VACUUM IN GAS AND FLUID DYNAMICS

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Abstract. In this paper, we review some interesting problems of vacuum states arising in hyperbolic conservations laws with applications to gas and fluid dynamics. We present the current status of the understanding of compressible Euler flows near vacuum and discuss related open problems.

Key words. Compressible Euler equations, physical vacuum, free boundary, energy estimates.

1. Introduction. A system of conservation laws is the collection of partial differential equations in divergence form, which describe the dynamics of continua such as fluids and plasmas, by means of the physical principles of conservation of mass, momentum, and energy with constitutive relations encoding the material properties of the medium. Systems of conservation laws identify the theory of mechanics, thermodynamics, electrodynamics and so on. Its canonical form is $\partial_t U + \sum_{j=1}^{d} \partial_{x_j} F_j(U) = 0$ where $U = U(t, x_1, \ldots, x_d) \in \mathbb{R}^n$ is a vector of conserved quantities and $F_j(U) \in \mathbb{R}^n$ is a flux function. This system is called hyperbolic in the t-direction if for any fixed U and $\nu \in S^{d-1}$, the $n \times n$ matrix $\sum_{j=1}^{d} \nu_j D_U F_j(U)$ has only real eigenvalues $\lambda_1, \ldots, \lambda_n$, called characteristic speeds, and it is diagonalizable [15].

One of the most fundamental examples of a system of hyperbolic conservation laws is the compressible Euler system of isentropic, ideal gas dynamics. In Eulerian coordinates, it takes the form:

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho u) &= 0\,, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \operatorname{grad} p &= 0\,. \end{aligned} \tag{1.1}$$

Here ρ, u and p denote respectively the density, velocity, and pressure of the gas. The first and second equations express respectively conservation of mass and momentum. In considering the polytropic gases, the constitutive relation which is also called the equation of state is given by

$$p = K\rho^\gamma \tag{1.2}$$

where K is an entropy constant and $\gamma > 1$ is the adiabatic gas exponent. The case of $\gamma = 1$ corresponds to the isothermal gas flow. In this article, we discuss isentropic compressible Euler equations (1.1) with (1.2) rather than general hyperbolic conservation laws. The main interest is to study vacuum states in the framework of classical solutions.

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When the initial density function contains a vacuum, the vacuum boundary Γ is defined as

$$\Gamma = cl\{(t,x) : \rho(t,x) > 0\} \cap cl\{(t,x) : \rho(t,x) = 0\}$$

where cl denotes the closure. We also introduce the sound speed c of Euler equations (1.1)

$$c = \sqrt{\frac{d}{d\rho}p(\rho)} \quad \Big(= \sqrt{A\gamma}\rho^{\frac{\gamma-1}{2}} \text{ for polytropic gases}\Big).$$

We recall that for one-dimensional Euler flows, $u \pm c$ are the characteristic speeds. If Γ is nonempty, Euler equations become degenerate along Γ, namely degenerate hyperbolic.

We briefly review some existence theories of compressible flows. In the absence of vacuum, namely if the density is bounded below from zero everywhere, then one can use the theory of symmetric hyperbolic systems developed by Friedrichs-Lax-Kato [19, 31, 33]; for instance, see Majda [46]. The breakdown of classical solutions was demonstrated by Sideris [61].

When the initial datum is compactly supported, there are at least three ways of looking at the problem. The first consists in solving the Euler equations in the whole space and requiring that the system (1.1) holds in the sense of distribution for all $x \in \mathbb{R}^d$ and $t \in [0,T]$. This is in particular the strategy used to construct global weak solutions (see for instance Diperna [16] and [10, 39]). The second way consists in symmetrizing the system first and then solving it using the theory of symmetric hyperbolic system. Again the symmetrized form has to be solved in the whole space. The third way is to require the Euler equations to hold on the set $\{(t,x) : \rho(t,x) > 0\}$ and write an equation for Γ. Here, the vacuum boundary Γ is part of the unknown: this is a free boundary problem and in this case, an appropriate boundary condition at vacuum is necessary.

