Chapter 4
Classes of hypercyclic and chaotic operators

In this chapter we study in detail some important classes of hypercyclic and chaotic operators. Each of them has its origin in the three classical hypercyclic operators. Rolewicz's multiples of backward shifts lead naturally to the study of arbitrary weighted shifts. MacLane's differentiation operator and Birkhoff's translation operators are both special cases of differential operators, while the translation operators can also be regarded as composition operators. Finally, Rolewicz's operators reappear as adjoint multipliers.

4.1 Weighted shifts

The basic model of all shifts is the *backward shift*

$$B(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

In order to distinguish this shift from the bilateral shift that we will discuss later one also speaks of the *unilateral backward shift*.

Rolewicz has shown that, for any λ with $|\lambda| > 1$, the multiples of B, $\lambda B(x_n)_n = (\lambda x_{n+1})_n$, are hypercyclic on the sequence space ℓ^2. It is then a small step to let the weights vary from coordinate to coordinate, which leads to the *(unilateral) weighted shift*

$$B_w(x_1, x_2, x_3, \ldots) = (w_2 x_2, w_3 x_3, w_4 x_4, \ldots),$$

where

$$w = (w_n)_n$$

is a sequence of nonzero scalars, called a *weight sequence*. Note that the value of w_1 is irrelevant.

We may also generalize these operators in a different direction. Rolewicz had already replaced ℓ^2 by any of the spaces ℓ^p, $1 \leq p < \infty$, and c_0. More

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generally, one may take as the underlying space an arbitrary *sequence space* X, that is, a linear space of sequences or, in other words, a subspace of $\omega = \mathbb{K}^{\mathbb{N}}$. In addition, X should carry a topology that is compatible with the sequence space structure of X. We interpret this as demanding that the embedding $X \to \omega$ is continuous, that is, convergence in X should imply coordinatewise convergence. A Banach (Fréchet, ...) space of this kind is called a *Banach (Fréchet, ...) sequence space.* The terms of a sequence $x, y, z, \ldots$ will be denoted by $x_n, y_n, z_n, \ldots$, $n \geq 1$.

By e_n, $n \in \mathbb{N}$,

$$e_n = (\delta_{n,k})_{k\in\mathbb{N}} = (0, \ldots, 0, \underset{n}{1}, 0, \ldots)$$

we denote the canonical unit sequences. If the e_n are contained in X and span a dense subspace then an alternative way of describing weighted shifts is by saying that

$$B_w e_n = w_n e_{n-1},\ n \geq 1, \quad \text{with } e_0 := 0.$$

The continuity of the embedding $X \to \omega$ amounts to requiring the continuity of each coordinate functional

$$X \to \mathbb{K},\ x \to x_n,\ n \geq 1,$$

which implies that each weighted shift has closed graph. From the closed graph theorem (see Appendix A) we thus obtain that a weighted shift defines an operator on a Fréchet sequence space X as soon as it maps X into itself.

Proposition 4.1. *Let X be a Fréchet sequence space. Then every weighted shift $B_w : X \to X$ is continuous.*

We start by studying the (unweighted) backward shift B. Our results will then extend immediately to all weighted shifts via a simple conjugacy.

The following technical result will help us simplify the condition characterizing hypercyclicity of B.

Lemma 4.2. *Let X be a metric space, $v_n \in X$, $n \geq 1$, and $v \in X$. Suppose that there is a strictly increasing sequence $(n_k)_k$ of positive integers such that*

$$v_{n_k - j} \to v \quad \text{for every } j \in \mathbb{N}.$$

Then there exists a strictly increasing sequence $(m_k)_k$ of positive integers such that

$$v_{m_k + j} \to v \quad \text{for every } j \in \mathbb{N}.$$

Proof. Let d denote the metric in X. It follows from the assumption that, for any $k \geq 1$, there is some $N_k \geq k + 2$ such that

$$d(v_{N_k - j}, v) < \frac{1}{k}, \quad j = 1, \ldots, k.$$

Setting $m_k = N_k - k - 1$, $k \geq 1$, we see that $d(v_{m_k+k+1-j}, v) < \frac{1}{k}$ for $j = 1, \ldots, k$, hence

$$d(v_{m_k+j}, v) < \frac{1}{k}, \quad j = 1, \ldots, k;$$

this implies the assertion when we pass to a strictly increasing subsequence of $(m_k)_k$, if necessary. □

We recall that the sequence $(e_n)_n$ is a basis in the space X if each e_n, $n \in \mathbb{N}$, belongs to X and, for any $x \in X$,

$$x = \lim_{N\to\infty} (x_1, x_2, \ldots, x_N, 0, 0, \ldots) = \sum_{n=1}^{\infty} x_n e_n.$$

Clearly, $(e_n)_n$ is a basis in any of the sequence spaces ℓ^p, $1 \leq p < \infty$, c_0 and ω.

Theorem 4.3. *Let X be a Fréchet sequence space in which $(e_n)_n$ is a basis. Suppose that the backward shift B is an operator on X. Then the following assertions are equivalent:*

(i) *B is hypercyclic;*

(ii) *B is weakly mixing;*

(iii) *there is an increasing sequence $(n_k)_k$ of positive integers such that $e_{n_k} \to 0$ in X as $k \to \infty$.*

Proof. Let $\|\cdot\|$ stand for an F-norm that induces the topology of X; see Section 2.1.

(i)$\Longrightarrow$(iii). Suppose that B is hypercyclic. Let $N \in \mathbb{N}$ and $\varepsilon > 0$. We show that there exists some $n \geq N$ with $\|e_n\| < \varepsilon$.

It follows from the basis assumption that, for any $x \in X$, the sequence $(x_n e_n)_n$ converges to 0 in X. By the Banach–Steinhaus theorem (see Appendix A), applied to the operators $x \to x_n e_n$, $n \geq 1$, there is some $\delta > 0$ such that, for all $x \in X$,

$$\|x\| < \delta \quad \Longrightarrow \quad \|x_n e_n\| < \tfrac{\varepsilon}{2} \quad \text{for all } n \geq 1. \tag{4.1}$$

Moreover, since convergence in X implies coordinatewise convergence, there is some $\eta > 0$ such that, for all $x \in X$,

$$\|x\| < \eta \quad \Longrightarrow \quad |x_1| \leq \tfrac{1}{2}. \tag{4.2}$$

Now, since B is hypercyclic and therefore topologically transitive, there are $x \in X$ and $n \geq N$ such that

$$\|x\| < \delta \quad \text{and} \quad \|B^{n-1}x - e_1\| < \eta.$$

Hence, by (4.1) and (4.2),

$$\|x_n e_n\| < \tfrac{\varepsilon}{2} \quad \text{and} \quad |x_n - 1| \le \tfrac{1}{2}; \tag{4.3}$$

the latter implies that x_n is closer to 1 than to 0 and hence that

$$\left|x_n^{-1} - 1\right| = \left|\frac{1 - x_n}{x_n}\right| \le 1.$$

From this and (4.3) we deduce, using the properties of an F-norm, that

$$\|e_n\| = \left\|(x_n^{-1} - 1)x_n e_n + x_n e_n\right\| \le \left\|(x_n^{-1} - 1)x_n e_n\right\| + \left\|x_n e_n\right\| < \varepsilon, \tag{4.4}$$

which had to be shown.

(iii)$\Longrightarrow$(ii). We apply the Hypercyclicity Criterion. For $X_0 = Y_0$ we take the set of finite sequences, which by the basis assumption is dense in X. For S_n we take the nth iterate of the forward shift

$$F : (x_1, x_2, x_3, \ldots) \to (0, x_1, x_2, \ldots),$$

that is, $S_n = F^n : Y_0 \to X$, $n \ge 1$. With this, conditions (i) and (iii) of the Hypercyclicity Criterion hold even for the full sequence (n).

As for condition (ii) note that, by continuity of B,

$$e_{n_k - j} = B^j e_{n_k} \to 0 \quad \text{as } k \to \infty,$$

for all $j \ge 1$. Since $(n_k)_k$ must be strictly increasing, it follows from Lemma 4.2 that there is an increasing sequence $(m_k)_k$ of positive integers such that

$$e_{m_k + j} \to 0 \quad \text{as } k \to \infty,$$

for all $j \ge 1$. But since $S_{m_k} e_j = e_{m_k + j}$, we have by linearity that

$$S_{m_k} y \to 0$$

for any $y \in Y_0$. This shows that conditions (i)–(iii) of the Hypercyclicity Criterion hold for the sequence $(m_k)_k$, so that B is weakly mixing.

(ii)$\Longrightarrow$(i) holds for all operators on X. □

We point out that B being an operator on X is part of the hypothesis. By Proposition 4.1 this can be restated simply as saying that $(x_{n+1})_n \in X$ whenever $(x_n)_n \in X$, which is usually easily verified for concrete spaces.

Example 4.4. (a) Let

$$\ell^p(v) = \Big\{(x_n)_{n\ge1} \;;\; \sum_{n=1}^{\infty} |x_n|^p v_n < \infty\Big\}, \quad 1 \le p < \infty,$$

be a weighted ℓ^p-space, where $v = (v_n)_n$ is a positive weight sequence. Then B is an operator on $\ell^p(v)$ if and only if there is an $M > 0$ such that, for all

$x \in \ell^p(v)$,

$$\Big(\sum_{n=1}^{\infty}|x_{n+1}|^p v_n\Big)^{1/p} \leq M\Big(\sum_{n=1}^{\infty}|x_n|^p v_n\Big)^{1/p},$$

which is equivalent to $\sup_{n\in\mathbb{N}} \frac{v_n}{v_{n+1}} < \infty$. Theorem 4.3 tells us that, under this condition, the hypercyclicity of B is characterized by

$$\inf_{n\in\mathbb{N}} v_n = 0.$$

The same conditions also characterize the continuity and hypercyclicity of the backward shift B on the weighted c_0-space

$$c_0(v) = \big\{(x_n)_{n\geq 1}\ ;\ \lim_{n\to\infty}|x_n|v_n = 0\big\}.$$

(b) Spaces of holomorphic functions constitute a rich and interesting source of sequence spaces via the identification of a holomorphic function with its sequence of Taylor coefficients. As a first example we consider here the *Bergman space* A^2 of all holomorphic functions f on the unit disk $\mathbb{D} = \{z \in \mathbb{C}\ ;\ |z| < 1\}$ such that

$$\|f\|^2 := \frac{1}{\pi}\int_{\mathbb{D}}|f(z)|^2\,d\lambda(z) < \infty,$$

where λ denotes two-dimensional Lebesgue measure. Using polar coordinates and writing $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we obtain that

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{D}}|f(z)|^2\,d\lambda(z) &= \frac{1}{\pi}\int_0^1\int_0^{2\pi}\Big|\sum_{n=0}^{\infty}a_n(re^{it})^n\Big|^2 dt\,r\,dr\\ &= 2\int_0^1\Big(\int_0^{2\pi}\Big|\sum_{n=0}^{\infty}a_n r^n\frac{1}{\sqrt{2\pi}}\,e^{int}\Big|^2dt\Big)r\,dr\\ &= 2\int_0^1\sum_{n=0}^{\infty}|a_n|^2r^{2n}r\,dr = \sum_{n=0}^{\infty}|a_n|^2\frac{1}{n+1},\end{aligned}$$

where we have applied Parseval's identity in $L^2[0,2\pi]$ for the orthonormal basis $(\frac{1}{\sqrt{2\pi}}\,e^{int})_{n\in\mathbb{Z}}$. As a consequence, A^2 is isometrically isomorphic to the weighted space $\ell^2(\frac{1}{n+1})$ (with indices running from 0). By (a), the backward shift is therefore hypercyclic on A^2. When acting on functions, B is the operator

$$Bf(z) = \sum_{n=0}^{\infty}a_{n+1}z^n = \frac{1}{z}(f(z) - f(0)) \quad \text{with } Bf(0) = f'(0).$$

Further Banach and Hilbert spaces of holomorphic functions will be studied in Section 4.4.

(c) As in (b) we can consider the space $H(\mathbb{C})$ of entire functions (see Example 2.1) as a sequence space by identifying the entire function $f(z) = \sum_{n=0}^\infty a_n z^n$ with the sequence $(a_n)_{n\geq 0}$. By the formula for the radius of convergence of Taylor series, this sequence space is given by

$$\Big\{(a_n)_{n\geq 0}\ ;\ \lim_{n\to\infty} |a_n|^{1/n} = 0\Big\} = \Big\{(a_n)_{n\geq 0}\ ;\ \sum_{n=0}^\infty |a_n| m^n < \infty,\ m \geq 1\Big\}.$$

Since $|a_{n+1}|^{1/n} = (|a_{n+1}|^{1/(n+1)})^{(n+1)/n} \to 0$ if $|a_n|^{1/n} \to 0$, we have that the backward shift B is an operator on $H(\mathbb{C})$; see Proposition 4.1. Moreover, the unit sequences e_n correspond to the monomials $z \to z^n$, $n \geq 0$. It then follows from Theorem 4.3 that B is not hypercyclic on $H(\mathbb{C})$.

Using the same arguments as in the proof of Theorem 4.3, but employing Kitai's criterion instead of the Hypercyclicity Criterion, we obtain a characterization of the mixing property for B.

Theorem 4.5. *Let X be a Fréchet sequence space in which $(e_n)_n$ is a basis. Suppose that the backward shift B is an operator on X. Then the following assertions are equivalent:*

(i) *B is mixing;*
(ii) *$e_n \to 0$ in X as $n \to \infty$.*

For chaos we have a curious phenomenon. Proceeding as before, but with somewhat stronger assumptions on the space X, we easily obtain a condition that characterizes chaos for B. But it turns out that this condition is already implied by the existence of a single *nontrivial periodic point*, that is, a periodic point other than 0. Hence, this fact alone implies chaos.

For this result we will require that $(e_n)_n$ is an unconditional basis, that is, it is a basis in X such that, for any $(x_n)_n \in X$ and any 0-1-sequence $(\varepsilon_n)_n$, the series

$$\sum_{n=1}^\infty \varepsilon_n x_n e_n$$

converges in X; see Appendix A.

Theorem 4.6. *Let X be a Fréchet sequence space in which $(e_n)_n$ is an unconditional basis. Suppose that the backward shift B is an operator on X. Then the following assertions are equivalent:*

(i) *B is chaotic;*
(ii) *$\sum_{n=1}^\infty e_n$ converges in X;*
(iii) *the constant sequences belong to X;*
(iv) *B has a nontrivial periodic point.*

Proof. (i)$\Longrightarrow$(iv) is trivial.

(iv)$\Longrightarrow$(iii). Let $x = (x_1, x_2, x_3, \ldots) \neq 0$ be periodic for B, that is, a periodic sequence. Let N be its period. Then there is some $j \leq N$ such that

$x_j \neq 0$, and we have $x_{j+\nu N} = x_j$ for $\nu \geq 0$. Setting all coordinates with indices other than $j + \nu N$ to zero and dividing the result by x_j we obtain, by unconditionality of the basis, that

$$\sum_{\nu=0}^{\infty} e_{j+\nu N} \in X.$$

Applying the backward shift $N-1$ times and adding the results we obtain (iii).

(iii)$\Longrightarrow$(ii) follows from our assumptions.

(ii)$\Longrightarrow$(i). First, by Theorem 4.3, condition (ii) implies that B is hypercyclic.

Next, since $(1,1,1,\ldots) \in X$, the unconditionality of the basis implies that all the periodic 0-1-sequences belongs to X, and hence also all the periodic sequences, which are exactly the periodic points for B. It remains to show that these form a dense set in X.

To see this, let $x = (x_n)_n \in X$ and $\varepsilon > 0$. Since $(e_n)_n$ is a basis, there is some $N \geq 1$ such that

$$\widetilde{x} := \sum_{n=1}^{N} x_n e_n$$

has distance less than $\varepsilon/2$ from x. The associated periodic sequence

$$\sum_{\nu=0}^{\infty} \sum_{n=1}^{N} x_n e_{n+\nu N}$$

belongs to X. The unconditionality of the basis implies that there is some $m \geq 1$ such that

$$\Big\| \sum_{\nu=m}^{\infty} \sum_{n=1}^{N} x_n \varepsilon_{n+\nu N} e_{n+\nu N} \Big\| < \frac{\varepsilon}{2}$$

for any 0-1-sequence $(\varepsilon_n)_n$; see Theorem A.16. In particular we have that

$$\Big\| \sum_{\mu=1}^{\infty} \sum_{n=1}^{N} x_n e_{n+\mu m N} \Big\| < \frac{\varepsilon}{2}.$$

This shows that the periodic point

$$\sum_{\mu=0}^{\infty} \sum_{n=1}^{N} x_n e_{n+\mu m N}$$

has distance less than $\varepsilon/2$ from $\widetilde{x}$, hence less than ε from x. □

Example 4.7. (a) We consider again the space $\ell^p(v)$ of Example 4.4(a). Under the assumption that B is an operator on $\ell^p(v)$ we have that B is mixing if and only if

$$\lim_{n\to\infty} v_n = 0,$$

and it is chaotic if and only if

$$\sum_{n=1}^{\infty} v_n < \infty.$$

In this example, mixing is implied by chaos. In particular, the backward shift on the Bergman space A^2 is mixing but not chaotic.

(b) It is not difficult to give an example that shows that Theorem 4.6 does not remain valid if one drops the unconditionality assumption on the basis $(e_n)_n$; see Exercise 4.1.3.

