguments above tell us that the function $\varrho(x, a)$ is continuous in $\mathbb{R}^n \setminus C$. Furthermore, we observe $0 \leq \varrho(x, a) \leq 2$ as well as

$$\begin{aligned} \varrho(x, a) &= 0 \qquad \text{for} \quad |a - x| \geq 2d(x), \\ \varrho(x, a) &\geq \frac{1}{2} \qquad \text{for} \quad |a - x| \leq \frac{3}{2}\, d(x). \end{aligned}$$

3. With $\left\{ a^{(k)} \right\} \subset C$ let us choose a sequence of points which is dense in C. Since the function $f(x) : C \to \mathbb{C}$ is bounded, the series below

$$\sum_{k=1}^{\infty} 2^{-k} \varrho\Big(x, a^{(k)}\Big) f\Big(a^{(k)}\Big) \quad \text{and} \quad \sum_{k=1}^{\infty} 2^{-k} \varrho\Big(x, a^{(k)}\Big)$$

converge uniformly for all $x \in \mathbb{R}^n \setminus C$, and represent continuous functions in the variable x there. Furthermore, we observe

$$\sum_{k=1}^{\infty} 2^{-k} \varrho\Big(x, a^{(k)}\Big) > 0 \quad \text{for} \quad x \in \mathbb{R}^n \setminus C,$$

since each point $x \in \mathbb{R}^n \setminus C$ possesses at least one index k with $\varrho(x, a^{(k)}) > 0$. Therefore, the function

$$h(x) := \frac{\sum\limits_{k=1}^{\infty} 2^{-k} \varrho\Big(x, a^{(k)}\Big) f\Big(a^{(k)}\Big)}{\sum\limits_{k=1}^{\infty} 2^{-k} \varrho\Big(x, a^{(k)}\Big)} = \sum_{k=1}^{\infty} \varrho_k(x) f\Big(a^{(k)}\Big), \qquad x \in \mathbb{R}^n \setminus C,$$

is continuous. Here we have set

$$\varrho_k(x) := \frac{2^{-k} \varrho\Big(x, a^{(k)}\Big)}{\sum\limits_{k=1}^{\infty} 2^{-k} \varrho\Big(x, a^{(k)}\Big)} \quad \text{for} \quad x \in \mathbb{R}^n \setminus C.$$

We have the identity

$$\sum_{k=1}^{\infty} \varrho_k(x) \equiv 1, \quad x \in \mathbb{R}^n \setminus C.$$

4. Now we define the function

$$g(x) := \begin{cases} f(x),\ x \in C \\ h(x),\ x \in \mathbb{R}^n \setminus C \end{cases}.$$

We have still to show the continuity of g on ∂C. We have the following estimate for $z \in C$ and $x \notin C$:

$$\begin{aligned} |h(x) - f(z)| &= \Big| \sum_{k=1}^{\infty} \varrho_k(x) \Big\{ f\Big(a^{(k)}\Big) - f(z) \Big\} \Big| \\ &\leq \sum_{k : |a^{(k)} - x| \leq 2d(x)} \varrho_k(x) \Big| f\Big(a^{(k)}\Big) - f(z) \Big| \\ &\leq \sup_{a \in C\,:\,|a-x| \leq 2d(x)} |f(a) - f(z)| \\ &\leq \sup_{a \in C\,:\,|a-z| \leq 2d(x) + |x-z|} |f(a) - f(z)| \\ &\leq \sup_{a \in C\,:\,|a-z| \leq 3|x-z|} |f(a) - f(z)|. \end{aligned}$$

Since the function $f : C \to \mathbb{C}$ is uniformly continuous, we infer

$$\lim_{\substack{x \to z \\ x \notin C}} h(x) = f(z) \quad \text{for} \quad z \in \partial C \quad \text{and} \quad x \notin C.$$

q.e.d.

The assumption of compactness for the subset C is decisive in the theorem above. The function $f(x) = \sin(1/x)$, $x \in (0, \infty)$ namely cannot be continuously extended into the origin 0.

Theorem 1.5 and Theorem 1.6 together yield

Theorem 1.7. *Let $f(x) \in C^0(C, \mathbb{C})$ denote a continuous function on the compact set $C \subset \mathbb{R}^n$. To each quantity $\varepsilon > 0$, we then find a polynomial $p_\varepsilon(x)$ with the property*

$$|p_\varepsilon(x) - f(x)| \leq \varepsilon \quad \textit{for all points} \quad x \in C.$$

We shall construct smoothing functions which turn out to be extremely valuable in the sequel. At first, we easily show that the function

$$\psi(t) := \begin{cases} \exp\Big(-\dfrac{1}{t}\Big), & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases} \tag{1}$$

belongs to the regularity class $C^\infty(\mathbb{R})$. We take $R > 0$ arbitrarily and consider the function

$$\varphi_R(x) := \psi\Big(|x|^2 - R^2\Big), \qquad x \in \mathbb{R}^n. \tag{2}$$

Then we observe $\varphi_R \in C^\infty(\mathbb{R}^n, \mathbb{R})$. We have $\varphi_R(x) > 0$ if $|x| > R$ holds true, $\varphi_R(x) = 0$ if $|x| \leq R$ holds true, and therefore

$$\operatorname{supp}(\varphi_R) = \Big\{x \in \mathbb{R}^n \;:\; |x| \geq R\Big\}.$$

Furthermore, we develop the following function out of $\psi(t)$, namely

$$\varrho = \varrho(t) : \mathbb{R} \to \mathbb{R} \;\in C^\infty(\mathbb{R}) \qquad \text{via} \quad t \mapsto \varrho(t) := \psi(1-t)\psi(1+t). \tag{3}$$

This function is symmetric, which means $\varrho(-t) = \varrho(t)$ for all $t \in \mathbb{R}$. Furthermore, we see $\varrho(t) > 0$ for all $t \in (-1, 1)$, $\varrho(t) = 0$ for all else, and consequently

$$\operatorname{supp}(\varrho) = [-1, 1].$$

Finally, we define the following ball for $\xi \in \mathbb{R}^n$ and $\varepsilon > 0$, namely

$$B_\varepsilon(\xi) := \Big\{x \in \mathbb{R}^n \;:\; |x - \xi| \leq \varepsilon\Big\} \tag{4}$$

as well as the functions

$$\varphi_{\xi,\varepsilon}(x) := \varrho\bigg(\frac{|x-\xi|^2}{\varepsilon^2}\bigg), \quad x \in \mathbb{R}^n. \tag{5}$$

Then the regularity property $\varphi_{\xi,\varepsilon} \in C^\infty(\mathbb{R}^n, \mathbb{R})$ is valid, and we deduce $\varphi_{\xi,\varepsilon}(x) > 0$ for all $x \in \overset{\circ}{B}_\varepsilon(\xi)$ as well as $\varphi_{\xi,\varepsilon}(x) = 0$ if $|x - \xi| \geq \varepsilon$ holds true. This implies

$$\operatorname{supp}(\varphi_{\xi,\varepsilon}) = B_\varepsilon(\xi).$$

A fundamental principle of proof is presented in the next

Theorem 1.8. (Partition of unity)
Let $K \subset \mathbb{R}^n$ denote a compact set, and to each point $x \in K$ the symbol $\mathcal{O}_x \subset \mathbb{R}^n$ indicates an open set with $x \in \mathcal{O}_x$. Then we can select finitely many points $x^{(1)}, x^{(2)}, \ldots, x^{(m)} \in K$ with the associate number $m \in \mathbb{N}$ such that the covering

$$K \subset \bigcup_{\mu=1}^{m} \mathcal{O}_{x^{(\mu)}}$$

holds true. Furthermore, we find functions $\chi_\mu = \chi_\mu(x) : \mathcal{O}_{x^{(\mu)}} \to [0, +\infty)$ satisfying $\chi_\mu \in C_0^\infty(\mathcal{O}_{x^{(\mu)}})$ for $\mu = 1, \ldots, m$ such that the function

$$\chi(x) := \sum_{\mu=1}^{m} \chi_\mu(x), \qquad x \in \mathbb{R}^n \tag{6}$$

has the following properties:

(a) The regularity $\chi \in C_0^\infty(\mathbb{R}^n)$ holds true.
(b) We have $\chi(x) = 1$ for all $x \in K$.
(c) The inequality $0 \leq \chi(x) \leq 1$ is valid for all $x \in \mathbb{R}^n$.

Proof:

1. Since the set $K \subset \mathbb{R}^n$ is compact, we find a radius $R > 0$ such that $K \subset B := B_R(0)$ holds true. To each point $x \in B$ we now choose an open ball $\overset{\circ}{B}_{\varepsilon_x}(x)$ of radius $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \subset \mathcal{O}_x$ for $x \in K$ and $B_{\varepsilon_x}(x) \subset \mathbb{R}^n \setminus K$ for $x \in B \setminus K$ is satisfied. The system of sets $\left\{\overset{\circ}{B}_{\varepsilon_x}(x)\right\}_{x \in B}$ yields an open covering of the compact set B. According to the Heine-Borel covering theorem, finitely many open sets suffice to cover B, let us say

$$\overset{\circ}{B}_{\varepsilon_1}(x^{(1)}), \overset{\circ}{B}_{\varepsilon_2}(x^{(2)}), \ldots, \overset{\circ}{B}_{\varepsilon_m}(x^{(m)}), \overset{\circ}{B}_{\varepsilon_{m+1}}(x^{(m+1)}), \ldots \overset{\circ}{B}_{\varepsilon_{m+M}}(x^{(m+M)}) \,.$$

Here we observe $x^{(\mu)} \in K$ for $\mu = 1, 2, \ldots, m$ and $x^{(\mu)} \in B \setminus K$ for $\mu = m+1, \ldots, m+M$, defining $\varepsilon_\mu := \varepsilon_{x^{(\mu)}}$ for $\mu = 1, \ldots, m+M$.
With the aid of the function from (5), we now consider the nonnegative functions $\varphi_\mu(x) := \varphi_{x^{(\mu)}, \varepsilon_\mu}(x)$. We note that the following regularity properties hold true: $\varphi_\mu \in C_0^\infty(\mathcal{O}_{x^{(\mu)}})$ for $\mu = 1, \ldots, m$ and $\varphi_\mu \in C_0^\infty(\mathbb{R}^n \setminus K)$ for $\mu = m+1, \ldots, m+M$, respectively. Furthermore, we define $\varphi_{m+M+1}(x) := \varphi_R(x)$, where we introduced φ_R already in (2).
Obviously, we arrive at the statement

$$\sum_{\mu=1}^{m+M+1} \varphi_\mu(x) > 0 \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

2. Now we define the functions χ_μ due to

$$\chi_\mu(x) := \Big[\sum_{\mu=1}^{m+M+1} \varphi_\mu(x)\Big]^{-1} \varphi_\mu(x), \quad x \in \mathbb{R}^n$$

for $\mu = 1, \dots, m + M + 1$. The functions χ_μ and φ_μ belong to the same classes of regularity, and we observe additionally

$$\sum_{\mu=1}^{m+M+1} \chi_\mu(x) = \Big[\sum_{\mu=1}^{m+M+1} \varphi_\mu(x)\Big]^{-1} \sum_{\mu=1}^{m+M+1} \varphi_\mu(x) \equiv 1 \quad \text{for all} \quad x \in \mathbb{R}^n.$$

The properties (a), (b), and (c) of the function $\chi(x) = \sum\limits_{\mu=1}^{m} \chi_\mu(x)$ are directly inferred from the construction above. q.e.d.

Definition 1.9. *We name the functions $\chi_1, \chi_2, \dots, \chi_m$ from Theorem 1.8 a* partition of unity subordinate to the open covering $\{\mathcal{O}_x\}_{x \in K}$ of the compact set K.

2 Parameter-invariant Integrals and Differential Forms

In the basic lectures of analysis the following fundamental result is established.

Theorem 2.1. (Transformation formula for multiple integrals)
Let $\Omega, \Theta \subset \mathbb{R}^n$ denote two open sets, where we take $n \in \mathbb{N}$. Furthermore, let $y = (y_1(x_1, \dots, x_n), \dots, y_n(x_1, \dots, x_n)) : \Omega \to \Theta$ denote a bijective mapping of the class $C^1(\Omega, \mathbb{R}^n)$ satisfying

$$J_y(x) := det\Big(\frac{\partial y_i(x)}{\partial x_j}\Big)_{i,j=1,\dots,n} \neq 0 \quad \textit{for all} \quad x \in \Omega.$$

Let the function $f = f(y) : \Theta \to \mathbb{R} \in C^0(\Theta)$ be given with the property

$$\int\limits_{\Theta} |f(y)|\, dy < +\infty$$

for the improper Riemannian integral of $|f|$. Then we have the transformation formula

$$\int\limits_{\Theta} f(y)\, dy = \int\limits_{\Omega} f(y(x))\, |J_y(x)|\, dx.$$

In the sequel, we shall integrate differential forms over m-dimensional surfaces in $\mathbb{R}^n$.

Definition 2.2. *Let the open set $T \subset \mathbb{R}^m$ with $m \in \mathbb{N}$ constitute the* parameter domain. *Furthermore, the symbol*

$$X(t) = \begin{pmatrix} x_1(t_1, \ldots, t_m) \\ \vdots \\ x_n(t_1, \ldots, t_m) \end{pmatrix} : T \longrightarrow \mathbb{R}^n \in C^k(T, \mathbb{R}^n)$$

represents a mapping - with $k, n \in \mathbb{N}$ and $m \leq n$ - whose functional matrix

$$\partial X(t) = \Big(X_{t_1}(t), \ldots, X_{t_m}(t)\Big), \quad t \in T$$

has the rank m for all $t \in T$. Then we call X a parametrized regular surface *with the* parametric representation $X(t) : T \to \mathbb{R}^n$.
When $X : T \to \mathbb{R}^n$ and $\widetilde{X} : \widetilde{T} \to \mathbb{R}^n$ are two parametric representations, we call them equivalent *if there exists a topological mapping*

$$t = t(s) = \Big(t_1(s_1, \ldots, s_m), \ldots, t_m(s_1, \ldots, s_m)\Big) : \widetilde{T} \longrightarrow T \in C^k(\widetilde{T}, T)$$

with the following properties:

1. $J(s) := \dfrac{\partial(t_1, \ldots, t_m)}{\partial(s_1, \ldots, s_m)}(s) = \begin{vmatrix} \frac{\partial t_1}{\partial s_1}(s) & \ldots & \frac{\partial t_1}{\partial s_m}(s) \\ \vdots & & \vdots \\ \frac{\partial t_m}{\partial s_1}(s) & \ldots & \frac{\partial t_m}{\partial s_m}(s) \end{vmatrix} > 0$ *for all* $s \in \widetilde{T}$;
2. $\widetilde{X}(s) = X\Big(t(s)\Big)$ *for all* $s \in \widetilde{T}$.

We say that $\widetilde{X}$ originates from X by an orientation-preserving reparametrization. *The equivalence class $[X]$ consisting of all those parametric representations which are equivalent to X is named an* open, oriented, m-dimensional, regular surface of the class C^k in $\mathbb{R}^n$. *We name a surface* embedded in the space $\mathbb{R}^n$ *if additionally the mapping $X : T \to \mathbb{R}^n$ is injective.*

Example 2.3. (Curves in $\mathbb{R}^n$)
On the interval $T = (a, b) \subset \mathbb{R}$ we consider the mapping

$$X = X(t) = \Big(x_1(t), \ldots, x_n(t)\Big) \in C^1(T, \mathbb{R}^n), \qquad t \in T$$

satisfying

$$|X'(t)| = \sqrt{\{x_1'(t)\}^2 + \ldots + \{x_n'(t)\}^2} > 0 \quad \text{for all} \quad t \in T.$$

Then the integral

$$L(X) = \int_a^b |X'(t)|\, dt$$

determines the *arc length of the curve* $X = X(t)$.

Example 2.4. (Classical surfaces in $\mathbb{R}^3$)
When $T \subset \mathbb{R}^2$ denotes an open parameter domain, we consider the Gaussian surface representation

$$X(u,v) = \Big(x(u,v), y(u,v), z(u,v)\Big) \,:\, T \longrightarrow \mathbb{R}^3 \in C^1(T, \mathbb{R}^3).$$

The vector in the direction of the *normal* to the surface is given by

$$\begin{aligned} X_u \wedge X_v &= \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right) \\ &= (y_u z_v - z_u y_v, z_u x_v - x_u z_v, x_u y_v - x_v y_u). \end{aligned}$$

The *unit normal vector* to the surface X is defined by the formula

$$N(u,v) := \frac{X_u \wedge X_v}{|X_u \wedge X_v|},$$

and we note that

$$|N(u,v)| = 1, \quad N(u,v)\cdot X_u(u,v) = N(u,v)\cdot X_v(u,v) = 0 \quad \text{for all} \quad (u,v) \in T.$$

Via the integral

$$A(X) := \iint\limits_T |X_u \wedge X_v| \, dudv$$

we determine the *area of the surface* $X = X(u,v)$. We evaluate

$$|X_u \wedge X_v|^2 = (X_u \wedge X_v) \cdot (X_u \wedge X_v) = |X_u|^2 |X_v|^2 - (X_u \cdot X_v)^2$$

such that

$$A(X) = \iint\limits_T \sqrt{|X_u|^2 |X_v|^2 - (X_u \cdot X_v)^2} \, dudv$$

follows.

Example 2.5. (Hypersurfaces in $\mathbb{R}^n$)
Let $X : T \to \mathbb{R}^n$ denote a regular surface - defined on the parameter domain $T \subset \mathbb{R}^{n-1}$. The $(n-1)$ vectors $X_{t_1}, \ldots, X_{t_{n-1}}$ are linearly independent for all $t \in T$; and they span the *tangential space to the surface* at the point $X(t) \in \mathbb{R}^n$. Now we shall construct the *unit normal vector* $\nu(t) \in \mathbb{R}^n$. Therefore, we require

$$|\nu(t)| = 1 \quad \text{and} \quad \nu(t) \cdot X_{t_k}(t) = 0 \quad \text{for all} \quad k = 1, \ldots, n-1$$

as well as

$$\det\Big(X_{t_1}(t), \ldots, X_{t_{n-1}}(t), \nu(t)\Big) > 0 \quad \text{for all} \quad t \in T.$$

Consequently, the vectors $X_{t_1}, \ldots, X_{t_{n-1}}$ and ν constitute a positive-oriented n-frame. In this context we define the functions

$$D_i(t) := (-1)^{n+i} \frac{\partial(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)}{\partial(t_1, \ldots, t_{n-1})}, \quad i = 1, \ldots, n.$$

Then we obtain the identity

$$\begin{vmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_1} \\ \vdots & & \vdots \\ \frac{\partial x_1}{\partial t_{n-1}} & \cdots & \frac{\partial x_n}{\partial t_{n-1}} \\ \lambda_1 & \cdots & \lambda_n \end{vmatrix} = \sum_{i=1}^{n} \lambda_i D_i \qquad \text{for all} \quad \lambda_1, \ldots, \lambda_n \in \mathbb{R}.$$

Now we introduce the *unit normal vector*

$$\nu(t) = \Big(\nu_1(t), \ldots, \nu_n(t)\Big) = \frac{1}{\sqrt{\sum\limits_{j=1}^{n} (D_j(t))^2}} \Big(D_1(t), \ldots, D_n(t)\Big), \qquad t \in T.$$

Evidently, the equation $|\nu(t)| = 1$ holds true and we calculate

$$\sum_{i=1}^{n} D_i \frac{\partial x_i}{\partial t_j} = \begin{vmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_1} \\ \vdots & & \vdots \\ \frac{\partial x_1}{\partial t_{n-1}} & \cdots & \frac{\partial x_n}{\partial t_{n-1}} \\ \frac{\partial x_1}{\partial t_j} & \cdots & \frac{\partial x_n}{\partial t_j} \end{vmatrix} = 0\,, \qquad 1 \le j \le n-1.$$

This implies the orthogonality relation $X_{t_j}(t) \cdot \nu(t) = 0$ for all $t \in T$ and $j = 1, \ldots, n-1$. The *surface element of the hypersurface in* $\mathbb{R}^n$ is given by

$$\begin{aligned} d\sigma &:= \begin{vmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_1} \\ \vdots & & \vdots \\ \frac{\partial x_1}{\partial t_{n-1}} & \cdots & \frac{\partial x_n}{\partial t_{n-1}} \\ \nu_1 & \cdots & \nu_n \end{vmatrix} dt_1 \ldots dt_{n-1} \\ &= \sum_{j=1}^{n} \nu_j D_j \, dt_1 \ldots dt_{n-1} \\ &= \sqrt{\sum_{j=1}^{n} (D_j(t))^2} \; dt_1 \ldots dt_{n-1}. \end{aligned}$$

Consequently, the surface area of X is determined by the improper integral

$$A(X) := \int_T \sqrt{\sum_{j=1}^{n} (D_j(t))^2}\, dt.$$

Example 2.6. An open set $\Omega \subset \mathbb{R}^n$ can be seen as a surface in $\mathbb{R}^n$ - via the mapping

$$X(t) := t, \quad \text{with} \quad t \in T \quad \text{and} \quad T := \Omega \subset \mathbb{R}^n.$$

Example 2.7. (An m-dimensional surface in $\mathbb{R}^n$)
Let $X(t) : T \to \mathbb{R}^n$ denote a surface with $T \subset \mathbb{R}^m$ as its parameter domain and the dimensions $1 \le m \le n$. By the symbols

$$g_{ij}(t) := X_{t_i} \cdot X_{t_j} \quad \text{for} \quad i,j = 1,\ldots,m$$

we define the *metric tensor* of the surface X. Furthermore, we call

$$g(t) := \det\Big(g_{ij}(t)\Big)_{i,j=1,\ldots,m}$$

its *Gramian determinant.* We complete the system $\{X_{t_i}\}_{i=1,\ldots,m}$ in $\mathbb{R}^n$ at each point $X(t)$ by the vectors ξ_j with $j = 1,\ldots,n-m$ such that the following properties are valid:

(a) We have $\xi_j \cdot \xi_k = \delta_{jk}$ for all $j,k = 1,\ldots,n-m$;
(b) The relations $X_{t_i} \cdot \xi_j = 0$ for $i = 1,\ldots,m$ and $j = 1,\ldots,n-m$ hold true;
(c) The condition $\det\Big(X_{t_1},\ldots,X_{t_m},\xi_1,\ldots,\xi_{n-m}\Big) > 0$ is correct.

Then we determine the surface element as follows:

$$\begin{aligned} d\sigma(t) &= \det\Big(X_{t_1},\ldots,X_{t_m},\xi_1,\ldots,\xi_{n-m}\Big)\, dt_1 \ldots dt_m \\ &= \sqrt{\det\Big\{(X_{t_1},\ldots,\xi_{n-m})^t \circ (X_{t_1},\ldots,\xi_{n-m})\Big\}}\, dt_1 \ldots dt_m \\ &= \sqrt{\det\Big(g_{ij}(t)\Big)_{i,j=1,\ldots,m}}\, dt_1 \ldots dt_m \\ &= \sqrt{g(t)}\, dt_1 \ldots dt_m. \end{aligned}$$

In order to evaluate our surface element via the Jacobi matrix $\partial X(t)$, we need the following

Proposition 2.8. *Let A and B denote two $n \times m$-matrices, where $m \le n$ holds true. For the numbers $1 \le i_1 < \ldots < i_m \le n$, let $A_{i_1 \ldots i_m}$ define the matrix consisting of those rows with the indices $i_1,\ldots,i_m$ from the matrix A.*

Correspondingly, we define the submatrices of the matrix B. Then we have the identity

$$det\,(A^t \circ B) = \sum_{1 \le i_1 < \ldots < i_m \le n} \det A_{i_1 \ldots i_m} \det B_{i_1 \ldots i_m}.$$

Proof: We fix A and show that the identity above holds true for all matrices B.

1. When we consider the unit vectors $e_1, \ldots, e_n$ as columns in $\mathbb{R}^n$, the formula above holds true for all $B = (e_{j_1}, \ldots, e_{j_m})$ with $j_1, \ldots, j_m \in \{1, \ldots, n\}$, at first.
2. When the formula above holds true for the matrix $B = (b_1, \ldots, b_m)$, this remains true for the matrix $B' = (b_1, \ldots, \lambda b_i, \ldots, b_m)$.
3. When we have our formula for the matrices $B' = (b_1, \ldots, b_i', \ldots, b_m)$ and $B'' = (b_1, \ldots, b_i'', \ldots, b_m)$, this remains true for the matrix $B = (b_1, \ldots, b_i' + b_i'', \ldots, b_m)$.

q.e.d.

Corollary: Given the $n \times m$-matrix A, we have the identity

$$\det\,(A^t \circ A) = \sum_{1 \le i_1 < \ldots < i_m \le n} (\det A_{i_1 \ldots i_m})^2.$$

We write the metric tensor in the form

$$\Big(g_{ij}(t)\Big)_{i,j=1,\ldots,m} = \partial X(t)^t \circ \partial X(t)$$

with the functional matrix $\partial X(t) = \Big(X_{t_1}(t), \ldots, X_{t_m}(t)\Big)$, and we deduce

$$g(t) = \det \Big(g_{ij}(t)\Big)_{i,j=1,\ldots,m}$$
$$= \sum_{1 \le i_1 < \ldots < i_m \le n} \left(\frac{\partial(x_{i_1}, \ldots, x_{i_m})}{\partial(t_1, \ldots, t_m)}\,(t) \right)^2.$$

Therefore, the *surface element* satisfies

$$d\sigma(t) = \sqrt{g(t)}\, dt_1 \ldots dt_m$$
$$= \sqrt{\sum_{1 \le i_1 < \ldots < i_m \le n} \left(\frac{\partial(x_{i_1}, \ldots, x_{i_m})}{\partial(t_1, \ldots, t_m)}(t) \right)^2}\, dt_1 \ldots dt_m.$$

Definition 2.9. *The* surface area *of an open, oriented, m-dimensional, regular C^1-surface in $\mathbb{R}^n$ with the parametric representation $X(t) : T \to \mathbb{R}^n$ is given by the improper Riemannian integral*

$$A(X) := \int\limits_T \sqrt{\sum_{1 \le i_1 < \ldots < i_m \le n} \left(\frac{\partial(x_{i_1}, \ldots, x_{i_m})}{\partial(t_1, \ldots, t_m)} \right)^2} \, dt_1 \ldots dt_m.$$

Here the parameter domain $T \subset \mathbb{R}^m$ *is open and the dimensions* $1 \le m \le n$ *are prescribed. If* $A(X) < +\infty$ *is valid, the surface* $[X]$ *possesses* finite area.

