Chapter 1
Topological dynamics

This chapter provides an introduction to the theory of (not necessarily linear) dynamical systems. Fundamental concepts such as topologically transitive, chaotic and (weakly) mixing maps are defined and illustrated with typical examples. The Birkhoff transitivity theorem is derived as a crucial tool for showing that a map has a dense orbit. Moreover we obtain several characterizations of weakly mixing maps that will be of great significance later on.

The interest in this chapter is that it derives results on dynamical systems that do not require linearity. From Chapter 2 onwards, all our systems will be linear.

1.1 Dynamical systems

The theory of dynamical systems studies the long-term behaviour of evolving systems.

As a motivating example we consider the size of a population, which we assume to be given by the value N_n at discrete times $n = 0, 1, 2, \ldots$. In a simple model the size at time $n + 1$ will only depend on the size at time n. The population is then described by a law

$$N_{n+1} = T(N_n), \quad n = 0, 1, 2, \ldots,$$

where T is a suitable map. It follows that

$$N_n = (T \circ \ldots \circ T)(N_0), \quad n = 1, 2, \ldots$$

with n applications of the map T. Thus the behaviour of the population is completely determined by the initial population N_0 and the map T.

More generally, we assume that the possible states of a (physical, biological, economic, ..., or abstract) system are described by the elements from a set X and that evolution of the system is described by a map $T : X \to X$;

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that is, if $x_n \in X$ is the state of the system at time $n \geq 0$, then

$$x_{n+1} = T(x_n), \quad n = 0, 1, 2, \ldots.$$

Since we want to measure changes in the values x_n, we require the underlying space to be a metric space. And since we want small changes in x_n only to result in small changes in x_{n+1} we require continuity of T.

Definition 1.1. A *(discrete) dynamical system* is a pair (X, T) consisting of a metric space X and a continuous map $T : X \to X$.

Often we will simply call T (when the underlying space X is taken for granted) or $T : X \to X$ a dynamical system. Moreover we adopt the notation used in operator theory to write Tx for $T(x)$.

What we are interested in is the evolution of the system that starts with a certain state x_0. For this we define the *iterates* $T^n : X \to X$, $n \geq 0$, by the n-fold iteration of T,

$$T^n = T \circ \ldots \circ T \quad (n \text{ times})$$

with

$$T^0 = I,$$

the identity on X.

Definition 1.2. Let $T : X \to X$ be a dynamical system. For $x \in X$ we call

$$\operatorname{orb}(x, T) = \{x, Tx, T^2x, \ldots\}$$

the *orbit* of x under T.

Returning to the previous discussion, suppose that the size N_n of a population changes proportionally to its actual size, that is, it follows the law

$$\frac{N_{n+1} - N_n}{N_n} = \gamma, \quad n \geq 0,$$

with some constant $\gamma > -1$. One may write this equivalently as

$$N_{n+1} = (1 + \gamma) N_n,$$

so that the corresponding dynamical system is given by

$$T : \mathbb{R}_+ \to \mathbb{R}_+, \quad Tx = (1 + \gamma)x.$$

The orbit of $x \in \mathbb{R}_+$ can be calculated explicitly as

$$\operatorname{orb}(x, T) = \{(1 + \gamma)^n x \ ; \ n \geq 0\}.$$

Thus, the orbit tends to 0, x and ∞ for $-1 < \gamma < 0$, $\gamma = 0$ and $\gamma > 0$, respectively.

As a more realistic model for the evolution of a population the following has been suggested. If we assume that the environment limits the size of the population by a certain number $L > 0$ then we might assume the law to be

$$\frac{N_{n+1} - N_n}{N_n} = \gamma(L - N_n), \quad \gamma > 0.$$

Rescaling by $M_n = (L + \gamma^{-1})^{-1} N_n$ and setting $\mu = \gamma L + 1$ we obtain that

$$M_{n+1} = \mu M_n(1 - M_n), \quad n \geq 0.$$

This refined model leads to the following dynamical system.

Example 1.3. **(Logistic map)** Let $\mu \in \mathbb{R}$. The *logistic map* $L_\mu : \mathbb{R} \to \mathbb{R}$ is given by $L_\mu x = \mu x(1-x)$, $x \in \mathbb{R}$. Figure 1.1 shows the graph of L_μ for $\mu = 3$.

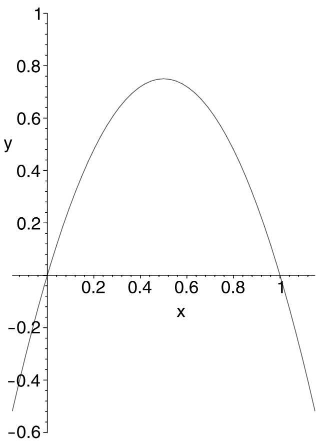

Fig. 1.1 The logistic map L_3

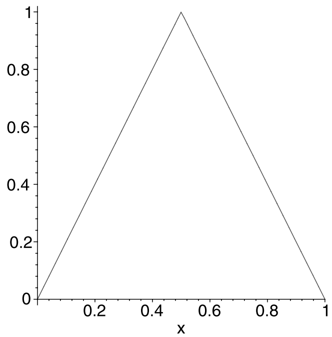

Fig. 1.2 The tent map T

We introduce several other popular dynamical systems.

Example 1.4. (a) **(Quadratic map)** A quadratic map is defined by the real dynamical system $Q_c : \mathbb{R} \to \mathbb{R}$, $x \to x^2 + c$, with a parameter $c \in \mathbb{R}$, or by the corresponding complex dynamical system $Q_c : \mathbb{C} \to \mathbb{C}$, $z \to z^2 + c$, with $c \in \mathbb{C}$.

(b) **(Doubling map on the circle)** Let $T : \mathbb{C} \to \mathbb{C}$ denote the square function $Tz = z^2$. Its iterates are $T^n z = z^{2^n}$. It follows that the orbits for points z with $|z| < 1$ tend to 0, while for $|z| > 1$ the orbits tend to infinity. As we will see later, the dynamics of T for points on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \,;\, |z| = 1\}$ are much more interesting. Since $T(\mathbb{T}) \subset \mathbb{T}$, we usually consider the dynamical system $T : \mathbb{T} \to \mathbb{T}$, $z \to z^2$, the so-called *doubling map*. The name refers to the fact that T doubles the argument of the complex number z.

(c) **(Circle rotation)** The system $T : \mathbb{T} \to \mathbb{T}$, $z \to e^{i\alpha}z$, $\alpha \in [0, 2\pi[$, describes the rotation of the point z on the unit circle by the angle α. We will see that its dynamical behaviour depends to a large extent on the question of whether the rotation is *rational* ($\alpha \in \pi\mathbb{Q}$) or *irrational* ($\alpha \notin \pi\mathbb{Q}$).

(d) **(Tent map)** The tent map is given by $T : [0,1] \to [0,1]$, $Tx = 2x$, if $x \in [0, \frac{1}{2}]$, and $Tx = 2 - 2x$, if $x \in]\frac{1}{2}, 1]$. The name derives from the shape of its graph; see Figure 1.2.

(e) **(Doubling map on the interval)** We consider the interval $[0,1]$ in which we identify 0 and 1; the metric on this space is given by $d(x,y) = \min(|x-y|, 1-|x-y|)$. Then $T : [0,1] \to [0,1]$, $x \to 2x \pmod 1$ describes a dynamical system.

(f) **(Shift on the interval)** When we identify again 0 and 1, the map $T : [0,1] \to [0,1]$, $x \to x + \alpha \pmod 1$ with $\alpha \in [0,1[$ describes the shift by α, modulo 1, of any point on the unit interval.

Every mathematical theory has its notion of isomorphism. When do we want to consider two dynamical systems $S : Y \to Y$ and $T : X \to X$ as equal? There should be a homeomorphism $\phi : Y \to X$ such that, when $x \in X$ corresponds to $y \in Y$ via ϕ then Tx should correspond to Sy via ϕ. In other words, if $x = \phi(y)$ then $Tx = \phi(Sy)$. This is equivalent to saying that $T \circ \phi = \phi \circ S$.

We recall that a homeomorphism is a bijective continuous map whose inverse is also continuous. In many applications, however, it is already enough to demand that ϕ is continuous with dense range.

Definition 1.5. Let $S : Y \to Y$ and $T : X \to X$ be dynamical systems.

(a) Then T is called *quasiconjugate* to S if there exists a continuous map $\phi : Y \to X$ with dense range such that $T \circ \phi = \phi \circ S$, that is, the diagram

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ \phi \downarrow & & \downarrow \phi \\ X & \xrightarrow{T} & X \end{array}$$

commutes.

(b) If ϕ can be chosen to be a homeomorphism then S and T are called *conjugate*.

Conjugacy is clearly an equivalence relation between dynamical systems, and conjugate dynamical systems have the same dynamical behaviour. What makes this notion even more interesting is the fact that it is by no means always obvious if two systems are conjugate or not.

Example 1.6. We refer to the various dynamical systems introduced above.

(a) For any $\mu \neq 0, 2$, the logistic maps L_μ and $L_{2-\mu}$ are conjugate; one can take an affine function $x \to ax + b$ for ϕ, as is easily verified. It therefore suffices to study these maps for $\mu \geq 1$.

(b) Any logistic map L_μ, $\mu \neq 0$, is conjugate to a suitable real quadratic map Q_c; again, the conjugacy can be given by an affine function.

(c) The logistic map L_4, when restricted to the interval $[0,1]$, is conjugate to the tent map. In fact, an easy calculation shows that one may take $\phi(x) = \sin^2(\frac{\pi}{2}x)$.

