Chapter 12
Linear dynamics in topological vector spaces

So far, we have been working with operators on Banach or Fréchet spaces. One of the main reasons was that we then had the Baire category theorem at our disposal, which is a basic tool in hypercyclicity.

We have made one exception. In Chapter 10, the left-multiplication operators that we needed were defined on the space $L(X)$ with the strong operator topology, which is not a Fréchet space unless X is finite dimensional. But even there, in the final analysis, we worked in a separable Fréchet space $\mathcal{K}$ of operators on X.

Dealing with more general spaces in which Baire category arguments cannot be applied makes life certainly more difficult for hypercyclicity; but there are several dynamical properties, like mixing or weak mixing, where the previous arguments extend, essentially unchanged, to arbitrary topological vector spaces. Also, several interesting and natural operators are defined on non-Fréchet topological vector spaces, which is a good motivation to study linear dynamics in a wider context. This is the purpose of this chapter.

12.1 Topological vector spaces

A *topological vector space* is a vector space X over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ endowed with a Hausdorff topology such that addition and scalar multiplication,

$$+ : X \times X \to X, \quad (x, y) \to x + y,$$
$$\cdot : \mathbb{K} \times X \to X, \quad (\lambda, x) \to \lambda x,$$

are continuous maps. We recall that a topology is Hausdorff if any two distinct points in the space have disjoint neighbourhoods.

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Many arguments in Banach and Fréchet spaces use the triangle inequality of the norm or the seminorms. In general topological vector spaces, such arguments are replaced by operations with 0-neighbourhoods.

A subset A of a vector space X is called *balanced* if $\lambda A \subset A$ whenever $\lambda \in \mathbb{K}$, $|\lambda| \leq 1$.

Proposition 12.1. *Let X be a topological vector space.*

(a) *A set U is a neighbourhood of a point $x \in X$ if and only if there is a 0-neighbourhood W such that*

$$x + W \subset U.$$

(b) *Let W be a 0-neighbourhood. For any $\lambda, \mu \in \mathbb{K}$ there is a 0-neighbourhood W_1 such that*

$$\lambda W_1 + \mu W_1 \subset W.$$

In particular, there is a 0-neighbourhood W_1 such that

$$W_1 + W_1 \subset W \quad \text{and} \quad W_1 - W_1 \subset W.$$

(c) *If W is a 0-neighbourhood and $M > 0$, then there is a 0-neighbourhood $W_1 \subset W$ such that $\lambda W_1 \subset W$ for every $\lambda \in \mathbb{K}$ with $|\lambda| \leq M$. In particular, every 0-neighbourhood contains a balanced 0-neighbourhood.*

Proof. Properties (a) and (b) are easy consequences of the continuity of the vector operations. For property (c), given W and M, by the continuity of scalar multiplication we can find $\varepsilon > 0$ and a 0-neighbourhood W_0 such that $\lambda W_0 \subset W$ for every $\lambda \in \mathbb{K}$ with $|\lambda| < \varepsilon$. Let $\delta = \varepsilon/(M+1)$, and consider $W_1 = \delta W_0$, which is a 0-neighbourhood since multiplication by a fixed nonzero scalar is a homeomorphism of X. If $|\lambda| \leq M$ then $\lambda W_1 = (\lambda\delta) W_0 \subset W$. As a consequence, $\bigcap_{|\lambda|\geq 1} \lambda W$ is a balanced 0-neighbourhood contained in W. □

Let us apply the proposition to obtain some basic facts.

Proposition 12.2. *Let X be a topological vector space.*

(a) *If A is an arbitrary subset of X and U an open set then $A+U$ is open.*

(b) *For any 0-neighbourhood W there is a 0-neighbourhood W_1 such that*

$$\overline{W_1} \subset W;$$

in particular, every 0-neighbourhood contains a closed 0-neighbourhood.

Proof. (a) Let $x = y + z$, $y \in A$, $z \in U$. Since U is open there is a 0-neighbourhood W such that $z + W \subset U$. Then $x + W = y + (z + W) \subset A + U$, so that x is an interior point of $A + U$. This proves the claim.

(b) By Proposition 12.1(b) there is a 0-neighbourhood W_1 such that $W_1 - W_1 \subset W$. Let $x \in \overline{W_1}$. Then $(x + W_1) \cap W_1 \neq \varnothing$, hence $x \in W_1 - W_1 \subset W$. □

In view of Propositions 12.1(c) and 12.2(b), the set of all closed and balanced 0-neighbourhoods of a topological vector space X is a base of 0-neighbourhoods in X, which will be denoted by $\mathcal{U}_0(X)$.

There are some classes of topological vector spaces that deserve special consideration.

To start with, any finite-dimensional topological vector space X is isomorphic to $\mathbb{K}^N$, for some $N \geq 0$, where $\mathbb{K}$ is the scalar field of X; see Exercise 12.1.2.

If a topological vector space X admits a countable base $(W_n)_n$ of 0-neighbourhoods, then there is a translation-invariant metric d on X generating the topology of X. If, moreover, (X, d) is complete, then X is called an *F-space*. Metrizable topological vector spaces are, thus, exactly the topological vector spaces admitting a countable base of 0-neighbourhoods, and the completion $\widehat{X}$ of a metrizable topological vector space is an F-space. See also the related discussion in Section 2.1.

A topological vector space X whose topology is defined by a family of seminorms is called a *locally convex space*; that is, X is locally convex if there is a family $(p_\alpha)_{\alpha\in\mathcal{A}}$ of seminorms on X such that a subset W of X is a 0-neighbourhood if and only if there are $\alpha_1, \ldots, \alpha_n \in \mathcal{A}$, $n \geq 1$, and $\varepsilon > 0$ such that

$$\{x \in X \ ; \ p_{\alpha_k}(x) < \varepsilon \ \text{ for } k = 1, \ldots, n\} \subset W.$$

Fréchet spaces are, precisely, the locally convex F-spaces.

A subset A of a vector space X is called *absolutely convex* if, for any $x_1, x_2 \in A$ and $\lambda_1, \lambda_2 \in \mathbb{K}$ with $|\lambda_1| + |\lambda_2| \leq 1$, the absolutely convex combination $\lambda_1 x_1 + \lambda_2 x_2$ belongs to A. An easy observation is that, if $A \subset X$ is absolutely convex, $x_k \in A$, $\lambda_k \in \mathbb{K}$, $k = 1, \ldots, n$, with $\sum_{k=1}^n |\lambda_k| \leq 1$, then

$$\sum_{k=1}^n \lambda_k x_k \in A.$$

Also, if p is a seminorm on X and $M \geq 0$, then the set $A = \{x \in X \ ; \ p(x) \leq M\}$ is absolutely convex. Conversely, if $A \subset X$ is an absolutely convex set, then the associated *gauge* of A, also called its *Minkowski functional*, is defined as

$$p_A(x) = \inf\{\lambda > 0 \ ; \ x \in \lambda A\}, \quad x \in \operatorname{span} A.$$

One can verify that p_A is a seminorm on $\operatorname{span} A$; see Exercise 12.1.5. Therefore, a topological vector space X is locally convex if, and only if, it has a base of 0-neighbourhoods $(W_\alpha)_{\alpha\in\mathcal{A}}$ consisting of absolutely convex sets.

Example 12.3. In Sections 8.3 and 10.2 we considered the space $L(X)$ of operators on a Fréchet space X, endowed with the strong operator topology (SOT). In this topology, a base of neighbourhoods of $T \in L(X)$ is given by

$$U_{x_1,\ldots,x_n}(T, \varepsilon) = \{S \in L(X) \ ; \ \|Tx_k - Sx_k\| < \varepsilon \ \text{ for } k = 1, \ldots, n\},$$

where $x_1, \ldots, x_n$, $n \geq 1$, is an arbitrary collection of linearly independent vectors of X, $\|\cdot\|$ is an F-norm defining the topology of X, and $\varepsilon > 0$. We immediately obtain that $L(X)$ with the strong operator topology is a locally convex space.

