Chapter 5
Zero Sets for Fock Spaces

In this chapter, we study zero sets for the Fock spaces F^p_α. Throughout this book, we say that a sequence $Z = \{z_n\} \subset \Omega$ is a zero set for a space X of analytic functions in Ω if there exists a function $f \in X$, not identically zero, such that Z is *exactly* the zero sequence of f, counting multiplicities.

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5.1 A Necessary Condition

Recall from Theorem 2.12 that every function $f \in F^p_\alpha$ is of order 2. Therefore, by Hadamard's factorization theorem, the zero sequence $\{z_n\}$ of f, with the origin removed, must satisfy

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^3} < \infty.$$

In this section, we improve upon this estimate and obtain the following necessary condition for a sequence $\{z_n\}$ to be a zero set for F^p_α.

Theorem 5.1. *Suppose* $0 < p \le \infty$ *and* $\{z_n\}$ *is the zero sequence of a function* $f \in F^p_\alpha$ *with* $f(0) \ne 0$. *Then there exist a positive constant* c *and a rearrangement of* $\{z_n\}$ *such that* $|z_n| \ge c\sqrt{n}$ *for all* n.

Proof. Without loss of generality, we may assume that $f(0) = 1$ and $p = \infty$. Let $\{z_n\}$ denote the zero sequence of f, repeated according to multiplicity and arranged so that $0 < |z_1| \le |z_2| \le |z_3| \le \cdots$.

Fix any positive radius r such that f has no zero on $|z| = r$ and let $n(r)$ denote the number of zeros of f in $|z| < r$. By Jensen's formula,

$$\sum_{k=1}^{n(r)} \log \frac{r}{|z_k|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta.$$

Since $f \in F^\infty_\alpha$, we have

$$|f(re^{i\theta})| \le \|f\|_{\infty,\alpha} e^{\frac{\alpha}{2} r^2}, \qquad 0 \le \theta \le 2\pi, r > 0.$$

It follows that

$$\sum_{k=1}^{n(r)} \log \frac{r}{|z_k|} \le \frac{\alpha}{2} r^2 + C,$$

where $C = \log \|f\|_{\infty,\alpha}$. Rewrite the above inequality as

$$\prod_{k=1}^{n(r)} \frac{r}{|z_k|} \le \exp\left(\frac{\alpha}{2} r^2 + C\right)$$

and observe that

$$\prod_{k=1}^{n} \frac{r}{|z_k|} \le \prod_{k=1}^{n(r)} \frac{r}{|z_k|}$$

for any positive integer n (independent of r). Then

$$\prod_{k=1}^{n} \frac{r}{|z_k|} \leq \exp\left(\frac{\alpha}{2}r^2 + C\right)$$

for all positive integers n and all $r > 0$ such that f has no zero on $|z| = r$. Since $\{|z_k|\}$ is nondecreasing, we have

$$\frac{r^n}{|z_n|^n} \leq \exp\left(\frac{\alpha}{2}r^2 + C\right),$$

or

$$\frac{1}{|z_n|} \leq \frac{1}{r}\exp\left(\frac{\alpha}{2n}r^2 + \frac{C}{n}\right), \tag{5.1}$$

where n is any positive integer and r is any radius such that f has no zero on $|z| = r$.

There are only a countable number of radius r such that f has zeros on $|z| = r$. Therefore, for any positive integer n, we can choose a sequence $\{r_k\}$ such that $r_k \to \sqrt{n}$ as $k \to \infty$ and f has no zero on each $|z| = r_n$. Combining this with (5.1), we conclude that

$$\frac{1}{|z_n|} \leq \frac{1}{\sqrt{n}}\exp\left(\frac{\alpha}{2} + \frac{\log\|f\|_{\infty,\alpha}}{n}\right), \qquad n \geq 1.$$

It is then clear that there is some positive constant c such that $|z_n| \geq c\sqrt{n}$ for all $n \geq 1$. □

Note that the assumption $f(0) \neq 0$ is not a critical one. In fact, if $f \in F^p_\alpha$ and it has a zero of order m at the origin, then the function g defined by $g(z) = f(z)/z^m$ is in F^p_α and does not vanish at the origin.