Remarks:

1. With the aid of the transformation formula for multiple integrals, we immediately verify that the value of our surface area is independent of the parametric representation.
2. In the case $m = 1$, we obtain by $A(X)$ the arc length of the curve $X : T \to \mathbb{R}^n$. The case $m = 2$ and $n = 3$ reduces to the classical area of a surface X in $\mathbb{R}^3$. In the case $m = n - 1$ we evaluate the area of hypersurfaces in $\mathbb{R}^n$.

In physics and geometry, we often meet with integrals which only depend on the m-dimensional surface and which are independent of their parametric representation. In this way, we are invited to consider integrals over so-called differential forms.

Definition 2.10. *On the open set* $\mathcal{O} \subset \mathbb{R}^n$, *let the functions* $a_{i_1 \ldots i_m} \in C^k(\mathcal{O})$ *with* $i_1, \ldots, i_m \in \{1, \ldots, n\}$ *and* $1 \le m \le n$ *be given; where* $k \in \mathbb{N}_0$ *holds true. Now we define the set*

$$\mathcal{F} := \Big\{ X \mid X : T \to \mathbb{R}^n \text{ is a regular, oriented, m-dimensional surface with finite area such that } X(T) \subset\subset \mathcal{O} \Big\}.$$

By a differential form of the degree m in the class $C^k(\mathcal{O})$, *namely*

$$\omega := \sum_{i_1, \ldots, i_m = 1}^{n} a_{i_1 \ldots i_m}(x) \, dx_{i_1} \wedge \ldots \wedge dx_{i_m},$$

or briefly an m-form of the class $C^k(\mathcal{O})$, *we comprehend the function* $\omega : \mathcal{F} \to \mathbb{R}$ *defined as follows:*

$$\omega(X) := \int\limits_T \sum_{i_1, \ldots, i_m = 1}^{n} a_{i_1 \ldots i_m}(X(t)) \, \frac{\partial(x_{i_1}, \ldots, x_{i_m})}{\partial(t_1, \ldots, t_m)} \, dt_1 \ldots dt_m, \quad X \in \mathcal{F}.$$

Remark:

1. We abbreviate $A \subset\subset \mathcal{O}$, if the set $\overline{A} \subset \mathbb{R}^n$ is compact and $\overline{A} \subset \mathcal{O}$ holds true.
2. Since the coefficient functions $a_{i_1 \ldots i_m}(X(t))$, $t \in T$ are bounded and the surface has finite area, the integral above converges absolutely.

3. When two differential symbols
$$\omega = \sum_{i_1,\dots,i_m=1}^{n} a_{i_1\dots i_m}(x)\, dx_{i_1} \wedge \dots \wedge dx_{i_m}$$
and
$$\widetilde{\omega} = \sum_{i_1,\dots,i_m=1}^{n} \widetilde{a}_{i_1\dots i_m}(x)\, dx_{i_1} \wedge \dots \wedge dx_{i_m}$$
are given, we introduce an equivalence relation between them as follows:
$$\omega \sim \widetilde{\omega} \quad \Longleftrightarrow \quad \omega(X) = \widetilde{\omega}(X) \quad \text{for all} \quad X \in \mathcal{F}.$$
Therefore, we comprehend a differential form as an *equivalence class* of differential symbols, where we choose a representative to characterize this differential form.
4. When $X, \widetilde{X} \in \mathcal{F}$ are two equivalent representations of the surface $[X]$, we observe
$$\begin{aligned}
\omega(\widetilde{X}) &= \int\limits_{\widetilde{T}} \sum_{i_1,\dots,i_m=1}^{n} a_{i_1\dots i_m}\Big(\widetilde{X}(s)\Big) \frac{\partial(\widetilde{x}_{i_1},\dots,\widetilde{x}_{i_m})}{\partial(s_1,\dots,s_m)}\, ds_1 \dots ds_m \\
&= \int\limits_{\widetilde{T}} \sum_{i_1,\dots,i_m=1}^{n} a_{i_1\dots i_m}\Big(X(t(s))\Big) \frac{\partial(x_{i_1},\dots,x_{i_m})}{\partial(t_1,\dots,t_m)} \frac{\partial(t_1,\dots,t_m)}{\partial(s_1,\dots,s_m)}\, ds_1 \dots ds_m \\
&= \int\limits_{T} \sum_{i_1,\dots,i_m=1}^{n} a_{i_1\dots i_m}\Big(X(t)\Big) \frac{\partial(x_{i_1},\dots,x_{i_m})}{\partial(t_1,\dots,t_m)}\, dt_1 \dots dt_m \\
&= \omega(X).
\end{aligned}$$
Therefore, ω is a mapping which is defined on the equivalence classes of the oriented surfaces $[X]$ with $X \in \mathcal{F}$.
5. An orientation-reversing parametric transformation $t = t(s)$ with $J(s) < 0$, $s \in \widetilde{T}$ induces the change of sign: $\omega(\widetilde{X}) = -\omega(X)$.

Definition 2.11. *A* 0-form *of the class $C^k(\mathcal{O})$ is simply a function $f(x) \in C^k(\mathcal{O})$ and more precisely*
$$\omega = f(x), \qquad x \in \mathcal{O}.$$
When $1 \le m \le n$ is fixed, we name
$$\beta^m := dx_{i_1} \wedge \dots \wedge dx_{i_m}, \qquad 1 \le i_1, \dots, i_m \le n$$
a basic m-form.

Definition 2.12. *Let $\omega, \omega_1, \omega_2$ represent three m-forms of the class $C^0(\mathcal{O})$ and choose $c \in \mathbb{R}$. Then we define the differential forms $c\omega$ and $\omega_1 + \omega_2$ by the prescription*

$$(c\omega)(X) := c\omega(X) \qquad \text{for all} \quad X \in \mathcal{F}$$

and

$$(\omega_1 + \omega_2)(X) := \omega_1(X) + \omega_2(X) \qquad \text{for all} \quad X \in \mathcal{F}$$

respectively.

The m-dimensional differential forms constitute a vector space with the null-element

$$o(X) = 0 \qquad \text{for all} \quad X \in \mathcal{F}.$$

Definition 2.13. (Exterior product of differential forms)
Let the differential forms

$$\omega_1 = \sum_{1 \le i_1, \ldots, i_l \le n} a_{i_1 \ldots i_l}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_l}$$

of degree l and

$$\omega_2 = \sum_{1 \le j_1, \ldots, j_m \le n} b_{j_1 \ldots j_m}(x)\, dx_{j_1} \wedge \ldots \wedge dx_{j_m}$$

of degree m in the class $C^k(\mathcal{O})$ with $k \in \mathbb{N}_0$ be given. Then we define the exterior product *of ω_1 and ω_2 as the $(l+m)$-form*

$$\omega = \omega_1 \wedge \omega_2 := \sum_{1 \le i_1, \ldots, i_l, j_1, \ldots, j_m \le n} a_{i_1 \ldots i_l}(x) b_{j_1 \ldots j_m}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_l} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_m}$$

of the class $C^k(\mathcal{O})$.

Remarks:

1. Arbitrary differential forms $\omega_1, \omega_2, \omega_3$ are subject to the associative law
$$(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3).$$
2. When two l-forms ω_1, ω_2 and one m-form ω_3 are given, we have the distributive law
$$(\omega_1 + \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3.$$
3. The alternating character of the determinant reveals
$$dx_{i_1} \wedge \ldots \wedge dx_{i_l} = \operatorname{sign}(\pi)\, dx_{i_{\pi(1)}} \wedge \ldots \wedge dx_{i_{\pi(l)}}.$$
Here the symbol $\pi : \{1, \ldots, l\} \to \{1, \ldots, l\}$ denotes a permutation with $\operatorname{sign}(\pi)$ as its sign.

4. In particular, when the two indices i_{j_1} and i_{j_2} coincide, we deduce
$$dx_{i_1} \wedge \ldots \wedge dx_{i_l} = 0.$$
Therefore, each m-form in $\mathbb{R}^n$ with the degree $m > n$ vanishes identically.
5. An l-form ω_1 and an m-form ω_2 are subject to the commutator relation
$$\omega_1 \wedge \omega_2 = (-1)^{lm} \omega_2 \wedge \omega_1 .$$
Therefore, the exterior product is not commutative.
6. We can represent each m-form in the following way:
$$\omega = \sum_{1 \le i_1 < \ldots < i_m \le n} a_{i_1 \ldots i_m}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_m} .$$
The basic m-forms $dx_{i_1} \wedge \ldots \wedge dx_{i_m}$, $1 \le i_1 < \ldots < i_m \le n$ constitute a basis for the space of all differential forms, with coefficient functions in the class $C^k(\mathcal{O})$, where $k \in \mathbb{N}_0$ holds true.

Definition 2.14. *Let the symbol*
$$\omega = \sum_{1 \le i_1 < \ldots < i_m \le n} a_{i_1 \ldots i_m}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_m}, \quad x \in \mathcal{O}$$
denote a continuous differential form on the open set $\mathcal{O} \subset \mathbb{R}^n$*, with* $1 \le m \le n$ *being fixed. Then we define the* improper Riemannian integral of the differential form ω *over the surface* $[X] \subset \mathcal{O}$ *via*
$$\int_{[X]} \omega := \int_T \sum_{1 \le i_1 < \ldots < i_m \le n} a_{i_1 \ldots i_m}\Big(X(t)\Big) \frac{\partial(x_{i_1}, \ldots, x_{i_m})}{\partial(t_1, \ldots, t_m)}\, dt_1 \ldots dt_m,$$
if ω is absolutely integrable over X *and consequently*
$$\int_{[X]} |\omega| := \int_T \Big| \sum_{1 \le i_1 < \ldots < i_m \le n} a_{i_1 \ldots i_m}\Big(X(t)\Big) \frac{\partial(x_{i_1}, \ldots, x_{i_m})}{\partial(t_1, \ldots, t_m)} \Big|\, dt_1 \ldots dt_m$$
$$< +\infty$$
is satisfied.

Remark: With the aid of the transformation formula, we show that these integrals are independent of the choice of the representatives for the surface. Therefore, we are allowed to write
$$\int_{[X]} |\omega| = \int_X |\omega|, \quad \int_{[X]} \omega = \int_X \omega.$$

Example 2.15. (Curvilinear integrals)
Let $a(x) = \Big(a_1(x_1, \ldots, x_n), \ldots, a_n(x_1, \ldots, x_n)\Big)$ denote a continuous vector-field and

$$\omega = \sum_{i=1}^{n} a_i(x)\, dx_i$$

the associate 1-form or *Pfaffian form.* Furthermore, let

$$X(t) = \Big(x_1(t), \ldots, x_n(t)\Big) \;:\; T \to \mathbb{R}^n \in C^1(T)$$

represent a regular C^1-curve defined on the parameter interval $T = (a, b)$. Then we observe

$$\int_X \omega = \int_a^b \left(\sum_{i=1}^{n} a_i\Big(X(t)\Big) x_i'(t)\right) dt.$$

We shall investigate curvilinear integrals in Section 6 more intensively.

Example 2.16. (Surface integrals)
Let the continuous vector-field $a(x) = \Big(a_1(x_1, \ldots, x_n), \ldots, a_n(x_1, \ldots, x_n)\Big)$ with the associate $(n-1)$-form

$$\omega = \sum_{i=1}^{n} a_i(x)(-1)^{n+i}\, dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n$$

be given. Furthermore, let $X(t_1, \ldots, t_{n-1}) \;:\; T \to \mathbb{R}^n$ represent a regular C^1-surface. Then we observe

$$\begin{aligned}
\int\limits_X \omega &= \int\limits_T \sum_{i=1}^{n} a_i\Big(X(t)\Big)(-1)^{n+i}\, \frac{\partial(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)}{\partial(t_1, \ldots, t_{n-1})}\, dt_1 \ldots dt_{n-1} \\
&= \int\limits_T \left(\sum_{i=1}^{n} a_i\Big(X(t)\Big) D_i(t)\right) dt_1 \ldots dt_{n-1} \\
&= \int\limits_T \{a(X(t)) \cdot \nu(t)\}\, d\sigma(t).
\end{aligned}$$

This surface integral will be studied more intensively in Section 5, when we prove the Gaussian integral theorem.

Example 2.17. (Domain integrals)
Let us consider the continuous function $f = f(x_1, \ldots, x_n)$ with the associate n-form

$$\omega = f(x)\, dx_1 \wedge \ldots \wedge dx_n.$$

Furthermore, $X = X(t) : T \to \mathbb{R}^n$ represents a regular C^1-surface. Then we infer the identity

$$\int_X \omega = \int_T f\Big(X(t)\Big) \frac{\partial(x_1, \ldots, x_n)}{\partial(t_1, \ldots, t_n)} \, dt_1 \ldots dt_n.$$

This parameter-invariant integral is well-suited for transformations of the domain.

3 The Exterior Derivative of Differential Forms

We begin with the fundamental

Definition 3.1. *For a* 0*-form* $f(x)$ *of the class* $C^1(\mathcal{O})$*, we define the* exterior derivative *as its differential*

$$df(x) = \sum_{i=1}^n f_{x_i}(x) \, dx_i, \qquad x \in \mathcal{O}.$$

When

$$\omega = \sum_{1 \le i_1 < \ldots < i_m \le n} a_{i_1 \ldots i_m}(x) \, dx_{i_1} \wedge \ldots \wedge dx_{i_m}$$

represents an m*-form of the class* $C^1(\mathcal{O})$*, we define its* exterior derivative *as the* $(m+1)$*-form*

$$d\omega := \sum_{1 \le i_1 < \ldots < i_m \le n} \Big(da_{i_1 \ldots i_m}(x)\Big) \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_m}.$$

Remarks:

1. When ω_1 and ω_2 are two m-forms in $\mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ are given, we have the identity

$$d(\alpha_1 \omega_1 + \alpha_2 \omega_2) = \alpha_1 d\omega_1 + \alpha_2 d\omega_2.$$

   Therefore, the differential operator d constitutes a linear operator.
2. When λ denotes an l-form and ω an m-form of the class $C^1(\mathcal{O})$, we infer the *product rule*

$$d(\omega \wedge \lambda) = (d\omega) \wedge \lambda + (-1)^m \omega \wedge d\lambda.$$

We shall prove only the last statement. Here it suffices to consider the situation

$$\omega = f(x)\beta^m, \quad \lambda = g(x)\beta^l,$$

where β^m and β^l are basic forms of the order m and l, respectively. Now we deduce

$$\omega \wedge \lambda = f(x)g(x)\beta^m \wedge \beta^l$$

and, moreover,

$$\begin{aligned} d(\omega \wedge \lambda) &= d\Big(f(x)g(x)\Big) \wedge \beta^m \wedge \beta^l \\ &= \Big(g(x)df(x) + f(x)dg(x)\Big) \wedge \beta^m \wedge \beta^l \\ &= d\omega \wedge \lambda + (-1)^m \omega \wedge d\lambda. \end{aligned}$$

Example 3.2. Taking the function $f(x) \in C^1(\mathcal{O})$, we can integrate immediately the differential form df over curves. With the curve

$$X(t) = \Big(x_1(t), \ldots, x_n(t)\Big) \in C^1([a,b], \mathbb{R}^n)$$

being given, we calculate

$$\begin{aligned} \int\limits_X df &= \int\limits_a^b \sum_{i=1}^n f_{x_i}\Big(X(t)\Big)\dot{x}_i(t)\,dt \\ &= \int\limits_a^b \frac{d}{dt} f\Big(X(t)\Big)\,dt \\ &= f\Big(X(b)\Big) - f\Big(X(a)\Big). \end{aligned}$$

Example 3.3. We consider the Pfaffian form

$$\omega = \sum_{i=1}^n a_i(x)\,dx_i$$

of the class $C^1(\mathcal{O})$ and determine its exterior derivative as follows:

$$\begin{aligned} d\omega = \sum_{j=1}^n da_j(x) \wedge dx_j &= \sum_{i,j=1}^n \frac{\partial a_j}{\partial x_i}\,dx_i \wedge dx_j \\ &= \sum_{1 \le i < j \le n} \left(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j}\right) dx_i \wedge dx_j. \end{aligned}$$

Obviously, the identity $d\omega = 0$ holds true if and only if the functional matrix $\left(\dfrac{\partial a_i}{\partial x_j}\right)_{i,j=1,\ldots,n}$ is symmetric. In the case $n = 3$, we evaluate

$$d\omega = \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right) dx_1 \wedge dx_2 + \left(\frac{\partial a_3}{\partial x_1} - \frac{\partial a_1}{\partial x_3}\right) dx_1 \wedge dx_3$$
$$+ \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right) dx_2 \wedge dx_3$$
$$= b_1(x)\, dx_2 \wedge dx_3 + b_2(x)\, dx_3 \wedge dx_1 + b_3(x)\, dx_1 \wedge dx_2.$$

Here we have defined the vector-field

$$\Big(b_1(x), b_2(x), b_3(x)\Big) = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}, \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}, \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)$$
$$= \nabla \wedge (a_1, a_2, a_3)(x) \ =: \ \mathrm{rot}\, a(x),$$

where $\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ denotes the *nabla-operator*. Integration of this differential form $d\omega$ over surfaces in $\mathbb{R}^3$ will be possible by the *classical Stokes integral theorem.*

Definition 3.4. *We name*

$$rot\, a(x) = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}, \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}, \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)$$

the rotation of the vector-field $a(x) = \Big(a_1(x), a_2(x), a_3(x)\Big) \in C^1(\mathcal{O}, \mathbb{R}^3)$.

Example 3.5. Now we consider a specific $(n-1)$-form in $\mathbb{R}^n$, namely

$$\omega = \sum_{i=1}^{n} a_i(x)(-1)^{i+1}\, dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n,$$

whose exterior derivative takes on the following form:

$$d\omega = \sum_{i=1}^{n} (-1)^{i+1} \Big(da_i(x)\Big) \wedge dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n$$
$$= \sum_{i,j=1}^{n} (-1)^{i+1} \frac{\partial a_i}{\partial x_j}(x)\, dx_j \wedge dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n$$
$$= \sum_{i=1}^{n} (-1)^{i+1} \frac{\partial a_i}{\partial x_i}(x)\, dx_i \wedge dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n$$
$$= \left(\sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i}(x)\right) dx_1 \wedge \ldots \wedge dx_n$$
$$= \Big(\mathrm{div}\, a(x)\Big)\, dx_1 \wedge \ldots \wedge dx_n.$$

Definition 3.6. *The vector-field* $a(x) = \Big(a_1(x), \ldots, a_n(x)\Big) \in C^1(\mathcal{O}, \mathbb{R}^n)$ *on the open set* $\mathcal{O} \subset \mathbb{R}^n$ *possesses the* divergence

$$div\, a(x) := \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i}(x), \qquad x \in \mathcal{O}.$$

Example 3.7. We can integrate the n-form

$$d\omega = (\operatorname{div} a(x))\, dx_1 \wedge \ldots \wedge dx_n$$

over an n-dimensional rectangle. This differential form can also be integrated over a substantially larger class of domains in $\mathbb{R}^n$ - bounded by finitely many hypersurfaces - with the aid of the *Gaussian integral theorem*, one of the most important theorems in the higher-dimensional analysis.
At first, we integrate $d\omega$ over the following standard domain: For $r > 0$ we define the *semidisc*

$$H := \Big\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \in (-r, 0),\ x_i \in (-r, +r),\ i = 2, \ldots, n\Big\}$$

with the upper bounding side

$$S := \Big\{x = (0, x_2, \ldots, x_n) \mid |x_i| < r,\ i = 2, \ldots, n\Big\}.$$

The exterior normal vector to the surface S is given by $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$ explicitly. Then we comprehend H and S as surfaces in $\mathbb{R}^n$ via the representations

$$H\,:\, X(t_1, \ldots, t_n) = (t_1, \ldots, t_n), \qquad (t_1, \ldots, t_n) \in H$$

and

$$S\,:\, Y(\widetilde{t}_1, \ldots, \widetilde{t}_{n-1}) = (0, \widetilde{t}_1, \ldots, \widetilde{t}_{n-1}), \qquad |\widetilde{t}_i| < r, \quad i = 1, \ldots, n-1,$$

respectively. With the assumption $\omega \in C_0^1(H \cup S)$, we obtain

$$\begin{aligned}\int_H d\omega = \int_X d\omega &= \int_{-r}^{0}\int_{-r}^{+r} \ldots \int_{-r}^{+r} \left(\frac{\partial a_1}{\partial x_1} + \ldots + \frac{\partial a_n}{\partial x_n}\right) dx_1 \ldots dx_n \\ &= \int_{-r}^{+r} \ldots \int_{-r}^{+r} a_1(0, x_2, \ldots, x_n)\, dx_2 \ldots dx_n \;=\; \int_S \omega.\end{aligned}$$

In the sequel, we shall investigate the behavior of differential forms with respect to transformations of the ambient space.

Definition 3.8. (Transformed differential form)
Let the symbol

$$\omega = \sum_{1 \le i_1 < \ldots < i_m \le n} a_{i_1 \ldots i_m}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_m}$$

denote a continuous m-form in an open set $\mathcal{O} \subset \mathbb{R}^n$. Furthermore, let $T \subset \mathbb{R}^l$ with $l \in \mathbb{N}$ describe an open set such that

$$\begin{aligned} x &= (x_1, \ldots, x_n) = \Phi(y) \\ &= (\varphi_1(y_1, \ldots, y_l), \ldots, \varphi_n(y_1, \ldots, y_l)) \,:\, T \to \mathcal{O} \end{aligned}$$

defines a mapping of the class $C^1(T, \mathbb{R}^n)$. With

$$d\varphi_i = \sum_{j=1}^{l} \frac{\partial \varphi_i}{\partial y_j}(y)\, dy_j, \qquad i = 1, \ldots, n$$

and

$$\omega_\Phi := \sum_{1 \le i_1 < \ldots < i_m \le n} a_{i_1 \ldots i_m}\Big(\Phi(y)\Big)\, d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_m},$$

we obtain the transformed m-form ω_Φ with respect to the mapping Φ.

Remarks:

1. When ω_1, ω_2 are two m-forms and $\alpha_1, \alpha_2 \in \mathbb{R}$ are given, we infer the identity
$$(\alpha_1 \omega_1 + \alpha_2 \omega_2)_\Phi = \alpha_1 (\omega_1)_\Phi + \alpha_2 (\omega_2)_\Phi.$$
2. When λ represents an l-form and ω an m-form, we have the rule
$$(\omega \wedge \lambda)_\Phi = \omega_\Phi \wedge \lambda_\Phi.$$

The following result is important for the evaluation of integrals for differential forms over surfaces.

Theorem 3.9. (Pull-back of differential forms)
Let ω denote a continuous m-form in the open set $\mathcal{O} \subset \mathbb{R}^n$. On the open set $T \subset \mathbb{R}^m$ we define a surface X by the parametric representation

$$x = \Phi(y) \,:\, T \longrightarrow \mathcal{O} \in C^1(T)$$

with $\Phi(T) \subset\subset \mathcal{O}$. Finally, we define the surface

$$Y(t) = (t_1, \ldots, t_m), \qquad t \in T$$

and note that

$$X(t) = \Phi \circ Y(t), \qquad t \in T.$$

Then the following identity holds true:

$$\int_X \omega = \int_Y \omega_\Phi.$$

Proof: We calculate

$$d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_m} = \left(\sum_{j_1=1}^{m} \frac{\partial \varphi_{i_1}}{\partial y_{j_1}} \, dy_{j_1} \right) \wedge \ldots \wedge \left(\sum_{j_m=1}^{m} \frac{\partial \varphi_{i_m}}{\partial y_{j_m}} \, dy_{j_m} \right)$$

$$= \frac{\partial(\varphi_{i_1}, \ldots, \varphi_{i_m})}{\partial(y_1, \ldots, y_m)} \, dy_1 \wedge \ldots \wedge dy_m,$$

as well as

$$\omega_\Phi = \sum_{1 \le i_1 < \ldots < i_m \le n} a_{i_1 \ldots i_m}(\Phi(y)) \, \frac{\partial(\varphi_{i_1}, \ldots, \varphi_{i_m})}{\partial(y_1, \ldots, y_m)} \, dy_1 \wedge \ldots \wedge dy_m.$$

This implies

$$\int_Y \omega_\Phi = \int_T \sum_{1 \le i_1 < \ldots < i_m \le n} a_{i_1 \ldots i_m}(X(t)) \, \frac{\partial(x_{i_1}, \ldots, x_{i_m})}{\partial(t_1, \ldots, t_m)} \, dt_1 \ldots dt_m$$

$$= \int_X \omega,$$

and our theorem is proved. q.e.d.

Theorem 3.10. *Let ω denote an m-form in the open set $\mathcal{O} \subset \mathbb{R}^n$ of the regularity class $C^1(\mathcal{O})$. Furthermore, let the mapping*

$$x = \Phi(y) \, : \, T \longrightarrow \mathcal{O} \in C^2(T)$$

be given on the open set $T \subset \mathbb{R}^l$, where $l \in \mathbb{N}$ holds true. Then we have the calculus rule

$$d(\omega_\Phi) = (d\omega)_\Phi.$$

Proof: At first, an arbitrary function $\Psi(y) \in C^2(\mathcal{O})$ satisfies the identity

$$d^2\Psi = d(d\Psi) = d\left(\sum_{i=1}^{n} \Psi_{y_i} \, dy_i \right) = \sum_{i,j=1}^{n} \Psi_{y_i y_j} \, dy_j \wedge dy_i = 0.$$

Now we note that

$$\omega_\Phi = \sum_{1 \le i_1 < \ldots < i_m \le n} a_{i_1 \ldots i_m}\Big(\Phi(y)\Big) \, d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_m},$$

and we arrive at

$$
\begin{aligned}
d\omega_\Phi &= \sum_{1 \le i_1 < \ldots < i_m \le n} da_{i_1 \ldots i_m}\Big(\Phi(y)\Big) \wedge d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_m} \\
&= \sum_{1 \le i_1 < \ldots < i_m \le n} \sum_{j=1}^{n} \sum_{k=1}^{l} \frac{\partial a_{i_1 \ldots i_m}}{\partial x_j}\Big(\Phi(y)\Big) \frac{\partial \varphi_j}{\partial y_k}\, dy_k \wedge d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_m} \\
&= \sum_{1 \le i_1 < \ldots < i_m \le n} \sum_{j=1}^{n} \frac{\partial a_{i_1 \ldots i_m}}{\partial x_j}\Big(\Phi(y)\Big)\, d\varphi_j \wedge d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_m},
\end{aligned}
$$

and consequently

$$d\omega_\Phi = (d\omega)_\Phi.$$

q.e.d.

