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Characterization of a Strongly Nonlinear Laboratory Benchmark System

D. Michael McFarland
Department of Aerospace Engineering
University of Illinois at Urbana-Champaign
Urbana, IL 61801
dmmcf@illinois.edu

Alexander F. Vakakis
Department of Mechanical Science and Engineering
University of Illinois at Urbana-Champaign
Urbana, IL 61801
avakakis@illinois.edu

Lawrence A. Bergman
Department of Aerospace Engineering
University of Illinois at Urbana-Champaign
Urbana, IL 61801
lbergman@illinois.edu

Timothy J. Copeland
m+p international, inc.
5375 Old Jacksonville Road
Springfield, IL 62711
tim.copeland@mpina.com

NOMENCLATURE

A	cross-sectional area
E	Young's modulus
F	static restoring force
L	wire half-span
T_0	wire initial tension
T	wire tension due to deformation
c	viscous damping coefficient
f	dynamic restoring force
k_1, k_2	linear spring stiffness
k_{lin}	coefficient of wire linear stiffness
k_{nl}	coefficient of wire nonlinear stiffness
m_1, m_2	masses of 2-DOF rig
x	wire static displacement
u	displacement of isolated mass
α	exponent of fitted nonlinear stiffness term
ϵ	strain in wire due to transverse displacement
$(\dot{\ })$	time differentiation

ABSTRACT

A two-degree-of-freedom test structure originally developed to exhibit essentially nonlinear (nonlinearizable) stiffness in one of its connections has been modified to display a variable linear term in parallel with the existing nonlinear spring. This structure will be used in generating data for input to new algorithms for nonlinear system identification. In this paper, we report results from experiments in controlling the linear term introduced by applying an initial tension (preload) to the nonlinear spring, including the estimation of the parameters of a nonlinear force-displacement relation. It is shown that a sufficiently large initial tension results in predominantly linear behavior, meaning the same test setup can be used to produce dynamic behavior ranging from nearly linear to essentially nonlinear.

1 Introduction

While the identification of models of linear structures from experimental data has become routine, the identification of nonlinear systems remains a much more challenging problem. To support recent theoretical and applied work reported in Refs. [1, 2, 3, 4, 5, 6, 7], we have developed a simple test structure in which the effects of nonlinear stiffness can be adjusted. This is achieved by varying one of the linear stiffness coefficients, from zero to a value large enough to reduce the influence of the nonlinear stiffness to a parasitic role.

The model structure taken as a starting point here was originally designed as a linear two-degree-of-freedom (2-DOF) model with an added essentially nonlinear spring (that is, a spring with no linear term in its force-displacement relation) between one of the structural masses and ground. That essentially nonlinear spring was formed from transversely deflected piano wire, and the essential nonlinearity was achieved only if the initial tension in the wire was zero. Several experiments were reported using this configuration and a similar arrangement in which the wire was connected between moving masses rather than between a mass and ground [4, 5, 6, 7]. In the present work, we exploit the linear stiffness terms introduced when a significant initial tension (preload) is established in the wire in its undeformed state.

2 Modeling

The geometry of the piano-wire spring is show in Fig. 1. The wire, of cross-sectional area A, Young's modulus E and total length $2L$, is deflected a distance x at its center in response to a load F. The resulting length of half of the wire is $\sqrt{L^2+x^2}$, and assuming linear material response we can compute the resulting increase in tension of the wire due to this stretching as

$$T = EA\epsilon \tag{1}$$

where

$$\epsilon = \frac{\sqrt{L^2+x^2}-L}{L} \tag{2}$$

is the (engineering) strain. Denoting by T_0 the initial tension in the wire (before deformation), we can express the equilibrium of the wire as

$$F = 2(T_0+T)\sin\theta, \tag{3}$$

where θ is the angle formed by the wire with its initial position and thus

$$\sin\theta = \frac{x}{\sqrt{L^2+x^2}}. \tag{4}$$

Substituting, we obtain an expression for the force F in terms of the transverse displacement x,

$$F = 2\left(T_0 + EA\frac{\sqrt{L^2+x^2}-L}{L}\right)\frac{x}{\sqrt{L^2+x^2}}. \tag{5}$$

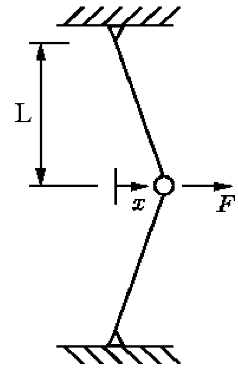

Fig. 1. Geometry of the deflected piano wire providing the nonlinear and variable linear stiffness between the second (NES) mass and ground.