In the first and second ways (called later the Cauchy problem), there is no need of knowing exactly the position of the vacuum boundary. DiPerna used the theory of compensated compactness to pass to the limit weakly in a parabolic approximation of the system and recovered a weak solution of the Euler system (see also [39] where a kinetic formulation of the system was also used). Makino, Ukai and Kawashima [51] wrote the system in a symmetric hyperbolic form which allows the density to vanish. The system they get is not equivalent to the Euler equations when the density vanishes. This special symmetrization was also used for the Euler-Poisson system. This formulation was also used by Chemin [9] to prove the local existence of regular solutions in the sense that $c, u \in C([0,T); H^m(\mathbb{R}^d))$ for some $m > 1 + d/2$ and d is the space dimension (see also Serre [59] and Grassin [22], for some global existence result of classical solutions under some special conditions on the initial data, by extracting a dispersive effect after some invariant transformation). However, it was noted in [47, 48] that the requirement that c is continuously differentiable excludes many interesting solutions such as the stationary solutions of the Euler-Poisson system which have a behavior of the type $\rho \sim |x - x_0|^{\frac{1}{\gamma-1}}$, namely $c^2 \sim |x - x_0|$ near the vacuum boundary. Indeed, Nishida in [55] suggested to consider a free boundary problem which includes this kind of singularity caused by vacuum, not shock wave singularity.

For the third way (called later the free boundary problem), we divide into a few cases according to the initial behavior of the sound speed c. For simplicity, let the origin be the initial vacuum contact point ($x_0 = 0$). And let $c \sim |x|^\alpha$. When $\alpha \geq 1$, namely initial contact to vacuum is smooth enough, Liu and Yang [43] constructed the local-in-time solutions to one-dimensional Euler equations with damping by using the energy method based on the adaptation of the theory of symmetric hyperbolic system and characteristic method. They also prove that c^2 can not be smooth across Γ after a finite time. We note that in these regimes there is no acceleration along the vacuum boundary.

For $0 < \alpha < 1$, the initial contact to vacuum is only Holder continuous. In particular, the corresponding behavior to $\alpha = 1/2$ can be realized by some self-similar solutions and stationary solutions for different physical systems such as Euler equations with damping, Navier-Stokes-Poisson or Euler-Poisson equations for gaseous stars [27, 28, 44, 64]. This motivates the following definition: A vacuum boundary Γ is called *physical* if the acceleration is bounded from below and above, namely

$$-\infty < \frac{\partial c^2}{\partial n} < 0 \tag{1.3}$$

in a small neighborhood of the boundary, where n is the outward unit normal to Γ. Despite its physical importance, the local existence theory of smooth solutions featuring the physical vacuum boundary even for one-dimensional flows was established only recently. This is because if the physical vacuum boundary condition (1.3) is assumed, the classical theory of hyperbolic systems can not be applied [44, 64]: the characteristic speeds $u \pm c$ become singular with infinite spatial derivatives at the vacuum boundary and this singularity creates an analytical difficulty. The first rigorous result regarding the physical vacuum was given by the authors [29] for one-dimensional Euler equations in mass Lagrangian coordinates based on extracting a new structure lying upon the physical vacuum. Recently Coutand and Shkoller [12] constructed more regular solutions in Lagrangian coordinates. For multi-dimensional case, Coutand, Lindblad and Shkoller [14] have a priori estimates by assuming smoothness of solutions. More recently, Coutand and Shkoller [13] extended their 1D methodology to construct solutions in 3D with physical vacuum and independently of this work, the authors [30] have established the well-posedness of 3D case. The methods are very different.

For $0 < \alpha < 1/2$ or $1/2 < \alpha < 1$, the corresponding boundary behavior is believed to be ill-posed and indeed, we think that it should instantaneously change into the physical vacuum. However, there is no mathematical justification available so far.

The case $\alpha = 0$ is when there is no continuous initial contact of the density with vacuum. It can be considered as either Cauchy problem or free boundary problem. An example of Cauchy problem when $\alpha = 0$ is the Riemann problem for genuinely discontinuous initial datum. For instance, see [24], which was brought up by Hunter [1] and Bouchut [7]. An example of a free boundary problem when $\alpha = 0$ is the work by Lindblad [36] where the density is positive at the vacuum boundary.

The paper proceeds as follows. In Section 2, we discuss the well-posedness of vacuum free boundary problems in detail. In Section 3, the comparison between Euler equations with damping and porus media equations and some open problems are presented. In Section 4, we present some analogues of vacuum singularity in other areas and further discuss other research directions.