Theorem 3.11. (Chain rule for differential forms)
Let ω denote a continuous m-form in an open set $\mathcal{O} \subset \mathbb{R}^n$. Furthermore, we consider the open sets $T' \subset \mathbb{R}^{l'}$ and $T'' \subset \mathbb{R}^{l''}$ - with $l', l'' \in \mathbb{N}$ - where the C^1-functions Φ, Ψ are defined due to

$$\Psi : T'' \to T', \quad \Phi : T' \to \mathcal{O} \quad \textit{with} \qquad z \overset{\Psi}{\longmapsto} y \overset{\Phi}{\longmapsto} x.$$

Then the following identity holds true:

$$(\omega_\Phi)_\Psi = \omega_{\Phi \circ \Psi}.$$

Proof: We calculate

$$
\begin{aligned}
\omega_{\Phi\circ\Psi} &= \sum_{i_1, \ldots, i_m} a_{i_1 \ldots i_m}\Big(\Phi \circ \Psi(z)\Big)\, d(\varphi_{i_1} \circ \Psi) \wedge \ldots \wedge d(\varphi_{i_m} \circ \Psi) \\
&= \sum_{\substack{i_1, \ldots, i_m \\ j_1, \ldots, j_m \\ k_1, \ldots, k_m}} a_{i_1 \ldots i_m}\Big(\Phi \circ \Psi(z)\Big) \left(\frac{\partial \varphi_{i_1}}{\partial y_{j_1}} \frac{\partial \psi_{j_1}}{\partial z_{k_1}}\, dz_{k_1}\right) \wedge \ldots \wedge \left(\frac{\partial \varphi_{i_m}}{\partial y_{j_m}} \frac{\partial \psi_{j_m}}{\partial z_{k_m}}\, dz_{k_m}\right) \\
&= \sum_{\substack{i_1, \ldots, i_m \\ j_1, \ldots, j_m}} a_{i_1 \ldots i_m}\Big(\Phi \circ \Psi(z)\Big) \left(\frac{\partial \varphi_{i_1}}{\partial y_{j_1}}\, d\psi_{j_1}\right) \wedge \ldots \wedge \left(\frac{\partial \varphi_{i_m}}{\partial y_{j_m}}\, d\psi_{j_m}\right) \\
&= \left(\sum_{i_1, \ldots, i_m} a_{i_1 \ldots i_m}\Big(\Phi(y)\Big)\, d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_m}\right)_{y = \Psi(z)},
\end{aligned}
$$

and consequently

$$\omega_{\Phi\circ\Psi} = (\omega_\Phi)_\Psi.$$

Here we perform our summation over the indices $i_1, \ldots, i_m \in \{1, \ldots, n\}$, $j_1, \ldots, j_m \in \{1, \ldots, l'\}$, and $k_1, \ldots, k_m \in \{1, \ldots, l''\}$. q.e.d.

4 The Stokes Integral Theorem for Manifolds

We choose $m \in \mathbb{N}$ and consider the m-dimensional plane

$$\mathbb{E}^m := \Big\{(0, y_1, \ldots, y_m) \in \mathbb{R}^{m+1} : (y_1, \ldots, y_m) \in \mathbb{R}^m\Big\}.$$

Parallel to the Example 3.7 from Section 3, we take the data $\eta \in \mathbb{R}^{m+1}$ and $r > 0$ in order to define the *semicube*

$$H_r(\eta) := \Big\{y \in \mathbb{R}^{m+1} : \, y_1 \in (\eta_1 - r, \eta_1), \; y_j \in (\eta_j - r, \eta_j + r) \text{ for } j = 2, \ldots, m+1\Big\}$$

with the lateral lengths $2r$. This object has the upper bounding side

$$S_r(\eta) := \Big\{y \in \mathbb{R}^{m+1} \, : \, y_1 = \eta_1, y_j \in (\eta_j - r, \eta_j + r) \text{ for } j = 2, \ldots, m+1\Big\}.$$

We comprehend $H_r(\eta)$ and $S_r(\eta)$ as surfaces in $\mathbb{R}^{m+1}$:

$$H_r(\eta) \, : \, Y(t_1, \ldots, t_{m+1}) = (\eta_1 + t_1, \ldots, \eta_{m+1} + t_{m+1})$$

$$\text{with} \quad -r < t_1 < 0, \quad |t_j| < r, \quad j = 2, \ldots, m+1$$

as well as

$$S_r(\eta) \, : \, Y(t_1, \ldots, t_m) := (\eta_1, \eta_2 + t_1, \ldots, \eta_{m+1} + t_m)$$

$$\text{with} \quad |t_j| < r, \quad j = 1, \ldots, m.$$

When $\eta \in \mathbb{E}^m$ and $r > 0$ are fixed, we define $H := H_r(\eta)$ and $S := S_r(\eta)$, respectively. With $n > m$ given, we denote by

$$\Phi = \Phi(y_1, \ldots, y_{m+1}) \, : \, \overline{H} \longrightarrow \mathbb{R}^n \in C^1(\overline{H}, \mathbb{R}^n)$$

a surface, which can be continued onto an open set containing $\overline{H}$ in $\mathbb{R}^{m+1}$. When we set

$$X(t_1, \ldots, t_{m+1}) := \Phi(t_1, \ldots, t_{m+1}), \quad (t_1, \ldots, t_{m+1}) \in \overline{H},$$

we obtain the following $(m+1)$-dimensional surface in $\mathbb{R}^n$, namely

$$\mathcal{F} := \Big\{X(t) \in \mathbb{R}^n \, : \, t \in H\Big\},$$

whose boundary contains the m-dimensional surface

$$\mathcal{S} := \Big\{X(t) \in \mathbb{R}^n \, : \, t \in S\Big\}.$$

Let the m-form be given on the set $\overline{\mathcal{F}} = \Phi(\overline{H})$ by the symbol

$$\omega = \sum_{i_1,\dots,i_m=1}^{n} a_{i_1\dots i_m}(x)\,dx_{i_1}\wedge\ldots\wedge dx_{i_m}, \quad x\in\overline{\mathcal{F}}$$

of the class $C_0^0(\mathcal{F}\cup\mathcal{S})\cap C^1(\mathcal{F})$. Here the symbol $\omega\in C^1(\mathcal{F})$ means that we have an open set $\mathcal{O}\subset\mathbb{R}^n$ with $\mathcal{F}\subset\mathcal{O}$ satisfying $\omega\in C^1(\mathcal{O})$. Finally, let $d\omega$ be absolutely integrable over $\mathcal{F}$ in the following sense:

$$\int_{\mathcal{F}} |d\omega| := \int_H \Big| \sum_{i_1,\dots,i_{m+1}=1}^{n} \frac{\partial a_{i_1\dots i_m}}{\partial x_{i_{m+1}}}\Big(X(t)\Big)\frac{\partial(x_{i_1},\dots,x_{i_{m+1}})}{\partial(t_1,\dots,t_{m+1})}\Big|\,dt_1\dots dt_{m+1} < +\infty.$$

Now we prove the basic

Proposition 4.1. (Local Stokes theorem)
Let the surface $\mathcal{F}$ with the boundary part $\mathcal{S}$ be given as above, and furthermore the symbol ω may denote an m-dimensional differential form of the class

$$C_0^0(\mathcal{F}\cup\mathcal{S})\cap C^1(\mathcal{F})$$

satisfying

$$\int_{\mathcal{F}} |d\omega| < +\infty.$$

Then we have the identity

$$\int_{\mathcal{F}} d\omega = \int_{\mathcal{S}} \omega.$$

Proof:

1. At first, we prove this formula under the stronger assumptions $\Phi\in C^2(\overline{H})$ and $\omega\in C_0^1(\mathcal{F}\cup\mathcal{S})$. Utilizing Theorem 3.10 and Example 3.7 from Section 3, we infer the identity

$$\int_{\mathcal{F}} d\omega = \int_X d\omega = \int_H (d\omega)_\Phi = \int_H d(\omega_\Phi) = \int_S \omega_\Phi = \int_{\mathcal{S}} \omega.$$

2. When $\Phi\in C^1(\overline{H})$ and $\omega\in C^1(\mathcal{F})\cap C_0^0(\mathcal{F}\cup\mathcal{S})$ hold true, we approximate Φ uniformly in H up to the first derivatives by the functions $\Phi^{(k)}(y)\in C^\infty$, due to the Weierstraß approximation theorem. Now we exhaust H by rectangles

$$H^{(l)} := H_{r-\frac{2}{l}}\Big(\eta_1-\frac{1}{l},\eta_2,\dots,\eta_{m+1}\Big)\subset H$$

with the upper bounding sides

$$S^{(l)} := S_{r-\frac{2}{l}}\Big(\eta_1-\frac{1}{l},\eta_2,\dots,\eta_{m+1}\Big).$$

The considerations in part 1.) reveal

$$\int\limits_{H^{(l)}} (d\omega)_{\Phi^{(k)}} = \int\limits_{S^{(l)}} \omega_{\Phi^{(k)}} \quad \text{for all} \quad k, l \geq N \in \mathbb{N}.$$

The transition to the limit $k \to \infty$ implies

$$\int\limits_{H^{(l)}} (d\omega)_{\Phi} = \int\limits_{S^{(l)}} \omega_{\Phi}.$$

On account of $\int_{\mathcal{F}} |d\omega| < +\infty$, the limit procedure $l \to \infty$ yields

$$\int\limits_{\mathcal{F}} d\omega = \int\limits_{H} (d\omega)_{\Phi} = \int\limits_{S} \omega_{\Phi} = \int\limits_{\mathcal{S}} \omega.$$

This is exactly the identity stated above. q.e.d.

Now we introduce the fundamental notion of a differentiable manifold.

Definition 4.2. *Let us fix the dimensions* $1 \leq m \leq n$ *as well as the set* $\mathcal{M} \subset \mathbb{R}^n$. *We name* $\mathcal{M}$ *an* m-dimensional C^k-manifold, *if each point* $\xi \in \mathcal{M}$ *possesses an element* $\eta \in \mathbb{R}^m$ *and open neighborhoods* $U \subset \mathbb{R}^n$ *of* $\xi \in U$ *and* $V \subset \mathbb{R}^m$ *of* $\eta \in V$ *as well as an embedded regular surface*

$$x = \Phi(y) \,:\, V \longrightarrow U \in C^k(V)$$

such that

$$\xi = \Phi(\eta) \quad \textit{and} \quad \Phi(V) = \mathcal{M} \cap U$$

is correct; here we have chosen $k \in \mathbb{N}$ *adequately. We call* (Φ, V) *a* chart of the manifold. *All charts together*

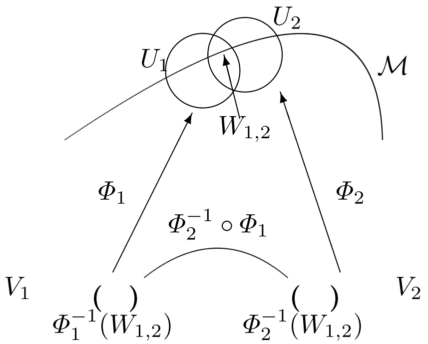
$$\mathcal{A} := \Big\{ (\Phi_\iota, V_\iota) \,:\, \iota \in J \Big\}$$

constitute an atlas of the manifold. *When* $\Phi_j : V_j \to U_j \cap \mathcal{M}$ *with* $j = 1, 2$ *represent two charts of the atlas* $\mathcal{A}$ *such that*

$$W_{1,2} := \mathcal{M} \cap U_1 \cap U_2 \neq \emptyset$$

is correct, then we consider the parameter transformation $\Phi_{2,1} := \Phi_2^{-1} \circ \Phi_1$. *If the functional determinant satisfies* $J_{\Phi_{2,1}} > 0$ *on* $\Phi_1^{-1}(W_{1,2})$ *for such ar-*

bitrarily chosen charts from the atlas, the manifold is oriented *by the atlas.*

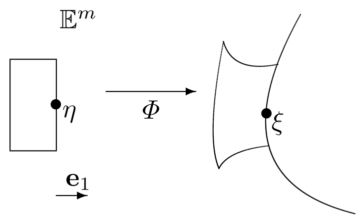


Definition 4.3. *Let $\mathcal{M}$ denote a bounded, $(m+1)$-dimensional, oriented C^1-manifold in $\mathbb{R}^n$ with $n > m$. We indicate the topological closure of the point set $\mathcal{M}$ by the symbol $\overline{\mathcal{M}}$ and the set of* boundary points *by the symbol $\dot{\mathcal{M}} := \overline{\mathcal{M}} \setminus \mathcal{M}$. We name $\xi \in \dot{\mathcal{M}}$ a* regular boundary point *of the manifold $\mathcal{M}$ if the following holds true:*

We have a semicube $H_r(\eta)$ in $\mathbb{R}^{m+1}$ with $\eta \in \mathbb{E}^m$ and $r > 0$, a regular embedded surface

$$\Phi(y) : \overline{H_r(\eta)} \to \mathbb{R}^n \in C^1(\overline{H_r(\eta)})$$

such that $\Phi|_{H_r(\eta)}$ belongs to the oriented atlas $\mathcal{A}$ of $\mathcal{M}$,



and an open neighborhood $U \subset \mathbb{R}^n$ of $\xi \in U$ with the following properties:

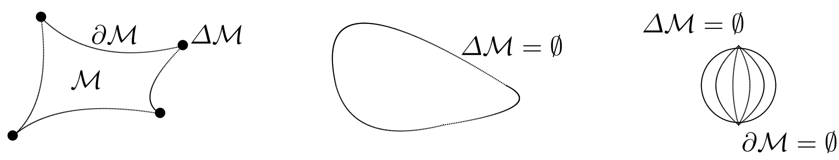
$$\Phi(\eta) = \xi, \quad \Phi\Big(S_r(\eta)\Big) = \dot{\mathcal{M}} \cap U, \quad \Phi\Big(H_r(\eta)\Big) = \mathcal{M} \cap U.$$

The set of regular boundary points *will be denoted by the symbol $\partial\mathcal{M}$.*

Definition 4.4. *For the bounded manifold $\mathcal{M}$ from Definition 4.3, we define the set of* singular boundary points $\triangle\mathcal{M}$ *according to*

$$\triangle\mathcal{M} := \dot{\mathcal{M}} \setminus \partial\mathcal{M}.$$

In the case $\triangle\mathcal{M} = \emptyset$, we obtain a compact manifold with regular boundary. *If the condition $\partial\mathcal{M} = \emptyset$ is fulfilled additionally, we speak of a* closed manifold.

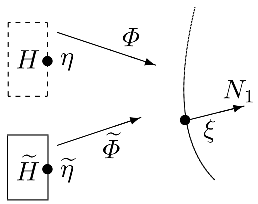


Proposition 4.5. (Induced orientation on $\partial\mathcal{M}$)
Let $\mathcal{M}$ and $\partial\mathcal{M}$ from Definition 4.3 with the charts $\Phi : \overline{H_r(\eta)} \to \mathbb{R}^n$ be given. Then the mappings

$$\left\{ \Phi\big|_{S_r(\eta)} : \Phi\big|_{H_r(\eta)} \text{ belongs to the oriented atlas } \mathcal{A} \text{ of } \mathcal{M} \right\} =: \partial\mathcal{A}$$

constitute an oriented atlas of $\partial\mathcal{M}$. Consequently, $\partial\mathcal{M}$ represents an oriented C^1-manifold.

Proof: We consider $\Phi(\eta) = \xi = \widetilde{\Phi}(\widetilde{\eta})$. The vectors $\Phi_{y_2}(\eta), \ldots, \Phi_{y_{m+1}}(\eta)$ and $\widetilde{\Phi}_{y_2}(\widetilde{\eta}), \ldots, \widetilde{\Phi}_{y_{m+1}}(\widetilde{\eta})$ span the m-dimensional tangential space $T_{\partial\mathcal{M}}(\xi)$ to $\partial\mathcal{M}$ at the point ξ. When we add the vectors $\Phi_{y_1}(\eta)$ and $\widetilde{\Phi}_{y_1}(\widetilde{\eta})$, respectively, the tangential space $T_{\mathcal{M}}(\xi)$ to $\mathcal{M}$ is generated.
Now we construct an orthonormal system $N^1, \ldots, N^{n-m} \in \mathbb{R}^n$ which is orthogonal to $T_{\partial\mathcal{M}}(\xi)$. We choose the vector $N^1 \in T_{\mathcal{M}}(\xi)$, directed out of the surface at the point ξ, and obtain



$$\Phi_{y_1}(\eta) \cdot N^1 > 0, \quad \widetilde{\Phi}_{y_1}(\widetilde{\eta}) \cdot N^1 > 0.$$

With the parameter $0 \le \tau \le 1$, we consider the matrices

$$M(\tau) := \begin{pmatrix} (1-\tau)\Phi_{y_1}(\eta) + \tau N^1 \\ \Phi_{y_2}(\eta) \\ \vdots \\ \Phi_{y_{m+1}}(\eta) \\ N^2 \\ \vdots \\ N^{n-m} \end{pmatrix}, \quad \widetilde{M}(\tau) := \begin{pmatrix} (1-\tau)\widetilde{\Phi}_{y_1}(\widetilde{\eta}) + \tau N^1 \\ \widetilde{\Phi}_{y_2}(\widetilde{\eta}) \\ \vdots \\ \widetilde{\Phi}_{y_{m+1}}(\widetilde{\eta}) \\ N^2 \\ \vdots \\ N^{n-m} \end{pmatrix}.$$

Furthermore, we define $\Psi := \Phi\big|_{S_r(\eta)}$ and $\widetilde{\Psi} := \widetilde{\Phi}\big|_{S_r(\widetilde{\eta})}$. Now the functions $\det M(\tau)$ and $\det \widetilde{M}(\tau)$ in [0,1] are continuous with $\det M(\tau) \neq 0$ and $\det \widetilde{M}(\tau) \neq 0$ for all $0 \le \tau \le 1$. Consequently, the following function is continuous in $[0,1]$, and we have

$$\det\left(\widetilde{M}(\tau)^{-1} \circ M(\tau)\right) \neq 0, \qquad 0 \le \tau \le 1.$$

By assumption we note that

$$\det\left(\widetilde{M}(0)^{-1}\circ M(0)\right)=\det\partial(\widetilde{\Phi}^{-1}\circ\Phi)\Big|_\eta>0,$$

and a continuity argument implies

$$\det\partial(\widetilde{\Psi}^{-1}\circ\Psi)\Big|_\eta=\det\left(\widetilde{M}(1)^{-1}\circ M(1)\right)>0.$$

Therefore, $\partial\mathcal{A}$ constitutes an oriented atlas of $\partial\mathcal{M}$. q.e.d.

We now intend to prove the Stokes integral theorem for manifolds $\mathcal{M}$ with the regular boundary $\partial\mathcal{M}$ and the singular boundary $\triangle\mathcal{M}$, namely the identity

$$\int\limits_{\mathcal{M}} d\omega=\int\limits_{\partial\mathcal{M}}\omega,$$

under weak assumptions. The transition from the local Stokes theorem to the global result is achieved by the *partition of unity.*

Let $\mathcal{M}$ denote an $(m+1)$-dimensional, bounded, oriented C^1-manifold in $\mathbb{R}^n$ with the regular boundary $\partial\mathcal{M}$. Furthermore, let the symbol

$$\lambda=\sum_{1\le i_1<\ldots<i_{m+1}\le n} b_{i_1\ldots i_{m+1}}(x)\,dx_{i_1}\wedge\ldots\wedge dx_{i_{m+1}},\quad x\in\mathcal{M}$$

represent a continuous differential form on $\mathcal{M}$.

We shall investigate which conditions for λ allow us to define the *improper integral*

$$\int\limits_{\mathcal{M}}\lambda$$

of the differential form λ over the manifold $\mathcal{M}$.

1. At first, let the set

$$\operatorname{supp}\lambda:=\overline{\{x\in\mathcal{M}\,:\,\lambda(x)\neq 0\}}\subset\mathcal{M}\cup\partial\mathcal{M}$$

be compact. Then we have open sets $V_\iota\subset\mathbb{R}^{m+1}$ and $U_\iota\subset\mathbb{R}^n\setminus\triangle\mathcal{M}$ with $\iota\in J$ and, moreover, charts $\Phi_\iota:V_\iota\to U_\iota\cap\mathcal{M}$ such that the open sets $\{U_\iota\}_{\iota\in J}$ cover the compact set $\operatorname{supp}\lambda$. Now we choose a partition of unity in $\mathbb{R}^n$ subordinate to the sets $\{U_\iota\}$ and obtain

$$\alpha_k(x)\,:\,\mathcal{M}\longrightarrow[0,1]\in C^1\quad\text{with}\quad\operatorname{supp}\alpha_k\subset U_{\iota_k}\qquad\text{for}\quad k=1,\ldots,k_0$$

as well as

$$\sum_{k=1}^{k_0}\alpha_k(x)=1\qquad\text{for all}\quad x\in\operatorname{supp}\lambda.$$

We define

$$\int\limits_{\mathcal{M}} \lambda := \sum_{k=1}^{k_0} \int\limits_{\mathcal{M}} \alpha_k \lambda = \sum_{k=1}^{k_0} \int\limits_{V_k} (\alpha_k \lambda)_{\Phi_k}, \tag{1}$$

if

$$\int\limits_{\mathcal{M}} \alpha_k |\lambda| < +\infty \quad \text{for} \quad k = 1, \ldots, k_0$$

is correct.

We still have to show that the integral, given in equation (1), is independent of the covering for the support of λ and of the partition of unity used.

When $\widetilde{\Phi}_\iota : \widetilde{V}_\iota \to \widetilde{U}_\iota \cap \mathcal{M}$ with $\iota \in \widetilde{J}$ represents an alternative system of charts covering $\operatorname{supp}\lambda$, we choose again a partition of unity for $\operatorname{supp}\lambda$ subordinate to the system $\{\widetilde{U}_\iota\}_\iota$. We obtain

$$\widetilde{\alpha}_l : \mathcal{M} \to [0,1] \in C^1, \quad \operatorname{supp}\widetilde{\alpha}_l \subset \widetilde{U}_{\iota_l}, \qquad l = 1, \ldots, l_0$$

as well as

$$\sum_{l=1}^{l_0} \widetilde{\alpha}_l(x) = 1 \qquad \text{for all} \quad x \in \operatorname{supp}\lambda.$$

We note that $\operatorname{supp}(\alpha_k \widetilde{\alpha}_l) \subset U_k \cap U_l \cap \mathcal{M}$ holds true. Under the mapping $\Phi_k^{-1} \circ \widetilde{\Phi}_l$ for all indices $k = 1, \ldots, k_0$ and $l = 1, \ldots, l_0$ we transform the integrals

$$\int\limits_{V_k} (\alpha_k \widetilde{\alpha}_l \lambda)_{\Phi_k} = \int\limits_{\widetilde{V}_l} (\alpha_k \widetilde{\alpha}_l \lambda)_{\widetilde{\Phi}_l}. \tag{2}$$

The summation yields

$$\sum_{k=1}^{k_0} \int\limits_{V_k} (\alpha_k \lambda)_{\Phi_k} = \sum_{k=1}^{k_0} \sum_{l=1}^{l_0} \int\limits_{V_k} (\alpha_k \widetilde{\alpha}_l \lambda)_{\Phi_k}$$

$$= \sum_{k=1}^{k_0} \sum_{l=1}^{l_0} \int\limits_{\widetilde{V}_l} (\alpha_k \widetilde{\alpha}_l \lambda)_{\widetilde{\Phi}_l} = \sum_{l=1}^{l_0} \int\limits_{\widetilde{V}_l} (\widetilde{\alpha}_l \lambda)_{\widetilde{\Phi}_l}.$$

Consequently, the integral given in (1) is independent of the choice of charts and the partition of unity. Correspondingly, we define $\int_{\mathcal{M}} |\lambda|$ and $\int_{\partial\mathcal{M}} \lambda$.

2. The differential form $\lambda \in C^0(\mathcal{M})$ is *absolutely integrable over* $\mathcal{M}$, symbolically

$$\int\limits_{\mathcal{M}} |\lambda| < +\infty,$$

if we have a constant $M \in [0, +\infty)$ such that the inequality

$$\int_{\mathcal{M}} |\beta\lambda| \le M \quad \text{for all} \quad \beta \in C_0^0(\mathcal{M} \cup \partial\mathcal{M}, [0,1])$$

is correct. We say that the sequence of functions $\beta_k \in C_0^0(\mathcal{M} \cup \partial\mathcal{M}, [0,1])$ is *exhausting the manifold*, when each compact set $K \subset \mathcal{M} \cup \partial\mathcal{M}$ possesses an index $k_0 = k_0(K) \in \mathbb{N}$ such that

$$\beta_k(x) = 1 \qquad \text{for all} \quad x \in K, \quad k \ge k_0.$$

When $\int_{\mathcal{M}} |\lambda| < +\infty$ holds true, we show as in the theory of improper integrals that for each exhausting sequence of functions $\{\beta_k\}_{k=1,2,\ldots}$ the following expression

$$\lim_{k\to\infty} \int_{\mathcal{M}} \beta_k \lambda$$

exists and has the same value. We set

$$\int_{\mathcal{M}} \lambda := \lim_{k\to\infty} \int_{\mathcal{M}} \beta_k \lambda. \tag{3}$$

In this sense, we comprehend all improper integrals appearing in the sequel.

