Chapter 6
Connectedness arguments in linear dynamics

This chapter is devoted to some of the most fundamental results in linear dynamics. What is particularly striking is that they hold for all operators, without further technical assumptions.

We have already obtained such a result in Chapter 2 . It says that every hypercyclic operator admits a dense subspace of hypercyclic vectors, except for the zero vector. Note that this property would not make sense in a nonlinear setting.

In this chapter we will consider the following problems, which, a priori, do not involve linearity.

- If T has a dense orbit, does then every power T^p also have a dense orbit?
- Suppose that the union of a finite collection of orbits is dense. Will then at least one of these orbits be actually dense?
- If an orbit is somewhere dense, is it (everywhere) dense?

Each of these questions has a negative answer for arbitrary, nonlinear maps. It is therefore even more surprising that they all have a positive answer for (linear) operators, and that without any restrictions. The proofs depend in a crucial way on connectedness arguments.

In the final section we will consider two more problems.

- Let T be a hypercyclic operator, and let $\lambda \in \mathbb{K}$ with $|\lambda| = 1$. Is then λT also hypercyclic?
- Let $(T_t)_{t\geq 0}$ be a hypercyclic C_0-semigroup on a Banach space. Is then every single operator T_t, $t > 0$, hypercyclic?

Again we will give positive answers to these questions. The proofs can be given within a common framework and use, once more, a connectedness argument, this time via a suitable homotopy.

K.-G. Grosse-Erdmann, A. Peris Manguillot, *Linear Chaos*, Universitext, DOI 10.1007/978-1-4471-2170-1_6,

6.1 Ansari's theorem

In this section we deal with the question of whether every power T^p, $p \in \mathbb{N}$, of a hypercyclic operator T is again hypercyclic. Since every sequence $(kp)_k$ is syndetic, Theorem 1.54 implies a positive answer if T is even a weakly mixing operator on a separable Fréchet space. We will show here that the answer is positive for all hypercyclic operators.

The following auxiliary result will be crucial.

Lemma 6.1. *The T be a continuous map on a metric space X without isolated points. Then the interiors of the closures of two orbits under T either coincide, or they are disjoint.*

Proof. Suppose that $\text{int}(\overline{\text{orb}(x,T)}) \cap \text{int}(\overline{\text{orb}(y,T)}) \neq \varnothing$, $x, y \in X$. Then there is some $n \in \mathbb{N}_0$ such that

$$T^n x \in \overline{\text{orb}(y,T)}.$$

Since $\overline{\text{orb}(y,T)}$ is T-invariant, we have that $T^k x \in \overline{\text{orb}(y,T)}$ for $k \geq n$ and therefore

$$\overline{\{T^k x \ ; \ k \geq n\}} \subset \overline{\text{orb}(y,T)}.$$

Since X has no isolated points, one shows easily that

$$\text{int}(\overline{\text{orb}(x,T)}) \subset \text{int}(\overline{\{T^k x \ ; \ k \geq n\}});$$

see also Exercise 6.2.1. Hence $\text{int}(\overline{\text{orb}(x,T)}) \subset \text{int}(\overline{\text{orb}(y,T)})$. By symmetry, we also have the converse inclusion, so that the two interiors coincide. □

Theorem 6.2 (Ansari). *Let T be an operator on a Fréchet space. Then, for any $p \in \mathbb{N}$, $HC(T) = HC(T^p)$. In particular, if T is hypercyclic then so is every power T^p.*

Proof. Let $p \in \mathbb{N}$. We clearly have that $HC(T^p) \subset HC(T)$.

For the converse inclusion we fix $x \in HC(T)$. From Proposition 1.15 and Corollary 2.56 we know that $D := HC(T)$ is a dense, T-invariant connected subset of X; in particular, it does not have isolated points. For the remainder of the proof we consider the map $T : D \to D$; the topological operations of closure and interior will be understood in D. Since D is dense in X it then suffices to show that $\overline{\text{orb}(x,T^p)} = D$.

To this end we define

$$D_j = \overline{\text{orb}(T^j x, T^p)}, \quad j = 0, \ldots, p-1.$$

We need to show that $D = D_0$. Observe that

$$D = \overline{\text{orb}(x,T)} = \overline{\bigcup_{j=0}^{p-1} \text{orb}(T^j x, T^p)} = \bigcup_{j=0}^{p-1} D_j$$

and

$$T(D_j) \subset D_{j+1(\mathrm{mod}\,p)}.$$

Let $F \subset \{0, \ldots, p-1\}$ be a set of minimal cardinality such that

$$D = \bigcup_{j \in F} D_j.$$

Suppose that F is not a singleton. Let, in addition, $\mathrm{int}(D_j) \cap \mathrm{int}(D_k) \neq \varnothing$ for some $j, k \in F$ with $j \neq k$. By Lemma 6.1, $\mathrm{int}(D_j) = \mathrm{int}(D_k)$. From minimality we deduce that

$$D \setminus \bigcup_{l \in F \setminus \{j\}} D_l$$

is nonempty, and it is an open set contained in D_j and thus in $\mathrm{int}(D_j) \subset D_k$, which is not possible. Therefore, $\mathrm{int}(D_j \cap D_k) = \varnothing$ for any $j, k \in F$ with $j \neq k$.

We now set $F_l = F + l(\mathrm{mod}\,p)$, $l = 0, \ldots, p-1$. We have $D = \overline{T^l(D)} = \overline{\bigcup_{j \in F} T^l(D_j)} = \bigcup_{k \in F_l} D_k$, $l = 0, \ldots, p-1$. Since $\mathrm{card}(F_l) = \mathrm{card}(F)$, which is minimal, we also get that $\mathrm{int}(D_j \cap D_k) = \varnothing$ for any $j, k \in F_l$ with $j \neq k$, $l = 0, \ldots, p-1$. As a consequence, the set

$$A := \bigcup_{l=0}^{p-1} \bigcup_{\substack{j,k \in F_l \\ j \neq k}} (D_j \cap D_k)$$

is nowhere dense as a finite union of nowhere dense sets; and it is T-invariant. If A were nonempty, with $y \in A$, say, then

$$D = \overline{\mathrm{orb}(y, T)} \subset \overline{A} = A,$$

which is a contradiction. Therefore $A = \varnothing$, which implies that

$$D = \bigcup_{j \in F} D_j$$

is a finite union of pairwise disjoint closed subsets. But this contradicts the connectedness of D.

In conclusion, $F = \{j\}$ is a singleton. Then $D = D_j$, and we obtain that $D = \overline{T^{p-j}(D_j)} = D_0$, which had to be shown. $\square$

The simple example $T : \{-1, 1\} \to \{-1, 1\}$, $Tx = -x$, shows that Ansari's theorem fails for nonlinear dynamical systems; see also Exercise 1.2.11 for an example on a metric space without isolated points. Ansari's theorem does extend to the nonlinear setting if the set of points with dense orbit is connected; see Exercise 6.1.7.

6.2 Somewhere dense orbits

We recall that a set is called *somewhere dense* if its closure contains a nonempty open set.