(d) When we identify the points 0 and 1, the map $\phi : [0,1] \to \mathbb{T}$, $x \to e^{2\pi i x}$ is a homeomorphism. It clearly defines a conjugacy between the doubling map on $[0,1]$ and the doubling map on the circle; and it defines a conjugacy between the shift by α on $[0,1]$ and the circle rotation by the angle $2\pi\alpha$.

Definition 1.7. We say that a property $\mathcal{P}$ for dynamical systems is *preserved under (quasi)conjugacy* if the following holds: if a dynamical system $S : Y \to Y$ has property $\mathcal{P}$ then every dynamical system $T : X \to X$ that is (quasi) conjugate to S also has property $\mathcal{P}$.

For example, the property of having dense range is clearly preserved under quasiconjugacy, while the property of being surjective is preserved under conjugacy but not under quasiconjugacy.

1.2 Topologically transitive maps

One way of defining a new dynamical system from a given dynamical system T is by restricting it to a subset. However, one has to ensure that T maps this subset into itself.

Definition 1.8. Let $T : X \to X$ be a dynamical system. Then a subset $Y \subset X$ is called *T-invariant* or *invariant under T* if $T(Y) \subset Y$.

Thus, if $Y \subset X$ is T-invariant, then $T|_Y : Y \to Y$ is also a dynamical system.

Example 1.9. The interval $[0,1]$ is invariant under the logistic map L_μ for $0 \leq \mu \leq 4$.

The study of a mathematical object is often simplified by breaking it up into smaller parts and by studying these separately. If such a splitting is not possible then one usually says that the object is irreducible. In the case of dynamical systems, we might regard $T : X \to X$ as irreducible if X cannot be divided into two T-invariant subsets with nonempty interior. In that direction we have the following result.

Proposition 1.10. *Let $T : X \to X$ be a dynamical system. Then we have the implications* (i) $\Longleftarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v), *where*

(i) *X cannot be written as $X = A \cup B$ with disjoint T-invariant subsets A, B such that A and B have nonempty interior;*

(ii) *X cannot be written as $X = A \cup B$ with disjoint subsets A, B such that A is T-invariant and A and B have nonempty interior;*
(iii) *for any pair U, V of nonempty open subsets of X there exists some $n \geq 0$ such that $T^n(U) \cap V \neq \varnothing$;*
(iv) *for any nonempty open subset U of X the set $\bigcup_{n=0}^{\infty} T^n(U)$ is dense in X;*
(v) *for any nonempty open subset U of X the set $\bigcup_{n=0}^{\infty} T^{-n}(U)$ is dense in X.*

Proof. (ii)$\Longrightarrow$(i) is trivial.

(ii)$\Longrightarrow$(iv). Let $A = \bigcup_{n=0}^{\infty} T^n(U)$ and $B = X \setminus A$. Then A is T-invariant, and it has nonempty interior since it contains U. By (ii), B must have empty interior, which implies that A is dense.

(iii)$\Longrightarrow$(ii). Suppose that $X = A \cup B$, $A \cap B = \varnothing$ and $T(A) \subset A$. Then $\text{int}(A)$ and $\text{int}(B)$ are open sets with $T^n(\text{int}(A)) \cap \text{int}(B) \subset A \cap B = \varnothing$ for all $n \geq 0$. By (iii) this can only be the case if either A or B has empty interior.

We clearly have that (iii)$\Longleftrightarrow$(iv). For (iii)$\Longleftrightarrow$(v) one need only note that $T^n(U) \cap V \neq \varnothing$ is equivalent to $U \cap T^{-n}(V) \neq \varnothing$. □

We see that condition (iii) is slightly stronger than the irreducibility of a dynamical system; see also Exercise 1.2.1. Since this condition will turn out to be of fundamental importance for the theory it is given its own name.

Definition 1.11. A dynamical system $T : X \to X$ is called *topologically transitive* if, for any pair U, V of nonempty open subsets of X, there exists some $n \geq 0$ such that $T^n(U) \cap V \neq \varnothing$.

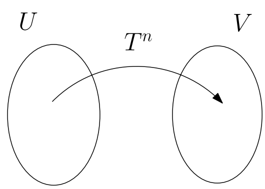

Fig. 1.3 Topological transitivity

Example 1.12. (a) The tent map is topologically transitive. To see this, note that T^n is the piecewise linear map with $T^n(\frac{2k}{2^n}) = 0$, $k = 0, 1, \ldots, 2^{n-1}$, and $T^n(\frac{2k-1}{2^n}) = 1$, $k = 1, \ldots, 2^{n-1}$; see Figure 1.4. Thus, let $U \subset [0,1]$ be nonempty and open. Then U contains some interval $J := [\frac{m}{2^n}, \frac{m+1}{2^n}]$. But since $[0,1] = T^n(J) \subset T^n(U)$, $T^n(U)$ in fact meets every nonempty set V.

(b) The doubling map on the circle, $T : \mathbb{T} \to \mathbb{T}$, $z \to z^2$, is also topologically transitive. In fact, every nonempty open set $U \subset \mathbb{T}$ contains a closed arc of angle $\frac{2\pi}{2^n}$, for some $n \geq 1$. Since the map T doubles angles, we have that $T^n(U)$ contains a closed arc of angle 2π, hence $T^n(U) = \mathbb{T}$ meets every nonempty set V.

(c) No rational rotation $T : \mathbb{T} \to \mathbb{T}$, $z \to e^{i\alpha}z$, is topologically transitive. For example, if $\alpha = \frac{2\pi}{n}$ then the iterates of an open arc γ of angle $\frac{\pi}{n}$ never meet the arc $e^{i\frac{\pi}{n}}\gamma$. In contrast, every irrational rotation is topologically transitive; see Exercise 1.2.9.

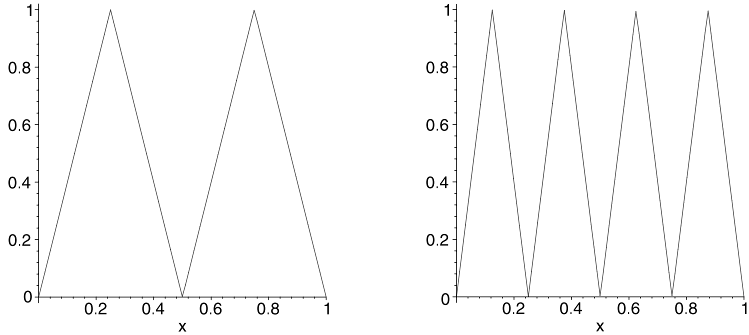

Fig. 1.4 The iterates T^2 and T^3 of the tent map

Proposition 1.13. *Topological transitivity is preserved under quasiconjugacy.*

Proof. Let $T : X \to X$ be quasiconjugate to $S : Y \to Y$ via $\phi : Y \to X$, and let U and V be nonempty open subsets of X. Since ϕ is continuous and of dense range, $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are nonempty and open. Thus there are $y \in \phi^{-1}(U)$ and $n \geq 0$ with $S^n y \in \phi^{-1}(V)$, which implies that $\phi(y) \in U$ and $T^n\phi(y) = \phi(S^n y) \in V$. $\square$

The equivalence of conditions (iv) and (v) in Proposition 1.10 implies the following.

Proposition 1.14. *Let $T : X \to X$ be a dynamical system with continuous inverse T^{-1}. Then T is topologically transitive if and only if T^{-1} is.*

Topological transitivity can be interpreted as saying that T connects all nontrivial parts of X. This is automatically the case whenever there is a point $x \in X$ with dense orbit under T.

Proposition 1.15. *Let T be a continuous map on a metric space X without isolated points.*

(a) *If $x \in X$ has dense orbit under T then so does each $T^n x$, $n \geq 1$.*

(b) *If T has a dense orbit then it is topologically transitive.*

Proof. (a) This follows easily from the fact that $\mathrm{orb}(x,T)\backslash\{x, Tx, \ldots, T^{n-1}x\}$ is contained in $\mathrm{orb}(T^n x, T)$ and that, in every metric space without isolated points, a dense set remains dense even after removing finitely many points.

(b) Suppose that $x \in X$ has dense orbit under T. Let U and V be nonempty open sets in X. Then there is some $n \geq 0$ such that $T^n x \in U$. By (a), also $T^n x$ has dense orbit, so that there is some $m \geq n$ such that $T^m x \in V$. This implies that $T^{m-n}(U) \cap V \neq \varnothing$. □

What is less obvious is that, in separable complete metric spaces, the converse of this result is also true: topologically transitive maps must have a dense orbit. The importance of this result for the theory of dynamical systems can hardly be overemphasized. It was first obtained in 1920 by G. D. Birkhoff in the context of maps on compact subsets of $\mathbb{R}^N$.

Theorem 1.16 (Birkhoff transitivity theorem). *Let T be a continuous map on a separable complete metric space X without isolated points. Then the following assertions are equivalent:*

(i) *T is topologically transitive;*

(ii) *there exists some $x \in X$ such that* $\mathrm{orb}(x,T)$ *is dense in X.*

If one of these conditions holds then the set of points in X with dense orbit is a dense G_δ-set.

Proof. By the previous proposition, (ii) implies (i). For the converse, let T be topologically transitive, and let $\mathcal{D}(T)$ denote the set of points in X that have dense orbit under T. Since X has a countable dense set $\{y_j\ ;\ j \geq 1\}$, the open balls of radius $\frac{1}{m}$ around the y_j, $m, j \geq 1$, form a countable base $(U_k)_{k\geq 1}$ of the topology of X. Hence, x belongs to $\mathcal{D}(T)$ if and only if, for every $k \geq 1$, there is some $n \geq 0$ such that $T^n x \in U_k$. In other words,

$$\mathcal{D}(T) = \bigcap_{k=1}^{\infty} \bigcup_{n=0}^{\infty} T^{-n}(U_k).$$

By continuity of T and Proposition 1.10, each set $\bigcup_{n=0}^{\infty} T^{-n}(U_k)$, $k \geq 0$, is open and dense. The Baire category theorem then implies that $\mathcal{D}(T)$ is a dense G_δ-set, and hence nonempty. □

We note that the absence of isolated points was not needed for the proof that (i) implies (ii).