Example 12.4. Given $0 < p < 1$ and $a < b$, let

$$L^p[a,b] = \Big\{ f : [a,b] \to \mathbb{K} \; ; f \text{ is measurable and } \int_a^b |f(t)|^p \, dt < \infty \Big\}.$$

We set $W_n = \{ f \in L^p[a,b] \; ; \; \int_a^b |f(t)|^p \, dt < 1/n \}$, $n \in \mathbb{N}$. Then the sequence $(W_n)_n$ defines a base of 0-neighbourhoods and, by translation, a topology on $L^p[a,b]$ that makes it an F-space that is not locally convex; see Exercise 12.1.3. Therefore, $L^p[a,b]$ is not a Fréchet space.

Example 12.5. Let $(X_n)_n$ be an increasing sequence of Banach spaces such that each inclusion map $i_n : X_n \to X_{n+1}$, $n \geq 1$, is continuous. We consider $X = \bigcup_{n=1}^\infty X_n$. For each sequence $\delta = (\delta_n)_n$ of strictly positive numbers, let

$$W_\delta = \bigcup_{n=1}^\infty \sum_{k=1}^n \delta_k B_k,$$

where B_k is the open unit ball of X_k, $k \in \mathbb{N}$. The family of absolutely convex sets $\{W_\delta \; ; \; \delta = (\delta_n)_n \in \,]0, \infty[^{\mathbb{N}}\}$ forms a base of 0-neighbourhoods for a locally convex topology on X, called the *inductive limit* of $(X_n)_n$; see Exercise 12.1.7.

12.2 Hypercyclicity, topological transitivity, and linear chaos

We are now in a position to study linear dynamics in its widest possible framework, that of operators on arbitrary topological vector spaces.

In the sequel we will not always define a notion when its generalization from the Fréchet space setting is evident. Still, we cannot help stating the following.

Definition 12.6. An operator T on a topological vector space X is called *hypercyclic* if there is some $x \in X$ whose orbit under T is dense in X. In such a case, x is called a *hypercyclic vector* for T. The set of hypercyclic vectors is denoted by $HC(T)$.

Clearly, separability of a space is again a necessary condition for the existence of a hypercyclic operator. Moreover, any finite-dimensional topological vector space is isomorphic to some $\mathbb{K}^N$ and therefore cannot support a hypercyclic operator. But unlike for the case of Fréchet spaces there are infinite-dimensional separable topological vector spaces that do not admit any hypercyclic operator.

Example 12.7. We consider the space φ of finite sequences,

$$\varphi = \{(x_n)_n \in \mathbb{K}^{\mathbb{N}} \ ; \text{ there is some } m \in \mathbb{N} \text{ such that } x_n = 0 \text{ for all } n > m\}.$$

The space φ has a natural locally convex topology, which is the strongest one that can be defined on it; it is generated by the family of norms

$$\|x\|_v = \sum_{n=1}^{\infty} |x_n| v_n, \quad x \in \varphi,$$

where $v = (v_n)_n$ is an arbitrary sequence of strictly positive numbers. Now let $x \in \varphi$ be a hypercyclic vector for an operator T on φ. We set $E_n = \{x \in \varphi \ ; \ x_k = 0 \text{ for } k > n\}$, $n \in \mathbb{N}$. Suppose that each E_n contains only a finite number of elements of the orbit of x. We define $F_1 = \operatorname{orb}(x, T) \cap (E_1 \setminus \{0\})$ and $F_n = \operatorname{orb}(x, T) \cap (E_n \setminus E_{n-1})$, $n > 1$. Then each F_n is finite, $\operatorname{orb}(x, T) = \bigcup_{n=1}^{\infty} F_n$, and every element $y \in F_n$ satisfies $y_n \neq 0$. We can therefore define

$$v_n = \frac{1}{\min\{|y_n| \ ; \ y \in F_n\}}, \quad n \in \mathbb{N},$$

if F_n is nonempty, and $v_n = 1$ otherwise. Considering the sequence $v = (v_n)_n$ we then find that $\|y\|_v \geq 1$ for every $y \in \operatorname{orb}(x, T)$, which contradicts the hypercyclicity of x. Therefore, some E_n, $n \geq 1$, must contain an infinite number of elements of $\operatorname{orb}(x, T)$, which is impossible because E_n is finite dimensional and the vectors in a dense orbit are linearly independent; note that Proposition 2.60 continues to hold. Consequently, φ admits no hypercyclic operator.

By the Birkhoff transitivity theorem, an operator on a separable Fréchet space is hypercyclic if and only if it is topologically transitive. One implication remains true since no topological vector space has isolated points.

Observation 12.8. *Any hypercyclic operator on a topological vector space is topologically transitive.*

But the converse is no longer true, as the following example shows.

Example 12.9. We consider again the space $X = \varphi$ of finite sequences, but this time endowed with the topology inherited from ℓ^2. Consider the multiple of the backward shift operator $T = 2B : X \to X$, $B(x_n)_n = 2(x_{n+1})_n$. Then T is topologically transitive, and even mixing, because Rolewicz's operator is. On the other hand, T cannot be hypercyclic since the orbit of any vector in X is finite.

Example 12.10. For a separable Banach space X, let us consider the space $L(X)$ of operators on X, endowed with the strong operator topology. In Theorem 10.20 we proved that an operator T on X satisfies the Hypercyclicity Criterion if, and only if, the left-multiplication operator $L_T : L(X) \to L(X)$, $S \to TS$, is hypercyclic. This characterization provides a good collection of hypercyclic operators on the non-metrizable locally convex space $L(X)$.

Since hypercyclicity and topological transitivity no longer coincide, we adopt Devaney's original definition of chaos in the general setting.

Definition 12.11 (Linear chaos). An operator T on a topological vector space X is said to be *chaotic* if it satisfies the following conditions:
(i) T is topologically transitive;
(ii) T has a dense set of periodic points.

We recall the useful result, Proposition 2.33, that the set $\mathrm{Per}(T)$ of periodic points for an operator T on a complex space X is given by

$$\mathrm{Per}(T) = \mathrm{span}\{x \in X\ ;\ Tx = e^{\alpha\pi i}x \text{ for some } \alpha \in \mathbb{Q}\},$$

whose density can often be checked easily for a concrete operator T.

Example 12.12. Let T be a chaotic operator on a separable Banach space X. Then there is a countable dense set $E \subset X$ such that each element of E is a periodic point for T. In the proof of Proposition 10.14 we showed that, if Φ is a countable weak-$*$-dense subset of X^*, then the countable set

$$\mathcal{F} = \mathcal{F}_{\Phi,E} = \Big\{ \sum_{j=1}^{m} \langle\,\cdot\,, y_j^*\rangle e_j\ ;\ y_j^* \in \Phi, e_j \in E, 1 \le j \le m \Big\}$$

of operators on X is SOT-dense in $L(X)$. We have that every element of $\mathcal{F}$ is periodic for the left-multiplication operator L_T on $L(X)$. On the other hand, hypercyclicity, and therefore topological transitivity, of L_T follows from Theorem 10.20 and the fact that chaotic operators satisfy the Hypercyclicity Criterion; see Theorem 3.18. We thus conclude that L_T is chaotic.

We next want to show that many fundamental results for hypercyclic operators on Fréchet spaces extend to arbitrary topological vector spaces. For this we need the notion of a quotient space.

Let X be a vector space and $L \subset X$ a subspace. Defining $x \sim y$ if $x - y \in L$, we obtain an equivalence relation on X. Let us denote by $[x] = x + L$ the equivalence class of $x \in X$, and by X/L the set of equivalence classes. Then X/L inherits in a natural way a vector space structure, and we denote by $q : X \to X/L$, $x \to [x]$, the *quotient map*, which is linear and surjective.

If, now, X is a topological vector space and $L \subset X$ is a closed subspace, then X/L becomes a topological vector space, called the *quotient space* of X modulo L, when endowed with the induced topology:

$U \subset X/L$ is open if and only if there is an open set $\widetilde{U} \subset X$ with $q(\widetilde{U}) = U$.

The quotient map q is then a continuous and open map. The requirement that L be closed is necessary for the Hausdorff property of X/L; see Exercise 12.2.5.

The following result, which generalizes Bourdon's theorem, is the key to the announced extensions.

Lemma 12.13 (Wengenroth). *Let T be an operator on topological vector space X. If either*
(i) *T is topologically transitive, or*
(ii) *T has a somewhere dense orbit,*
then, for any nonzero polynomial p, the operator $p(T)$ has dense range.