Definition 4.6. *The singular boundary $\triangle\mathcal{M}$ of the manifold $\mathcal{M}$ has* capacity zero *if we can find a function*

$$\chi \in C_0^1(\mathcal{M} \cup \partial\mathcal{M}, [0,1])$$

for each $\varepsilon > 0$ and each compact set $K \subset \mathcal{M} \cup \partial\mathcal{M}$ with the following properties:

1. *We have $\chi(x) = 1$ for all $x \in K$;*
2. *The following condition holds true:*

$$\int_{\mathcal{M}} \sqrt{\boldsymbol{\nabla}(\chi,\chi)}\, d^{m+1}\sigma \le \varepsilon.$$

Here $d^{m+1}\sigma$ denotes the $(m+1)$-dimensional surface element on $\mathcal{M}$, and we set

$$|\boldsymbol{\nabla}(\chi)|^2\Big|_x = \boldsymbol{\nabla}(\chi,\chi)\Big|_x := sup\Big\{|\nabla\chi \cdot \xi|^2 \,:\, \xi \in T_{\mathcal{M}}(x),\ |\xi| = 1\Big\}.$$

Now we arrive at our central result, namely

Theorem 4.7. (The Stokes integral theorem for manifolds)
Assumptions:

1. *Let $\mathcal{M}$ represent a bounded, oriented, $(m+1)$-dimensional C^1-manifold in $\mathbb{R}^n$ - where $n > m$ is correct - with the atlas $\mathcal{A}$. Via the induced atlas $\partial\mathcal{A}$, the regular boundary $\partial\mathcal{M}$ becomes a bounded, oriented, m-dimensional C^1-manifold. We assume that the regular boundary possesses finite surface area as follows:*
$$\int_{\partial\mathcal{M}} d^m\sigma < +\infty.$$
Furthermore, the singular boundary $\triangle\mathcal{M}$ has capacity zero.
2. *Let the symbol*
$$\omega = \sum_{1\le i_1<\ldots<i_m\le n} a_{i_1\ldots i_m}(x)\, dx_{i_1}\wedge\ldots\wedge dx_{i_m}, \quad x\in\overline{\mathcal{M}}$$
denote an m-dimensional differential form of the class $C^1(\mathcal{M})\cap C^0(\overline{\mathcal{M}})$, such that its exterior derivative $d\omega$ is absolutely integrable in the following sense:
$$\int_{\mathcal{M}} |d\omega| < +\infty.$$

Statement: Then we have the identity
$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega.$$

Proof:

1. At first, let the condition $\omega \in C^1(\mathcal{M})\cap C^0_0(\mathcal{M}\cup\partial\mathcal{M})$ be fulfilled. As above we choose a partition of unity $\{\alpha_k\}$ with $k = 1,\ldots,k_0$ on the set $\operatorname{supp}\omega \subset \mathcal{M}\cup\partial\mathcal{M}$ subordinate to the covering system of the charts. We utilize Proposition 4.1 and deduce
$$\int_{\partial\mathcal{M}} \omega = \sum_{k=1}^{k_0}\int_{\partial\mathcal{M}} \alpha_k\omega = \sum_{k=1}^{k_0}\int_{\mathcal{M}} d(\alpha_k\omega) = \int_{\mathcal{M}} d\omega.$$
2. Let the differential form ω be arbitrary now. Then we choose a sequence $\{\beta_k\}_{k=1,2,\ldots}$ of functions exhausting the manifold $\mathcal{M}$ with the property
$$\int_{\mathcal{M}} \sqrt{\boldsymbol{\nabla}(\beta_k,\beta_k)}\, d^{m+1}\sigma \to 0 \qquad \text{for} \quad k\to\infty.$$
According to part 1, we obtain the following identities for $k = 1,2,\ldots$, namely

$$\int\limits_{\partial\mathcal{M}} \beta_k\omega = \int\limits_{\mathcal{M}} d(\beta_k\omega) = \int\limits_{\mathcal{M}} \beta_k\, d\omega + \int\limits_{\mathcal{M}} d\beta_k \wedge \omega. \tag{4}$$

At first, we see

$$\Big|\int\limits_{\mathcal{M}} d\beta_k \wedge \omega\Big| \le c \int\limits_{\mathcal{M}} \sqrt{\boldsymbol{\nabla}(\beta_k, \beta_k)}\, d^{m+1}\sigma \to 0 \quad \text{for} \quad k \to \infty.$$

Furthermore, we estimate

$$\int\limits_{\partial\mathcal{M}} |\beta_k\omega| \le \int\limits_{\partial\mathcal{M}} |\omega| \le c \int\limits_{\partial\mathcal{M}} d^{m+1}\sigma < +\infty \quad \text{for} \quad k = 1, 2, \ldots$$

Therefore, we comprehend

$$\lim_{k\to\infty} \int\limits_{\partial\mathcal{M}} \beta_k\omega =: \int\limits_{\partial\mathcal{M}} \omega < +\infty.$$

On account of $\int_{\mathcal{M}} |d\omega| < +\infty$, we infer

$$\lim_{k\to\infty} \int\limits_{\mathcal{M}} \beta_k\, d\omega =: \int\limits_{\mathcal{M}} d\omega < +\infty.$$

The transition to the limit $k \to \infty$ in (4) reveals the identity

$$\int\limits_{\partial\mathcal{M}} \omega = \int\limits_{\mathcal{M}} d\omega,$$

which corresponds to the statement above. q.e.d.

5 The Integral Theorems of Gauß and Stokes

We endow the bounded open set $\Omega \subset \mathbb{R}^n$ with the chart $X(t) = t$, $t \in \Omega$ generating an atlas $\mathcal{A}$. In this way, we obtain a bounded oriented n-dimensional manifold $\mathcal{M} = \Omega$ in $\mathbb{R}^n$. When

$$f(x) = \Big(f_1(x), \ldots, f_n(x)\Big) \,:\, \Omega \longrightarrow \mathbb{R}^n \in C^1(\Omega, \mathbb{R}^n)$$

denotes an n-dimensional vector-field in $\mathbb{R}^n$ with its divergence

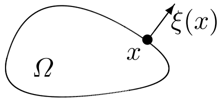
$$\operatorname{div} f(x) = \frac{\partial}{\partial x_1} f_1(x) + \ldots + \frac{\partial}{\partial x_n} f_n(x), \quad x \in \Omega,$$

we consider the $(n-1)$-form

$$\omega = \sum_{i=1}^{n} f_i(x)(-1)^{i+1}\, dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n.$$

The set of regular points $\partial\Omega$, endowed by the induced atlas $\partial\mathcal{A}$, becomes an $(n-1)$-dimensional bounded oriented manifold in $\mathbb{R}^n$. We show the identity

$$\int\limits_{\partial\Omega} \omega = \int\limits_{\partial\Omega} \Big(f(x)\cdot\xi(x)\Big)\, d^{n-1}\sigma$$



later, where $\xi(x)$ denotes the exterior normal to the domain Ω at the point x. When we take the relation

$$d\omega = \Big(\operatorname{div} f(x)\Big)\, dx_1 \wedge \ldots \wedge dx_n$$

into account, Theorem 4.7 from Section 4 reveals the fundamental *identity of Gauß:*

$$\int\limits_{\Omega} \operatorname{div} f(x)\, d^n x = \int\limits_{\partial\Omega} \Big(f(x)\cdot\xi(x)\Big)\, d^{n-1}\sigma. \tag{1}$$

With the aid of Theorem 4.7 from Section 4, we shall derive the identity (1) under very general conditions to Ω and f which are relevant for the applications in this textbook. Thus we shall obtain the *Gaussian integral theorem.*

Assumption (A):

Let $\Omega \subset \mathbb{R}^n$ denote a bounded open set, with the topological boundary $\dot{\Omega} = \overline{\Omega} \setminus \Omega$. For each boundary point $x \in \dot{\Omega}$, we can find a sequence of points

$$\Big\{x^{(p)}\Big\} \subset \mathbb{R}^n \setminus \overline{\Omega}, \qquad p = 1, 2, \ldots$$

satisfying $x^{(p)} \to x$ for $p \to \infty$; this means *each boundary point is attainable from outside.*

Assumption (B):

We choose $N \in \mathbb{N}$ bounded domains $T_i \subset \mathbb{R}^{n-1}$ with $i = 1, 2, \ldots, N$ as our parameter domains. Then we consider N regular hypersurfaces in $\mathbb{R}^n$ as follows:

$$\mathcal{F}_i \,:\, X^{(i)}(t) = \Big(x_1^{(i)}(t_1,\ldots,t_{n-1}),\ldots,x_n^{(i)}(t_1,\ldots,t_{n-1})\Big) \,:\, \overline{T}_i \to \mathbb{R}^n.$$

Here the mapping $X^{(i)}(t) \in C^1(T_i) \cap C^0(\overline{T}_i)$ is injective, and the rank of its functional matrix satisfies the condition $\operatorname{rg} \partial X^{(i)}(t) = n-1$ for all points $t \in T_i$ and the indices $i = 1, \ldots, N$. Furthermore, their surface areas fulfill

$$A(\mathcal{F}_i) := \int\limits_{T_i} d^{n-1}\sigma^{(i)}(t) < +\infty \quad \text{for} \quad i = 1, \ldots, N.$$

We define

$$F_i := X^{(i)}(T_i), \quad \overline{F}_i := X^{(i)}(\overline{T}_i), \quad \dot{F}_i := X^{(i)}(\dot{T}_i)$$

with $i = 1, \ldots, N$. Let the union of these finitely many hypersurfaces F_i constitute the boundary of Ω; more precisely

$$\dot{\Omega} = \overline{F}_1 \cup \ldots \cup \overline{F}_N.$$

Furthermore, we require the condition

$$\overline{F}_i \cap \overline{F}_j = \dot{F}_i \cap \dot{F}_j \quad \text{for all} \quad i, j \in \{1, \ldots, N\} \quad \text{with} \quad i \neq j.$$

Therefore, two different hypersurfaces possess common boundary points at most.

We need the following two auxiliary lemmas:

Proposition 5.1. *The point set $\Omega \subset \mathbb{R}^n$ may satisfy the assumptions (A) and (B). Furthermore, let $x^0 \in F_l$ denote an arbitrary point of the surface F_l with $l \in \{1, \ldots, N\}$. Then we find an index $k = k(x^0) \in \{1, \ldots, n\}$ as well as two positive numbers $\varrho = \varrho(x^0)$ and $\sigma = \sigma(x^0)$, such that the rectangle*

$$Q(x^0, \varrho, \sigma) := \Big\{x \in \mathbb{R}^n \,:\, |x_i - x_i^0| < \varrho,\ i = 1, \ldots, n \text{ with } i \neq k;\ |x_k - x_k^0| < \sigma\Big\}$$

is subject to the following conditions:

$$\dot{\Omega} \cap Q = \Big\{x \in \mathbb{R}^n \,:\, |x_i - x_i^0| < \varrho,\ i \neq k;\ x_k = \Phi(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)\Big\}.$$

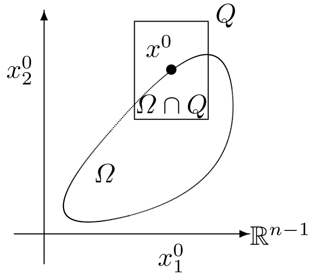
Here Φ denotes a C^1-function on the domain of definition being given, such that $|\Phi - x_k^0| < \frac{1}{2}\sigma$ holds true. Furthermore, we have the alternative

$$\Omega \cap Q = \Big\{x \in \mathbb{R}^n \,:\, |x_i - x_i^0| < \varrho \quad \text{for } i \neq k, \\ |x_k - x_k^0| < \sigma,\ x_k < \Phi(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)\Big\}$$

or

$$\Omega \cap Q = \Big\{x \in \mathbb{R}^n \,:\, |x_i - x_i^0| < \varrho \quad \text{for } i \neq k, \\ |x_k - x_k^0| < \sigma,\ x_k > \Phi(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)\Big\}.$$

The adjacent diagram illustrates the statement of our proposition.



Proof:

1. With the open set $T \subset \mathbb{R}^{n-1}$, let us represent our surface $F = F_l$ by the mapping

$$X(t) = \Big(x_1(t_1, \ldots, t_{n-1}), \ldots, x_n(t_1, \ldots, t_{n-1})\Big) : T \longrightarrow \mathbb{R}^n.$$

On account of $\operatorname{rg} \partial X(t) = n-1$ for all points $t \in T$, we find an index $k = k(x^0) \in \{1, \ldots, n\}$ with $x^0 = X(t^0)$, such that

$$\frac{\partial(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)}{\partial(t_1, \ldots, t_{n-1})}\bigg|_{t=t^0} \neq 0$$

is correct. Now the theorem of the inverse mapping provides an open set $U \subset \mathbb{R}^{n-1}$ and a rectangle

$$\begin{aligned} R_\varrho := (x_1^0 - \varrho, x_1^0 + \varrho) \times \ldots \times (x_{k-1}^0 - \varrho, x_{k-1}^0 + \varrho) \\ \times (x_{k+1}^0 - \varrho, x_{k+1}^0 + \varrho) \times \ldots\ldots \times (x_n^0 - \varrho, x_n^0 + \varrho) \end{aligned}$$

with a sufficiently small quantity $\varrho = \varrho(x^0) > 0$, such that

$$f(t_1, \ldots, t_{n-1}) := \Big(x_1(t), \ldots, x_{k-1}(t), x_{k+1}(t), \ldots, x_n(t)\Big) : U \longrightarrow R_\varrho$$

constitutes a C^1-diffeomorphism. This means that f is bijective, f as well as f^{-1} are continuously differentiable, and we have the condition $J_f(t) \neq 0$ for all $t \in U$. We define

$$\overset{\overset{k}{\vee}}{x} := (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in R_\varrho \subset \mathbb{R}^{n-1}$$

and introduce the function

$$\Phi(\overset{\overset{k}{\vee}}{x}) := x_k\Big(f^{-1}(\overset{\overset{k}{\vee}}{x})\Big), \qquad \overset{\overset{k}{\vee}}{x} \in R_\varrho.$$

Then we observe

$$\Phi \in C^1(R_\varrho, \mathbb{R}), \quad X(U) = \Big\{(x_1, \ldots, x_n) : \overset{\overset{k}{\vee}}{x} \in R_\varrho,\ x_k = \Phi(\overset{\overset{k}{\vee}}{x})\Big\}.$$

Now we see

$$x^0 \in \dot{\Omega} \setminus \bigcup_{\substack{m=1 \\ m \neq l}}^{N} \overline{F}_m,$$

and consequently

$$\operatorname{dist}(x^0, \bigcup_{\substack{m=1 \\ m \neq l}}^{N} \overline{F}_m) > 0.$$

We choose the quantities $\varrho > 0$ and $\sigma > 0$ sufficiently small, such that

$$Q(x^0,\varrho,\sigma)\cap\dot{\Omega}=Q(x^0,\varrho,\sigma)\cap F_l \quad \text{as well as} \quad |\Phi(\overset{k}{\vee}{x})-x_k^0|<\frac{1}{2}\,\sigma$$

holds true for all $\overset{k}{\vee}{x}\in R_\varrho$. We summarize our considerations and obtain

$$\dot{\Omega}\cap Q(x^0,\varrho,\sigma)=\Big\{x\in\mathbb{R}^n\;:\overset{k}{\vee}{x}\in R_\varrho,\;x_k=\Phi(\overset{k}{\vee}{x})\Big\}.$$

2. Now we define the point sets

$$P^+:=\Big\{x\in Q(x^0,\varrho,\sigma)\;:\;x_k>\Phi(\overset{k}{\vee}{x})\Big\},$$

$$P^0:=\Big\{x\in Q(x^0,\varrho,\sigma)\;:\;x_k=\Phi(\overset{k}{\vee}{x})\Big\},$$

$$P^-:=\Big\{x\in Q(x^0,\varrho,\sigma)\;:\;x_k<\Phi(\overset{k}{\vee}{x})\Big\}.$$

These sets above decompose the set $Q(x^0,\varrho,\sigma)$ according to the prescription

$$Q(x^0,\varrho,\sigma)=P^-\cup P^0\cup P^+. \tag{2}$$

From the first part of our proof we infer

$$\dot{\Omega}\cap Q(x^0,\varrho,\sigma)=P^0. \tag{3}$$

On account of $x^0\in\dot{\Omega}$ and the assumption (A), we can find the two points $y\in\Omega\cap Q$ and $z\in(\mathbb{R}^n\setminus\overline{\Omega})\cap Q$. We distinguish between two possible cases, namely the case 1: $y\in P^-$ and the case 2: $y\in P^+$.

Case 1. When we consider with $\widetilde{y}\in P^-$ an arbitrary further point, we find a continuous curve $\Gamma\subset P^-$ from y to $\widetilde{y}$, which does not intersect the surface P^0. Since $y\in\Omega$ holds true and the curve Γ does not intersect the set $\dot{\Omega}$ due to (3), we infer $\widetilde{y}\in\Omega$. We finally obtain the inclusion

$$P^-\subset\Omega\cap Q. \tag{4}$$

Now we arrive at $z\in P^+$. Each further point $\widetilde{z}\in P^+$ can be connected by a curve Γ in P^+ with the point z. Since this curve does not intersect $\dot{\Omega}$, the condition $z\in\mathbb{R}^n\setminus\overline{\Omega}$ implies $\widetilde{z}\in\mathbb{R}^n\setminus\overline{\Omega}$ as well. We conclude

$$P^+\subset(\mathbb{R}^n\setminus\overline{\Omega})\cap Q. \tag{5}$$

Furthermore, we observe

$$Q(x^0,\varrho,\sigma)=(\Omega\cap Q)\cup(\dot{\Omega}\cap Q)\cup\Big((\mathbb{R}^n\setminus\overline{\Omega})\cap Q\Big). \tag{6}$$

We deduce $P^-=\Omega\cap Q$ and $P^+=(\mathbb{R}^n\setminus\overline{\Omega})\cap Q$ from the equations (2) to (6).

Case 2. In the same way as in the first case, we show $P^+ = \Omega \cap Q$ and $P^- = (\mathbb{R}^n \setminus \overline{\Omega}) \cap Q$.

q.e.d.

Remark: In the neighborhood of a regular boundary point

$$x^0 \in \bigcup_{i=1}^{N} F_i$$

we choose the function

$$\Psi(x) := \pm\Big(x_k - \Phi(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)\Big)$$

due to Proposition 5.1. Thus we can characterize the set Ω in this neighborhood by the inequality $\Psi(x) < 0$.

Proposition 5.2. *The set $\Omega \subset \mathbb{R}^n$ may satisfy the assumptions (A) and (B); let $x^0 \in F_l$ with $l \in \{1, \ldots, N\}$ denote a point of the surface F_l. Furthermore, we have an open set $U = U(x^0) \subset \mathbb{R}^n$ containing the point x^0 and a function $\Psi(x) \in C^1(U)$ with $|\nabla\Psi(x)| > 0$ for all points $x \in U$, such that*

$$\Omega \cap U = \{x \in U \,:\, \Psi(x) < 0\}.$$

Then the vector

$$\xi(x) := |\nabla\Psi(x)|^{-1}\nabla\Psi(x), \quad x \in \dot{\Omega} \cap U$$

has the following properties:

1. *We have $\xi\Big(X(t)\Big) \cdot X_{t_i}(t) = 0$ for $i = 1, \ldots, n-1$ near $t = t^0$;*
2. *The condition $|\xi| = 1$ on $\dot{\Omega} \cap U$ holds true;*
3. *For each point $x \in \dot{\Omega} \cap U$, we can find a number $\varrho_0(x) > 0$ such that*

$$x + \varrho\xi \in \begin{cases} \Omega & \text{for } -\varrho_0 < \varrho < 0 \\ \mathbb{R}^n \setminus \overline{\Omega} & \text{for } 0 < \varrho < +\varrho_0 \end{cases}.$$

The vector ξ is uniquely determined by these conditions.

Definition 5.3. *The function $\xi = \xi(x)$, defined in Proposition 5.2 for all points $x \in F_1 \cup \ldots \cup F_N$, is named the* exterior normal of $\dot{\Omega}$ at the point x.

Proof of Proposition 5.2: The uniqueness of ξ follows from the properties 1 to 3 above. Now we shall prove the properties given for the function ξ. At first, $\Psi = 0$ on $\dot{\Omega} \cap U$ holds true, and we infer

$$0 = \Psi\Big(x_1(t), \ldots, x_n(t)\Big), \quad t = (t_1, \ldots, t_{n-1}) \in V(t_1^0, \ldots, t_{n-1}^0) \subset \mathbb{R}^{n-1} \text{ open},$$

and consequently

$$0 = \sum_{i=1}^{n} \Psi_{x_i}\Big(X(t)\Big) \frac{\partial x_i}{\partial t_j}, \quad j = 1, \ldots, n-1.$$

This implies $\xi \cdot X_{t_j} = 0$ in V for $j = 1, \ldots, n-1$ and the property 1. Evidently, the condition $|\xi| = 1$ is valid on $\dot{\Omega} \cap U$. Therefore, it remains to show the property 3. When $0 < |\varrho| < \varrho_0$ holds true, we infer the inequality

$$\begin{aligned}\Psi(x+\varrho\xi) &= \Psi(x+\varrho\xi) - \Psi(x) = \varrho \sum_{i=1}^{n} \Psi_{x_i}(x+\kappa\varrho\xi)\xi_i \\ &= \varrho \frac{1}{|\nabla\Psi(x)|} \sum_{i=1}^{n} \Psi_{x_i}(x+\kappa\varrho\xi)\Psi_{x_i}(x) \begin{cases} < 0 \text{ if } -\varrho_0 < \varrho < 0 \\ > 0 \text{ if } 0 < \varrho < \varrho_0 \end{cases}\end{aligned}$$

for all points $x \in \dot{\Omega} \cap U$; with a number $\kappa = \kappa(\varrho) \in (0,1)$. This implies

$$x + \varrho\xi \in \begin{cases} \Omega & \text{if } -\varrho_0 < \varrho < 0 \\ \mathbb{R}^n \setminus \overline{\Omega} & \text{if } 0 < \varrho < \varrho_0 \end{cases}.$$

q.e.d.

Remark: Let the surface patch $F = F_l$ bounding Ω be given by the parametric representation

$$X(t) = X(t_1, \ldots, t_{n-1}) : T \longrightarrow \mathbb{R}^n \quad \text{on the domain} \quad T \subset \mathbb{R}^{n-1}$$

with the normal

$$\begin{aligned}\nu(t) &= |X_{t_1} \wedge \ldots \wedge X_{t_{n-1}}|^{-1} X_{t_1} \wedge \ldots \wedge X_{t_{n-1}}(t) \\ &= \left[\sum_{j=1}^{n} \Big(D_j(t)\Big)^2\right]^{-\frac{1}{2}} (D_1(t), \ldots, D_n(t)), \qquad t \in T.\end{aligned}$$

With a fixed $\varepsilon \in \{\pm 1\}$, we observe

$$\xi\Big(X(t)\Big) = \varepsilon\nu(t) \qquad \text{for all} \quad t \in T.$$

Proof: At first, we see $\xi\Big(X(t)\Big) = \varepsilon(t)\nu(t)$, $t \in T$ with the orientation factor $\varepsilon(t) \in \{\pm 1\}$. Now the function

$$\varepsilon(t) = \xi\Big(X(t)\Big) \cdot \nu(t), \quad t \in T$$

is continuous on the domain T, and we obtain $\varepsilon(t) \equiv +1$ or $\varepsilon(t) \equiv -1$ on T.

q.e.d.

Definition 5.4. *The set $\Omega \subset \mathbb{R}^n$ may satisfy the assumptions (A) and (B). Then we define*

$$\partial\Omega := \bigcup_{j=1}^{N} F_j$$

as the regular boundary of Ω. *Furthermore, let* $g(x) : \partial\Omega \to \mathbb{R}$ *denote a continuous bounded function on* $\partial\Omega$. *We define the* surface integral of g over the regular boundary $\partial\Omega$ *by the expression*

$$\int\limits_{\partial\Omega} g(x)\, d^{n-1}\sigma := \sum_{j=1}^{N} \int\limits_{F_j} g(x)\, d^{n-1}\sigma_j.$$

Now we formulate the assumption for our vector-fields to be integrated.

Assumption (C):
The function $f(x) = (f_1(x), \ldots, f_n(x))$, $x \in \overline{\Omega}$ belongs to the regularity class $C^1(\Omega, \mathbb{R}^n) \cap C^0(\overline{\Omega}, \mathbb{R}^n)$, and we require

$$\int\limits_{\Omega} |\operatorname{div} f(x)|\, dx < +\infty.$$

We present a condition on the singular boundary $\dot{F}_1 \cup \ldots \cup \dot{F}_N$, which guarantees the validity of the Gaussian identity (1):

Assumption (D):
The set $\dot{F}_1 \cup \ldots \cup \dot{F}_N$ has the $(n-1)$-dimensional Hausdorff content zero or equivalently represents an $(n-1)$-dimensional Hausdorff null-set. More precisely, for each quantity $\varepsilon > 0$ we have finitely many balls

$$K_j := \Big\{x \in \mathbb{R}^n \,:\, |x - x^{(j)}| \le \varrho_j\Big\} \quad \text{for} \quad j = 1, \ldots, J$$

with the centers $x^{(j)} \in \mathbb{R}^n$ and radii $\varrho_j > 0$, such that the following conditions hold true:

1. $\dot{F}_1 \cup \ldots \cup \dot{F}_N \subset \bigcup_{j=1}^{J} K_j$ (Covering property);
2. $\sum_{j=1}^{J} \varrho_j^{n-1} \le \varepsilon$ (Smallness of the total area).

Remark: The condition (D) is valid, if all surface patches F_l with $l = 1, \ldots, N$ fulfill the subsequent assumptions: When F_l is parametrized by the representation $X = X(t) : \overline{T}_l \to \overline{F}_l$, we require the following:

1. The set $\overline{T}_l$ constitutes a Jordan domain in $\mathbb{R}^{n-1}$, which means that T_l is compact and its boundary $\dot{T}_l$ represents a Jordan null-set in $\mathbb{R}^{n-1}$;

2. The mapping $X(t)$ satisfies a *Lipschitz condition* on $\overline{T}_l$, namely
$$|X(t') - X(t'')| \le L|t' - t''| \qquad \text{for all} \quad t', t'' \in \overline{T}_l,$$
with the Lipschitz constant $L > 0$.

We now arrive at the central theorem of the n-dimensional integral-calculus.

Theorem 5.5. (Gaussian integral theorem)
Let $\Omega \subset \mathbb{R}^n$ denote a bounded open set satisfying the assumptions (A), (B), and (D). Furthermore, the vector-valued function $f(x)$ fulfills the assumption (C). Then we have the identity
$$\int\limits_{\Omega} \operatorname{div} f(x)\,dx = \int\limits_{\partial\Omega} f(x)\cdot\xi(x)\,d^{n-1}\sigma.$$

Proof: (E. Heinz)
We shall prove this statement by referring to Theorem 4.7 from Section 4.

1. We comprehend $\mathcal{M} = \Omega \subset \mathbb{R}^n$ as an n-dimensional manifold in $\mathbb{R}^n$ with the atlas $\mathcal{A} : X(t) = t,\ t \in \Omega$. For each point
$$x^0 \in \bigcup_{l=1}^{N} F_l \subset \dot{\Omega}$$
we now find a rectangle $Q(x^0, \varrho, \sigma)$ due to Proposition 5.1, such that
$$\Omega \cap Q = \Big\{ x \in \mathbb{R}^n \,:\, |x_i - x_i^0| < \varrho\,(i \ne k),$$
$$x_k \lesseqgtr \Phi(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n),\ |x_k - x_k^0| < \sigma \Big\}.$$
On the semicube
$$H := \Big\{ t \in \mathbb{R}^n \,:\, t_1 \in (-\varrho, 0),\ |t_i| < \varrho,\ i = 2, \ldots, n \Big\}$$
with the upper bounding side
$$S := \Big\{ t \in \mathbb{R}^n \,:\, t_1 = 0,\ |t_i| < \varrho,\ i = 2, \ldots, n\}$$
in the direction of e_1, we consider the transformation
$$Y(t) = \Big(x_1^0 + \varepsilon_2 t_2, \ldots, x_{k-1}^0 + \varepsilon_k t_k, \Phi(x_1^0 + \varepsilon_2 t_2, \ldots, x_{k-1}^0 + \varepsilon_k t_k,$$
$$x_{k+1}^0 + \varepsilon_{k+1} t_{k+1}, \ldots, x_n^0 + \varepsilon_n t_n) + \varepsilon_1 t_1, x_{k+1}^0 + \varepsilon_{k+1} t_{k+1}, \ldots, x_n^0 + \varepsilon_n t_n \Big)$$
where $\varepsilon_k \in \{\pm 1\}$ for $k = 1, \ldots, n$ holds true. Choosing the sign factors $\varepsilon_1, \ldots, \varepsilon_n$ suitably, we attain the conditions

$$Y(H) \subset \Omega \cap Q, \quad Y(S) = \dot{\Omega} \cap Q, \quad \text{and} \quad J_Y(0) = +1$$

for the functional determinant of Y. Therefore, the mapping Y is compatible with the chart X from above, and we endow $\partial\mathcal{M} = \partial\Omega$ with the induced atlas. On account of the condition $J_Y(0) > 0$, the normal $\nu(t)$ to a surface patch oriented by $\partial\Omega$ points in the direction of the exterior normal ξ to $\partial\Omega$.

