Chapter 9
Frequently hypercyclic operators

The theory of linear dynamical systems has its roots in topological dynamics. But there is also a parallel theory of measurable dynamics, which is better known under the name of ergodic theory. In this chapter we show how concepts and results from that theory lead to a deepened understanding of linear dynamics. More specifically, we will see how the celebrated Birkhoff ergodic theorem suggests an interesting and rather strong variant of hypercyclicity, that of frequently hypercyclic operators. We point out that, while ergodic theory has turned out to be a most powerful tool in linear dynamics, we will use it here only for motivating the new concept.

Having introduced frequently hypercyclic operators, we then derive a Frequent Hypercyclicity Criterion and an eigenvalue criterion that allow us to show that, quite surprisingly, many of the hypercyclic operators met so far are in fact frequently hypercyclic. In the final section we revisit several of the structural properties of hypercyclicity within the new framework.

9.1 Frequently recurrent orbits

Let T be an operator on a separable Fréchet space X. In order to look at T from the point of view of ergodic theory we need to have a probability measure μ on X. Since we are in a topological situation it is natural to assume that μ is defined on the Borel σ-algebra $\mathcal{B}(X)$, that is, the smallest σ-algebra containing the open subsets of X; the elements of $\mathcal{B}(X)$ are called the Borel sets of X. Since T is continuous, it is then also measurable. We assume that T satisfies the minimum requirement in ergodic theory, namely, that it is μ-invariant, that is, $\mu(T^{-1}(A)) = \mu(A)$ for every Borel set A.

Of course, μ-invariance alone does not yet give us interesting dynamics since, for example, the identity operator is automatically μ-invariant for any measure μ. This changes when we inject ergodicity: one way of defining this notion is by demanding that, for any Borel sets A and B with $\mu(A) > 0$

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and $\mu(B) > 0$, there is some $n \in \mathbb{N}_0$ such that $\mu(T^{-n}(A) \cap B) > 0$. This notion is not only formally similar to topological transitivity. Suppose that the measure μ has the additional property that $\mu(U) > 0$ for any nonempty open set U; μ is then said to be of full (topological) support. Under this assumption, ergodicity obviously implies topological transitivity.

But there is an added bonus in the form of the Birkhoff ergodic theorem. It tells us that if T is ergodic with respect to μ then, for any μ-integrable function f on X, its time average with respect to T coincides with its space average; more precisely we have that

$$\frac{1}{N+1}\sum_{n=0}^{N} f(T^n x) \to \int_X f\,d\mu, \quad \text{for } \mu\text{-almost all } x \in X, \tag{9.1}$$

as $N \to \infty$. This then implies an interesting topological property for T. Indeed, since X is separable, its topology has a countable base $(U_k)_k$. When we apply (9.1) to the indicator functions $\mathbf{1}_{U_k}$, $k \geq 1$, the left-hand side turns out to be

$$\frac{1}{N+1}\sum_{n=0}^{N} \mathbf{1}_{U_k}(T^n x) = \frac{\operatorname{card}\{0 \leq n \leq N \ ; \ T^n x \in U_k\}}{N+1},$$

while the right-hand side is simply $\int_X \mathbf{1}_{U_k}\,d\mu = \mu(U_k) > 0$, where we have assumed again that μ is of full support. Thus there are subsets $A_k \subset X$, $k \geq 1$, of full measure such that, for any $x \in A_k$,

$$\lim_{N\to\infty} \frac{\operatorname{card}\{0 \leq n \leq N \ ; \ T^n x \in U_k\}}{N+1} > 0.$$

Since every nonempty open set contains some U_k and since $\bigcap_{k\geq 1} A_k$ has full measure we obtain that, for μ-almost all $x \in X$ and every nonempty open subset U of X,

$$\liminf_{N\to\infty} \frac{\operatorname{card}\{0 \leq n \leq N \ ; \ T^n x \in U\}}{N+1} > 0.$$

What we have found here is that, under the mentioned assumptions, the operator T has a property that is much stronger than hypercyclicity. There must even be an $x \in X$ whose orbit meets every nonempty open set very often, in the sense given above. Let us recall here the following.

Definition 9.1. The *lower density* of a subset $A \subset \mathbb{N}_0$ is defined as

$$\underline{\operatorname{dens}}(A) = \liminf_{N\to\infty} \frac{\operatorname{card}\{0 \leq n \leq N \ ; \ n \in A\}}{N+1}.$$

Our discussion so far leads us to the following concept.

Definition 9.2. An operator T on a Fréchet space X is called *frequently hypercyclic* if there is some $x \in X$ such that, for any nonempty open subset U of X,

$$\underline{\text{dens}}\,\{n \in \mathbb{N}_0 \ ; \ T^n x \in U\} > 0.$$

In this case, x is called a *frequently hypercyclic vector* for T. The set of frequently hypercyclic vectors for T is denoted by $FHC(T)$.

The orbit of a frequently hypercyclic vector is therefore, in the specified sense, frequently recurrent. Obviously, frequent hypercyclicity is a stronger notion than hypercyclicity.

There is an equivalent formulation of frequent hypercyclicity that nicely differentiates it from hypercyclicity. Let A be a subset of $\mathbb{N}_0$; if $(n_k)_{k\geq 1}$ is the increasing sequence of integers forming A and $n_k \leq N < n_{k+1}$ then

$$\frac{k}{n_{k+1}} \leq \frac{\text{card}\{0 \leq n \leq N \ ; \ n \in A\}}{N+1} \leq \frac{k}{n_k},$$

which implies that $\underline{\text{dens}}\,(A) = \liminf_{k\to\infty} \frac{k}{n_k}$. Thus A has positive lower density if and only if $(\frac{n_k}{k})_k$ is bounded; in other words, if $n_k = O(k)$.

Proposition 9.3. *A vector $x \in X$ is frequently hypercyclic for T if and only if, for any nonempty open subset U of X, there is a strictly increasing sequence $(n_k)_k$ of positive integers such that*

$$T^{n_k} x \in U \quad \textit{for all } k \in \mathbb{N}, \qquad \textit{and } n_k = O(k).$$

By contrast, T is hypercyclic if and only if the same is true *for some* $(n_k)_k$, not necessarily of order $O(k)$. This seems to indicate that our new notion requires much more than mere hypercyclicity.

We have the usual behaviour under quasiconjugacies, which can be proved as in Proposition 1.19.

Proposition 9.4. *Frequent hypercyclicity is preserved under quasiconjugacy.*

Our first task will be to show that frequently hypercyclic operators exist. We saw above that an operator T on a separable Fréchet space X is frequently hypercyclic if one can find a Borel probability measure μ of full support on X with respect to which T is ergodic. However, in order to keep our introduction to frequent hypercyclicity simple we will not pursue this circle of ideas any further. Instead, we will favour a constructive approach to frequent hypercyclicity.

So, what does it take for a vector x to be frequently hypercyclic for an operator T? Let $\|\cdot\|$ denote an F-norm defining the topology of X, and let $(y_l)_l$ be a dense sequence in X. Then there are subsets $A(l,\nu)$, $l,\nu \geq 1$, of $\mathbb{N}_0$ of positive lower density such that, for any $n \in A(l,\nu)$,

$$\|T^n x - y_l\| < \frac{1}{\nu}.$$

Moreover, if $y_l \neq y_k$ then the sets $A(l,\nu)$ and $A(k,\mu)$ are disjoint if ν and μ are big. In fact, in the sequel we will need the existence of sets $A(l,\nu)$ with a stronger separation property.

Lemma 9.5. *There exist pairwise disjoint subsets $A(l,\nu)$, $l,\nu \geq 1$, of $\mathbb{N}_0$ of positive lower density such that, for any $n \in A(l,\nu)$ and $m \in A(k,\mu)$, we have that $n \geq \nu$ and*

$$|n-m| \geq \nu+\mu \quad \textit{if } n \neq m.$$

Proof. We start by partitioning $\mathbb{N}$ in a very natural fashion by using the dyadic representation

$$n = \sum_{j=0}^{\infty} a_j 2^j =: (a_0, a_1, a_2, \ldots)$$

of any positive integer n. We define $I(l,\nu)$, $l,\nu \geq 1$, as the set of all $n \in \mathbb{N}$ whose dyadic representation has the form

$$n = (0,\ldots,0,1,\ldots,1,0,*)$$

with $l-1$ leading zeros, followed by ν ones, then one zero, followed by an arbitrary tail. It is clear that the sets $I(l,\nu)$ form a partition of $\mathbb{N}$, but they do not satisfy the required separation property. To achieve this we let $\delta_k = \nu$ if $k \in I(l,\nu)$ for some $l \geq 1$, and we define

$$n_k = 2\sum_{i=1}^{k-1} \delta_i + \delta_k, \quad k \geq 1,$$

which is a strictly increasing sequence. We claim that

$$A(l,\nu) = \{n_k \ ; \ k \in I(l,\nu)\}, \quad l,\nu \geq 1$$

has the desired properties. First, these sets are pairwise disjoint. Moreover, if $n_k \in A(l,\nu)$ then $n_k \geq \delta_k = \nu$; and if $n_j \in A(l,\nu)$, $n_m \in A(k,\mu)$ with $n_j \neq n_m$, where we can assume that $j > m$, then

$$n_j - n_m = \delta_m + 2\sum_{i=m+1}^{j-1} \delta_i + \delta_j \geq \mu + \nu.$$

It remains to show that each set $A(l,\nu)$ has positive lower density. We begin by proving that there is some $M > 0$ such that

$$n_k \leq Mk, \quad k \geq 1. \tag{9.2}$$

It suffices to do this for $k = 2^N$, $N \geq 1$, because we then have for $2^{N-1} \leq k < 2^N$ that

$$n_k \leq n_{2^N} \leq M2^N \leq 2Mk.$$

Thus let $k = 2^N$. A simple but tedious enumeration shows that, if $l + \nu \leq N + 2$, then $I(l, \nu)$ contains at most $2^{N+2-l-\nu}$ elements that do not exceed 2^N, and none if $l + \nu > N + 2$. Hence we have that

$$n_{2^N} \leq 2\sum_{i=1}^{2^N} \delta_i \leq 2 \sum_{l+\nu \leq N+2} 2^{N+2-l-\nu}\nu \leq \Big(8 \sum_{l,\nu\geq 1} \frac{\nu}{2^{l+\nu}}\Big)2^N,$$

so that (9.2) holds for some $M > 0$.

Now let $l, \nu \geq 1$. Let $(k_j)_j$ be the increasing sequence of elements of $I(l, \nu)$. Since the latter set has positive lower density, the argument leading up to Proposition 9.3 shows that there is some constant $K > 0$ such that

$$k_j \leq Kj, \quad j \geq 1.$$

It then follows that $A(l, \nu) = \{n_{k_j}\ ;\ j \geq 1\}$ and

$$n_{k_j} \leq Mk_j \leq MKj, \quad j \geq 1.$$

Hence each set $A(l, \nu)$ has positive lower density. □

This result allows us to obtain a first example of a frequently hypercyclic operator.

Example 9.6. **(Birkhoff's operators)** The translation operators $T_a : f \to f(\cdot + a)$, $a \neq 0$, on the space $H(\mathbb{C})$ of entire functions are frequently hypercyclic. By Proposition 9.4 and Example 4.26 it suffices to consider $a = 1$.