It was a key point in the proof of Ansari's theorem to write the space D as a finite union of closures of orbits. Then one of these closures must have an interior point, which means that the corresponding orbit is somewhere dense. In the end we concluded that this orbit is, in fact, (everywhere) dense. Do we have a general principle here, that is, is every somewhere dense orbit necessarily dense? We will give a positive answer to this question.

Thus, let T be an operator on a Fréchet space X. For $x \in X$ we write

$$D(x) = \overline{\mathrm{orb}(x,T)} \quad \text{and} \quad U(x) = \mathrm{int}\, D(x).$$

The following properties can be easily deduced from the continuity of T and the fact that X has no isolated points (see Exercise 6.2.1):

(i) if $y \in D(x)$, then $D(y) \subset D(x)$;
(ii) $U(x) = U(T^k x)$ for each $k \in \mathbb{N}$;
(iii) if $R : X \to X$ is a continuous map that commutes with T, then $R(D(x)) \subset D(Rx)$.

We first need a generalization of Theorem 2.54. An easy adaptation of the argument used there gives the result; see Exercise 6.2.2.

Lemma 6.3. *If T admits a somewhere dense orbit and p is a nonzero polynomial, then the operator $p(T)$ has dense range.*

Before proving that a vector whose orbit is somewhere dense is necessarily hypercyclic, we will show that it is cyclic, that is, the linear span of its orbit is dense in X.

Lemma 6.4. *If $\mathrm{orb}(x,T)$ is somewhere dense, then the set $\{p(T)x\ ;\ p \neq 0$ a polynomial$\}$ is connected and dense in X.*

Proof. The set $A := \{p(T)x\ ;\ p \neq 0 \text{ a polynomial}\}$ is path connected. Indeed, let p, q be nonzero polynomials. If q is not a multiple of p then the straight path $t \to tp(T)x + (1-t)q(T)x$, $t \in [0,1]$, is contained in A. Otherwise we select a third nonzero polynomial r that is not a multiple of p, and therefore not of q, and we take the union of the straight paths connecting $p(T)x$ and $q(T)x$ with $r(T)x$.

On the other hand, $\overline{A}$ is a subspace of X that contains $\overline{\mathrm{orb}(x,T)}$. It follows from the hypothesis that there is some $x_0 \in X$ and a 0-neighbourhood W such that $x_0 + W \subset \overline{A}$. Thus, for any $y \in X$, there is a scalar λ with $y \in \lambda W$; hence $y \in \lambda(x_0 + W) - \lambda x_0 \subset \overline{A}$. Consequently, A is dense in X. $\square$

Theorem 6.5 (Bourdon–Feldman). *Let T be an operator on a Fréchet space X and $x \in X$. If $\mathrm{orb}(x,T)$ is somewhere dense in X, then it is dense in X.*

Proof. We have to show that if $U(x) \neq \varnothing$ then $D(x) = X$. The proof will be split into four steps.

Step 1. We have that $T(X \setminus U(x)) \subset X \setminus U(x)$.

We show, equivalently, that $T^{-1}(U(x)) \subset U(x)$. First, since $U(x) \neq \varnothing$ there is some $m \in \mathbb{N}_0$ with $x_m := T^m x \in U(x)$.

Now let $y \in T^{-1}(U(x))$, and let V be an arbitrary neighbourhood of y. Since, by property (ii), x_m also has a somewhere dense orbit, Lemma 6.4 implies that we can find a polynomial p such that $p(T)x_m \in V \cap T^{-1}(U(x))$.

We have, using property (ii), that

$$p(T)x_m \in p(T)(U(x)) = p(T)(U(T^{m+1}x)) \subset p(T)(D(T^{m+1}x)).$$

Moreover, since $Tp(T)x_m \in U(x) \subset D(x)$, properties (iii) and (i) yield that

$$p(T)(D(T^{m+1}x)) \subset D(Tp(T)x_m) \subset D(x).$$

We have therefore shown that $V \cap D(x) \neq \varnothing$. Since V was arbitrary and $D(x)$ is closed, we deduce that $y \in D(x)$ and hence $T^{-1}(U(x)) \subset D(x)$. Continuity of T implies that $T^{-1}(U(x)) \subset U(x)$.

Step 2. For any $z \in X \setminus U(x)$, $D(z) \subset X \setminus U(x)$.

By Step 1, $X \setminus U(x)$ is T-invariant, and it is closed. The claim then follows from the definition of $D(z)$.

Step 3. For any polynomial $p \neq 0$, $p(T)x \in X \setminus \partial D(x)$, where $\partial D(x)$ denotes the boundary of $D(x)$; see Figure 6.1.

Suppose that $p(T)x \in \partial D(x)$ for some polynomial $p \neq 0$. By Lemma 6.3 there is some $y \in X$ such that $p(T)y \in U(x)$. Since $p(T)x \notin U(x)$, property (iii) and Step 2 imply that

$$p(T)(D(x)) \subset D(p(T)x) \subset X \setminus U(x).$$

We therefore have that $y \in X \setminus D(x)$. By Lemma 6.4 there then exists a polynomial q such that $q(T)x$ is close enough to y to satisfy $q(T)x \in X \setminus D(x) \subset X \setminus U(x)$ and $p(T)q(T)x \in U(x)$. Since $p(T)x \in D(x)$, property (iii) and Step 2 imply that

$$p(T)q(T)x = q(T)p(T)x \in q(T)(D(x)) \subset D(q(T)x) \subset X \setminus U(x),$$

which is a contradiction. This proves the claim.

Step 4. We have that $D(x) = X$.

By Step 3,

$$A := \{p(T)x \; ; \; p \neq 0 \text{ a polynomial}\} \subset U(x) \cup (X \setminus D(x)),$$

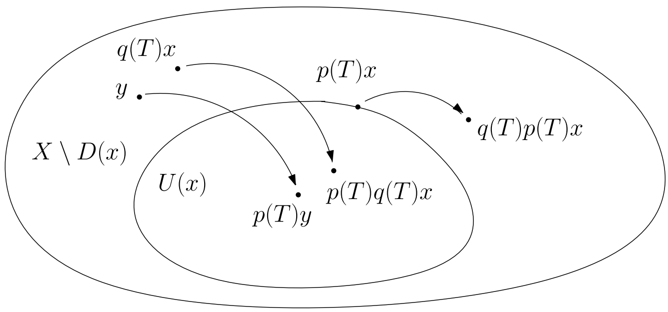

Fig. 6.1 Step 3

which is a disjoint union of open sets. Since, by Lemma 6.4, A is connected and, by density of A, $A \cap U(x) \neq \varnothing$, we must have that $A \cap (X \setminus D(x)) = \varnothing$. Hence, $A \subset D(x)$, which implies that $D(x) = X$. □

6.3 Multi-hypercyclic operators

The Bourdon–Feldman theorem provides us with a very powerful tool for obtaining dense orbits. A particular case occurs when the union of a finite number of orbits under T is dense in X. In this case the operator T is called *multi-hypercyclic*.

Theorem 6.6 (Costakis–Peris). *Let T be an operator on a Fréchet space X and $x_1, \ldots, x_n \in X$. If*

$$\bigcup_{j=1}^{n} \operatorname{orb}(x_j, T)$$

is dense in X, then there is some $j \in \{1, \ldots, n\}$ such that $\operatorname{orb}(x_j, T)$ is dense in X. In particular, every multi-hypercyclic operator is hypercyclic.