It is now an easy matter to transfer our results so far to arbitrary weighted shifts by means of a suitable conjugacy. Let B_w be a weighted shift on some sequence space X. We define new weights v_n by

$$v_n = \Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1}, \quad n \geq 1,$$

and consider the sequence space

$$X_v = \{(x_n)_n \;;\; (x_n v_n)_n \in X\}.$$

The map $\phi_v : X_v \to X$, $(x_n)_n \to (x_n v_n)_n$ is a vector space isomorphism.

We may use ϕ_v to transfer a topology from X to X_v: a set U is open in X_v if and only if $\phi_v(U)$ is open in X. If X is a Banach (Fréchet, ...) sequence space then so is X_v. And if $(e_n)_n$ is a basis in X then it is also a basis in X_v.

Finally, a simple calculation shows that $B_w \circ \phi_v = \phi_v \circ B$, that is, the following diagram commutes:

$$\begin{array}{ccc} X_v & \xrightarrow{B} & X_v \\ {\scriptstyle\phi_v}\downarrow & & \downarrow{\scriptstyle\phi_v} \\ X & \xrightarrow{B_w} & X. \end{array}$$

Thus $B_w : X \to X$ and $B : X_v \to X_v$ are conjugate operators.

Since conjugacies preserve hypercyclicity, (weak) mixing and chaos, our previous results immediately yield the following.

Theorem 4.8. *Let X be a Fréchet sequence space in which $(e_n)_n$ is a basis. Suppose that the weighted shift B_w is an operator on X.*

(a) *The following assertions are equivalent:*

  (i) *B_w is hypercyclic;*

  (ii) *B_w is weakly mixing;*

(iii) *there is an increasing sequence* $(n_k)_k$ *of positive integers such that*

$$\Big(\prod_{\nu=1}^{n_k} w_\nu\Big)^{-1} e_{n_k} \to 0$$

in X *as* $k \to \infty$.

(b) *The following assertions are equivalent:*

(i) B_w *is mixing;*

(ii) *we have that*

$$\Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} e_{n} \to 0$$

in X *as* $n \to \infty$.

(c) *Suppose that the basis* $(e_n)_n$ *is unconditional. Then the following assertions are equivalent:*

(i) B_w *is chaotic;*

(ii) *the series*

$$\sum_{n=1}^{\infty}\Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} e_{n}$$

converges in X*;*

(iii) *the sequence*

$$\bigg(\Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1}\bigg)_n$$

belongs to X*;*

(iv) B_w *has a nontrivial periodic point.*

Example 4.9. (a) A weighted shift B_w is an operator on a sequence space ℓ^p, $1 \leq p < \infty$, or c_0 if and only if the weights w_n, $n \geq 1$, are bounded. The respective characterizing conditions for B_w to be hypercyclic, mixing or chaotic on ℓ^p, $1 \leq p < \infty$, are

$$\sup_{n\geq 1}\prod_{\nu=1}^{n}|w_\nu| = \infty, \quad \lim_{n\to\infty}\prod_{\nu=1}^{n}|w_\nu| = \infty, \quad \sum_{n=1}^{\infty}\frac{1}{\prod_{\nu=1}^{n}|w_\nu|^p} < \infty.$$

We remark that only the third condition depends on the parameter p. The first condition also characterizes when B_w is hypercyclic on c_0, and the second when it is mixing or, equivalently, chaotic on c_0.

In particular, for Rolewicz's operator $T = \lambda B$, $|\lambda| > 1$, we have that $\prod_{\nu=1}^{n}|w_\nu| = \lambda^n$, which implies once more that this operator is chaotic.

As another specific example we consider, for $\alpha > 0$, the weights $w_n = (\frac{n+1}{n})^\alpha$, $n \geq 1$. Then $\prod_{\nu=1}^{n}|w_\nu| = (n+1)^\alpha$, and the corresponding weighted shift is mixing; it is even chaotic on c_0, and it is chaotic on ℓ^p exactly when $\alpha > 1/p$. We note that, for $w_n = (\frac{n+1}{n})^{1/2}$, $n \geq 1$, the weighted shift B_w

on ℓ^2 is conjugate to the backward shift on the Bergman space; see Example 4.4(b).

(b) We consider the Fréchet space $H(\mathbb{C})$ of all entire functions as a sequence space; see Example 4.4(c). It is easy to see that a weighted shift B_w defines an operator on $H(\mathbb{C})$ if and only if $\sup_{n\geq 1} |w_n|^{1/n} < \infty$; moreover, we have that $a_n e_n \to 0$ in $H(\mathbb{C})$ if and only if $|a_n|^{1/n} \to 0$; see Exercise 4.1.1. Theorem 4.8 then shows that a weighted shift B_w on $H(\mathbb{C})$ is mixing if and only if it is chaotic, and that the characterizing conditions for hypercyclicity and mixing/chaos are, respectively,

$$\sup_{n\geq 1} \Big(\prod_{\nu=1}^{n} |w_\nu| \Big)^{1/n} = \infty, \quad \lim_{n\to\infty} \Big(\prod_{\nu=1}^{n} |w_\nu| \Big)^{1/n} = \infty.$$

In particular, for the differentiation operator we have that $D(\sum_{n=0}^\infty a_n z^n) = \sum_{n=0}^\infty (n+1) a_{n+1} z^n$, so that D is a weighted shift with weight sequence $w_n = n$, $n \geq 1$. Since $(n!)^{1/n} \to \infty$ we obtain MacLane's theorem that D is hypercyclic; in fact, it is even a mixing and chaotic operator.

But in order to prove chaos for D it suffices, as we have seen, to come up with a nontrivial periodic point; the easiest such example is $f(z) = e^z$. Thus, one might be tempted to say that *the exponential function makes D chaotic.*

(c) In the space $\omega = \mathbb{K}^{\mathbb{N}}$, every series $\sum_{n=1}^\infty a_n e_n$ converges. Thus, every weighted shift B_w defines an operator on ω and is, indeed, mixing and chaotic on ω.

Remark 4.10. By Example 4.9(a), any bounded weight sequence $(w_n)_n$ with

$$\liminf_{n\to\infty} \prod_{\nu=1}^{n} |w_\nu| < \limsup_{n\to\infty} \prod_{\nu=1}^{n} |w_\nu| = \infty$$

defines a weighted shift B_w on ℓ^p, $1 \leq p < \infty$, or c_0 that is weakly mixing but not mixing. This provides us with a large supply of operators of this kind; see also Example 3.11.

Remark 4.11. One might wonder why we have studied backward shifts and not forward shifts. The simple truth is that a forward shift is never hypercyclic. More precisely, a *(unilateral) weighted forward shift* is given by

$$F_w(x_1, x_2, x_3, \ldots) = (0, w_1 x_1, w_2 x_2, \ldots)$$

with a weight sequence $w = (w_n)_n$. The first coordinate of every point in the orbit of x is either x_1 or 0. By the assumption that convergence in the space implies coordinatewise convergence no orbit can be dense.

Some new and interesting phenomena arise when we study shifts on sequence spaces indexed over $\mathbb{Z}$. The *bilateral backward shift* is given by

$$B(x_n)_{n\in\mathbb{Z}} = (x_{n+1})_{n\in\mathbb{Z}},$$

and the *bilateral weighted backward shifts* are given by

$$B_w(x_n)_{n\in\mathbb{Z}} = (w_{n+1}x_{n+1})_{n\in\mathbb{Z}},$$

where $w = (w_n)_{n\in\mathbb{Z}}$ is a *weight sequence*, that is, a sequence of nonzero scalars. The underlying space is then supposed to be a *Banach (Fréchet, ...) sequence space over* $\mathbb{Z}$, that is, a subspace of $\omega(\mathbb{Z}) := \mathbb{K}^{\mathbb{Z}}$ that carries a Banach (Fréchet, ...) space topology under which the embedding $X \to \omega(\mathbb{Z})$ is continuous.

In this new setting, we say that the unit sequences

$$e_n = (\delta_{n,k})_{k\in\mathbb{Z}}, \quad n \in \mathbb{Z},$$

form a basis in X if they are contained in X and if every sequence $x = (x_n)_{n\in\mathbb{Z}} \in X$ satisfies

$$x = \lim_{M,N\to\infty} (\ldots, 0, 0, x_{-M}, x_{-M+1}, \ldots, x_{N-1}, x_N, 0, 0, \ldots).$$

The *finite sequences* are the sequences in $\operatorname{span}\{e_n \ ; \ n \in \mathbb{Z}\}$.

Theorem 4.12. *Let X be a Fréchet sequence space over $\mathbb{Z}$ in which $(e_n)_{n\in\mathbb{Z}}$ is a basis. Suppose that the bilateral shift B is an operator on X.*

(a) *The following assertions are equivalent:*
  - (i) *B is hypercyclic;*
  - (ii) *B is weakly mixing;*
  - (iii) *there is an increasing sequence $(n_k)_k$ of positive integers such that, for any $j \in \mathbb{Z}$, $e_{j-n_k} \to 0$ and $e_{j+n_k} \to 0$ in X as $k \to \infty$.*

(b) *The following assertions are equivalent:*
  - (i) *B is mixing;*
  - (ii) *$e_{-n} \to 0$ and $e_n \to 0$ in X as $n \to \infty$.*

(c) *Suppose that the basis $(e_n)_n$ is unconditional. Then the following assertions are equivalent:*
  - (i) *B is chaotic;*
  - (ii) *$\sum_{n=-\infty}^{\infty} e_n$ converges in X;*
  - (iii) *the constant sequences belong to X;*
  - (iv) *B has a nontrivial periodic point.*

Proof. (a), (i)$\Longrightarrow$(iii). Let $\|\cdot\|$ be an F-norm that induces the topology of X. We will derive the following equivalent formulation of (iii): *for any $\varepsilon > 0$ and any $N \in \mathbb{N}$ there exists some $n \geq N$ such that if $|j| \leq N$, then*

$$\|e_{j-n}\| < \varepsilon \quad \textit{and} \quad \|e_{j+n}\| < \varepsilon.$$

To this end, we fix $\varepsilon > 0$ and $N \in \mathbb{N}$. As in the unilateral case we can find some $\delta > 0$ such that, for all $x \in X$,

$$\|x\| < \delta \quad \Longrightarrow \quad \|x_n e_n\| < \tfrac{\varepsilon}{2} \quad (n \in \mathbb{Z}) \quad \text{and} \quad |x_j| \leq \tfrac{1}{2} \quad (|j| \leq N). \tag{4.5}$$

Now, by the topological transitivity of B, we can find some $x \in X$ and $n > 2N$ such that

$$\Big\| x - \sum_{|j|\le N} e_j \Big\| < \delta \quad \text{and} \quad \Big\| B^n x - \sum_{|j|\le N} e_j \Big\| < \delta. \tag{4.6}$$

From (4.5) and (4.6) we obtain that

$$\|x_n e_n\| < \tfrac{\varepsilon}{2} \quad (|n| > N) \quad \text{and} \quad |x_{n+j} - 1| \le \tfrac{1}{2} \quad (|j| \le N),$$

hence

$$\|x_{j+n} e_{j+n}\| < \tfrac{\varepsilon}{2} \quad (|j| \le N) \quad \text{and} \quad |(x_{n+j})^{-1} - 1| \le 1 \quad (|j| \le N);$$

here we have used that $n > 2N$. As in (4.4) this implies that

$$\|e_{j+n}\| < \varepsilon \quad \text{for } |j| \le N.$$

On the other hand, (4.5) and (4.6) yield that

$$|x_j - 1| \le \tfrac{1}{2} \quad (|j| \le N) \quad \text{and} \quad \|x_{n+k} e_k\| < \tfrac{\varepsilon}{2} \quad (|k| > N),$$

hence

$$\big|(2x_j)^{-1}\big| \le 1 \quad (|j| \le N) \quad \text{and} \quad \|x_j e_{j-n}\| < \tfrac{\varepsilon}{2} \quad (|j| \le N),$$

whence

$$\|e_{j-n}\| = \big\|(2x_j)^{-1} 2x_j\, e_{j-n}\big\| < \varepsilon \quad \text{for } |j| \le N.$$

(iii)$\Longrightarrow$(ii). One need only observe that for the forward shift

$$F(x_n)_{n\in\mathbb{Z}} = (x_{n-1})_{n\in\mathbb{Z}}$$

we have that $BFx = x$ for any finite sequence x, and for any $j \in \mathbb{Z}$

$$B^{n_k} e_j = e_{j-n_k} \to 0, \quad F^{n_k} e_j = e_{j+n_k} \to 0,$$

so that the Hypercyclicity Criterion gives the required implication.

The implication (ii)$\Longrightarrow$(i) holds for all operators on X.

(b) The proof here is the same as that for hypercyclicity; for the sufficiency of condition (ii) one applies Kitai's criterion instead of the Hypercyclicity Criterion, while the proof of the necessity of this condition simplifies as we have only to consider the case of $j = 0$.

(c) This proof is much the same as that in the unilateral case. □

Using a suitable conjugacy this result can again be generalized immediately to weighted shifts. The conjugacy here is given by

$$\begin{array}{ccc} X_v & \xrightarrow{B} & X_v \\ {\scriptstyle \phi_v}\downarrow & & \downarrow{\scriptstyle \phi_v} \\ X & \xrightarrow{B_w} & X, \end{array}$$

where

$$X_v = \{(x_n)_{n\in\mathbb{Z}} \ ; \ (x_n v_n)_n \in X\}$$

and $\phi_v : X_v \to X$, $(x_n)_{n\in\mathbb{Z}} \to (x_n v_n)_{n\in\mathbb{Z}}$ with

$$v_n = \Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} \quad \text{for } n \geq 1, \quad v_n = \prod_{\nu=n+1}^{0} w_\nu \quad \text{for } n \leq -1, \quad v_0 = 1.$$

Theorem 4.13. *Let X be a Fréchet sequence space over $\mathbb{Z}$ in which $(e_n)_{n\in\mathbb{Z}}$ is a basis. Suppose that the weighted shift B_w is an operator on X.*

(a) *The following assertions are equivalent:*
  (i) *B_w is hypercyclic;*
  (ii) *B_w is weakly mixing;*
  (iii) *there is an increasing sequence $(n_k)_k$ of positive integers such that, for any $j \in \mathbb{Z}$,*

$$\Big(\prod_{\nu=j-n_k+1}^{j} w_\nu\Big) e_{j-n_k} \to 0 \quad \textit{and} \quad \Big(\prod_{\nu=j+1}^{j+n_k} w_\nu\Big)^{-1} e_{j+n_k} \to 0$$

  in X as $k \to \infty$.

(b) *The following assertions are equivalent:*
  (i) *B_w is mixing;*
  (ii) *we have that*

$$\Big(\prod_{\nu=-n+1}^{0} w_\nu\Big) e_{-n} \to 0 \quad \textit{and} \quad \Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} e_n \to 0$$

  in X as $n \to \infty$.

(c) *Suppose that the basis $(e_n)_{n\in\mathbb{Z}}$ is unconditional. Then the following assertions are equivalent:*
  (i) *B_w is chaotic;*
  (ii) *the series*

$$\sum_{n=-\infty}^{0} \Big(\prod_{\nu=n+1}^{0} w_\nu\Big) e_n + \sum_{n=1}^{\infty} \Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} e_n$$

  converges in X;
  (iii) *the sequence $(x_n)_{n\in\mathbb{Z}}$ with*

$$x_n = \prod_{\nu=n+1}^{0} w_\nu \ (n \le 0), \quad x_n = \Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} \ (n \ge 1)$$

belongs to X;

(iv) *B_w has a nontrivial periodic point.*

We see that the absence of an analogue of Lemma 4.2 leads to a more complicated characterization of hypercyclic bilateral shifts. However, for invertible bilateral shifts a simplified characterization is available; see Exercises 4.1.4 and 4.1.5.

Remark 4.14. In the bilateral case, forward shifts can be hypercyclic. A *bilateral weighted forward shift* is given by an operator

$$F_w : X \to X, \quad (x_n)_{n\in\mathbb{Z}} \to (w_{n-1}x_{n-1})_{n\in\mathbb{Z}},$$

where $w = (w_n)_{n\in\mathbb{Z}}$ is a weight sequence. It is easily seen to be conjugate to a suitable backward shift. As a result one obtains, under the same assumptions as in Theorem 4.13, that F_w is hypercyclic if and only if there is an increasing sequence $(n_k)_k$ of positive integers such that, for any $j \in \mathbb{Z}$,

$$\Big(\prod_{\nu=j-n_k}^{j-1} w_\nu\Big)^{-1} e_{j-n_k} \to 0 \quad \text{and} \quad \Big(\prod_{\nu=j}^{j+n_k-1} w_\nu\Big) e_{j+n_k} \to 0$$

in X as $k \to \infty$. The corresponding characterizations hold for the mixing property and chaos.

Example 4.15. A weighted backward shift B_w is an operator on a sequence space $\ell^p(\mathbb{Z})$, $1 \le p < \infty$ if and only if the weights w_n, $n \in \mathbb{Z}$, are bounded. Such an operator is then hypercyclic, mixing or chaotic if and only if the following conditions, respectively, are satisfied:

$$\exists (n_k)_k\ \forall j \in \mathbb{Z} : \lim_{k\to\infty} \prod_{\nu=j-n_k+1}^{j} w_\nu = 0 \quad \text{and} \quad \lim_{k\to\infty} \prod_{\nu=j+1}^{j+n_k} |w_\nu| = \infty;$$

$$\lim_{n\to\infty} \prod_{\nu=-n+1}^{0} w_\nu = 0 \quad \text{and} \quad \lim_{n\to\infty} \prod_{\nu=1}^{n} |w_\nu| = \infty;$$

$$\sum_{n=0}^{\infty} \prod_{\nu=-n+1}^{0} |w_\nu|^p < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\prod_{\nu=1}^{n} |w_\nu|^p} < \infty.$$

In particular, a symmetric weight (that is, one with $w_{-n} = w_n$ for all $n \ge 0$) never defines a hypercyclic weighted shift B_w on these spaces.