Thus, let $A(l, \nu)$, $l, \nu \geq 1$, be subsets of $\mathbb{N}_0$ as given by Lemma 9.5, and let $(P_l)_l$ be a dense sequence of polynomials. Let $(n_k)_k$ be the increasing sequence of elements of $\bigcup_{l,\nu\geq 1} A(l, \nu)$. If $n_k \in A(l, \nu)$ then we define B_k as the closed ball around n_k of radius $r_k := \nu/2$, and on this ball we consider the function $g_k := P_l(z - n_k)$; see Figure 9.1. It follows from the lemma that

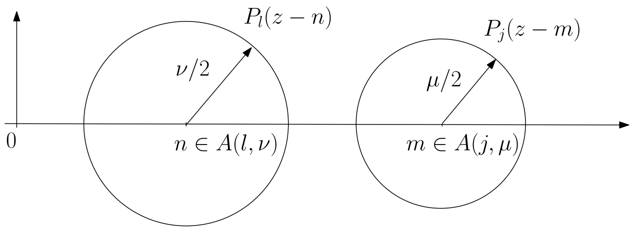

Fig. 9.1 Constructing Birkhoff frequently hypercyclic functions

the B_k are pairwise disjoint. We now apply Runge's theorem recursively. We start with $f_1 = g_1$. If entire functions $f_1, \ldots, f_k$, $k \geq 1$, have been constructed then we consider the function that is defined as f_k on $|z| \leq n_k + r_k$ and as g_{k+1} on B_{k+1}. Let $\varepsilon_k > 0$ be numbers that will be specified later. By Runge's approximation theorem there is an entire function f_{k+1} such that

$$\sup_{|z|\leq n_k+r_k} |f_{k+1}(z) - f_k(z)| < \varepsilon_k \quad \text{and} \quad \sup_{z\in B_{k+1}} |f_{k+1}(z) - g_{k+1}(z)| < \varepsilon_k.$$

If $\sum_{k=1}^{\infty} \varepsilon_k < \infty$ then it follows from the first inequality and the fact that $n_k \to \infty$ that

$$f(z) := f_1(z) + \sum_{k=1}^{\infty}(f_{k+1}(z) - f_k(z)) = \lim_{k\to\infty} f_k(z)$$

defines an entire function. Moreover we have with $\varepsilon_0 = 0$ that

$$\begin{aligned}\sup_{z\in B_k} |f(z) - g_k(z)| &\leq \sup_{z\in B_k} |f_k(z) - g_k(z)| + \sum_{j=k}^{\infty} \sup_{z\in B_k} |f_{j+1}(z) - f_j(z)| \\ &\leq \sum_{j=k-1}^{\infty} \varepsilon_j,\end{aligned}$$

that is,

$$\sup_{|z-n_k|\leq\nu/2} |f(z) - P_l(z - n_k)| \leq \sum_{j=k-1}^{\infty} \varepsilon_j$$

for $n_k \in A(l,\nu)$. It is easy to see that we can choose the ε_j in such a way that $\sum_{j=k-1}^{\infty} \varepsilon_j < \frac{1}{\nu}$ whenever $n_k \in A(l,\nu)$. We therefore have that

$$\sup_{|z|\leq\nu/2} |T_1^{n_k} f(z) - P_l(z)| < \frac{1}{\nu}$$

for $n_k \in A(l,\nu)$. Since the sets $\{g \in H(\mathbb{C})\ ;\ \sup_{|z|\leq\nu/2} |g(z) - P_l(z)| < \frac{1}{\nu}\}$, $l, \nu \geq 1$, form a basis of the topology of $H(\mathbb{C})$ and since each set $A(l,\nu)$ has positive lower density, it follows that T_1 is frequently hypercyclic.

It is of interest to compare the new, and strong, form of hypercyclicity with other strong forms such as weak mixing, mixing and chaos. We start here by showing that every frequently hypercyclic operator is weakly mixing. For the proof we need a property of sets of positive lower density. For any subset A of $\mathbb{N}_0$ its *difference set* is defined as

$$A - A = \{n - m\ ;\ n, m \in A, n \geq m\};$$

it should be noted that we consider here only nonnegative differences. We recall that a subset B of $\mathbb{N}_0$ is called syndetic if its complement does not contain intervals of arbitrary length; one also says that it has bounded gaps.

Theorem 9.7 (Erdős–Sárközy). *Let $A \subset \mathbb{N}_0$ be a set of positive lower density. Then the difference set $A - A$ is syndetic.*

Proof. Suppose that the difference set D of A is not syndetic. In particular, there exists some $n_1 \notin D$. Moreover, since $\mathbb{N}_0 \setminus D$ contains intervals of arbitrary length, there is some $n_2 \notin D$ such that also $n_2 + n_1 \notin D$. Hence $\{n_1, n_2, n_1 + n_2\} \subset \mathbb{N}_0 \setminus D$. Similarly, there is some $n_3 \notin D$ such that $n_3+n_1, n_3+n_2 \notin D$, which implies that $\{n_1, n_2, n_3, n_1+n_2, n_1+n_3, n_2+n_3\} \subset \mathbb{N}_0 \setminus D$. Continuing in this way we obtain a sequence $(n_k)_k$ in $\mathbb{N}_0$ such that any finite sum of elements in the sequence belongs to $\mathbb{N}_0 \setminus D$.

We now fix a positive integer m such that $\underline{\text{dens}}(A) > \frac{1}{m}$, and we consider the sets

$$A_k = A + (n_1 + \ldots + n_k), \quad k \in \mathbb{N}.$$

Since each set A_k also has lower density larger than $\frac{1}{m}$, there is some $N \geq 1$ such that, for any $k \leq m$,

$$\text{card}\{n \leq N \; ; \; n \in A_k\} > \frac{N+1}{m}.$$

If the A_k, $k = 1, \ldots, m$, were pairwise disjoint, we would have that

$$\text{card}\{n \leq N \; ; \; n \in A_1 \cup \ldots \cup A_m\} > m\frac{N+1}{m} = N + 1,$$

which is impossible. Hence there are $j < k$ with $A_j \cap A_k \neq \varnothing$, which implies that

$$n_{j+1} + \ldots + n_k \in A - A = D.$$

This contradicts the construction of the n_k. □

With this we can prove the announced result.

Theorem 9.8. *Any frequently hypercyclic operator on a Fréchet space is weakly mixing.*

Proof. Let T be a frequently hypercyclic operator on a Fréchet space X. We want to show that the condition of Theorem 2.47 is satisfied. Thus, let W be a 0-neighbourhood and U and V nonempty open subsets of X.

First, since T is hypercyclic and therefore topologically transitive, there is some $n_0 \geq 0$ such that $T^{n_0}(U) \cap W \neq \varnothing$. By continuity there is a nonempty open subset U_0 of U such that $T^{n_0}(U_0) \subset W$. Now let x be an arbitrary frequently hypercyclic vector for T. Then there is a set $A \subset \mathbb{N}_0$ of positive lower density such that

$$T^n x \in U_0 \;\text{ for any } n \in A.$$

For $m, n \in A$, $m \geq n$, we then find that

$$T^{n_0+m-n}(T^n x) = T^{n_0}(T^m x) \in W.$$

We thus have that

$$n_0 + (A - A) \subset N(U_0, W) \subset N(U, W).$$

It follows from Theorem 9.7 that $N(U, W)$ is syndetic.

Secondly, by continuity and linearity, each set $T^{-k}(W)$ is a 0-neighbourhood. Thus, given any $m \geq 1$, there is a 0-neighbourhood W_0 such that $T^k(W_0) \subset W$ for $k = 1, \ldots, m$. Again by topological transitivity there is some $K > m$ and some $y \in W_0$ such that $T^K y \in V$ and hence, for $1 \leq k \leq m$,

$$T^{K-k}(T^k y) \in T^{K-k}(W) \cap V.$$

This shows that, for any $m \geq 1$, $N(W, V)$ contains m consecutive integers.

Our two conclusions imply that $N(U, W) \cap N(W, V) \neq \varnothing$, so that, by Theorem 2.47, T is weakly mixing. $\square$

9.2 The Frequent Hypercyclicity Criterion

In order to obtain further examples of frequently hypercyclic operators, we derive here a sufficient condition for frequent hypercyclicity that resembles the Hypercyclicity Criterion. Its proof is inspired by Kitai's constructive approach to that criterion; see the alternative proof of Theorem 3.12. However, at a crucial point we have to depart from that proof: since we require approximation on sets of positive lower density we can no longer define the k_j inductively. On the other hand, fixing the k_j in advance is no option either as they will necessarily depend on the chosen operator. Lemma 9.5 provides us with exactly the right tool for the construction.

We refer to Appendix A for the notion of unconditionally convergent series.

Theorem 9.9 (Frequent Hypercyclicity Criterion). *Let T be an operator on a separable Fréchet space X. If there is a dense subset X_0 of X and a map $S : X_0 \to X_0$ such that, for any $x \in X_0$,*

(i) $\sum_{n=0}^{\infty} T^n x$ *converges unconditionally,*

(ii) $\sum_{n=0}^{\infty} S^n x$ *converges unconditionally,*

(iii) $TSx = x$,

then T is frequently hypercyclic.

Proof. Since X is separable we can choose a sequence $(y_j)_j$ from X_0 that is dense in X. Let $\|\cdot\|$ denote an F-norm that defines the topology of X. Conditions (i) and (ii) imply that there are $N_l \in \mathbb{N}$, $l \geq 1$, such that, for any $j \leq l$ and any finite set $F \subset \{N_l, N_l+1, N_l+2, \ldots\}$ we have that

$$\Big\| \sum_{n\in F} T^n y_j \Big\| < \frac{1}{l2^l}, \tag{9.3}$$

$$\Big\| \sum_{n\in F} S^n y_j \Big\| < \frac{1}{l2^l}. \tag{9.4}$$

Now let $A(l,\nu)$, $l,\nu \geq 1$, be subsets of $\mathbb{N}_0$ as given by Lemma 9.5. We set

$$A = \bigcup_{l=1}^{\infty} A(l, N_l)$$

and

$$z_n = y_l \quad \text{if} \quad n \in A(l, N_l).$$

We then consider

$$x = \sum_{n\in A} S^n z_n. \tag{9.5}$$

First we want to verify that this series converges unconditionally. Let us fix $l \geq 1$. For any finite set $F \subset \mathbb{N}_0$ we have that

$$\sum_{\substack{n\in A\\ n\in F}} S^n z_n = \sum_{j=1}^{\infty} \sum_{\substack{n\in A(j,N_j)\\ n\in F}} S^n y_j = \sum_{j=1}^{l} \sum_{\substack{n\in A(j,N_j)\\ n\in F}} S^n y_j + \sum_{j=l+1}^{\infty} \sum_{\substack{n\in A(j,N_j)\\ n\in F}} S^n y_j.$$

It follows from (9.4) that, for $j \leq l$ and $F \subset \{N_l, N_l+1, N_l+2, \ldots\}$ finite,

$$\Big\| \sum_{\substack{n\in A(j,N_j)\\ n\in F}} S^n y_j \Big\| < \frac{1}{l2^l};$$

moreover, since $n \geq N_j$ for any $n \in A(j, N_j)$ by Lemma 9.5, we also have by (9.4) that, for any $j \geq 1$ and any finite set F,

$$\Big\| \sum_{\substack{n\in A(j,N_j)\\ n\in F}} S^n y_j \Big\| < \frac{1}{j2^j} \leq \frac{1}{2^j}.$$

Altogether we have that, for any finite set $F \subset \{N_l, N_l+1, N_l+2, \ldots\}$,

$$\Big\| \sum_{\substack{n\in A\\ n\in F}} S^n z_n \Big\| < \sum_{j=1}^{l} \frac{1}{l2^l} + \sum_{j=l+1}^{\infty} \frac{1}{2^j} = \frac{2}{2^l}.$$

Since l was arbitrary we have proved that the series (9.5) converges unconditionally.