Proof. The hypothesis says that

$$\bigcup_{j=1}^{n} \overline{\operatorname{orb}(x_j, T)} = \overline{\bigcup_{j=1}^{n} \operatorname{orb}(x_j, T)} = X.$$

Since a finite union of nowhere dense sets is nowhere dense, $\operatorname{orb}(x_j, T)$ must be somewhere dense in X for some $j \in \{1, \ldots, n\}$. By the Bourdon–Feldman theorem, x_j then has a dense orbit. □

Ansari's result can easily be derived from this theorem. Let $x \in HC(T)$ and $p \in \mathbb{N}$. Since

$$\operatorname{orb}(x,T) = \bigcup_{j=0}^{p-1} \operatorname{orb}(T^j x, T^p)$$

is dense in X, Theorem 6.6 implies that there is some $j \in \{0,\dots,p-1\}$ such that $T^j x$ is hypercyclic for T^p. Since T^{p-j} has dense range and

$$T^{p-j}(\operatorname{orb}(T^j x, T^p)) \subset \operatorname{orb}(x, T^p)$$

we obtain that $x \in HC(T^p)$.

These arguments also imply that because Ansari's theorem fails for nonlinear dynamical systems the same is true for the theorems of Costakis–Peris and Bourdon–Feldman; see also Exercise 1.2.11.

6.4 Hypercyclic semigroup actions

In this section we will be dealing with two additional important problems in linear dynamics.

The *problem of unimodular multiples* asks whether, given a hypercyclic operator T, is every multiple λT with $\lambda \in \mathbb{K}$, $|\lambda| = 1$, also hypercyclic? The operator λT is called a *rotation* of T. In the real setting the answer is positive. Indeed, one only needs to show that if T is hypercyclic then so is $-T$. But this follows from Ansari's theorem because the two operators have a common square, $T^2 = (-T)^2$. Thus we will concentrate here on the complex setting.

The *problem of hypercyclic discretizations of semigroups* asks whether, given a hypercyclic C_0-semigroup $(T_t)_{t\geq 0}$ on a Banach space, is every single operator T_t, $t > 0$, also hypercyclic? Although C_0-semigroups will only be treated in the next chapter (and we ask the reader to consult the relevant definitions there), there will be no harm in already considering the discretization problem here. The (very basic) proof that a hypercyclic C_0-semigroup satisfies the assumptions imposed in this section will be postponed to Chapter 7.

The main aim of this section is to show that both problems have a positive answer. In analogy with Ansari's theorem, it will even be proved that the corresponding sets of hypercyclic vectors coincide.

Theorem 6.7 (León–Müller). *Let T be an operator on a complex Fréchet space X. If $x \in X$ is such that $\{\lambda T^n x \,;\, \lambda \in \mathbb{C},\ |\lambda| = 1,\ \text{and}\ n \in \mathbb{N}_0\}$ is dense in X then* $\operatorname{orb}(x, \lambda T)$ *is dense in X for each $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.*

In particular, for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, T and λT have the same hypercyclic vectors, that is,

$$HC(T) = HC(\lambda T).$$

In Exercise 2.5.1 we saw that rotations of mixing (or weakly mixing) operators are mixing (or weakly mixing, respectively). But that result did not say anything about the sets of hypercyclic vectors.

Theorem 6.8 (Conejero–Müller–Peris). *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a Banach space X. If $x \in X$ is hypercyclic for $(T_t)_{t\geq 0}$, then it is hypercyclic for each operator T_t, $t > 0$.*

It is particularly gratifying that the two problems can be treated within a common framework, that of semigroup actions. We will also show that a variant of the method leads to a new proof of Ansari's theorem; see Exercise 6.4.5.

Throughout this section we will write

$$G = \mathbb{N}_0 \times \mathbb{R}_+,$$

which is a semigroup under addition. If X is a Fréchet space then a map

$$\Psi : G \to L(X)$$

is called a *(continuous and linear) semigroup action* of G on X if the following properties hold:

(i) $\Psi(0) = I$;
(ii) for any $g_1, g_2 \in G$, $\Psi(g_1 + g_2) = \Psi(g_1)\Psi(g_2)$;
(iii) the map $G\times X \to X$, $(g, x) \to \Psi(g)x$, is continuous, where $G = \mathbb{N}_0 \times \mathbb{R}_+$ and $G \times X$ carry the product topology.

Definition 6.9. A semigroup action Ψ on a Fréchet space X is called *hypercyclic* if there is some $x \in X$ such that $\{\Psi(g)x\ ;\ g \in G\}$ is dense in X. The vector x is then called *hypercyclic* for Ψ, and we write $x \in HC(\Psi)$.

Let us see how our two problems fit into this framework. If T is an operator on a complex Fréchet space X, then we define

$$\Psi(n, t) = e^{2\pi ti}T^n, \quad n \in \mathbb{N}_0,\ t \geq 0.$$

In the second case, if $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X, then we define

$$\Psi(n, t) = T_t, \quad n \in \mathbb{N}_0,\ t \geq 0.$$

It is easy to see that these are semigroup actions of G on X; we refer to Chapter 7 for the definition of a C_0-semigroup.

Moreover, in both cases, the following properties are satisfied:

(α) either $\Psi(1, 0) = I$ or $\Psi(0, 1) = I$;
(β) if the semigroup action is hypercyclic then each convex combination of $\Psi(0, s)$ and $\Psi(1, t)$, $s, t \geq 0$, has dense range.

That property (β) is satisfied follows from a simple generalization of Theorem 2.54 and by Theorem 7.16, respectively.

The following theorem therefore immediately implies the Theorems of León–Müller and Conejero–Müller–Peris.

Theorem 6.10. *Let Ψ be a semigroup action on an infinite-dimensional Fréchet space X satisfying properties (α) and (β). If $x \in X$ is hypercyclic for Ψ then it is hypercyclic for each operator $\Psi(1,t)$, $t > 0$.*

Proof. We first note that it suffices to prove the claim for $t = 1$. Indeed, let x be hypercyclic for Ψ, and let $t > 0$ be arbitrary. We distinguish the two subcases of (α). If $\Psi(1,0) = I$ then

$$\widetilde{\Psi}(n,s) := \Psi(n,st)$$

defines a semigroup action that satisfies (α) and (β). Since x is also hypercyclic for $\widetilde{\Psi}$ we can conclude that x is hypercyclic for $\widetilde{\Psi}(1,1) = \Psi(1,t)$. If $\Psi(0,1) = I$ then we define

$$\widetilde{\Psi}(n,s) = \Psi(n,nt+s),$$

and we can conclude as before that x is hypercyclic for $\widetilde{\Psi}(1,1) = \Psi(1,t+1) = \Psi(1,t)$.

As usual, $\mathbb{T} = \{z \in \mathbb{C}\ ;\ |z| = 1\}$ is the unit circle. For ease of notation we introduce the map $\rho : \mathbb{R}_+ \to \mathbb{T}$, given by $\rho(t) := e^{2\pi ti}$. We then define, for every pair $u, v \in X$, the subset $F_{u,v}$ of $\mathbb{T}$ by

$$F_{u,v} := \big\{\lambda \in \mathbb{T}\ ;\ \exists\,((n_k,t_k))_k \subset G \text{ with } \Psi(n_k,t_k)u \to v \ \text{ and } \rho(t_k) \to \lambda\big\}.$$

The remainder of the proof will be divided into several steps.