As a concrete example, the weight

$$w = \big(\ldots, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 2, 2, 2, \ldots\big)$$

induces a chaotic weighted backward shift on each $\ell^p(\mathbb{Z})$.

One reason for studying shifts is that they provide a rich source of examples. As a first illustration we construct a hypercyclic operator whose adjoint is also hypercyclic.

Proposition 4.16. *There exists an operator T on $\ell^2(\mathbb{Z})$ such that T and its adjoint T^* are weakly mixing, and hence hypercyclic.*

Proof. As usual, we identify the dual of $\ell^2(\mathbb{Z})$ with itself; indeed, every continuous linear functional x^* on $\ell^2(\mathbb{Z})$ is of the form

$$x^*(x) = \langle x, x^* \rangle = \sum_{n\in\mathbb{Z}} x_n y_n, \quad (x_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$$

for a suitable sequence $y = (y_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$.

Now let $T = B_w$ be a bilateral shift. It defines an operator on $\ell^2(\mathbb{Z})$ if and only if the w_n, $n \in \mathbb{Z}$, are bounded. Since

$$\langle B_w x, y \rangle = \sum_{n\in\mathbb{Z}} w_{n+1} x_{n+1} y_n = \sum_{n\in\mathbb{Z}} x_n w_n y_{n-1} = \langle x, F_{(w_{n+1})} y \rangle,$$

we see that the adjoint $T^* = B_w^*$ of B_w is the forward shift $F_{(w_{n+1})}$.

When we define

$$v_n = \Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} \ (n \geq 1), \quad v_n = \prod_{\nu=n+1}^{0} w_\nu \ (n \leq -1), \quad v_0 = 1,$$

then Theorem 4.13 and Remark 4.14 tell us that B_w and $F_{(w_{n+1})}$ are weakly mixing if and only if there are increasing sequences $(n_k)_k$ and $(m_k)_k$ of positive integers such that, for any $j \in \mathbb{Z}$,

$$v_{j-n_k} \to 0, \quad v_{j+n_k} \to 0,$$
$$v_{j-m_k} \to \infty, \quad v_{j+m_k} \to \infty,$$

and the continuity of B_w requires that v_n/v_{n+1}, $n \in \mathbb{Z}$, is bounded. But such a sequence is easy to find: we choose the symmetric sequence $(v_n)_{n\in\mathbb{Z}}$ with

$$(v_n)_{n\geq 0} = \left(1, 1, 2, 1, \tfrac{1}{2}, 1, 2, 4, 2, 1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{2}, 1, 2, 4, 8, 4, 2, 1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{4}, \ldots\right),$$

and the n_k are the indices of the local minima, the m_k the indices of the local maxima of this sequence. Note that B_w is even invertible. □

Remark 4.17. This proposition provides us with an example of two weakly mixing, hence hypercyclic operators $S, T : X \to X$ whose direct sum $S \oplus T$ is not hypercyclic.

We show more generally that for any operator T on a Banach space X the operator $T \oplus T^*$ cannot be hypercyclic on $X \oplus X^*$. Indeed, suppose that (x, x^*) is a hypercyclic vector for $T \oplus T^*$. If we consider $-x \in X$ as a continuous linear functional on X^* then we have for $n \geq 0$ that

$$\langle (T^n x, (T^*)^n x^*), (x^*, -x)\rangle = \langle T^n x, x^*\rangle - \langle x, (T^*)^n x^*\rangle = 0,$$

which is impossible since the left-hand side must be dense in $\mathbb{K}$; note that $(x^*, -x)$ cannot be the zero vector.

It was this observation that motivated Herrero's problem if $T \oplus T$ is hypercyclic whenever T is; see Section 2.5.

4.2 Differential operators

As the last section demonstrates, Rolewicz's result on the hypercyclicity of multiples of the backward shift has seen far-reaching generalizations. Let us turn, in the same spirit, to Birkhoff's theorem and MacLane's theorem. At first glance, the operators

$$Df(z) = f'(z) \quad \text{and} \quad T_a f(z) = f(z+a),\ a \in \mathbb{C},$$

on the space $H(\mathbb{C})$ of entire functions have little in common. But there is a surprisingly simple connection. Since

$$f(z+a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!} a^n = \sum_{n=0}^{\infty} \frac{a^n D^n f}{n!}(z)$$

we have, at least formally, that

$$T_a = e^{aD}.$$

In fact, this representation can be justified rigorously. We will need the following notion from complex analysis: an entire function φ is said to be of *exponential type* if there are constants $M, A > 0$ such that

$$|\varphi(z)| \leq M e^{A|z|} \quad \text{for all } z \in \mathbb{C}. \tag{4.7}$$

Lemma 4.18. *An entire function $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ is of exponential type if and only if there are $M, R > 0$ such that, for $n \geq 0$,*

$$|a_n| \leq M \frac{R^n}{n!}. \tag{4.8}$$

Proof. On the one hand, if (4.7) holds, then by the Cauchy estimates we have for any $\rho > 0$ that

$$|a_n| = \left|\frac{\varphi^{(n)}(0)}{n!}\right| \le \frac{1}{\rho^n}\max_{|z|\le\rho}|\varphi(z)| \le \frac{M}{\rho^n}e^{A\rho}.$$

Setting $\rho = n/A$ and using Stirling's formula we get, with some $C > 0$,

$$|a_n| \le \frac{MA^n}{n^n}e^n \le CM\frac{\sqrt{n}A^n}{n!} \le CM\frac{(2A)^n}{n!}.$$

Conversely, if (4.8) holds then

$$|\varphi(z)| \le \sum_{n=0}^{\infty}|a_n z^n| \le M\sum_{n=0}^{\infty}\frac{(R|z|)^n}{n!} = Me^{R|z|},$$

so that φ is of exponential type. □

Proposition 4.19. *Let*

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function of exponential type. Then

$$\varphi(D)f = \sum_{n=0}^{\infty} a_n D^n f$$

converges in $H(\mathbb{C})$ for every entire function f and defines an operator on $H(\mathbb{C})$.

Proof. Let $f \in H(\mathbb{C})$ and $|z| \le m$. By the Cauchy estimates and Lemma 4.18 there are $M, R > 0$ such that

$$\left|a_n f^{(n)}(z)\right| \le |a_n|\frac{n!}{m^n}\max_{|\zeta|\le 2m}|f(\zeta)| \le M\left(\frac{R}{m}\right)^n \max_{|\zeta|\le 2m}|f(\zeta)|. \tag{4.9}$$

Therefore, if $m > R$ then $\sum_{n=0}^{\infty} a_n f^{(n)}(z)$ converges uniformly on $|z| \le m$. Hence

$$\varphi(D)f = \sum_{n=0}^{\infty} a_n D^n f$$

converges in $H(\mathbb{C})$. Moreover, by (4.9), writing $p_m(f) = \max_{|z|\le m}|f(z)|$, we have for $m > R$ that

$$p_m(\varphi(D)f) \le M\frac{1}{1-R/m}p_{2m}(f).$$

This shows that $\varphi(D)$ is an operator on $H(\mathbb{C})$; see Proposition 2.11. □

We will call the operators $\varphi(D)$ simply *differential operators* on $H(\mathbb{C})$. They include all finite-order differential operators

$$T = a_0 I + a_1 D + \ldots + a_m D^m.$$

Proposition 4.19, in particular, justifies our earlier calculation concerning Birkhoff's operators that

$$T_a = \varphi(D) \quad \text{with } \varphi(z) = e^{az}. \tag{4.10}$$

The following result gives a useful description of the differential operators $\varphi(D)$ among the operators on $H(\mathbb{C})$.

Proposition 4.20. *Let T be an operator on $H(\mathbb{C})$. Then the following assertions are equivalent:*

(i) *$T = \varphi(D)$ for some entire function φ of exponential type;*
(ii) *T commutes with D;*
(iii) *T commutes with each $T_a, a \in \mathbb{C}$.*

Proof. (i)$\Longrightarrow$(ii). Let $T = \varphi(D)$. By the continuity of D we have for $f \in H(\mathbb{C})$

$$TDf = \sum_{n=0}^{\infty} a_n D^n(Df) = \sum_{n=0}^{\infty} D(a_n D^n f) = DTf.$$

(ii)$\Longrightarrow$(iii). By the same token, using (4.10), we obtain for $f \in H(\mathbb{C})$ that

$$TT_a f = T \sum_{n=0}^{\infty} \frac{a^n}{n!} D^n f = \sum_{n=0}^{\infty} \frac{a^n}{n!} TD^n f = \sum_{n=0}^{\infty} \frac{a^n}{n!} D^n(Tf) = T_a T f.$$

(iii)$\Longrightarrow$(i). By continuity of $f \to (Tf)(0)$ there is some $M > 0$ and some $R \in \mathbb{N}$ such that

$$|(Tf)(0)| \leq M \max_{|z| \leq R} |f(z)|, \quad f \in H(\mathbb{C}).$$

Denoting by e_n, $n \geq 0$, the monomials $e_n(z) = z^n$ we define

$$a_n = \frac{(Te_n)(0)}{n!}$$

and deduce that

$$|a_n| \leq M \frac{R^n}{n!}.$$

It follows from Lemma 4.18 and Proposition 4.19 that $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ defines an entire function φ of exponential type and that $\varphi(D) = \sum_{n=0}^{\infty} a_n D^n$ defines an operator on $H(\mathbb{C})$. Then

$$(\varphi(D) e_n)(0) = a_n n! = (Te_n)(0), \ n \geq 0.$$

Since the monomials span a dense subspace of $H(\mathbb{C})$, we obtain that

$$(\varphi(D)f)(0) = (Tf)(0) \quad \text{for } f \in H(\mathbb{C}).$$

By what we have shown above we also know that $\varphi(D)$ commutes with each T_a. Thus we get with (iii) for any $z \in \mathbb{C}$ and $f \in H(\mathbb{C})$, using the definition of T_z,

$$\begin{aligned}(\varphi(D)f)(z) &= (T_z\varphi(D)f)(0) = (\varphi(D)T_zf)(0)\\ &= (TT_zf)(0) = (T_zTf)(0) = Tf(z),\end{aligned}$$

so that $T = \varphi(D)$. $\square$

With this in hand we can prove a remarkably general common extension of the theorems of Birkhoff and MacLane.

Theorem 4.21 (Godefroy–Shapiro). *Suppose that $T : H(\mathbb{C}) \to H(\mathbb{C})$, $T \neq \lambda I$, is an operator that commutes with D, that is,*

$$TD = DT.$$

Then T is mixing and chaotic.

Proof. By Proposition 4.20 we can write $T = \varphi(D)$ for some entire function

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$$

of exponential type. Our additional assumption implies that φ is nonconstant. It is now easy to verify that T satisfies the conditions of the Godefroy–Shapiro criterion. In fact, considering the exponential functions

$$e_\lambda(z) = e^{\lambda z}, \quad \lambda \in \mathbb{C},$$

we calculate that

$$Te_\lambda = \varphi(D)e_\lambda = \sum_{n=0}^{\infty} a_n\lambda^n e_\lambda = \varphi(\lambda)e_\lambda.$$

Thus each e_λ is an eigenvector of T to the eigenvalue $\varphi(\lambda)$. Consequently,

$$\text{span}\{f \in H(\mathbb{C}) \ ; \ Tf = \mu f \text{ for some } \mu \in \mathbb{C} \text{ with } |\mu| < 1\}$$

contains $\text{span}\{e_\lambda \ ; \ |\varphi(\lambda)| < 1\}$, which is dense in $H(\mathbb{C})$ by Lemma 2.34; indeed, since any nonconstant entire function has dense range (see Appendix A), $\{\lambda \in \mathbb{C} \ ; \ |\varphi(\lambda)| < 1\}$ is a nonempty open set and therefore has an accumulation point. For the same reason, the eigenvectors of T to eigenvalues μ with $|\mu| > 1$ span a dense set in $H(\mathbb{C})$. For the density of

$$\text{span}\{f \in H(\mathbb{C}) \ ; \ Tf = e^{\alpha\pi i}f \text{ for some } \alpha \in \mathbb{Q}\}$$

it suffices to observe that also the set $\{\lambda \in \mathbb{C} \; ; \; \varphi(\lambda) = e^{\alpha\pi i} \text{ for some } \alpha \in \mathbb{Q}\}$ has an accumulation point. Indeed, since $\varphi(\mathbb{C})$ is connected and dense, it must intersect the unit circle. And since nonconstant holomorphic functions are open mappings, infinitely many preimages under φ of roots of unity lie in some bounded subset of $\mathbb{C}$ and therefore have an accumulation point. □

Having established the hypercyclicity of every differential operator $T = \varphi(D) \neq \lambda I$ we now want to focus our attention on properties of the corresponding hypercyclic functions. MacLane had already addressed such a problem: he showed that there exists a D-hypercyclic entire function f of exponential type 1, which means that for every $\varepsilon > 0$ there is some $M > 0$ such that

$$|f(z)| \leq Me^{(1+\varepsilon)r} \quad \text{for all } z \in \mathbb{C}.$$

Here we follow the usual convention of writing $r = |z|$. MacLane's growth condition can be improved, and one can even determine the least possible rate of growth.

Theorem 4.22. (a) *Let $\phi :]0, \infty[\; \to [1, \infty[$ be a function with $\phi(r) \to \infty$ as $r \to \infty$. Then there exists an entire function f that is hypercyclic for D and that satisfies*

$$|f(z)| \leq M\phi(r)\frac{e^r}{\sqrt{r}} \quad \textit{for } |z| = r > 0$$

with some $M > 0$.

(b) *There is no entire function f that is hypercyclic for D and that satisfies*

$$|f(z)| \leq M\frac{e^r}{\sqrt{r}} \quad \textit{for } |z| = r > 0$$

with some $M > 0$.

Proof. (a) The assertion suggests consideration of the space

$$X = \Big\{ f \in H(\mathbb{C}) \; ; \; \|f\| := \sup_{r=|z|>0} \frac{\sqrt{r}\,|f(z)|}{\phi(r)e^r} < \infty \Big\}.$$

Proving our assertion then amounts to showing that the sequence of operators

$$T_n : X \to H(\mathbb{C}), \quad f \to f^{(n)}, \quad n \geq 0$$

admits a dense orbit in the sense of Section 3.4; note that the T_n are indeed operators because the inclusion map $X \to H(\mathbb{C})$ is obviously continuous.

To prove that $(T_n)_n$ is hypercyclic we apply the Hypercyclicity Criterion for sequences of operators, Theorem 3.24.

It is an easy exercise to see that X is a Banach space. We would like to take as X_0 the set of polynomials, but we cannot guarantee that the polynomials are dense in X. Thus we replace X by the closure $\overline{X_0}$ of X_0 in X. Clearly $T_n f \to 0$ for any $f \in X_0$. For Y_0 we take the set of polynomials in $H(\mathbb{C})$, and we define $S_n = S^n : Y_0 \to \overline{X}_0$ using the antiderivative operator $Sf(z) = \int_0^z f(\zeta)d\zeta$. Then $T_n S_n f = f$, $n \in \mathbb{N}_0$, for any $f \in Y_0$.

It remains to show that $S_n f \to 0$ in $\overline{X}_0$ for any polynomial f. By linearity we may assume that f is a monomial $e_n(z) = z^n$, and because $S_n e_k = \frac{k!}{(n+k)!} e_{n+k} = k! S_{n+k} e_0$ it suffices to consider $f = e_0$. For this we find that

$$\|S_n e_0\| = \left\| \frac{e_n}{n!} \right\| = \sup_{r>0} \frac{r^{n+1/2}}{n!\phi(r)e^r}.$$

A simple calculation shows that

$$\sup_{r>0} \frac{r^{n+1/2}}{n!e^r} = \frac{(n+1/2)^{n+1/2}}{n!e^{n+1/2}},$$

and Stirling's formula implies that this is bounded in $n \geq 0$ by some constant C. Fixing $\varepsilon > 0$, and letting $R > 0$ be such that $\phi(r) > 1/\varepsilon$ for $r \geq R$ we obtain that

$$\|S_n e_0\| \leq \frac{R^{n+1/2}}{n!} + \sup_{r \geq R} \frac{r^{n+1/2}\varepsilon}{n!e^r} \leq \frac{R^{n+1/2}}{n!} + C\varepsilon,$$

which implies that $S_n e_0 \to 0$ in X and therefore in $\overline{X}_0$.

(b) Let $f \in H(\mathbb{C})$. Under the assumed growth condition we have by the Cauchy estimates that, for any $n \in \mathbb{N}_0$ and $\rho > 0$,

$$\left|f^{(n)}(0)\right| \leq \frac{n!}{\rho^n} \max_{|z| \leq \rho} |f(z)| \leq M \frac{n!}{\rho^n \sqrt{\rho}} e^{\rho}.$$

Choosing $\rho = n$ we get

$$\left|f^{(n)}(0)\right| \leq M \frac{n!}{n^{n+1/2}} e^n,$$

which is bounded by Stirling's formula. Thus, f cannot be hypercyclic for D. □

Exercise 4.2.5 explains how the critical rate of growth $e^r/\sqrt{r}$ is related to the differentiation operator.