Corollary 5.2. *Suppose* $0 < p \leq \infty$ *and* $\{z_n\}$ *is the zero sequence of some* $f \in F^p_\alpha$ *with* $f(0) \neq 0$. *Then*

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^r} < \infty$$

for every $r > 2$.

The function

$$f(z) = \frac{\sin(\delta z^2)}{\delta z^2}$$

used in the proof of Theorem 5.4 shows that the estimate in Theorem 5.1 is best possible. More specifically, we can find a positive constant C in this case such that

$$C^{-1}\sqrt{n} \leq |z_n| \leq C\sqrt{n}$$

for all $n \geq 1$.

5.2 A Sufficient Condition

The purpose of this section is to prove the following sufficient condition for zero sequences of F^p_α.

Theorem 5.3. *Suppose that $\{z_n\}$ is a sequence of complex numbers such that*

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^2} < \infty. \tag{5.2}$$

Then $\{z_n\}$ is a zero set for F^p_α, where $0 < p \le \infty$.

Proof. Suppose that $\{z_n\}$ satisfies condition (5.2). We may also assume that the sequence $\{z_n\}$ has been ordered in such a way that $\{|z_n|\}$ is nondecreasing. Consider the Weierstrass product

$$f(z) = \prod_{n=1}^{\infty} E_1\left(\frac{z}{z_n}\right),$$

where $E_1(z) = (1-z)e^z$. By Theorem 1.6, f is entire, and $\{z_n\}$ is the zero sequence of f. We will show that this function f belongs to all the Fock spaces F^p_α, where $0 < p \le \infty$ and $\alpha > 0$.

If $|z| < 1/2$, we have

$$\begin{aligned}
\log|E_1(z)| &= \text{Re}\,[\log(1-z)+z] \\
&= \text{Re}\left[-\frac{z^2}{2} - \frac{z^3}{3} - \frac{|z|^4}{4} - \cdots\right] \\
&\le |z|^2\left[\frac{1}{2} + \frac{|z|}{3} + \frac{|z|^2}{4} + \cdots\right] \\
&\le \frac{1}{2}|z|^2\left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right] \\
&= |z|^2.
\end{aligned}$$

On the other hand, we have

$$|E_1(z)| \le (1+|z|)e^{|z|}, \qquad \log|E_1(z)| \le |z| + \log(1+|z|), \tag{5.3}$$

for all z. It follows that for any positive A, there exists a positive number R such that

$$\log|E_1(z)| \le A|z|^2, \qquad |z| > R.$$

On the annulus $1/2 \le |z| \le R$, the function $|z|^2 \log|E_1(z)|$ is continuous except at $z = 1$, where it tends to $-\infty$. Hence, there is a constant B such that

$$\log|E_1(z)| \le B|z|^2, \qquad \frac{1}{2} \le |z| \le R.$$

Combining the estimates from the last three paragraphs, we conclude that

$$\log|E_1(z)| \le M|z|^2, \qquad z \in \mathbb{C},$$

where $M = \max(1, A, B)$.