2. Euler equations with vacuum free boundary.

2.1. Mass Lagrangian formulation in 1D flows and $\alpha \geq 1$. Let the initial density $\rho_0(0,x)$ for $a \leq x \leq b$ be given so that $\rho_0(a) = 0$ and $\rho_0(x) > 0$ for $a < x \leq b$. Let $a(t)$ be the particle path from $x = a$. We seek $\rho(t,x)$, $u(t,x)$, and $a(t)$ for $t \in [0,T]$, $T > 0$ and $x \in [a(t), b]$ so that for such t and x, $\rho(t,x)$ and $u(t,x)$ satisfy Euler equations (1.1) with boundary conditions

$$\rho(t,a(t)) = 0\,,\ u(t,b) = 0\,,\ 0 < \frac{\partial}{\partial x}\rho^{\frac{\gamma-1}{2\alpha}}\Big|_{x=a(t)} < \infty\,.$$

In the following, we present the work of Liu and Yang [43] which concerns the well-posedness of the boundary condition of the case $\alpha \geq 1$. For one-dimensional Euler equations, there is a natural transformation which fixes the free boundary, mass Lagrangian coordinates:

$$y \equiv \int_{a(t)}^{x} \rho(t,z)dz, \quad x \in [a(t), b].$$

Note that the vacuum free boundary $x = a(t)$ corresponds to $y = 0$, and $x = b$ to $y = M$ where M is the total mass of the gas, and thus both boundaries are fixed in (t,y). The Euler equations (1.1) can be rewritten as a symmetric hyperbolic system

$$\phi_t + \mu u_y = 0\,;\ \ u_t + \mu\phi_y = 0 \tag{2.1}$$

in the variables $\phi \sim \rho^{(\gamma-1)/2}$ and $\mu \sim \rho^{(\gamma+1)/2}$. Since $\phi \sim y^{\alpha(\gamma-1)/(2\alpha+\gamma-1)}$ vanishes algebraically at $y = 0$, one can try to find another change of variables so that (2.1) reduces to the system in non-vanishing unknown and the corresponding propagation speed becomes smooth in a new variable. To this end, let $z \equiv y^q$, $\phi \equiv z^\beta \eta$ and $u \equiv C + z^\beta \zeta$ where C is a given constant, $\beta = \alpha(\gamma-1)/q(2\alpha+\gamma-1)$ and q is a constant to be determined. Now the interest is when $\eta > 0$. Write (2.1) for η, ζ:

$$\begin{aligned} &\eta_t + dz^{(\alpha-1)(\gamma-1)/q(2\alpha+\gamma-1)}\eta^{(\gamma+1)/(\gamma-1)}(z\zeta_z + \beta\zeta) = 0 \\ &\zeta_t + dz^{(\alpha-1)(\gamma-1)/q(2\alpha+\gamma-1)}\eta^{(\gamma+1)/(\gamma-1)}(z\eta_z + \beta\eta) = 0 \end{aligned} \tag{2.2}$$

where d is a constant depending on q, γ. We now state the result in [43].

THEOREM 2.1. *Let $\alpha \geq 1$. There exist solutions for the system* (2.1) *of the following form locally in time $\phi(t,y) = y^{\alpha(\gamma-1)/(2\alpha+\gamma-1)}\eta$, $u(t,y) = C + y^{\alpha(\gamma-1)/(2\alpha+\gamma-1)}\zeta$, where $\eta > 0$, ζ satisfies* (2.2).

Here the point is that for $\alpha \geq 1$, one can choose a positive constant q so that the propagation speed $z^{(\alpha-1)(\gamma-1)/q(2\alpha+\gamma-1)+1}$ of (2.2) is smooth, indeed either z or z^2 and thus Kato's theorem [31] on a symmetric quasi-linear hyperbolic system for η, ζ can be applied. However, this method is not applied to other singular cases $0 < \alpha < 1$.

2.2. Well-posedness of physical vacuum. We fix $\alpha = 1/2$. Let us start with (2.1). Note that the degeneracy for the physical singularity is given by $\phi \sim y^{(\gamma-1)/2\gamma}$ and $\mu \sim y^{(\gamma+1)/2\gamma}$. In order to get around this degeneracy, the following change of variables was introduced by Liu and Yang [44]:

$$\xi \equiv \frac{2\gamma}{\gamma-1} y^{\frac{\gamma-1}{2\gamma}} \text{ such that } \partial_y = y^{-\frac{\gamma+1}{2\gamma}}\partial_\xi \, .$$

After normalizing A and M, the equations (2.1) take the form:

$$\phi_t + (\frac{\phi}{\xi})^{2k} u_\xi = 0\,; \quad u_t + (\frac{\phi}{\xi})^{2k}\phi_\xi = 0\,, \quad k \equiv \frac{1}{2}\frac{\gamma+1}{\gamma-1} \tag{2.3}$$

for $t \geq 0$ and $0 \leq \xi \leq 1$. The physical singularity condition (1.3) is written as $0 < |\phi_\xi| < \infty$. Thus we expect $\phi \sim \xi$ for a short time near 0. The propagation speed is now non-degenerate. However, its behavior is different in the interior and on the boundary, which makes it hard to apply any standard energy method to construct solutions in the current formulation.