We now consider the $(n-1)$-form

$$\omega = \sum_{i=1}^{n} (-1)^{i+1} f_i(x)\, dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n \in C^1(\mathcal{M}) \cap C^0(\overline{\mathcal{M}}).$$

From our considerations above we infer

$$\int_{\partial\Omega} \omega = \int_{\partial\Omega} f(x) \cdot \xi(x)\, d^{n-1}\sigma.$$

2. Due to the assumption (D), we have finitely many balls to each quantity $\varepsilon > 0$, namely

$$K_j := \Big\{ x \in \mathbb{R}^n \,:\, |x - x^{(j)}| \le \varrho_j \Big\} \quad \text{for} \quad j = 1, \ldots, J,$$

satisfying

$$\dot{F}_1 \cup \ldots \cup \dot{F}_N \subset \bigcup_{j=1}^{J} K_j \quad \text{and} \quad \sum_{j=1}^{J} \rho_j^{n-1} \le \epsilon.$$

Now we show that the capacity of the singular boundary vanishes. In this context we construct a function $\Psi(r) : [0, +\infty) \to [0, 1] \in C^1$ with

$$\Psi(r) = \begin{cases} 0, \; 0 \le r \le 2 \\ 1, \quad 3 \le r \end{cases} \quad \text{and} \quad M := \sup_{r \ge 0} |\Psi'(r)| < +\infty.$$

For the indices $j = 1, \ldots, J$ we consider the functions

$$\chi_j(x) := \Psi\Big(|x - x^{(j)}| \,/\, \varrho_j \Big), \qquad x \in \mathbb{R}^n,$$

satisfying $\chi_j \in C^1(\mathbb{R}^n)$ and

$$\chi_j(x) = \begin{cases} 1, \; |x - x^{(j)}| \ge 3\varrho_j \\ 0, \; |x - x^{(j)}| \le 2\varrho_j \end{cases}.$$

When E_n denotes the volume of the n-dimensional unit ball, we evaluate

$$\int_{\mathbb{R}^n} |\nabla \chi_j(x)|\, dx = \int_{2\varrho_j \le |x-x^{(j)}| \le 3\varrho_j} \left| \Psi'\Big(\frac{1}{\varrho_j}|x - x^{(j)}|\Big) \right| \frac{1}{\varrho_j}\, dx$$
$$\le \frac{M}{\varrho_j} E_n (3^n \varrho_j^n - 2^n \varrho_j^n)$$
$$= M E_n (3^n - 2^n) \varrho_j^{n-1}$$

for $j = 1, \ldots, J$. We obtain a function

$$\chi(x) := \chi_1(x) \cdot \ldots \cdot \chi_J(x) \in C_0^1\Big(\overline{\Omega} \setminus (\dot{F}_1 \cup \ldots \cup \dot{F}_N)\Big)$$

with

$$\int_{\Omega} |\nabla \chi(x)|\, dx \le \sum_{j=1}^{J} \int_{\mathbb{R}^n} |\nabla \chi_j(x)|\, dx$$
$$\le M E_n (3^n - 2^n) \sum_{j=1}^{J} \varrho_j^{n-1}$$
$$\le M E_n (3^n - 2^n) \varepsilon.$$

Therefore, the set $\dot{F}_1 \cup \ldots \cup \dot{F}_n \subset \dot{\Omega}$ has capacity zero.

3. The Stokes integral theorem for manifolds finally reveals

$$\int_{\partial\Omega} f(x) \cdot \xi(x)\, d^{n-1}\sigma = \int_{\partial\mathcal{M}} \omega = \int_{\mathcal{M}} d\omega = \int_{\Omega} \operatorname{div} f(x)\, dx.$$

This corresponds to the statement above. q.e.d.

We obtain immediately *Green's formula* from Theorem 5.5, which is fundamental for the *potential theory* presented in Chapter 5.

Theorem 5.6. (Green's formula)
Let $\Omega \subset \mathbb{R}^n$ denote an open bounded set in $\mathbb{R}^n$ satisfying the assumptions (A), (B), and (D). Furthermore, let the functions $f(x)$ and $g(x)$ belong to the class $C^1(\overline{\Omega}) \cap C^2(\Omega)$ subject to the integrability condition

$$\int_{\Omega} \Big(|\Delta f(x)| + |\Delta g(x)|\Big)\, dx < +\infty.$$

Here the symbol $\triangle$ denotes the Laplace operator due to

$$\triangle f(x) := \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_i}(x).$$

Then we have the identity

$$\int_{\Omega} \Big(f\Delta g - g\Delta f\Big)\,dx = \int_{\partial\Omega} \Big(f\,\frac{\partial g}{\partial\xi} - g\,\frac{\partial f}{\partial\xi}\Big)\,d^{n-1}\sigma$$

using the notations

$$\frac{\partial f}{\partial\xi} := \nabla f(x)\cdot\xi(x), \quad \frac{\partial g}{\partial\xi} := \nabla g(x)\cdot\xi(x), \qquad x\in\partial\Omega.$$

Proof: We apply the Gaussian integral theorem to the vector-field

$$h(x) := f(x)\nabla g(x) - g(x)\nabla f(x).$$

Now we deduce

$$\operatorname{div} h(x) = \nabla h(x) = f(x)\Delta g(x) - g(x)\Delta f(x),$$

and we obtain

$$\int_{\Omega} \Big(f(x)\Delta g(x) - g(x)\Delta f(x)\Big)\,dx = \int_{\partial\Omega} h(x)\cdot\xi(x)\,d^{n-1}\sigma$$

$$= \int_{\partial\Omega} \Big(f(x)\frac{\partial g}{\partial\xi}(x) - g(x)\frac{\partial f}{\partial\xi}(x)\Big)\,d^{n-1}\sigma,$$

which implies the statement above. q.e.d.

We specialize the Stokes integral theorem for manifolds onto 2-dimensional surfaces in the Euclidean space $\mathbb{R}^3$. Since we even prove this theorem for surfaces with singular boundaries, we need the following result which is important to construct conformal mappings (in Chapter 4) and central within the theory of *Nonlinear Elliptic Systems* (in Chapter 12).

Theorem 5.7. (Oscillation lemma of R. Courant and H. Lebesgue) *Let*

$$B := \Big\{w = u + iv = (u,v)\in\mathbb{C}\cong\mathbb{R}^2\,:\,|w|<1\Big\}$$

denote the open unit disc and

$$X(u,v) = \Big(x_1(u,v),\ldots,x_n(u,v)\Big)\,:\,B\to\mathbb{R}^n\in C^1(B)$$

a vector-valued function with finite Dirichlet integral $D(X)$; more precisely

$$D(X) := \iint_B \Big(|X_u(u,v)|^2 + |X_v(u,v)|^2\Big)\,dudv \le N < +\infty.$$

For each point $w_0 = u_0 + iv_0 \in \overline{B}$ and each quantity $\delta\in(0,1)$, we then find a number $\delta^*\in[\delta,\sqrt{\delta}]$, such that the estimate

$$L := \int\limits_{\substack{|w-w_0|=\delta^*\\ w\in B}} d\sigma(w) \;\le\; 2\sqrt{\frac{\pi N}{\log\frac{1}{\delta}}}$$

is valid for the length L of the curve $X(w)$, $|w-w_0| = \delta^*$, $w \in B$.

For the proof of this theorem we add the elementary

Proposition 5.8. *Let the numbers $a < b$ be given and the function $f(x) : [a,b] \to \mathbb{R}$ be continuous. Then we have the estimate*

$$\int\limits_a^b |f(x)|\,dx \le \sqrt{b-a}\,\sqrt{\int\limits_a^b |f(x)|^2\,dx}.$$

Proof: Let $\mathcal{Z} : a = x_0 < x_1 < \ldots < x_N = b$ represent an equidistant decomposition of the interval $[a,b]$ - with the partitioning points $x_j := a + j\frac{b-a}{N}$ for $j = 0, 1, \ldots, N$. When $\xi_j \in [x_j, x_{j+1}]$ denote arbitrary intermediate points, the Cauchy-Schwarz inequality reveals

$$\sum_{j=0}^{N-1} |f(\xi_j)|(x_{j+1}-x_j) \le \sqrt{\sum_{j=0}^{N-1} |f(\xi_j)|^2(x_{j+1}-x_j)}\,\sqrt{\sum_{j=0}^{N-1}(x_{j+1}-x_j)}$$

$$= \sqrt{b-a}\,\sqrt{\sum_{j=0}^{N-1} |f(\xi_j)|^2(x_{j+1}-x_j)}.$$

The transition to the limit $N \to \infty$ yields the inequality

$$\int\limits_a^b |f(x)|\,dx \le \sqrt{b-a}\,\sqrt{\int_a^b |f(x)|^2\,dx},$$

which has been stated above. q.e.d.

Proof of Theorem 5.7: We introduce polar coordinates about the point $w_0 = u_0 + iv_0$ as follows:

$$u = u_0 + \varrho\cos\varphi, \quad v = v_0 + \varrho\sin\varphi, \qquad 0 \le \varrho \le \sqrt{\delta}, \quad \varphi_1(\varrho) \le \varphi \le \varphi_2(\varrho).$$

Furthermore, we define the function

$$\Psi(\varrho,\varphi) := X(u_0 + \varrho\cos\varphi, v_0 + \varrho\sin\varphi)$$

and calculate

$$\Psi_\varrho = X_u \cos\varphi + X_v \sin\varphi,$$
$$\Psi_\varphi = -X_u \varrho \sin\varphi + X_v \varrho \cos\varphi$$

as well as

$$|\Psi_\varrho|^2 + \frac{1}{\varrho^2}\,|\Psi_\varphi|^2 = |X_u|^2 + |X_v|^2.$$

Using the intermediate value theorem of the integral-calculus in combination with Proposition 5.8, we obtain

$$N \geq D(X) = \iint\limits_B (|X_u|^2 + |X_v|^2)\,dudv \geq \int\limits_\delta^{\sqrt{\delta}} \int\limits_{\varphi_1(\varrho)}^{\varphi_2(\varrho)} \Big(|\Psi_\varrho|^2 + \frac{1}{\varrho^2}\,|\Psi_\varphi|^2\Big)\,\varrho\,d\varrho d\varphi$$

$$\geq \int\limits_\delta^{\sqrt{\delta}} \frac{1}{\varrho}\left(\int\limits_{\varphi_1(\varrho)}^{\varphi_2(\varrho)} |\Psi_\varphi|^2\,d\varphi\right) d\varrho = \left(\int\limits_{\varphi_1(\delta^*)}^{\varphi_2(\delta^*)} |\Psi_\varphi(\delta^*,\varphi)|^2\,d\varphi\right) \int\limits_\delta^{\sqrt{\delta}} \frac{d\varrho}{\varrho}$$

$$\geq \frac{1}{2}\left(\log\frac{1}{\delta}\right) \frac{1}{\varphi_2(\delta^*) - \varphi_1(\delta^*)} \left(\int\limits_{\varphi_1(\delta^*)}^{\varphi_2(\delta^*)} |\Psi_\varphi(\delta^*,\varphi)|\,d\varphi\right)^2$$

$$\geq \frac{1}{4\pi}\log\left(\frac{1}{\delta}\right) \left(\int\limits_{\varphi_1(\delta^*)}^{\varphi_2(\delta^*)} |\Psi_\varphi(\delta^*,\varphi)|\,d\varphi\right)^2$$

for a number $\delta^* \in [\delta, \sqrt{\delta}]$. Finally, we infer the inequality

$$L = \int\limits_{\varphi_1(\delta^*)}^{\varphi_2(\delta^*)} |\Psi_\varphi(\delta^*,\varphi)|\,d\varphi \leq \sqrt{\frac{4\pi N}{\log\frac{1}{\delta}}} = 2\sqrt{\frac{\pi N}{\log\frac{1}{\delta}}}$$

and arrive at the statement above. q.e.d.

Remark: When we choose $w_0 \in B$ in Theorem 5.7, we have only to require the regularity $X \in C^1(B \setminus \{w_0\})$.

We are now prepared to prove the interesting

Theorem 5.9. (Classical Stokes integral theorem with singular boundary)

1. *On the boundary of the closed unit disc* $\overline{B}$ *we have given* $k_0 \in \mathbb{N} \cup \{0\}$ *points* $w_k = exp\,(i\varphi_k)$ *for* $k = 1, \ldots, k_0$ *with their associate angles* $0 \leq \varphi_1 < \ldots < \varphi_{k_0} < 2\pi$. *When we exempt the points* w_k *for* $k = 1, \ldots, k_0$ *from the sets* $\overline{B}$ *and* ∂B, *we obtain the sets* $\overline{B}'$ *and* $\partial B'$, *respectively.*

2. *Furthermore, let the injective mapping*

$$X(u,v) = \Big(x_1(u,v), x_2(u,v), x_3(u,v)\Big) \,:\, \overline{B} \longrightarrow \mathbb{R}^3 \in C^1(\overline{B}\,') \cap C^0(\overline{B})$$

with the property $X_u \wedge X_v \neq 0$ *for all* $(u,v) \in \overline{B}\,'$ *and finite Dirichlet integral* $D(X) < +\infty$ *be given. Let the surface be* conformally parametrized, *which means the conformality relations*

$$|X_u| = |X_v|, \quad X_u \cdot X_v = 0 \qquad \text{for all} \quad (u,v) \in B$$

are satisfied. Denoting by

$$\overline{X}(\varphi) := X\Big(e^{i\varphi}\Big), \qquad 0 \le \varphi \le 2\pi$$

the restriction of X *onto* ∂B*, we obtain the line element*

$$d^1\sigma(\varphi) = |\overline{X}'(\varphi)|\, d\varphi, \qquad 0 \le \varphi \le 2\pi, \quad \varphi \notin \{\varphi_1, \ldots, \varphi_{k_0}\}.$$

We require finite length for the curve $\overline{X}(\varphi)$*; and more precisely*

$$L(\overline{X}) = \sum_{k=0}^{k_0-1} \int\limits_{\varphi_k}^{\varphi_{k+1}} d^1\sigma(\varphi) < +\infty,$$

where we defined $\varphi_0 := \varphi_{k_0} - 2\pi$.

3. *By the symbol*

$$\nu(u,v) := |X_u \wedge X_v|^{-1} X_u \wedge X_v\,, \qquad (u,v) \in \overline{B}\,'$$

we denote the unit normal vector and by

$$d^2\sigma(u,v) := |X_u \wedge X_v|\, dudv$$

the surface element of the surface $X(u,v)$*. The tangential vector to the boundary curve is abbreviated by*

$$T(\varphi) := \frac{\overline{X}\,'(\varphi)}{|\overline{X}\,'(\varphi)|}.$$

4. *Let* $\mathcal{O} \supset X(B) =: \mathcal{M}$ *constitute an open set in* $\mathbb{R}^3$*, and let the vector-field*

$$a(x) = \Big(a_1(x_1,x_2,x_3), a_2(x_1,x_2,x_3), a_3(x_1,x_2,x_3)\Big) \in C^1(\mathcal{O}) \cap C^0(\overline{\mathcal{M}})$$

be prescribed with the integrability property

$$\iint\limits_B |\mathrm{rot}\, a(X(u,v))|\, d^2\sigma(u,v) < +\infty.$$

Then we have the Stokes identity

$$\iint\limits_B \Big\{\operatorname{rot} a\Big(X(u,v)\Big)\cdot \nu(u,v)\Big\}\, d^2\sigma(u,v) = \int\limits_0^{2\pi} \Big\{a\Big(\overline{X}(\varphi)\Big)\cdot T(\varphi)\Big\}\, d^1\sigma(\varphi). \quad (7)$$

Remarks: Since the surface is *conformally parametrized*, our condition $D(X) < +\infty$ is equivalent to the finiteness of the surface area of X, on account of the relation

$$D(X) = 2\iint\limits_B d^2\sigma(u,v) =: 2A(X).$$

The introduction of isothermal parameters in the large is treated in Section 8 of Chapter 12.

Proof of Theorem 5.9:

1. We intend to apply the Stokes integral theorem for manifolds: The set $\mathcal{M} := X(B)$ constitutes a bounded oriented 2-dimensional C^1-manifold in $\mathbb{R}^3$ with the chart $X(u,v) : B \to \mathcal{M}$. The regular boundary $\partial\mathcal{M} := X(\partial B')$ inherits its orientation by the mapping $\overline{X}(\varphi)$, $0 \le \varphi \le 2\pi$ and possesses finite length $L(\overline{X}) < +\infty$. At first, we show that the singular boundary $\Delta\mathcal{M} := X(\{w_1,\ldots,w_{k_0}\}) \subset \dot{\mathcal{M}} \subset \mathbb{R}^3$ has capacity zero.
2. When $w^* \in \partial B$ is a singular point of the surface, we introduce polar coordinates in a neighborhood of w^* as follows:

$$w = w^* + \varrho e^{i\varphi}, \qquad 0<\varrho<\varrho^*, \quad \varphi_1(\varrho)<\varphi<\varphi_2(\varrho).$$

For the quantity $\eta > 0$ being given, the Courant-Lebesgue oscillation lemma provides a number $\delta \in (0,\rho^*)$ with the following property: Defining the function $Y(\varrho,\varphi) := X(w^*+\varrho e^{i\varphi})$, $0<\rho<\rho^*$, $\varphi_1(\rho)<\varphi<\varphi_2(\rho)$, we have the inequality

$$\int\limits_{\varphi_1(\delta^*)}^{\varphi_2(\delta^*)} |Y_\varphi(\delta^*,\varphi)|\, d\varphi \le 2\sqrt{\frac{\pi D(X)}{\log\frac{1}{\delta}}} \le \eta \qquad (8)$$

for one number $\delta^* \in [\delta,\sqrt{\delta}]$ at least. Consequently, we find two numbers $0<\varrho_1<\delta^*<\varrho_2<\varrho^*$ with the property

$$\int\limits_{\varphi_1(\varrho)}^{\varphi_2(\varrho)} |Y_\varphi(\varrho,\varphi)|\, d\varphi \le 2\eta \qquad \text{for all} \quad \varrho\in[\varrho_1,\varrho_2].$$

Now we consider the weakly monotonic function

$$\Psi(\varrho) : [0,\varrho^*] \longrightarrow [0,1] \in C^1$$

with the properties

$$\Psi(\varrho) = \begin{cases} 0, \; 0 \le \varrho \le \varrho_1 \\ 1, \; \varrho_2 \le \varrho \le \varrho^* \end{cases} .$$

In a neighborhood of the surface $\mathcal{M}$, we now construct a function

$$\chi = \chi(x_1, x_2, x_3) \in C^1(\mathcal{M})$$

satisfying

$$\Psi(\varrho) = \chi \circ Y(\varrho, \varphi), \;\; 0 < \varrho < \varrho^*, \;\; \varphi_1(\varrho) < \varphi < \varphi_2(\varrho).$$

This implies

$$\Psi'(\varrho) = \nabla\chi\big|_{Y(\varrho,\varphi)} \cdot Y_\varrho(\varrho, \varphi) = |\boldsymbol{\nabla}\chi(Y(\varrho, \varphi))||Y_\varrho(\varrho, \varphi)|.$$

We conclude

$$\begin{aligned} &\iint\limits_{w \in B \cap B_{\varrho^*}(w^*)} |\boldsymbol{\nabla}\chi| \, d^2\sigma(u, v) \\ &\qquad \le \int_0^{\varrho^*} \left(\int_{\varphi_1(\varrho)}^{\varphi_2(\varrho)} |\boldsymbol{\nabla}\chi(Y(\varrho, \varphi))||Y_\varrho||Y_\varphi| \, d\varphi \right) d\varrho \\ &\qquad = \int_0^{\varrho^*} \Psi'(\varrho) \left(\int_{\varphi_1(\varrho)}^{\varphi_2(\varrho)} |Y_\varphi(\varrho, \varphi)| \, d\varphi \right) d\varrho \\ &\qquad = \int_{\varrho_1}^{\varrho_2} \Psi'(\varrho) \left(\int_{\varphi_1(\varrho)}^{\varphi_2(\varrho)} |Y_\varphi(\varrho, \varphi)| \, d\varphi \right) d\varrho \;\le\; 2\eta \int_{\varrho_1}^{\varrho_2} \Psi'(\varrho) \, d\varrho \;=\; 2\eta \end{aligned}$$

for all $\eta > 0$. In this way, we see that the boundary point $X(w^*) \in \dot{\mathcal{M}}$ has capacity zero, and the finitely many boundary points $X(\{w_1, \ldots, w_{k_0}\})$ share this property.

3. Now we consider the Pfaffian form

$$\omega = a_1(x) \, dx_1 + a_2(x) \, dx_2 + a_3(x) \, dx_3 \in C^1(\mathcal{M}) \cap C^0(\overline{\mathcal{M}})$$

satisfying

$$\int\limits_{\mathcal{M}} |d\omega| \le \iint\limits_{B} |\mathrm{rot}\, a\Big(X(u, v)\Big)| \, d^2\sigma(u, v) < +\infty.$$

Theorem 4.7 from Section 4 yields the identity

$$\iint\limits_B \left\{\operatorname{rot} a\Big(X(u,v)\Big)\cdot\nu\right\} d^2\sigma$$

$$= \int\limits_{\mathcal{M}} d\omega = \int\limits_{\partial\mathcal{M}} \omega = \int\limits_0^{2\pi} \left\{a\Big(\overline{X}(\varphi)\Big)\cdot T(\varphi)\right\} d^1\sigma(\varphi),$$

and our theorem is proved. q.e.d.

6 Curvilinear Integrals

We begin with the fundamental

Example 6.1. (Gravitational potentials)
Let the solid of the mass $M > 0$ and another solid of the mass $m > 0$ with $m \ll M$ be given (imagine the system *Sun - Earth*). Based on the *theory of gravitation* by I. Newton, the movement in the arising force-field can be described by the *Newtonian potential*

$$F(x) = \gamma\,\frac{mM}{r}, \qquad r = r(x) = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad x \in \mathbb{R}^3 \setminus \{0\};$$

here $\gamma > 0$ denotes the gravitational constant. We determine the work being performed during the movement from a given point P to another point Q in the Euclidean space by the formula $W = F(Q) - F(P)$. We can deduce the *force-field* by differentiation from the potential as follows:

$$f(x) = \Big(f_1(x), f_2(x), f_3(x)\Big) = \nabla F(x)$$

$$= -\gamma\,\frac{mM}{r^3}\,(x_1, x_2, x_3) = -\gamma\,\frac{mM}{r^3}\,x.$$

Now we associate the Pfaffian form

$$\omega = f_1(x)\,dx_1 + f_2(x)\,dx_2 + f_3(x)\,dx_3$$

$$= -\gamma\,\frac{mM}{r^3}(x_1\,dx_1 + x_2\,dx_2 + x_3\,dx_3).$$

When

$$X(t) \,:\, [a,b] \longrightarrow \mathbb{R}^3 \setminus \{0\} \in C^1([a,b])$$

denotes an arbitrary path satisfying $X(a) = P$ and $X(b) = Q$, we infer

$$\int_X \omega = \int_a^b \Big(F_{x_1} x_1'(t) + F_{x_2} x_2'(t) + F_{x_3} x_3'(t)\Big)\,dt$$

$$= \int_a^b \frac{d}{dt}\Big(F(X(t))\Big)\,dt$$

$$= F\Big(X(b)\Big) - F\Big(X(a)\Big).$$

Consequently, this integral depends only on the end-points - and does not depend on the path chosen. Then we speak of a *conservative force-field*; movements along closed curves do not require energy.

We intend to present the theory of curvilinear integrals in the sequel.

Definition 6.2. *Let $\Omega \subset \mathbb{R}^n$ - with $n \geq 2$ - denote a domain and $P, Q \in \Omega$ two points. Then we define the* class $\mathcal{C}(\Omega, P, Q)$ of piecewise continuously differentiable paths (or synonymously, curves) in Ω from P to Q *as follows:*

$$\mathcal{C}(\Omega,P,Q) := \Big\{X(t)\,:\,[a,b] \longrightarrow \Omega \in C^0([a,b])\;: \\ -\infty < a < b < +\infty,\; X(a) = P,\; X(b) = Q; \\ \textit{We have } a = t_0 < t_1 < \ldots < t_N = b \textit{ such that} \\ X\big|_{[t_i,t_{i+1}]} \in C^1([t_i,t_{i+1}],\Omega) \textit{ for } i = 0,\ldots,N-1 \textit{ holds true}\Big\}.$$

With the set

$$\mathcal{C}(\Omega) := \bigcup_{P\in\Omega} \mathcal{C}(\Omega,P,P),$$

we obtain the class of closed paths (or synonymously, closed curves) in Ω. *When $X(t) \equiv P$, $a \leq t \leq b$ holds true, we speak of a* point-curve.

Remark: In particular, the polygonal paths from P to Q are contained in $\mathcal{C}(\Omega, P, Q)$.

Definition 6.3. *Let*

$$\omega = \sum_{i=1}^{n} f_i(x)\,dx_i\,, \qquad x \in \Omega$$

denote a continuous Pfaffian form in the domain Ω and $X \in \mathcal{C}(\Omega, P, Q)$ a piecewise continuously differentiable path between the two points $P, Q \in \Omega$. Introducing

$$X^{(j)} := X\big|_{[t_j,t_{j+1}]} \in C^1([t_j,t_{j+1}]) \quad \textit{for} \quad j = 0,\ldots,N-1,$$

we define by

$$\int_X \omega := \sum_{j=0}^{N-1} \int_{X^{(j)}} \omega = \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{i=1}^{n} f_i\Big(X(t)\Big) x_i'(t)\, dt$$

the curvilinear integral of ω over X.