This expression, while exact, is not convenient to work with. If x is small (compared to the half-length L), we can expand $F(x)$ in a Taylor series about $x = 0$, with the result

$$F(x) \approx \frac{2T_0}{L}x - \frac{T_0 - EA}{L^3}x^3 + \frac{3(T_0 - EA}{4L^5}x^5 + O(x^7). \tag{6}$$

In practice, it is sufficient to retain only the leading terms of this expression to represent the force-displacement relation of a real spring,

$$F(x) = \frac{2T_0}{L}x - \frac{T_0 - EA}{L^3}x^3, \tag{7}$$

from which form it is easy to see that the linear term will exist only when there is a preload in the wire. When $T_0 > 0$, we obtain the simple expression

$$F(x) = \frac{EA}{L^3}x^3, \tag{8}$$

which has been used successfully to represent this type of essentially nonlinear spring in several experiments.

3 Experiments and Data Reduction

The physical system incorporating two linear degrees of freedom along with the piano-wire spring of the previous section is show schematically in Fig. 3, where the wire connects the second mass to ground. Both masses are made of aluminum angle stock and run on a straight air track, connected by linear leaf springs. The laboratory model, set up atop an optical table and instrumented with force transducers and accelerometers, is shown in the photograph of Fig. 2. The wire was in this case 0.020 in. in diameter and of total length $2L = 20.0$in.

The parameters of the linear system resulting when the wire spring was disconnected were determined by standard techniques of linear modal analysis, and are summarized in Table 1. This structure is very similar to that described in Ref. [3], where is was configured with the wire installed with zero initial tension and thus providing an essentially nonlinear restoring force. The focus of that work was targeted energy transfer ("energy pumping") from the primary mass to the secondary (the nonlinear energy sink, or NES), a phenomenon that depends on the essential nonlinearity.

m_1, kg	m_2, kg	c_1, N/m/s	c_2, N/m/s	k_1, N/m	k_2, N/m
1.288	518	0.112	0.079	1228	680

Table 1. Identified values of the parameters of the linear 2-DOF structure (with the piano-wire spring removed).

The nonlinear spring was identified here using the restoring force surface method, in which the ring-down response of (a portion of) the structure is fitted with a model equation including viscous linear damping as well as