Proof. We will only show the complex case. The real case can be deduced in a similar way after some minor considerations; see Exercise 12.2.6.

As in the proof of Bourdon's theorem it suffices to show that $T - \lambda I$ has dense range for every $\lambda \in \mathbb{C}$. Let $L = \overline{(T - \lambda I)(X)}$, which is a closed subspace of X, and suppose that $L \neq X$. We then consider the quotient space X/L, which is nontrivial, and the quotient map $q : X \to X/L$. Since, for any $x \in X$, $q((T - \lambda I)x) = 0$ we have that $q(Tx) = \lambda q(x)$. Hence the operator S on X/L given by $S[x] = \lambda[x]$ is quasiconjugate to T via q and therefore inherits the stated properties from T; see the following section.

Under assumption (i), S is topologically transitive. On the other hand, let $[x] \in X/L$, $[x] \neq 0$. By the Hausdorff property there is an open neighbourhood U of $[x]$ and a balanced 0-neighbourhood W such that $U \cap W = \varnothing$. Now, if $|\lambda| \geq 1$, then $W \subset \lambda^n W$, and therefore $S^n(U) \cap W = \varnothing$ for all $n \in \mathbb{N}_0$. And if $|\lambda| < 1$, then $\lambda^n W \subset W$, and therefore $S^n(W) \cap U = \varnothing$ for all $n \in \mathbb{N}_0$. Thus, for any $\lambda \in \mathbb{C}$, S is not topologically transitive, a contradiction.

Under assumption (ii), S has a somewhere dense orbit $\{\lambda^n[x]\ ;\ n \in \mathbb{N}_0\}$. Then $\operatorname{span}\{[x]\} = \overline{\operatorname{span}}\{[x]\} = X/L$ (see Exercises 12.1.1(v) and 12.1.2), so that X/L is isomorphic to $\mathbb{C}$. But every orbit $\{\lambda^n z\ ;\ n \in \mathbb{N}_0\}$, $z \in \mathbb{C}$, is nowhere dense, a contradiction. □

Now, looking back at the proofs of the following fundamental results we see that they work unrestrictedly once one has Wengenroth's lemma at hand. They therefore hold for operators on all topological vector spaces.

Herrero–Bourdon theorem. *Any hypercyclic operator admits a dense invariant subspace consisting, except for zero, of hypercyclic vectors.*

Ansari's theorem. *Any power of a hypercyclic operator is hypercyclic.*

Costakis–Peris theorem. *Any multi-hypercyclic operator is hypercyclic.*

Bourdon–Feldman theorem. *Any somewhere dense orbit is (everywhere) dense.*

Indeed, each result holds in the more detailed form given in Chapters 2 and 6.

12.3 Dynamical transference principles

Working with general topological vector spaces instead of F-spaces sometimes requires some abstract considerations; for example, instead of sequences, balls, or the distance one needs to use notions like nets or neighbourhoods. In addition, of course, one has to do without the Baire category theorem. There do exist topological vector spaces beyond F-spaces in which Baire's theorem holds but they are rare.

In this section we want to discuss three techniques that allow us to transfer dynamical properties from operators on F-spaces to operators on general topological vector spaces.

The first technique is by now well known, that of quasiconjugacies. As before, if X and Y are topological vector spaces then an operator T on X is called *quasiconjugate* to an operator S on Y via a continuous map $\phi : Y \to X$ with dense range if $T \circ \phi = \phi \circ S$. Then the usual notions of linear dynamics are preserved under quasiconjugacy: hypercyclicity, topological transitivity, (weak) mixing, chaos, frequent hypercyclicity, etc. Moreover, if $y \in Y$ is a hypercyclic vector for S, then $x := \phi(y)$ is hypercyclic for T.

A particular case of quasiconjugacy that frequently occurs naturally is when one can find a T-invariant dense subspace $Y \subset X$ that carries its own, not necessarily the induced, vector space topology such that the restriction $T|_Y$ is an operator Y. If, in addition, the embedding $Y \to X$ is continuous, then T is quasiconjugate to $T|_Y$, so that T inherits dynamical properties from $T|_Y$. This is commonly known as the *hypercyclic comparison principle*; see Exercise 2.2.6.

Now, if Y is, in particular, an F-space, then the results of the previous chapters can be applied. We illustrate this by an example.

Example 12.14. Let $(X_n)_n$ be an increasing sequence of Banach spaces with continuous inclusions, and let X be the inductive limit of $(X_n)_n$; see Example 12.5. Suppose that T is an operator on X such that, for some $n \geq 1$, X_n is dense in X, $T(X_n) \subset X_n$ and $T|_{X_n}$ is continuous and hypercyclic. Then, by the comparison principle, T is hypercyclic.

As a particular case, let $1 < p \leq \infty$, and consider the space $\ell^{p-} := \bigcup_{q<p} \ell^q$. Obviously, $\ell^{p-} = \bigcup_{n=1}^{\infty} \ell^{p_n}$ for any strictly increasing sequence $(p_n)_n$ in $]1, p[$ tending to p. A natural topology on ℓ^{p-} is the corresponding inductive limit topology. If $\lambda \in \mathbb{K}$ is any scalar with $|\lambda| > 1$, then the multiple $T = \lambda B$ of the backward shift satisfies the above requirements and is therefore hypercyclic on ℓ^{p-}.

The second method is a kind of converse to the first technique. Of course, if, for a given operator T, all operators S that are quasiconjugate to T are hypercyclic then T itself must be hypercyclic; one may simply take $S = T$. It is, however, remarkable that the result remains true when we only admit operators S defined on F-spaces. In addition, the map ϕ defining the quasiconjugacy may be required to be linear.

Proposition 12.15. *Let T be an operator on a topological vector space X, and $x \in X$.*

(a) *If every operator S, defined on an F-space and quasiconjugate to T via a linear map, is hypercyclic (topologically transitive, weakly mixing, or mixing), then the same holds for T.*

(b) *If for any operator S, defined on an F-space and quasiconjugate to T via a linear map ϕ, $\phi(x)$ is (frequently) hypercyclic for S, then x is (frequently) hypercyclic for T.*

(c) *If X is a locally convex space then it suffices in* (a) *and* (b) *to allow operators S on Fréchet spaces.*

Proof. We will only show assertion (a) for the mixing property; the remaining cases follow similarly.

Let $U, V \subset X$ be arbitrary nonempty open subsets. Let $x_1 \in U$, $x_2 \in V$, and choose $W \in \mathcal{U}_0(X)$ such that $x_1 + W \subset U$ and $x_2 + W \subset V$. By continuity, we obtain a decreasing sequence $(W_n)_n$ of closed balanced 0-neighbourhoods such that $W_1 = W$, $W_{n+1} + W_{n+1} \subset W_n$, and $T(W_{n+1}) \subset W_n$, $n \in \mathbb{N}$. Let $L := \bigcap_{n=1}^{\infty} W_n$, which is easily seen to be a closed T-invariant subspace of X. We set $Y' = X/L$, and endow it with the topology τ generated by the family of neighbourhoods $\{y' + \widetilde{W}_n \ ; \ y' \in Y', \ n \in \mathbb{N}\}$, where $\widetilde{W}_n$ is the image of W_n under the quotient map $q : X \to Y'$, $n \in \mathbb{N}$. It is routine to verify that (Y', τ) is a topological vector space, which is metrizable since it has a countable base of 0-neighbourhoods, and that the operator T induces an operator $S' : Y' \to Y'$ that is quasiconjugate to T via q.

Now let Y be the completion of (Y', τ), which is an F-space, $S : Y \to Y$ the extension of S' to the completion, and $\phi : X \to Y$ the operator induced by q, which has dense range. It is clear that S is quasiconjugate to T via the linear map ϕ. It then follows from the assumption that S is a mixing operator. Therefore, also S' is mixing, so that there is some $N \in \mathbb{N}_0$ such that

$$q(T^n(x_1 + W_2)) \cap q(x_2 + W_2) = (S')^n \left(q(x_1) + \widetilde{W}_2\right) \cap \left(q(x_2) + \widetilde{W}_2\right) \neq \varnothing$$

for every $n \geq N$. This implies that

$$T^n(U) \cap V \supset T^n(x_1 + W) \cap (x_2 + W) \supset T^n(x_1 + W_2) \cap (x_2 + W_2 + L) \neq \varnothing$$

for every $n \geq N$, so that T is mixing.