Example 1.17. It follows from Example 1.12 that the tent map and the doubling map have dense orbits. Irrational rotations (that is, with $\alpha \notin \pi\mathbb{Q}$) have the stronger property that each of their orbits is dense; see Exercise 1.2.9.

Let us briefly reflect on the usefulness of the transitivity theorem. Suppose that we are interested in the existence of a dense orbit under a given map. Sometimes such a point presents itself with little effort, as is the case for irrational rotations. But what if this is not the case? Without further information we will never be likely to stumble on a point with dense orbit. In contrast, topological transitivity seems much easier to prove, as we have seen, for example, in the case of the tent map: we need only connect any two nonempty open sets by suitable iterates.

We stress, however, that in more general spaces topological transitivity and the existence of a dense orbit need not coincide.

Example 1.18. Let X be the set of all points on the unit circle that are 2^nth roots of unity, for some $n \geq 1$. By Example 1.12(b) the doubling map, restricted to X, is topologically transitive, but clearly has no dense orbits.

This example shows that the completeness assumption in the Birkhoff transitivity theorem cannot be dropped.

Proposition 1.19. *The property of having a dense orbit is preserved under quasiconjugacy.*

Proof. Let $T : X \to X$ be quasiconjugate to $S : Y \to Y$ via $\phi : Y \to X$, and let $y \in Y$ have dense orbit under S. If U is a nonempty open subset of X then $\phi^{-1}(U)$ is nonempty and open, so that some $S^n y$, $n \geq 0$, belongs to $\phi^{-1}(U)$. But then $T^n \phi(y) = \phi(S^n y)$ belongs to U. □

Example 1.20. By Examples 1.6(c) and 1.17, the logistic map L_4 on $[0, 1]$ has a dense orbit.

1.3 Chaos

What is chaos? Even when we restrict the meaning of this word to deterministic chaos, that is, chaotic behaviour of a dynamical system, mathematicians have come up with different answers to this question. We will follow here the definition that was suggested by Devaney in 1986. It has three ingredients, which we discuss in turn.

The first ingredient tries to capture the idea of the so-called butterfly effect: small changes in the initial state may lead, after some time, to large discrepancies in the orbit. In order to be able to perturb points we consider only spaces without isolated points.

Definition 1.21. Let (X, d) be a metric space without isolated points. Then a dynamical system $T : X \to X$ is said to have *sensitive dependence on initial conditions* if there exists some $\delta > 0$ such that, for every $x \in X$ and $\varepsilon > 0$, there exists some $y \in X$ with $d(x, y) < \varepsilon$ such that, for some $n \geq 0$, $d(T^n x, T^n y) > \delta$. The number δ is called a *sensitivity constant* for T.

We stress that the definition involves the metric of the space. In the following examples we will always work with the usual metric.

Example 1.22. (a) Using our knowledge of the iterates of the tent map (see Example 1.12), we can easily show that it has sensitive dependence on initial conditions with sensitivity constant 1/4, say. Indeed, if $x \in [0, 1]$ and $\varepsilon > 0$ then there is some $n \geq 0$ such that the open ball of radius ε around x contains points y_1 and y_2 with $T^n y_1 = 0$ and $T^n y_2 = 1$; thus $|T^n x - T^n y_j| \geq 1/2$ for some $j \in \{1, 2\}$.

(b) A similar argument, based on the fact that the doubling map doubles angles, shows that it has sensitive dependence on initial conditions.

(c) No circle rotation has sensitive dependence on initial conditions because we clearly have that $|T^n z_1 - T^n z_2| = |z_1 - z_2|$ for any $z_1, z_2 \in \mathbb{T}$.

The second ingredient of chaos demands that the system is irreducible in the sense that the map T connects any nontrivial parts of the space. We saw in Section 1.2 that this idea is well captured by the notion of topological transitivity of the system.

The third ingredient demands that the system has many orbits with a regular behaviour; more precisely, there should be a dense set of points with periodic orbit.

Definition 1.23. Let $T : X \to X$ be a dynamical system.

(a) A point $x \in X$ is called a *fixed point* of T if $Tx = x$.

(b) A point $x \in X$ is called a *periodic point* of T if there is some $n \geq 1$ such that $T^n x = x$. The least such number n is called the *period* of x. The set of periodic points is denoted by $\mathrm{Per}(T)$.

A point is periodic if and only if it is a fixed point of some iterate T^n, $n \geq 1$. Thus, for real functions T, one easily detects them by searching for the points where the graphs of T^n and the identity function meet.

Example 1.24. (a) Considering the iterates of the tent map (see Example 1.12), we find that in every interval $[\frac{m}{2^n}, \frac{m+1}{2^n}]$ there is a periodic point of period n. Thus, the tent map has a dense set of periodic points.

(b) The periodic points of the doubling map on the circle are exactly the $(2^n - 1)$st roots of unity, $n \geq 1$, so that also the doubling map has a dense set of periodic points.

(c) For any rational rotation T there is some $N \geq 1$ such that $T^N = I$, so that every point is periodic. In contrast, irrational rotations have no periodic points at all.

Proposition 1.25. *The property of having a dense set of periodic points is preserved under quasiconjugacy.*

Proof. Let $T : X \to X$ be quasiconjugate to $S : Y \to Y$ via $\phi : Y \to X$, and let $U \subset X$ be a nonempty open set. Then $\phi^{-1}(U)$, being also open and nonempty, contains a point y with $S^n y = y$ for some $n \geq 1$. Hence $\phi(y) \in U$ and $T^n \phi(y) = \phi(S^n y) = \phi(y)$. □

Summarizing, we are led to Devaney's definition of chaos.

Definition 1.26 (Devaney chaos – preliminary version). Let (X, d) be a metric space without isolated points. Then a dynamical system $T : X \to X$ is said to be *chaotic (in the sense of Devaney)* if it satisfies the following conditions:
(i) T has sensitive dependence on initial conditions;
(ii) T is topologically transitive;
(iii) T has a dense set of periodic points.

Example 1.27. By Examples 1.12, 1.22 and 1.24, the tent map and the doubling map are chaotic, but no circle rotation is chaotic.

The definition of chaos has a serious blemish: sensitive dependence on initial conditions is not preserved under conjugacy, or, which is the same, it depends on the metric on the underlying space. We illustrate this by an example.

Example 1.28. Let $T :]1,\infty[\to]1,\infty[$ be given by $Tx = 2x$. Since $|T^n x - T^n y| = 2^n|x - y| \to \infty$ whenever $x \neq y$, we have that T has sensitive dependence on initial conditions with respect to the usual metric on $]1,\infty[$. But if we define $d(x, y) = |\log x - \log y|$ then d is an equivalent metric for which $d(T^n x, T^n y) = d(x, y)$ for all $x, y \in]1,\infty[$, which shows that T does not have sensitive dependence on initial conditions with respect to d. On the other hand, the two versions of T are conjugate when we take the identity map as the linking homeomorphism.

Fortunately, one can drop sensitive dependence from Devaney's definition because it is implied by the other two conditions.

Theorem 1.29 (Banks–Brooks–Cairns–Davis–Stacey). *Let X be a metric space without isolated points. If a dynamical system $T : X \to X$ is topologically transitive and has a dense set of periodic points then T has sensitive dependence on initial conditions with respect to any metric defining the topology of X.*

Proof. We fix a metric d defining the topology of X. We first show that there exists some constant $\eta > 0$ such that, for any point $x \in X$ there is a periodic point p such that
$$d(x, T^n p) \geq \eta \quad \text{for all } n \in \mathbb{N}_0.$$
Indeed, since X has no isolated points it is an infinite set, so that we can find two periodic points p_1, p_2 whose orbits are disjoint. Hence,
$$\eta := \inf_{m,n\in\mathbb{N}_0} d(T^m p_1, T^n p_2)/2 > 0.$$
It then follows from the triangle inequality that, for any $x \in X$, either for $j = 1$ or for $j = 2$ we have that $d(x, T^n p_j) \geq \eta$ for all $n \in \mathbb{N}_0$.

We now claim that T has sensitive dependence on initial conditions with sensitivity constant $\delta := \eta/4 > 0$. To this end, let $x \in X$ and $\varepsilon > 0$. By assumption there is a periodic point q such that

$$d(x,q) < \min(\varepsilon,\delta). \tag{1.1}$$

Let q have period N. As we have seen above there is also a periodic point p such that

$$d(x,T^n p) \geq \eta = 4\delta \quad \text{for } n \in \mathbb{N}_0. \tag{1.2}$$

Since T is continuous there is some neighbourhood V of p such that

$$d(T^n p, T^n y) < \delta \quad \text{for } n = 0,1,\ldots,N \text{ and } y \in V. \tag{1.3}$$

Finally, by topological transitivity of T we can find a point z and some $k \in \mathbb{N}_0$ such that $d(x,z) < \varepsilon$ and $T^k z \in V$. Let $j \in \mathbb{N}_0$ be such that $k \leq jN < k+N$. The triangle inequality, together with (1.2), (1.3) and (1.1), then yields that

$$\begin{aligned} d\left(T^{jN}q, T^{jN}z\right) &= d\left(T^{jN}q, T^{jN-k}T^k z\right) = d\left(q, T^{jN-k}T^k z\right) \\ &\geq d\left(x, T^{jN-k}p\right) - d\left(T^{jN-k}p, T^{jN-k}T^k z\right) - d(x,q) \\ &> 4\delta - \delta - \delta = 2\delta. \end{aligned}$$

This implies that either $d(T^{jN}x, T^{jN}q) > \delta$ or $d(T^{jN}x, T^{jN}z) > \delta$. Since both z and q have a distance less than ε from x, the claim follows. □

This allows us to drop sensitive dependence from Devaney's definition of chaos; for simplicity we also extend it to all metric spaces.