Definition 6.4. *Let*

$$\omega = \sum_{i=1}^{n} f_i(x)\, dx_i, \qquad x \in \Omega$$

represent a continuous Pfaffian form in the domain $\Omega \subset \mathbb{R}^n$. Then we call $F(x) \in C^1(\Omega)$ a primitive *of ω, if the identity*

$$dF = \omega \qquad in \quad \Omega$$

or equivalently the equations

$$F_{x_i}(x) = f_i(x) \qquad for \quad x \in \Omega \quad and \quad i = 1, \ldots, n$$

hold true. When ω possesses a primitive, we speak of an exact Pfaffian form.

Theorem 6.5. (Curvilinear integrals)
Let $\Omega \subset \mathbb{R}^n$ denote a domain and ω a continuous Pfaffian form in Ω. Then ω possesses a primitive F in Ω if and only if we have the identity $\int\limits_X \omega = 0$ for each closed curve $X \in \mathcal{C}(\Omega, P, P)$ - with a point $P \in \Omega$. In the latter case, we obtain a primitive as follows: We take a fixed point $P \in \Omega$ and have the following representation for all arbitrary points $Q \in \Omega$, namely

$$F(Q) := \gamma + \int_Y \omega \quad with \quad Y \in \mathcal{C}(\Omega, P, Q),$$

where $\gamma \in \mathbb{R}$ is a constant.

Proof:

1. When ω possesses a primitive F, we infer

$$\omega = \sum_{i=1}^{n} f_i(x)\, dx_i = \sum_{i=1}^{n} F_{x_i}(x)\, dx_i, \qquad x \in \Omega.$$

Let us consider $X \in \mathcal{C}(\Omega, P, P)$ with $P \in \Omega$ and

$$X^{(j)} := X\Big|_{[t_j, t_{j+1}]} \in C^1([t_j, t_{j+1}]) \quad \text{for} \quad j = 0, \ldots, N-1.$$

This implies

$$\int\limits_X \omega = \sum_{j=0}^{N-1} \int\limits_{X^{(j)}} \omega = \sum_{j=0}^{N-1} \int\limits_{t_j}^{t_{j+1}} \left(\sum_{i=1}^{n} F_{x_i}\Big(X(t)\Big) x_i'(t)\, dt \right)$$

$$= \sum_{j=0}^{N-1} \int\limits_{t_j}^{t_{j+1}} \frac{d}{dt} F\Big(X(t)\Big)\, dt = \sum_{j=0}^{N-1} \Big\{ F\Big(X(t_{j+1})\Big) - F\Big(X(t_j)\Big) \Big\}$$

$$= F\Big(X(t_N)\Big) - F\Big(X(t_0)\Big) = F(P) - F(P) = 0.$$

2. Now we start with the assumption

$$\int\limits_X \omega = 0 \quad \text{for all curves} \quad X \in \mathcal{C}(\Omega, P, P) \quad \text{with} \quad P \in \Omega.$$

The point $P \in \Omega$ being fixed, we choose a path $X \in \mathcal{C}(\Omega, P, Q)$ for an arbitrary $Q \in \Omega$ and define $F(Q) := \int\limits_X \omega$. Then we have to show the independence of this definition from the choice of the curve X: When $Y \in \mathcal{C}(\Omega, P, Q)$ represents another curve, we have to establish the identity

$$\int\limits_X \omega = \int\limits_Y \omega.$$

We associate the following closed curve to the curves $X : [a, b] \to \mathbb{R}^n$ and $Y : [c, d] \to \mathbb{R}^n$, namely

$$Z(t) := \begin{cases} X(t),\ t \in [a, b] \\ Y(b + d - t),\ t \in [b, b + d - c] \end{cases}.$$

Evidently, $Z \in \mathcal{C}(\Omega, P, P)$ holds true and

$$0 = \int\limits_Z \omega = \int\limits_X \omega - \int\limits_Y \omega$$

follows, which implies

$$\int\limits_X \omega = \int\limits_Y \omega.$$

3. Finally, we have to deduce the formulas

$$F_{x_i}(Q) = f_i(Q) \quad \text{for} \quad i = 1, \dots, n.$$

Here we proceed from Q to the point

$$Q_\varepsilon := Q + \varepsilon e_i, \quad e_i := (0, \ldots, \underbrace{1}_{i-th}, \ldots, 0)$$

along the path

$$Y(t) : [0, \varepsilon] \to \mathbb{R}^n, \quad Y(t) = Q + te_i$$

for a fixed index $i \in \{1, \ldots, n\}$. Now we evaluate

$$\begin{aligned} F(Q_\varepsilon) = F(Q) + F(Q_\varepsilon) - F(Q) &= F(Q) + \int_Y \omega \\ &= F(Q) + \int_0^\varepsilon \sum_{j=1}^n f_j\Big(Y(t)\Big) y_j'(t)\, dt \\ &= F(Q) + \int_0^\varepsilon f_i(Q + te_i)\, dt. \end{aligned}$$

Finally, we obtain

$$\frac{d}{dx_i} F\Big|_Q = \frac{d}{d\varepsilon} F(Q_\varepsilon)\Big|_{\varepsilon=0} = f_i(Q), \qquad i = 1, \ldots, n$$

proving the statement above. q.e.d

Let

$$\omega = \sum_{i=1}^n f_i(x)\, dx_i$$

represent an exact differential form of the class $C^1(\Omega)$ in a domain $\Omega \subset \mathbb{R}^n$. Then we have a function $F(x) : \Omega \longrightarrow \mathbb{R} \in C^2(\Omega)$ with the property

$$dF = \omega \quad \text{or equivalently} \quad f_i(x) = F_{x_i}(x).$$

Furthermore, we infer the identity

$$d\omega = d^2 F = d \sum_{i=1}^n F_{x_i}\, dx_i = \sum_{i,j=1}^n F_{x_i x_j}\, dx_j \wedge dx_i = 0,$$

since the Hessian matrix $(F_{x_i x_j})_{i,j=1,\ldots,n}$ is symmetric.

Definition 6.6. *We name an m-form $\omega \in C^1(\Omega)$ in a domain $\Omega \subset \mathbb{R}^n$ as being* closed, *if the identity $d\omega = 0$ in Ω holds true.*

Remark: The Pfaffian form $\omega = \sum_{i=1}^n f_i(x)\, dx_i, \quad x \in \Omega$ is closed if and only if the matrix $\left(\frac{\partial f_i(x)}{\partial x_j}\right)$ is symmetric.

The considerations above show that an exact Pfaffian form is always closed. We shall now answer the question, which conditions guarantee that a closed Pfaffian form is necessarily exact - and consequently has a primitive.

Example 6.7. In the pointed plane $\mathbb{R}^2 \setminus \{(0,0)\}$, we consider the Pfaffian form

$$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy, \qquad x^2+y^2>0.$$

This 1-form is closed, since we have

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-(x^2+y^2)-(-y)2y}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

as well as

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

and consequently

$$d\omega = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) dy\wedge dx + \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) dx \wedge dy = 0.$$

We observe the closed curve

$$X(t) := (\cos t, \sin t), \qquad 0 \le t \le 2\pi$$

and evaluate

$$\int\limits_X \omega = \int\limits_0^{2\pi} \Big(-\sin t(-\sin t) + \cos t \cos t \Big)\, dt = 2\pi.$$

According to Theorem 6.5, a primitive to ω in $\mathbb{R}^2 \setminus \{0,0\}$ does not exist - and the differential form is not exact there.

The nonvanishing of this curvilinear integral is caused by the fact that the curve X in $\mathbb{R}^2 \setminus \{(0,0)\}$ cannot be contracted to a point-curve.

Definition 6.8. *Let $\Omega \subset \mathbb{R}^n$ denote a domain. Two* closed curves

$$X(t) \,:\, [a,b] \longrightarrow \Omega \quad \textit{and} \quad Y(t) \,:\, [a,b] \longrightarrow \Omega, \qquad X, Y \in \mathcal{C}(\Omega)$$

are named homotopic in *Ω, if we have a mapping*

$$Z(t,s) \,:\, [a,b]\times[0,1] \longrightarrow \Omega \in C^0([a,b]\times[0,1], \mathbb{R}^n)$$

with the properties

$$Z(a,s) = Z(b,s) \qquad \textit{for all} \quad s \in [0,1]$$

as well as

$$Z(t,0) = X(t), \quad Z(t,1) = Y(t) \qquad \textit{for all} \quad t \in [a,b].$$

Now we establish the profound

Theorem 6.9. (Curvilinear integrals)
Let $\Omega \subset \mathbb{R}^n$ constitute a domain, where the two closed curves $X, Y \in \mathcal{C}(\Omega)$ are homotopic to each other. Finally, let

$$\omega = \sum_{i=1}^{n} f_i(x)\,dx_i, \qquad x \in \Omega$$

represent a closed Pfaffian form of the class $C^1(\Omega)$. Then we have the identity

$$\int_X \omega = \int_Y \omega.$$

For our proof we need the following

Proposition 6.10. (Smoothing of a closed curve)
Let

$$X(t)\,:\,[a,b] \longrightarrow \mathbb{R}^n \in \mathcal{C}(\Omega)$$

represent a closed curve, which is continued periodically via

$$X\Big(t + k(b-a)\Big) = X(t), \qquad t \in \mathbb{R}, \quad k \in \mathbb{Z}$$

onto the entire real line $\mathbb{R}$ with the period $(b-a)$. Furthermore, let the function

$$\chi(t) \in C_0^\infty((-1,+1),[0,\infty))$$

give us a mollifier with the properties

$$\chi(-t) = \chi(t) \qquad \textit{for all} \quad \in (-1,1)$$

and

$$\int_{-1}^{+1} \chi(t)\,dt = 1.$$

When we define

$$\chi_{t,\varepsilon}(\tau) := \frac{1}{\varepsilon}\,\chi\left(\frac{\tau - t}{\varepsilon}\right), \qquad \tau \in \mathbb{R},$$

we obtain the smoothed function

$$X^\varepsilon(t) := \int_{-\infty}^{+\infty} X(\tau)\chi_{t,\varepsilon}(\tau)\,d\tau = \int_{-\infty}^{+\infty} X(\tau)\frac{1}{\varepsilon}\,\chi\left(\frac{\tau - t}{\varepsilon}\right)\,d\tau,$$

which has the period $(b-a)$ again. Then we observe

$$\lim_{\varepsilon\to 0+} X^\varepsilon(t) = X(t) \qquad \textit{uniformly on} \quad [a,b].$$

Furthermore, the function $X^\varepsilon(t)$ *belongs to the class* $C^\infty(\mathbb{R})$*, and we obtain the estimate*

$$\left|\frac{d}{dt}X^\varepsilon(t)\right| \le C \qquad \textit{for all} \quad t\in[a,b], \quad 0<\varepsilon<\varepsilon_0,$$

with a constant $C>0$ *and a sufficiently small* ε_0*. For all compact subsets*

$$T\subset (t_0,t_1)\cup(t_1,t_2)\cup\ldots\cup(t_{N-1},t_N)\subset(a,b)$$

we infer

$$\frac{d}{dt}\,X^\varepsilon(t)\longrightarrow X'(t) \qquad \textit{for} \quad \varepsilon\to 0+ \quad \textit{uniformly in} \quad T.$$

Proof: We show parallel to Proposition 1.2 in Section 1 that

$X^\varepsilon(t)\longrightarrow X(t)$ for all $t\in[a,b]$ uniformly, where $\varepsilon\to 0+$ holds true.

Since X is piecewise differentiable and continuous, a partial integration yields

$$\begin{aligned}\frac{d}{dt}\,X^\varepsilon(t) &= \int\limits_{-\infty}^{+\infty} X(\tau)\frac{d}{dt}\,\chi_{t,\varepsilon}(\tau)\,d\tau = \int\limits_{-\infty}^{+\infty} X(\tau)\left(-\frac{d}{d\tau}\,\chi_{t,\varepsilon}(\tau)\right)\,d\tau\\ &= \int\limits_{-\infty}^{+\infty} X'(\tau)\chi_{t,\varepsilon}(\tau)\,d\tau.\end{aligned}$$

Therefore, we obtain

$$\left|\frac{d}{dt}X^\varepsilon(t)\right| \le \int\limits_{-\infty}^{+\infty} |X'(\tau)|\chi_{t,\varepsilon}(\tau)\,d\tau \le C\int\limits_{-\infty}^{+\infty}\chi_{t,\varepsilon}(\tau)\,d\tau = C \qquad \text{for all} \quad t\in\mathbb{R},$$

using the estimate $|X'(\tau)|\le C$ on $\mathbb{R}$. Finally, we show - parallel to Proposition 1.2 in Section 1 again - the relation

$$\lim_{\varepsilon\to 0+}\frac{d}{dt}\,X^\epsilon(t) = X'(t) \qquad \text{uniformly in} \quad T\subset(t_0,t_1)\cup\ldots\cup(t_{N-1},t_N),$$

which had to be proved. q.e.d.

Proof of Theorem 6.9:

1. Let $X,Y\in\mathcal{C}(\Omega)$ represent two homotopic closed curves. Then we have a continuous function

$$Z(t,s)\,:\,[a,b]\times[0,1]\longrightarrow\Omega\in C^0([a,b]\times[0,1],\mathbb{R}^n)$$

with the properties

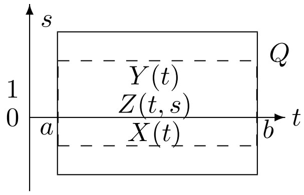
$$Z(a,s) = Z(b,s) \qquad \text{for all} \quad s \in [0,1]$$

and

$$Z(t,0) = X(t), \quad Z(t,1) = Y(t) \qquad \text{for all} \quad t \in [a,b].$$

We continue Z onto the rectangle $[a,b] \times [-2,3]$ to the function

$$\Phi(t,s) := \begin{cases} X(t),\ (t,s) \in [a,b] \times [-2,0] \\ Z(t,s),\ (t,s) \in [a,b] \times [0,1] \\ Y(t),\ (t,s) \in [a,b] \times [1,3] \end{cases} .$$



Via the prescription

$$\Phi\Big(t + k(b-a), s\Big) = \Phi(t,s) \quad \text{for} \quad t \in \mathbb{R}, \quad s \in [-2,3] \quad \text{and} \quad k \in \mathbb{Z},$$

we extend the function onto the stripe $\mathbb{R} \times [-2,3]$ to a continuous function, which is periodic in the first variable with the period $(b-a)$.

2. On the rectangle $Q := [a,b] \times [-1,2]$ we consider the function

$$\Phi^\varepsilon(u,v) := \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \Phi(\xi,\eta)\chi_{u,\varepsilon}(\xi)\chi_{v,\varepsilon}(\eta)\, d\xi d\eta \qquad \text{for all} \quad 0 < \varepsilon < 1.$$

Now the regularity $\Phi^\varepsilon \in C^\infty(Q)$ is fulfilled, and we have the limit relation

$$\Phi^\varepsilon(u,v) \longrightarrow \Phi(u,v) \quad \text{for} \quad \varepsilon \to 0 \quad \text{uniformly in} \quad [a,b] \times [-1,2].$$

This implies the property $\Phi^\varepsilon(Q) \subset \Omega$, $0 < \varepsilon < \varepsilon_0$ and the periodicity

$$\Phi^\varepsilon\Big(u + k(b-a), v\Big) = \Phi^\varepsilon(u,v) \quad \text{for all} \quad (u,v) \in \mathbb{R} \times [-1,2], \quad k \in \mathbb{Z}.$$

For all parameters $a \le u \le b$ we have

$$\begin{aligned} \Phi^\varepsilon(u,-1) &= \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \Phi(\xi,\eta)\chi_{u,\epsilon}(\xi)\chi_{-1,\varepsilon}(\eta)\, d\xi d\eta \\ &= \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} X(\xi)\chi_{u,\varepsilon}(\xi)\chi_{-1,\varepsilon}(\eta)\, d\xi d\eta \\ &= \int\limits_{-\infty}^{+\infty} X(\xi)\chi_{u,\epsilon}(\xi)\, d\xi \;=\; X^\epsilon(u) \end{aligned}$$

and additionally

$$\Phi^\varepsilon(u,2) = Y^\varepsilon(u).$$

3. By the Stokes integral theorem on the rectangle Q, we obtain the following identity for all $0 < \varepsilon < \varepsilon_0$, namely

$$\int_{X^\varepsilon} \omega - \int_{Y^\varepsilon} \omega = \oint_{\partial Q} \omega_{\Phi^\varepsilon} = \int_Q d(\omega_{\Phi^\varepsilon}) = \int_Q (d\omega)_{\Phi^\varepsilon} = 0.$$

We observe $\varepsilon \to 0+$, and Proposition 6.10 yields

$$0 = \lim_{\varepsilon \to 0+} \left(\int_{X^\varepsilon} \omega - \int_{Y^\varepsilon} \omega \right) = \int_X \omega - \int_Y \omega$$

and therefore our statement above. q.e.d

Definition 6.11. *Let the domain $\Omega \subset \mathbb{R}^n$ as well as the points $P, Q \in \Omega$ be given. We name two curves*

$$X(t), Y(t) \,:\, [a,b] \longrightarrow \Omega \in \mathcal{C}(\Omega, P, Q)$$

as being homotopic in Ω with the fixed start-point P and end-point Q, *if we have a continuous mapping*

$$Z(t,s) \,:\, [a,b] \times [0,1] \longrightarrow \Omega$$

with the following properties:

$$Z(a,s) = P, \quad Z(b,s) = Q \qquad \text{for all} \quad s \in [0,1]$$

as well as

$$Z(t,0) = X(t), \quad Z(t,1) = Y(t) \qquad \text{for all} \quad t \in [a,b].$$

We deduce immediately the following result from Theorem 6.9.

Theorem 6.12. (Monodromy)
Let $\Omega \subset \mathbb{R}^n$ denote a domain and $P, Q \in \Omega$ two arbitrary points. Furthermore, let the two curves $X(t), Y(t) \in \mathcal{C}(\Omega, P, Q)$ be homotopic to each other with fixed start- and end-point. Finally, let

$$\omega = \sum_{i=1}^n f_i(x)\, dx_i, \qquad x \in \Omega$$

represent a closed Pfaffian form of the class $C^1(\Omega)$. Then we have the identity

$$\int_X \omega = \int_Y \omega.$$

Proof: We consider the following homotopy of closed curves in Ω, namely

$$\Phi(t,s)\,:\,[a,2b-a]\times[0,1]\longrightarrow\Omega$$

with

$$\Phi(t,s)=\begin{cases} X(t),\, a\le t\le b \\ Z(2b-t,s),\, b\le t\le 2b-a \end{cases}.$$

Now we note that

$$\Phi(t,0)=\begin{cases} X(t),\, a\le t\le b \\ X(2b-t),\, b\le t\le 2b-a \end{cases}.$$

Here the curve X is run through from P to Q and then backwards from Q to P. Therefore, we infer

$$\int\limits_{\Phi(\cdot,0)}\omega=0.$$

Furthermore, we deduce

$$\Phi(t,1)=\begin{cases} X(t),\, a\le t\le b \\ Y(2b-t),\, b\le t\le 2b-a \end{cases}.$$

Here the curve X is run through from P to Q at first, and the curve Y is run through from Q to P afterwards. Finally, Theorem 6.9 reveals the identity

$$0=\int\limits_{\Phi(\cdot,0)}\omega=\int\limits_{\Phi(\cdot,1)}\omega=\int\limits_{X}\omega-\int\limits_{Y}\omega.$$

q.e.d.

The study of curvilinear integrals becomes very simple in the following domains.

Definition 6.13. *A domain $\Omega\subset\mathbb{R}^n$ is named* simply connected*, if each closed curve $X(t)\in\mathcal{C}(\Omega)$ is homotopic to a point-curve in Ω. This means geometrically that each closed curve is contractible to one point.*

Theorem 6.14. (Curvilinear integrals in simply connected domains) *Let $\Omega\subset\mathbb{R}^n$ constitute a simply connected domain and*

$$\omega=\sum_{i=1}^{n}f_i(x)\,dx_i,\qquad x\in\Omega$$

a Pfaffian form of the class $C^1(\Omega)$. Then the following statements are equivalent:

1. *The Pfaffian form ω is exact, and therefore possesses a primitive F.*
2. *For all curves $X \in \mathcal{C}(\Omega, P, P)$ - with a point $P \in \Omega$ - we have the identity $\int\limits_X \omega = 0$.*
3. *The Pfaffian form ω is closed, which means*

$$d\omega = 0 \quad in \quad \Omega$$

or equivalently that the matrix $\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{i,j=1,\ldots,n}$ is symmetric for all points $x \in \Omega$.

Proof: From the first theorem on curvilinear integrals we infer the equivalence '1. ⇔ 2.'. The statement '1. ⇒ 3.' is revealed by the considerations preceding Definition 6.6. We only have to show the direction '3. ⇒ 2.': Here we choose an arbitrary closed curve $X(t) \in \mathcal{C}(\Omega, P, P)$, which is homotopic to the closed curve $Y(t) \equiv P, \quad a \leq t \leq b$, due to the assumption on the domain Ω. The application of Theorem 6.9 yields

$$\int\limits_X \omega = \int\limits_Y \omega = \int\limits_a^b \sum_{i=1}^n f_i\Big(Y(t)\Big) y_i'(t)\, dt = 0,$$

which implies our theorem. q.e.d.

Remark: In the Euclidean space $\mathbb{R}^3$, our condition 3 from Theorem 6.14 implies that the vector-field $f(x) = \Big(f_1(x), f_2(x), f_3(x)\Big), \quad x \in \Omega$ is irrotational, which means

$$\operatorname{rot} f(x) = 0 \qquad \text{in} \quad \Omega.$$

In simply connected domains $\Omega \subset \mathbb{R}^3$, Theorem 6.14 guarantees the existence of a primitive $F : \Omega \to \mathbb{R} \in C^2(\Omega)$ with the property $\nabla F(x) = f(x), \quad x \in \Omega$.

7 The Lemma of Poincaré

The theory of curvilinear integrals was transferred to the higher-dimensional situation of surface-integrals especially by de Rham (compare G. de Rham: *Varietés differentiables*, Hermann, Paris 1955). In this context we refer the reader to Paragraph 20 in the textbook by H. Holmann and H. Rummler: *Alternierende Differentialformen*, BI-Wissenschaftsverlag, 2.Auflage, 1981.

We shall construct primitives for arbitrary m-forms, which correspond to vector-potentials - however, in 'contractible domains' only. Here we do not need the Stokes integral theorem!

Definition 7.1. *A continuous m-form with $1 \leq m \leq n$ in an open set $\Omega \subset \mathbb{R}^n$ with $n \in \mathbb{N}$, namely*

$$\omega = \sum_{1\le i_1<\ldots<i_m\le n} a_{i_1\ldots i_m}(x)\,dx_{i_1}\wedge\ldots\wedge dx_{i_m}, \qquad x\in\Omega,$$

is named exact *if we have an* $(m-1)$*-form*

$$\lambda = \sum_{1\le i_1<\ldots<i_{m-1}\le n} b_{i_1\ldots i_{m-1}}(x)\,dx_{i_1}\wedge\ldots\wedge dx_{i_{m-1}}, \qquad x\in\Omega$$

of the class $C^1(\Omega)$ *with the property*

$$d\lambda = \omega \qquad in \quad \Omega.$$

We begin with the easy

Theorem 7.2. *An exact differential form* $\omega \in C^1(\Omega)$ *is closed.*

Proof: We calculate

$$d\omega = d(d\lambda) = d \sum_{1\le i_1<\ldots<i_{m-1}\le n} db_{i_1\ldots i_{m-1}}(x)\wedge dx_{i_1}\wedge\ldots\wedge dx_{i_{m-1}}$$

$$= \sum_{1\le i_1<\ldots<i_{m-1}\le n} \Big(d\,db_{i_1\ldots i_{m-1}}(x)\Big)\wedge dx_{i_1}\wedge\ldots\wedge dx_{i_{m-1}} \;=\; 0,$$

which implies the statement above. q.e.d.

We now provide a condition on the domain Ω, which guarantees that a closed differential form is necessarily exact.

Definition 7.3. *Let* $\Omega\subset\mathbb{R}^n$ *denote a domain with the associate cylinder*

$$\widehat{\Omega} := \Omega\times[0,1]\subset\mathbb{R}^{n+1}.$$

Furthermore, we have a point $x_0\in\Omega$ *and a mapping*

$$F = F(x,t) = \Big(f_1(x_1,\ldots,x_n,t),\ldots,f_n(x_1,\ldots,x_n,t)\Big) \,:\, \widehat{\Omega}\longrightarrow\Omega$$

of the class $C^2(\widehat{\Omega},\mathbb{R}^n)$ *as follows:*

$$F(x,0) = x_0\,,\quad F(x,1) = x \qquad for\ all \quad x\in\Omega.$$

Then we name the domain Ω contractible *(onto the point* x_0*).*

Remarks:

1. Let the domain Ω be *star-shaped* with respect to the point $x_0\in\Omega$, which means
$$(tx+(1-t)x_0)\in\Omega \qquad \text{for all} \quad t\in[0,1], \quad x\in\Omega.$$
Then Ω is contractible with the contraction-mapping
$$F(x,t) := tx+(1-t)x_0\,, \qquad x\in\Omega, \quad t\in[0,1].$$

2. Each contractible domain $\Omega \subset \mathbb{R}^n$ is simply connected as well. When $X(s)$, $0 \le s \le 1$ with $X(0) = X(1)$ represents a closed curve in Ω, it is contractible onto the point x_0 via
$$Y(s,t) := F\Big(X(s),t\Big), \qquad 0 \le s \le 1, \quad 0 \le t \le 1.$$
In a contractible domain, we can perform the contraction of an arbitrary curve $X(s)$ by the joint mapping F. Therefore, the contraction is independent from the choice of the curve X.
3. The following chain of implications for domains in $\mathbb{R}^n$ holds true:
$$\begin{aligned}\text{convex} &\Longrightarrow \text{star-shaped}\\ &\Longrightarrow \text{contractible}\\ &\Longrightarrow \text{simply connected.}\end{aligned}$$

On the cylinder $\widehat{\Omega}$ we consider the l-form
$$\gamma(x,t) := \sum_{1 \le i_1 < \ldots < i_l \le n} c_{i_1 \ldots i_l}(x,t)\, dx_{i_1} \wedge \ldots \wedge dx_{i_l}$$
of the class $C^1(\widehat{\Omega})$. We use the abbreviation $\frac{d}{dt} := \dot{}$ for the time-derivative and define
$$\dot{\gamma}(x,t) := \sum_{1 \le i_1 < \ldots < i_l \le n} \dot{c}_{i_1 \ldots i_l}(x,t)\, dx_{i_1} \wedge \ldots \wedge dx_{i_l}.$$
Furthermore, we set
$$\int_0^1 \gamma(x,t)\, dt := \sum_{1 \le i_1 < \ldots < i_l \le n} \left(\int_0^1 c_{i_1 \ldots i_l}(x,t)\, dt \right) dx_{i_1} \wedge \ldots \wedge dx_{i_l}.$$
The fundamental theorem of the differential- and integral-calculus reveals
$$\int_0^1 \dot{\gamma}(x,t)\, dt = \gamma(x,1) - \gamma(x,0). \tag{1}$$
The function $g(x,t) : \widehat{\Omega} \to \mathbb{R} \in C^1(\widehat{\Omega})$ being given, we determine its exterior derivative
$$dg = \sum_{k=1}^n \frac{\partial g}{\partial x_k}\, dx_k + \dot{g}(x,t)\, dt =: d_x g + \dot{g}\, dt.$$
Consequently, we obtain
$$d\gamma = d_x \gamma + dt \wedge \dot{\gamma}$$
abbreviating

$$d_x\gamma := \sum_{1\le i_1<\ldots<i_l\le n} \Big(d_x c_{i_1\ldots i_l}(x,t)\Big) \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_l}.$$

Finally, we deduce the identity

$$d\left(\int_0^1 \gamma(x,t)\,dt\right) = \int_0^1 \Big(d_x\gamma(x,t)\Big)\,dt. \qquad (2)$$

Therefore, we calculate

$$\begin{aligned}
& d\left(\int_0^1 \gamma(x,t)\,dt\right) \\
&= \sum_{1\le i_1<\ldots<i_l\le n} \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\int_0^1 c_{i_1\ldots i_l}(x,t)\,dt\right) dx_i \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_l} \\
&= \sum_{1\le i_1<\ldots<i_l\le n} \sum_{i=1}^n \left(\int_0^1 \frac{\partial}{\partial x_i} c_{i_1\ldots i_l}(x,t)\,dt\right) dx_i \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_l} \\
&= \int_0^1 \left\{ \sum_{1\le i_1<\ldots<i_l\le n} \left(\sum_{i=1}^n \frac{\partial}{\partial x_i} c_{i_1\ldots i_l}(x,t)\,dx_i\right) \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_l} \right\} dt \\
&= \int_0^1 \Big(d_x\gamma(x,t)\Big)\,dt.
\end{aligned}$$

We are now prepared to prove the central result of this section.