In the case that X is a locally convex space, the 0-neighbourhoods W_n, $n \in \mathbb{N}$, can be chosen to be absolutely convex, and Y is a Fréchet space. □

An application of this result yields the generalization of the León–Müller theorem to arbitrary complex topological vector spaces.

Corollary 12.16. *Let T be an operator on a complex topological vector space X. Then, for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, T and λT have the same hypercyclic vectors, that is, $HC(T) = HC(\lambda T)$.*

Proof. Let $x \in HC(T)$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. By Proposition 12.15 it suffices to show that every operator S, defined on an arbitrary F-space Y and quasiconjugate to λT via a linear map $\phi : X \to Y$, has $\phi(x)$ as a hypercyclic vector. But under these assumptions, $\lambda^{-1}S$ is quasiconjugate to T via ϕ, so that $\phi(x)$ is a hypercyclic vector for $\lambda^{-1}S$. Since Y is an F-space, we can apply Theorem 6.7; note that its proof also works in arbitrary F-spaces. We then obtain that $\phi(x)$ is hypercyclic for S, as demanded. □

As another application of Proposition 12.15 we show that the space φ of finite sequences supports a mixing operator, which is surprising in view of the fact that φ does not admit any hypercyclic operator.

Example 12.17. We claim that the operator $T = I + B$ on φ is mixing, where B is the backward shift. Thus, let $S : Y \to Y$ be an operator on an arbitrary Fréchet space Y that is quasiconjugate to T via a linear map $\phi : \varphi \to Y$. We fix an increasing sequence of seminorms $(p_n)_n$ on Y generating its topology; by the continuity of ϕ there are strictly positive sequences $v(n) = (v_{n,k})_k$, $n \in \mathbb{N}$, such that $p_n(\phi(x)) \leq \|x\|_{v(n)}$, for any $n \in \mathbb{N}$ and $x \in \varphi$; see Example 12.7. Defining $v = (v_k)_k$ by $v_k = \max_{n,m\leq k} v_{n,m}$, $k \in \mathbb{N}$, a simple calculation shows that there are constants $M_n > 0$ such that $p_n(\phi(x)) \leq M_n\|x\|_v$ for any $n \in \mathbb{N}$, $x \in \varphi$. Since Y is complete, there is a continuous extension $\phi : \ell^1(v) \to Y$, and S is quasiconjugate to $I + B : \ell^1(v) \to \ell^1(v)$ via ϕ. Since, by Theorem 8.2, $I + B$ is mixing on $\ell^1(v)$, so is S on Y. Proposition 12.15 then implies that T is mixing on φ.

We turn to the third transference principle. For this we need a new concept. A *projective spectrum* $\mathcal{X}$ of Fréchet spaces consists of a family $(X_\alpha)_{\alpha\in I}$ of Fréchet spaces, where I is a directed index set, and operators $\varrho^\alpha_\beta : X_\beta \to X_\alpha$ for $\alpha \leq \beta$, called the *spectral maps*, that satisfy $\varrho^\alpha_\beta \circ \varrho^\beta_\gamma = \varrho^\alpha_\gamma$ and $\varrho^\alpha_\alpha = I_{X_\alpha}$, the identity on X_α, for any $\alpha \leq \beta \leq \gamma$. The *projective limit* of $\mathcal{X}$ is defined as

$$\operatorname{proj} \mathcal{X} = \Big\{(x_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} X_\alpha \ ; \ \varrho^\alpha_\beta x_\beta = x_\alpha \text{ for all } \alpha \leq \beta\Big\},$$

endowed with the topology inherited from the product topology on $\prod_{\alpha\in I} X_\alpha$; in this way, $\operatorname{proj} \mathcal{X}$ is a locally convex space. We denote by $\varrho^\alpha : \operatorname{proj} \mathcal{X} \to X_\alpha$ the projection onto the component with index α. It is not difficult to see that the sets $(\varrho^\alpha)^{-1}(W_\alpha)$, $W_\alpha \in \mathcal{U}_0(X_\alpha)$, $\alpha \in I$, form a base of 0-neighbourhoods for the topology of $\operatorname{proj} \mathcal{X}$. We say that $\mathcal{X}$ is *strongly reduced* if for each α there is a larger β such that $\varrho^\alpha_\beta(X_\beta)$ is contained in the closure of $\varrho^\alpha(\operatorname{proj} \mathcal{X})$ in X_α.

Now, a family $(T_\alpha)_{\alpha\in I}$ of operators T_α on X_α is called an *endomorphism* of $\mathcal{X}$ if their elements commute with the spectral maps in the sense that, for any $\alpha \leq \beta$, $T_\alpha \circ \varrho^\alpha_\beta = \varrho^\alpha_\beta \circ T_\beta$. The *projective limit* of the endomorphism is the operator T on $\operatorname{proj} \mathcal{X}$ defined by $T(x_\alpha)_{\alpha\in I} = (T_\alpha x_\alpha)_{\alpha\in I}$.

Proposition 12.18. *Let $\mathcal{X}$ be a strongly reduced projective spectrum of Fréchet spaces, $(T_\alpha)_{\alpha\in I}$ an endomorphism of $\mathcal{X}$, and T its projective limit.*

(a) *If every T_α, $\alpha \in I$, is topologically transitive (mixing, weakly mixing) on X_α then T is topologically transitive (mixing, weakly mixing) on* $\operatorname{proj}\mathcal{X}$.

(b) *If $x \in \operatorname{proj}\mathcal{X}$ is such that, for every $\alpha \in I$, $\varrho^\alpha x \in X_\alpha$ is (frequently) hypercyclic for T_α then x is (frequently) hypercyclic for T.*

Proof. (a) We will only show the result for the mixing property. Let $x, y \in X := \operatorname{proj}\mathcal{X}$ and $W_0 \in \mathcal{U}_0(X)$ be given. Then there are $\alpha \in I$ and $W_1 \in \mathcal{U}_0(X_\alpha)$ with $W_0 \supset (\varrho^\alpha)^{-1}(W_1)$, and there is some $\beta \geq \alpha$ with $\varrho^\alpha_\beta X_\beta \subset \overline{\varrho^\alpha(X)}$. For each $W \in \mathcal{U}_0(X_\alpha)$ we obtain that $\varrho^\alpha_\beta(X_\beta) \subset \varrho^\alpha(X) + W$, and thus $X_\beta \subset \varrho^\beta(X) + (\varrho^\alpha_\beta)^{-1}(W)$. This means that the image of ϱ^β is dense in X_β with respect to the vector space topology τ having $\{(\varrho^\alpha_\beta)^{-1}(W) \ ; \ W \in \mathcal{U}_0(X_\alpha)\}$ as a base of 0-neighbourhoods. Moreover, T_β is continuous on (X_β, τ) since $T_\beta^{-1}((\varrho^\alpha_\beta)^{-1}(W)) = (\varrho^\alpha_\beta)^{-1}(T_\alpha^{-1}(W)) \in \tau$ for every $W \in \mathcal{U}_0(X_\alpha)$, and T_β is mixing on (X_β, τ) since τ is coarser than the original topology on X_β.

Hence there is some $N \in \mathbb{N}_0$ such that

$$U_n := (\varrho^\beta x + (\varrho^\alpha_\beta)^{-1}(W_1)) \cap (T_\beta^{-n}(\varrho^\beta y + (\varrho^\alpha_\beta)^{-1}(W_1))) \neq \varnothing$$

for all $n \geq N$. Since U_n is open with respect to τ, there are $z_n \in X$ such that $\varrho^\beta z_n \in U_n$ for all $n \geq N$. Then we have that $\varrho^\alpha(z_n - x) = \varrho^\alpha_\beta(\varrho^\beta z_n - \varrho^\beta x) \in W_1$, that is, $z_n \in x + W_0$, and

$$\varrho^\alpha(T^n z_n) = \varrho^\alpha_\beta(\varrho^\beta(T^n z_n)) = \varrho^\alpha_\beta(T^n_\beta(\varrho^\beta z_n)) \in \varrho^\alpha y + W_1,$$

which gives that $T^n z_n \in y + W_0$ for every $n \geq N$. This proves that

$$(x + W_0) \cap (T^{-n}(y + W_0)) \neq \varnothing,$$

for each $n \geq N$, and therefore that T is mixing on X.