Definition 1.30 (Devaney chaos). A dynamical system $T : X \to X$ is said to be *chaotic (in the sense of Devaney)* if it satisfies the following conditions:
(i) T is topologically transitive;
(ii) T has a dense set of periodic points.

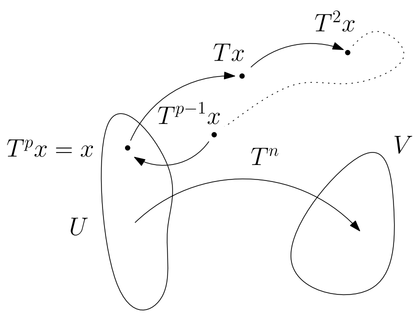

Fig. 1.5 Devaney chaos

By Theorem 1.29, this is consistent with Definition 1.26. Proposition 1.13 and 1.25 immediately show the following.

Proposition 1.31. *Devaney chaos is preserved under quasiconjugacy.*

As an application of this fact we obtain the following from Examples 1.6 and 1.27.

Example 1.32. The logistic map L_4 is chaotic on $[0, 1]$.

For the tent map and the doubling map we have shown explicitly that they satisfy the defining conditions of chaos. But their dynamics remain mysterious. To get a better feeling for the origin of chaos in a dynamical system we now want to discuss briefly a special chaotic dynamical system whose dynamical behaviour is quite transparent. On the space

$$\Sigma_2 = \{(x_n)_{n\in\mathbb{N}_0}\ ;\ x_n \in \{0, 1\}\}$$

of all 0-1-sequences we study the map

$$\sigma : \Sigma_2 \to \Sigma_2, \quad \sigma(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots).$$

As usual, sequences $(x_n)_n, (y_n)_n, \ldots$ will be denoted by $x, y, \ldots$. We define a topology on Σ_2 by the metric

$$d(x, y) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}.$$

Under this metric, σ is clearly a continuous map.

Definition 1.33. The dynamical system (Σ_2, σ) is called the *shift on two symbols.*

For topological considerations in Σ_2 the following easy result will be of constant use.

Lemma 1.34. *Let $x, y \in \Sigma_2$.*
(a) *If $x_j = y_j$ for $j = 0, 1, \ldots, n$ then $d(x, y) \leq \frac{1}{2^n}$.*
(b) *If $d(x, y) < \frac{1}{2^n}$ then $x_j = y_j$ for $j = 0, 1, \ldots, n$.*

In particular, a sequence of points in Σ_2 converges if and only if each coordinate converges. It also follows easily that Σ_2 is a compact metric space without isolated points; in fact it is homeomorphic to the Cantor set.

As promised, the dynamics of this system are completely transparent.

Proposition 1.35. (a) *A point $x \in \Sigma_2$ is periodic under σ if and only if the sequence $x = (x_n)_n$ is periodic.*

(b) *A point $x \in \Sigma_2$ has dense orbit under σ if and only if every finite* 0-1-*sequence appears as a block in x.*

Proof. (a) is obvious from the definition of the map σ.

For (b), suppose that x has dense orbit. Let $(y_0, \dots, y_m)$ be a finite 0-1-sequence and let $y = (y_0, \dots, y_m, 0, 0, \dots)$. Since there is some $n \geq 0$ such that $d(y, \sigma^n x) < \frac{1}{2^m}$, Lemma 1.34(b) implies that $(x_n, \dots, x_{n+m}) = (y_0, \dots, y_m)$. The converse follows similarly with Lemma 1.34(a). □

Theorem 1.36. *The shift on two symbols is chaotic.*

Proof. Since the set of all finite 0-1-sequences is countable, one can construct a 0-1-sequence that contains each finite 0-1-sequence as a block. Hence, by Proposition 1.35(b), σ has a dense orbit, which shows that it is topologically transitive because Σ_2 has no isolated points.

Now, let $x \in \Sigma_2$ and $m \in \mathbb{N}$. Then the sequence y defined by $y_{j(m+1)+k} = x_k$, $0 \leq k \leq m$, $j \geq 0$ is periodic, hence a periodic point for σ. Moreover, by Lemma 1.34, we have that $d(x, y) \leq \frac{1}{2^m}$. This shows that the periodic points are dense in Σ_2.

Altogether, σ is chaotic. □

It will be a recurrent theme in this book that shifts create chaos. And in many cases maps are chaotic precisely because there is an underlying shift. We illustrate this by the doubling map.

Example 1.37. The doubling map is given by $Tz = z^2$, $z \in \mathbb{T}$. If we write $z = \exp(2\pi i\alpha)$ with $0 \leq \alpha < 1$ then we may represent α in binary form as

$$\alpha = \sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}}, \quad x_n \in \{0, 1\}.$$

In this representation we have that T maps z into

$$z^2 = \exp(2\pi i 2\alpha) = \exp\Big(2\pi i x_0 + 2\pi i \sum_{n=1}^{\infty} \frac{x_n}{2^n}\Big) = \exp\Big(2\pi i \sum_{n=0}^{\infty} \frac{x_{n+1}}{2^{n+1}}\Big).$$

In other words, the doubling map acts as a shift on the binary representation of the argument of z. In this form the dynamics of the doubling map become much clearer. To put it formally, the map

$$\phi : \Sigma_2 \to \mathbb{T}, \quad (x_n)_n \to \exp\Big(2\pi i \sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}}\Big)$$

provides a quasiconjugacy from the shift on two symbols to the doubling map. This proves once more that the doubling map is chaotic.

1.4 Mixing maps

We return to the discussion of topologically transitive maps. As we saw in Example 1.12, the tent map and the doubling map have a strong form of transitivity: $T^n(U)$ intersects V not only for some n but for all sufficiently large $n \in \mathbb{N}_0$. This property carries a special name.

Definition 1.38. A dynamical system $T : X \to X$ is called *mixing* if, for any pair U, V of nonempty open subsets of X, there exists some $N \geq 0$ such that

$$T^n(U) \cap V \neq \varnothing \quad \text{for all } n \geq N.$$

Example 1.39. As noted above, both the tent map and the doubling map are mixing. An example of a topologically transitive system that is not mixing will be given in Example 1.43.

As in the case of topological transitivity one obtains the following.

Proposition 1.40. *The mixing property is preserved under quasiconjugacy.*

Mixing maps have a remarkable permanence property. In order to describe this we need to define products of maps.

Let $S : X \to X$ and $T : Y \to Y$ be dynamical systems. The Cartesian product $X \times Y$ is endowed with the product topology, which is induced by the metric $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$, where d_X and d_Y denote the metrics in X and Y, respectively. A base for the topology is formed by the products $U \times V$ of nonempty open sets $U \subset X$ and $V \subset Y$.

Definition 1.41. Let $S : X \to X$ and $T : Y \to Y$ be dynamical systems. Then the map $S \times T$ is defined by

$$S \times T : X \times Y \to X \times Y, \quad (S \times T)(x, y) = (Sx, Ty).$$

Then $S \times T$ is clearly continuous, and for the iterates we have that

$$(S \times T)^n = S^n \times T^n.$$

Products of more than two spaces or maps are defined similarly.

Proposition 1.42. *Let $S : X \to X$ and $T : Y \to Y$ be dynamical systems. Then we have the following:*

(i) *if $S \times T$ has a dense orbit then so do S and T;*
(ii) *if $S \times T$ is topologically transitive then so are S and T;*
(iii) *if $S \times T$ is chaotic then so are S and T;*
(iv) *if S and T are topologically transitive and at least one of them is mixing then $S \times T$ is topologically transitive;*
(v) *$S \times T$ is mixing if and only if both S and T are.*

Proof. Obviously, S and T are quasiconjugate to $S \times T$ under the maps $(x, y) \to x$ and $(x, y) \to y$, respectively. Thus the preservation results obtained so far imply (i), (ii) and (iii).

The assertions (iv) and (v) follow directly from the identity

$$(S \times T)^n(U_1 \times U_2) \cap (V_1 \times V_2) = (S^n(U_1) \cap V_1) \times (T^n(U_2) \cap V_2),$$

the fact that topological transitivity and the mixing property can be tested on a base of the topology, and, for (iv), the observation of Exercise 1.2.4. □

In general, however, the product of two topologically transitive maps need not be topologically transitive, even if $S = T$, as the following example shows; hence some extra condition, as in the previous proposition, is needed.

Example 1.43. Let $T : \mathbb{T} \to \mathbb{T}$, $z \to e^{i\alpha}z$ be a circle rotation. Then $(T \times T)^n(z_1, z_2) = (e^{in\alpha}z_1,\ e^{in\alpha}z_2)$, and we observe that the quotient of the two coordinates is z_1/z_2, independent of n. This is easily seen to imply that $T \times T$ cannot be topologically transitive. On the other hand we know that any irrational rotation is topologically transitive; see Exercise 1.2.9. This also shows that no irrational rotation is mixing.

1.5 Weakly mixing maps

Having looked at the question of when a product of two topologically transitive systems is topologically transitive, one may wonder when the product of a topologically transitive map with itself is again topologically transitive. We saw in Example 1.43 that this is not always the case. On the other hand, for any mixing map T the product $T \times T$ is topologically transitive. This leads us to the following notion.

Definition 1.44. A dynamical system $T : X \to X$ is called *weakly mixing* if $T \times T$ is topologically transitive.