In contrast to the result for MacLane's operator, entire functions that are hypercyclic for Birkhoff's operators can grow arbitrarily slowly. The proof requires a different technique and will be provided in Chapter 8; see Exercise 8.1.3.

Theorem 4.23 (Duyos-Ruiz). *Let $a \neq 0$. Let $\phi :]0,\infty[\to [1,\infty[$ be a function so that, for any $N \geq 1$, $\phi(r)/r^N \to \infty$. Then there exists an entire function f that is hypercyclic for T_a and that satisfies*

$$|f(z)| \leq M\phi(r) \quad \text{for } |z| = r > 0$$

with some $M > 0$.

By the method used in the proof of Theorem 4.21 one can derive certain possible rates of growth for an arbitrary operator $\varphi(D)$; see Exercise 4.2.4.

4.3 Composition operators I

As we have seen, operators may often be interpreted in various ways. MacLane's operator is both a differential operator and a weighted shift. Birkhoff's operators are differential operators as well. Here now we have another interpretation of Birkhoff's operators T_a: they are special composition operators. Writing

$$\tau_a(z) = z + a$$

we see that τ_a is an entire function such that

$$T_a f = f \circ \tau_a.$$

In fact, τ_a is even an automorphism of $\mathbb{C}$, that is, a bijective entire function. These observations serve as the starting point of another major investigation: the hypercyclicity of general composition operators.

Let Ω be an arbitrary domain in $\mathbb{C}$, that is, a nonempty connected open set. An *automorphism* of Ω is a bijective holomorphic function

$$\varphi : \Omega \to \Omega;$$

its inverse is then also holomorphic. The set of all automorphisms of Ω is denoted by $\mathrm{Aut}(\Omega)$. Now, for $\varphi \in \mathrm{Aut}(\Omega)$ the corresponding *composition operator* is defined as

$$C_\varphi f = f \circ \varphi,$$

that is, $(C_\varphi f)(z) = f(\varphi(z))$, $z \in \Omega$.

What about the underlying space? Following Birkhoff we consider the space $H(\Omega)$ of all holomorphic functions on Ω which we endow, as in the case $\Omega = \mathbb{C}$, with the topology of local uniform convergence. To describe this topology by seminorms we need an *exhaustion* of Ω by compact sets, that is, an increasing sequence of compact sets $K_n \subset \Omega$ such that each compact set $K \subset \Omega$ is contained in some K_n.

Lemma 4.24. *Every domain $\Omega \subset \mathbb{C}$ has an exhaustion of compact sets.*

Proof. For each $n \in \mathbb{N}$ we consider the grid of all points $x + iy$ in $\mathbb{C}$ so that either x or y is an integer multiple of $\frac{1}{2^n}$; then let K_n be the (finite) union of all closed squares that have their sides lying on the grid and that lie entirely in $\Omega \cap \{z : |z| < n\}$. It is obvious that $(K_n)_n$ is an exhaustion of Ω. $\square$

Now, if $(K_n)_n$ is an exhaustion of Ω then we endow $H(\Omega)$ with the topology induced by the seminorms

$$p_n(f) = \sup_{z \in K_n} |f(z)|, \quad n \in \mathbb{N}.$$

In this way $H(\Omega)$ turns into a Fréchet space; note that the topology is independent of the chosen exhaustion. Moreover, by Runge's theorem, $H(\Omega)$ is separable; see Exercise 4.3.1.

Clearly, for any automorphism φ of Ω the composition operator C_φ is continuous on $H(\Omega)$. Let us first note that conformal maps, that is, holomorphic bijections between two domains, induce conjugacies between the corresponding composition operators; the proof is immediate.

Proposition 4.25. *Let Ω_1 and Ω_2 be domains in $\mathbb{C}$ and $\psi : \Omega_1 \to \Omega_2$ a conformal map. If φ_1 and φ_2 are automorphisms of Ω_1 and Ω_2, respectively, such that $\varphi_2 \circ \psi = \psi \circ \varphi_1$ then C_{φ_2} and C_{φ_1} are conjugate via the map $J : H(\Omega_2) \to H(\Omega_1), f \to f \circ \psi$, that is, the diagram*

$$\begin{array}{ccc} H(\Omega_2) & \xrightarrow{C_{\varphi_2}} & H(\Omega_2) \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ H(\Omega_1) & \xrightarrow{C_{\varphi_1}} & H(\Omega_1) \end{array}$$

commutes.

Example 4.26. Any two Birkhoff operators T_a, T_b, $a, b \neq 0$, are conjugate. This follows immediately by taking $\psi(z) = \frac{b}{a}z$, $z \in \mathbb{C}$, since $\tau_b \circ \psi = \psi \circ \tau_a$.

We turn to the problem of determining which composition operators are hypercyclic. The crucial concept will be the notion of a run-away sequence.

Definition 4.27. Let Ω be a domain in $\mathbb{C}$ and $\varphi_n : \Omega \to \Omega$, $n \geq 1$, holomorphic maps. Then the sequence $(\varphi_n)_n$ is called a *run-away sequence* if, for any compact subset $K \subset \Omega$, there is some $n \in \mathbb{N}$ such that

$$\varphi_n(K) \cap K = \varnothing.$$

We will usually apply this definition to the sequence $(\varphi^n)_n$ of iterates of an automorphism φ on Ω. Let us consider two examples. Another important example will be studied below; see Proposition 4.36.

Example 4.28. (a) Let $\Omega = \mathbb{C}$. Then the automorphisms of $\mathbb{C}$ are the functions

$$\varphi(z) = az + b, \quad a \neq 0, \, b \in \mathbb{C},$$

and $(\varphi^n)_n$ is run-away if and only if $a = 1$, $b \neq 0$.

Indeed, let φ be an automorphism of $\mathbb{C}$. If φ is not a polynomial then, by the Casorati–Weierstrass theorem, $\varphi(\{z \in \mathbb{C} \; ; \; |z| > 1\})$ is dense in $\mathbb{C}$ and therefore intersects the set $\varphi(\mathbb{D})$, which is open by the open mapping theorem. Since this contradicts injectivity, φ must be a polynomial. Again by injectivity, its degree must be one, so that φ is of the stated form. Now, if $a = 1$ then $\varphi^n(z) = z + nb$, so that we have the run-away property if and only if $b \neq 0$; while if $a \neq 1$ then $(1-a)^{-1}b$ is a fixed point of φ so that $(\varphi^n)_n$ cannot be run-away.

(b) Let $\Omega = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, the punctured plane. An argument as in (a) shows that the automorphisms of $\mathbb{C}^*$ are the functions

$$\varphi(z) = az \quad \text{or} \quad \varphi(z) = \frac{a}{z}, \quad a \neq 0.$$

Then $(\varphi^n)_n$ is run-away if and only if $\varphi(z) = az$ with $|a| \neq 1$.

We first show that the run-away property is a necessary condition for the hypercyclicity of the composition operator.

Proposition 4.29. *Let Ω be a domain in $\mathbb{C}$ and $\varphi \in \mathrm{Aut}(\Omega)$. If C_φ is hypercyclic then $(\varphi^n)_n$ is a run-away sequence.*

Proof. If $(\varphi^n)_n$ is not run-away then there exists a compact set $K \subset \Omega$ and elements $z_n \in K$ such that

$$\varphi^n(z_n) \in K, \quad n \in \mathbb{N}. \tag{4.11}$$

Now suppose that $f \in H(\Omega)$ is a hypercyclic vector for C_φ. Let $M = \sup_{z\in K} |f(z)|$. Then, by (4.11), we have that

$$\inf_{z\in K} |((C_\varphi)^n f)(z)| \leq |((C_\varphi)^n f)(z_n)| = |f(\varphi^n(z_n))| \leq M,$$

so that the functions $(C_\varphi)^n f$ cannot approximate, for example, the constant function $M + 1$ uniformly on K, a contradiction. □

Corollary 4.30. *There is no automorphism of $\mathbb{C}^*$ whose composition operator is hypercyclic.*

Proof. By Proposition 4.29 and Example 4.28(b), C_φ can only be hypercyclic on $H(\mathbb{C}^*)$ if $\varphi(z) = az$ with $|a| \neq 1$. Suppose that $f \in H(\mathbb{C}^*)$ is hypercyclic for such a function φ. If $f(z) = \sum_{n\in\mathbb{Z}} c_n z^n$ then

$$\int_{\mathbb{T}} ((C_\varphi)^n f - \tfrac{1}{z})\, dz = \int_{\mathbb{T}} f(a^n z)\, dz - \int_{\mathbb{T}} \tfrac{1}{z}\, dz = 2\pi i\Big(\frac{c_{-1}}{a^n} - 1\Big),$$

where the unit circle $\mathbb{T}$ is positively oriented. By hypercyclicity, there is a sequence $(n_k)_k$ for which the left-hand side converges to zero, unlike the right-hand side, which is a contradiction. □

Thus, when Ω is $\mathbb{C}^*$ then the run-away property is not a sufficient condition for hypercyclicity. Our goal now is to show that in essentially all other cases, hypercyclicity is characterized by the run-away property. To this end we need to introduce some topological properties of plane sets.

We denote by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the one-point compactification of $\mathbb{C}$. A domain Ω is called *simply connected* if $\widehat{\mathbb{C}} \setminus \Omega$ is connected. A domain Ω is called *finitely connected* if $\widehat{\mathbb{C}} \setminus \Omega$ contains at most finitely many connected components, otherwise it is *infinitely connected*. If M is any set in $\mathbb{C}$ then one also speaks of a bounded component of $\widehat{\mathbb{C}} \setminus M$ as a *hole*. In that sense, finitely connected domains have only finitely many holes, a simply connected domain has no hole.

We first deal with finitely, not simply connected domains Ω. One can show that unless such a domain is conformally equivalent to $\mathbb{C}^*$, that is, unless there is a holomorphic bijection between Ω and $\mathbb{C}^*$, Ω does not admit an automorphism φ so that $(\varphi^n)_n$ is run-away; we omit the proof. By Proposition 4.29 and Corollary 4.30 we therefore have the following.

Proposition 4.31. *Let Ω be a finitely connected but not simply connected domain in $\mathbb{C}$. Then C_φ is not hypercyclic for any automorphism of Ω.*

In all other cases we have the following.

Theorem 4.32. *Let Ω be a domain in $\mathbb{C}$ that is either simply connected or infinitely connected. Let $\varphi \in \mathrm{Aut}(\Omega)$. Then C_φ is hypercyclic if and only if $(\varphi^n)_n$ is a run-away sequence.*

In view of Proposition 4.29 we only have to prove sufficiency of the run-away property. For this we need to study the geometry of domains more closely, at least for infinitely connected domains. A compact subset K of a domain Ω will be called *Ω-convex* if every hole of K contains a point of $\mathbb{C} \setminus \Omega$; see Figure 4.1. Of course, in a simply connected domain, Ω-convexity only says that K has no holes, in other words, that its complement is connected.

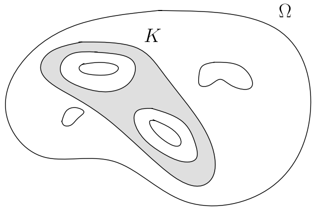

Fig. 4.1 An Ω-convex set K

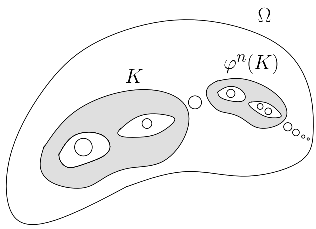

Fig. 4.2 $\varphi^n(K) \cup K$ is Ω-convex

The following auxiliary result will be crucial for the proof of sufficiency in Theorem 4.32. However, since its proof is rather technical we will postpone it to the end of the section. For later use we formulate the lemma for arbitrary sequences $(\varphi_n)_n$ of automorphisms.

Lemma 4.33. *Let Ω be an infinitely connected domain in $\mathbb{C}$ and $(\varphi_n)_n$ a run-away sequence of automorphisms of Ω. Then every compact subset of Ω is contained in some Ω-convex compact subset K of Ω for which there is some $n \in \mathbb{N}$ such that $\varphi_n(K) \cap K = \varnothing$ and $\varphi_n(K) \cup K$ is Ω-convex.*

Proof of Theorem 4.32 *(sufficiency).* Suppose that $(\varphi^n)_n$ is a run-away sequence. We want to show that then C_φ is topologically transitive. Let $f, g \in H(\Omega)$, let L be a compact subset of Ω and $\varepsilon > 0$. Then there is a compact subset K of Ω containing L and an $n \in \mathbb{N}$ such that $\varphi^n(K) \cap K = \varnothing$ and $\varphi^n(K) \cup K$ is Ω-convex (see Figure 4.2); in the simply connected case one can take any Ω-convex compact set K containing L, in the infinitely connected case one applies Lemma 4.33. Then the function $g \circ (\varphi^n)^{-1}$ is holomorphic on some neighbourhood of $\varphi^n(K)$, and f is holomorphic on some neighbourhood of K. It follows from Runge's theorem that there is a function $h \in H(\Omega)$ such that

$$\sup_{z \in K} |f(z) - h(z)| < \varepsilon \quad \text{and} \quad \sup_{z \in \varphi^n(K)} \left| g \circ (\varphi^n)^{-1}(z) - h(z) \right| < \varepsilon,$$

hence

$$\sup_{z \in L} |f(z) - h(z)| < \varepsilon \quad \text{and} \quad \sup_{z \in L} |g(z) - h(\varphi^n(z))| < \varepsilon.$$

As in Example 2.20 this implies that C_φ is topologically transitive. □

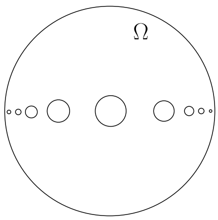

Fig. 4.3 The set Ω (Example 4.34)

Example 4.34. We give an example of a hypercyclic composition operator on an infinitely connected domain. We start with the unit disk $\mathbb{D}$ and the automorphism

$$\varphi(z) = \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}$$

of $\mathbb{D}$; see also Proposition 4.36. Let $A = \{z : |z| \leq \frac{1}{10}\}$. It is easy to see that the forward and backward iterates $\varphi^n(A)$, $n \in \mathbb{Z}$, of A are pairwise disjoint. Then

$$\Omega := \mathbb{D} \setminus \bigcup_{n\in\mathbb{Z}} \varphi^n(A)$$

is an infinitely connected domain (see Figure 4.3), and the restriction of φ to Ω is an automorphism of Ω. Moreover, a simple calculation shows that

$$\varphi^n(z) = \frac{z-a_n}{1-a_n z} \quad \text{with} \quad a_n = \frac{3^n-1}{3^n+1},\ n\geq 0,$$

so that $\lim_{n\to\infty}\varphi^n(z) = -1$, uniformly on compact subsets of Ω. Hence, $(\varphi^n)_n$ is a run-away sequence on Ω, which implies that C_φ is hypercyclic on $H(\Omega)$.

Remark 4.35. Any hypercyclic composition operator C_φ on a domain Ω is even weakly mixing. To see this, let $(K_n)_n$ be an exhaustion of Ω by compact sets. Since $(\varphi^n)_n$ is run-away, there is some m_1 such that $\varphi^{m_1}(K_1)\cap K_1 = \varnothing$. If $L = K_2 \cup \bigcup_{k=1}^{m_1}\varphi^k(K_1)$, there is some m_2 such that $\varphi^{m_2}(L)\cap L = \varnothing$. Then, in particular, $\varphi^{m_2}(K_2)\cap K_2 = \varnothing$; moreover, since $\varphi^{m_2}(L)$ contains $\varphi^{m_2}(K_1)$ and L contains $\varphi^k(K_1)$ for $k = 1,\ldots,m_1$, we must have that $m_2 > m_1$. Proceeding inductively we obtain a strictly increasing sequence $(m_n)_n$ such that $\varphi^{m_n}(K_n)\cap K_n = \varnothing$, for any $n\in\mathbb{N}$; as a consequence, $(\varphi^{m_n})_n$ and any of its subsequences is run-away. The proofs in this section then show that every subsequence of $(C_{\varphi^{m_n}})_n$ admits a dense orbit. This tells us that C_φ is hereditarily hypercyclic, and hence weakly mixing by Theorem 3.15.

We want to study the case of simply connected domains in greater detail. If $\Omega = \mathbb{C}$, the automorphisms are given by

$$\varphi(z) = az+b, \quad a\neq 0, b\in\mathbb{C},$$

and C_φ is hypercyclic if and only if $a = 1$, $b\neq 0$; see Example 4.28(a) and Theorem 4.32. Thus the hypercyclic composition operators on $\mathbb{C}$ are precisely Birkhoff's translation operators.

Let us now consider the simply connected domains Ω other than $\mathbb{C}$. By the Riemann mapping theorem, Ω is conformally equivalent to the unit disk, that is, there is a conformal map $\psi : \mathbb{D}\to\Omega$. By Proposition 4.25 it suffices to study the case when $\Omega = \mathbb{D}$.