In [29], we proposed a new formulation to (2.3) so that some energy estimates can be closed in the appropriate energy space. As a preparation, first define the operators V and V^* associated to (2.3) as follows:

$$V(f) \equiv \frac{1}{\xi^k}\partial_\xi \left[\frac{\phi^{2k}}{\xi^k} f\right], \quad V^*(g) \equiv -\frac{\phi^{2k}}{\xi^k}\partial_\xi \left[\frac{1}{\xi^k} g\right] \tag{2.4}$$

for $f,\ g \in L^2_\xi$. In terms of V and V^*, the Euler equations (2.3) can be rewritten as follows:

$$\partial_t(\xi^k\phi) - V^*(\xi^k u) = 0\,;\ \ \partial_t(\xi^k u) + V(\xi^k\phi)/(2k+1) = 0 \tag{2.5}$$

with the boundary conditions $\phi(t,0)=0$ and $u(t,1)=0$. Equations (2.5) look like a symmetric linear hyperbolic system with respect to nonlinear operators V, V^*. Define the associated energy

$$\mathcal{E}(\phi,u) \equiv \sum_{i=0}^{\lceil k\rceil+3} \int |(V)^i(\xi^k\phi)|^2 + |(V^*)^i(\xi^k u)|^2 d\xi$$

where $\lceil k\rceil = \min\{n\in\mathbb{Z} : k\le n\}$. The study of V, V^* operators and V, V^* energy estimates, which require Hardy-type inequalities, leads to the well-posedness of physical vacuum [29].

Theorem 2.2. *Fix k, $1/2<k<\infty$. Suppose given initial data ϕ_0 and u_0 have finite energy $\mathcal{E}(\phi_0,u_0)<\infty$ and ϕ_0 satisfies the physical vacuum singularity condition: $\phi_0/\xi\sim 1$ near $\xi\sim 0$. Then there exist a time $T>0$ only depending on the initial data, and a unique solution (ϕ,u) to the reformulated Euler equations* (2.5) *on the time interval* $[0,T]$ *so that $\mathcal{E}(\phi,u)<\infty$ and $\phi/\xi\sim 1$ near $\xi\sim 0$.*

We remark that the above energy $\mathcal{E}$ is designed to guarantee such *minimal* regularity that $\frac{\phi}{\xi}$ and ϕ_ξ are bounded away from zero and from above, and continuous. In particular, since $\frac{\partial c^2}{\partial x} = (\frac{\phi}{\xi})^{2k}\phi_\xi$, up to constant, the solution constructed in Theorem 2.2 satisfies the physical vacuum boundary condition (1.3). The energy $\mathcal{E}$ is a ρ-weighted energy in Eulerian coordinates where the weights and the derivatives are carefully combined and the smaller γ, the more derivatives are needed. And the energy may be finite for initial data which is regular inside but not very regular at the boundary. The evolution of the vacuum boundary $x=a(t)$ is given by $\dot a(t) = u(t,\xi=0)$. By Theorem 2.2, we deduce that the vacuum interface is well-defined.

2.3. Lagrangian formulation and physical vacuum. Parallel to the recent progress in free surface boundary problems with geometry involved (see for instance [2, 3, 4, 11, 60, 67]), one-dimensional result is expected to be generalized to multi-dimensional case, since the difficulty of the physical singularity lies in how the solution behaves with respect to the normal direction to the boundary. Indeed, by assuming to have smooth solutions in hand, Coutand, Shkoller and Lindblad [14] have a priori estimates for 3D compressible Euler equations for $\gamma=2$. In this section, we discuss their result established in real Lagrangian coordinates. This Lagrangian approach leads to the study of Lagrangian velocity and flow map only, since the density is constructed from the initial datum and the Jacobian determinant of the deformation gradient.