Since the products $U \times V$ of nonempty open sets $U, V \subset X$ form a base of the topology of $X \times X$, T is weakly mixing if and only if, for any 4-tuple $U_1, U_2, V_1, V_2 \subset X$ of nonempty open sets, there exists some $n \geq 0$ such that

$$T^n(U_1) \cap V_1 \neq \varnothing \quad \text{and} \quad T^n(U_2) \cap V_2 \neq \varnothing.$$

Observation 1.45. *For any dynamical system,*

$$\textit{mixing} \implies \textit{weak mixing} \implies \textit{topological transitivity.}$$

Remark 1.46. By Example 1.43, any irrational circle rotation is topologically transitive but not weakly mixing. On the other hand, it is not an easy matter to construct examples of weakly mixing maps that are not mixing. However, in the context of linear operators that will be discussed in the following chapters, such examples will appear abundantly as consequences of our investigations there; see, for example, Remark 4.10.

Proposition 1.47. *The weak mixing property is preserved under quasiconjugacy.*

Proof. If $\phi : Y \to X$ defines a quasiconjugacy from $S : Y \to Y$ to $T : X \to X$, then $\phi \times \phi$ defines a quasiconjugacy from $S \times S$ to $T \times T$. Now the result follows from Proposition 1.13. □

As a consequence, we have the following as in Proposition 1.42.

Proposition 1.48. *Let $S : X \to X$ and $T : Y \to Y$ be dynamical systems. If $S \times T$ is weakly mixing then so are S and T.*

In order to formulate the arguments involving weakly mixing maps more succinctly we introduce the following useful concept.

Definition 1.49. Let $T : X \to X$ be a dynamical system. Then, for any sets $A, B \subset X$, the *return set from A to B* is defined as

$$N_T(A, B) = N(A, B) = \{n \in \mathbb{N}_0 \ ; \ T^n(A) \cap B \neq \varnothing\}.$$

We usually drop the index T when this causes no ambiguity. In this notation, T is topologically transitive (or mixing) if and only if, for any pair U, V of nonempty open subsets of X, the return set

$$N(U, V) \ \text{ is nonempty (or cofinite, respectively);}$$

and T is weakly mixing if and only if, for any 4-tuple $U_1, U_2, V_1, V_2 \subset X$ of nonempty open sets,

$$N(U_1, V_1) \cap N(U_2, V_2) \neq \varnothing.$$

Note also that if T is topologically transitive then the return sets $N(U, V)$ are even infinite for any nonempty open sets U, V; see Exercise 1.2.4.

Incidentally, we observe that the larger the sets A and B are, the larger also is the return set $N(A, B)$.

It is our aim to give several characterizations of the weak mixing property. To do this we will provide a useful lemma that, due to its form, we will call the *4-set trick.* We note that the return sets $N(A, B)$ refer to T.

Lemma 1.50 (4-set trick). *Let $T : X \to X$ be a dynamical system, and let $U_1, V_1, U_2, V_2 \subset X$ be nonempty open sets.*

(a) *If there is a continuous map $S : X \to X$ commuting with T such that*

$$S(U_1) \cap U_2 \neq \varnothing \quad \textit{and} \quad S(V_1) \cap V_2 \neq \varnothing,$$

then there exist nonempty open sets $U_1' \subset U_1$, $V_1' \subset V_1$ *such that*

$$N(U_1', V_1') \subset N(U_2, V_2) \quad \text{and} \quad N(V_1', U_1') \subset N(V_2, U_2).$$

If, moreover, T *is topologically transitive then* $N(U_1, V_1) \cap N(U_2, V_2) \neq \varnothing$.

(b) *If* T *is topologically transitive then*

$$N(U_1, U_2) \cap N(V_1, V_2) \neq \varnothing \Longrightarrow N(U_1, V_1) \cap N(U_2, V_2) \neq \varnothing.$$

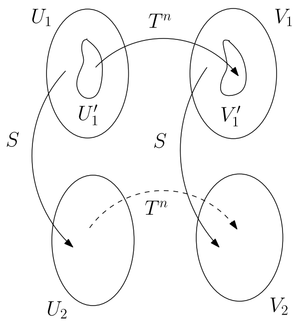

Fig. 1.6 The 4-set trick

Proof. (a) Since S is continuous, it follows from the hypothesis that we can find nonempty open sets $U_1' \subset U_1$ and $V_1' \subset V_1$ such that $S(U_1') \subset U_2$ and $S(V_1') \subset V_2$. If $n \in N(U_1', V_1')$, then there exists some $x \in U_1'$ with $T^n x \in V_1'$. Therefore $T^n Sx = ST^n x \in V_2$ and $Sx \in U_2$, which yields that $n \in N(U_2, V_2)$. By symmetry we also obtain that $N(V_1', U_1') \subset N(V_2, U_2)$. If, moreover, T is topologically transitive then

$$\varnothing \neq N(U_1', V_1') \subset N(U_1, V_1) \cap N(U_2, V_2).$$

(b) If T is topologically transitive and $n \in N(U_1, U_2) \cap N(V_1, V_2)$, then $N(U_1, V_1) \cap N(U_2, V_2) \neq \varnothing$ follows if (a) is applied to $S := T^n$. □

The 4-set trick, simple as it is, already implies an important result that is at first sight quite surprising: as soon as the product $T \times T$ is topologically transitive, every higher product $T \times \cdots \times T$ also is.

Theorem 1.51 (Furstenberg). *Let* $T : X \to X$ *be a weakly mixing dynamical system. Then the* n*-fold product* $T \times \cdots \times T$ *is weakly mixing for each* $n \geq 2$.

Proof. Since the n-fold product being weakly mixing amounts to the $2n$-fold product being topologically transitive, it suffices to show that every n-fold product $T \times \cdots \times T$ is topologically transitive for $n \geq 2$.

We proceed by induction, the case of $n = 2$ being trivial by definition. Thus, suppose that the n-fold product $T \times \ldots \times T$ is topologically transitive. To prove topological transitivity of the corresponding $(n+1)$-fold product we need to show that, given nonempty open sets $U_k, V_k \subset X$, $k = 1, \ldots, n+1$, we have that

$$\bigcap_{k=1}^{n+1} N(U_k, V_k) \neq \varnothing. \tag{1.4}$$

Indeed, since T is weakly mixing there is some $m \in \mathbb{N}_0$ such that $T^m(U_n) \cap U_{n+1} \neq \varnothing$ and $T^m(V_n) \cap V_{n+1} \neq \varnothing$. The 4-set trick then yields the existence of nonempty open sets $U_n' \subset U_n$, $V_n' \subset V_n$ with $N(U_n', V_n') \subset N(U_n, V_n) \cap N(U_{n+1}, V_{n+1})$. On the other hand, the induction hypothesis implies that

$$\bigcap_{k=1}^{n-1} N(U_k, V_k) \cap N(U_n', V_n') \neq \varnothing,$$

which implies (1.4). □

The next results show that in the definition of weak mixing one may reduce the four open sets to three, and then to two open sets.

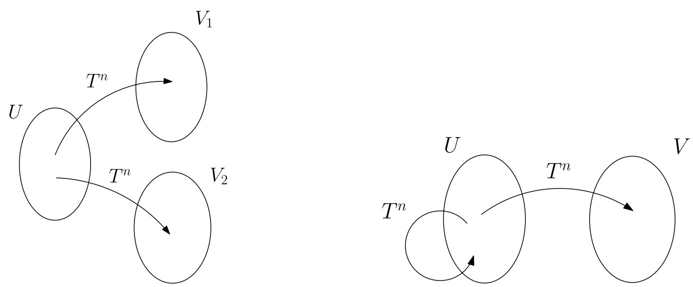

Fig. 1.7 Weak mixing (Propositions 1.52 and 1.53)

Proposition 1.52. *A dynamical system $T : X \to X$ is weakly mixing if and only if, for any nonempty open sets $U, V_1, V_2 \subset X$, we have that*

$$N(U, V_1) \cap N(U, V_2) \neq \varnothing.$$

Proof. We need only show sufficiency of the condition. Thus, let $U_1, U_2, V_1, V_2 \subset X$ be arbitrary nonempty open sets. By hypothesis there is some $n \in N(U_1, U_2) \cap N(U_1, V_2)$. In particular $U := U_1 \cap T^{-n}(U_2)$ and $T^{-n}(V_2)$ are nonempty open sets. By another application of the hypothesis we find some $m \in N(U, V_1) \cap N(U, T^{-n}(V_2))$. In particular, there exists some $x \in U$ with $T^m x \in T^{-n}(V_2)$. We then conclude that $T^m T^n x = T^n T^m x \in V_2$ and $T^n x \in U_2$, which yields that $m \in N(U_1, V_1) \cap N(U_2, V_2)$. □

Proposition 1.53. *A dynamical system $T : X \to X$ is weakly mixing if and only if, for any nonempty open sets $U, V \subset X$, we have that*

$$N(U,U) \cap N(U,V) \neq \varnothing.$$

Proof. It suffices to show that the stated condition implies the condition of Proposition 1.52. Thus, let $U, V_1, V_2 \subset X$ be arbitrary nonempty open sets. By hypothesis there exists some $n \in \mathbb{N}_0$ such that $U_1 := U \cap T^{-n}(V_1)$ is a nonempty open set. Since topologically transitive maps have dense range, the hypothesis also implies that $T^{-n}(V_2)$ is nonempty and open, so that there exists some $m \in N(U_1, U_1) \cap N(U_1, T^{-n}(V_2))$. Therefore there are $x, y \in U_1$ with $T^m x \in U_1$ and $T^n T^m y \in V_2$. We then have that $T^n T^m x \in V_1$, which implies that $n + m \in N(U, V_1) \cap N(U, V_2)$, as desired. □

For more characterizations of weak mixing in the same spirit we refer to Exercise 1.5.1.