Step 1. If $u \in HC(\Psi)$, then $F_{u,v} \neq \varnothing$ for all $v \in X$.

Since $u \in HC(\Psi)$, we can find sequences $(n_k)_k$ in $\mathbb{N}_0$ and $(t_k)_k$ in $\mathbb{R}_+$ such that $\Psi(n_k,t_k)u \to v$. By passing to a subsequence if necessary, we may assume that $(\rho(t_k))_k$ is convergent. Its limit is an element of $F_{u,v}$.

Step 2. If $\lambda_k \in F_{u,v_k}$, $v_k \to v$ and $\lambda_k \to \lambda$, then $\lambda \in F_{u,v}$. In particular, $F_{u,v}$ is a closed set for each $u, v \in X$.

Let W be a 0-neighbourhood of X and $\varepsilon > 0$. There is a 0-neighbourhood W_1 such that $W_1 + W_1 \subset W$; see Lemma 2.36. By assumption, there is some $k \in \mathbb{N}$ with $v - v_k \in W_1$ and $|\lambda - \lambda_k| < \varepsilon$. Now, by definition, there are $n_k \in \mathbb{N}_0$ and $t_k \in \mathbb{R}_+$ such that $v_k - \Psi(n_k,t_k)u \in W_1$ and $|\lambda_k - \rho(t_k)| < \varepsilon$. We then get that $v - \Psi(n_k,t_k)u \in W_1 + W_1 \subset W$ and $|\lambda - \rho(t_k)| < 2\varepsilon$, so that $\lambda \in F_{u,v}$.

Step 3. If $u, v, w \in X$, $\lambda \in F_{u,v}$, and $\mu \in F_{v,w}$, then $\lambda\mu \in F_{u,w}$.

Given a 0-neighbourhood W, take a 0-neighbourhood W_1 such that $W_1 + W_1 \subset W$. Let $\varepsilon > 0$. Then there are $n_1 \in \mathbb{N}_0$ and $t_1 \in \mathbb{R}_+$ such that $w - \Psi(n_1,t_1)v \in W_1$ and $|\mu - \rho(t_1)| < \varepsilon$. One can then find a 0-neighbourhood

V, $n_2 \in \mathbb{N}_0$ and $t_2 \in \mathbb{R}_+$ satisfying $\Psi(n_1,t_1)(V) \subset W_1$, $v - \Psi(n_2,t_2)u \in V$, and $|\lambda - \rho(t_2)| < \varepsilon$. Consequently we have for $n_3 := n_1 + n_2$ and $t_3 := t_1 + t_2$ that

$$w - \Psi(n_3,t_3)u = w - \Psi(n_1,t_1)v + \Psi(n_1,t_1)(v - \Psi(n_2,t_2)u) \in W_1 + W_1 \subset W,$$

and

$$|\lambda\mu - \rho(t_3)| \leq |\lambda|\,|\mu - \rho(t_1)| + |\rho(t_1)|\,|\lambda - \rho(t_2)| < 2\varepsilon.$$

Hence $\lambda\mu \in F_{u,w}$.

We now fix $x \in HC(\Psi)$. Our aim is to show that $x \in HC(\Psi(1,1))$. By Steps 1, 2 and 3, $F_{x,x}$ is a nonempty closed subsemigroup of the multiplicative group $\mathbb{T}$.

Step 4. If $F_{x,x} = \mathbb{T}$ then x is hypercyclic for $\Psi(1,1)$.

Suppose that $F_{x,x} = \mathbb{T}$. Given any $y \in X$, Steps 1 and 3 imply that $F_{x,y} = \mathbb{T}$. In particular $1 \in F_{x,y}$, which yields the existence of sequences $(n_k)_k$ in $\mathbb{N}_0$ and $(t_k)_k$ in $\mathbb{R}_+$ such that $\Psi(n_k,t_k)x \to y$ and $\rho(t_k) = e^{2\pi t_k i} \to 1$. We can then write $t_k = j_k - 1 + \varepsilon_k$ with $j_k \in \mathbb{N}$ and $\varepsilon_k \in [-1/2, 1/2]$, where $\varepsilon_k \to 0$.

Let W be a 0-neighbourhood, and let W_1 be a 0-neighbourhood such that $W_1 + W_1 \subset W$. By a standard compactness argument, the continuity of the semigroup action implies that there is a 0-neighbourhood V such that $\Psi(0,t)(V) \subset W_1$ if $0 \leq t \leq 2$. Moreover, there is some $k \in \mathbb{N}$ such that $\Psi(n_k,t_k)x - y \in V$ and $\Psi(0,1-\varepsilon_k)y - \Psi(0,1)y \in W_1$. Therefore

$$\begin{aligned}
&\Psi(n_k,j_k)x - \Psi(0,1)y \\
&\qquad = \Psi(0,1-\varepsilon_k)(\Psi(n_k,t_k)x - y) + (\Psi(0,1-\varepsilon_k) - \Psi(0,1))y \\
&\qquad \in \Psi(0,1-\varepsilon_k)(V) + W_1 \subset W_1 + W_1 \subset W.
\end{aligned}$$

Observe that, by property (α), $\Psi(n_k,j_k)x \in \mathrm{orb}(x,\Psi(1,1))$. Thus $\Psi(0,1)y \in \mathrm{orb}(x,\Psi(1,1))$. Since $\Psi(0,1)$ has dense range by property (β), and $y \in X$ is arbitrary, x is hypercyclic for $\Psi(1,1)$.

For the rest of the proof we can now assume that $F_{x,x} \neq \mathbb{T}$, and we will show that this leads to a contradiction.

Step 5. There exists some $m \in \mathbb{N}$ such that, for each $y \in HC(\Psi)$, there is $\lambda \in \mathbb{T}$ satisfying $F_{x,y} = \{\lambda z\ ;\ z^m = 1\}$.

We first note that $F_{x,x}$ must be of the form $F_{x,x} = \{z \in \mathbb{T}\ ;\ z^m = 1\}$ for some $m \in \mathbb{N}$. Indeed, if $F_{x,x}$ contained points $z = e^{2\pi t i}$ with $t > 0$ arbitrarily small then, being a closed subsemigroup of $\mathbb{T}$, $F_{x,x}$ would be dense, and hence coincide with $\mathbb{T}$, which was excluded. Hence there is a minimal $t_0 \in \,]0,1]$ such that $z_0 = e^{2\pi t_0 i} \in F_{x,x}$. By the same argument, t_0 cannot be irrational; see

Example 1.17. There is then a minimal $m \in \mathbb{N}$ with $z_0^m = 1$. The minimality of t_0 and m easily imply that $F_{x,x} = \{z \in \mathbb{T}\ ;\ z^m = 1\}$.