We now show that x is frequently hypercyclic for T. To this end, fix $l \geq 1$. Then, for $n \in A(l, N_l)$,

$$T^n x - y_l = \sum_{\substack{k \in A \\ k < n}} T^n S^k z_k + \sum_{\substack{k \in A \\ k > n}} T^n S^k z_k + T^n S^n z_n - y_l.$$

For the second sum we have, for any $m \geq n$, using condition (iii),

$$\sum_{\substack{k \in A \\ n < k \leq m}} T^n S^k z_k = \sum_{j=1}^{l} \sum_{\substack{k \in A(j, N_j) \\ n < k \leq m}} S^{k-n} y_j + \sum_{j=l+1}^{\infty} \sum_{\substack{k \in A(j, N_j) \\ n < k \leq m}} S^{k-n} y_j.$$

Note that, by Lemma 9.5, $k - n \geq N_l$ in the first sum and $k - n \geq N_j$ in the second sum. Therefore, the same argument as above shows that

$$\Big\| \sum_{\substack{k \in A \\ n < k \leq m}} T^n S^k z_k \Big\| < \sum_{j=1}^{l} \frac{1}{l 2^l} + \sum_{j=l+1}^{\infty} \frac{1}{2^j} = \frac{2}{2^l},$$

hence

$$\Big\| \sum_{\substack{k \in A \\ k > n}} T^n S^k z_k \Big\| \leq \frac{2}{2^l}.$$

In the same way, but using (9.3) instead of (9.4), we obtain that also

$$\Big\| \sum_{\substack{k \in A \\ k < n}} T^n S^k z_k \Big\| \leq \frac{2}{2^l}.$$

Finally, since $n \in A(l, N_l)$ we have that

$$T^n S^n z_n = y_l.$$

Altogether we find that for all $n \in A(l, N_l)$

$$\|T^n x - y_l\| \leq \frac{4}{2^l}.$$

Since the y_l form a dense set in X and since each set $A(l, N_l)$ is of positive lower density we conclude that x is frequently hypercyclic for T. □

Remark 9.10. For a later application we note that the same proof works when we replace conditions (ii) and (iii) by the following:

For any $x \in X_0$ there is a sequence $(u_n)_{n\geq 0}$ in X with $u_0 = x$ such that $\sum_{n=0}^{\infty} u_n$ converges unconditionally and $T^n u_k = u_{k-n}$ if $n \leq k$.

Let us also note here that the Frequent Hypercyclicity Criterion not only implies frequent hypercyclicity but also two other strong forms of hypercyclicity.

Proposition 9.11. *An operator on a separable Fréchet space that satisfies the Frequent Hypercyclicity Criterion is also chaotic and mixing.*

Proof. The mixing property follows immediately from Kitai's criterion.

As for chaos, we have from conditions (i) and (ii) that, for any $x \in X_0$ and $N \geq 1$,

$$y_{x,N} := \sum_{j=1}^{\infty} S^{jN}x + x + \sum_{j=1}^{\infty} T^{jN}x$$

converges in X. Moreover, by condition (iii), $T^N y_{x,N} = y_{x,N}$, and by (i) and (ii) we have that $y_{x,N} \to x$ as $N \to \infty$. Since X_0 is dense, the $y_{x,N}$ therefore form a dense set of periodic points for T. Knowing already that T is hypercyclic we deduce that T is even chaotic. □

In the last section we saw that Birkhoff's operators are frequently hypercyclic. The Frequent Hypercyclicity Criterion allows us to show that also the other two classical hypercyclic operators, the operators of MacLane and Rolewicz, are in fact frequently hypercyclic; see also Exercise 9.2.2.

Example 9.12. **(MacLane's operator)** The differentiation operator D on $H(\mathbb{C})$ is frequently hypercyclic. To see this we proceed as in Example 3.7. Let X_0 be the set of polynomials and S the operator $Sf(z) = \int_0^z f(\zeta)\,d\zeta$. Condition (i) of the Frequent Hypercyclicity Criterion is satisfied since any finite series converges unconditionally, and (iii) is trivial. For (ii) we need only consider the monomials, for which we find that $\sum_{n=0}^{\infty} S^n(z^k) = k!\sum_{n=0}^{\infty} \frac{1}{(k+n)!} z^{k+n}$, which converges uniformly and unconditionally on any compact set.

We study Rolewicz's operators in the broader context of general weighted shifts. We will use the notation and terminology of Section 4.1. In particular, we consider weighted (backward) shifts

$$B_w : (x_1, x_2, x_3, \ldots) \to (w_2x_2, w_3x_3, w_4x_4, \ldots),$$

where $w = (w_n)_n$ is a weight sequence. By e_n, $n \geq 1$, we denote the canonical unit sequences.

Proposition 9.13. *Let B_w be a weighted shift on a Fréchet sequence space X in which* $\operatorname{span}\{e_n \;;\; n \geq 1\}$ *is dense. If the series*

$$\sum_{n=1}^{\infty}\Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} e_n$$

converges unconditionally in X then B_w is frequently hypercyclic.

Proof. We apply the Frequent Hypercyclicity Criterion. We choose X_0 as the set of finite sequences, which is dense by assumption, and for S we consider the weighted forward shift $(x_1, x_2, x_3, \ldots) \to (0, x_1/w_2, x_2/w_3, \ldots)$. Then condition (i) holds because any finite series converges unconditionally, and condition (iii) is obvious. By linearity, we need to confirm (ii) only for the sequences e_k, $k \geq 1$. But then

$$\sum_{n=0}^{\infty} S^n e_k = \sum_{n=0}^{\infty} \frac{e_{k+n}}{w_{k+1}\cdots w_{k+n}} = \Big(\prod_{\nu=1}^{k} w_\nu\Big) \sum_{n=0}^{\infty}\Big(\prod_{\nu=1}^{k+n} w_\nu\Big)^{-1} e_{k+n},$$

which converges unconditionally by hypothesis. □

In particular, by Theorem 4.8 we have the following.

Corollary 9.14. *On a Fréchet sequence space X in which $(e_n)_n$ is an unconditional basis, every chaotic weighted shift is frequently hypercyclic.*

The result covers some interesting special cases.

Example 9.15. **(Rolewicz's operators)** For $\lambda \in \mathbb{K}$ we consider the multiples $T = \lambda B$ of the shift operator B with $|\lambda| > 1$. Then T is frequently hypercyclic on any Fréchet sequence space on which it is defined, in which $(e_n)_n$ is an unconditional basis and that contains the sequence $(1/\lambda^n)_n$. This includes, in particular, the spaces ℓ^p, $1 \leq p < \infty$, and c_0.

Example 9.16. (a) We follow Example 4.9(b) and consider weighted shifts $T = B_w$ on $H(\mathbb{C})$, or rather its corresponding sequence space. Since the sequences e_n, $n \geq 0$, correspond to the monomials z^n, B_w turns out to be frequently hypercyclic if $\sum_{n=1}^{\infty}(\prod_{\nu=1}^{n} w_\nu)^{-1} z^n$ converges unconditionally in $H(\mathbb{C})$, which is equivalent to

$$\lim_{n\to\infty}\Big(\prod_{\nu=1}^{n} |w_\nu|\Big)^{1/n} = \infty.$$

Since the differentiation operator D corresponds to the weights $w_n = n$, we also get a new proof that D is frequently hypercyclic.

(b) In the space $\omega = \mathbb{K}^{\mathbb{N}}$, every series $\sum_{n=1}^{\infty} a_n e_n$ converges unconditionally. As a consequence, every weighted shift is frequently hypercyclic on ω.

One is still far away from a characterization of frequently hypercyclic weighted shifts. We complement here the sufficient condition derived above by a necessary condition.

Proposition 9.17. *Let B_w be a weighted shift on a Fréchet sequence space X in which $(e_n)_n$ is an unconditional basis. If B_w is frequently hypercyclic then there exists a subset $A \subset \mathbb{N}_0$ of positive lower density such that*

$$\sum_{n\in A} \Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} e_n \quad \textit{converges.}$$

Proof. Let x be a frequently hypercyclic vector for $T = B_w$. Since

$$T^n x = (w_2 w_3 \cdots w_{n+1} x_{n+1}, \ldots)$$

and since the projection onto the first coordinate is continuous on X, there is a set $B \subset \mathbb{N}_0$ of positive lower density such that, for any $n \in B$,

$$|w_2 w_3 \cdots w_{n+1} x_{n+1} - 2| < 1,$$

hence

$$|x_{n+1}| > \frac{1}{|w_2 w_3 \cdots w_{n+1}|}.$$

Together with the unconditional convergence of $\sum_{n=1}^{\infty} x_n e_n$ this implies that

$$\sum_{n\in B} \frac{1}{w_1 w_2 \cdots w_{n+1}}\, e_{n+1}$$

converges; see Theorem A.16. This proves the claim for $A = \{n+1\ ;\ n \in B\}$. □

We note that this condition is not, in general, a sufficient condition; see Exercise 9.2.5.

While at the outset it was not even clear if frequently hypercyclic operators exist, we have now actually seen that all the classical hypercyclic operators and many others have this strong form of hypercyclicity. Although it is not to be expected that hypercyclicity and frequent hypercyclicity coincide, we are also in the position to give an example that differentiates the two concepts.

Example 9.18. On $X = \ell^2$ we consider the weighted shift B_w with weights $w_n = (\frac{n+1}{n})^{1/2}$. It follows from Example 4.9(a) that B_w is hypercyclic and even mixing. However, if B_w were frequently hypercyclic then by the previous proposition we could find a set $A = \{n_k\ ;\ k \geq 1\}$ of positive lower density such that $\sum_{k=1}^{\infty} \frac{1}{n_k+1} < \infty$, which is impossible since $n_k = O(k)$; see the discussion before Proposition 9.3. Note that B_w is conjugate to the shift operator B on the Bergman space A^2; see Example 4.4(b).

This example takes us back to the problem of comparing frequent hypercyclicity with other forms of hypercyclicity. We saw in Theorem 9.8 that every frequently hypercyclic operator is weakly mixing. Some other implications have turned out to be false. We have just seen that the mixing property does not imply frequent hypercyclicity. But there is also a frequently hypercyclic operator on c_0 that is neither mixing nor chaotic. In particular, by Proposition 9.11, this operator does not satisfy the Frequent Hypercyclicity Criterion. The construction of this example goes well beyond the scope of this book.

The reader may have noticed that in frequent hypercyclicity so far we have not made use of the Baire category theorem. This is unlike the situation in hypercyclicity where the existence of a hypercyclic vector was deduced from the fact that, in the sense of Baire category, there must be many of them; see the proof of the Birkhoff transitivity theorem. In fact, this procedure is ruled out in frequent hypercyclicity because, in general, the set $FHC(T)$ of frequently hypercyclic vectors for an operator T is only of first Baire category.