(b) Now let $x \in X$ be such that, for every $\alpha \in I$, $\varrho^\alpha x \in X_\alpha$ is hypercyclic for T_α. Let $y \in X$ and $W_0 \in \mathcal{U}_0(X)$ be given. Then there are $\alpha \in I$ and $W_1 \in \mathcal{U}_0(X_\alpha)$ with $W_0 \supset (\varrho^\alpha)^{-1}(W_1)$. It follows that there is some $n \in \mathbb{N}_0$ such that $\varrho^\alpha T^n x = T^n_\alpha \varrho^\alpha x \in \varrho^\alpha y + W_1$, hence $T^n x - y \in (\varrho^\alpha)^{-1}(W_1) \subset W_0$. This shows that x is hypercyclic for T. The same argument also shows the claim for frequent hypercyclicity. □

We single out a particular case of this result. Let $(X_n)_n$ be a decreasing sequence of Fréchet spaces such that each inclusion map $i_n : X_{n+1} \to X_n$, $n \geq 1$, is continuous. For any $n \geq 1$, let $(p_{n,k})_k$ be an increasing sequence of seminorms defining the topology of X_n. We then consider the space $X := \bigcap_{n=1}^\infty X_n$ with the locally convex topology induced by the seminorms $p_k(x) := \max_{n\leq k} p_{n,k}(x)$, $x \in X$. Then X is a Fréchet space, also called the *projective limit* of $(X_n)_n$.

Taking the inclusion maps as spectral maps, we see that $(X_n)_n$ is also a projective spectrum, and $\operatorname{proj} \mathcal{X}$ is the space of constant sequences with entries from $X = \bigcap_{n=1}^\infty X_n$. It is then clear that $\operatorname{proj} \mathcal{X}$ is isomorphic to X. Moreover, the projective spectrum is strongly reduced if, for example, X is dense in each X_n, $n \geq 1$.

Corollary 12.19. *Let $(X_n)_n$ be a decreasing sequence of Fréchet spaces with continuous inclusion maps such that $X = \bigcap_{n=1}^\infty X_n$ is dense in X_n for all $n \geq 1$. Let $T : X \to X$ be an operator that can be extended to an operator $T_n : X_n \to X_n$ for any $n \geq 1$.*

(a) *If every T_n, $n \geq 1$, is topologically transitive (mixing, weakly mixing) on X_n then T is topologically transitive (mixing, weakly mixing) on X.*

(b) *If $x \in X$ is (frequently) hypercyclic for every T_n, $n \geq 1$, then x is (frequently) hypercyclic for T.*

Proof. This follows directly from Proposition 12.18 because, by the assumptions, $(T_n)_n$ is an endomorphism of $\operatorname{proj} \mathcal{X}$ and the projective limit of $(T_n)_n$ on $\operatorname{proj} \mathcal{X}$ turns into T via the identification of $\operatorname{proj} \mathcal{X}$ with X. □

We note that, by quasiconjugacy, the conditions in the corollary are also necessary.

Example 12.20. Let $X = L^{\infty-}[0,1] := \bigcap_{p<\infty} L^p[0,1]$ be endowed with the Fréchet space topology induced by the increasing sequence of norms $(p_n)_n$, where p_n is the norm of $L^n[0,1]$, $n \in \mathbb{N}$. Let $C : L^{\infty-}[0,1] \to L^{\infty-}[0,1]$ be the Cesàro operator given by $Cf(t) = \frac{1}{t} \int_0^t f(s)\, ds$. By Exercise 3.1.4, C is mixing on $L^p[0,1]$ for any $1 < p < \infty$. Hence C is mixing on $L^{\infty-}[0,1]$.

12.4 Mixing and weakly mixing operators

In this section we will convince ourselves that the central results of Sections 2.4 and 2.5 remain true in general topological vector spaces when we replace the assumption of hypercyclicity there by topological transitivity. We will omit the proofs when the arguments given in those sections translate directly to the general situation.

Proposition 12.21. *An operator T on a topological vector space X is mixing if and only if, for any nonempty open set $U \subset X$ and any 0-neighbourhood W, the sets*

$$N(U,W) \text{ and } N(W,U)$$

are cofinite.

Example 12.22. We pointed out in Example 12.17 that there are mixing operators on φ. Another easy example of a mixing operator is $T = \lambda B$, $|\lambda| > 1$, on the inductive limit ℓ^{p-} for $1 < p \leq \infty$; see Example 12.14. Also, if X is an arbitrary topological vector space, then the product space $X^{\mathbb{N}}$, endowed with the product topology, is a topological vector space, and the backward shift $B : X^{\mathbb{N}} \to X^{\mathbb{N}}$, $(x_1, x_2, \dots) \to (x_2, x_3, \dots)$ is a mixing operator. Thus, if I is an infinite set then the space X^I, endowed with the product topology, is isomorphic to $(X^I)^{\mathbb{N}}$ and therefore admits a mixing operator.

We turn to the weak mixing property. As before, under mild additional assumptions, topologically transitive operators turn out to be weakly mixing. We start with a useful auxiliary result.

Lemma 12.23. *Let T be a topologically transitive operator on a topological vector space X. Then, for any nonempty open sets U and V in X and for any 0-neighbourhood W, there is a nonempty open set $U_1 \subset U$ and a 0-neighbourhood $W_1 \subset W$ such that*

$$N(U_1, W_1) \subset N(V, W) \quad \textit{and} \quad N(W_1, U_1) \subset N(W, V).$$

From this we can deduce the main result of this section.

Theorem 12.24. *Let T be a topologically transitive operator on a topological vector space X. If, for any nonempty open set $U \subset X$ and any 0-neighbourhood W, there is a continuous map $S : X \to X$ commuting with T such that*

$$S(U) \cap W \neq \varnothing \textit{ and } S(W) \cap U \neq \varnothing,$$

then T is weakly mixing.

Recall that an operator T is flip transitive if, for any pair $U, V \subset X$ of nonempty open sets, $N(U, V) \cap N(V, U) \neq \varnothing$. Thus we have in particular:

Corollary 12.25. *Every flip transitive operator is weakly mixing.*

Theorem 12.24 also implies that Theorem 2.47 extends to general topological vector spaces.

Theorem 12.26. *An operator T on a topological vector space X is weakly mixing if and only if, for any nonempty open sets $U, V \subset X$ and any 0-neighbourhood W,*

$$N(U, W) \cap N(W, V) \neq \varnothing.$$

Another application of Theorem 12.24 provides us with a useful sufficient condition for a topologically transitive operator to be weakly mixing; see Exercise 12.2.4 for the notion of a bounded set.

Theorem 12.27. *Let T be a topologically transitive operator on a topological vector space X. If there exists a dense subset X_0 of X such that the orbit of each $x \in X_0$ is bounded, then T is weakly mixing.*

Proof. In order to adapt the proof of Theorem 2.48 one need only note that, for any $x \in X_0$ and any 0-neighbourhood W, there is some $\varepsilon > 0$ such that $\varepsilon T^n x \in W$ for all $n \in \mathbb{N}_0$. □

Recall that the generalized kernel of an operator T is given by $\bigcup_{n=0}^{\infty} \ker T^n$.

Corollary 12.28. *Let T be a topologically transitive operator on a topological vector space X. If one of the following conditions is satisfied:*
(i) *T is chaotic;*
(ii) *T has a dense set of points for which the orbits converge;*
(iii) *T has dense generalized kernel;*
then T is weakly mixing.

As a final application of Theorem 12.24 we will characterize weakly mixing operators by the behaviour of multiples of iterates of T. For the notion of topological transitivity for sequences of operators we refer to Section 1.6.