Now let $y \in HC(\Psi)$. By Step 1, there exist $\lambda \in F_{x,y}$ and $\mu \in F_{y,x}$. Then, by Step 3, $\lambda F_{x,x} \subset F_{x,y}$ and $\mu F_{x,y} \subset F_{x,x}$, so that $\mathrm{card}(F_{x,y}) = \mathrm{card}(F_{x,x})$. This implies that $F_{x,y} = \lambda F_{x,x}$.

Step 6. There is a continuous function $f : HC(\Psi) \to \mathbb{T}$ *such that* $f(\Psi(0,t)x) = e^{2\pi mti}$ *for every* $t \geq 0$.

Let $m \in \mathbb{N}$ be the integer given by Step 5. Then, for any $y \in HC(\Psi)$, we define

$$f(y) = \lambda^m \quad \text{if } \lambda \in F_{x,y}.$$

By Step 5, this is well defined. Moreover, f is continuous. Otherwise there are $y_k \in HC(\Psi)$ and $y \in HC(\Psi)$ such that $y_k \to y$ but $f(y_k) \not\to f(y)$. We choose $\lambda_k \in F_{x,y_k}$. Passing to a subsequence if necessary, we can assume that $f(y_k) \to \mu \neq f(y)$ and $\lambda_k \to \lambda$ for some $\lambda, \mu \in \mathbb{T}$. It follows from Step 2 that $\lambda \in F_{x,y}$ and hence that $f(y_k) = \lambda_k^m \to \lambda^m = f(y)$, which is a contradiction.

Now let $t \geq 0$. By property (β), $\Psi(0,t)$ has dense range and therefore $\Psi(0,t)x \in HC(\Psi)$. Since, by definition, $\rho(t) = e^{2\pi ti} \in F_{x,\Psi(0,t)x}$ we conclude that $f(\Psi(0,t)x) = e^{2\pi mti}$.

Step 7. There is a continuous function $h : \overline{\mathbb{D}} \to \mathbb{T}$, *whose restriction to the unit circle is homotopically nontrivial. A contradiction.*

This is the decisive, and most difficult part of the proof. We will use here the terminology and some results of homotopy theory; see Appendix A. In order to define the function h we will first define a function $g : \mathbb{T} \to HC(\Psi)$, where we will distinguish the two subcases of (α).

Case 1: $\Psi(0,1) = I$. Here we define $g : \mathbb{T} \to HC(\Psi)$ by

$$g(e^{2\pi ti}) = \Psi(0,t)x, \quad 0 \leq t < 1,$$

which is well defined by property (β), and g is continuous because $\Psi(0,1) = I$. By Step 6, the function $f \circ g : \mathbb{T} \to \mathbb{T}$ satisfies $f(g(e^{2\pi ti})) = e^{2\pi mti}$, $0 \leq t < 1$, so that the index of $f \circ g$ is $m \geq 1$.

We extend the function g to the closed unit disk $\overline{\mathbb{D}}$ by defining $g(z) = (1-r)\Psi(1,0)x + rg(e^{2\pi ti})$ for $z = re^{2\pi ti} \in \overline{\mathbb{D}}$, $r \geq 0$. This extension is clearly continuous on $\overline{\mathbb{D}}$. Since $g(z)$ is a convex combination of $\Psi(1,0)x$ and $\Psi(0,t)x$ for some $t \geq 0$, property (β) implies that $g(z) \in HC(\Psi)$ for every $z \in \overline{\mathbb{D}}$.

To summarize, we have found a continuous function $h := f \circ g : \overline{\mathbb{D}} \to \mathbb{T}$ whose restriction to the unit circle is homotopically nontrivial. In other words, the map $H : \mathbb{T} \times [0,1] \to \mathbb{T}$, $(e^{2\pi ti}, r) \to h(re^{2\pi ti})$ defines a homotopy between the function h on $\mathbb{T}$, which is homotopically nontrivial, and a constant function. This is the desired contradiction.

Case 2: $\Psi(1,0) = I$. Here the construction of g is slightly more delicate. First, since f is continuous and $f(x) = 1$, we can find a 0-neighbourhood W

such that $|f(y) - 1| < 1$ if $y \in HC(\Psi)$ and $y - x \in W$. We can assume that W is balanced, that is, $\mu W \subset W$ whenever $|\mu| \leq 1$; see Lemma 2.6(iii).

Since no 0-neighbourhood in an infinite-dimensional Fréchet space can be relatively compact (see Appendix A), the set $U := W \setminus \{x - \Psi(0,t)x \ ; \ 0 \leq t \leq 1\}$ is open and nonempty. By the hypercyclicity of x there are $n_0 \in \mathbb{N}_0$ and $t_0 \geq 0$ such that $x - \Psi(n_0, t_0)x \in U$. Since $\Psi(1,0) = I$ we also have that $x - \Psi(0, t_0)x \in U$, and therefore $t_0 > 1$ and $x - \Psi(0, t_0)x \in W$. We can now define $g : \mathbb{T} \to HC(\Psi)$ by

$$g(e^{2\pi ti}) = \begin{cases} \Psi(0, 2tt_0)x & \text{if } 0 \leq t < 1/2, \\ (2t-1)x + (2-2t)\Psi(0,t_0)x & \text{if } 1/2 \leq t < 1, \end{cases}$$

which is clearly continuous. The fact that g is well defined is a consequence of property (β); note that $x = \Psi(1,0)x$.

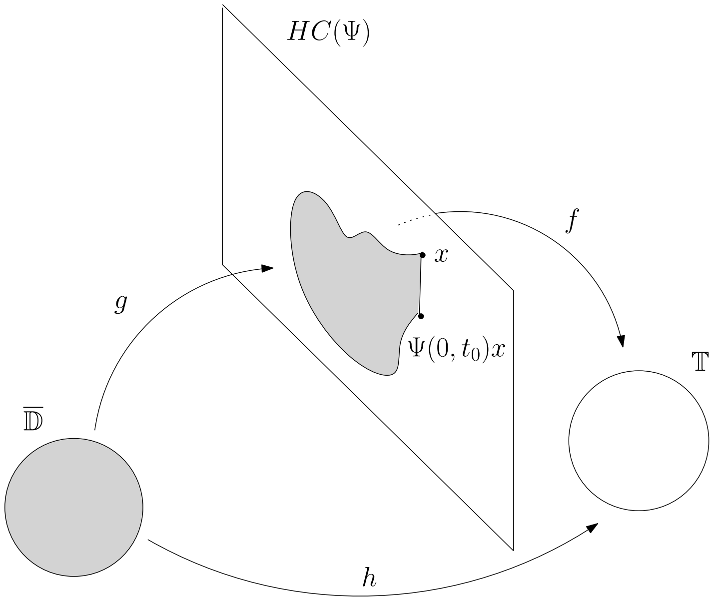

Fig. 6.2 The map h, Case 2

We consider the function $f \circ g : \mathbb{T} \to \mathbb{T}$. Then $f(g(e^{2\pi ti})) = e^{4\pi m t t_0 i}$ for $0 \leq t < 1/2$. Moreover, by the selection of t_0, and since W is balanced, we obtain that $|f(g(e^{2\pi ti})) - 1| < 1$ for $1/2 \leq t < 1$. Thus, as t moves from 0 to 1/2, $f(g(e^{2\pi ti}))$, starting from 1, moves along the unit circle in a positive direction and covers it $[mt_0]$ times, finishing inside the disk of radius 1 around 1. As t then moves from 1/2 to 1, $f(g(e^{2\pi ti}))$ stays completely in that disk, returning to 1 for $t = 1$. As a consequence, the path $t \to f(g(e^{2\pi ti}))$ can be deformed homotopically to the path $t \to e^{2\pi ni}$ with either $n = [mt_0] \geq 1$ if

$\mathrm{Im}(e^{2\pi m t_0 i}) \geq 0$ or with $n = [mt_0] + 1 \geq 2$ if $\mathrm{Im}(e^{2\pi m t_0 i}) < 0$. In any case, the index of $f \circ g$ is nonzero.