Proposition 9.19. *Let T be an operator on a Fréchet space X. If there is a dense set X_0 such that $T^n x \to 0$ for all $x \in X_0$ then $FHC(T)$ is of first Baire category. This is true, in particular, for all operators satisfying the Frequent Hypercyclicity Criterion.*

Proof. Let $\|\cdot\|$ be an F-norm defining the topology of X, and choose $\delta > 0$ such that $\{x \in X\ ;\ \|x\| > \delta\}$ is nonempty. Then every frequently hypercyclic vector for T belongs to the set

$$E := \{x \in X\ ;\ \underline{\mathrm{dens}}\{n \in \mathbb{N}_0\ ;\ \|T^n x\| \geq \delta\} > 0\}.$$

We have that

$$E = \bigcup_{k\geq 1} \bigcup_{M \geq 1} E_{k,M},$$

where

$$E_{k,M} = \bigcap_{N \geq M} \left\{x \in X\ ;\ \ \mathrm{card}\{n \leq N\ ;\ \|T^n x\| \geq \delta\} \geq \tfrac{N+1}{k}\right\}.$$

The continuity of T implies that the complement of $E_{k,M}$,

$$X \setminus E_{k,M} = \bigcup_{N \geq M} \left\{x \in X\ ;\ \ \mathrm{card}\{n \leq N\ ;\ \|T^n x\| < \delta\} > (N+1)(1 - \tfrac{1}{k})\right\},$$

is open, and it contains the dense set X_0. Hence each set $E_{k,M}$ is nowhere dense, so that E is of first Baire category. □

Thus one cannot argue as in the case of hypercyclicity (see Proposition 2.52), that any vector in the underlying space is the sum of two frequently hypercyclic vectors. Indeed, there are frequently hypercyclic operators T for which $X \neq FHC(T) + FHC(T)$; see Exercise 9.2.6. On the other hand, there are operators T for which the set $FHC(T)$ is sufficiently large to ensure that $X = FHC(T) + FHC(T)$; see Exercise 9.1.4.

We end this section with another interesting phenomenon. Chapter 11 will be devoted to the question of whether an uncountable family of hypercyclic operators on a given space can have a common hypercyclic vector. The answer is positive, for example, for the Rolewicz operators λB, $\lambda > 1$, on any of the spaces $X = \ell^p$, $1 \le p < \infty$, or c_0; see Example 11.11. The corresponding result is false, however, for frequent hypercyclicity.

Example 9.20. Let X be one of the spaces ℓ^p, $1 \le p < \infty$, or c_0. Then the Rolewicz operators λB, $\lambda > 1$, on X have no common frequently hypercyclic vector. Indeed, suppose that x was such a vector. By the proof of Proposition 9.17 it then follows that, for any $\lambda > 1$,

$$\delta_\lambda := \underline{\text{dens}}\{n \in \mathbb{N}_0 \ ; \ |\lambda^n x_{n+1} - 2| < 1\} > 0.$$

Since there are uncountably many λ, one can find a finite subset, $\lambda_1 < \lambda_2 < \ldots < \lambda_K$ say, such that

$$\sum_{k=1}^{K} \delta_{\lambda_k} > 2.$$

Let $\rho = \min_{1\le k<K} \frac{\lambda_{k+1}}{\lambda_k}$, and choose $M \in \mathbb{N}$ such that $\rho^M \ge 3$. We then have for N sufficiently large that, for any $k = 1, \ldots, K$,

$$\text{card}\{M \le n \le N \ ; \ |\lambda_k^n x_{n+1} - 2| < 1\} \ge \tfrac{1}{2}\delta_{\lambda_k} N.$$

Since $\sum_{k=1}^{K} \frac{1}{2}\delta_{\lambda_k} N > N$, the corresponding sets cannot be pairwise disjoint. Hence there are $1 \le k < l \le K$ and $n \ge M$ such that $|\lambda_k^n x_{n+1} - 2| < 1$ and $|\lambda_l^n x_{n+1} - 2| < 1$. Thus, $\lambda_k^n |x_{n+1}| > 1$ and $\lambda_l^n |x_{n+1}| < 3$, which implies that $\rho^M \le (\lambda_l/\lambda_k)^n < 3$, a contradiction.

9.3 An eigenvalue criterion for frequent hypercyclicity

By the Godefroy–Shapiro criterion, eigenvalues inside and outside the unit disk with many associated eigenvectors are useful for proving an operator to be hypercyclic. Additional eigenvectors to certain unimodular eigenvalues are responsible for chaos. We recall that an eigenvalue λ is called *unimodular* if $|\lambda| = 1$.

In this section we will see that, rather surprisingly, a large supply of eigenvectors to unimodular eigenvalues by itself may lead to hypercyclicity, and in some cases to frequent hypercyclicity. Let us only mention that the correct interpretation of largeness in this context was again motivated by ergodic theoretic considerations.

Suppose for the moment that T is an operator on a complex Fréchet space X whose eigenspaces to unimodular eigenvalues all have dimension at most one. One can then define an eigenvector field $E : \mathbb{T} \to X$ so that, for any

$\lambda \in \mathbb{T}$, $E(\lambda)$ is either an eigenvector to the eigenvalue λ, or 0. Since we want a large supply of eigenvectors we would demand that $\operatorname{span}\{E(\lambda)\ ;\ \lambda \in \mathbb{T}\}$ is dense in X, in which case E is called spanning. In order to capture the situation where eigenspaces are higher-dimensional one has to allow for a collection of eigenvector fields. In the sequel, J is a nonempty index set.

Definition 9.21. Let T be an operator on a complex Fréchet space X. Then a collection of functions $E_j : \mathbb{T} \to X$, $j \in J$, is called a *spanning eigenvector field associated to unimodular eigenvalues* if $E_j(\lambda) \in \ker(\lambda I - T)$ for any $\lambda \in \mathbb{T}$, $j \in J$, and

$$\operatorname{span}\{E_j(\lambda)\ ;\ \lambda \in \mathbb{T}, j \in J\} \quad \text{is dense in } X.$$

In addition, the vector field is said to be *continuous* (or C^2) if each function $E_j : \mathbb{T} \to X$, $j \in J$, is continuous (or C^2, respectively).

As usual, a function $E : \mathbb{T} \to X$, is called C^2 if it is twice continuously differentiable, where differentiation is defined as in the scalar-valued case.

We now have the announced eigenvalue criterion.

Theorem 9.22. *Let T be an operator on a complex separable Fréchet space.*

(a) *If T has a spanning continuous eigenvector field associated to unimodular eigenvalues then it is mixing and chaotic.*

(b) *If T has a spanning C^2-eigenvector field associated to unimodular eigenvalues then it is frequently hypercyclic.*

The proof is similar to that of Theorem 7.32. We will need the Riemann integral

$$\int_0^{2\pi} f(t)\,dt$$

for a continuous function $f : [0, 2\pi] \to X$; see Appendix A for details and basic properties.

Lemma 9.23. *Let X be a complex Fréchet space and $f : [0, 2\pi] \to X$ a continuous function.*

(a) **(Riemann–Lebesgue lemma)** *Then $\int_0^{2\pi} e^{int} f(t)\,dt \to 0$ as $n \to \pm\infty$.*

(b) *If f is twice continuously differentiable with $f(0) = f(2\pi)$ and $f'(0) = f'(2\pi)$ then $\sum_{n=0}^{\infty} \int_0^{2\pi} e^{int} f(t)\,dt$ and $\sum_{n=0}^{\infty} \int_0^{2\pi} e^{-int} f(t)\,dt$ converge unconditionally.*

Proof. Let $(p_k)_k$ be an increasing sequence of seminorms defining the topology of X.

(a) As in the proof of Lemma 7.31 one shows that, for any $k \geq 1$, $p_k\left(\int_0^{2\pi} e^{int} f(t)\,dt\right) \to 0$ as $n \to \pm\infty$. This implies the claim.

(b) Upon integrating by parts twice we obtain that, for any $k \geq 1$, $n \neq 0$,

$$p_k\left(\int_0^{2\pi} e^{int} f(t)\, dt\right) = p_k\left(-\frac{1}{n^2}\int_0^{2\pi} e^{int} f''(t)\, dt\right) \le \frac{1}{n^2}\int_0^{2\pi} p_k(f''(t))\, dt,$$

which implies the claim. □

We are now in a position to prove the eigenvalue criterion.

Proof of Theorem 9.22. (a) Let $(E_j)_{j\in J}$ be the given eigenvector field of T. Since each $E_j : \mathbb{T} \to X$ is continuous the integrals

$$x_{k,j} := \int_0^{2\pi} e^{ikt} E_j(e^{it})\, dt \in X, \quad k \in \mathbb{Z}, j \in J,$$

are defined. In order to apply Kitai's criterion we set

$$X_0 = Y_0 = \operatorname{span}\{x_{k,j}\ ;\ k \in \mathbb{Z}, j \in J\}.$$

We will use the Hahn–Banach theorem to show that this set is dense. Thus, let x^* be a continuous linear functional on X so that, for all $k \in \mathbb{Z}$, $j \in J$,

$$\langle x_{k,j}, x^*\rangle = \int_0^{2\pi} e^{ikt}\langle E_j(e^{it}), x^*\rangle\, dt = 0.$$

The functions $t \to \langle E_j(e^{it}), x^*\rangle$ are continuous and therefore belong to $L^2[0, 2\pi]$. Since $(\frac{1}{\sqrt{2\pi}} e^{ikt})_{k\in\mathbb{Z}}$ is an orthonormal basis in this Hilbert space, we deduce that, by continuity,

$$\langle E_j(e^{it}), x^*\rangle = 0 \quad \text{for all } t \in [0, 2\pi],\ j \in J.$$

Hence x^* vanishes on the set $\operatorname{span}\{E_j(\lambda)\ ;\ \lambda \in \mathbb{T}, j \in J\}$, which is dense by assumption, so that x^* itself must vanish. Thus $X_0 = Y_0$ is dense.

Now, since each $E_j(\lambda)$ is in the eigenspace of λ we have for any $k \in \mathbb{Z}$ and $j \in J$ that

$$T^n x_{k,j} = \int_0^{2\pi} e^{ikt} T^n E_j(e^{it})\, dt = \int_0^{2\pi} e^{i(k+n)t} E_j(e^{it})\, dt \to 0$$

as $n \to \infty$, as a result of the Riemann–Lebesgue lemma. By linearity, we conclude that $T_n x \to 0$ for all $x \in X_0$.

It would seem natural to define the mapping $S : Y_0 \to Y_0$ by

$$x_{k,j} = \int_0^{2\pi} e^{ikt} E_j(e^{it})\, dt \to \int_0^{2\pi} e^{i(k-1)t} E_j(e^{it})\, dt = x_{k-1,j},$$

followed by linear extension to Y_0. Since this may lead to a conflict if the $x_{k,j}$ are not linearly independent we apply, instead, the variant, Exercise 3.1.1, of Kitai's criterion. Thus, for any $y \in Y_0$, we consider a representation $y = \sum_{l=1}^m a_l x_{k_l,j_l}$ and define

$$u_n = \sum_{l=1}^{m} a_l x_{k_l-n,j_l}, \quad n \geq 0.$$

We then have $T^n u_n = y$ and, again by the Riemann–Lebesgue lemma, that $x_{k_l-n,j} \to 0$ as $n \to \infty$, so that $u_n \to 0$. We can therefore conclude that T is mixing.

Moreover, by continuity of the eigenvector field, also

$$\text{span}\{E_j(\lambda)\ ;\ j \in J, \lambda = e^{\alpha\pi i} \text{ for some } \alpha \in \mathbb{Q}\}$$

is dense in X, and each vector in this span is a periodic point for T. Consequently, T is chaotic.

(b) The proof follows the same lines, this time using Lemma 9.23(b) and the Frequent Hypercyclicity Criterion in the form of Remark 9.10. □

The eigenvalue criterion provides a new proof that the three classical hypercyclic operators are even frequently hypercyclic.

Example 9.24. **(Rolewicz' operators)** We consider the Rolewicz operators $T = \mu B$, $|\mu| > 1$, on one of the complex spaces $X = \ell^p$, $1 \leq p < \infty$, or c_0. Then

$$E : \mathbb{T} \to X, \quad \lambda \to (\lambda^n/\mu^n)_n$$

is an eigenvector field associated to unimodular eigenvalues. An elementary but tedious calculation shows that the field is C^2 (see Exercise 9.3.2), while the spanning property was proved in Example 3.2.