We finally characterize the weak mixing property in terms of the size of the return sets $N(U, V)$, or equivalently, in terms of the topological transitivity of certain subsequences $(T^{n_k})_k$; for the notion of topological transitivity for a sequence of maps we refer to the next section.

A strictly increasing sequence $(n_k)_k$ of positive integers is called *syndetic* if

$$\sup_{k \geq 1}(n_{k+1} - n_k) < \infty.$$

Likewise, a subset A of $\mathbb{N}_0$ is called *syndetic* if the increasing sequence of positive integers forming A is syndetic, or equivalently, if its complement does not contain arbitrarily long intervals.

Theorem 1.54. *Let $T : X \to X$ be a dynamical system. Then the following assertions are equivalent:*

(i) *T is weakly mixing;*
(ii) *for any pair $U, V \subset X$ of nonempty open sets, $N(U, V)$ contains arbitrarily long intervals;*
(iii) *for any syndetic sequence $(n_k)_k$, the sequence $(T^{n_k})_k$ is topologically transitive.*

Proof. (i)$\Longrightarrow$(ii). Let $U, V \subset X$ be nonempty open sets, and let $m \in \mathbb{N}$. As in the proof of Proposition 1.53, each set $T^{-k}(V)$, $k = 1, \ldots, m$, is nonempty and open. Since, by Furstenberg's theorem, the m-fold product map $T \times \cdots \times T$ is topologically transitive, there is some $n \in \mathbb{N}$ such that

$$T^n(U) \cap T^{-k}(V) \neq \varnothing \quad \text{for } k = 1, \ldots, m.$$

This implies that $T^{n+k}(U) \cap V \neq \varnothing$ for $k = 1, \ldots, m$.

(ii)$\Longrightarrow$(i). By Proposition 1.52 it suffices to show that, given any nonempty open subsets $U, V_1, V_2 \subset X$, $N(U, V_1) \cap N(U, V_2) \neq \varnothing$. First, by (ii) there is some $m \in N(V_1, V_2)$ and therefore a nonempty open set $V_3 \subset V_1$ such that

$T^m(V_3) \subset V_2$. Also by (ii) there is some $k \in \mathbb{N}_0$ such that $k + j \in N(U, V_3)$ for $j = 0, 1, \ldots, m$. In particular we have that $k + m \in N(U, V_1)$ and

$$T^{k+m}(U) \cap V_2 \supset T^{k+m}(U) \cap T^m(V_3) \supset T^m(T^k(U) \cap V_3) \neq \varnothing.$$

We conclude that $k + m \in N(U, V_1) \cap N(U, V_2)$.

(ii)$\Longleftrightarrow$(iii). This follows immediately from the definitions and the fact that a subset of $\mathbb{N}_0$ contains arbitrarily long intervals if and only if it meets every syndetic sequence. □

Condition (ii) in this result shows nicely how weak mixing sits between topological transitivity and mixing.

1.6 Universality

The basic concepts introduced so far in this chapter allow a far-reaching generalization. The orbit of a point x under a map T is obtained by applying the iterates T^n, $n = 0, 1, 2, \ldots$, of T to x. Instead, one could think of applying arbitrary maps T_n, $n = 0, 1, 2, \ldots$, to x; in this case we need not even have that the T_n are self-maps.

Definition 1.55. Let X and Y be metric spaces, and let $T_n : X \to Y$, $n \in \mathbb{N}_0$, be continuous maps. Then the *orbit* of x under $(T_n)_n$ is defined as

$$\mathrm{orb}(x, (T_n)) = \{T_n x \ ; \ n \in \mathbb{N}_0\}.$$

An element $x \in X$ is called *universal for* $(T_n)_n$ if it has dense orbit under $(T_n)_n$.

An interesting and nontrivial example is provided by *universal Taylor series*: it can be shown that there exists an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ such that, for any continuous function $g : \mathbb{R} \to \mathbb{R}$ with $g(0) = 0$, there exists an increasing sequence $(n_k)_k$ of positive integers such that

$$\sum_{\nu=0}^{n_k} \frac{f^{(\nu)}(0)}{\nu!} x^\nu \to g(x) \quad \text{uniformly on any compact subset of } \mathbb{R}.$$

In this case, T_n is the map that associates to f its Taylor polynomial of degree n at 0.

The theory of universality will not be developed in any depth in this book. We note that there is a difference in philosophy between universality and topological dynamics: in the former one is interested in the universal elements and their properties while in the latter the focus is rather on the map and its properties.

However, occasionally the study of the dynamics of a single map requires looking at orbits under general sequences of maps; Theorem 1.54 has already provided such an example. For this reason we consider here briefly how the concepts and results of this chapter can be generalized to universality.

Definition 1.56. Let $T_n : X \to Y$, $n \in \mathbb{N}_0$, be continuous maps between metric spaces X and Y. Then $(T_n)_n$ is called *topologically transitive* if, for any pair $U \subset X$ and $V \subset Y$ of nonempty open sets, there is some $n \geq 0$ such that

$$T_n(U) \cap V \neq \varnothing;$$

it is *mixing* if the same holds for all sufficiently large n, and it is *weakly mixing* if $(T_n \times T_n)_n$ is topologically transitive on $X \times X$.

Now, many of the results in this chapter extend, at least under suitable assumptions, to general sequences. We will content ourselves here with some examples.

Theorem 1.57 (Universality Criterion). *Let X be a complete metric space, Y a separable metric space and $T_n : X \to Y$, $n \in \mathbb{N}_0$, continuous maps. Then the following assertions are equivalent:*
(i) *$(T_n)_n$ is topologically transitive;*
(ii) *there exists a dense set of points $x \in X$ such that $\mathrm{orb}(x, (T_n))$ is dense in Y.*

If one of these conditions holds then the set of points in X with dense orbit is a dense G_δ-set.

Proof. Suppose that (ii) holds. If U and V are nonempty open sets of X and Y, respectively, then there exists some $x \in U$ with dense orbit under $(T_n)_n$, so that there exists some $n \geq 0$ with $T_n x \in V$. This implies (i).

The converse implication and the fact that the set of points with dense orbit is a dense G_δ-set can be proved exactly as in the proof of the Birkhoff transitivity theorem. □

Typically, results on iterates of maps have a good chance of extending to sequences $(T_n)_n$ if they consist of commuting self-maps $T_n : X \to X$ of dense range. For example, for such sequences the Birkhoff transitivity theorem has a perfect analogue; see Exercise 1.6.2.

Remark 1.58. If we define the return sets

$$N(A, B) = \{n \in \mathbb{N}_0 \ ; \ T_n(A) \cap B \neq \varnothing\}$$

then part (a) of the 4-set trick remains valid for sequences $(T_n)_n$ of self-maps if the map S commutes with all T_n, $n \geq 0$, as does part (b) for commuting self-maps. As a consequence, Furstenberg's theorem also holds for commuting sequences $(T_n)_n$, as does Proposition 1.52.

Exercises

Exercise 1.1.1. Show that for $\mu = 2$ the iterates of the logistic map L_2 are given by

$$L_2^n x = \tfrac{1}{2}\left(1 - (1-2x)^{2^n}\right), \quad n \geq 0.$$

Deduce from this the long-term behaviour of the orbits $\mathrm{orb}(x, L_2)$ for $x \in \mathbb{R}$.

Exercise 1.1.2. Consider the dynamical system $T :]0,\infty[\to]0,\infty[$, $Tx = \frac{1}{2}(x + \frac{2}{x})$. Show that there is some $q \in]0,1[$ such that $|Tx - Ty| \leq q|x-y|$ for $x, y \geq 1$. Deduce that $|T^n x - \sqrt{2}| \leq q^n|x - \sqrt{2}|$ and hence that $T^n x \to \sqrt{2}$ for all $x \geq 1$.

Exercise 1.1.3. Prove the statements in Example 1.6(a)–(c).

Exercise 1.1.4. Show that the logistic map L_4, when restricted to the interval $[0,1]$, is quasiconjugate to the doubling map on the interval via $\phi(x) = \sin^2(\pi x)$. Also show that the maps are not conjugate.

Exercise 1.2.1. Show that, in general, the implication (i)$\Longrightarrow$(ii) does not hold in Proposition 1.10.

Exercise 1.2.2. Show that the following assertions on a dynamical system $T : X \to X$ are equivalent:

(i) T is topologically transitive;
(ii) for any open set $U \subset X$ with $T^{-1}(U) \subset U$, either $U = \varnothing$ or U is dense in X;
(iii) for any closed set $E \subset X$ with $T(E) \subset E$, either $E = X$ or E is nowhere dense.

Exercise 1.2.3. Suppose that X has at least one isolated point. Prove that, if there is any topologically transitive map $T : X \to X$, then X is finite and $X = \mathrm{orb}(x, T)$ for any $x \in X$.

Exercise 1.2.4. Show that, if $T : X \to X$ is topologically transitive, then for any pair U, V of nonempty open subsets of X, the return set $N(U, V)$ is infinite; see Definition 1.49. (*Hint*: For the trivial case in which X has isolated points apply Exercise 1.2.3. If X has no isolated points, then given $m \in N(U,V)$ and $W := U \cap T^{-m}(V)$, observe that $N(W,W) \cap \mathbb{N} \neq \varnothing$ and $m + N(W,W) \subset N(U,V)$.)