Given any positive ε, we can find a positive integer N such that

$$\sum_{n=N+1}^{\infty} \frac{1}{|z_n|^2} < \frac{\varepsilon}{2M}.$$

From this, we deduce that

$$\sum_{n=N+1}^{\infty} \log|E_1(z/z_n)| \le M \sum_{n=N+1}^{\infty} \left|\frac{z}{z_n}\right|^2 \le \frac{\varepsilon}{2}|z|^2$$

for all $z \in \mathbb{C}$. Using (5.3) again, we can find some $r_1 > 0$ such that

$$\log|E_1(z)| \le \frac{\varepsilon}{2S}|z|^2, \qquad |z| > r_1,$$

where

$$S = \sum_{n=1}^{N} \frac{1}{|z_n|^2}.$$

Set $r_2 = r_1|z_N|$. Then $|z| > r_2$ implies that $|z/z_n| > r_1$ for $1 \le n \le N$. It follows that

$$\sum_{n=1}^{N} \log|E_1(z/z_n)| \le \frac{\varepsilon}{2}|z|^2, \qquad |z| > r_2.$$

Therefore,

$$\log|f(z)| = \sum_{n=1}^{\infty} \log|E_1(z/z_n)| < \varepsilon|z|^2$$

for all $|z| > r_2$, or $|f(z)| < e^{\varepsilon|z|^2}$ for all $|z| > r_2$. Since ε is arbitrary, we see that $f \in F^p_\alpha$ for all $\alpha > 0$ and $0 < p \le \infty$. □

Note that the proof above can easily be adapted to show that the function $P(z)f(z)$ belongs to F^p_α for any polynomial $P(z)$. Therefore, if $\{z_n\}$ satisfies (5.2), then $\{z_n\} \cup F$ is also a zero set for F^p_α, where F is any finite set. It is permitted to have the origin contained in F.

5.3 Pathological Properties

In this section, we present examples to show certain pathological properties of zero sequences of Fock spaces. More specifically, we will show that:

(i) The union of two zero sequences for F^p_α is not necessarily a zero sequence for F^p_α again.
(ii) A subsequence of a zero sequence for F^p_α is not necessarily a zero sequence for F^p_α again.
(iii) If $\alpha \neq \beta$, then the spaces F^p_α and F^q_β have different zero sequences.
(iv) An interpolating sequence for F^p_α is not necessarily a zero sequence for F^p_α.

Theorem 5.4. *Suppose $\alpha > 0$ and $0 < p \le \infty$. There exist two zero sequences for F^p_α whose union is no longer a zero sequence for F^p_α.*

Proof. Fix $\delta \in (\pi\alpha/8, \alpha/2)$ and consider the sequence

$$Z = \left\{ e^{k\pi i/2} \sqrt{n\pi/\delta} : k = 0,1,2,3; n = 1,2,3,\cdots \right\}.$$

It is easy to see that Z is the zero sequence of the entire function

$$f(z) = \frac{\sin(\delta z^2)}{\delta z^2}.$$

Converting the sine function above to complex exponential functions and using the assumption that $\delta < \alpha/2$, we easily check that $f \in F^p_\alpha$. Therefore, Z is a zero sequence for F^p_α.

Let $Z' = \{e^{\pi i/4} z : z \in Z\}$ be a rotation of the sequence Z above. Then Z' is also an F^p_α zero sequence. Clearly, Z and Z' are disjoint. We now arrange $Z \cup Z'$ into a single sequence $\{z_n\}$ such that

$$|z_1| \le |z_2| \le |z_3| \le \cdots .$$

If $\{z_n\}$ is a zero sequence for $F^p_\alpha \subset F^\infty_\alpha$, it follows from the proof of Theorem 5.1 that there exists a positive constant C such that

$$\prod_{k=1}^{n} \frac{r}{|z_k|} \le C e^{\frac{\alpha}{2} r^2}$$

for all $n \ge 1$ and $r > 0$. Square both sides, replace n by $8n$, and integrate from 0 to ∞ with respect to the measure $re^{-\beta r^2}$, where $\beta > \alpha$. We obtain another positive constant C such that

$$\frac{(8n)!}{\beta^{8n}} \prod_{k=1}^{8n} \frac{1}{|z_k|^2} \le C$$

for all $n \geq 1$. It is easy to see that this reduces to

$$\left(\frac{\delta}{\pi\beta}\right)^{8n} \frac{(8n)!}{(n!)^8} \leq C, \qquad n \geq 1.$$