Concerning MacLane's and Birkhoff's operators we will show a much more general result, namely that Theorem 4.21 by Godefroy and Shapiro also holds for frequent hypercyclicity.

Theorem 9.25. *Suppose that $T : H(\mathbb{C}) \to H(\mathbb{C})$, $T \neq \lambda I$, is an operator that commutes with D, that is,*

$$TD = DT.$$

Then T is frequently hypercyclic.

Proof. Following the proof of Theorem 4.21 we can write $T = \varphi(D)$ with a nonconstant entire function φ of exponential type, which also implies that every function $e_\lambda(z) = e^{\lambda z}$, $\lambda \in \mathbb{C}$, is an eigenvector of T to the eigenvalue $\varphi(\lambda)$. Since $\varphi(\mathbb{C})$ is connected and dense (see Appendix A), there is a point $z \in \mathbb{C}$ with $w := \varphi(z) \in \mathbb{T}$; and since $\varphi(\mathbb{C})$ is open and the zeros of φ' are isolated points we can also achieve that $\varphi'(z) \neq 0$. Thus φ maps a neighbourhood of z conformally onto a neighbourhood U of w; let ψ be the inverse map, which is holomorphic. Fix a nontrivial closed subarc $\gamma \subset U$ of $\mathbb{T}$ containing w and a C^2-function $f : \mathbb{T} \to \mathbb{C}$ with $f(w) \neq 0$ that vanishes outside γ. It follows that $E : \mathbb{T} \to H(\mathbb{C})$ with $E(\lambda) = f(\lambda)e_{\psi(\lambda)}$ if $\lambda \in \gamma$ and $E(\lambda) = 0$, else, defines an eigenvector field associated to unimodular eigenvalues for T. It was shown in the proof of Lemma 2.34 that the function $\mathbb{C} \to H(\mathbb{C})$, $\lambda \to e_\lambda$ is, in fact, infinitely differentiable, so that E is a C^2-field. Finally, E is spanning by Lemma 2.34. Now the eigenvalue criterion for frequent hypercyclicity implies the result. □

As in the case of hypercyclicity one may ask how slowly a frequently hypercyclic entire function can grow at infinity. The eigenvalue criterion allows us to deduce corresponding results for any operator $T = \varphi(D)$; see Exercise 9.3.3. Here we consider only the special case of Birkhoff's operators $T_a f(z) = f(z+a)$, $a \neq 0$. The theorem of Duyos-Ruiz tells us that corresponding hypercyclic functions can grow arbitrarily slowly. This is no longer true in the frequent context.

Theorem 9.26. *Let $a \neq 0$.*

(a) *Let $\varepsilon > 0$. Then there exists an entire function f that is frequently hypercyclic for T_a and that satisfies*

$$|f(z)| \leq M e^{\varepsilon r} \quad \textit{for } |z| = r > 0$$

with some $M > 0$.

(b) *Let $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ be a function with* $\liminf_{r\to\infty} \varepsilon(r) = 0$. *Then there is no entire function f that is frequently hypercyclic for T_a and that satisfies*

$$|f(z)| \leq M e^{\varepsilon(r) r} \quad \textit{for all } |z| = r > 0 \textit{ sufficiently large}$$

with some $M > 0$.

Proof. (a) This result follows from a general growth result for all operators that commute with D (see Exercise 9.3.3) because $T_a = e^{aD}$ and $e^{az} = 1$ for $z = 0$.

(b) We will assume that $a = 1$; see Example 4.26. Suppose, on the contrary, that f is a frequently hypercyclic entire function with the stated growth condition; by adding a constant, if necessary, we can assume that $f(0) = 1$. Then there is a strictly increasing sequence $(n_k)_k$ of positive integers with $n_k = O(k)$ such that, for any $k \geq 1$,

$$|f(z + n_k) - z| < \frac{1}{2} \text{ for } |z| \leq \frac{1}{2}.$$

Thus, by Rouché's theorem (see Appendix A), f has a zero in $|z - n_k| < \frac{1}{2}$. If $N(r)$ denotes the number of zeros of f in $|z| < r$, counting multiplicity, then

$$N(n_k + 1) \geq k, \quad k \geq 1.$$

On the other hand, it follows from Jensen's formula (see Theorem A.23) and the growth assumption on f that

$$N(r)\log 2 \le \log M + \varepsilon(2r)2r$$

for r sufficiently large.

Now let $r_\nu \to \infty$ be such that $\varepsilon(2r_\nu) \to 0$ as $\nu \to \infty$. For sufficiently large ν choose k_ν such that $n_{k_\nu} + 1 \le r_\nu \le n_{k_\nu+1}$. Altogether we conclude that

$$\frac{k_\nu}{n_{k_\nu+1}} \le \frac{N(n_{k_\nu}+1)}{n_{k_\nu+1}} \le \frac{N(r_\nu)}{r_\nu} \le \frac{\log M + \varepsilon(2r_\nu)2r_\nu}{r_\nu \log 2} \to 0,$$

hence that $\frac{n_{k_\nu+1}}{k_\nu+1} = \frac{k_\nu}{k_\nu+1}\frac{n_{k_\nu+1}}{k_\nu} \to \infty$, which is a contradiction. □

9.4 Structural properties

In the previous chapters we derived various structural properties of hypercyclicity. In this section we revisit several of them in the context of frequent hypercyclicity.

We begin by looking at the main results of Chapter 6. Ansari's theorem says that every power T^p, $p \ge 1$, of a hypercyclic operator is again hypercyclic; in fact, T and T^p have the same hypercyclic vectors. For frequent hypercyclicity we have the corresponding property, but its proof relies on very different techniques than Ansari's theorem.

Theorem 9.27. *Let T be an operator on a Fréchet space. Then, for any $p \in \mathbb{N}$, $FHC(T) = FHC(T^p)$. In particular, if T is frequently hypercyclic then so is every power T^p.*

Proof. Since every orbit $\mathrm{orb}(x, T^p)$ is obtained from the orbit $\mathrm{orb}(x, T)$ by retaining only the powers $T^{np}x$, $n \ge 0$, it is clear that every frequently hypercyclic vector for T^p is also frequently hypercyclic for T.

Conversely, let $x \in X$ be a frequently hypercyclic vector for T and $p \ge 1$. In order to show that x is also frequently hypercyclic for T^p we fix a nonempty open subset U of X. Since the sequence $(kp-1)_{k\ge1}$ is syndetic, we can deduce from Theorems 9.8 and 1.54 that there is some $m_1 \ge 0$ of the form $m_1 = k_1p - 1$ such that $U_1 := U \cap T^{-m_1}(U) \ne \varnothing$. For the same reason, there is some $m_2 \ge 0$ of the form $m_2 = k_2p-2$ such that $U_2 := U_1 \cap T^{-m_2}(U_1) \ne \varnothing$. Proceeding inductively we find, for $j = 1, \dots, p-1$, integers $m_j \ge 0$ of the form $m_j = k_jp - j$ such that $U_j := U_{j-1} \cap T^{-m_j}(U_{j-1}) \ne \varnothing$, where $U_0 := U$. Moreover we set $k_0 = 0$.

Now let $V = U_{p-1}$, which clearly satisfies $V \subset U$ and $T^{k_jp-j}(V) \subset U$, for $j = 0, 1, \dots, p-1$. Since x is frequently hypercyclic there is a subset $A \subset \mathbb{N}_0$ of positive lower density such that $T^n x \in V$ for all $n \in A$. We then define the function $f : \mathbb{N}_0 \to \mathbb{N}_0$ by $f(n) = \frac{n-j}{p} + k_j$ if $n = j \,(\mathrm{mod}\, p)$, $j = 0, \dots, p-1$; note that this is well defined.

We finally set $B = f(A)$. It is easy to show that $\underline{\mathrm{dens}}(B) \ge \underline{\mathrm{dens}}(A) > 0$; see Exercise 9.4.1. Moreover, if $m \in B$, then $m = \frac{n-j}{p} + k_j$ for some $n \in A$

with $n = j \pmod p$, and hence

$$(T^p)^m x = T^{n-j+k_j p} x = T^{k_j p-j}(T^n x) \in T^{k_j p-j}(V) \subset U.$$

This proves that x is frequently hypercyclic for T^p. $\square$

We saw in Section 6.3 how Ansari's theorem follows from the fact that if the union of the orbits of finitely many vectors is dense then one of these orbits must already be dense. The corresponding result fails for frequent hypercyclicity.

Example 9.28. We consider the Rolewicz operator $T = 2B$ on ℓ^1. We claim that there are two vectors $v, w \in \ell^1$ such that, for any nonempty open subset $U \subset \ell^1$,

$$\underline{\text{dens}}\{n \in \mathbb{N}_0 \ ; \ T^n v \in U \text{ or } T^n w \in U\} > 0, \tag{9.6}$$

but neither v nor w is frequently hypercyclic for T.

To see this, let $(y_j)_j$ be a dense sequence in ℓ^1 consisting of finite sequences. The proof of the Frequent Hypercyclicity Criterion, with S being half the forward shift, then constructs a frequently hypercyclic vector x for T given by

$$x = \sum_{n \in A} S^n z_n,$$

where A is the union of certain pairwise disjoint sets $A(l, N_l)$, $l \geq 1$, of positive lower density, and $z_n = y_l$ for $n \in A(l, N_l)$. For an increasing sequence $(m_k)_{k \geq 0}$ of positive integers with $m_0 = 0$, that will be determined later, we split the set A into two subsets

$$B = \{n \in A \ ; \ \exists\, k \geq 0 : m_{2k} \leq n < m_{2k+1}\},$$
$$C = \{n \in A \ ; \ \exists\, k \geq 0 : m_{2k+1} \leq n < m_{2k+2}\}.$$

The proof of the Frequent Hypercyclicity Criterion then shows that the series

$$v := \sum_{n \in B} S^n z_n, \quad w := \sum_{n \in C} S^n z_n$$

converge and that, for any $n \in A(l, N_l)$,

$$\|T^n v - y_l\| \leq \frac{4}{2^l} \quad \text{or} \quad \|T^n w - y_l\| \leq \frac{4}{2^l}$$

depending on whether $n \in B$ or $n \in C$. This implies that the joint orbits of v and w are frequently recurrent, in the sense of (9.6).

On the other hand, let l_n be the length of the finite sequence z_n, $n \in A$. Let $k \geq 0$. If $n \in \mathbb{N}_0$ satisfies

$$M_{2k+1} := \max_{\nu \in B, \nu < m_{2k+1}} (\nu + l_\nu) \leq n < m_{2k+2}$$

then the sequence $T^n v$ starts with a 0 and hence $T^n v \notin U$, where $U = \{x \in \ell^1 \; ; \; \|x - e_1\| < 1\}$. Now, if we choose the m_{2k+2}, $k \geq 1$, such that

$$\frac{M_{2k+1}}{m_{2k+2}} \leq \frac{1}{k}$$

then $\underline{\text{dens}}\{n \geq 0 \; ; \; T^n v \in U\} = 0$, which shows that v is not frequently hypercyclic for T. Imposing, in addition, a similar condition on the m_{2k+1} one can also achieve that w is not frequently hypercyclic for T.

For a variant of the Bourdon–Feldman theorem for frequent hypercyclicity see Exercise 9.4.4.

We next turn to the results of Section 6.4. For this we need to define frequent hypercyclicity for C_0-semigroups. The *lower density* of a measurable subset $A \subset \mathbb{R}_+$ is given by

$$\underline{\text{dens}}(A) := \liminf_{T\to\infty} \frac{\lambda\{t \in [0,T] \; ; \; t \in A\}}{T},$$

where λ denotes the Lebesgue measure.