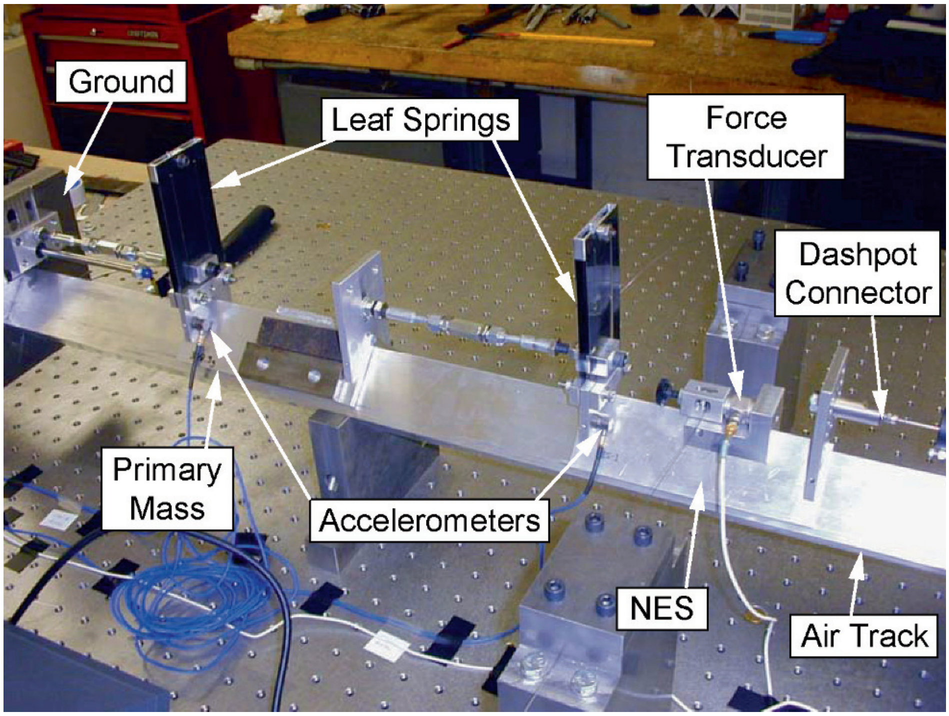

Fig. 2. The 2-DOF system of Fig. 3 realized as two cars running on a rectilinear air track. The car labeled "NES" here is the secondary mass, connected to ground by both nonlinear and (with preload on the piano wire) linear springs.

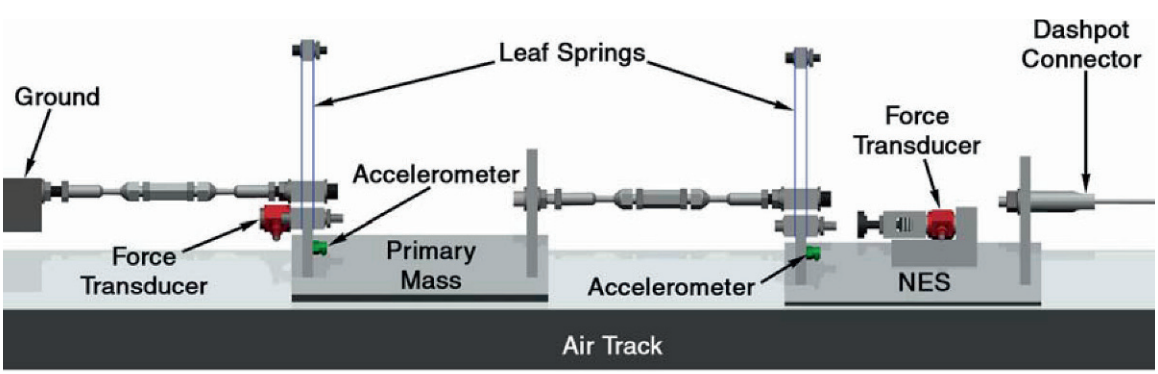

Fig. 3. Schematic of 2-DOF system with variable linear stiffness in parallel with cubic-hardening spring between second (right-hand) mass and ground. The nonlinear, piano-wire spring is normal to the page in this view.

linear and nonlinear (cubic) terms. Specifically, we assumed the restoring force was of the form

$$f = c\dot{u} + k_{\text{lin}} u + k_{\text{nl}} |u|^{\alpha} \operatorname{sgn} \ u, \tag{9}$$

where u is the displacement of the mass attached to the wire spring, $\dot{u}$ is its velocity, c is the damping coefficient, k_{lin} and k_{nl} are the coefficients of the linear and nonlinear stiffness terms, and α is the exponent of the nonlinear term (nominally equal to 3 for a purely cubic spring). The results obtained with several values of initial tension T_0 are reported in Table 2, where it may be seen that the results of increasing preload are generally in agreement with the predictions of eq. (7). A linear term does appear even when $T_0 = 0$, reflecting the finite bending stiffness of the piano wire, but insofar as our goal here was to add a significant linear stiffness to an essentially nonlinear spring this linear term is, if anything, beneficial.

Preload, lb	c, N/m/s	k_{lin}, N/m	k_{nl}, N/m$^{\alpha} \times 10^3$	α	k_{lin}, N/m
5.0	1.02	245	1138	2.73	—
7.5	1.03	355	891	2.70	—
10.0	1.44	584	50	2.07	—
20.0	3.04	1507	-0.073	0.72	1335

Table 2. Parameters for the nonlinear model of eq. (7), estimated with various preloads applied to the piano-wire spring.