Theorem 12.29. *Let T be an operator on a topological vector space X and $\lambda, \mu \in \mathbb{K} \setminus \{0\}$ with $\lambda \neq \mu$. Then the following assertions are equivalent:*
(i) *T is weakly mixing;*
(ii) *for any $M > \delta > 0$ and for any $(\lambda_n)_n$ with $\delta \leq |\lambda_n| \leq M$, $n \in \mathbb{N}_0$, the sequence $(\lambda_n T^n)_n$ is topologically transitive;*
(iii) *for any $(\lambda_n)_n$ with $\{\lambda_n \ ; \ n \in \mathbb{N}_0\} \subset \{\lambda, \mu\}$, the sequence $(\lambda_n T^n)_n$ is topologically transitive.*

Proof. (i) $\Longrightarrow$ (ii). Given $(\lambda_n)_n$ with $\delta \leq |\lambda_n| \leq M$, $n \in \mathbb{N}_0$, we let $U, V \subset X$ be nonempty open sets. By Exercise 12.1.1 there are nonempty open sets U_1 and V_1 and a 0-neighbourhood W such that $U_1 + W \subset U$ and $V_1 + W \subset V$. By Proposition 12.1, if $L = \max(\frac{1}{\delta}, M)$, then there is a 0-neighbourhood W_1 such that $\lambda W_1 \subset W$, for any $\lambda \in \mathbb{K}$ with $|\lambda| \leq L$. Now, since T is weakly mixing, there are $n \in \mathbb{N}_0$, $u \in U_1$ and $w \in W_1$ such that $T^n u \in W_1$ and $T^n w \in V_1$. Thus $u + \lambda_n^{-1} w \in U_1 + W \subset U$ and $\lambda_n T^n(u + \lambda_n^{-1} w) = \lambda_n T^n u + T^n w \in W + V_1 \subset V$. Hence $(\lambda_n T^n)_n$ is topologically transitive.

(ii) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (i). By taking $(\lambda_n)_n$ to be a constant sequence, T is easily seen to be topologically transitive. It therefore suffices to verify the hypothesis of Theorem 12.24. Thus let W be a 0-neighbourhood and $U \subset X$ a nonempty open set. Let $x \in U$. Using the properties of Proposition 12.1 we can find some $M > 0$ and an open neighbourhood $U_1 \subset U$ of x such that

$$U_1 \subset \frac{M(\lambda - \mu)}{\lambda} W, \quad U_1 - U_1 \subset M^{-1} W, \ \text{ and } \frac{\lambda}{\lambda - \mu} U_1 - \frac{\mu}{\lambda - \mu} U_1 \subset U.$$

Let $\alpha = M(\lambda - \mu)$. The hypothesis implies that there is some $n \in \mathbb{N}_0$ such that $\lambda T^n(U_1) \cap \alpha U_1 \neq \varnothing$ and $\mu T^n(U_1) \cap \alpha U_1 \neq \varnothing$; otherwise there would exist a sequence $(\lambda_n)_n$ with entries λ or μ such that $\lambda_n T^n(U_1) \cap \alpha U_1 = \varnothing$ for all $n \in \mathbb{N}_0$. Thus there are $u_1, u_2 \in U_1$ with $T^n(\alpha^{-1}\lambda u_1) \in U_1$ and $T^n(\alpha^{-1}\mu u_2) \in U_1$. Then

$$\alpha^{-1}\lambda u_1 \in \frac{\lambda}{M(\lambda-\mu)}U_1 \subset W, \quad T^n(\alpha^{-1}\lambda u_1) \in U,$$

and

$$M\alpha^{-1}(\lambda u_1 - \mu u_2) \in U, \quad T^n(M\alpha^{-1}(\lambda u_1 - \mu u_2)) \in M(U_1 - U_1) \subset W.$$

We then conclude with Theorem 12.24. □

12.5 Criteria for weak mixing, mixing and chaos

We extend here the criteria of Chapter 3 to general topological vector spaces. Since the arguments given there can be adapted directly to the general situation we omit the proofs.

Following the same order, we start with the criterion based on a large supply of eigenvectors.

Theorem 12.30 (Godefroy–Shapiro criterion). *Let T be an operator on a topological vector space X. Suppose that the subspaces*

$$X_0 := \operatorname{span}\{x \in X \;;\; Tx = \lambda x \textit{ for some } \lambda \in \mathbb{K} \textit{ with } |\lambda| < 1\},$$

$$Y_0 := \operatorname{span}\{x \in X \;;\; Tx = \lambda x \textit{ for some } \lambda \in \mathbb{K} \textit{ with } |\lambda| > 1\}$$

are dense in X. Then T is mixing.

If, moreover, X is a complex space and also the subspace

$$Z_0 := \operatorname{span}\{x \in X \;;\; Tx = e^{\alpha\pi i}x \textit{ for some } \alpha \in \mathbb{Q}\}$$

is dense in X, then T is chaotic.

Kitai's Criterion for mixing extends likewise.

Theorem 12.31 (Kitai's criterion). *Let T be an operator on a topological vector space X. If there are dense subsets $X_0, Y_0 \subset X$ and a map $S : Y_0 \to Y_0$ such that, for any $x \in X_0$, $y \in Y_0$,*
(i) $T^n x \to 0$,
(ii) $S^n y \to 0$,
(iii) $TSy = y$,
then T is mixing.

Example 12.32. Let $X = L^p[0,1]$, $0 < p < 1$, be the space of p-integrable functions on $[0,1]$; see Example 12.4. Let $\varphi : [0,1] \to [0,1]$ be the invertible function given by $\varphi(t) = t/2$ if $t \in [0,1/2]$, and $\varphi(t) = (3/2)t - 1/2$ if $t \in]1/2,1]$. We then consider the composition operator $C_\varphi : X \to X$, $C_\varphi f = f \circ \varphi$. The set

$$X_0 = Y_0 = \{f \in X \ ; \ f \text{ is continuous and } f(0) = f(1) = 0\},$$

is dense in X, and for $S : Y_0 \to Y_0$ we choose the composition operator $S = C_{\varphi^{-1}}$, so that $TSf = f$ for all $f \in Y_0$. Moreover, it is easy to check that, if $t \in [0,1[$, then $\lim_{n\to\infty} \varphi^n(t) = 0$, which implies that $C_\varphi^n f \to 0$ for every $f \in X_0$ by the dominated convergence theorem. Analogously, $\lim_{n\to\infty}(\varphi^{-1})^n(t) = 1$ for all $t \in]0,1]$, so that $C_{\varphi^{-1}}^n f \to 0$ for every $f \in Y_0$. An application of Kitai's criterion shows that T is mixing.

Finally, the Hypercyclicity Criterion turns out to be a weak mixing criterion within the general framework.

Theorem 12.33. *Let T be an operator on a topological vector space X. If there are dense subsets $X_0, Y_0 \subset X$, an increasing sequence $(n_k)_k$ of positive integers, and maps $S_{n_k} : Y_0 \to X$, $k \geq 1$, such that, for any $x \in X_0$, $y \in Y_0$,*
(i) $T^{n_k} x \to 0$,
(ii) $S_{n_k} y \to 0$,
(iii) $T^{n_k} S_{n_k} y \to y$,
then T is weakly mixing.

By Example 12.9 then, an operator satisfying this criterion need not be hypercyclic. With this realization that the Hypercyclicity Criterion is a misnomer we conclude the book.

Exercises

Exercise 12.1.1. Let X be a topological vector space. Prove the following assertions:
(i) if $x \in X$ and W is a 0-neighbourhood, then there exists some $M > 0$ and a neighbourhood U of x such that $U \subset MW$;
(ii) if U is a nonempty open set, then there is a 0-neighbourhood W and a nonempty open set $U_1 \subset U$ such that $U_1 + W \subset U$;
(iii) if U is a nonempty open set, $\lambda, \mu \in \mathbb{K}$ with $\lambda + \mu \neq 0$, and $x \in U$, then there is a neighbourhood $U_1 \subset U$ of x such that $\lambda U_1 + \mu U_1 \subset (\lambda + \mu)U$;
(iv) for any $\lambda \in \mathbb{K} \setminus \{0\}$ and $y \in X$, the operators $M_\lambda : X \to X$, $x \to \lambda x$, and $T_y : X \to X$, $x \to x + y$, are homeomorphisms;
(v) if $A \subset X$ is somewhere dense in X then $\operatorname{span} A$ is dense in X.

Exercise 12.1.2. Show that every finite-dimensional topological vector space X over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is isomorphic to $\mathbb{K}^N$, where N is the dimension of X. Deduce that finite-dimensional subspaces of topological vector spaces are closed. Here, as usual, an isomorphism between two topological vector spaces is, by definition, a linear homeomorphism.