Definition 9.29. A C_0-semigroup $(T_t)_{t\geq 0}$ on a Banach space X is called *frequently hypercyclic* if there is a vector $x \in X$ such that, for any nonempty open subset U of X,

$$\underline{\text{dens}}\,\{t \in \mathbb{R}_+ \; ; \; T_t x \in U\} > 0.$$

In this case, x is called a *frequently hypercyclic vector* for $(T_t)_{t\geq 0}$.

As before we will treat the problems of unimodular multiples and of discretizations of semigroups within the common framework of semigroup actions; see Section 6.4. We recall that if T is an operator on a complex Fréchet space X then

$$\Psi(n,t) = e^{2\pi t i} T^n, \quad n \in \mathbb{N}_0, \; t \geq 0, \tag{9.7}$$

defines a semigroup action. Similarly, if $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X then

$$\Psi(n,t) = T_t, \quad n \in \mathbb{N}_0, \; t \geq 0, \tag{9.8}$$

defines a semigroup action. In both cases, properties (α) and (β) of Section 6.4 are satisfied.

We then also need a concept of frequent hypercyclicity for semigroup actions. The natural notion of *lower density* on $G = \mathbb{N}_0 \times \mathbb{R}_+$ is given by

$$\underline{\text{dens}}(A) := \liminf_{N\to\infty} \frac{1}{N(N+1)} \sum_{n=0}^{N} \lambda\{t \in [0,N] \; ; \; (n,t) \in A\},$$

where $A \subset G$ is such that $\{t \geq 0 \; ; \; (n,t) \in A\}$ is measurable for each $n \geq 0$; λ denotes the Lebesgue measure.

Definition 9.30. A semigroup action $\Psi : G \to L(X)$ is called *frequently hypercyclic* if there is some $x \in X$ such that, for any nonempty open subset U of X,

$$\underline{\mathrm{dens}}\,\{g \in G\ ;\ \Psi(g)x \in U\} > 0.$$

In this case, x is called a *frequently hypercyclic vector* for Ψ.

Now, frequent hypercyclicity of some operator $\Psi(n,t)$, $n,t > 0$, implies frequent hypercyclicity of Ψ.

Proposition 9.31. *Let Ψ be a semigroup action on a Fréchet space X satisfying property (α). If $x \in X$ is frequently hypercyclic for some operator $\Psi(n,t)$, $n,t > 0$, then it is frequently hypercyclic for Ψ.*

Proof. Let U be a nonempty open subset of X. Since $\Psi(0,0) = I$ and Ψ is continuous, there is a nonempty open subset V of U and some $\eta > 0$ such that $\Psi(0,s)V \subset U$ if $0 \le s < \eta$. By assumption, there is some $(n,t) \in G$ such that $\underline{\mathrm{dens}}(A) = \delta > 0$, where $A = \{k \in \mathbb{N}_0\ ;\ \Psi(n,t)^k x \in V\}$. Now, if $k \in A$ and $0 \le s < \eta$ then $\Psi(kn, kt+s)x = \Psi(0,s)\Psi(n,t)^k x \in U$. In view of property (α), if $\Psi(1,0) = I$ then also $\Psi(kn+m, kt+s)x \in U$ for $m \in \mathbb{Z}$ with $kn+m \ge 0$; if $\Psi(0,1) = I$ then $\Psi(kn, kt+s+m)x \in U$ for $m \in \mathbb{Z}$ with $kt+s+m \ge 0$. In both cases a simple count reveals that $\underline{\mathrm{dens}}\{(k,s) \in G\ ;\ \Psi(k,s)x \in U\} \ge \eta\delta/\max(n,t) > 0$. □

Our main aim is to prove the converse statement. The following will be crucial.

Lemma 9.32. *Let Ψ be a semigroup action on an infinite-dimensional Fréchet space X satisfying properties (α) and (β). If $x \in X$ is frequently hypercyclic for Ψ then, for any $k \in \mathbb{N}$ and any nonempty open subset U of X, we have that*

$$\underline{\mathrm{dens}}\,\big\{(n,t) \in G\ ;\ \Psi(n,t)x \in U,\ t \in \textstyle\bigcup_{m=1}^{\infty}[m - \frac{1}{k}, m[\big\} > 0.$$

Proof. We fix $k \in \mathbb{N}$ and a nonempty open subset U of X. For $j = 1, \ldots, k$ we define the sets

$$I_j = \bigcup_{m=1}^{\infty}[m - \tfrac{j}{k}, m - \tfrac{j-1}{k}[.$$

By Theorem 6.10, x is hypercyclic for $\Psi(1,1)$. It follows from property (β) that, for any $t \ge 0$, also $\Psi(0,t)x$ is hypercyclic for $\Psi(1,1)$. Therefore there are $n_j \in \mathbb{N}_0$, $j = 1, \ldots, k$, such that

$$\Psi(n_j, n_j + \tfrac{j-1}{k})x = \Psi(1,1)^{n_j}\Psi(0, \tfrac{j-1}{k})x \in U.$$

By continuity there is a neighbourhood V of x such that

$$\Psi(n_j, n_j + \tfrac{j-1}{k})(V) \subset U, \quad j = 1, \ldots, k.$$

Let $N_0 = \max(n_1, \ldots, n_k) + 1$. It follows from frequent hypercyclicity of x for Ψ that there are $\delta > 0$ and $N_1 \geq N_0$ such that, if $N \geq N_1$, then

$$\frac{1}{N(N+1)} \sum_{n=0}^{N} \lambda\{t \in [0, N] \ ; \ \Psi(n,t)x \in V\} \geq \delta.$$

We now fix $N \geq N_1$. Since the I_j, $j = 1, \ldots, k$, form a partition of $\mathbb{R}_+$, there is some j such that

$$\sum_{n=0}^{N} \lambda\{t \in [0, N] \ ; \ \Psi(n,t)x \in V,\ t \in I_j\} \geq \frac{1}{k} \sum_{n=0}^{N} \lambda\{t \in [0, N] \ ; \ \Psi(n,t)x \in V\}.$$

We fix such a j. If $0 \leq n \leq N$, $0 \leq t \leq N$, $t \in I_j$, and $\Psi(n,t)x \in V$ then $\nu := n + n_j \leq 2N$, $\tau := t + n_j + \frac{j-1}{k} \leq 2N$, $\tau \in I_1$, and

$$\Psi(\nu,\tau)x = \Psi\left(n_j, n_j + \tfrac{j-1}{k}\right)\Psi\big(n,t\big)x \in \Psi\left(n_j, n_j + \tfrac{j-1}{k}\right)\big(V\big) \subset U.$$

We conclude that

$$\sum_{\nu=0}^{2N} \lambda\{\tau \in [0, 2N] \ ; \ \Psi(\nu,\tau)x \in U, \tau \in I_1\}$$
$$\geq \frac{1}{k} \sum_{n=0}^{N} \lambda\{t \in [0, N] \ ; \ \Psi(n,t)x \in V\},$$

so that

$$\frac{1}{2N(2N+1)} \sum_{\nu=0}^{2N} \lambda\{\tau \in [0, 2N] \ ; \ \Psi(\nu,\tau)x \in U,\ \tau \in I_1\} \geq \frac{\delta}{4k}.$$

Since $N \geq N_1$ was arbitrary, the claim follows. □

We can now prove the analogue of Theorem 6.10 for frequent hypercyclicity.

Theorem 9.33. *Let Ψ be a semigroup action on an infinite-dimensional Fréchet space X satisfying properties (α) and (β). If $x \in X$ is frequently hypercyclic for Ψ then it is frequently hypercyclic for every operator $\Psi(1,t)$, $t > 0$.*

Proof. We first prove the case when $t = 1$. Thus, let U be a nonempty open subset of X. Since $\Psi(0,0) = I$, continuity of Ψ implies that there is a nonempty open subset V of U and some $\eta > 0$ such that $\Psi(0,s)V \subset U$ if $0 \leq s < \eta$. Let $k \in \mathbb{N}$ be such that $\frac{1}{k} < \eta$. Then, by Lemma 9.32, there are $\delta > 0$ and $N_0 \in \mathbb{N}$ such that, for any $N \geq N_0$,

$$r := \sum_{n=0}^{N} \lambda\{t \in [0,N] \ ; \ \Psi(n,t)x \in V, \ t \in \textstyle\bigcup_{m=1}^{\infty}[m - \frac{1}{k}, m[\} \geq N(N+1)\delta.$$

Now, if $\Psi(n,t)x \in V$ and $t \in [m-\frac{1}{k}, m[$ then $\Psi(n,m)x = \Psi(0,m-t)\Psi(n,t)x \in U$. Thus, for

$$p := \text{card}\{(n,m) \ ; \ 0 \leq n,m \leq N, \Psi(n,m)x \in U\}$$

we have that $p\frac{1}{k} \geq r$.

Next, let $\Psi(1,1)^n x \in U$. We distinguish the two cases described by (α). If $\Psi(1,0) = I$ then $\Psi(m,n)x = \Psi(n,n)x = \Psi(1,1)^n x \in U$ for any $m \in \mathbb{Z}$; and if $\Psi(0,1) = I$ then $\Psi(n,m)x = \Psi(n,n)x \in U$ for any $m \in \mathbb{Z}$. Thus, for

$$q := \text{card}\{0 \leq n \leq N \ ; \ \Psi(1,1)^n x \in U\}$$

we have that $p = (N+1)q$.

Altogether we find that, for any $N \geq N_0$,

$$\frac{\text{card}\{0 \leq n \leq N \ ; \ \Psi(1,1)^n x \in U\}}{N+1} = \frac{p}{(N+1)^2} \geq \frac{kr}{(N+1)^2} \geq \frac{kN}{N+1}\delta.$$

Hence x is frequently hypercyclic for $\Psi(1,1)$.

Now, if $t > 0$ is arbitrary then we rescale the semigroup action as in the proof of Theorem 6.10. It is then not difficult to see, using property (α), that x is also frequently hypercyclic for $\widetilde{\Psi}$ and thus frequently hypercyclic for $\widetilde{\Psi}(1,1) = \Psi(1,t)$. □

If we combine Theorem 9.33 with Theorem 9.27, noting that $\psi(n,t) = \Psi(1,t/n)^n$, we obtain the announced converse of Proposition 9.31.

Corollary 9.34. *Let Ψ be a semigroup action on a Fréchet space X satisfying properties (α) and (β). If $x \in X$ is frequently hypercyclic for Ψ then it is frequently hypercyclic for every operator $\Psi(n,t)$, $n,t > 0$.*

Proposition 9.31 and Theorem 9.33, applied to the semigroup action (9.7), immediately imply a version of the León–Müller theorem for frequent hypercyclicity.

Theorem 9.35. *Let T be an operator on a complex Fréchet space and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Then T and λT have the same frequently hypercyclic vectors, that is, $FHC(T) = FHC(\lambda T)$.*

Similarly, applying Theorem 9.33 to the semigroup action (9.8) yields an analogue of the Conejero–Müller–Peris theorem.

Theorem 9.36. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a Banach space X. If $x \in X$ is frequently hypercyclic for $(T_t)_{t\geq 0}$, then it is frequently hypercyclic for every operator T_t, $t > 0$.*

Apart from being interesting in its own right, Theorem 9.35 has an important application. We saw in Chapter 5 that the spectrum of a hypercyclic operator on a complex Banach space has the property that each of its connected components meets the unit circle; this is the content of Kitai's theorem. In particular, the spectrum cannot have isolated points outside the unit circle. We will now show that the spectrum of frequently hypercyclic operators, just like that of chaotic operators (see Proposition 5.7), cannot even have isolated points on the unit circle.