Exercise 1.2.5. Prove that a dynamical system $T : X \to X$ on a metric space X is topologically transitive if and only if, for any $\varepsilon > 0$ and any pair of points $x, y \in X$, we can find $z \in X$ and $n, m \in \mathbb{N}_0$ satisfying $d(T^n z, x) < \varepsilon$ and $d(T^m z, y) < \varepsilon$. (*Hint*: First observe that the above condition is equivalent to the fact that for any pair U, V of nonempty open subsets of X one can find $n, m \in \mathbb{N}_0$ with $T^{-n}(U) \cap T^{-m}(V) \neq \varnothing$. This condition is obviously implied by topological transitivity. For the converse, given nonempty open sets $U, V \subset X$, either find $k \in N(U,V)$ (in that case you are done) or, if $k \in N(V,U)$, set $W = V \cap T^{-k}(U)$, note that $N(W,W)$ is infinite by Exercise 1.2.4, and then find some $j \in N(U,V)$.)

Exercise 1.2.6. Let T be a topologically transitive dynamical system on a separable complete metric space X without isolated points. Prove constructively, not using the Baire category theorem, that T has a dense set of points with dense orbit. (*Hint*: Let $(y_n)_n$ be a dense sequence in X. Start with $x_0 \in X$. Then find x_1 close to x_0 and a positive integer m_1 so that $T^{m_1}x_1$ is close to y_1. Then find x_2 close to x_1 and a positive integer m_2 so that $T^{m_1}x_2$ is close to $T^{m_1}x_1$ and $T^{m_1+m_2}x_2$ is close to y_2. Continue.)

Exercise 1.2.7. Let T be a dynamical system on a metric space X without isolated points. A *backward orbit* of a vector x is a sequence $(x_n)_{n\geq 0}$ in X (if it exists!) such that $x_0 = x$ and $Tx_n = x_{n-1}$, $n \geq 1$. Show the following:

(i) if T is topologically transitive and X is separable and complete then there exists a dense set of points with dense backward orbits;

(ii) if T has a dense backward orbit then T is topologically transitive.

(*Hint*: See the previous exercise.)

Exercise 1.2.8. Let $T : X \to X$ be a dynamical system. For $x \in X$ the *J-set* $J_T(x) = J(x)$ is defined as the set of all points $y \in X$ for which there is a strictly increasing sequence $(n_k)_k$ of positive integers and a sequence $(x_k)_k$ in X such that $x_k \to x$ and $T^{n_k}x_k \to y$ as $k \to \infty$.

(a) Show that $J(x)$ is a closed T-invariant set.

(b) Suppose that X has no isolated points. Show that $J(x) = X$ if and only if, for any pair U, V of nonempty open subsets of X with $x \in U$, there exists some $n \geq 0$ such that $T^n(U) \cap V \neq \varnothing$.

(c) Suppose that X has no isolated points. Show that the following assertions are equivalent:

(i) T is topologically transitive;

(ii) for any $x \in X$, $J(x) = X$;

(iii) there is a dense set of points $x \in X$ such that $J(x) = X$.

Exercise 1.2.9. Show that every orbit under an irrational rotation is dense. (*Hint*: Use the pigeonhole principle to show that, for any $\varepsilon > 0$, some arc of angle ε must contain two iterates of 1, $T^m 1$ and $T^n 1$, $m > n$. Then look at the iterates of T^{m-n}.)

Exercise 1.2.10. A dynamical system $T : X \to X$ is called *minimal* if every orbit under T is dense. Find a characterization of minimality in the spirit of Exercise 1.2.2.

Exercise 1.2.11. Consider the dynamical system $T : [-1, 1] \to [-1, 1]$ given by

$$Tx = \begin{cases} 2 + 2x, & \text{if } -1 \leq x < -1/2, \\ -2x, & \text{if } -1/2 \leq x < 1/2, \\ -2 + 2x, & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

(a) Show that T has a dense orbit but that T^2 does not.

(b) Show that there are two points $x, y \in [-1, 1]$ such that $\operatorname{orb}(x, T^2) \cup \operatorname{orb}(y, T^2)$ is dense in $[-1, 1]$ but neither of them has a dense orbit under T^2.

(c) Show that there is a point $x \in [-1, 1]$ such that $\overline{\operatorname{orb}(x, T^2)}$ contains a nonempty open set but x does not have a dense orbit under T^2.

(*Remark*: We will prove in Chapter 6 that none of these properties can hold in a linear setting.)

Exercise 1.3.1. Show that none of the three conditions in Definition 1.26 alone implies chaos.

Exercise 1.3.2. Let X be a finite set, endowed with the discrete metric. Describe all maps on X that are chaotic. Do the same for countably infinite sets under the discrete metric.

Exercise 1.3.3. Suppose that (X, d) is a metric space without isolated points and $T : X \to X$ is a contracting map, that is $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$. Show that if T has one dense orbit then T is minimal (see Exercise 1.2.10); in particular, it cannot be chaotic.

Exercise 1.3.4. Show that $T : X \to X$ is chaotic if and only if every finite family of nonempty open sets shares a periodic orbit, in the following sense: for each finite family $U_j \subset X$, $j = 1, \ldots, n$, of nonempty open sets there is a periodic point $x \in U_1$ such that $T^{k_j}x \in U_j$ for some $k_j \geq 0$, $j = 2, \ldots, n$; see Figure 1.8. (*Hint:* One implication is trivial; for the other one use continuity of T and an induction process.)

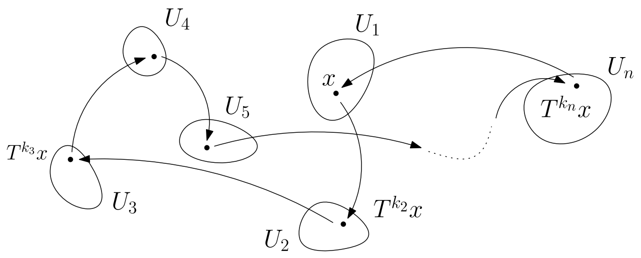

Fig. 1.8 Exercise 1.3.4

Exercise 1.3.5. Show that the space Σ_2 is a complete metric space without isolated points. Show also that the sequences with only finitely many nonzero entries form a dense set.

Exercise 1.3.6. Why is the quasiconjugacy of Example 1.37 between the shift on two symbols and the doubling map not a conjugacy? Use this quasiconjugacy to find a representation of the periodic points and the points with dense orbit for the doubling map.

Exercise 1.4.1. Show that the shift on two symbols is mixing.

Exercise 1.4.2. Prove that a dynamical system T is mixing if and only if, for any strictly increasing sequence $(n_k)_k$ of positive integers, the sequence $(T^{n_k})_k$ is topologically transitive; see Definition 1.56 for the notion of topological transitivity for sequences of maps.

Exercise 1.4.3. Let X be a complete metric space. Prove that a dynamical system $T : X \to X$ is mixing if and only if, for every sequence $(x_n)_n$ in X and for every strictly increasing sequence $(n_k)_k$ of positive integers for which $\{x_{n_k}\ ;\ k \in \mathbb{N}\}$ is relatively compact there exists a dense G_δ-set of points $y \in X$ such that $\liminf_{k\to\infty} d(x_{n_k}, T^{n_k}y) = 0$. (*Hint*: Use the previous exercise; a subset A of a metric space X is relatively compact if and only if every sequence in A has a subsequence that converges in X.)

Exercise 1.4.4. Let $T : X \to X$ be a dynamical system. For $x \in X$, the set $J_T^{\text{mix}}(x) = J^{\text{mix}}(x)$ is defined as the set of all points $y \in X$ for which there is a sequence $(x_n)_n$ in X such that $x_n \to x$ and $T^n x_n \to y$ as $n \to \infty$; see also Exercise 1.2.8.

(a) Show that $J^{\text{mix}}(x)$ is a closed T-invariant set.

(b) Show that $J^{\text{mix}}(x) = X$ if and only if, for any pair U, V of nonempty open subsets of X with $x \in U$, there exists some $N \geq 0$ such that $T^n(U) \cap V \neq \varnothing$ for all $n \geq N$.

(c) Show that the following assertions are equivalent:

(i) T is mixing;
(ii) for any $x \in X$, $J^{\text{mix}}(x) = X$;
(iii) there is a dense set of points $x \in X$ such that $J^{\text{mix}}(x) = X$.

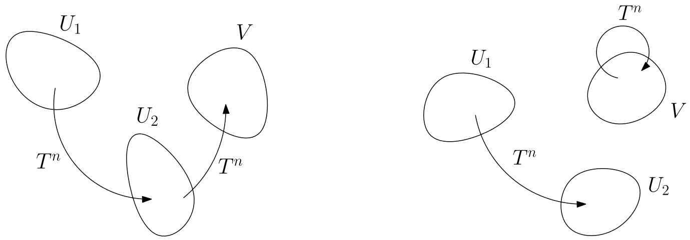

Fig. 1.9 Weak mixing (Exercise 1.5.1(iii) and (iv))

Exercise 1.4.5. Show that minimality and mixing are independent properties of a dynamical system.

Exercise 1.5.1. Let $T : X \to X$ be a dynamical system. In the following, U, U_1, U_2, V will denote arbitrary nonempty open subsets of X. Prove that any of the following conditions is equivalent to T being weakly mixing:
(i) for any $U, V \subset X$ we have $N(U, V) \cap N(V, V) \neq \varnothing$;
(ii) for any $U_1, U_2, V \subset X$ we have $N(U_1, V) \cap N(U_2, V) \neq \varnothing$;
(iii) for any $U_1, U_2, V \subset X$ we have $N(U_1, U_2) \cap N(U_2, V) \neq \varnothing$;
(iv) for any $U_1, U_2, V \subset X$ we have $N(U_1, U_2) \cap N(V, V) \neq \varnothing$;
see Figure 1.9. (*Hint*: To prove sufficiency of (i) use the 4-set trick and Proposition 1.53; this will then imply the sufficiency of the other conditions.)