By Stirling's formula, there exists yet another positive constant C, independent of n, such that

$$\left(\frac{8\delta}{\pi\beta}\right)^{8n} \frac{\sqrt{n}}{n^4} \leq C$$

for all $n \geq 1$. This clearly implies that $8\delta \leq \pi\beta$. Since β can be arbitrarily close to α, we have $\delta \leq \pi\alpha/8$, which is a contradiction. This shows that $\{z_n\}$ is not an F^p_α zero set and completes the proof of the theorem. □

Theorem 5.5. *Let $\alpha > 0$ and $0 < p \leq \infty$. There exists an F^p_α zero sequence $\{z_n\}$ and a subsequence $\{z_{n_k}\}$ which is not an F^p_α zero sequence.*

Proof. Fix a positive constant δ such that $\delta < \alpha/2$ and consider the following entire function:

$$f(z) = \frac{e^{i\delta z^2} - 1}{i\delta z^2}.$$

It is easy to check that $f \in F^p_\alpha$. Thus, its zero set

$$\left\{\pm\sqrt{\frac{2n\pi}{\delta}} : n = 1,2,3,\cdots\right\} \cup \left\{\pm i\sqrt{\frac{2n\pi}{\delta}} : n = 1,2,3,\cdots\right\}$$

is an F^p_α zero sequence. Let $\{z_n\}$ denote the subsequence consisting of real elements in the above set. We proceed to show that $\{z_n\}$ is not an F^p_α zero set.

Again, aiming to arrive at a contradiction later, we assume that g is a function in F^p_α that vanishes precisely on $\{z_n\}$. It is clear that $\rho_1(g) = m(g) = 2$; see Sect. 1.1 for definitions and properties of these numbers. By Theorem 1.10, we always have $\rho(g) \geq \rho_1(g)$, so g must be of order greater than or equal to 2. Combining this with Theorem 2.12, we conclude that g must be of order 2. By Lindelöf's theorem (see Theorem 1.11), the function g must be of maximum (infinite) type since the sums

$$S(r) = \sum_{|z_n| \leq r} \frac{1}{z_n^2} \sim \log r, \quad r > 1,$$

are clearly unbounded. By Theorem 2.12 again, the function g cannot possibly be in F^p_α. This contradiction shows that $\{z_n\}$ is not an F^p_α zero set. □

We now consider zero sets for different Fock spaces. The Weierstrass σ-functions play a significant role here.

Recall that for any positive α,

$$\Lambda_\alpha = \left\{ \omega_{mn} = \sqrt{\frac{\pi}{\alpha}}(m + \mathrm{i}n) : m \in \mathbb{Z}, n \in \mathbb{Z} \right\}$$

is the square lattice in the complex plane with fundamental region

$$\Omega_\alpha = \left\{ z = x + \mathrm{i}y : |x| < \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}, |y| < \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \right\}.$$

The Weierstrass σ-function associated to Λ_α is the following infinite product:

$$\sigma_\alpha(z) = z \prod_{m,n}{}' \left(1 - \frac{z}{\omega_{mn}}\right) \exp\left(\frac{z}{\omega_{mn}} + \frac{1}{2}\frac{z^2}{\omega_{mn}^2}\right),$$

where the product is taken over all integers m and n with $\omega_{mn} \neq 0$.

Lemma 5.6. *Let* $0 < \alpha_1 < \alpha < \alpha_2 < \infty$. *We have:*

(a) $\sigma_\alpha \in F^p_{\alpha_2}$ *for all* $0 < p \leq \infty$.
(b) $\sigma_\alpha \notin F^p_{\alpha_1}$ *for any* $0 < p \leq \infty$.
(c) $\sigma_\alpha \in F^\infty_\alpha$.
(d) $\sigma_\alpha \notin f^\infty_\alpha$, *and so* $\sigma_\alpha \notin F^p_\alpha$ *for any* $0 < p < \infty$.