We extend the function g continuously to the closed unit disk $\overline{\mathbb{D}}$ by defining $g(z) = (1 - r)x + rg(e^{2\pi ti})$ for $z = re^{2\pi ti} \in \overline{\mathbb{D}}$, $r \geq 0$. Since $g(z)$ is a convex combination of $x = \Psi(1, 0)x$ and $\Psi(0, s)x$ for some $s \geq 0$, property (β) implies that $g(z) \in HC(\Psi)$ for every $z \in \overline{\mathbb{D}}$. We define again the map $h : \overline{\mathbb{D}} \to \mathbb{T}$ by $h = f \circ g$; see Figure 6.2. Since the restriction of h to the unit circle is homotopically nontrivial we obtain a contradiction as in Case 1. □

When we combine Theorem 6.10 with Ansari's theorem, by which $\Psi(n, t) = \Psi(1, t/n)^n$ is hypercyclic whenever $\Psi(1, t/n)$ is, we obtain the following.

Corollary 6.11. *Let Ψ be a semigroup action on a Fréchet space X satisfying properties (α) and (β). If $x \in X$ is hypercyclic for Ψ then it is hypercyclic for each operator $\Psi(n, t)$, $n, t > 0$.*

Exercises

Exercise 6.1.1. In a metric space, show that a finite union of nowhere dense sets is nowhere dense.

Exercise 6.1.2. Let $T : X \to X$ be a (not necessarily linear) weakly mixing dynamical system. Show that any T^p, $p \in \mathbb{N}$, is also weakly mixing. (*Hint*: Theorem 1.54.)

Exercise 6.1.3. Let T be an operator on a separable Fréchet space X that satisfies the Hypercyclicity Criterion. Give two proofs of the fact that any T^p, $p \in \mathbb{N}$, also satisfies the Hypercyclicity Criterion.

Exercise 6.1.4. Let T be a chaotic operator on a Fréchet space X. Without the use of Theorem 6.2, show that any T^p, $p \in \mathbb{N}$, is also chaotic. This is not true for nonlinear maps by the example of Exercise 1.2.11.

Exercise 6.1.5. Let $S : X \to X$, $T : Y \to Y$ be topologically ergodic operators on Fréchet spaces X and Y. Show that any operator $S^p \oplus T^q$, $p, q \in \mathbb{N}$, is topologically ergodic on $X \oplus Y$. (*Hint*: Exercises 2.5.5 and 2.5.6.)

Exercise 6.1.6. Let T be an operator on a Fréchet space X and x a hypercyclic vector for T. Show that there exists an increasing sequence $(n_k)_k$ of positive integers with $\sup_{k\geq 1}(n_{k+1} - n_k) = 2$ such that x does not have dense orbit under $(T^{n_k})_k$. (*Hint*: Show that there is some $y \in X$ and a 0-neighbourhood W such that $z \in y + W$ implies that $Tz \notin y + W$.)

In the following two exercises, let $T : X \to X$ be a (not necessarily linear) dynamical system, that is, a continuous map T on a metric space X. Suppose that X does not have isolated points, and let $D = \{x \in X\ ;\ \mathrm{orb}(x, T)$ is dense in $X\}$.

Exercise 6.1.7. Show the following generalization of Ansari's theorem. If D contains a connected and dense set then T and T^p, $p \in \mathbb{N}$, have the same points of dense orbits. (*Hint*: Follow the proof of Ansari's theorem and note that D itself must be connected; see the proof of Corollary 2.56.)

Exercise 6.1.8. An alternative proof of Exercise 6.1.7 (and thus of Ansari's theorem) is the following. With the notation of Theorem 6.2, let

$$A_k := \bigcup_{0\le j_1<\cdots<j_k\le p-1} \left(D_{j_1}\cap\cdots\cap D_{j_k}\right),$$

where $k=1,\dots,p$. Prove the following assertions:
(i) $A_1=D$, $A_p=\bigcap_{j=0}^{p-1} D_j$, and $A_{k+1}\subset A_k$, $k=1,\dots,p-1$;
(ii) $T(A_k)\subset A_k$, $k=1,\dots,p$;
(iii) if $A_k=D$, then $A_{k+1}=D$, $k=1,\dots,p-1$.
In particular, $\mathrm{orb}(x,T^p)$ is dense in X for every $x\in D$. (*Hint*: For (iii), observe that if $A_{k+1}\neq D$, then $A_{k+1}=\varnothing$, since it is closed, T-invariant, and $T:D\to D$ is minimal; hence A_k is a finite union of pairwise disjoint closed sets.)

Exercise 6.2.1. Prove assertions (i), (ii) and (iii) before Lemma 6.3. (*Hint*: See the proof of Proposition 1.15.)

Exercise 6.2.2. Prove Lemma 6.3. (*Hint*: Follow the argument of Theorem 2.54.)

Exercise 6.2.3. Let T be a continuous map on a metric space X without isolated points, and let $x\in X$. With the notation of this section, prove that if $U(x)\neq\varnothing$ and $T(X\setminus U(x))\subset X\setminus U(x)$, then $D(x)=\overline{U(x)}$. (*Hint*: Show that $\mathrm{orb}(x,T)\subset U(x)$.)

Exercise 6.2.4. Let T be an operator on a Fréchet space X and $x\in X$. With the notation of this section, prove directly that if X is a complex (or real) space, then $U(x)=U(\lambda x)=\lambda U(x)$ for $\lambda\neq 0$ (or for $\lambda>0$, respectively). Deduce that $D(x)=X$ if $0\in U(x)$. (*Hint*: Use Lemma 6.1.)

Exercise 6.2.5. Let $T:X\to X$ be a (not necessarily linear) topologically transitive dynamical system and $x\in X$. Show that if $\mathrm{orb}(x,T)$ is somewhere dense in X, then it is dense in X.

Exercise 6.2.6. Let T be an operator on a Fréchet space X and x a hypercyclic vector for T. Show that there exists an increasing sequence $(n_k)_k$ of positive integers with $\sup_{k\ge1}(n_{k+1}-n_k)=2$ such that the orbit of x under $(T^{n_k})_k$ is somewhere dense but not dense. (*Hint*: See Exercise 6.1.6.)