4 Conclusion

A structure used in a series of bench-top experiments comprises a linear oscillator of one degree of freedom coupled to a strongly nonlinear, single-degree-of-freedom substructure. The nonlinear stiffness of the attachment, created by the transverse deflection of a piano-wire spring, is essential (lacking a linear part) if the wire is installed with no pretension. If the wire is preloaded with an initial tension, the spring will exhibit a linear term in its force-displacement relation in addition to a strong cubic term and generally negligible higher-order terms.

While in previous work we have sought to minimize the pretension on the wire spring and thus produce a nearly pure, cubic-hardening spring characteristic, here we intentionally apply a significant preload to create a linear restoring force comparable to the nonlinear spring force for displacements of the order observed in tests on the combined system. Taken as a whole, the structure displays strong, controllable, but not essential nonlinearity. We have demonstrated that the linear term produced by the pretension can be varied and can in fact be used to produce essentially nonlinear, strongly nonlinear, or effectively linear dynamics in the same test rig. In future work, this variability will be used to generate experimental data for input to nonlinear system identification algorithms that do not require isolating the nonlinear degree(s) of freedom.

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References

[1] Jiang, X., D. M. McFarland, L. A. Bergman and A. F. Vakakis, “Steady State Passive Nonlinear Energy Pumping in Coupled Oscillators: Theoretical and Experimental Results,” *Nonlinear Dynamics,* 33(1):87102, 2003.

[2] Vakakis, A. F., D. M. McFarland, L. A. Bergman, L. I. Manevitch and O. Gendelman, “Isolated Resonance Captures and Resonance Capture Cascades Leading to Single- or Multi-Mode Passive Energy Pumping in Damped Coupled Oscillators,” *ASME Journal of Vibration and Acoustics,* 126(2):235244, April 2004.

[3] McFarland, D. M. , L. A. Bergman and A. F. Vakakis, “Experimental Study of Nonlinear Energy Pumping Occurring at a Single Fast Frequency,” *International Journal of Nonlinear Mechanics,* 40(6):891899, July 2005.

[4] McFarland, D. M., G. Kerschen, J. J. Kowtko, L. A. Bergman and A. F. Vakakis, “Experimental Investigation of Targeted Energy Transfers in Strongly and Nonlinearly Coupled Oscillators,” *Journal of the Acoustical Society of America,* 118(2):791799, August 2005.

[5] Kerschen, G., D. M. McFarland, J. J. Kowtko, Y. S. Lee, L. A. Bergman and A. F. Vakakis, “Experimental Demonstration of Transient Resonance Capture in a System of Two Coupled Oscillators with Essential Stiffness Nonlinearity,” *Journal of Sound and Vibration,* 299(45):822838, 2007.

[6] Kerschen, G., J. J. Kowtko, D. M. McFarland, L. A. Bergman and A. F. Vakakis, “Theoretical and Experimental Study of Multimodal Targeted Energy Transfer in a System of Coupled Oscillators,” *Nonlinear Dynamics*, 47(13):285309, 2007.

[7] Vakakis, A. F., O. Gendelman, L. A. Bergman, D. M. McFarland, G. Kerschen and Y. S. Lee, *Nonlinear Targeted Energy Transfer in Mechanical and Structural Systems,* Volume 156 of the series Solid Mechanics and Its Applications, Springer-Verlag, New York, 2008.

[8] Kerschen, G., A. F. Vakakis, Y. S. Lee, D. M. McFarland and L. A. Bergman, “Toward a Fundamental Understanding of the Hilbert-Huang Transform in Nonlinear Structural Dynamics,” *Journal of Vibration and Control*, 14(12):77105, 2008.

[9] Lee, Y. S., A. F. Vakakis, L. A. Bergman, D. M. McFarland, G. Kerschen, F. Nucera, S. Tsakirtzis and P. N. Panagopoulos, “Passive Nonlinear Targeted Energy Transfer (TET) and Its Applications to Vibration Absorption: A Review,” Proceedings of the Institution of Mechanical Engineers, Part K: *Journal of Multi-body Dynamics,* 222(2):77134, 2008.

[10] Y. S. Lee, A. F. Vakakis, D. M. McFarland and L. A. Bergman, “Time-Domain Nonlinear System Identification of the Dynamics of Aeroelastic Instability Suppression Based on Targeted Energy Transfers ,” *The Aeronautical Journal,* in press.