Let $\eta(t,x)$ be the position of the gas particle x at time t so that

$$\eta_t = u(t,\eta(t,x)) \text{ for } t>0 \text{ and } \eta(0,x) = x \text{ in } \Omega\,.$$

Defining the following Lagrangian quantities

$$v(t,x) \equiv u(t,\eta(t,x))\,,\ f(t,x) \equiv \rho(t,\eta(t,x))\,,$$
$$A \equiv [D\eta]^{-1}\,,\ J \equiv \det D\eta\,,\ a \equiv JA\,,$$

and using Einstein's summation convention and the notation $F,_k$ to denote the kth-partial derivative of F, Euler equations (1.1) for $\gamma = 2$ read as follows:

$$f_t + fA^j_i v^i,_j = 0\,;\ \ fv^i_t + A^k_i f^2,_k = 0\,. \tag{2.6}$$

Since $J_t = JA^j_i v^i,_j$ and $J(0) = 1$, together the equation for f, we find that $fJ = \rho_0$ where ρ_0 is given initial density function. Thus, using $A^k_i = J^{-1}a^k_i$, (2.6) reduce to the following:

$$\rho_0 v^i_t + a^k_i({\rho_0}^2 J^{-2}),_k = 0. \tag{2.7}$$

For simplicity, $\Omega = \mathbb{T}^2 \times (0,1)$, $\Gamma(0) = \{x_3 = 1\}$ as the reference vacuum boundary and $\rho_0 = 1 - x_3$ are considered. The moving vacuum boundary is given by $\Gamma(t) = \eta(t)(\Gamma(0))$. Define the higher-order energy function

$$E(t) = \sum_{i=0}^{4} \|\partial_t^{2i}\eta(t)\|^2_{4-i} + \sum_{i=0}^{4}\left\{\|\rho_0\overline{\partial}^{4-i}\partial_t^{2i}D\eta(t)\|^2_0 + \|\sqrt{\rho_0}\overline{\partial}^{4-i}\partial_t^{2i}v(t)\|^2_0\right\}$$
$$+\sum_{i=0}^{3}\|\rho_0\partial_t^{2i}J^{-2}(t)\|^2_{4-i} + \|\mathrm{curl}_\eta v(t)\|^2_3 + \|\rho_0\overline{\partial}^4\mathrm{curl}_\eta v(t)\|^2_0$$

where $\overline{\partial} = (\partial_{x_1}, \partial_{x_2})$ tangential derivatives. We now state the result [14].

THEOREM 2.3. *Suppose $\eta(t)$ is a smooth solution of* (2.7). *Then there exists a sufficiently small $T_0 > 0$ depending on $E(0)$ such that for $0 < T < T_0$, the energy function $E(t)$ constructed from $\eta(t)$ satisfies the a priori estimate* $\sup_{t\in[0,T]} E(t) \le M_0$ *where M_0 depends on $E(0)$.*

The proof consists of a few different key ingredients such as weighted Sobolev embedding, curl estimates, *time differentiated* energy estimates, and elliptic estimates for normal derivatives. The additional estimates for normal derivatives are obtained by inverting time derivatives via the equation. As noted in [14], in order to do so legally, sufficient smoothness of solutions was assumed and its justification by their energy should require additional work.

Recently Coutand and Shkoller [12] constructed H^2-type solutions for one-dimensional Euler equations based on the a priori estimates given in Theorem 2.3 and Hardy inequalities by degenerate parabolic regularization. This work provides an answer to the regularity question; these solutions are smoother than the solutions constructed in [29], of course, with smoother initial data. Due to the regularized approximations, the initial data need to have more regularity than the solutions to guarantee the uniqueness.

More recently, there are many works trying to prove local existence in 3D: Coutand and Shkoller [13] have a way of constructing solutions in 3D with the same method used in 1D case. Also, Lindblad [37] has a similar result using the linearized compressible Euler equations with a Nash-Moser iteration. Independently of these works, the authors [30] have constructed solutions to 3D compressible Euler equations in Lagrangian coordinates by a new analysis of physical vacuum. We briefly discuss the analysis of [30] for general γ.