Proof. It follows from the quasiperiodicity of σ_α that if $z = \omega_{mn} + w$ and $w \in \Omega_\alpha$, then

$$|\sigma_\alpha(z)|\mathrm{e}^{-\frac{\alpha}{2}|z|^2} = |\sigma_\alpha(w)|\mathrm{e}^{-\frac{\alpha}{2}|w|^2}. \tag{5.4}$$

Since the function $|\sigma_\alpha(w)|\mathrm{e}^{-\alpha|w|^2/2}$ is bounded on the relatively compact set Ω_α, there exists a positive constant C such that

$$|\sigma_\alpha(z)| \leq C\mathrm{e}^{\frac{\alpha}{2}|z|^2}, \qquad z \in \mathbb{C}.$$

This clearly implies that $\sigma_\alpha \in F^\infty_\alpha$ and $\sigma_\alpha \in F^p_{\alpha_2}$ for all $0 < p \leq \infty$.

If S is any compact set contained in the fundamental region of Λ_α, then there exists a positive constant δ such that

$$|\sigma_\alpha(w)|\mathrm{e}^{-\frac{\alpha}{2}|w|^2} \geq \delta, \qquad w \in S.$$

This together with (5.4) shows that

$$|\sigma_\alpha(z)|\mathrm{e}^{-\frac{\alpha}{2}|z|^2} \geq \delta, \qquad z \in S + \omega_{mn},$$

for all (m,n). This clearly shows that $\sigma_\alpha \notin f^\infty_\alpha$. Since $F^p_\alpha \subset f^\infty_\alpha$ for $0 < p < \infty$, we have $\sigma_\alpha \notin F^p_\alpha$ for any $0 < p < \infty$. Also, $F^p_{\alpha_1} \subset f^\infty_\alpha$ for all $0 < p \le \infty$. So $\sigma_\alpha \notin F^p_{\alpha_1}$ for all $0 < p \le \infty$. □

Lemma 5.7. *Suppose* $0 < p < \infty$ *and* $f \in F^p_\alpha$. *If* $f(z) = 0$ *for all* $z \in \Lambda_\alpha$, *then* f *is identically zero.*

Proof. By the Weierstrass factorization theorem, we can write $f = h\sigma_\alpha$, where h is an entire function. In view of the quasiperiodicity of σ_α, we have

$$\int_{\mathbb{C}} \left| f(z)\mathrm{e}^{-\frac{\alpha}{2}|z|^2} \right|^p \mathrm{d}A(z) = \sum_{m,n} \int_{\Omega_\alpha} |h(z+\omega_{mn})|^p \left| \sigma_\alpha(z)\mathrm{e}^{-\frac{\alpha}{2}|z|^2} \right|^p \mathrm{d}A(z),$$

where Ω_α is the fundamental region of σ_α. Let D be any small disk centered at 0 and contained in $\frac{1}{2}\Omega_\alpha$. Then by Corollary 1.21, there exists a positive constant C such that

$$\int_{\mathbb{C}} \left| f(z)\mathrm{e}^{-\frac{\alpha}{2}|z|^2} \right|^p \mathrm{d}A(z) \ge C \sum_{m,n} \int_{\Omega_\alpha - D} |h(z+\omega_{mn})|^p \,\mathrm{d}A(z).$$

Since the function $z \mapsto |h(z+\omega_{mn})|^p$ is subharmonic, there exists a positive constant δ (independent of (m,n)) such that

$$\int_{\Omega_\alpha - D} |h(z+\omega_{mn})|^p \,\mathrm{d}A(z) \ge \delta \int_{\Omega_\alpha} |h(z+\omega_{mn})|^p \,\mathrm{d}A(z)$$

for all (m,n). It follows that there is another positive constant C such that

$$\int_{\mathbb{C}} \left| f(z)\mathrm{e}^{-\frac{\alpha}{2}|z|^2} \right|^p \mathrm{d}A(z) \ge C \int_{\mathbb{C}} |h(z)|^p \,\mathrm{d}A(z).$$