We start with a crucial lemma whose proof uses complex analysis in a very clever way.

Lemma 9.37. *Let T be an operator on a real Fréchet space X. Let $x \in X$ and $x^* \in X^*$ with $\langle x, x^*\rangle \neq 0$ be such that*

$$|\langle (T-I)^n x, x^*\rangle|^{1/n} \to 0$$

as $n \to \infty$. Then x is not frequently hypercyclic for T.

Proof. First, we may assume that $\langle x, x^*\rangle = 1$. Suppose that x is frequently hypercyclic for T. Then $(\langle T^n x, x^*\rangle)_{n\geq 0}$ is dense in $\mathbb{R}$. Thus there must be some $n \geq 0$ such that $\langle T^n x, x^*\rangle \leq 0$ and $\langle T^{n+1}x, x^*\rangle > 0$. Then, for $\alpha > 0$ sufficiently small, $\langle T^n x - \alpha x, x^*\rangle < 0$ and $\langle T^{n+1}x - \alpha T x, x^*\rangle > 0$, so that the open set

$$U = \{y \in X \ ; \ \langle y, x^*\rangle < 0 \quad \text{and} \quad \langle Ty, x^*\rangle > 0\}$$

is nonempty.

We now consider the series

$$f(z) = \sum_{k=0}^{\infty} \langle (T-I)^k x, x^*\rangle \frac{z(z-1)\cdots(z-k+1)}{k!}, \quad z \in \mathbb{C},$$

where we regard the quotient as 1 if $k = 0$. We claim that this defines an entire function. Indeed, it follows from the assumption that, for any $\varepsilon \in]0,1[$, there is some $M > 0$ such that

$$|\langle (T-I)^n x, x^*\rangle| \leq M\varepsilon^n, \quad n \geq 0,$$

so that, for any $R > 0$ and $|z| \leq R$,

$$\begin{aligned}\sum_{k=0}^{\infty} |\langle (T-I)^k x, x^*\rangle| &\left|\frac{z(z-1)\cdots(z-k+1)}{k!}\right| \\ &\leq M \sum_{k=0}^{\infty} \varepsilon^k \frac{R(R+1)\cdots(R+k-1)}{k!} \\ &= M \sum_{k=0}^{\infty} \binom{-R}{k} (-\varepsilon)^k = \frac{M}{(1-\varepsilon)^R} < \infty,\end{aligned}$$

where we have used the binomial theorem. Moreover, setting $\eta = -\log(1-\varepsilon)$, this inequality implies that

$$|f(z)| \leq M e^{\eta|z|}, \quad z \in \mathbb{C}.$$

In addition, $f(0) = \langle x, x^* \rangle = 1$. It follows from Jensen's formula (see Theorem A.23) that if $N(r)$ denotes the number of zeros of f in $|z| < r$, counting multiplicity, then

$$N(r) \log 2 \leq \log M + 2r\eta, \quad r > 0. \tag{9.9}$$

On the other hand we have that for $n \in \mathbb{N}_0$,

$$\begin{aligned} f(n) &= \sum_{k=0}^{n} \langle (T-I)^k x, x^* \rangle \frac{n(n-1)\cdots(n-k+1)}{k!} \\ &= \Big\langle \sum_{k=0}^{n} \binom{n}{k} (T-I)^k x, x^* \Big\rangle = \langle T^n x, x^* \rangle. \end{aligned}$$

Thus, if $T^n x \in U$ then $f(n) < 0$ and $f(n+1) > 0$, so that f, being real on the real axis, has a zero in the interval $]n, n+1[$. It follows with (9.9) that

$$\frac{\operatorname{card}\{0 \leq n \leq m \,;\, T^n x \in U\}}{m+1} \leq \frac{N(m+1)}{m+1} \leq \frac{\log M + 2(m+1)\eta}{(m+1)\log 2} \to \frac{2\eta}{\log 2}$$

as $m \to \infty$. Since $\eta > 0$ is arbitrary, we deduce that x is not frequently hypercyclic. □

As an immediate consequence we have the following. Recall that an operator T on a Banach space is called quasinilpotent if $\|T^n\|^{1/n} \to 0$ as $n \to \infty$.

Lemma 9.38. *Let T be an operator on a Banach space X of the form $T = \lambda I + S$ with $|\lambda| = 1$ and S quasinilpotent. Then T is not frequently hypercyclic.*

Proof. By Theorem 9.35 we may assume that $\lambda = 1$, so that $\|(T-I)^n\|^{1/n} \to 0$ as $n \to \infty$. Moreover, we can regard X as a real Banach space and $T - I$ as a (real-linear) operator on X. We then have that, for any $x \in X$ and any (real-linear) continuous linear functional x^* on X,

$$|\langle (T-I)^n x, x^* \rangle|^{1/n} \leq \big(\|(T-I)^n\| \|x\| \|x^*\|\big)^{1/n} \to 0.$$

By the previous lemma, T cannot be frequently hypercyclic on X; note that this notion does not depend on the scalar field. □

We can now prove the mentioned spectral property of frequently hypercyclic operators.

Theorem 9.39. *Let T be a frequently hypercyclic operator on a complex Banach space. Then its spectrum $\sigma(T)$ has no isolated points.*

Proof. Suppose that $\lambda \in \mathbb{C}$ is an isolated point of the spectrum. Then $\sigma(T)$ can be partitioned into some closed subset and the singleton $\{\lambda\}$. By the Riesz decomposition theorem (see Appendix B) there are nontrivial T-invariant closed subspaces M_1 and M_2 of X such that $X = M_1 \oplus M_2$ and $\sigma(T|_{M_2}) = \{\lambda\}$. By Exercise 2.2.8 and Proposition 9.4, $T|_{M_2}$ is frequently hypercyclic. By Kitai's theorem we have that $|\lambda| = 1$, and the spectral radius formula (see Appendix B) implies that $T|_{M_2} = \lambda I + S$ with a quasinilpotent operator S. This contradicts Lemma 9.38. □

Lemma 9.38 has another application. First, combining it with Lemma 5.19 yields the following.

Proposition 9.40. *No compact perturbation of a multiple of the identity on a Banach space is frequently hypercyclic.*

We can then apply the Argyros–Haydon theorem; see Theorem 8.11.

Corollary 9.41. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then there exists an infinite-dimensional separable Banach space over $\mathbb{K}$ that supports no frequently hypercyclic operator.*

With this we end our introduction to frequent hypercyclicity.

Exercises

Exercise 9.1.1. Show that the Herrero–Bourdon theorem also holds for frequent hypercyclicity. In particular, every frequently hypercyclic operator on a Fréchet space admits a dense T-invariant subspace consisting, except for 0, of frequently hypercyclic vectors.

Exercise 9.1.2. Using Lemma 9.5, show that every weighted shift is frequently hypercyclic on the space $\omega = \mathbb{K}^{\mathbb{N}}$.

Exercise 9.1.3. Show that every frequently hypercyclic operator on a Fréchet space is topologically ergodic; see Exercise 1.5.6. Deduce that if T is a frequently hypercyclic operator on a Banach space then its adjoint T^* cannot be frequently hypercyclic. (*Hint*: Exercise 2.5.5, Remark 4.17.)

Exercise 9.1.4. Show that every entire function is the sum of two functions that are frequently hypercyclic for the translation operator $T_1 f(z) = f(z+1)$. (*Hint*: Use a variant of the construction in Example 9.6.)

Exercise 9.2.1. Let $T_n : X \to Y$, $n \geq 0$, be operators between separable Fréchet spaces X and Y. The definition of frequent hypercyclicity for the sequence $(T_n)_{n\geq 0}$ is obvious. Prove the following version of the Frequent Hypercyclicity Criterion for $(T_n)_n$. For the notion of uniformly unconditionally convergent series see Definition 11.7 below.

If there is a dense subset Y_0 of Y and maps $S_n : Y_0 \to X$, $n \geq 0$, such that, for any $y \in Y_0$,

(i) $\displaystyle\sum_{n=0}^{m} T_m S_{m-n} y$ converges unconditionally in Y, uniformly for $m \geq 0$,

(ii) $\sum_{n=0}^{\infty} T_m S_{m+n} y$ converges unconditionally in Y, uniformly for $m \geq 0$,

(iii) $\sum_{n=0}^{\infty} S_n y$ converges unconditionally in X,

(iv) $T_n S_n y \to y$, as $n \to \infty$,

then $(T_n)_n$ is frequently hypercyclic.

Note that, in (i), the finite sums can be understood as infinite series by adding 0 terms.

Exercise 9.2.2. Use the Frequent Hypercyclicity Criterion to give a new proof that Birkhoff's operators are frequently hypercyclic. (*Hint*: Example 3.8.)

Exercise 9.2.3. Formulate and prove an analogue of Proposition 9.13 for weighted bilateral shifts.

Exercise 9.2.4. Let B_w be a frequently hypercyclic weighted shift on ℓ^p, $1 \leq p < \infty$. Show that, for any $\varepsilon > 0$, there exists a subset $A \subset \mathbb{N}_0$ of positive lower density such that, for any $m \in A$,

$$\sum_{\substack{n \in A \\ n > m}} \frac{1}{|w_2 w_3 \cdots w_{n-m+1}|^p} < \varepsilon.$$

(*Hint*: Proceed as in the proof of Proposition 9.17 and consider the coordinates of index $n - m + 1$ in $B_w^m x - e_1$.)

Exercise 9.2.5. Let $N_j = 2j^2 - 2j + 1$, $j \geq 1$. Define w_n as $(\frac{n+1}{n})^2$ for $N_j \leq n < N_j + j$, as $(N_j + j)^{-2/j}$ for $N_j + j \leq n < N_j + 2j$, as 1 for $N_j + 2j \leq n < N_j + 3j$, and as $(N_{j+1})^{2/j}$ for $N_j + 3j \leq n < N_j + 4j = N_{j+1}$. Show that B_w is a weighted shift on ℓ^p, $1 \leq p < \infty$, that satisfies the condition given in Proposition 9.17 but not the condition in the previous exercise. Thus, the condition in Proposition 9.17 does not characterize frequent hypercyclicity of weighted shifts on ℓ^p. (*Hint*: Use the result by Erdős and Sárközy.)

Exercise 9.2.6. The aim of this exercise is to show that not every vector $x \in \ell^1$ is the sum of two frequently hypercyclic vectors for the Rolewicz operator $2B$. Suppose that $x = y + z$ with y and z frequently hypercyclic. Then there is an increasing sequence $(m_k)_k$ of positive integers such that $\|T^{m_k} y\| < 1$ for $k \geq 1$; further let $\underline{\mathrm{dens}}\{n \in \mathbb{N}_0 \ ; \ \|T^n z\| < 1\} =: 2\delta > 0$. Deduce that there are positive integers n_k such that $\|T^{n_k} z\| < 1$ and $\delta m_k \leq n_k \leq m_k$, $k \geq 1$ sufficiently large, and hence that $\|T^{m_k} x\| \leq 1 + 2^{(1-\delta) m_k}$. Finally find some $x \in \ell^1$ that fails this inequality for any $\delta > 0$ and any increasing sequence $(m_k)_k$ of positive integers.

Exercise 9.2.7. Generalize Example 9.20: let T be an operator on a Fréchet space X and $\Lambda \subset \,]0, \infty[$ an uncountable set such that λT is frequently hypercyclic for any $\lambda \in \Lambda$. Show that these operators have no common frequently hypercyclic vector. (*Hint*: Consider $U = \{x \in X \ ; \ |\langle x, x^* \rangle - 2| < 1\}$.)