Theorem 7.4. (Lemma of Poincaré)
Let $\Omega \subset \mathbb{R}^n$ denote a contractible domain, and choose a dimension $1 \le m \le n$. Then each closed m-form ω in Ω is exact.

Proof (A. Weil):

1. Since Ω is contractible, we have a mapping

$$F = F(x,t) \,:\, \widehat{\Omega} \longrightarrow \Omega \in C^2(\widehat{\Omega})$$

satisfying

$$F(x,0) = x_0, \quad F(x,1) = x \qquad \text{for all} \quad x \in \Omega.$$

On the set $\widehat{\Omega} = \Omega \times [0,1]$, we consider the transformed differential form

$$\begin{aligned}\widehat{\omega}(x,t) &:= \omega \circ F(x,t)\\ &= \sum_{1\le i_1<\ldots<i_m\le n} a_{i_1\ldots i_m}(F(x,t))\, df_{i_1}\wedge\ldots\wedge df_{i_m}\\ &= \sum_{1\le i_1<\ldots<i_m\le n} a_{i_1\ldots i_m}(F(x,t))\, d_x f_{i_1}\wedge\ldots\wedge d_x f_{i_m} + dt\wedge\omega_2(x,t)\\ &= \omega_1 + dt\wedge\omega_2.\end{aligned}$$

Here we used the identities

$$df_{i_k} = d_x f_{i_k} + \dot{f}_{i_k}\, dt \qquad \text{for} \qquad k=1,\ldots,m.$$

The differential forms $\omega_1(x,t)$ and $\omega_2(x,t)$ are independent of dt and have the degrees m and $(m-1)$, respectively. Furthermore, we note that

$$\omega_1(x,0)=0 \quad \text{and} \quad \omega_1(x,1)=\omega(x).$$

2. We evaluate

$$\begin{aligned}0 = (d\omega)\circ F = d(\omega\circ F) &= d\widehat{\omega}\\ &= d\omega_1 + d(dt\wedge\omega_2) = d_x\omega_1 + dt\wedge\dot{\omega}_1 - dt\wedge d\omega_2\\ &= d_x\omega_1 + dt\wedge\dot{\omega}_1 - dt\wedge(d_x\omega_2 + dt\wedge\dot{\omega}_2)\\ &= d_x\omega_1 + dt\wedge(\dot{\omega}_1 - d_x\omega_2).\end{aligned}$$

This implies

$$\dot{\omega}_1 = d_x\omega_2. \tag{3}$$

3. Now we define the $(m-1)$-form

$$\lambda := \int_0^1 \omega_2(x,t)\, dt.$$

With the aid of the identities (1), (2), and (3) we calculate

$$d\lambda = \int_0^1 \Big(d_x\omega_2(x,t)\Big)\, dt = \int_0^1 \dot{\omega}_1(x,t)\, dt = \omega_1(x,1)-\omega_1(x,0) = \omega(x),$$

which completes the proof. q.e.d.

Example 7.5. In a star-shaped domain $\Omega\subset\mathbb{R}^3$, let the source-free vector-field

$$b(x) = \Big(b_1(x), b_2(x), b_3(x)\Big) \,:\, \Omega \longrightarrow \mathbb{R}^3 \in C^1(\Omega,\mathbb{R}^3)$$

with

$$\operatorname{div} b(x) = 0$$

be given. Then its associate 2-form

$$\omega = b_1(x)\,dx_2 \wedge dx_3 + b_2(x)\,dx_3 \wedge dx_1 + b_3(x)\,dx_1 \wedge dx_2$$

is closed. Theorem 7.4 gives us a Pfaffian form

$$\lambda = a_1(x)\,dx_1 + a_2(x)\,dx_2 + a_3(x)\,dx_3 \in C^2(\Omega)$$

satisfying $d\lambda = \omega$. The calculations in Section 3 imply the following identity for the vector-field $a(x) = (a_1(x), a_2(x), a_3(x))$, namely

$$\operatorname{rot} a(x) = b(x) \qquad \text{for all} \quad x \in \Omega.$$

Therefore, we have constructed a vector-potential $a(x)$ for the source-free vector-field $b(x)$.

8 Co-derivatives and the Laplace-Beltrami Operator

In this section we introduce an inner product for differential forms. We consider the space

$$\mathbb{R}^n := \Big\{ \overline{x} = (\overline{x}_1, \ldots, \overline{x}_n) \,:\, \overline{x}_i \in \mathbb{R},\ i = 1, \ldots, n \Big\}$$

with the subset $\Theta \subset \mathbb{R}^n$. Furthermore, we have given two continuous m-forms on Θ, namely

$$\overline{\alpha} := \sum_{1 \le i_1 < \ldots < i_m \le n} \overline{a}_{i_1 \ldots i_m}(\overline{x})\, d\overline{x}_{i_1} \wedge \ldots \wedge d\overline{x}_{i_m}, \qquad \overline{x} \in \Theta,$$

as well as

$$\overline{\beta} := \sum_{1 \le i_1 < \ldots < i_m \le n} \overline{b}_{i_1 \ldots i_m}(\overline{x})\, d\overline{x}_{i_1} \wedge \ldots \wedge d\overline{x}_{i_m}, \qquad \overline{x} \in \Theta.$$

We define an *inner product* between the m-forms $\overline{\alpha}$ and $\overline{\beta}$ as follows:

$$(\overline{\alpha}, \overline{\beta})_m := \sum_{1 \le i_1 < \ldots < i_m \le n} \overline{a}_{i_1 \ldots i_m}(\overline{x})\, \overline{b}_{i_1 \ldots i_m}(\overline{x}), \qquad m = 0, 1, \ldots, n. \tag{1}$$

Consequently, the inner product attributes a 0-form to a pair of m-forms. It represents a symmetric bilinear form on the vector space of m-forms.

Now we consider the parameter transformation

$$\overline{x} = \Phi(x) = \Big(\Phi_1(x_1, \ldots, x_n), \ldots, \Phi_n(x_1, \ldots, x_n) \Big) \,:\, \Omega \longrightarrow \Theta \in C^2(\Omega)$$

on the open set $\Omega \subset \mathbb{R}^n$. The mapping Φ satisfies

$$J_\Phi(x) = \det\Big(\partial\Phi(x)\Big) \neq 0 \qquad \text{for all} \quad x \in \Omega. \tag{2}$$

We set

$$g(x) := \Big(J_\Phi(x)\Big)^2 = \det\Big(\partial\Phi(x)^t \circ \partial\Phi(x)\Big), \qquad x \in \Omega.$$

The volume form

$$\omega = \sqrt{g(x)}\, dx_1 \wedge \ldots \wedge dx_n, \qquad x \in \Omega \tag{3}$$

is associated with the transformation $\overline{x} = \Phi(x)$ in a natural way. The m-forms $\overline{\alpha}$ and $\overline{\beta}$ are transformed into the m-forms

$$\begin{aligned}\alpha := \overline{\alpha}_\Phi &= \sum_{1\le i_1<\ldots<i_m\le n} \overline{a}_{i_1\ldots i_m}\Big(\Phi(x)\Big)\, d\Phi_{i_1}(x) \wedge \ldots \wedge d\Phi_{i_m}(x) \\ &=: \sum_{1\le i_1<\ldots<i_m\le n} a_{i_1\ldots i_m}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_m}\end{aligned}$$

and

$$\begin{aligned}\beta := \overline{\beta}_\Phi &= \sum_{1\le i_1<\ldots<i_m\le n} \overline{b}_{i_1\ldots i_m}\Big(\Phi(x)\Big)\, d\Phi_{i_1}(x) \wedge \ldots \wedge d\Phi_{i_m}(x) \\ &=: \sum_{1\le i_1<\ldots<i_m\le n} b_{i_1\ldots i_m}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_m},\end{aligned}$$

respectively. We shall define an inner product $(\alpha, \beta)_m$ between the transformed m-forms α and β such that it is parameter-invariant:

$$(\alpha, \beta)_m(x) = (\overline{\alpha}, \overline{\beta})_m\Big(\Phi(x)\Big), \qquad x \in \Omega. \tag{4}$$

We shall explicitly represent this inner product for differential forms of the orders $0, 1, n-1, n$ in the sequel.

1. Let $m = 0$ hold true. We consider the 0-forms

$$\overline{\alpha} = \overline{a}(\overline{x}), \quad \overline{\beta} = \overline{b}(\overline{x}).$$

Then we see

$$\alpha = \overline{\alpha}_\Phi = \overline{a}\Big(\Phi(x)\Big), \quad \beta = \overline{\beta}_\Phi = \overline{b}\Big(\Phi(x)\Big).$$

Setting

$$(\alpha, \beta)_0(x) := a(x)b(x),$$

we obtain

$$\begin{aligned}(\alpha, \beta)_0(x) = a(x)b(x) &= \overline{a}\Big(\Phi(x)\Big)\, \overline{b}\Big(\Phi(x)\Big) \\ &= (\overline{\alpha}, \overline{\beta})_0\Big(\Phi(x)\Big), \qquad x \in \Omega.\end{aligned}$$

2. Let $m = n$ hold true. We consider the n-forms

$$\overline{\alpha} = \overline{a}(\overline{x})\, d\overline{x}_1 \wedge \ldots \wedge d\overline{x}_n, \quad \overline{\beta} = \overline{b}(\overline{x})\, d\overline{x}_1 \wedge \ldots \wedge d\overline{x}_n.$$

We calculate

$$\begin{aligned} \alpha = \overline{\alpha}_\Phi &= \overline{a}\Big(\Phi(x)\Big)\, d\Phi_1 \wedge \ldots \wedge d\Phi_n \\ &= \overline{a}\Big(\Phi(x)\Big) \left(\sum_{i_1=1}^{n} \frac{\partial \Phi_1}{\partial x_{i_1}}\, dx_{i_1}\right) \wedge \ldots \wedge \left(\sum_{i_n=1}^{n} \frac{\partial \Phi_n}{\partial x_{i_n}}\, dx_{i_n}\right) \\ &= \overline{a}\Big(\Phi(x)\Big) J_\Phi(x)\, dx_1 \wedge \ldots \wedge dx_n. \end{aligned}$$

Therefore, we have

$$a(x) = \overline{a}\Big(\Phi(x)\Big) J_\Phi(x), \quad b(x) = \overline{b}\Big(\Phi(x)\Big) J_\Phi(x), \qquad x \in \Omega.$$

Now we set

$$(\alpha, \beta)_n(x) := \frac{1}{g(x)}\, a(x) b(x), \qquad x \in \Omega,$$

observe $g(x) = \Big(J_\Phi(x)\Big)^2$, and infer

$$\begin{aligned} (\alpha, \beta)_n(x) &= \frac{1}{\Big(J_\Phi(x)\Big)^2}\, \overline{a}\Big(\Phi(x)\Big)\, J_\Phi(x)\, \overline{b}\Big(\Phi(x)\Big)\, J_\Phi(x) \\ &= \overline{a}\Big(\Phi(x)\Big)\, \overline{b}\Big(\Phi(x)\Big) = (\overline{\alpha}, \overline{\beta})_n\Big(\Phi(x)\Big). \end{aligned}$$

3. Let $m = 1$ hold true. We consider the Pfaffian forms

$$\overline{\alpha} = \sum_{i=1}^{n} \overline{a}_i(\overline{x})\, d\overline{x}_i, \quad \overline{\beta} = \sum_{i=1}^{n} \overline{b}_i(\overline{x})\, d\overline{x}_i$$

and calculate

$$\begin{aligned} \alpha = \overline{\alpha}_\Phi &= \sum_{i=1}^{n} \overline{a}_i\Big(\Phi(x)\Big)\, d\Phi_i \\ &= \sum_{i=1}^{n} \overline{a}_i\Big(\Phi(x)\Big) \left(\sum_{j=1}^{n} \frac{\partial \Phi_i}{\partial x_j}\, dx_j\right) \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \overline{a}_i\Big(\Phi(x)\Big) \frac{\partial \Phi_i}{\partial x_j}\right) dx_j. \end{aligned}$$

Thus we obtain

$$\alpha = \overline{\alpha}_\Phi = \sum_{j=1}^n a_j(x)\,dx_j \quad \text{with} \quad a_j(x) = \sum_{i=1}^n \overline{a}_i\Big(\Phi(x)\Big)\frac{\partial \Phi_i}{\partial x_j},$$

$$\beta = \overline{\beta}_\Phi = \sum_{j=1}^n b_j(x)\,dx_j \quad \text{with} \quad b_j(x) = \sum_{i=1}^n \overline{b}_i\Big(\Phi(x)\Big)\frac{\partial \Phi_i}{\partial x_j},$$

where $j = 1, \ldots, n$ is valid. We introduce the following abbreviation for the *functional matrix*

$$F(x) := \left(\frac{\partial \Phi_i}{\partial x_j}(x)\right)_{i,j=1,\ldots,n}, \quad x \in \Omega.$$

The vectors

$$a(x) = \Big(a_1(x), \ldots, a_n(x)\Big), \quad \overline{a}(x) = \Big(\overline{a}_1(\overline{x}), \ldots, \overline{a}_n(\overline{x})\Big)$$

and

$$b(x) = \Big(b_1(x), \ldots, b_n(x)\Big), \quad \overline{b}(x) = \Big(\overline{b}_1(\overline{x}), \ldots, \overline{b}_n(\overline{x})\Big)$$

are subject to the transformation laws

$$a(x) = \overline{a}\Big(\Phi(x)\Big) \circ F(x), \quad b(x) = \overline{b}\Big(\Phi(x)\Big) \circ F(x),$$

and

$$a(x) \circ F^{-1}(x) = \overline{a}\Big(\Phi(x)\Big), \quad b(x) \circ F^{-1}(x) = \overline{b}\Big(\Phi(x)\Big),$$

respectively. We define the *transformation matrix*

$$G(x) = \Big(g_{ij}(x)\Big)_{i,j=1,\ldots,n} := F(x)^t \circ F(x)$$

with the inverse matrix

$$G^{-1}(x) = \Big(g^{ij}(x)\Big)_{i,j=1,\ldots,n} = F^{-1}(x) \circ \Big(F^{-1}(x)\Big)^t.$$

Evidently, we have

$$\sum_{j=1}^n g^{ij}(x) g_{jk}(x) = \delta^i_k, \qquad i,k = 1, \ldots, n$$

and

$$g(x) = \Big(J_\Phi(x)\Big)^2 = \det G(x).$$

Now we define

$$(\alpha,\beta)_1(x) := \sum_{i,j=1}^{n} g^{ij}(x)a_i(x)b_j(x).$$

Then we infer

$$\begin{aligned}(\alpha,\beta)_1(x) &= a(x)\circ G^{-1}(x)\circ \Big(b(x)\Big)^t \\ &= \overline{a}\Big(\Phi(x)\Big)\circ F(x)\circ F^{-1}(x)\circ\Big(F^{-1}(x)\Big)^t\circ\Big(F(x)\Big)^t\circ\Big(\overline{b}(\Phi(x))\Big)^t \\ &= \overline{a}\Big(\Phi(x)\Big)\circ\Big(\overline{b}(\Phi(x))\Big)^t \\ &= (\overline{\alpha},\overline{\beta})_1\Big(\Phi(x)\Big).\end{aligned}$$

4. Let $m = n-1$ hold true. We define the $(n-1)$-forms

$$\overline{\theta}_i := (-1)^{i-1}\, d\overline{x}_1\wedge\ldots\wedge d\overline{x}_{i-1}\wedge d\overline{x}_{i+1}\wedge\ldots\wedge d\overline{x}_n$$

for $1\le i\le n$ and consider the $(n-1)$-forms

$$\overline{\alpha} = \sum_{i=1}^{n}\overline{a}_i(\overline{x})\overline{\theta}_i,\quad \overline{\beta} = \sum_{i=1}^{n}\overline{b}_i(\overline{x})\overline{\theta}_i.$$

We use the symbol $\check{}$ to indicate that we omit this factor. Defining

$$\theta_j := (-1)^{j-1}\, dx_1\wedge\ldots\wedge dx_{j-1}\wedge dx_{j+1}\wedge\ldots\wedge dx_n$$

for $j = 1,\ldots,n$, we calculate

$$\begin{aligned}\alpha = \overline{\alpha}_\Phi &= \sum_{i=1}^{n}\overline{a}_i\Big(\Phi(x)\Big)(-1)^{i-1}\, d\Phi_1\wedge\ldots\wedge d\Phi_{i-1}\wedge d\Phi_{i+1}\wedge\ldots\wedge d\Phi_n \\ &= \sum_{i=1}^{n}\overline{a}_i\Big(\Phi(x)\Big)(-1)^{i-1}\left(\sum_{j_1=1}^{n}\frac{\partial\Phi_1}{\partial x_{j_1}}\,dx_{j_1}\right)\wedge\ldots\wedge\left(\sum_{j_{i-1}=1}^{n}\frac{\partial\Phi_{i-1}}{\partial x_{j_{i-1}}}\,dx_{j_{i-1}}\right) \\ &\wedge\left(\sum_{j_{i+1}=1}^{n}\frac{\partial\Phi_{i+1}}{\partial x_{j_{i+1}}}\,dx_{j_{i+1}}\right)\wedge\ldots\wedge\left(\sum_{j_n=1}^{n}\frac{\partial\Phi_n}{\partial x_{j_n}}\,dx_{j_n}\right) \\ &= \sum_{i=1}^{n}\overline{a}_i\Big(\Phi(x)\Big)(-1)^{i-1}\sum_{j=1}^{n}\frac{\partial(\Phi_1,\ldots,\check{\Phi}_i,\ldots,\Phi_n)}{\partial(x_1,\ldots,\check{x}_j,\ldots,x_n)}\cdot \\ &\cdot dx_1\wedge\ldots\wedge d\check{x}_j\wedge\ldots\wedge dx_n \\ &= \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\overline{a}_i\Big(\Phi(x)\Big)(-1)^{i+j}\frac{\partial(\Phi_1,\ldots,\check{\Phi}_i,\ldots,\Phi_n)}{\partial(x_1,\ldots,\check{x}_j,\ldots,x_n)}\right)\theta_j \;=:\; \sum_{j=1}^{n}a_j(x)\theta_j.\end{aligned}$$

Correspondingly, we define $b_j(x)$ for $j = 1, \ldots, n$. The matrix of adjoints for $F(x)$, namely

$$E(x) := \left((-1)^{i+j} \frac{\partial(\Phi_1, \ldots, \check{\Phi}_i, \ldots, \Phi_n)}{\partial(x_1, \ldots, \check{x}_j, \ldots, x_n)} \right)_{i,j=1,\ldots,n},$$

satisfies the identity

$$\Big(F(x)^t\Big)^{-1} = \left(\left(\frac{\partial \Phi_j}{\partial x_i}(x) \right)_{i,j=1,\ldots,n} \right)^{-1} = \frac{1}{J_\Phi(x)} E(x),$$

and equivalently

$$E(x) = J_\Phi(x) \Big(F(x)^t\Big)^{-1}. \tag{5}$$

When

$$\overline{\alpha}_\Phi = \alpha = \sum_{j=1}^{n} a_j(x)\theta_j, \quad \overline{\beta}_\Phi = \beta = \sum_{j=1}^{n} b_j(x)\theta_j$$

denote the transformed $(n-1)$-forms, their coefficient vectors

$$a(x) = \Big(a_1(x), \ldots, a_n(x)\Big), \quad \overline{a}(x) = \Big(\overline{a}_1(\overline{x}), \ldots, \overline{a}_n(\overline{x})\Big)$$

and

$$b(x) = \Big(b_1(x), \ldots, b_n(x)\Big), \quad \overline{b}(x) = \Big(\overline{b}_1(\overline{x}), \ldots, \overline{b}_n(\overline{x})\Big)$$

are subject to the transformation laws

$$a(x) = \overline{a}\Big(\Phi(x)\Big) \circ E(x) = J_\Phi(x) \overline{a}\Big(\Phi(x)\Big) \circ \Big(F(x)^t\Big)^{-1},$$

$$b(x) = \overline{b}\Big(\Phi(x)\Big) \circ E(x) = J_\Phi(x) \overline{b}\Big(\Phi(x)\Big) \circ \Big(F(x)^t\Big)^{-1}.$$

Now we define as the inner product

$$(\alpha, \beta)_{n-1}(x) := \frac{1}{g(x)} \sum_{i,j=1}^{n} g_{ij}(x) a_i(x) b_j(x).$$

Finally, we infer

$$\begin{aligned}
(\alpha, \beta)_{n-1}(x) &= \frac{1}{\Big(J_\Phi(x)\Big)^2} a(x) \circ G(x) \circ \Big(b(x)\Big)^t \\
&= \overline{a}\Big(\Phi(x)\Big) \circ \Big(F(x)^t\Big)^{-1} \circ F(x)^t \circ F(x) \circ \Big(F(x)\Big)^{-1} \circ \Big(\overline{b}(\Phi(x))\Big)^t \\
&= \overline{a}\Big(\Phi(x)\Big) \circ \Big(\overline{b}(\Phi(x))\Big)^t = (\overline{\alpha}, \overline{\beta})_{n-1}\Big(\Phi(x)\Big).
\end{aligned}$$

Now we introduce another operation in the set of differential forms.