Exercise 12.1.3. Given $0 < p < 1$ and $a < b$, show that the vector space

$$X = L^p[a,b] = \left\{ f : [a,b] \to \mathbb{K} \ ; f \text{ is measurable and } \int_a^b |f(t)|^p \, dt < \infty \right\}$$

is an F-space if we endow it with the base of neighbourhoods $(g + W_n)_n$, $g \in X$, where $W_n = \{f \in X \ ; \ \int_a^b |f(t)|^p \, dt < 1/n\}$, $n \in \mathbb{N}$. Prove that X is not a Fréchet space.

Exercise 12.1.4. (a) Let X be a vector space. Show that a subset A of X is absolutely convex if and only if it is convex and balanced.

(b) Show that a topological vector space is locally convex if, and only if, it has a base of 0-neighbourhoods consisting of convex sets.

Exercise 12.1.5. Given a vector space X, prove that, if $A \subset X$ is an absolutely convex set, the associated gauge of A,

$$p_A(x) = \inf\{\lambda > 0 \ ; \ x \in \lambda A\}, \quad x \in \operatorname{span} A,$$

is a seminorm on $\operatorname{span} A$.

Exercise 12.1.6. If X is an infinite-dimensional Fréchet space, then show that $L(X)$ endowed with the strong operator topology is not metrizable.

Exercise 12.1.7. Let $(X_n)_n$ be an increasing sequence of Banach spaces such that each inclusion map $i_n : X_n \to X_{n+1}$, $n \geq 1$, is continuous, and set $X = \bigcup_{n=1}^\infty X_n$. Show that the inductive limit topology defined in Example 12.5 is a locally convex topology on X. Prove that it is not metrizable unless there is some $m \in \mathbb{N}$ such that $X_n = X_m$ for all $n \geq m$.

Exercise 12.2.1. The *weak topology* on a Banach space X is the locally convex topology on X defined by the seminorms $x \to |\langle x, x^* \rangle|$, $x^* \in X^*$; that is, it is the topology of pointwise convergence on X^*. An operator T on a Banach space X is called *weakly hypercyclic* if it is hypercyclic on X endowed with the weak topology.

Let $X = \ell^p$, $1 \leq p < \infty$, or $X = c_0$. Show that a weighted backward shift B_w on X is hypercyclic if and only if it is weakly hypercyclic.

Exercise 12.2.2. Construct a chaotic and mixing operator that is not hypercyclic. (*Hint*: Enlarge the space in Example 12.9.)

Exercise 12.2.3. An operator T on a locally convex space X is called *compact* if there exists some $W \in \mathcal{U}_0(X)$ such that $\overline{T(W)}$ is compact. Show that no compact operator on a locally convex space is hypercyclic, and that no compact perturbation of a multiple of the identity is chaotic. (*Hint*: If $W \in \mathcal{U}_0(X)$ is absolutely convex and p_W is the gauge of W, then p_W induces a norm on $X/\ker p_W$. The completion of this normed space is called the *local Banach space* X_W. If $W \in \mathcal{U}_0(X)$ is absolutely convex such that $\overline{T(W)}$ is compact, then consider the operator T_W on X_W induced by T.)

Exercise 12.2.4. A subset B of a topological vector space X is called *bounded* if, for any $W \in \mathcal{U}_0(X)$, there is some $M > 0$ such that $B \subset MW$. If, in addition, B is absolutely convex then $X_B := \operatorname{span} B$ is a normed space when endowed with the gauge of B.

An operator T on X is called *bounded* if there is some $U \in \mathcal{U}_0(X)$ such that $T(U)$ is a bounded subset of X. Show that a bounded operator T with dense range is hypercyclic (or mixing, weakly mixing or chaotic) if and only if there is a bounded absolutely convex set $B \subset X$ such that X_B is a T-invariant dense subspace of X and the induced operator $T_B : X_B \to X_B$ is hypercyclic (or mixing, weakly mixing or chaotic, respectively).

Moreover, if T is a bounded operator and $\lambda \in \mathbb{K}$, show that there is some $M > 0$ such that $M(\lambda I + T)$ is not hypercyclic.

Exercise 12.2.5. Let X be a topological vector space and L a subspace of X. Show that X/L is Hausdorff if and only if L is closed. (*Hint*: Use some properties of Exercise 12.1.1.)

Exercise 12.2.6. Let X be a topological vector space over $\mathbb{R}$, T an operator on X that is topologically transitive or has a somewhere dense orbit, and p a nonzero polynomial over $\mathbb{R}$. Prove that $p(T)$ has dense range. (*Hint*: Define, as for Fréchet spaces, the *complexifications* $\widetilde{X}$ of X and $\widetilde{T}$ of T. As in the proof of Theorem 2.54 it suffices to show that $\widetilde{T}-\lambda I$ has dense range for all $\lambda \in \mathbb{C}$. *First case:* proceed as in the proof of Theorem 2.54, taking into account that $\widetilde{T}$ has the property that, for any $U, V \subset X$ open and nonempty, there is $n \in \mathbb{N}$ with $\widetilde{T}^n(U + iU) \cap (V + iV) \neq \varnothing$. *Second case:* there is $x \in X$ such that $\{T^n x + iT^m x \; ; \; n, m \geq 0\}$ is somewhere dense in $\widetilde{X}$. Applying the quotient map q, deduce that $\widetilde{X}$ can be identified with $\mathbb{C}$. By continuity and openness of $|q| : X \to \mathbb{R}_+$, $\{|q(T^n x)| \; ; \; n \geq 0\}$ is somewhere dense in $\mathbb{R}_+$, but $|q(T^n x)| = |q(\widetilde{T}^n x)| = |\lambda|^n |q(x)|$.)

Exercise 12.3.1. Given a compact subset $K \subset \mathbb{C}$, a *holomorphic germ* on K is a function that is defined and holomorphic on an open set U containing K. Let $H(K)$ be the space of holomorphic germs on K, and let $A(K)$ be the Banach space of continuous functions on K that are holomorphic on the interior of K, endowed with the sup-norm.

If $(K_n)_n$ is a decreasing sequence of compact sets such that the interior of each K_n contains K and $\bigcap_{n=1}^{\infty} K_n = K$, then $H(K)$ can be viewed as the inductive limit of the increasing sequence $(A(K_n))_n$ of Banach spaces. Prove that the differentiation operator D is a well-defined operator on $H(K)$, and that it is hypercyclic and chaotic if K is connected with connected complement $\mathbb{C} \setminus K$.

Exercise 12.3.2. Consider the weighted Banach space of holomorphic functions on the unit disk,

$$Hv_n(\mathbb{D}) := \Big\{ f \in H(\mathbb{D}) \; ; \; \|f\| := \sup_{z\in\mathbb{D}} |f(z)| v_n(z) < \infty \Big\},$$

where $v_n(z) := (1 - |z|)^n$, $n \in \mathbb{N}$; the inclusions $Hv_n(\mathbb{D}) \hookrightarrow Hv_{n+1}(\mathbb{D})$, $n \geq 1$, are continuous. The *Korenblum space* $A^{-\infty}$ is defined as the inductive limit of $(Hv_n(\mathbb{D}))_n$. Show that the differentiation operator D is a well-defined operator on $A^{-\infty}$, and prove that any finite-order differential operator on $A^{-\infty}$ that is not a multiple of the identity is chaotic. In contrast, observe that $D(Hv_n(\mathbb{D})) \not\subset Hv_n(\mathbb{D})$ for any $n \in \mathbb{N}$, so that the argument in Example 12.14 cannot be applied.

Exercise 12.3.3. Let $w = (w_n)_n$ be a weight sequence and B_w the corresponding weighted backward shift. Show that $T := I + B_w$ is mixing on φ. (*Hint*: Show that T is quasiconjugate to $I + B$ on φ via a suitable diagonal operator.)

Exercise 12.3.4. Let X be a *topological sequence space*, that is, a topological vector space X such that $X \subset \omega = \mathbb{K}^{\mathbb{N}}$ with continuous inclusion. Suppose that φ is contained and dense in X. If the weighted backward shift B_w is a well-defined operator on X, prove that $T := I + B_w$ is mixing on X.