Let $w \equiv K\rho_0^{\gamma-1}$ satisfying (1.3). For instance, one can take $w = x_3(1-x_3)$ for the domain $\Omega = \mathbb{T}^2 \times (0,1)$. Note that the equations (1.1) or (2.7) can be viewed as a degenerate nonlinear acoustic equation for η:

$$w^\alpha \eta^i_{tt} + (w^{1+\alpha}\, A^k_i J^{-1/\alpha})_{,k} = 0 \;\text{ and }\; \eta^i_t = v^i \tag{2.8}$$

where $\alpha \equiv 1/(\gamma-1)$. Define the instant energy and the total energy:

$$\mathcal{E}^N \equiv \sum_{m+n=0}^{N} \frac{1}{2}\int_\Omega w^{\alpha+n}|\overline{\partial}^m \partial_3^n v|^2 dx + \frac{1}{2}\int_\Omega w^{1+\alpha+n} J^{-1/\alpha}|D_\eta \overline{\partial}^m \partial_3^n \eta|^2 dx$$

$$\mathcal{TE}^N \equiv \mathcal{E}^N + \sum_{m+n=1}^{N} \frac{1}{2}\int_\Omega w^{1+\alpha+n} J^{-1/\alpha}|\mathrm{curl}_\eta \overline{\partial}^m \partial_3^n v|^2 dx.$$

Theorem 2.4 ([30]). *Let $\alpha > 0$ be fixed and $N \geq 2[\alpha]+9$ be given. Let $\mathcal{TE}^N(0) < \infty$. Then there exist a time $T > 0$ depending only on $\mathcal{TE}^N(0)$ and a unique solution (η, v) to the Euler equation (2.8) on the time interval $[0,T]$ satisfying $\mathcal{TE}^N(\eta,v) \leq 2\mathcal{TE}^N(0)$ and $\|A-I\|_\infty \leq 1/8$.*

The proof is based on a hyperbolic type of new energy estimates which consist in the instant energy estimates and the curl estimates. The new key is to extract *right algebraic weighted structure* for the linearization in the normal direction such that one can directly estimate normal derivatives via the energy estimates and thus the energy estimates provide a unified, systematic way of treating all the spatial derivatives. The solutions are constructed by the duality argument and the degenerate elliptic regularity.

Theorem 2.4 indicates that the minimal number of derivatives needed to capture the physical vacuum (1.3) depends on γ as captured in the 1D result [29]. Besides boundary geometry, a critical difference between 1D case and the new analysis is that while in V, V^* framework, the energy space is nonlinear and the number of V, V^* is rigid, the energy spaces given in the above are indeed equivalent to the standard linear weighted Sobolev spaces and also the higher regularity can be readily established.

3. Open problems.

3.1. Long time behavior with or without damping. Having the local existence theory of vacuum states, the next important question is whether such a local solution exists globally in time or how it breaks down. We note that study of vacuum free boundary problems automatically excludes the breakdown of solutions caused by vacuum, which is one possible senario for positive solutions to compressible Euler equations (1.1). It was shown in [41] that the shock waves vanish at the vacuum and the singular behavior is similar to the behavior of the centered rarefaction waves corresponding to the case when c is regular [44], which indicates that vacuum has a regularizing effect. Therefore it would be interesting to investigate the long time behavior of vacuum states.

When there is damping, based on self-similar behavior, Liu conjectured [40] that time asymptotically, solutions to Euler equations with damping

$$\rho_t + (\rho u)_x = 0\,; \quad \rho(u_t + u u_x) + (A\rho^\gamma)_x = -\rho u$$

should behave like, via Darcy's law, the ones to the porus media equation:

$$\rho_t - (A\rho^\gamma)_{xx} = 0 \tag{3.1}$$

where the canonical boundary is characterized by the physical vacuum condition (1.3). The statement on the canonical boundary for porus media equations will be clear in Section 3.2. This conjecture was established by Huang, Marcati and Pan [26] in the framework of the entropy solution where the method of compensated compactness yields a global weak solution in L^∞. But in their work, there is no way of tracking the vacuum boundary. Thus it would be interesting to investigate the asymptotic relationship between regular solutions of physical vacuum and regular solutions to the porus media equation. Of course, prior to it, one should have global solutions in hand.

3.2. Ill-posedness and change of behavior. We now go back to other vacuum states of compressible Euler equations: $c \sim x^\alpha$ for $0 < \alpha < 1/2$ or $1/2 < \alpha < 1$ described in the introduction. Recall that the physical vacuum corresponds to $\alpha = 1/2$. The question is whether the boundary condition $c \sim x^\alpha$ is well-posed or not for Euler equations. Indeed, for such a fixed α, if assuming that it were well-posed for a short time and tracking the behavior $c \sim x^\alpha$ near vacuum boundary through both equations in (2.3) within that time, one can see that always the more singular mode will be created. The conjecture is that the behavior should instantaneously change into the physical vacuum, namely from $\alpha \in (0, 1/2) \cup (1/2, 1)$ to $\alpha = 1/2$, but its mathematical justification is an open problem.