Exercise 9.3.1. Let $L^p(\mathbb{T})$, $1 \leq p < \infty$, be the space of all complex-valued functions f on $\mathbb{T}$ such that $\|f\|_p := (\int_0^{2\pi} |f(e^{it})|^p \, dt)^{1/p} < \infty$. Show that $Tf(\lambda) = \lambda f(\lambda) - \int_{(1,\lambda)} f(\zeta) \, d\zeta$ defines a mixing and chaotic operator on $L^p(\mathbb{T})$, where (λ_1, λ_2) denotes the positively oriented arc from λ_1 to λ_2. (*Hint*: Consider the indicator functions $f = \mathbf{1}_{(\lambda,1)}$.)

Exercise 9.3.2. Let X be one of the complex spaces ℓ^p, $1 \leq p < \infty$, or c_0. Show that the map $\mathbb{D} \to X$, $\lambda \to (\lambda^n)_n$, is infinitely differentiable. Deduce that also the maps $\mathbb{D} \to H^2$, $\lambda \to k_{\overline{\lambda}}$ (see Proposition 4.38) and $\mathbb{D}_\tau \to E^2_\tau$, $\lambda \to e_\lambda$ (see Exercise 4.2.4) are infinitely differentiable.

Exercise 9.3.3. Let φ be a nonconstant entire function of exponential type and $A = \min\{|z| \;;\; z \in \mathbb{C}, |\varphi(z)| = 1\}$. Show that, for any $\varepsilon > 0$, there is an entire function f that is frequently hypercyclic for $\varphi(D)$ such that

$$|f(z)| \leq Me^{(A+\varepsilon)r} \quad \text{for } |z| = r > 0$$

with some $M > 0$. (*Hint*: Combine the ideas of Exercise 4.2.4 and the proof of Theorem 9.25.)

Exercise 9.3.4. Let D be the differentiation operator on $H(\mathbb{C})$. Let $\phi :]0, \infty[\to [1, \infty[$ be a function with $\phi(r) \to \infty$ as $r \to \infty$. Show that there exists an entire function f that is frequently hypercyclic for D and that satisfies

$$|f(z)| \leq M\phi(r)e^r \quad \text{for } |z| = r > 0$$

with some $M > 0$. (*Hint*: Look at the proof of Theorem 4.22, using the Frequent Hypercyclicity Criterion in the version of Exercise 9.2.1.)

Exercise 9.3.5. Let φ be a nonconstant bounded holomorphic function on $\mathbb{D}$ and let M^*_φ be the corresponding adjoint multiplication operator on H^2; see Section 4.4. Show that M^*_φ is frequently hypercyclic if and only if it is hypercyclic, that is, if $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \varnothing$. (*Hint*: Look at the proofs of Theorems 4.42 and 9.25.)

Exercise 9.4.1. In the proof of Theorem 9.27, show that $\underline{\text{dens}}(B) \geq \underline{\text{dens}}(A)$.

Exercise 9.4.2. Let T be a frequently hypercyclic operator on a Fréchet space X. Show that then $T^p \oplus T^q$ is hypercyclic on $X \oplus X$ for any $p, q \in \mathbb{N}$. (*Hint*: Exercises 2.5.5 and 9.1.3.)

Exercise 9.4.3. Let T be a topologically ergodic operator on a separable Fréchet space; see Exercise 1.5.6. Show that T^p is then also topologically ergodic for any $p \geq 1$. (*Hint*: Follow the proof of Theorem 9.27, using Exercise 2.5.5; see Exercise 6.1.5 for an alternative proof.)

Exercise 9.4.4. Let T be an operator on a separable Fréchet space X. Suppose that there is a vector $x \in X$ and a nonempty open subset U of X such that $\underline{\text{dens}}\{n \in \mathbb{N}_0 \;;\; T^n x \in V\} > 0$ for all nonempty open subsets V of U. Show that x is frequently hypercyclic for T. (*Hint*: Use the Bourdon–Feldman theorem.)

Exercise 9.4.5. Let T be an operator on a (real or complex) Banach space X. Show that if there is some $x^* \in X^*$, $x^* \neq 0$, and some λ with $|\lambda| = 1$ such that $\|(\lambda I - T^*)^n x^*\|^{1/n} \to 0$ as $n \to \infty$ then T is not frequently hypercyclic.

Sources and comments

Section 9.1. Frequently hypercyclic operators were introduced by Bayart and Grivaux [38], [40]. The idea of using ergodic theory to obtain the dynamical properties of linear operators seems to be due to Rudnicki [272] and Flytzanis [152, 153]. Bayart and Grivaux

[40] obtained Lemma 9.5 (see also Bonilla and Grosse-Erdmann [87]) as well as the frequent hypercyclicity of the Birkhoff operators. The theorem of Erdős and Sárközy can be found in [296]. Theorem 9.8 is due to Grosse-Erdmann and Peris [185]; Bayart and Matheron [45] show that this result is essentially optimal.

For an introduction to ergodic theory we refer to Walters [300].

Section 9.2. The Frequent Hypercyclicity Criterion was obtained by Bayart and Grivaux [38, 40]; the form given here is due to Bonilla and Grosse-Erdmann [87]. Grivaux [173] also provided a probabilistic version of it. Proposition 9.11 is due to Bonilla and Grosse-Erdmann [87]. The remaining results in this section can essentially be found in Bayart and Grivaux [40]; see also Bonilla and Grosse-Erdmann [87]. The latter paper also contains further conditions under which the set $FHC(T)$ of frequently hypercyclic operators is of first Baire category, or when $FHC(T)+FHC(T)$ does or does not coincide with the full space.

Bayart and Grivaux [41] constructed a weighted shift on c_0 that is frequently hypercyclic, but neither chaotic nor mixing; this also shows that not every frequently hypercyclic operator satisfies the Frequent Hypercyclicity Criterion, and that Proposition 9.13 does not characterize frequently hypercyclic weighted shifts on c_0. Badea and Grivaux [19] found operators on a Hilbert space that are frequently hypercyclic and chaotic but not mixing.

It remains an open problem whether every chaotic operator is frequently hypercyclic, and to find a characterization of frequently hypercyclic weighted shifts, even on ℓ^2 or on c_0.

Section 9.3. The proof of Theorem 9.22 follows Bayart and Grivaux [38]; see also [39]. Theorems 9.25 and 9.26 are due to Blasco, Bonilla and Grosse-Erdmann [86, 76]; these authors also show that the operators of differentiation and translation on the space of harmonic functions on $\mathbb{R}^N$ are frequently hypercyclic, and they obtain some related growth results.

In order to keep the presentation simple we have imposed rather strong assumptions on the eigenvector fields. A much deeper analysis leads to one of the most striking results in linear dynamics.

To be more specific, an operator T on a complex separable Banach space X is said to have a *perfectly spanning set of eigenvectors associated to unimodular eigenvalues* if one of the following two equivalent conditions holds:

(i) there exists an atomless probability measure σ on $\mathbb{T}$ such that, for any measurable set $A \subset \mathbb{T}$ with $\sigma(A) = 1$, $\operatorname{span}\{\ker(\lambda I - T)\ ;\ \lambda \in A\}$ is dense in X;

(ii) for any countable set $D \subset \mathbb{T}$, $\operatorname{span}\{\ker(\lambda I - T)\ ;\ \lambda \in \mathbb{T} \setminus D\}$ is dense in X.

These conditions were first introduced by Flytzanis [152, 153]. Their equivalence was shown by Grivaux [174], who also obtained the following fundamental principle.

Theorem 9.42. *Any operator on a complex separable Banach space with a perfectly spanning set of eigenvectors associated to unimodular eigenvalues is frequently hypercyclic.*

When the underlying space is even a Hilbert space then one can show that there exists a Borel probability measure of full support on X with respect to which T is ergodic (see Bayart and Grivaux [40]); as explained in Section 9.1, this immediately implies that T is frequently hypercyclic. The measure can even be a so-called Gaussian measure. A similar result for nuclear Fréchet spaces is due to Grosse-Erdmann [182]. For surveys on the application of ergodic theory to linear dynamics we refer to Godefroy [164] and Grosse-Erdmann [182]. A detailed treatment can be found in Bayart and Matheron [44].

Bayart and Grivaux [40, 41] have applied their results to various operators. In particular they have shown that if φ is an automorphism of the unit disk $\mathbb{D}$ then the corresponding composition operator C_φ (see Section 4.5) is frequently hypercyclic on the Hardy space H^2 if and only if it is hypercyclic, that is, if and only if φ is parabolic or hyperbolic.

The example of Bayart and Grivaux [41] of a frequently hypercyclic weighted shift on c_0, mentioned above, has no unimodular eigenvalues, so that the approach chosen in this section is not always possible. Moreover, their operator does not possess any invariant Gaussian measure of full support.

Section 9.4. Theorem 9.27 is due to Bayart and Grivaux [40], whose proof uses Ansari's theorem. The alternative proof given in Grosse-Erdmann and Peris [185] contains an error; in fact, Example 9.28 contradicts Theorem 1.4 in that paper. The proof given here is due to Grosse-Erdmann and Peris [186].

Theorem 9.33 provides a new common approach to Theorems 9.35 and Theorem 9.36 that were previously obtained by Bayart and Matheron [44] and by Conejero, Müller and Peris [110], respectively. The remainder of the section, including Theorem 9.39 and Corollary 9.41, is due to Shkarin [287]. Grivaux [174] has recently shown that the necessary spectral conditions of Theorems 5.6 and 9.39 actually characterize spectra of frequently hypercyclic operators on Hilbert spaces.

Theorem 9.43. *Let $K \subset \mathbb{C}$ be a nonempty compact set. There exists a frequently hypercyclic operator T on a complex Hilbert space such that $\sigma(T) = K$ if and only if K has no isolated points and each of its connected components meets the unit circle.*

Further interesting results on frequent hypercyclicity include the facts that every operator on an infinite-dimensional complex separable Hilbert space is the sum of two frequently hypercyclic operators (Bayart and Grivaux [40]) and that every infinite-dimensional complex Fréchet space with an unconditional basis supports a frequently hypercyclic and chaotic operator (De la Rosa, Frerick, Grivaux, and Peris [127]).

Many questions concerning frequently hypercyclic operators remain open. For example (see Bayart and Grivaux [40]), whether the frequent hypercyclicity of an operator T is inherited by its direct sum $T \oplus T$; and whether it is inherited by its inverse T^{-1}, if it exists.

Exercises. Exercise 9.1.1 is taken from Bayart and Grivaux [40], Exercises 9.1.4, 9.2.1 and 9.2.6 from Bonilla and Grosse-Erdmann [87], and Exercises 9.2.4 and 9.2.5 from Grosse-Erdmann and Peris [185]. For Exercise 9.3.1 we refer to Bayart and Grivaux [39], for Exercise 9.3.3 to Bonilla and Grosse-Erdmann [86]. Exercise 9.3.4 is taken from Blasco, Bonilla and Grosse-Erdmann [76] who also show that, in the converse direction, given any function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{r\to\infty} \phi(r) = 0$ there is no entire function f that is frequently hypercyclic for D such that $|f(z)| \leq \phi(r)\frac{e^r}{r^{1/4}}$ for $|z| = r$ sufficiently large. Exercise 9.3.5 is taken from Bayart and Grivaux [40], Exercise 9.4.2 from Costakis and Ruzsa [122], Exercise 9.4.4 from Grosse-Erdmann and Peris [185], and Exercise 9.4.5 from Shkarin [287].