Definition 8.1. *When $k \in K := \{0, 1, n-1, n\}$ holds true, we attribute to each k-form α its dual $(n-k)$-form $*\alpha$ as follows:*

1. *Let $k = 0$ and $\alpha = a(x)$ be given. Then we define*
$$*\alpha := a(x)\omega,$$
where
$$\omega = \sqrt{g(x)}\, dx_1 \wedge \ldots \wedge dx_n$$
denotes the volume form (compare (3)).
2. *Let $k = 1$ and*
$$\alpha = \sum_{i=1}^{n} a_i(x)\, dx_i$$
be given. Then we define
$$*\alpha := \sqrt{g(x)}\, \sum_{i=1}^{n} \left(\sum_{j=1}^{n} g^{ij}(x) a_j(x) \right) \theta_i.$$
3. *Let $k = n-1$ and*
$$\alpha = \sum_{i=1}^{n} a_i(x) \theta_i$$
be given. Then we define
$$*\alpha := \frac{(-1)^{n-1}}{\sqrt{g(x)}}\, \sum_{i=1}^{n} \left(\sum_{j=1}^{n} g_{ij}(x) a_j(x) \right)\, dx_i.$$
4. *Let $k = n$ and $\alpha = a(x)\omega$ be given. Then we define*
$$*\alpha := a(x).$$

We collect some *properties of the $*$-operator.*

1. The $*$-operator represents a linear operator from the vector space of k-forms into the vector space of $(n-k)$-forms. It gives us an *involution*, which means
$$* * \alpha = (-1)^{k(n-k)} \alpha$$
for all k-forms α with $k \in K$.
2. The k-form α and the $(n-k)$-form β fulfill the identity
$$(\alpha, *\beta)_k = (*\alpha, \beta)_{n-k} (-1)^{k(n-k)}, \qquad k \in K.$$

We prove this statement for all $k \in K$:

a) Let $k = 0$, $\alpha = a(x)$, $\beta = b(x)\omega$, $*\beta = b(x)$, $*\alpha = a(x)\omega$ be given. Then we obtain

$$(\alpha, *\beta)_0 = a(x)b(x) = a(x)b(x)(\omega, \omega)_n = (a(x)\omega, b(x)\omega)_n = (*\alpha, \beta)_n.$$

b) Let $k = n$, α an n-form, β a 0-form be given. We calculate with the aid of property 1 and (a) as follows:

$$(\alpha, *\beta)_n = (*(*\alpha), *\beta)_n = (*\alpha, *(*\beta))_0 = (*\alpha, \beta)_0.$$

c) Let $k = 1$ be given. We consider the forms

$$\alpha = \sum_{i=1}^{n} a_i(x)\, dx_i, \quad \beta = \sum_{i=1}^{n} b_i(x)\theta_i.$$

Then we obtain

$$\begin{aligned}
(\alpha, *\beta)_1 &= \frac{(-1)^{n-1}}{\sqrt{g(x)}} \sum_{i,j=1}^{n} g^{ij}(x)a_i(x) \left(\sum_{k=1}^{n} g_{jk}(x)b_k(x) \right) \\
&= \frac{(-1)^{n-1}}{\sqrt{g(x)}} \sum_{i,j=1}^{n} a_i(x) \left(\sum_{k=1}^{n} g^{ij}(x)g_{jk}(x)b_k(x) \right) \\
&= \frac{(-1)^{n-1}}{\sqrt{g(x)}} \sum_{i=1}^{n} a_i(x) \left(\sum_{k=1}^{n} \delta_k^i b_k(x) \right) \\
&= \frac{(-1)^{n-1}}{\sqrt{g(x)}} \sum_{i=1}^{n} a_i(x)b_i(x),
\end{aligned}$$

as well as

$$\begin{aligned}
(*\alpha, \beta)_{n-1} &= \frac{\sqrt{g(x)}}{g(x)} \sum_{i,j=1}^{n} g_{ij}(x) \left(\sum_{k=1}^{n} g^{ik}(x)a_k(x) \right) b_j(x) \\
&= \frac{1}{\sqrt{g(x)}} \sum_{i,j=1}^{n} b_j(x) \left(\sum_{k=1}^{n} g_{ij}(x)g^{ik}(x)a_k(x) \right) \\
&= \frac{1}{\sqrt{g(x)}} \sum_{j,k=1}^{n} b_j(x) \left(\delta_j^k a_k(x) \right) \\
&= \frac{1}{\sqrt{g(x)}} \sum_{i=1}^{n} a_i(x)b_i(x).
\end{aligned}$$

This implies $(\alpha, *\beta)_1 = (-1)^{n-1}(*\alpha, \beta)_{n-1}$.

d) The case $k = n-1$ remains. With the aid of property 1 and (c), we deduce for the $(n-1)$-form α and the 1-form β as follows:

$$\begin{aligned}(\alpha, *\beta)_{n-1} &= (-1)^{n-1}(*(*\alpha), *\beta)_{n-1} \\ &= (*\alpha, *(*\beta))_1 = (-1)^{n-1}(*\alpha, \beta)_1.\end{aligned}$$

3. Taking the two k-forms α and β with $k \in K$, we infer

$$\begin{aligned}(*\alpha, *\beta)_{n-k} &= (-1)^{k(n-k)}(*(*\alpha), \beta)_k \\ &= \left((-1)^{k(n-k)}\right)^2 (\alpha, \beta)_k = (\alpha, \beta)_k.\end{aligned}$$

Consequently, the $*$-operator represents an *isometry.*

4. Two k-forms α and β satisfy the identity

$$\alpha \wedge (*\beta) = (-1)^{k(n-k)}(*\alpha) \wedge \beta = (\alpha, \beta)_k \omega, \qquad k \in K.$$

For the proof, we show the relation

$$\alpha \wedge (*\beta) = (\alpha, \beta)_k \omega. \tag{6}$$

Then the $(n-k)$-form $*\alpha$ and the k-form β satisfy

$$(-1)^{k(n-k)}(*\alpha) \wedge \beta = \beta \wedge (*\alpha) = (\beta, \alpha)_k \omega = (\alpha, \beta)_k \omega = \alpha \wedge (*\beta).$$

a) Let $k = 0$, $\alpha = a(x)$, $\beta = b(x)$, $*\beta = b(x)\omega$ be given. Then we see

$$\alpha \wedge (*\beta) = a(x)b(x)\omega = (\alpha, \beta)_0 \omega.$$

b) Let $k = 1$ as well as

$$\alpha = \sum_{i=1}^{n} a_i(x)\, dx_i, \quad \beta = \sum_{i=1}^{n} b_i(x)\, dx_i$$

and

$$*\beta = \sqrt{g(x)} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} g^{ij}(x) b_j(x) \right) \theta_i$$

be given. Now we evaluate

$$\alpha \wedge (*\beta) = \sqrt{g(x)} \left(\sum_{i,j=1}^{n} g^{ij}(x) a_i(x) b_j(x) \right) dx_1 \wedge \ldots \wedge dx_n = (\alpha, \beta)_1 \omega.$$

c) For $k = n-1$ and

$$\alpha = \sum_{i=1}^{n} a_i(x)\theta_i, \quad \beta = \sum_{i=1}^{n} b_i(x)\theta_i$$

as well as

$$*\beta = \frac{(-1)^{n-1}}{\sqrt{g(x)}} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} g_{ij}(x) b_j(x) \right) dx_i,$$

we infer

$$\alpha \wedge (*\beta) = \left(\sum_{i=1}^{n} a_i(x) \theta_i \right) \wedge \left(\frac{(-1)^{n-1}}{\sqrt{g(x)}} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} g_{ij}(x) b_j(x) \right) dx_i \right)$$

$$= \left(\frac{1}{\sqrt{g(x)}} \sum_{i,j=1}^{n} g_{ij}(x) a_i(x) b_j(x) \right) dx_1 \wedge \ldots \wedge dx_n$$

$$= (\alpha, \beta)_{n-1} \sqrt{g(x)}\, dx_1 \wedge \ldots \wedge dx_n = (\alpha, \beta)_{n-1} \omega.$$

d) Finally, let $k = n$, $\alpha = a(x)\omega$, and $\beta = b(x)\omega$ be given. This implies

$$\alpha \wedge (*\beta) = a(x)\omega b(x) = a(x)b(x)\omega = (\alpha, \beta)_n \omega.$$

5. Let

$$\alpha = \sum_{i=1}^{n} a_i(x) dx_i$$

denote a Pfaffian form and

$$x = \Phi(\overline{x}) = \Big(\Phi_1(\overline{x}_1, \ldots, \overline{x}_n), \ldots, \Phi_n(\overline{x}_1, \ldots, \overline{x}_n) \Big)$$

a parameter transformation. Then we observe $(*\alpha)_\Phi = *(\alpha_\Phi)$.

We use the invariance of the inner product as well as the property 4: For an arbitrary 1-form

$$\beta = \sum_{i=1}^{n} b_i(x)\, dx_i$$

with the transformed 1-form β_Φ, we infer the identity

$$\beta_\Phi \wedge *(\alpha_\Phi) = (\beta_\Phi, \alpha_\Phi)_1 \omega_\Phi = \{(\beta, \alpha)_1\}_\Phi \omega_\Phi$$

$$= \{(\beta, \alpha)_1 \omega\}_\Phi = \{\beta \wedge (*\alpha)\}_\Phi = \beta_\Phi \wedge (*\alpha)_\Phi.$$

Then we obtain

$$\beta_\Phi \wedge (*(\alpha_\Phi) - (*\alpha)_\Phi) = 0 \qquad \text{for all} \quad \beta,$$

and consequently

$$*(\alpha_\Phi) = (*\alpha)_\Phi.$$

Definition 8.2. *Given a* 1*-form*

$$\alpha = \sum_{i=1}^{n} a_i(x)\, dx_i\,, \qquad x \in \Omega$$

of the class $C^1(\Omega)$*, we define the* co-derivative $\delta\alpha$ *due to*

$$\delta\alpha := *d*\alpha.$$

Remark: Now δ represents a parameter-invariant differential operator of first order - and attributes a 0-form to each 1-form. We determine the operator δ in arbitrary coordinates. Let us consider

$$\alpha = \sum_{i=1}^{n} a_i(x)\, dx_i, \quad *\alpha = \sqrt{g(x)} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} g^{ij}(x) a_j(x) \right) \theta_i.$$

Then we evaluate

$$d*\alpha = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{g(x)} \sum_{j=1}^{n} g^{ij}(x) a_j(x) \right) dx_1 \wedge \ldots \wedge dx_n$$

$$= \frac{1}{\sqrt{g(x)}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{g(x)} \sum_{j=1}^{n} g^{ij}(x) a_j(x) \right) \omega.$$

The application of the $*$-operator on $d*\alpha$ yields

$$\delta\alpha = *d*\alpha = \frac{1}{\sqrt{g(x)}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{g(x)} \sum_{j=1}^{n} g^{ij}(x) a_j(x) \right). \tag{7}$$

Theorem 8.3. (Partial integration in arbitrary parameters)
Let $\Omega \subset \mathbb{R}^n$ *denote a domain satisfying the assumptions* (A)*,* (B)*, and* (D) *for the Gaussian integral theorem. The parameter transformation*

$$\overline{x} = \Phi(x) \,:\, \Omega \longrightarrow \Theta \in C^1(\overline{\Omega})$$

may be bijective and subject to the condition

$$J_\Phi(x) \geq \eta > 0 \qquad \text{for all points} \quad x \in \overline{\Omega}.$$

Furthermore, let a 1*-form*

$$\alpha = \sum_{i=1}^{n} a_i(x)\, dx_i, \qquad x \in \overline{\Omega}$$

and a 0*-form* $\beta = b(x)$, $x \in \overline{\Omega}$ *of the class* $C^1(\overline{\Omega})$ *be given. Then we have the identity*

$$\int\limits_{\Omega} (\alpha, d\beta)_1 \omega + \int\limits_{\Omega} (\delta\alpha, \beta)_0 \omega = \int\limits_{\partial\Omega} (*\alpha) \wedge \beta.$$

Here the boundary $\partial\Omega$ *is endowed with the induced canonical orientation of* $\mathbb{R}^n$.

Proof: The assumptions on the parameter transformation Φ guarantee that all functions appearing belong to the regularity class $C^1(\overline{\Omega})$. We apply the Stokes integral theorem and obtain - with the aid of (6) - our statement as follows:

$$\begin{aligned}
\int\limits_{\Omega} (\alpha, d\beta)_1 \omega &= \int\limits_{\Omega} \alpha \wedge (*d\beta) = (-1)^{n-1} \int\limits_{\Omega} (*\alpha) \wedge d\beta \\
&= \int\limits_{\Omega} d\Big((*\alpha) \wedge \beta\Big) - \int\limits_{\Omega} (d * \alpha) \wedge \beta \\
&= \int\limits_{\partial\Omega} (*\alpha) \wedge \beta - \int\limits_{\Omega} (d * \alpha) \wedge (* * \beta) \\
&= \int\limits_{\partial\Omega} (*\alpha) \wedge \beta - \int\limits_{\Omega} (d * \alpha, *\beta)_n \omega \\
&= \int\limits_{\partial\Omega} (*\alpha) \wedge \beta - \int\limits_{\Omega} (*d * \alpha, \beta)_0 \omega \\
&= \int\limits_{\partial\Omega} (*\alpha) \wedge \beta - \int\limits_{\Omega} (\delta\alpha, \beta)_0 \omega.
\end{aligned}$$

q.e.d.

Corollary: When we require zero-boundary-values in Theorem 8.3 for the function β, or more precisely $\beta \in C_0^1(\Omega)$, we deduce the identity

$$\int\limits_{\Omega} (\alpha, d\beta)_1 \omega + \int\limits_{\Omega} (\delta\alpha, \beta)_0 \omega = 0.$$

Therefore, we name δ the *adjoint derivative* to the exterior derivative d.

Definition 8.4. *The two functions* $\psi(x)$ *and* $\chi(x)$ *of the class* $C^1(\Omega)$ *with their associate differentials*

$$d\psi = \sum_{i=1}^{n} \psi_{x_i}\, dx_i, \quad d\chi = \sum_{i=1}^{n} \chi_{x_i}\, dx_i$$

being given, we define the Beltrami operator of first order *via*

$$\boldsymbol{\nabla}(\psi,\chi) := (d\psi, d\chi)_1(x) = \sum_{i,j=1}^{n} g^{ij}(x)\psi_{x_i}(x)\chi_{x_j}(x).$$

Remark: Evidently, the property

$$\boldsymbol{\nabla}(\psi,\chi)(x) = \boldsymbol{\nabla}(\overline{\psi},\overline{\chi})\Big(\Phi(x)\Big)$$

holds true, where we note that

$$\overline{\psi}\Big(\Phi(x)\Big) = \psi(x), \quad \overline{\chi}\Big(\Phi(x)\Big) = \chi(x).$$

Consequently, $\boldsymbol{\nabla}$ represents a parameter-invariant differential operator of first order.

Definition 8.5. *We define the* Laplace-Beltrami operator

$$\boldsymbol{\Delta}\psi(x) := \delta d\psi(x), \qquad x \in \Omega$$

for functions $\psi(x) \in C^2(\Omega)$.

Remark: Since the operators d and δ are parameter-invariant, the operator $\boldsymbol{\Delta}$ is parameter-invariant as well:

$$\boldsymbol{\Delta}\psi(x) = \boldsymbol{\Delta}\overline{\psi}\Big(\Phi(x)\Big), \qquad x \in \Omega.$$

Using (7), we now describe $\boldsymbol{\Delta}$ in coordinates:

$$\begin{aligned}\boldsymbol{\Delta}\psi = \delta d\psi &= \delta\left(\sum_{j=1}^{n}\psi_{x_j}\,dx_j\right)\\ &= \frac{1}{\sqrt{g(x)}}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sqrt{g(x)}\sum_{j=1}^{n}g^{ij}(x)\psi_{x_j}\right).\end{aligned} \tag{8}$$

Theorem 8.6. *Let* $\Omega \subset \mathbb{R}^n$ *denote a domain satisfying the assumptions* (A), (B), *and* (D) *of the Gaussian integral theorem. Furthermore, the parameter transformation*

$$\overline{x} = \Phi(x) \,:\, \overline{\Omega} \longrightarrow \overline{\Theta}$$

belongs to the class $C^2(\overline{\Omega})$ *and is bijective subject to the condition*

$$J_\Phi(x) \geq \eta > 0 \qquad \text{for all points} \quad x \in \overline{\Omega}.$$

Finally, let the functions $\psi(x) \in C^2(\overline{\Omega})$ *as well as* $\chi(x) \in C^1(\overline{\Omega})$ *be given. Then we have the identity*

$$\int\limits_{\Omega}\boldsymbol{\nabla}(\psi,\chi)\omega + \int\limits_{\Omega}(\boldsymbol{\Delta}\psi,\chi)_0\omega = \int\limits_{\partial\Omega}(*d\psi)\chi.$$

Proof: We apply Theorem 8.3 and insert

$$\alpha = d\psi \in C^1(\overline{\Omega}), \quad \beta = \chi(x) \in C^1(\overline{\Omega}).$$

At first, we obtain

$$\int\limits_{\Omega} (d\psi, d\chi)_1 \omega + \int\limits_{\Omega} (\delta d\psi, \beta)_0 \omega = \int\limits_{\partial\Omega} (*d\psi)\chi.$$

Using the Definitions 8.4 and 8.5, we infer the identity

$$\int\limits_{\Omega} \boldsymbol{\nabla}(\psi, \chi)\omega + \int\limits_{\Omega} (\boldsymbol{\Delta}\psi, \chi)_0 \omega = \int\limits_{\partial\Omega} (*d\psi)\chi$$

stated above. q.e.d.

Remark:

1. We evaluate the Laplace operator in cylindrical coordinates,

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = h,$$

where $0 < r < +\infty$, $0 \le \varphi < 2\pi$, $-\infty < h < +\infty$ hold true. Therefore, we consider the case $n = 3$ and choose

$$x_1 = r, \quad x_2 = \varphi, \quad x_3 = h.$$

The fundamental tensor appears in the following form:

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This implies

$$g(x) = \det(g_{ij}) = r^2.$$

In our calculations we have to respect only those elements on the principal diagonal. With the aid of (7), we then obtain

$$\begin{aligned}\boldsymbol{\Delta} &= \frac{1}{r}\left\{\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial\varphi}\left(\frac{1}{r}\frac{\partial}{\partial\varphi}\right) + \frac{\partial}{\partial h}\left(r\frac{\partial}{\partial h}\right)\right\} \\ &= \frac{1}{r}\left(\frac{\partial}{\partial r} + r\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial^2}{\partial\varphi^2} + r\frac{\partial^2}{\partial h^2}\right) \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial h^2}.\end{aligned}$$

For plane polar coordinates we set $z \equiv 0$, and the expression above is reduced to

$$\boldsymbol{\Delta} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}.$$

Defining

$$\boldsymbol{\Lambda} := \frac{\partial^2}{\partial \varphi^2}$$

for the angular expression, we rewrite $\boldsymbol{\Delta}$ into the form

$$\boldsymbol{\Delta} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\boldsymbol{\Lambda}.$$

(compare the Laplace operator in spherical coordinates).

2. We introduce spherical coordinates

$$x = r\cos\varphi\sin\theta, \quad y = r\sin\varphi\sin\theta, \quad z = r\cos\theta$$

with $0 < r < +\infty$, $0 \le \varphi < 2\pi$, and $0 < \theta < \pi$. Calculations parallel to Remark 1 yield

$$\begin{aligned}
\boldsymbol{\Delta} &= \frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}\\
&= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}\\
&=: \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\boldsymbol{\Lambda}.
\end{aligned}$$

Here the operator $\boldsymbol{\Lambda}$ does not depend on r again. However, it is only dependent on the angles φ, θ.

When we investigate *spherical harmonic functions* in Chapter 5, we need the Laplace operator for spherical coordinates in n dimensions. Now we treat this general case.

Let the unit sphere in $\mathbb{R}^n$, namely

$$\Sigma = \Big\{\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \, : \, |\xi| = 1\Big\},$$

by parametrized by

$$\xi = \xi(t) = \Big(\xi_1(t_1, \ldots, t_{n-1}), \ldots, \xi_n(t_1, \ldots, t_{n-1})\Big)^t \, : \, T \longrightarrow \Sigma \in C^2(T),$$

with the open set $T \subset \mathbb{R}^{n-1}$. Via the mapping

$$X(r, t) := r\xi(t_1, \ldots, t_{n-1}), \qquad r \in (0, +\infty), \quad t \in T,$$

we obtain polar coordinates in $\mathbb{R}^n$. Furthermore, the functional matrix appears in the form

$$\partial X(r,t) = (X_r, X_{t_1}, \ldots, X_{t_{n-1}}) = (\xi, r\xi_{t_1}, \ldots, r\xi_{t_{n-1}}).$$

We determine the metric tensor as follows:

$$G(r,t) = \Big(g_{ij}(r,t)\Big)_{i,j} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & r^2 h_{11} & \cdots & r^2 h_{1,n-1} \\ \vdots & \cdots & & \vdots \\ 0 & r^2 h_{n-1,1} & \cdots & r^2 h_{n-1,n-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & r^2 H(t) & \\ 0 & & & \end{pmatrix},$$

where we abbreviate

$$H(t) = \Big(h_{ij}(t)\Big)_{i,j=1,\ldots,n-1} := \Big(\xi_{t_i}(t) \cdot \xi_{t_j}(t)\Big)_{i,j=1,\ldots,n-1}.$$

Using the convention

$$H^{-1}(t) = \Big(h^{ij}(t)\Big)_{i,j=1,\ldots,n-1}, \quad G^{-1}(r,t) = \Big(g^{ij}(r,t)\Big)_{i,j=1,\ldots,n},$$

we infer

$$G^{-1}(r,t) = \Big(g^{ij}(r,t)\Big)_{i,j} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \dfrac{H^{-1}(t)}{r^2} & \\ 0 & & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{h^{11}}{r^2} & \cdots & \frac{h^{1,n-1}}{r^2} \\ \vdots & \cdots & & \vdots \\ 0 & \frac{h^{n-1,1}}{r^2} & \cdots & \frac{h^{n-1,n-1}}{r^2} \end{pmatrix}.$$

Furthermore, we define

$$g(r,t) := \det G(r,t), \quad h(t) := \det H(t)$$

and obtain

$$g(r,t) = r^{2(n-1)} h(t).$$

When $u = u(r,t)$ and $v = v(r,t)$ are two functions, we determine the Beltrami differential operator of first order due to

$$\begin{aligned} \boldsymbol{\nabla}(u,v) &= \sum_{i,j=1}^{n} g^{ij}(x) u_{x_i} v_{x_j} \\ &= \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + \frac{1}{r^2} \sum_{i,j=1}^{n-1} h^{ij}(t) \frac{\partial u}{\partial t_i}\frac{\partial v}{\partial t_j}. \end{aligned}$$

We express the *invariant Beltrami operator of first order on the sphere Σ* via

$$\boldsymbol{\Gamma}(u,v) := \sum_{i,j=1}^{n-1} h^{ij}(t) \frac{\partial u}{\partial t_i}\frac{\partial v}{\partial t_j}$$

and deduce

$$\boldsymbol{\nabla}(u,v)=\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\boldsymbol{\Gamma}(u,v)\qquad\text{for all}\quad u=u(r,t),\quad v=v(r,t).\tag{9}$$

Now we represent the Laplace-Beltrami operator in spherical coordinates: We take the function

$$u=u(r,t)=u(r,t_1,\ldots,t_{n-1}),$$

utilize the identity $\sqrt{g(r,t)}=r^{n-1}\sqrt{h(t)}$ as well as formula (8), and obtain

$$\begin{aligned}\boldsymbol{\Delta}u&=\frac{1}{\sqrt{g(r,t)}}\,\mathrm{div}_{(r,t)}\left\{\sqrt{g(r,t)}\;G^{-1}(r,t)\circ\begin{pmatrix}u_r\\u_{t_1}\\\vdots\\u_{t_{n-1}}\end{pmatrix}\right\}\\
&=\frac{1}{\sqrt{g(r,t)}}\frac{\partial}{\partial r}\left(\sqrt{g(r,t)}\,\frac{\partial u}{\partial r}\right)\\
&\quad+\frac{1}{\sqrt{g(r,t)}}\,\mathrm{div}_t\left\{r^{n-1}\sqrt{h(t)}\,\frac{1}{r^2}H^{-1}(t)\circ\begin{pmatrix}u_{t_1}\\\vdots\\u_{t_{n-1}}\end{pmatrix}\right\}\\
&=\frac{\partial^2 u}{\partial r^2}+\frac{n-1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{1}{\sqrt{h(t)}}\,\mathrm{div}_t\left\{\sqrt{h(t)}\;H^{-1}(t)\circ\begin{pmatrix}u_{t_1}\\\vdots\\u_{t_{n-1}}\end{pmatrix}\right\}.\end{aligned}$$

Defining the *Laplace-Beltrami operator on the sphere* Σ by

$$\boldsymbol{\Lambda}u:=\frac{1}{\sqrt{h(t)}}\sum_{i=1}^{n-1}\frac{\partial}{\partial t_i}\left(\sqrt{h(t)}\sum_{j=1}^{n-1}h^{ij}(t)\frac{\partial u}{\partial t_j}\right),\qquad t\in T,$$

we obtain the following identity

$$\boldsymbol{\Delta}u=\frac{\partial^2 u}{\partial r^2}+\frac{n-1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\boldsymbol{\Lambda}u\qquad\text{for all}\quad u=u(r,t)\in C^2((0,+\infty)\times T).\tag{10}$$

We still show the symmetry of the Laplace-Beltrami operator on the sphere for later use.

Theorem 8.7. *Taking the functions* $f,g\in C^2(\Sigma)$, *we have the relation*

$$\int\limits_{\Sigma}f(\xi)\Big(\boldsymbol{\Lambda}g(\xi)\Big)\,d\sigma(\xi)=-\int\limits_{\Sigma}\boldsymbol{\Gamma}(f,g)\,d\sigma(\xi)=\int\limits_{\Sigma}\Big(\boldsymbol{\Lambda}f(\xi)\Big)g(\xi)\,d\sigma(\xi).$$

Here $d\sigma$ *denotes the surface element on* Σ.

Proof: Let $0 < \varepsilon < 1$ be given, and we consider the domain

$$\Omega_\varepsilon := \Big\{x \in \mathbb{R}^n \,:\, 1-\varepsilon < |x| < 1+\varepsilon\Big\}.$$

Furthermore, we have

$$u(r,\xi) := f(\xi), \quad v(r,\xi) := g(\xi), \qquad r \in (1-\varepsilon, 1+\varepsilon), \quad \xi \in \Sigma.$$

Theorem 8.6 yields

$$\int\limits_{\Omega_\varepsilon} \boldsymbol{\nabla}(u,v)\,\omega + \int\limits_{\Omega_\varepsilon} (\boldsymbol{\Delta} u, v)_0\,\omega = \int\limits_{\partial\Omega_\varepsilon} (*du)v = \int\limits_{\partial\Omega_\varepsilon} v\,\frac{\partial u}{\partial\nu}\,d\sigma,$$

where ν denotes the exterior normal to $\partial\Omega_\varepsilon$. These parameter-invariant integrals are evaluated in (r,ξ)-coordinates: Via the identities (9) as well as (10) and noting that

$$\frac{\partial u}{\partial\nu} = \pm\frac{\partial u}{\partial r} \equiv 0 \quad \text{on} \quad \partial\Omega_\varepsilon,$$

we arrive at the relation

$$0 = \int\limits_{1-\varepsilon}^{1+\varepsilon} \left(\int\limits_{\Sigma} \frac{1}{r^2}\,\boldsymbol{\Gamma}(f,g)\,d\sigma(\xi)\,r^{n-1} \right) dr + \int\limits_{1-\varepsilon}^{1+\varepsilon} \left(\int\limits_{\Sigma} \frac{1}{r^2}\,\boldsymbol{\Lambda}(f)\,g\,d\sigma(\xi)\,r^{n-1} \right) dr$$

$$= \left(\int\limits_{1-\epsilon}^{1+\epsilon} r^{n-3}\,dr \right) \int\limits_{\Sigma} \Big(\boldsymbol{\Gamma}(f,g) + \boldsymbol{\Lambda}(f)\,g \Big) d\sigma(\xi).$$

This implies

$$\int_\Sigma \Big(\boldsymbol{\Lambda} f(\xi)\Big) g(\xi)\,d\sigma(\xi) = -\int_\Sigma \boldsymbol{\Gamma}(f,g)\,d\sigma(\xi).$$

Correspondingly, we deduce the second identity stated above. q.e.d.

9 Some Historical Notices to Chapter 1

The theory of partial differential equations in the classical sense is treated within the framework of the continuously differentiable functions. The profound integral theorem of Gauß constitutes the center for the classical investigations of partial differential equations. This might explain the title *Princeps Mathematicorum* attributed to him. His tomb in Göttingen and the monument for him, together with the physicist W. Weber, express the great respect, which is given to C.F. Gauß.

Our treatment within the framework of differential forms, created by E. Cartan (1869–1961), simplifies the various integral theorems and classifies them geometrically. Though differential forms are systematically used, with great success, in differential geometry, analysts mostly refrain from their application in the theory of partial differential equations. We owe the introduction of invariant differential operators to E. Beltrami (1835–1900) – the first representative of a great differential-geometric tradition in Italy.

Figure 1.1 PORTRAIT OF CARL FRIEDRICH GAUSS (1777–1855)
Lithography by Siegried Detlef Bendixen published in Schumacher's *Astronomische Nachrichten* in 1828; taken from the inner titel-page of the biography by *Horst Michling: Carl Friedrich Gauß – Aus dem Leben des Princeps Mathematicorum*, Verlag Göttinger Tageblatt, Göttingen (1976).