Exercise 6.3.1. Let T be an invertible operator on a Fréchet space X and $x\in X$ such that $\{T^n x\ ;\ n\in\mathbb{Z}\}$ is dense in X. Show that x is either hypercyclic for T or for T^{-1}; in particular, both T and T^{-1} are hypercyclic. For the proof,
(i) either use the Bourdon–Feldman theorem,
(ii) or proceed directly.
(*Hint*: For (i); see the proof of Theorem 6.6. For (ii), suppose that $T^n x\notin U$ for all $n\ge 0$; for any V, find $k\in\mathbb{Z}$ and $U'\subset U$ such that $T^k(U')\subset V$; find $m<-|k|$ such that $T^m x\in U'$; then $T^{m+k}x\in V$, $m+k<0$.)

Exercise 6.3.2. Let T be an operator on a Fréchet space X admitting a countable set $\{x_1,x_2,\dots\}$ of vectors such that

$$\bigcup_{j=1}^{\infty}\overline{\mathrm{orb}(x_j,T)}=X.$$

Show that some vector x_j, $j\ge1$, is hypercyclic for T. Give an example of an operator on a normed space for which this assertion fails.

Exercise 6.3.3. An operator T on a Banach space X is called *countably hypercyclic* if it admits a countable bounded set $\{x_1, x_2, \dots\}$ of vectors with $\inf_{j\neq k} \|x_j - x_k\| > 0$ such that

$$\overline{\bigcup_{j=1}^{\infty} \mathrm{orb}(x_j, T)} = X.$$

Show that the operator $T = 2(I \oplus B)$ on $X = \ell^2 \oplus \ell^2$ is countably hypercyclic but not hypercyclic, where B is the backward shift. (*Hint*: Take $x_j = (0, e_j) + 2^{-n_j}(I \oplus F)^{n_j} y_j$, where F is the forward shift.)

Exercise 6.3.4. Let T be a countably hypercyclic operator on a Banach space X. Show the following.

(a) The spectrum $\sigma(T)$ meets the unit circle.

(b) The orbit of every $x^* \neq 0$ in X^* under T^* is unbounded.

(*Hint*: See Section 5.1.)

Exercise 6.3.5. Let $T = B_w$ be a weighted backward shift on $X = \ell^p$, $1 \leq p < \infty$, or c_0; see Section 4.1. Show that if T is countably hypercyclic then it is hypercyclic. (*Hint*: Apply Exercise 6.3.4(b) to $x^* = e_1$.)

Exercise 6.4.1. Let $T = B_w$ be the weighted bilateral backward shift on $\ell^2(\mathbb{Z})$ with weights $w_n = \frac{n+1}{n}$ if $n \geq 1$ and $w_{-n} = \frac{n+1}{n+2}$ if $n \geq 0$; see Section 4.1. Show that λT, $\lambda \in \mathbb{C}$, is hypercyclic if and only if $|\lambda| = 1$. Discuss this result in the light of Kitai's theorem, showing first that $\sigma(T) \subset \mathbb{T}$. (*Hint*: For the first part note that $\lambda B_w = B_{\lambda w}$; for the second part use the spectral radius formula for T and T^{-1} and Exercise 5.0.7.)

Exercise 6.4.2. Let T_j be operators on complex Fréchet spaces X_j, $j = 1, \dots, n$, such that $T_1 \oplus \dots \oplus T_n$ is hypercyclic. Show that, for any $\lambda_j \in \mathbb{C}$ with $|\lambda_j| = 1$, $j = 1, \dots, n$, the operator $\lambda_1 T_1 \oplus \dots \oplus \lambda_n T_n$ is also hypercyclic and, moreover, that it shares the set of hypercyclic vectors with $T_1 \oplus \dots \oplus T_n$. (*Hint*: Set $\Psi(n,t) = S_1^n \oplus e^{2\pi t i} S_2^n$ for suitable operators S_1, S_2, deduce that $HC(\Psi) = HC(S_1 \oplus S_2)$, and apply this result repeatedly.)

Exercise 6.4.3. Let T_j be operators on complex Fréchet spaces X_j, $j = 1, \dots, n$, and let $x_j \in X_j$, $j = 1, \dots, n$, be such that

$$\left\{(\lambda_1 T_1^k x_1, \dots, \lambda_n T_n^k x_n) \ ; \ k \in \mathbb{N}_0, \ (\lambda_1, \dots, \lambda_n) \in \mathbb{T}^n \right\}$$

is dense in $X_1 \oplus \dots \oplus X_n$. Show that $x := (x_1, \dots, x_n)$ is hypercyclic for $T_1 \oplus \dots \oplus T_n$. (*Hint*: Let $(U_m)_m$ be a countable base of open sets in $X_1 \oplus \dots \oplus X_n$. Show that the sets $\{(\mu_1, \dots, \mu_n) \in \mathbb{T}^n \ ; \ \exists\, k \in \mathbb{N}_0$ with $(\mu_1^k T_1^k x_1, \dots, \mu_n^k T_n^k x_n) \in U_m\}$ are open and dense in $\mathbb{T}^n$. By a Baire argument, find $(\mu_1, \dots, \mu_n) \in \mathbb{T}^n$ such that x is hypercyclic for $\mu_1 T_1 \oplus \dots \oplus \mu_n T_n$, and conclude by using Exercise 6.4.2.)

Exercise 6.4.4. Let $X = C_0(\mathbb{R}_+)$, the space of continuous functions on $\mathbb{R}_+$ that vanish at ∞, endowed with the sup-norm. Consider the semigroup action $\Psi(n,t)f(x) := 2^{n-t} f(x+t)$, $n \in \mathbb{N}_0$, $t \in \mathbb{R}_+$. Then Ψ is hypercyclic but the operator $\Psi(1,1)$ is not hypercyclic. Which hypothesis of Theorem 6.10 is not satisfied?

Exercise 6.4.5. Give a new proof of Ansari's theorem by proceeding as follows. Let T be a hypercyclic operator on a Fréchet space X, x a hypercyclic vector for T and $p \in \mathbb{N}$. For $u, v \in X$ define the subset $F_{u,v}$ of $\mathbb{T}$ by

$$F_{u,v} = \left\{ e^{2\pi j i/p} \ ; \ \exists\, (n_k)_k \subset \mathbb{N}_0 \text{ with } T^{n_k p + j} u \to v, j = 0, \dots, p-1 \right\}.$$

Show the following:

(i) if $u \in HC(T)$, then $F_{u,v} \neq \varnothing$ for all $v \in X$;
(ii) if $u, v, w \in X$, $\lambda \in F_{u,v}$, and $\mu \in F_{v,w}$, then $\lambda\mu \in F_{u,w}$;
(iii) there is a divisor $m \geq 1$ of p such that $F_{x,x} = \{e^{2\pi mi/p}\ ;\ j = 0, \dots, p/m - 1\}$;
(iv) for every $y \in HC(T)$ there is some j, $0 \leq j \leq m-1$, such that $F_{x,y} = e^{2\pi ji/p} F_{x,x}$.

Now let $D_j = \{y \in HC(T)\ ;\ F_{x,y} = e^{2\pi ji/p} F_{x,x}\}$, $j = 0, \dots, m-1$. Then finish the proof as follows:

(v) show that the D_j form a partition of $HC(T)$ into closed (and open) sets;
(vi) deduce that $m = 1$ and hence that $x \in HC(T^p)$.

In the following two exercises, let $T : X \to X$ be a (not necessarily linear) dynamical system, where X does not have isolated points.