This is impossible unless h is identically zero. □

Theorem 5.8. *Suppose* $0 < p \le \infty$, $0 < q \le \infty$, *and* $\alpha_1 \ne \alpha_2$. *Then* $F^p_{\alpha_1}$ *and* $F^q_{\alpha_2}$ *have different zero sets.*

Proof. Without loss of generality, let us assume that $\alpha_1 < \alpha < \alpha_2$. By Lemma 5.6, the Weierstrass function σ_α belongs to $F^q_{\alpha_2}$, so its zero sequence Λ_α is a zero set for $F^q_{\alpha_2}$. On the other hand, if $f \in F^p_{\alpha_1} \subset F^2_\alpha$ and f vanishes on Λ_α, then it follows from Lemma 5.7 that f is identically zero. Therefore, Λ_α cannot possibly be a zero set for $F^p_{\alpha_1}$. □

The remaining question for us now is this: do F^p_α and F^q_α have different zero sets whenever $p \ne q$? As of this writing, there is no complete answer, but it is easy to produce examples of such pairs that do not have the same zero sets. The simplest example is $Z = \Lambda_\alpha$, which is a zero set for F^∞_α, but not a zero set for any F^p_α when $0 < p < \infty$. This again follows from Lemmas 5.6 and 5.7.

Similarly, the sequence $Z = \Lambda_\alpha - \{0\}$ is a zero set for F^p_α when $p > 2$ because the function $f(z) = \sigma_\alpha(z)/z$ belongs to F^p_α if and only if $p > 2$. However, this sequence Z is not a zero set for F^2_α. To see this, suppose f is a function in F^2_α, not identically zero, such that f vanishes on Z. By Weierstrass factorization, we have $f(z) = [\sigma_\alpha(z)/z]g(z)$ for some entire function g that is not identically zero. Mimicking the proof of Lemma 5.7, we can show that

$$\int_{|z|>1} \left| \frac{g(z)}{z} \right|^2 \mathrm{d}A(z) < \infty.$$

It follows from polar coordinates and the Taylor expansion of g that this is impossible unless g is identically zero. This actually shows that $Z = \Lambda_\alpha - \{0\}$ is a uniqueness set for F^2_α. In the above arguments, the point 0 can be replaced by any other point in Λ_α.

On the other hand, if Z is the resulting sequence when two points a and b are removed from Λ_α, then the function

$$f(z) = \frac{\sigma_\alpha(z)}{(z-a)(z-b)}$$

belongs to F^2_α and has Z as its zero sequence. Therefore, Z is a zero set for F^2_α. Consequently, it is possible to go from a uniqueness set to a zero set by removing just one point. Equivalently, it is possible to add just a single point to a zero set of F^2_α so that the resulting sequence becomes a uniqueness set for F^2_α. This shows how delicate the problem of characterizing zero sets for F^p_α is.

We can also show by an example that it is generally very difficult to distinguish between zero sets for F^p_α and F^q_α. More specifically, for any positive integer N with $Np > 2$, if Z is an F^q_α zero set and if N points $\{z_1, \cdots, z_N\}$ are removed from Z, then the remaining sequence Z' is an F^p_α zero set. To see this, let Z be the zero sequence of a function $f \in F^q_\alpha$, not identically zero, then Z' is the zero sequence of the function

$$g(z) = \frac{f(z)}{(z-z_1)\cdots(z-z_N)},$$

which is easily seen to be in F^p_α. Therefore, zero sets for F^p_α and F^q_α may be different, but they are not too much different.

Let Z be a zero sequence for F^p_α and let I_Z denote the set of functions f in F^p_α such that f vanishes on Z. In the classical theories of Hardy and Bergman spaces, the space I_Z is always infinite dimensional. This is no longer true for Fock spaces.