In Section 3.1, we discussed some connection between Euler equations with damping and porus media equations. For the porus media equations (3.1), which are nonlinear degenerate parabolic equations, the initial boundary value problem including the behavior of solutions and the regularity of the vacuum free boundary has been well studied. In particular, in the work of Knerr, Cafarelli and Friedman [32, 8], it was shown that if the initial data $\rho_0^{\gamma-1} \leq Cx^2$, then there exists a waiting time $t^* > 0$ such that the boundary starts moving after this waiting time and $(\rho^{\gamma-1})_x$ is bounded away from zero and infinity for $t > t^*$; on the other hand, if the initial data $\rho_0^{\gamma-1} \geq Cx^{2\alpha}$ where $0 < \alpha < 1$, the boundary moves instantaneously and $(\rho^{\gamma-1})_x$ is bounded away from zero and infinity for $t > 0$. We note that the canonical boundary behavior $\rho^{\gamma-1} \sim x$ for the porus media equations corresponds to the physical vacuum $c^2 \sim x$ for Euler equations. One expects to have this kind of waiting time behavior or instantaneous change of behavior for Euler equations with damping, but again it is an open question.

3.3. Relativistic fluids. The relativistic Euler equations for a perfect fluid in four-dimensional Minkowski spacetime are given by

$$\begin{aligned} \partial_t\left(\frac{\rho+\varepsilon^2 p}{1-\varepsilon^2u^2} - \varepsilon^2 p\right) + \partial_x\left(\frac{\rho+\varepsilon^2 p}{1-\varepsilon^2u^2}u\right) &= 0\,, \\ \partial_t\left(\frac{\rho+\varepsilon^2 p}{1-\varepsilon^2u^2}u\right) + \partial_x\left(\frac{\rho+\varepsilon^2 p}{1-\varepsilon^2u^2}u^2 + p\right) &= 0\,, \end{aligned}$$

where the parameter $1/\varepsilon$ represents the speed of light. The range of physical interest is $\rho \geq 0$, $|u| < 1/\varepsilon$, and $c < 1/\varepsilon$. Letting $\varepsilon \to 0$, non-relativistic Euler equations (1.1) are formally recovered. For the physical background, see [23, 34, 49, 50, 57, 58] and the references therein.

The relativistic Euler equations is known to be symmetric hyperbolic in the so-called entropy variables introduced by Makino and Ukai [49, 50] away from vacuum. A particular interest is compactly supported relativistic flows which for instance can be applied to the dynamics of stars in the context of special relativity. LeFloch and Ukai [34] proposed a new symmetrization, which makes sense for solutions containing vacuum states and generalizes the theory in [51] for non-relativistic Euler equations as Cauchy problem. Whether one can extend the theory of free boundary problems including physical vacuum developed for non-relativistic Euler equations to relativistic case is an open problem.

4. Further discussions.

4.1. Viscous flows and vacuum. Compressible Navier-Stokes equations

$$\begin{aligned} \partial_t\rho + \mathrm{div}(\rho u) &= 0\,, \\ (\rho u)_t + \mathrm{div}(\rho u\otimes u) + \nabla p &= \mathrm{div}(\mu[\nabla u + \nabla u^T]) + \nabla(\delta \mathrm{div} u)\,, \end{aligned}$$

where the viscosity coefficients are assumed to satisfy $\mu \geq 0$ and $2\mu/d + \delta \geq 0$, describe the dynamics of viscous gases and fluids. Often $\mathrm{div}(\mu\nabla u)$ is used as the viscosity. There is huge literature on the studies of the existence and the behavior of solutions; we will not attempt to address exhaustive references: for classical works, see [18, 38]. One of the major difficulties when trying to prove global existence of weak solutions and strong regularity results is the possible appearance of vacuum.