Proposition 4.36. *The automorphisms of* $\mathbb{D}$ *are the linear fractional transformations*

$$\varphi(z) = b\frac{a-z}{1-\overline{a}z}, \quad |a|<1,\ |b| = 1.$$

Moreover, φ *maps* $\mathbb{T}$ *bijectively onto itself.*

Proof. We first consider the maps

$$h_a(z) = \frac{a-z}{1-\overline{a}z}, \quad |a|<1.$$

A simple calculation shows that, for $w = h_a(z)$,

$$1 - |w|^2 = \frac{1 - |a|^2}{|1 - \overline{a}z|^2}\,(1 - |z|^2). \tag{4.12}$$

Hence $\mathbb{D}$ and $\mathbb{T}$ are invariant under h_a. Moreover one finds that $h_a \circ h_a = I$ on $\overline{\mathbb{D}}$. This implies that h_a is an automorphism of $\mathbb{D}$ that maps $\mathbb{T}$ bijectively onto itself. The same is then true for bh_a, $|b| = 1$.

Conversely, let φ be an automorphism of $\mathbb{D}$, and let $0 = \varphi(a)$ with $|a| < 1$. Then the map $f := \varphi \circ h_a^{-1}$ is also an automorphism of $\mathbb{D}$ with $f(0) = 0$. The Schwarz lemma then implies that $|f(z)| \leq |z|$ for $z \in \mathbb{D}$. The same argument applied to the inverse of f shows that $|f^{-1}(z)| \leq |z|$, hence $|z| \leq |f(z)|$ for $z \in \mathbb{D}$. Altogether we have that $|f(z)| = |z|$ for $z \in \mathbb{D}$. Again by the Schwarz lemma, f can only be a rotation, that is, there is some b with $|b| = 1$ such that $f(z) = bz$ and therefore $\varphi = bh_a$. □

Now, linear fractional transformations are a very well understood class of holomorphic maps; see Appendix A. Using their properties it is not difficult to determine the dynamical behaviour of the corresponding composition operators; via conjugacy these results can then be carried over to arbitrary simply connected domains.

Theorem 4.37. *Let Ω be a simply connected domain and $\varphi \in \mathrm{Aut}(\Omega)$. Then the following assertions are equivalent:*

(i) *C_φ is hypercyclic;*
(ii) *C_φ is mixing;*
(iii) *C_φ is chaotic;*
(iv) *$(\varphi^n)_n$ is a run-away sequence;*
(v) *φ has no fixed point in Ω;*
(vi) *C_φ is quasiconjugate to a Birkhoff operator.*

Proof. The implications (vi)$\Longrightarrow$(iii) and (vi)$\Longrightarrow$(ii) follow from known properties of the Birkhoff operators, (iii)$\Longrightarrow$(i) and (ii)$\Longrightarrow$(i) hold for all operators on $H(\Omega)$, and (i)$\Longleftrightarrow$(iv) was proved in Theorem 4.32.

(i)$\Longrightarrow$(v). If φ has a fixed point $z_0 \in \Omega$ then, for any $f \in H(\Omega)$ and $n \geq 0$, $((C_\varphi)^n f)(z_0) = f(\varphi^n(z_0)) = f(z_0)$, so that f cannot have a dense orbit.

It remains to prove that (v)$\Longrightarrow$(vi). In the case $\Omega = \mathbb{C}$ the result was shown in Example 4.28(a). In the case $\Omega \neq \mathbb{C}$ we can assume by the discussion leading up to Proposition 4.36 that $\Omega = \mathbb{D}$. The proof then requires certain properties of linear fractional transformations. Since we will have occasion to study them in Section 4.5 we will postpone the proof to the end of that section. □

In particular the final condition in the theorem is of great interest. Any property of the Birkhoff operators that is preserved under quasiconjugacies will transmit to all composition operators on simply connected domains.

It remains to give the proof of Lemma 4.33.

Proof of Lemma 4.33. We consider the exhaustion of Ω by compact sets K_n constructed in the proof of Lemma 4.24. Then each K_n is automatically Ω-convex. But also $\psi(K_n)$ is Ω-convex for every automorphism ψ. Indeed, if some hole of $\psi(K_n)$ contained only points of Ω then one could deform the boundary of that hole continuously in Ω to a point in Ω; applying the map ψ^{-1}, the same would then be true for the corresponding hole of K_n, contradicting the Ω-convexity of K_n.

By the run-away property, we can find a strictly increasing sequence $(m_n)_n$ of positive integers such that $\varphi_{m_n}(K_n) \cap K_n = \varnothing$ for $n \geq 1$; see Remark 4.35.

Now, every compact subset of Ω is contained in some K_N, and since Ω is infinitely connected we can assume that K_N has at least two holes. Then, for all $n \geq N$, $\varphi_{m_n}(K_N) \cap K_N = \varnothing$. Also, each $\varphi_{m_n}(K_N)$ is Ω-convex. To finish the proof it suffices to show that there is some $n \geq N$ such that, in addition, $\varphi_{m_n}(K_N) \cup K_N$ is Ω-convex.

We distinguish three cases. First, if there is some $n \geq N$ such that $\varphi_{m_n}(K_N)$ lies in the unbounded component of the complement of K_N and K_N lies in the unbounded component of the complement of $\varphi_{m_n}(K_N)$ then clearly $\varphi_{m_n}(K_N) \cup K_N$ is Ω-convex.

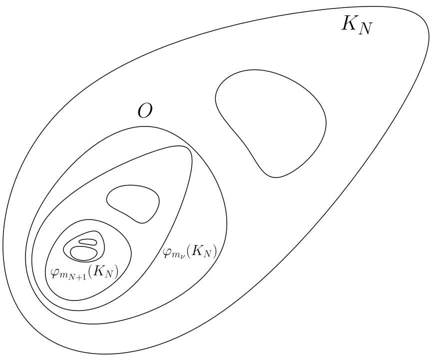

Fig. 4.4 $\varphi_{m_{N+1}}(K_N) \cup K_N$ is Ω-convex

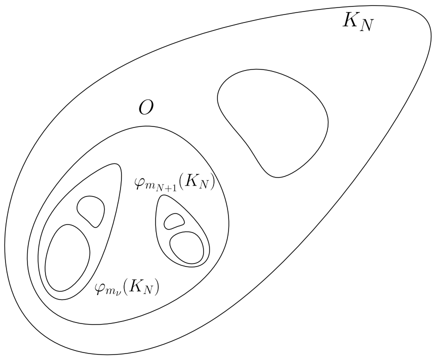

Fig. 4.5 Both $\varphi_{m_{N+1}}(K_N) \cup K_N$ and $\varphi_{m_\nu}(K_N) \cup K_N$ are Ω-convex

Secondly, suppose that infinitely many $\varphi_{m_n}(K_N)$, $n \geq N$, lie in holes of K_N. Since K_N only has a finite number of holes, infinitely many $\varphi_{m_n}(K_N)$, $n \geq N$, must lie in some fixed hole O of K_N; by passing to a subsequence we may assume that this is true for all $n > N$. We then choose some $\nu > N$ such that $\varphi_{m_{N+1}}(K_N) \subset K_\nu$. Since $\varphi_{m_\nu}(K_\nu) \cap K_\nu = \varnothing$ we have that $\varphi_{m_{N+1}}(K_N)$ and $\varphi_{m_\nu}(K_N)$ are disjoint subsets of O. Now one has three possibilities: either $\varphi_{m_{N+1}}(K_N)$ lies in a hole of $\varphi_{m_\nu}(K_N)$ (see Figure 4.4), or $\varphi_{m_\nu}(K_N)$ lies in a hole of $\varphi_{m_{N+1}}(K_N)$, or both sets lie in the unbounded component of the complement of the other set (see Figure 4.5). Since both sets have at least two holes one finds that in each of these cases either $\varphi_{m_{N+1}}(K_N) \cup K_N$ or $\varphi_{m_\nu}(K_N) \cup K_N$ is Ω-convex.

Finally, the remaining case is when for infinitely many $n \geq N$, K_N lies in a hole of $\varphi_{m_n}(K_N)$. Again we can assume that this is true for all $n > N$. We then choose some $\nu > N$ such that $\varphi_{m_{N+1}}(K_N) \subset K_\nu$. As above we find that $\varphi_{m_{N+1}}(K_N)$ and $\varphi_{m_\nu}(K_N)$ are disjoint sets. Since both these sets contain K_N in one of their holes, we have that either $\varphi_{m_{N+1}}(K_N)$ lies in a hole of $\varphi_{m_\nu}(K_N)$, or vice versa. Since both sets have two holes we find that either $\varphi_{m_\nu}(K_N) \cup K_N$ or $\varphi_{m_{N+1}}(K_N) \cup K_N$ is Ω-convex. □

4.4 Adjoint multipliers

In this section we consider an interesting generalization of the backward shift operator. The underlying space will be the Hardy space H^2. Arguably its easiest definition is the following. If $(a_n)_{n\geq 0}$ is a complex sequence such that

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty,$$

then it is, in particular, bounded, and hence

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C}, |z| < 1,$$

defines a holomorphic function on the complex unit disk $\mathbb{D}$. The Hardy space is then defined as the space of these functions, that is,

$$H^2 = \Big\{ f : \mathbb{D} \to \mathbb{C} \ ; \ f(z) = \sum_{n=0}^{\infty} a_n z^n, z \in \mathbb{D}, \text{ with } \sum_{n=0}^{\infty} |a_n|^2 < \infty \Big\}.$$

In other words, the Hardy space is simply the sequence space $\ell^2(\mathbb{N}_0)$, with its elements written as holomorphic functions. It is then clear that H^2 is a Banach space under the norm

$$\|f\| = \Big(\sum_{n=0}^{\infty} |a_n|^2 \Big)^{1/2} \quad \text{when } f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

and it is even a Hilbert space under the inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} \quad \text{when } f(z) = \sum_{n=0}^{\infty} a_n z^n, \ g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

The polynomials form a dense subspace of H^2.

The following result is an immediate consequence of the definitions.

Proposition 4.38. *For any $\lambda \in \mathbb{D}$ define $k_\lambda : \mathbb{D} \to \mathbb{C}$ by*

$$k_\lambda(z) = \sum_{n=0}^{\infty} \overline{\lambda}^n z^n = \frac{1}{1 - \overline{\lambda} z}.$$

Then $k_\lambda \in H^2$ and, for any $f \in H^2$,

$$f(\lambda) = \langle f, k_\lambda \rangle.$$

This implies that for any $\lambda \in \mathbb{D}$ the point evaluation

$$f \to f(\lambda)$$

is a continuous linear functional on H^2. The functions k_λ, $\lambda \in \mathbb{D}$, are called *reproducing kernels*. They will play the same role here as the exponential functions $e_\lambda \in H(\mathbb{C})$, $\lambda \in \mathbb{C}$, in Section 4.2. In particular we have the following analogue of Lemma 2.34.

Lemma 4.39. *Let $\Lambda \subset \mathbb{D}$ be a set with an accumulation point in $\mathbb{D}$. Then the set*

$$\operatorname{span}\{k_\lambda \ ; \ \lambda \in \Lambda\}$$

is dense in H^2.

Proof. It suffices to show that only the zero function can be orthogonal to $\operatorname{span}\{k_\lambda \ ; \ \lambda \in \Lambda\}$. But that is immediate by the identity theorem for holomorphic functions: if, for $f \in H^2$, $\langle f, k_\lambda \rangle = f(\lambda)$ vanishes for all $\lambda \in \Lambda$, then $f = 0$. □

The operators that we want to study are those that map $f \in H^2$ to φf, where φ is a bounded holomorphic function on $\mathbb{D}$. In order to see that this defines an operator on H^2 we need another representation of the space.

Proposition 4.40. *A holomorphic function $f : \mathbb{D} \to \mathbb{C}$ belongs to H^2 if and only if*

$$\sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{it})|^2 \, dt < \infty.$$

Moreover, for any $f, g \in H^2$,

$$\|f\| = \Big(\sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 \, dt \Big)^{1/2} = \Big(\lim_{r \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 \, dt \Big)^{1/2}$$

and

$$\langle f, g \rangle = \lim_{r \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) \overline{g(re^{it})} \, dt.$$

Proof. Writing $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we obtain that

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{it})|^2\,dt = \frac{1}{2\pi}\int_0^{2\pi}\Big|\sum_{n=0}^{\infty}a_n(re^{it})^n\Big|^2\,dt$$
$$= \int_0^{2\pi}\Big|\sum_{n=0}^{\infty}a_n r^n\frac{1}{\sqrt{2\pi}}e^{int}\Big|^2\,dt = \sum_{n=0}^{\infty}|a_n|^2 r^{2n},$$

where we have used Parseval's identity in $L^2[0,2\pi]$ for the orthonormal basis $(\frac{1}{\sqrt{2\pi}}\,e^{int})_{n\in\mathbb{Z}}$; see also Example 4.4(b). Since

$$\sup_{0\le r<1}\sum_{n=0}^{\infty}|a_n|^2 r^{2n} = \lim_{r\nearrow 1}\sum_{n=0}^{\infty}|a_n|^2 r^{2n} = \sum_{n=0}^{\infty}|a_n|^2 = \|f\|^2,$$

the first part of the assertion follows. In the same way, one obtains the second part by using Parseval's identity for the inner product. □

Now let φ be a bounded holomorphic function on $\mathbb{D}$. Then, for any $f\in H^2$, φf is holomorphic on $\mathbb{D}$, and we have that

$$\sup_{0\le r<1}\frac{1}{2\pi}\int_0^{2\pi}|(\varphi f)(re^{it})|^2\,dt \le \sup_{z\in\mathbb{D}}|\varphi(z)|^2\sup_{0\le r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{it})|^2\,dt,$$

so that also $\varphi f\in H^2$ by the previous proposition. Moreover, we see that

$$M_\varphi f = \varphi f$$

defines an operator on H^2 with $\|M_\varphi\|\le\sup_{z\in\mathbb{D}}|\varphi(z)|$. The function φ is called a *multiplier* of H^2, M_φ is called the corresponding *multiplication operator* or briefly *multiplier*.

Clearly, multiplication operators are never hypercyclic. For if $(M_\varphi)^n f = \varphi^n f$, $n\ge 0$, formed a dense set in H^2 then, by continuity of point evaluations, the same would be true of the sequence $(\varphi(0)^n f(0))_{n\ge 0}$ in $\mathbb{C}$, which is never the case. Instead, we will consider the (Hilbert space) adjoint $M_\varphi^*: H^2\to H^2$ of M_φ, called an *adjoint multiplication operator* or *adjoint multiplier*.

In fact, we already know that some operators M_φ^* are hypercyclic, as we will see now. As usual, B and F denote the backward and forward shifts on $\ell^2(\mathbb{N}_0)$, respectively, which are operators of norm 1. Hence, if $\varphi(z)=\sum_{n=0}^\infty a_n z^n$ is holomorphic on some neighbourhood of $\overline{\mathbb{D}}$ then $\sum_{n=0}^\infty\|a_nB^n\|\le\sum_{n=0}^\infty|a_n|<\infty$, so that

$$\varphi(B)=\sum_{n=0}^{\infty}a_nB^n$$

defines an operator on $\ell^2(\mathbb{N}_0)$, and the same is true for $\varphi(F)=\sum_{n=0}^\infty a_nF^n$; see also Appendix B.

Proposition 4.41. *Let $\varphi(z)=\sum_{n=0}^\infty a_n z^n$ be holomorphic on a neighbourhood of $\overline{\mathbb{D}}$, and set $\varphi^*(z)=\sum_{n=0}^\infty\overline{a_n}z^n$. Then, via the identification of H^2 with $\ell^2(\mathbb{N}_0)$:*

(i) *the multiplier* M_φ *corresponds to the operator* $\varphi(F)$ *on* $\ell^2(\mathbb{N}_0)$*;*
(ii) *the adjoint multiplier* M_φ^* *corresponds to the operator* $\varphi^*(B)$ *on* $\ell^2(\mathbb{N}_0)$*.*

Proof. (i) On the one hand we have that for $f \in H^2$, $f(z) = \sum_{k=0}^\infty b_k z^k$,

$$M_\varphi f(z) = \sum_{n=0}^\infty a_n z^n \sum_{k=0}^\infty b_k z^k = \sum_{n=0}^\infty \Big(\sum_{k=0}^n a_{n-k} b_k\Big) z^n.$$

On the other hand we have for $(b_k)_{k\geq 0} \in \ell^2(\mathbb{N}_0)$,

$$\begin{aligned}\varphi(F)(b_k)_k &= \sum_{n=0}^\infty a_n F^n (b_k)_k = \sum_{n=0}^\infty a_n (0,\ldots,0,b_0,b_1,\ldots)\\ &= (a_0b_0, a_1b_0 + a_0b_1, a_2b_0 + a_1b_1 + a_0b_2,\ldots) = \Big(\sum_{k=0}^n a_{n-k}b_k\Big)_{n\geq 0}.\end{aligned}$$

This implies the result.

(ii) A simple calculation shows that B is the adjoint of F. Hence, by (i), the adjoint M_φ^* corresponds to $\varphi(F)^* = (\sum_{n=0}^\infty a_n F^n)^* = \sum_{n=0}^\infty \overline{a_n} B^n$, where we have used properties of the adjoint; see Proposition A.8. □

In particular, the adjoint multipliers M_φ^* with $\varphi(z) = \lambda z$, $|\lambda| > 1$, correspond to the Rolewicz operators $\overline{\lambda} B$ and are therefore hypercyclic.

The Godefroy–Shapiro criterion allows us to characterize the hypercyclic adjoint multipliers. We can exclude constant multipliers because their adjoint multiplication operators are multiples of the identity.