Exercise 1.5.2. A dynamical system $T : X \to X$ is called *totally transitive* if every power T^p, $p \in \mathbb{N}$, is topologically transitive. Show that any weakly mixing map is totally transitive.

Exercise 1.5.3. Prove that every chaotic and totally transitive dynamical system $T : X \to X$ is weakly mixing. (*Hint*: Verify the hypothesis of Proposition 1.53 by finding a periodic point in U, of period k say, and then by using topological transitivity of T^k.)

Exercise 1.5.4. A dynamical system $T : X \to X$ is called *flip transitive* if, for any pair $U, V \subset X$ of nonempty open sets, $N(U, V) \cap N(V, U) \neq \varnothing$. Show that the map T of Exercise 1.2.11 is flip transitive but not weakly mixing.

Exercise 1.5.5. Show that T is weakly mixing if and only if it is flip transitive and T^2 is topologically transitive. (*Hint*: To prove the weak mixing property use condition (i) in Exercise 1.5.1. To do this, given nonempty open sets $U, V \subset X$, find $k \in \mathbb{N}_0$ with $U' := U \cap T^{-2k}(V) \neq \varnothing$ and then some $m \in N(U', T^{-k}(V)) \cap N(T^{-k}(V), U')$. Consider $m + k$.)

Exercise 1.5.6. A dynamical system $T : X \to X$ is called *topologically ergodic* if, for any pair $U, V \subset X$ of nonempty open sets, $N(U, V)$ is syndetic. Prove the following:
(i) any irrational rotation is topologically ergodic but not weakly mixing;
(ii) every mixing and every chaotic dynamical system is topologically ergodic;
(iii) if $T : X \to X$ is topologically ergodic and $S : Y \to Y$ is weakly mixing, then $T \times S$ is topologically transitive.

Exercise 1.5.7. Let $T : X \to X$ be a dynamical system. Show that any of the following conditions is equivalent to T being weakly mixing:
(i) for any nonempty open sets $U, V \subset X$ and any $m \in \mathbb{N}$ there is some k with $k, k + m \in N(U, V)$;

(ii) for any $m \in \mathbb{N}$ and for any increasing sequence $(n_k)_k$ with $n_{k+1} - n_k \in \{m, 2m\}$, $k \in \mathbb{N}$, we have that $(T^{n_k})_k$ is topologically transitive.

(*Hint*: See the proof of Theorem 1.54.)

Exercise 1.5.8. Let $T : X \to X$ be a dynamical system. Establish the equivalence of the following assertions:

(i) T is weakly mixing;

(ii) for any pair $U, V \subset X$ of nonempty open sets, $N(U, V)$ contains two consecutive integers.

(*Hint*: Proceeding by induction, show that $N(U, V)$ contains arbitrarily long intervals: if $k, k+1 \in N(U, U)$, set $U_1 := U \cap T^{-k}(U)$, $U_2 := U \cap T^{-k-1}(U)$, apply the inductive hypothesis to the pair (U_1, U_2) to find an interval $[j, j+m]$ contained in $N(U_1, U_2)$. By the selection of U_1, obtain that $[j, j+m+1] \subset N(U, U)$.)

Exercise 1.5.9. Given a metric space X, the corresponding *hyperspace* is defined as $\mathcal{K}(X) = \{K \subset X \ ; \ K \text{ is compact}\}$. The space $\mathcal{K}(X)$ is endowed with the (metrizable) *Vietoris topology*, for which a base of open sets is given by the family of sets

$$\mathcal{V}(U_1, \ldots, U_k) := \Big\{K \in \mathcal{K}(X) \ ; \ K \subset \bigcup_{j=1}^{k} U_j \text{ and } K \cap U_j \neq \varnothing, \ j = 1, \ldots, k\Big\},$$

where $U_1, \ldots, U_k$, $k \in \mathbb{N}$, are nonempty open sets in X. If $T : X \to X$ is continuous, then it naturally induces a continuous *hyperextension* $\overline{T} : \mathcal{K}(X) \to \mathcal{K}(X)$ defined by $\overline{T}(K) = T(K) = \{Tx \ ; \ x \in K\}$.

We say that a dynamical system $T : X \to X$ is *hypertransitive* if its hyperextension $\overline{T}$ is topologically transitive. Prove that T is hypertransitive if and only if it is weakly mixing. (*Hint*: For the sufficiency of weak mixing use Furstenberg's theorem; for the necessity use Proposition 1.52.)

Exercise 1.6.1. Let $(x_n)_n$ be a dense sequence in $\mathbb{R}^2$, and let $y_n \in \mathbb{R}^2$, $n \geq 1$, be vectors of length n that are orthogonal to x_n. Consider the maps $T_n : \mathbb{R}^2 \to \mathbb{R}^2$ with $T_n(\alpha, \beta) = \alpha x_n + \beta y_n$. Determine all points in $\mathbb{R}^2$ with dense orbit under $(T_n)_n$. Deduce that $(T_n)_n$ has a dense orbit but is not topologically transitive.

Exercise 1.6.2. Prove the *Birkhoff transitivity theorem* for commuting continuous maps $T_n : X \to X$, $n \in \mathbb{N}_0$, of dense range on a separable complete metric space X, that is, that the following assertions are equivalent:

(i) $(T_n)_n$ is topologically transitive;

(ii) there exists some $x \in X$ such that $\operatorname{orb}(x, (T_n))$ is dense in X.

If one of these conditions holds then the set of points in X with dense orbit is a dense G_δ-set.

Exercise 1.6.3. Let S be a mixing map on a separable complete metric space X and T a map on a metric space Y without isolated points that admits a dense orbit $\operatorname{orb}(y, T)$, $y \in Y$. Show that there exists some $x \in X$ such that (x, y) has a dense orbit under the map $S \times T$. (*Hint*: Use the previous exercise.)

Exercise 1.6.4. A sequence $(T_n)_n$ of continuous maps on a metric space X is called *hereditarily transitive* with respect to an increasing sequence $(n_k)_k$ of positive integers if $(T_{m_k})_k$ is topologically transitive for every subsequence $(m_k)_k$ of $(n_k)_k$. The sequence $(T_n)_n$ is called *hereditarily transitive* if it is so with respect to some sequence $(n_k)_k$. Prove that, if X is separable, then a commuting sequence $(T_n)_n$ is hereditarily transitive if and only if $(T_n)_n$ is weakly mixing. (*Hint*: Note that X has a countable base; use Furstenberg's theorem for sequences of maps.)

Exercise 1.6.5. Show that Theorem 1.54 does not hold for sequences $(T_n)_n$ even if the maps $T_n : X \to X$ commute and have dense range. (*Hint*: A weakly mixing sequence $(T_n)_n$ remains weakly mixing when adding arbitrary maps.)

Sources and comments

Section 1.1. A standard reference for the theory of dynamical systems is Devaney [132]. For more recent textbooks we refer to Brin and Stuck [97] and Robinson [267], while Gulick [188] provides an elementary introduction.

Section 1.2. Kolyada and Snoha [217] give an excellent survey on topological transitivity, with many additional equivalent conditions. The original version of the Birkhoff transitivity theorem can be found in [74, § 62].

Section 1.3. Chaos in the sense of Devaney was introduced in [132]. While there are many other definitions of chaos (see for example Kolyada [216] or Forti [154]), Devaney's definition has become very popular. The theorem of Banks et al. was obtained in [31]; see also Silverman [294] and Glasner and Weiss [163].

Sections 1.4 and 1.5. For Furstenberg's theorem see [157], which also contains the 4-set trick implicitly. Theorem 1.54 is due, independently, to Akin [4], Glasner [162], and Peris and Saldivia [257]; in the context of linear operators on Banach spaces it was also obtained by Grivaux [172]. The remaining characterizations of weak mixing in Section 1.5, including Exercise 1.5.1, are due to Banks [29] and Akin [4]; more precisely, Proposition 1.53 is called the "Furstenberg Intersection Lemma" in Akin's book.

Section 1.6. The universal Taylor series mentioned in Section 1.6 is essentially due to Fekete; see the discussion in [179, Section 3a]. The Universality Criterion was obtained by Grosse-Erdmann [177].

Exercises. Exercises 1.2.8 and 1.4.4 are taken from Costakis and Manoussos [118, 119]. For Exercise 1.3.4 we refer to Touhey [299], for Exercise 1.4.3 to Moothathu [244], for Exercise 1.5.3 to Bauer and Sigmund [32] and to Banks (attributed to Stacey) [28], for Exercise 1.5.5 to Banks [29], and for Exercise 1.5.6 to Moothathu [245]. The two parts of Exercise 1.5.7 are taken from Grivaux [172] and Peris and Saldivia [257], respectively; Exercise 1.5.8 is from Grosse-Erdmann and Peris [187] (see also Grivaux [172] and Bayart and Matheron [45]). The assertion of Exercise 1.5.9 is due to Bauer and Sigmund [32] (one direction), and, independently, to Banks [30] and Peris [256] (the other direction). Exercise 1.6.1 is from Godefroy and Shapiro [165], Exercise 1.6.4 from Bès and Peris [71].

Extensions. Let us add a word on the setting chosen in this chapter. Since the overwhelming majority of linear dynamical systems studied in the literature acts on metric spaces we have restricted our attention to such spaces. In general, however, a *dynamical system* is given by a continuous map $T : X \to X$ on a topological space X. The definitions of *topologically transitive*, *(weakly) mixing* and *chaotic maps* extend verbatim to such systems. The same applies to sequences $(T_n)_n$ of continuous maps $T_n : X \to Y$ between arbitrary topological spaces X and Y.

Then, as the proofs show, all the results in this chapter on general dynamical systems remain true in the setting of arbitrary topological spaces. To be more specific, this concerns all results apart from Proposition 1.15 and the Theorems 1.16, 1.29 and 1.57.