When vacuum is allowed to appear initially, studying Cauchy problems for compressible Navier-Stokes equations with constant viscosity coefficients yields somewhat negative results: for instance, a finite time blow-up for nontrivial compactly supported initial density [63] and a failure of continuous dependence on initial data [25]. There are some existence theories available with the physical vacuum boundary: the vacuum interface behavior as well as the regularity to one-dimensional Navier-Stokes free boundary problems were investigated in [45]. And the local-in-time well-posedness of Navier-Stokes-Poisson equations in three dimensions with radial symmetry featuring the physical vacuum boundary was established in [28]. Global existence of such strong solutions is an open problem. See also [17, 53, 56] for related models. On the other hand, to resolve the issue of no continuous dependence on initial data in [25] for constant viscosity coefficient, a density-dependent viscosity coefficient was introduced in [42]. Since then, there has been a lot of studies on global weak solutions for various models and stabilization results under gravitation and external forces: see [35, 54, 65, 66] and the references therein. Despite the significant progress over the years, many interesting and important questions are still unanswered especially for general multi-dimensional flows.

4.2. Magnetohydrodynamics and vacuum. The theory of magnetohydrodynamics (MHD), another interesting system of hyperbolic conservation laws arising in electromechanical phenomena, describes the interaction of a magnetic field with an electrically conducting thermoelastic fluid. The equations, also called Lundquist's equations, read as follows:

$$
\begin{aligned}
&\partial_t \rho + \mathrm{div}(\rho u) = 0\,,\\
&(\rho u)_t + \mathrm{div}(\rho u \otimes u - H \otimes H) + \mathrm{grad}\left(p + \frac{|H|^2}{2}\right) = 0\,,\\
&\partial_t H - \mathrm{curl}(u \times H) = 0\,, \quad \mathrm{div} H = 0\,,
\end{aligned}
$$

where H is the magnetic field and the electric field is given by $E = H \times u$. See [15, 21] for more physical background. Vacuum states of MHD can be realized by seeking equilibria for the system and for example, static axisymmetric equilibria are obtained by solving the Grad-Shafranov equation which is a nonlinear elliptic partial differential equation [6]. Due to the interplay between the scalar pressure of the fluid and the anisotropic magnetic stress, vacuum states are richer than in hydrodynamics and their rigorous study in the context of nonlinear partial differential equations seems to be widely open.

4.3. Degenerate elliptic equations. One of the main difficulty of studying the Euler system with a free boundary is that it leads to a degenerate hyperbolic equation due to the fact that the density vanishes at the free boundary. For this we need techniques coming from elliptic degenerate equations (see for instance Baouendi [5]). Also, similar problems arise in degenerate parabolic equations (in porous medium equations [8], in thin film equations [20], in the study of polymeric flows [52] and so on).

4.4. Validity of vacuum in hydrodynamics and kinetic theory. The following Von Neumann's comment on vacuum of hydrodynamics in 1949 [62] was brought up and read by Serre at the meeting [1].

There is a further difficulty in the expansion case considered by Burgers. It was accepted that the front advances into a vacuum. It is evident that you cannot get the normal conditions of kinetic theory here either, because the density of the gas goes to zero at the front, which means that the mean free path of the molecules will go to infinity. This means that if we are in the expanding gas and approach the (theoretical) front, we will necessarily come to regions where the mean free path is larger than the distance from the front. In such regions one cannot use the hydrodynamical equations. But, as in the case of the shock wave, where ordinary conditions are reached at a distance of a few mean free paths from the shock itself, so in the case of expansion into a vacuum, at a short distance from the theoretical front, one comes into regions where the mean free path is considerably smaller than the distance from the front, and where again the classical hydrodynamical equations can be applied. If this is applied to expanding interstellar clouds, I think that in order that the classical theory be true down to 1/1000 of the density of the clouds, it is necessary that the distance from the theoretical front should be of the order of a percent of a parsec.

As to the issue of validity of vacuum in hydrodynamics, Serre [1] suggested to consider ρ^ε problem for Euler equations (1.1) where $\inf \rho^\varepsilon \geq \varepsilon$ and to study the limit $\varepsilon \to 0$ to test the stability of vacuum.

The above comment also suggests the importance of the modeling of the vacuum boundary. It suggests the existence of a layer where we need to use kinetic theory, in particular Boltzmann equations. This poses an other question, namely the boundary condition between the region where the kinetic theory is necessary and the region where hydrodynamic equations are used. This study would not be only important for the mathematical theory of vacuum but it can also be applicable to other physical problems for instance the modeling of stellar structure by using both kinetic equations and hydrodynamical equations in rarefied gas region and vacuum region.

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