Theorem 4.42. *Let* φ *be a nonconstant bounded holomorphic function on* $\mathbb{D}$ *and let* M_φ^* *be the corresponding adjoint multiplier on* H^2*. Then the following assertions are equivalent:*
(i) M_φ^* *is hypercyclic;*
(ii) M_φ^* *is mixing;*
(iii) M_φ^* *is chaotic;*
(iv) $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \varnothing$*.*

Proof. Suppose that condition (iv) holds. Considering the reproducing kernels k_λ, $\lambda \in \mathbb{D}$, we find that, for all $f \in H^2$,

$$\langle f, M_\varphi^* k_\lambda\rangle = \langle \varphi f, k_\lambda\rangle = (\varphi f)(\lambda) = \langle f, \overline{\varphi(\lambda)}\ k_\lambda\rangle,$$

which shows that

$$M_\varphi^*\ k_\lambda = \overline{\varphi(\lambda)}\ k_\lambda.$$

Consequently,

$$\operatorname{span}\{f \in H^2\ ;\ M_\varphi^* f = \mu f \text{ for some } \mu \in \mathbb{C} \text{ with } |\mu| < 1\}$$

contains $\operatorname{span}\{k_\lambda\ ;\ |\varphi(\lambda)| < 1\}$, which is dense in H^2 by Lemma 4.39; indeed, since nonconstant holomorphic functions are open mappings, condition (iv) implies that $\{\lambda \in \mathbb{D}\ ;\ |\varphi(\lambda)| < 1\}$ is nonempty and open and therefore contains an accumulation point in $\mathbb{D}$. For the same reason the eigenvectors of M_φ^* to eigenvalues of modulus greater than 1 span a dense set in H^2. Finally, the same is true for the eigenvectors of M_φ^* to eigenvalues that are roots of unity. For this it suffices to show that $\{\lambda \in \mathbb{D}\ ;\ \varphi(\lambda) \text{ is a root of unity}\}$ has an accumulation point. But since φ is an open mapping, condition (iv) implies that infinitely many preimages of roots of unity lie in some relatively compact subset of $\mathbb{D}$ and therefore have an accumulation point in $\mathbb{D}$. By the Godefroy–Shapiro criterion, therefore, (iv) implies (ii) and (iii).

To finish the proof it suffices to show that (i) implies (iv). Let us suppose that $\varphi(\mathbb{D})$ does not intersect the unit circle. Since $\varphi(\mathbb{D})$ is connected, it must lie entirely inside or entirely outside $\mathbb{D}$. If $\varphi(\mathbb{D}) \subset \mathbb{D}$ then

$$\|M_\varphi^*\| = \|M_\varphi\| \le \sup_{z\in\mathbb{D}} |\varphi(z)| \le 1$$

(see Proposition A.8), and hence M_φ^* cannot be hypercyclic. On the other hand, if $\varphi(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ then $\psi := 1/\varphi$ is a bounded holomorphic function on $\mathbb{D}$ with $\psi(\mathbb{D}) \subset \mathbb{D}$, which implies that M_ψ^* cannot be hypercyclic. But M_φ is the inverse of M_ψ and therefore M_φ^* is the inverse of M_ψ^*; see Proposition A.8. By Proposition 2.23, M_φ^* cannot be hypercyclic. □

This result can easily be extended to more general Hilbert spaces of holomorphic functions, for example the Bergman space (see Exercise 4.4.3); but see also Exercise 4.4.4. An extension to some Banach spaces X of holomorphic functions is also possible, in which case, of course, M_φ^* is the Banach space adjoint on the dual X^*; see Exercise 4.4.5.

We pass on to another, partial generalization of Theorem 4.42 that is motivated by Proposition 4.41. Under the assumptions of that proposition, $\varphi(B) = \sum_{n=0}^\infty a_n B^n$ defines an operator on each of the spaces ℓ^p, $1 \le p < \infty$, and c_0.

Theorem 4.43. *Let X be one of the complex sequence spaces ℓ^p, $1 \le p < \infty$, or c_0. Furthermore, let φ be a nonconstant holomorphic function on a neighbourhood of $\overline{\mathbb{D}}$. Then the following assertions are equivalent:*
(i) *$\varphi(B)$ is chaotic;*
(ii) *$\varphi(\mathbb{D}) \cap \mathbb{T} \neq \varnothing$;*
(iii) *$\varphi(B)$ has a nontrivial periodic point.*

Proof. (ii)$\Longrightarrow$(i). We saw in Example 3.2 that any sequence

$$e_\lambda := (\lambda, \lambda^2, \lambda^3, \ldots), \quad |\lambda| < 1$$

is an eigenvector of B to the eigenvalue λ and that, for any set $\Lambda \subset \mathbb{D}$ that has an accumulation point in $\mathbb{D}$, the set

$$\text{span}\{e_\lambda\ ;\ \lambda \in \Lambda\}$$

is dense in X. Now, for any $\lambda \in \mathbb{D}$ we have that

$$\varphi(B)e_\lambda = \sum_{n=0}^{\infty} a_n B^n e_\lambda = \sum_{n=0}^{\infty} a_n \lambda^n e_\lambda = \varphi(\lambda) e_\lambda,$$

so that each e_λ is also an eigenvector of $\varphi(B)$ to the eigenvalue $\varphi(\lambda)$. From here we proceed exactly as in the proof of Theorem 4.42 to show that $\varphi(B)$ is chaotic.

(i)$\Longrightarrow$(iii) is trivial.

(iii)$\Longrightarrow$(ii). For this implication we need to use results from spectral theory; see Appendix B. By condition (iii) there is some point $x \neq 0$ from X and some $N \geq 1$ such that $\varphi^N(B)x = \varphi(B)^N x = x$. Thus, $1 \in \sigma_p(\varphi^N(B))$, the point spectrum of $\varphi^N(B)$. It follows from the point spectral mapping theorem (see Theorem B.7) that $1 = \varphi^N(\lambda)$ for some $\lambda \in \sigma_p(B) = \mathbb{D}$; see Example 3.2. Hence $\varphi(\lambda) \in \mathbb{T}$, which implies (ii). □

Example 4.44. The theorem shows in particular that any operator

$$I + \lambda B \quad \text{and} \quad e^{\lambda B}, \quad \lambda \neq 0$$

is hypercyclic (and even chaotic) on $X = \ell^p$, $1 \leq p < \infty$, or c_0. In Section 8.1 we will see that much more is true: for any weight sequence w for which the backward shift B_w is an operator on X, the operators $I + B_w$ and e^{B_w} are hypercyclic (and even mixing); see Theorems 8.1 and 8.2.

4.5 Composition operators II

In this section we return to the composition operators studied in Section 4.3, but we consider them now on the Hardy space H^2. Thus, let φ be an automorphism of the unit disk $\mathbb{D}$ and let

$$C_\varphi f = f \circ \varphi$$

be the corresponding composition operator, where we now demand that f belongs to H^2. The first problem is that of determining if this defines an operator on H^2.

Proposition 4.45. *For any $\varphi \in \text{Aut}(\mathbb{D})$, C_φ defines an operator on H^2.*

Proof. By Proposition 4.36 there are $a, b \in \mathbb{C}$ with $|a| < 1$ and $|b| = 1$ such that

$$\varphi(z) = b\frac{a - z}{1 - \overline{a}z}.$$

First, let f be a polynomial. Then f and $f \circ \varphi$ are continuous on $\overline{\mathbb{D}}$ so that, by Proposition 4.40,

$$\|f \circ \varphi\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \left| f(\varphi(e^{it})) \right|^2 dt, \tag{4.13}$$

and similarly for $\|f\|^2$. Also by Proposition 4.36, φ is a bijective self-map on $\mathbb{T}$ so that there is some $u_0 \in \mathbb{R}$ and a continuously differentiable function $u : [0, 2\pi] \to [u_0, u_0 + 2\pi]$ such that

$$e^{iu(t)} = \varphi(e^{it}), \quad t \in [0, 2\pi].$$

Differentiating with respect to t we obtain that

$$ie^{iu(t)} \frac{du}{dt} = ie^{it} \varphi'(e^{it}), \quad t \in [0, 2\pi],$$

so that (4.13) and the substitution $u = u(t)$ yield

$$\|f \circ \varphi\|^2 = \frac{1}{2\pi} \int_{u_0}^{u_0+2\pi} |f(e^{iu})|^2 \frac{1}{|\varphi'(e^{it(u)})|} \, du.$$

Since, for $|z| = 1$,

$$|\varphi'(z)| = \left| b \frac{a\overline{a} - 1}{(1 - \overline{a}z)^2} \right| \geq \frac{1 - |a|^2}{(1 + |a|)^2} = \frac{1 - |a|}{1 + |a|},$$

we deduce that

$$\|f \circ \varphi\|^2 \leq \frac{1 + |a|}{1 - |a|} \cdot \frac{1}{2\pi} \int_{u_0}^{u_0+2\pi} |f(e^{iu})|^2 \, du = \frac{1 + |a|}{1 - |a|} \cdot \|f\|^2. \tag{4.14}$$

Now let $f \in H^2$ be arbitrary, and let f_n, $n \geq 0$, be the partial sums of its Taylor series. By Proposition 4.40 and (4.14) we have for $n \geq 0$ and $0 \leq r < 1$ that

$$\frac{1}{2\pi} \int_0^{2\pi} |f_n(\varphi(re^{it}))|^2 \, dt \leq \|f_n \circ \varphi\|^2 \leq \frac{1 + |a|}{1 - |a|} \cdot \|f_n\|^2.$$

Letting $n \to \infty$ and noting that $f_n \to f$ in H^2 and locally uniformly on $\mathbb{D}$ we deduce that, for $0 \leq r < 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\varphi(re^{it}))|^2 \, dt \leq \frac{1 + |a|}{1 - |a|} \cdot \|f\|^2.$$

By Proposition 4.40 this shows that $f \circ \varphi \in H^2$ and that C_φ is continuous on H^2. $\square$

The proof also gives us a norm estimate on the operator C_φ, namely

$$\|C_\varphi\| \le \Big(\frac{1+|a|}{1-|a|}\Big)^{1/2}.$$

Our aim now is to characterize when C_φ is hypercyclic on H^2. To this end we need to study the (nonlinear) dynamical system that is described by the automorphism φ on $\mathbb{D}$. It will be convenient to consider φ as a particular linear fractional transformation; see Appendix A.

Indeed, let

$$\varphi(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0,$$

be an arbitrary linear fractional transformation, which we consider as a map on the extended complex plane $\widehat{\mathbb{C}}$. Then φ has either one or two fixed points in $\widehat{\mathbb{C}}$, or it is the identity.

Suppose that φ has a single fixed point z_0, and let σ be a linear fractional transformation that maps z_0 to ∞. Then $\psi := \sigma\circ\varphi\circ\sigma^{-1}$ has ∞ as a unique fixed point, which easily implies that $\psi(z) = z + c$ for some $c \neq 0$.

Now suppose that φ has two distinct fixed points z_0 and z_1, and let σ be a linear fractional transformation that maps z_0 to 0 and z_1 to ∞. Then $\psi := \sigma\circ\varphi\circ\sigma^{-1}$ has fixed points 0 and ∞, which easily implies that $\psi(z) = \lambda z$ for some $\lambda \neq 0$. The constant λ is called the *multiplier* of φ. Replacing σ by $1/\sigma$ one sees that also $1/\lambda$ is a multiplier, which, however, causes no problem in the following.

Definition 4.46. Let φ be a linear fractional transformation that is not the identity.

(a) If φ has a single fixed point then it is called *parabolic*.

(b) Suppose that φ has two fixed points, and let λ be its multiplier. If $|\lambda| = 1$ then φ is called *elliptic*; if $\lambda > 0$ then φ is called *hyperbolic*; in all other cases, φ is called *loxodromic*.

It is now easy to deduce some important dynamical properties of automorphisms φ of $\mathbb{D}$.

Proposition 4.47. *Let $\varphi \in \mathrm{Aut}(\mathbb{D})$, not the identity. Then we have the following:*

(i) *if φ is parabolic then its fixed point z_0 lies in $\mathbb{T}$, and $\varphi^n(z) \to z_0$, $\varphi^{-n}(z) \to z_0$ for all $z \in \widehat{\mathbb{C}}$;*
(ii) *if φ is elliptic then it has a fixed point in $\mathbb{D}$;*
(iii) *if φ is hyperbolic then it has distinct fixed points z_0 and z_1 in $\mathbb{T}$ such that $\varphi^n(z) \to z_0$ for all $z \in \widehat{\mathbb{C}}$, $z \neq z_1$, and $\varphi^{-n}(z) \to z_1$ for all $z \in \widehat{\mathbb{C}}$, $z \neq z_0$;*
(iv) *φ cannot be loxodromic.*

Proof. In the various cases, let σ and ψ be the linear fractional transformations given above. Then σ provides a conjugacy between φ and ψ.

(i) If φ is parabolic then $\psi^n(z) = z + nc \to \infty$ for all $z \in \widehat{\mathbb{C}}$ and hence $\varphi^n(z) \to \sigma^{-1}(\infty) = z_0$ for all $z \in \widehat{\mathbb{C}}$. In the same way, $\varphi^{-n}(z) \to z_0$ for all

$z \in \widehat{\mathbb{C}}$. Since φ maps $\mathbb{T}$ into $\mathbb{T}$ (see Proposition 4.36), we must have that $z_0 \in \mathbb{T}$.

(ii) Let φ be elliptic. Since φ maps $\mathbb{D}$ onto itself, ψ maps $\sigma(\mathbb{D})$ onto itself, which is either a half-plane, or the interior or the exterior of a disk U. Since ψ is a rotation and $\lambda \neq 1$, the first alternative is excluded and U must be centred at 0. Thus, either 0 or ∞ lies in $B = \sigma(\mathbb{D})$, so that either z_0 or z_1 belongs to $\mathbb{D}$.

(iii) Let φ be hyperbolic. Then $\lambda > 0$, and since with λ also $1/\lambda$ is a multiplier we can assume that $\lambda < 1$. Then $\psi^n(z) = \lambda^n z \to 0$ for all $z \in \widehat{\mathbb{C}}$, $z \neq \infty$, and therefore $\varphi^n(z) \to \sigma^{-1}(0) = z_0$ for all $z \in \widehat{\mathbb{C}}$, $z \neq \sigma^{-1}(\infty) = z_1$. It follows as in (i) that $z_0 \in \mathbb{T}$. Moreover, we find that $\psi^{-n}(z) = \lambda^{-n} z \to \infty$ for all $z \in \widehat{\mathbb{C}}$, $z \neq 0$, and hence $\varphi^{-n}(z) \to \sigma^{-1}(\infty) = z_1$ for all $z \in \widehat{\mathbb{C}}$, $z \neq \sigma^{-1}(0) = z_0$. Since $\mathbb{T}$ is also invariant under φ^{-1} we find that also $z_1 \in \mathbb{T}$.

(iv) Let φ be loxodromic. As in (ii), ψ maps $\sigma(\mathbb{D})$ onto itself, which is either a half-plane, or the interior or the exterior of a disk. But this is incompatible with the fact that $|\lambda| \neq 1$ and $\lambda \not> 0$. □

The dynamical properties of φ imply the dynamical properties of C_φ.

Theorem 4.48. *Let $\varphi \in \mathrm{Aut}(\mathbb{D})$ and C_φ be the corresponding composition operator on H^2. Then the following assertions are equivalent:*
(i) *C_φ is hypercyclic;*
(ii) *C_φ is mixing;*
(iii) *φ has no fixed point in $\mathbb{D}$.*

Proof. The implication (ii)$\Longrightarrow$(i) holds for all operators on H^2, and (i)$\Longrightarrow$(iii) follows as in the proof of Theorem 4.37, using the fact that point evaluations are continuous on H^2.

(iii)$\Longrightarrow$(ii). Suppose that φ has no fixed point in $\mathbb{D}$. It suffices to show that C_φ satisfies Kitai's criterion. By Proposition 4.47, φ is either parabolic or hyperbolic, and in both cases there are $z_0, z_1 \in \mathbb{T}$ (possibly with $z_0 = z_1$) such that $\varphi^n(z) \to z_0$ for all $z \in \mathbb{T}\backslash\{z_1\}$ and $\varphi^{-n}(z) \to z_1$ for all $z \in \mathbb{T}\backslash\{z_0\}$.

Now, for X_0 we will take the subspace of H^2 of all functions that are holomorphic on a neighbourhood of $\overline{\mathbb{D}}$ and that vanish at z_0. To see that X_0 is dense in H^2, let $f \in H^2$, $f(z) = \sum_{n=0}^\infty a_n z^n$, be orthogonal to any $g \in X_0$. Since, for any $n \geq 0$, the functions $g_n : z \to z_0 z^n - z^{n+1}$ belong to X_0 we have that $\langle f, g_n \rangle = \overline{z_0} a_n - a_{n+1} = 0$ and hence $a_n = a_0 \overline{z_0}^n$, $n \geq 0$. Since $(a_n)_n$ is square summable and $|z_0| = 1$ we must have that $a_0 = 0$, hence $f = 0$. This implies that X_0 is dense in H^2. Moreover, let $f \in X_0$. As in (4.13) we have that

$$\|(C_\varphi)^n f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(\varphi^n\left(e^{it}\right)\right)\right|^2 dt.$$

Since the integrands are uniformly bounded and convergent to $|f(z_0)|^2 = 0$, for every t with possibly one exception, the dominated convergence theorem implies that $(C_\varphi)^n f \to 0$ for all $f \in X_0$.