Exercise 6.4.6. Show the following *separation theorem.* If $x \in X$ has dense orbit under T but not under T^p, $p > 1$, then there is a divisor $m > 1$ of p and a partition $D_0, \dots, D_{m-1}$ of $D = \{x \in X\ ;\ \text{orb}(x, T) \text{ is dense in } X\}$ into closed (and open) subsets with the following properties:

(i) $T(D_j) \subset D_{j+1 (\text{mod}\, m)}$, $j = 0, \dots, m-1$;
(ii) for $j = 0, \dots, m-1$, the orbit of $T^j x$ under T^p is contained and dense in D_j.

(*Hint*: Proceed as in the previous exercise.)

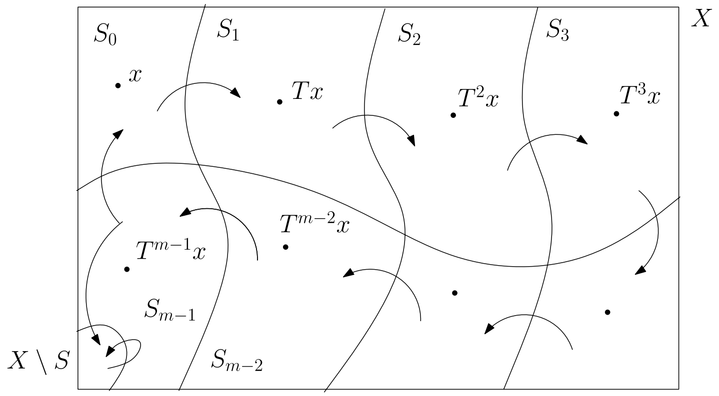

Fig. 6.3 Nonlinear dynamics if x has dense orbit under T but not under T^p, $m|p$

Exercise 6.4.7. Show the following *decomposition theorem.* If $x \in X$ has dense orbit under T but not under T^p, $p > 1$, then there is a divisor $m > 1$ of p and pairwise disjoint open subsets $S_0, S_1, \dots, S_{m-1}$ of X with the following properties:

(i) $S := \bigcup_{j=0}^{m-1} S_j$ is dense in X;
(ii) $T(S_j) \subset S_{j+1}$, $j = 0, \dots, m-2$, and $T(S_{m-1}) \subset S_0 \cup (X \setminus S)$;
(iii) $X \setminus S$ is invariant under T;
(iv) for $j = 0, \dots, m-1$, the orbit of $T^j x$ under T^p is contained and dense in S_j;

see Figure 6.3.

(*Hint*: Consider the sets D_j of the previous exercise; set $S_{m-1} = X \setminus \bigcup_{j=0}^{m-2} \overline{D_j}$, with closure in X, and $S_j = T^{-m+j+1}(S_{m-1})$; show first that $T^{-m}(S_{m-1}) \subset S_{m-1}$, $D_j \subset S_j$, and that $T^n x \in S_j$ if and only if $n = j (\text{mod}\, m)$.)

Exercise 6.4.8. Verify the results of the previous two exercises in the case of the map of Exercise 1.2.11.

Sources and comments

The results in this chapter have in common that their proofs use connectedness arguments. But their relationship runs deeper than that. As we have seen, Ansari's theorem is a consequence of the Costakis–Peris theorem, which in turn follows from the Bourdon–Feldman theorem. Moreover, the theorems of León–Müller and Conejero–Müller–Peris are proved by a common approach. Recently, Shkarin [284] was able to unify the latter two theorems with Ansari's theorem by deriving them as consequences of a single, quite general result. An alternative common framework was developed by Bayart and Matheron [44], which was further generalized by Matheron [235] to include Shkarin's result.

Section 6.1. Ansari [9] showed that powers of hypercyclic operators are hypercyclic. Independently, Banks [28] proved a more general result: any power of a minimal map on a connected topological space is also minimal (see Exercise 6.1.7). We combine ideas from Banks [28] and Peris [254] for the proof of Theorem 6.2. Lemma 6.1 is from Peris [254].

Section 6.2. Theorem 6.5 is due to Bourdon and Feldman [93], answering a question from Peris [254]. The corresponding result for semigroups of operators (see the next chapter for this notion) is due to Costakis and Peris [121]. It is interesting to note that for a weighted backward shift on ℓ^p, $1 \leq p < \infty$, to be hypercyclic it already suffices to have an orbit with a nonzero limit point, as was shown by Chan and Seceleanu [105]; such an orbit though, need not be dense.

Section 6.3. The fact that multi-hypercyclic operators are hypercyclic was independently proved by Costakis [117] and Peris [254], answering a question raised by Herrero [195]. The original proofs motivated the question leading to the Bourdon–Feldman theorem.

Section 6.4. Theorem 6.7 on rotations of hypercyclic operators is due to León and Müller [222]. Bayart and Bermúdez [37] show that the corresponding result for chaos fails. Badea, Grivaux and Müller [20] characterize the subsets of $\mathbb{C}$ that can appear as $\{\lambda \in \mathbb{C}\ ;\ \lambda T \text{ hypercyclic}\}$ for invertible operators T on a complex Hilbert space.

Theorem 6.8 on discretizations of hypercyclic C_0-semigroups is due to Conejero, Müller and Peris [110]. Exercise 6.4.4 shows that the result fails for semigroups indexed over $\mathbb{N}_0 \times \mathbb{R}_+$; see also Shkarin [284] and Exercise 7.3.1. Bayart [36] shows that it even fails for holomorphic groups over $\mathbb{C}$. And by Bayart and Bermúdez [37] there are chaotic C_0-semigroups on a Hilbert space for which no individual operator is chaotic.

The unified proof of Theorem 6.10 essentially follows the argument of [110]. The related approach to Ansari's theorem in Exercise 6.4.5 is due to Grosse-Erdmann, León and Piqueras [183]. As mentioned above, Shkarin [284], Bayart and Matheron [44] and Matheron [235] obtain much more general results that contain the theorems of Ansari, León–Müller and Conejero–Müller–Peris as special cases. Shkarin and Matheron point out that the main common idea in all these proofs can already be found in a paper by Furstenberg [156].

Exercises. Exercise 6.1.6 is taken from Montes and Salas [243], Exercise 6.1.7 from Banks [28]. Exercise 6.1.8 outlines essentially the original proof of Ansari [9]. Exercises 6.2.2 and 6.2.3 are taken from Bourdon and Feldman [93], while the result of Exercise 6.2.6 is due to Peris and Saldivia [257]. The result of Exercise 6.3.1 is due to Herrero and Kitai [196]. The notion of a countably hypercyclic operator (see Exercise 6.3.3), as well as the results of Exercises 6.3.4 and 6.3.5 are due to Feldman [149]. Exercise 6.4.1 is taken from León and Müller [222], Exercise 6.4.2 from Shkarin [284]. Exercises 6.4.6 and 6.4.7 are due to Grosse-Erdmann, León and Piqueras [183] (see also Marano and

Salas [226]); the case of $p = 2$ was previously obtained by Bourdon [91].

Extensions. We will show in Chapter 12 that the theorems of Ansari, Bourdon–Feldman, Costakis–Peris and León–Müller continue to hold in arbitrary topological vector spaces.