Exercise 12.3.5. For this exercise we will need *Young's inequality:* given any $x \in \ell^p(\mathbb{Z})$ and $y \in \ell^q(\mathbb{Z})$, $1 \leq p, q < 2$, the *convolution product*

$$x * y := \Big(\sum_{k\in\mathbb{Z}} x_k y_{n-k} \Big)_{n\in\mathbb{Z}},$$

exists and belongs to $\ell^r(\mathbb{Z})$, where $1/p + 1/q = 1/r + 1$. Moreover, $\|x * y\|_r \leq \|x\|_p \|y\|_q$. We consider the Fréchet space $\ell^{1+}(\mathbb{Z}) = \bigcap_{p>1} \ell^p(\mathbb{Z})$. The space ℓ^{1+} is defined similarly.

(a) Show that, for any $y \in \ell^{1+}(\mathbb{Z})$, the map $y * \cdot : \ell^{1+}(\mathbb{Z}) \to \ell^{1+}(\mathbb{Z})$ given by $x \to y * x$ defines an operator on $\ell^{1+}(\mathbb{Z})$. Deduce that, for any function $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ with $(\alpha_{n-1})_n \in \ell^{1+}$, $T = f(B) = \sum_{n=0}^{\infty} \alpha_n B^n$ defines an operator on ℓ^{1+}, where B is the (unweighted) backward shift.

(b) Let $\lambda \in \mathbb{K} \setminus \{0\}$, and suppose that $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ with $(\alpha_{n-1})_n \in \ell^{1+}$ is a nonconstant function such that there exist $z_1, z_2 \in \mathbb{K}$ with $|z_1|, |z_2| < 1$ such that $|\lambda f(z_1)| < 1$ and $|\lambda f(z_2)| > 1$. Show that $\lambda f(B)$ is hypercyclic on ℓ^{1+}. If $\mathbb{K} = \mathbb{C}$ and, moreover, there exists $z \in \mathbb{C}$ with $|z| < 1$ such that $|\lambda f(z)| = 1$, show that $\lambda f(B)$ is chaotic on ℓ^{1+}. As a consequence obtain that λT is chaotic on ℓ^{1+} for every $\lambda \neq 0$, where $T := \sum_{n=1}^{\infty} \frac{1}{n} B^n$. (*Hint*: If $|\lambda| < 1$, then $e_\lambda := (\lambda^n)_n$ is an eigenvector of $f(B)$.)

Observe that $T(\ell^p) \not\subset \ell^p$ for any $p \geq 1$, which shows that the above result cannot be transferred from the Banach spaces defining the projective spectrum of ℓ^{1+}.

Exercise 12.4.1. Let $(T_n)_n$ be a topologically transitive commuting sequence of operators on a topological vector space X such that, for any nonempty open set $U \subset X$ and any 0-neighbourhood W, there is a continuous map $S : X \to X$ commuting with all T_n, $n \in \mathbb{N}_0$, such that

$$S(U) \cap U \neq \varnothing \quad \text{and} \quad S(U) \cap W \neq \varnothing.$$

Show that $(T_n)_n$ is weakly mixing. (*Hint*: Given U, W, find $U_1 \subset U$ and $W_1 \subset W$ with $U_1 - U_1 \subset W$, $U_1 - W_1 \subset U$. Apply the hypothesis and the 4-set trick to get that $N(U_1, U_1) \cap N(U_1, W_1) \neq \varnothing$. Then deduce the result from the analogue of Theorem 12.24 for sequences of operators.)

Exercise 12.4.2. Inspired by Theorems 1.54 and 12.29, prove that an operator T on a topological vector space X is weakly mixing if and only if, for any $M > \delta > 0$, for any $(\lambda_n)_n$ with $\delta \leq |\lambda_n| \leq M$, $n \in \mathbb{N}_0$, and for any syndetic sequence $(n_k)_k$, the sequence $(\lambda_k T^{n_k})_k$ is topologically transitive.

Exercise 12.5.1. Let $\varphi : [0,1] \to [0,1]$ be a continuous, surjective, and strictly increasing function such that $\varphi(t) \neq t$ for all $t \in]0,1[$. Show that the composition operator C_φ is mixing on $L^p[0,1]$ for any $p > 0$.

Exercise 12.5.2. Let T be an operator on a separable Banach space X satisfying the Godefroy–Shapiro criterion (or the Hypercyclicity Criterion with respect to $(n_k)_k$). Prove that the left-multiplication operator L_T on $L(X)$, endowed with the strong operator topology, satisfies the hypotheses of Theorem 12.30 (or the hypotheses of Theorem 12.33 with respect to $(n_k)_k$, respectively).

Exercise 12.5.3. Let X be a Banach space with separable dual X^* and T an operator on X whose adjoint $T^* : X^* \to X^*$ satisfies the Hypercyclicity Criterion (or is chaotic). Show that the right-multiplication operator $R_T : L(X) \to L(X)$, $S \to ST$, is hypercyclic and weakly mixing (or is hypercyclic and chaotic, respectively) on $L(X)$, endowed with the strong operator topology; see also Exercise 10.2.7.

Sources and comments

Section 12.1. All the basic results of this section can be found in the books by Meise and Vogt [237] and Rudin [271]. For F-spaces we also refer to Kalton, Peck and Roberts [212].

Section 12.2. Dynamical properties of linear operators on topological vector spaces beyond F-spaces were apparently first studied by Ansari [10]. The fact that the space φ admits no hypercyclic operators was obtained by Bonet and Peris [85] and Grosse-Erdmann [179]. The definition of chaos in general topological vector spaces was proposed by Bonet [78], where one also finds Example 12.9. The crucial Lemma 12.13 is due to Wengenroth [301].

We want to mention an interesting result on *weakly topologically transitive operators*, that is, operators that are topologically transitive with respect to the weak topology; it is due to Desch and Schappacher [130] (for Banach spaces) and Shkarin [286].

Theorem 12.34. *An operator T on a complex locally convex space is weakly topologically transitive if and only if T^* has no eigenvalues.*

Section 12.3. Proposition 12.15 is new, while Corollary 12.16 is due to Shkarin [284] and Bayart and Matheron [44].

Example 12.17 and Proposition 12.18 are due to Bonet, Frerick, Peris and Wengenroth [81]. Since φ is thus a topological vector space without hypercyclic operators, but that admits a mixing, and therefore topologically transitive, operator, one might wonder if every infinite-dimensional topological vector space necessarily admits a topologically transitive operator. This is not the case, as Bermúdez and Kalton [52] have shown: there are (non-separable) Banach spaces, like ℓ^∞, or $L(\ell^2)$ with the operator norm, without any topologically transitive operators. Further existence and nonexistence results concerning hypercyclic or topologically transitive operators on locally convex spaces beyond Fréchet spaces are due to Bonet and Peris [85], Bonet, Frerick, Peris and Wengenroth [81], and Shkarin [286, 288, 293].

Example 12.20 solves a problem from León, Piqueras and Seoane [224].

Section 12.4. For the results of this section we refer to Grosse-Erdmann and Peris [187]. Similar investigations can be found in Bayart and Matheron [44], [45] and Moothathu [245].

Section 12.5. Example 12.32 is from Grosse-Erdmann [179].

Exercises. Exercise 12.2.1 is taken from Chan and Sanders [101], where the authors also show that there are weakly hypercyclic bilateral shifts that are not hypercyclic. Exercise 12.2.2 is taken from Bonet [78], Exercise 12.2.3 from Bonet and Peris [85] and Martínez and Peris [229]. The first part of Exercise 12.2.4 is from Bonet and Peris [85]. The second part is extracted from Bonet [80]; this paper studies the open problem of the existence of non-normable Fréchet spaces X such that every operator on X is of the form $\lambda I + T$ with T a bounded operator. It is also asked whether, for every infinite-dimensional separable non-normable Fréchet space X, there exists an operator T on X such that λT is hypercyclic for any $\lambda \neq 0$. Exercise 12.2.6 is taken from Wengenroth [301]. For Exercise 12.3.2 we refer to Bonet [78], for Exercises 12.3.3 and 12.3.4 to Bonet, Frerick, Peris and Wengenroth [81], and for Exercise 12.3.5 to Frerick and Peris [155]. Exercise 12.5.3 is taken from Bonet, Martínez and Peris [84].