Next, for Y_0 we take the subspace of H^2 of all functions that are holomorphic on a neighbourhood of $\overline{\mathbb{D}}$ and that vanish at z_1, and for S we take the map $S = C_{\varphi^{-1}}$. Since z_1 is a fixed point of φ^{-1}, S maps Y_0 into itself, and clearly $TS = I$. It follows as above that Y_0 is dense in H^2 and that $S^n f \to 0$ for all $f \in Y_0$.

Therefore the conditions of Kitai's criterion are satisfied, so that C_φ is mixing. □

Some concrete instances of this result are treated in Exercise 4.5.2.

We end this chapter by returning to Theorem 4.37 of Section 4.3. Proposition 4.47 allows us to give the missing proof of the implication (v)$\Longrightarrow$(vi) for $\Omega = \mathbb{D}$.

Conclusion of the proof of Theorem 4.37. Let φ be an automorphism of $\mathbb{D}$ without fixed points in $\mathbb{D}$. Again, φ is either parabolic or hyperbolic.

First, let φ be parabolic. By the discussion before Definition 4.46 there is a linear fractional transformation σ that provides a conjugacy between φ and $\psi(z) = z + c$, $c \neq 0$. Then ψ is an automorphism of $\sigma(\mathbb{D})$, so that C_φ is conjugate to the operator C_ψ on $H(\sigma(\mathbb{D}))$ by Proposition 4.25. By Runge's theorem, the continuous restriction map $H(\mathbb{C}) \to H(\sigma(\mathbb{D}))$, $f \to f|_{\sigma(\mathbb{D})}$ has dense range. Hence C_φ is quasiconjugate to the Birkhoff operator C_ψ on $H(\mathbb{C})$.

In the hyperbolic case, there is a linear fractional transformation σ such that φ is conjugate to a dilation $\psi(z) = \lambda z$, $\lambda \neq 1$ strictly positive, and ψ is an automorphism of $\sigma(\mathbb{D})$, which therefore must be a half-plane with 0 on its boundary. After conjugation with a suitable rotation we can assume that it is the right half-plane $\mathbb{C}_+$, and ψ remains unchanged. Now, the principal branch log of the logarithm is a conformal map from $\mathbb{C}_+$ to the strip $S = \{z \in \mathbb{C} \ ; \ |\mathrm{Im}(z)| < \frac{\pi}{2}\}$, and conjugating ψ with log gives us the translation $\tau(z) = z + \log\lambda$, $\log\lambda \neq 0$, on S. We conclude as in the parabolic case that C_φ is quasiconjugate to the Birkhoff operator C_τ on $H(\mathbb{C})$. □

Exercises

Exercise 4.1.1. Show that a weighted shift B_w defines an operator on $H(\mathbb{C})$ if and only if $\sup_{n\geq 1} |w_n|^{1/n} < \infty$, and that $a_n e_n \to 0$ in $H(\mathbb{C})$ if and only if $|a_n|^{1/n} \to 0$.

Exercise 4.1.2. Let $T := B_w$ be a weighted shift on ℓ^p, $1 \leq p < \infty$.

(a) Given an increasing sequence $(n_k)_k$ of positive integers, show that the sequence of operators $(T^{n_k})_k$ is hypercyclic if and only if, for each $j \in \mathbb{N}$,

$$\sup_{k\geq 1} \prod_{\nu=1}^{j+n_k} |w_\nu| = \infty.$$

(b) Show that T is hereditarily hypercyclic with respect to $(n_k)_k$ if and only if, for each $j \in \mathbb{N}$,

$$\lim_{k\to\infty} \prod_{\nu=1}^{j+n_k} |w_\nu| = \infty.$$

Exercise 4.1.3. Let X be the Banach space of all sequences $(x_n)_n$ satisfying

$$\|x\| = \sum_{n=1}^{\infty} \left| \frac{x_n}{n} - \frac{x_{n+1}}{n+1} \right| < \infty \quad \text{and} \quad \frac{x_n}{n} \to 0 \text{ as } n \to \infty.$$

Show that the backward shift B is a hypercyclic operator on X and that conditions (ii)–(iv) of Theorem 4.6 are satisfied, but that the only periodic points of B are the constant sequences and that B is therefore not chaotic.

Exercise 4.1.4. Let X be a Fréchet sequence space over $\mathbb{Z}$ in which $(e_n)_{n\in\mathbb{Z}}$ is a basis. Suppose that the bilateral weighted shift B_w is an invertible operator on X. Show that B_w is hypercyclic if and only if there is an increasing sequence $(n_k)_k$ of positive integers such that

$$\Big(\prod_{\nu=-n_k+1}^{0} w_\nu \Big) e_{-n_k} \to 0 \quad \text{and} \quad \Big(\prod_{\nu=1}^{n_k} w_\nu \Big)^{-1} e_{n_k} \to 0$$

in X as $k \to \infty$. (*Hint:* For the sufficiency, look at the proof of (iii)$\Longrightarrow$(ii) in Theorem 4.3.)

Exercise 4.1.5. Find a (necessarily non-invertible) bilateral weighted shift that satisfies the condition stated in the previous exercise but that is not hypercyclic. (*Hint*: See the proof of Proposition 4.16, but choose nonsymmetric v_n.)

Exercise 4.1.6. Prove the results stated in Remark 4.14; instead of using a conjugacy one may also observe that a forward shift on the basis $(e_n)_n$ is a backward shift on the basis $(e_{-n})_n$.

Exercise 4.1.7. Show that the characterizing conditions on a weight w to define a hypercyclic bilateral weighted shift B_w on $\ell^p(\mathbb{Z})$ can also be written as follows: for any $\varepsilon > 0$ and any $M, N \geq 1$ there exists some $n \geq N$ such that whenever $|j| \leq M$ then

$$\prod_{\nu=j-n+1}^{j} |w_\nu| < \varepsilon, \qquad \prod_{\nu=j+1}^{j+n} |w_\nu| > \frac{1}{\varepsilon}.$$

Exercise 4.2.1. An entire function φ is of *exponential type* 0 if for any $\varepsilon > 0$ there is some $M > 0$ such that

$$|\varphi(z)| \leq M e^{\varepsilon|z|} \quad \text{for all } z \in \mathbb{C}.$$

For example, any polynomial but no exponential function $z \to e^{\lambda z}$, $\lambda \neq 0$, is of exponential type 0.

For a domain $\Omega \subset \mathbb{C}$, let $H(\Omega)$ denote the Fréchet space of holomorphic functions on Ω; see Section 4.3. Show the following:

(i) an entire function $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ is of exponential type 0 if and only if, for any $\varepsilon > 0$, there is some $M > 0$ such that $|a_n| \leq M \frac{\varepsilon^n}{n!}$;

(ii) for any domain $\Omega \subset \mathbb{C}$ and any entire function $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ of exponential type 0, $\varphi(D) = \sum_{n=0}^{\infty} a_n D^n$ defines an operator on $H(\Omega)$;

(iii) for any simply connected domain $\Omega \subset \mathbb{C}$ and any nonconstant entire function φ of exponential type 0, $\varphi(D)$ is chaotic on $H(\Omega)$. (*Hint*: Use the Godefroy–Shapiro theorem and the fact, that, by Runge's theorem, $H(\mathbb{C})$ is dense in $H(\Omega)$.)

Exercise 4.2.2. Let Ω be a domain and P a nonconstant polynomial. Show that the following assertions are equivalent:
(i) $P(D)$ is chaotic on $H(\Omega)$;
(ii) $P(D)$ is hypercyclic on $H(\Omega)$;
(iii) Ω is simply connected.
(*Hint*: If Ω is not simply connected then there is a smooth Jordan curve Γ in Ω surrounding some $a \notin \Omega$. Show that $f \to \int_\Gamma f(\zeta)\,d\zeta$ is an eigenvector of $P(D)^*$, and use Lemma 2.53.)

Exercise 4.2.3. Let $X = C^\infty_{\mathbb{R}}(\mathbb{R})$ be the space of infinitely differentiable real functions $f : \mathbb{R} \to \mathbb{R}$; see Exercise 2.1.5. Show that every (real) differential operator $T : X \to X$, $Tf = \sum_{n=0}^{N} a_n f^{(n)}$, $T \neq a_0 I$, is chaotic. (*Hint*: See Exercise 2.2.5.)

Exercise 4.2.4. Let φ be a nonconstant entire function of exponential type and $A = \min\{|z| \; ; \; z \in \mathbb{C}, |\varphi(z)| = 1\}$. Show that, for any $\varepsilon > 0$, there is an entire function f that is hypercyclic for $\varphi(D)$ such that

$$|f(z)| \leq M e^{(A+\varepsilon)r} \quad \text{for } |z| = r > 0$$

with some $M > 0$.

For the proof consider the Hilbert spaces

$$E^2_\tau = \left\{ f \in H(\mathbb{C}) \; ; \; f(z) = \sum_{n=0}^{\infty} a_n z^n, \; \sum_{n=0}^{\infty} \left(\frac{n!}{\tau^n}\right)^2 |a_n|^2 < \infty \right\}, \quad \tau > 0.$$

Show that any $f \in E^2_\tau$ satisfies $|f(z)| \leq M e^{\tau r}$; use ideas from Example 3.2 to show that for any $\Lambda \subset \mathbb{D}_\tau$ with an accumulation point, $\operatorname{span}\{e_\lambda \; ; \; \lambda \in \Lambda\}$ is dense in E^2_τ (see Appendix A for the dual of E^2_τ); show that $\varphi(D)$ is an operator on any E^2_τ, and that $\varphi(D)$ is hypercyclic on $E^2_{A+\varepsilon}$ for any $\varepsilon > 0$.

Apply the result to MacLane's and Birkhoff's operators.

Exercise 4.2.5. Let B_w be a chaotic weighted shift on $H(\mathbb{C})$; see Example 4.9(b). Then $\sum_{n=0}^{\infty} (\prod_{\nu=1}^{n} w_\nu)^{-1} z^n$ is an entire function, and its *maximum term* is defined by

$$\mu_w(r) = \max_{n \geq 0} \frac{r^n}{\prod_{\nu=1}^{n} |w_\nu|}, \quad r \geq 0.$$

(a) Let $\phi :]0, \infty[\to [1, \infty[$ be a function with $\phi(r) \to \infty$ as $r \to \infty$. Show that there exists an entire function f that is hypercyclic for B_w and that satisfies

$$|f(z)| \leq M \phi(r) \mu_w(r) \quad \text{for } |z| = r > 0$$

with some $M > 0$.

(b) Suppose that $|w_n| \to \infty$ monotonically. Show that there is no entire function f that is hypercyclic for B_w and that satisfies

$$|f(z)| \leq M \, \mu_w(r) \quad \text{for } |z| = r > 0$$

with some $M > 0$. (*Hint*: Determine $\mu_w(\rho)$ for $\rho = |w_n|$.)

Deduce Theorem 4.22 from this.

Exercise 4.3.1. Show in detail that $H(\Omega)$ is a separable Fréchet space and that its topology is independent of the exhaustion chosen. (*Hint*: For separability, fix an exhaustion $(K_n)_n$ of Ω by Ω-convex compact sets. In each connected component of the complements of the K_n fix one point outside Ω, possibly ∞; this set is denumerable. Now use Runge's theorem.)

Exercise 4.3.2. Let Ω be a domain and $\varphi : \Omega \to \Omega$ a holomorphic self-map that is not necessarily an automorphism of Ω. Show that $C_\varphi : H(\Omega) \to H(\Omega)$, $C_\varphi f = f \circ \varphi$, defines an operator on $H(\Omega)$, also called a *composition operator*. Moreover, show the following:

(i) if C_φ is hypercyclic then φ is injective;

(ii) let Ω be simply connected; then C_φ is hypercyclic if and only if φ is injective and $(\varphi^n)_n$ is a run-away sequence.

Exercise 4.3.3. Let Ω be a domain, $\varphi : \Omega \to \Omega$ an injective holomorphic self-map and $C_\varphi f = f \circ \varphi$ the corresponding composition operator on $H(\Omega)$; see the previous exercise.

(a) Based on the proof of Theorem 4.32, find a sufficient condition under which C_φ is hypercyclic.

(b) Let $\varphi : \mathbb{D} \to \mathbb{D}$ be given by $\varphi(z) = \frac{z}{4} + \frac{3}{4}$. Let $K = \{z \in \mathbb{C}\ ;\ |z| \leq \frac{1}{2}\}$ and $\Omega = \mathbb{D} \setminus \bigcup_{n=0}^{\infty} \varphi^n(K)$. Show that the restriction of φ to Ω defines a hypercyclic composition operator C_φ on $H(\Omega)$.

Exercise 4.3.4. Let $\Omega = \mathbb{C} \setminus \mathbb{Z}$. Then $\varphi(z) = z + 1$ is an automorphism of Ω. Show that the composition operator C_φ is chaotic on $H(\Omega)$. (*Hint*: Show that the linear span of the functions $e^{\lambda z}$, $e^{\lambda N} = 1$ for some $N \geq 1$, and $\lim_{m\to\infty} \sum_{\nu=-m}^{m} \frac{1}{(z-k-\nu N)^\alpha}$, $k \in \mathbb{Z}$, $\alpha > 1$, $N \geq 1$, forms a dense set of periodic points.)

Exercise 4.4.1. Let φ be a holomorphic function on $\mathbb{D}$ such that $\varphi f \in H^2$ for all $f \in H^2$. Use the closed graph theorem to show that the mapping $M_\varphi : f \to \varphi f$ is continuous. Deduce that φ is necessarily bounded, and $\|M_\varphi\| = \sup_{z\in\mathbb{D}} |\varphi(z)|$. (*Hint*: $\varphi^n(z) = \langle (M_\varphi)^n 1, k_z\rangle$.)

Exercise 4.4.2. Let $\Omega \subset \mathbb{C}$ be a domain and $H \neq \{0\}$ a Hilbert space of holomorphic functions on Ω. Suppose that each point evaluation $f \to f(\lambda)$, $\lambda \in \Omega$, is a continuous linear functional on H. Use the closed graph theorem to prove that the canonical embedding $H \hookrightarrow H(\Omega)$ is continuous, so that convergence in H implies locally uniform convergence on Ω.

Exercise 4.4.3. Let $\Omega \subset \mathbb{C}$ be a domain and $H \neq \{0\}$ a Hilbert space of holomorphic functions on Ω such that each point evaluation $f \to f(\lambda)$, $\lambda \in \Omega$, is continuous on H.

(a) By the Riesz representation theorem (see Appendix A), there is a unique function $k_\lambda \in H$, again called a reproducing kernel, such that

$$f(\lambda) = \langle f, k_\lambda\rangle, \quad f \in H.$$

Prove an analogue of Lemma 4.39 and deduce that H is separable.

(b) Now let φ be a nonconstant bounded holomorphic function on Ω for which $M_\varphi f = \varphi f$ defines an operator on H. Let M_φ^* be the corresponding adjoint multiplier on H. Show that M_φ^* is chaotic and mixing as soon as $\varphi(\Omega) \cap \mathbb{T} \neq \varnothing$. Show that, in this case, for some $\lambda \in \mathbb{C}$, λM_φ^* is chaotic. Deduce that M_φ^* is supercyclic.

(c) Finally, suppose that every bounded holomorphic function φ on Ω defines a multiplication operator with $\|M_\varphi\| \leq \sup_{z\in\Omega} |\varphi(z)|$. Show that if φ is a nonconstant bounded holomorphic function on Ω such that M_φ^* is hypercyclic then $\varphi(\Omega) \cap \mathbb{T} \neq \varnothing$.

(d) Deduce that Theorem 4.42 holds also for the Bergman space A^2; see Example 4.4(b).

Exercise 4.4.4. The *Dirichlet space* $\mathcal{D}$ is defined as the space of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|^2 := |f(0)|^2 + \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2 \, d\lambda(z) < \infty,$$

where λ denotes two-dimensional Lebesgue measure. Show the following:

(i) if $f(z)=\sum_{n=0}^{\infty} a_n z^n$ then $\|f\|^2=|a_0|^2+\sum_{n=1}^{\infty} n|a_n|^2$;
(ii) $\mathcal{D}$ is a Hilbert space with continuous point evaluations;
(iii) $\mathcal{D}\subset H^2\subset A^2$, where A^2 is the Bergman space (see Example 4.4);
(iv) if φ is a bounded holomorphic function on $\mathbb{D}$ such that φ' is also bounded then M_φ defines an operator on $\mathcal{D}$;
(v) if $\varphi(z)=\sum_{n=0}^{\infty} b_n z^n$ with $\sum_{n=0}^{\infty}|b_n|<\infty$ and $\sum_{n=0}^{\infty} n|b_n|^2=\infty$ (existence?) then φ is a bounded holomorphic function on $\mathbb{D}$ for which M_φ does not define an operator on $\mathcal{D}$;
(vi) if φ is a nonconstant bounded holomorphic function on $\mathbb{D}$ such that M_φ defines an operator on $\mathcal{D}$ and if $\varphi(\mathbb{D})\cap\mathbb{T}\neq\varnothing$ then M_φ^* is mixing and chaotic;
(vii) the function $\varphi(z)=z$, $z\in\mathbb{D}$, defines a hypercyclic adjoint multiplier M_φ^* on $\mathcal{D}$, but $\varphi(\mathbb{D})\cap\mathbb{T}=\varnothing$; is M_φ^* mixing or chaotic? (*Hint*: Identify M_φ^* with a weighted shift on a weighted ℓ^2-space.)

Exercise 4.4.5. Let $X\neq\{0\}$ be a Banach space of holomorphic functions on a domain $\Omega\subset\mathbb{C}$. Suppose that X is reflexive, that is, $X^{**}=X$. Suppose further that the point-evaluations $f\to f(\lambda)$, $\lambda\in\Omega$, are continuous on X and that every bounded holomorphic function φ on Ω defines a multiplication operator M_φ with $\|M_\varphi\|\leq\sup_{z\in\Omega}|\varphi(z)|$. Then show the analogue of Theorem 4.42 for M_φ^*, the (Banach space) adjoint of M_φ. (*Hint*: Use reflexivity and the Hahn–Banach theorem to obtain the analogue of Lemma 4.39.)

Exercise 4.4.6. Let X be one of the complex spaces ℓ^p, $1\leq p<\infty$, or c_0. Let $a,b\in\mathbb{C}$, $b\neq 0$. Show that the following assertions are equivalent:
(i) $aI+bB$ is chaotic on X;
(ii) $|b|>|1-|a||$.

Exercise 4.4.7. Generalize part of Theorem 4.43 as follows: let $X=\ell^p(v)=\{(x_n)_n\in\mathbb{C}^{\mathbb{N}};\ \sum_{n=1}^{\infty}|x_n|^p v_n<\infty\}$, $1\leq p<\infty$, where $v=(v_n)_n$ is a positive weight sequence such that $M:=\sup_{n\in\mathbb{N}}\frac{v_n}{v_{n+1}}<\infty$. Let $R:=(\limsup_{n\to\infty} v_n^{1/n})^{-1}>0$, which is finite, and let $\varphi(z)=\sum_{n=0}^{\infty}a_n z^n$ be a nonconstant function that is holomorphic in $\mathbb{D}_r$ for some $r>M$. Then $\varphi(B)=\sum_{n=0}^{\infty}a_n B^n$ defines an operator on X, and if

$$\varphi\Big(R^{1/p}\mathbb{D}\Big)\cap\mathbb{T}\neq\varnothing \tag{4.15}$$

then $\varphi(B)$ is a chaotic operator on X. (*Hint*: See Appendix A for the dual of X.)

Exercise 4.4.8. In the setting of Exercise 4.4.7, let $v_n=1/n^2$, $n\geq 1$.
(a) Show that B is chaotic but condition (4.15) does not hold.
(b) Show that the operator $\frac{1}{2}(I+B)$ has a nontrivial periodic point but is not chaotic.

Exercise 4.4.9. Let $w=(w_n)_n$ be a weight sequence, and let f be a nonconstant polynomial. Show that $f(B_w)$ is (well defined and) chaotic on $\omega=\mathbb{K}^{\mathbb{N}}$. (*Hint*: For the density of eigenvectors for B_w show that they are contained and dense in a suitable weighted ℓ^1-space; see Appendix A.)

Exercise 4.5.1. The aim of this exercise is to prove *Littlewood's subordination principle*: if $\varphi:\mathbb{D}\to\mathbb{D}$ is a holomorphic self-map then $C_\varphi: f\to f\circ\varphi$ defines an operator on H^2 with $\|C_\varphi\|\leq(\frac{1+r}{1-r})^{1/2}$, where $r=|\varphi(0)|$.
(a) First prove the result when $\varphi(0)=0$ by proceeding as follows:
  (i) show that, for any $f\in H^2$, $C_\varphi f=f(0)+M_\varphi C_\varphi Bf$, where we write B for M_z^* (see Proposition 4.41(ii));
  (ii) using orthogonality, deduce that, for any polynomial f, $\|C_\varphi f\|^2\leq|f(0)|^2+\|C_\varphi Bf\|^2$;
  (iii) deduce that $\|C_\varphi f\|^2\leq\sum_{k=0}^{n}|(B^k f)(0)|^2+\|C_\varphi B^{n+1}f\|^2$;

(iv) deduce that $\|C_\varphi f\| \le \|f\|$;
(v) conclude that $C_\varphi f \in H^2$ and $\|C_\varphi f\| \le \|f\|$ for all $f \in H^2$.

(b) Prove the result by factorizing $C_\varphi = C_{\varphi_1} C_{\varphi_2}$ with $\varphi_1(0) = 0$ and $\varphi_2 \in \mathrm{Aut}(\mathbb{D})$.

Exercise 4.5.2. For the following linear fractional transformations decide if they are automorphisms of $\mathbb{D}$; in the case of an automorphism, determine if the corresponding composition operator C_φ is hypercyclic on H^2:

(i) $\varphi(z) = \dfrac{2z-1}{2-z}$;
(ii) $\varphi(z) = \dfrac{1+(i-1)z}{(i+1)-z}$;
(iii) $\varphi(z) = \dfrac{4-5z}{5-4z}$;
(iv) $\varphi(z) = \dfrac{z+1}{2}$.

Exercise 4.5.3. Let $\alpha > -1$. Then the *weighted Bergman space* A^2_α is defined as the space of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|^2 := \frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^2 \left(1-|z|^2\right)^\alpha \, d\lambda(z) < \infty,$$

where λ denotes two-dimensional Lebesgue measure; see Example 4.4(b).

(a) Let f be a holomorphic function on $\mathbb{D}$ with $f(z) = \sum_{n=0}^\infty a_n z^n$, $z \in \mathbb{D}$. Show that $\|f\|^2 = \sum_{n=0}^\infty |a_n|^2 \frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(\alpha+n+2)}$ and deduce that $f \in A^2_\alpha$ if and only if $\sum_{n=0}^\infty |a_n|^2 \frac{1}{(n+1)^{\alpha+1}} < \infty$. (*Hint*: Stirling's formula.)

(b) Let $\varphi \in \mathrm{Aut}(\mathbb{D})$. Show that C_φ is an operator on A^2_α with $\|C_\varphi\| \le (\frac{1+r}{1-r})^{1+\alpha/2}$, where $r = |\varphi(0)|$. (*Hint*: Show that if $w = \varphi(z)$ then $d\lambda(z) = |\varphi'(z)|^{-2}\, d\lambda(w)$, and use (4.12).)

Exercise 4.5.4. Let $\alpha > -1$. Then the *weighted Dirichlet space* $\mathcal{D}_\alpha$ is defined as the space of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|^2 := |f(0)|^2 + \frac{1}{\pi}\int_{\mathbb{D}} \left|f'(z)\right|^2 \left(1-|z|^2\right)^\alpha \, d\lambda(z) < \infty,$$

where λ denotes two-dimensional Lebesgue measure; see Exercise 4.4.4 and the previous exercise.

(a) Let f be a holomorphic function on $\mathbb{D}$ with $f(z) = \sum_{n=0}^\infty a_n z^n$, $z \in \mathbb{D}$. Show that $f \in \mathcal{D}_\alpha$ if and only if $\sum_{n=0}^\infty |a_n|^2 (n+1)^{1-\alpha} < \infty$.

(b) Let $\varphi \in \mathrm{Aut}(\mathbb{D})$. Show that C_φ is an operator on $\mathcal{D}_\alpha$.

(c) Let $\alpha > 0$. Show that Theorem 4.48 remains true for $\mathcal{D}_\alpha$. (*Hint*: Proceed as in the proof of that theorem; use the change of variables $w = \varphi^n(z)$; note that $1-|\varphi^{-n}(w)|^2 \to 0$.)

Exercise 4.5.5. Let $\varphi \in \mathrm{Aut}(\mathbb{D})$. By the previous exercise, C_φ is an operator on the Dirichlet space $\mathcal{D}$. Show that, for any $f \in \mathcal{D}$,

$$\|C_\varphi f\|^2 \ge \frac{1}{\pi}\int_{\mathbb{D}} \left|f'(z)\right|^2 \, d\lambda(z).$$

Deduce that C_φ is not hypercyclic on $\mathcal{D}$. (*Hint*: Change of variables $w = \varphi(z)$.)

Exercise 4.5.6. Let $\beta = (\beta_n)_{n\geq 0}$ be a sequence of strictly positive numbers such that $\sum_{n=0}^{\infty} \beta_n^{-2} r^n < \infty$ whenever $0 \leq r < 1$. Then the *weighted Hardy space* $H^2(\beta)$ is defined as the space of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|^2 := \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty,$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$. By the assumption on $(\beta_n)_n$, this condition alone implies that $f \in H(\mathbb{D})$.

Let $\varphi \in \text{Aut}(\mathbb{D})$ and suppose that C_φ defines an operator on $H^2(\beta)$. Show the following:

(i) if $\sum_{n=0}^{\infty} \beta_n^{-2} < \infty$ then C_φ is never hypercyclic on $H^2(\beta)$;

(ii) if $\sum_{n=0}^{\infty} \beta_n^{-2} = \infty$ and φ is elliptic then C_φ is not hypercyclic on $H^2(\beta)$.

(*Hint* for (i): Show that all functions in $H^2(\beta)$ have a continuous extension to $\overline{\mathbb{D}}$; and use the fact that φ has a fixed point in $\overline{\mathbb{D}}$.)

Exercise 4.5.7. Let $\nu \in \mathbb{R}$. Then the space $\mathcal{S}_\nu$ is defined as the space of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|^2 := \sum_{n=0}^{\infty} |a_n|^2 (n+1)^{2\nu} < \infty,$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$. In particular, $\mathcal{S}_0$ is the Hardy space H^2, $\mathcal{S}_{-1/2}$ is the Bergman space A^2, and $\mathcal{S}_{1/2}$ is the Dirichlet space $\mathcal{D}$ under an equivalent norm.

(a) Show that $f \in \mathcal{S}_\nu$ if and only if $f' \in \mathcal{S}_{\nu-1}$. If $\nu \in \mathbb{N}$, show that $f \in \mathcal{S}_\nu$ if and only if $f^{(\nu)} \in H^2$.

(b) Show that the multiplier M_z is an operator on each $\mathcal{S}_\nu$, and calculate the norm of M_z^n, $n \geq 0$. More generally, let φ be holomorphic on $\mathbb{D}$, $\varphi(z) = \sum_{n=0}^{\infty} b_n z^n$, such that $\sum_{n=0}^{\infty} |b_n|(n+1)^{\nu} < \infty$. Show that M_φ is an operator on $\mathcal{S}_\nu$.

(c) Let $\varphi \in \text{Aut}(\mathbb{D})$. Deduce from Exercise 4.5.3 that C_φ defines an operator on $\mathcal{S}_\nu$ for $\nu < 0$. Use parts (a) and (b) to conclude that C_φ defines an operator on $\mathcal{S}_\nu$ for any $\nu \in \mathbb{R}$. (*Hint*: $(C_\varphi f)' = M_{\varphi'} C_\varphi f'$.)

(d) Let $\varphi(z) = \sum_{n=0}^{\infty} b_n z^n$ with $\sum_{n=0}^{\infty} |b_n| \leq 1$ and $\sum_{n=0}^{\infty} |b_n|^2 n = \infty$ (existence?). Show that φ is a holomorphic self-map of $\mathbb{D}$ for which C_φ does not define an operator on the Dirichlet space $\mathcal{D}$.

Exercise 4.5.8. Let $\nu \in \mathbb{R}$ and $\varphi \in \text{Aut}(\mathbb{D})$. By Exercise 4.5.7, C_φ is an operator on S_ν. Deduce the following from the previous exercises:

(i) if $\nu \geq \frac{1}{2}$ then C_φ is never hypercyclic on S_ν;

(ii) if $\nu < \frac{1}{2}$ then C_φ is hypercyclic on S_ν if and only if φ is not the identity and non-elliptic.

Spell out these results for the (weighted) Bergman and Dirichlet spaces.

Sources and comments

Section 4.1. Rolewicz's multiples of the backward shift were the first Banach space operators to be proved hypercyclic [268]. Due to its simple structure, *the class of weighted shifts is a favorite testing ground for operator-theorists* (Salas [274]). Accordingly, whenever a new notion in linear dynamics is introduced it is usually first tested on weighted shifts.

Salas [274] characterized hypercyclic and weakly mixing unilateral and bilateral weighted shifts on ℓ^2 and $\ell^2(\mathbb{Z})$, respectively. The characterizations for more general sequence spaces and of chaos are due to Grosse-Erdmann [180], see also Martínez and Peris [229] in the special case of Köthe sequence spaces, while mixing shifts on ℓ^2 and $\ell^2(\mathbb{Z})$ were characterized by Costakis and Sambarino [124]. The approach chosen here of first studying the unweighted shift and then using suitable conjugacies is due to Martínez and Peris [229].

The first example of a hypercyclic operator whose adjoint is also hypercyclic (see Proposition 4.16) was found by Salas [273]. He later showed [276] that every separable Banach space with separable dual supports such an operator. The observation that $T \oplus T^*$ is never hypercyclic is due to Deddens; see [273].

Section 4.2. The investigation of hypercyclicity for differential operators $\varphi(D)$ is due to Godefroy and Shapiro [165]. Theorem 4.22 on the rate of growth of MacLane's operator was obtained independently by Grosse-Erdmann [178] and Shkarin [283]. The corresponding result for Birkhoff's operators was obtained by Duyos-Ruiz [137]; alternative proofs can be found in Chan and Shapiro [106] and in Exercise 8.1.3.

Translation and differentiation operators have also been studied on spaces of harmonic functions on $\mathbb{R}^N$, $N \geq 2$. Hypercyclicity of these operators and corresponding growth results have been obtained by Dzagnidze [138], Aldred and Armitage [6], [7], [13], and Gómez, Martínez, Peris and Rodenas [166].

Section 4.3. This section draws heavily on the work of Bernal and Montes [64], who also coined the term "run-away sequence", and the work of Shapiro [281]. The material up to Theorem 4.32 can be found in [64], while most of Theorem 4.37 is implicit in [281]. We mention that Seidel and Walsh [278] were the first to study the analogue of Birkhoff's result in the unit disk.

For two different proofs of Proposition 4.31 we refer to Bernal and Montes [64] and to Grosse-Erdmann and Mortini [184]. Example 4.34 is taken from Kim and Krantz [214]; see also Gorkin, León and Mortini [168].

Section 4.4. The study of the dynamical properties of adjoint multipliers was initiated by Godefroy and Shapiro [165], who also obtained Theorem 4.42. Functions of the backward shift on the spaces ℓ^p and c_0 were studied by deLaubenfels and Emamirad [128], who also obtained Theorem 4.43. An interesting related investigation of functions of the backward shift on the Bergman space is due to Bourdon and Shapiro [96].

For a more detailed introduction to Hardy spaces we refer to Duren [136] and Rudin [270].

Section 4.5. In this section we closely follow the book of Shapiro [279]; see also Shapiro [281]. Proposition 4.45 is a special case of the Littlewood subordination principle; see Exercise 4.5.1. Theorem 4.48 is due to Bourdon and Shapiro [94], [95]. This result is only the beginning of a fascinating story on the interplay between operator theory and complex function theory. The extension of Theorem 4.48, first to non-automorphic linear fractional transformations and then to more arbitrary holomorphic self-maps of $\mathbb{D}$, can be found in the cited work of Bourdon and Shapiro. The proofs, however, require a much deeper understanding, for example, of the (nonlinear) dynamics of self-maps of $\mathbb{D}$.

Hosokawa [204] proved that, for any automorphism φ of $\mathbb{D}$, C_φ is chaotic whenever it is hypercyclic; see also Taniguchi [298]. Thus one can add chaos to the equivalent conditions in Theorem 4.48.

We note that Gallardo and Montes [158] have obtained a complete characterization of the cyclic, supercyclic and hypercyclic composition operators C_φ for linear fractional self-maps φ of $\mathbb{D}$ on any of the spaces $\mathcal{S}_\nu$, $\nu \in \mathbb{R}$ (see Exercises 4.5.7 and 4.5.8).

For a more detailed introduction to composition operators on weighted Hardy, Bergman and Dirichlet spaces we refer to Cowen and MacCluer [125].

Exercises. Exercise 4.1.2 is taken from Bès and Peris [71], Exercise 4.1.3 from Grosse-Erdmann [180]. For Exercises 4.1.4 and 4.1.5 we refer to Feldman [150], Exercise 4.1.7 states the condition in the form found originally by Salas [274]. But note that the weighted shifts considered by Feldman and Salas are forward shifts. For Exercise 4.2.1 we refer to Bernal [55] and Shapiro [280], for Exercise 4.2.2 to Shapiro [280], for Exercise 4.2.4 to Chan and Shapiro [106] and to Bernal and Bonilla [60], for Exercise 4.2.5 to Grosse-Erdmann [181]. Exercises 4.3.2 and 4.3.3 follow Montes [241] and Grosse-Erdmann and Mortini [184], while Exercise 4.3.4 is taken from Shapiro [280]. The material for Exercises 4.4.1–4.4.4 can be found in Godefroy and Shapiro [165], with Exercise 4.4.4(vii) being taken from Chan and Seceleanu [104]; for Exercises 4.4.6–4.4.8 we refer to deLaubenfels and Emamirad [128], for Exercise 4.4.9 to Martínez [228]. For Exercise 4.5.1 we have again followed Shapiro [279]; Exercises 4.5.4(c), 4.5.5 and 4.5.8 are taken from Gallardo and Montes [158], Exercise 4.5.6 from Zorboska [304] and Exercise 4.5.7 from Hurst [205].