Theorem 5.9. *For any* $0 < p \leq \infty$ *and* $k \in \{1, 2, \cdots\} \cup \{\infty\}$, *there exists a zero set* Z *for* F^p_α *such that* $\dim(I_Z) = k$.

Proof. The case $k = \infty$ is trivial; any finite sequence Z will work. So we assume that k is a positive integer in the rest of the proof.

We first consider the case $p=\infty$ and $k>1$. In this case, we consider $Z=\Lambda_\alpha-\{a_1,\cdots,a_{k-1}\}$, where $a_1,\cdots,a_{k-1}$ are (any) distinct points in Λ_α and

$$f(z)=\frac{\sigma_\alpha(z)}{(z-a_1)\cdots(z-a_{k-1})}.$$

It follows from Corollary 1.21 that $f\in F^\infty_\alpha$ and Z is exactly the zero sequence of f. Furthermore, if h is a polynomial of degree less than or equal to $k-1$, then the function $f(z)h(z)$ is still in F^∞_α.

On the other hand, if F is any function in F^∞_α that vanishes on Z, then we can write

$$F(z)=f(z)g(z)=\frac{\sigma_\alpha(z)g(z)}{(z-a_1)\cdots(z-a_{k-1})},$$

where g is an entire function. For any positive integer n, let C_n be the boundary of the square centered at 0 with horizontal and vertical side length $(2n+1)\sqrt{\pi/\alpha}$. It is clear that

$$d(C_n,\Lambda_\alpha)\geq\sqrt{\pi/\alpha}/2,\qquad n\geq 1.$$

So there exists a positive constant C such that

$$|\sigma_\alpha(z)|e^{-\frac{\alpha}{2}|z|^2}\geq C,\qquad z\in C_n, n\geq 1.$$

This together with the assumption that $F\in F^\infty_\alpha$ implies that there exists another positive constant C such that

$$|g(z)|\leq C|z-a_1|\cdots|z-a_{k-1}| \tag{5.5}$$

for all $z\in C_n$ and $n\geq 1$. By Cauchy's integral estimates, the function g must be a polynomial of degree at most $k-1$.

Therefore, when $p=\infty$, $k>1$, and $Z=\Lambda_\alpha-\{a_1,\cdots,a_{k-1}\}$, we have shown that a function $F\in F^\infty_\alpha$ vanishes on Z if and only if

$$F(z)=\frac{\sigma_\alpha(z)h(z)}{(z-a_1)\cdots(z-a_{k-1})},$$

where h is a polynomial of degree less than or equal to $k-1$. This shows that $\dim(I_Z)=k$.

When $p=\infty$ and $k=1$, we simply take $Z=\Lambda_\alpha$. The arguments above can be simplified to show that a function $F\in F^\infty_\alpha$ vanishes on Z if and only if $F=c\sigma_\alpha$ for some constant c.

Next, we assume that $0 < p < \infty$ and k is a positive integer. In this case, we let N denote the smallest positive integer such that $Np > 2$, or equivalently,

$$\int_{|z|>1} \left| \frac{\sigma_\alpha(z) e^{-\frac{\alpha}{2}|z|^2}}{z^N} \right|^p dA(z) < \infty. \tag{5.6}$$

Remove any $N+k-1$ points $\{a_1, \cdots, a_{N+k-1}\}$ from Λ_α and denote the remaining sequence by Z. Then Z is the zero sequence of the function

$$\frac{\sigma_\alpha(z)}{(z-a_1)\cdots(z-a_{N+k-1})},$$

which belongs to F^p_α in view of (5.6). In fact, if g is any polynomial of degree less than or equal to $k-1$, then it follows from (5.6) that g times the above function belongs to I_Z.

Conversely, if f is any function in F^p_α that vanishes on Z, then we can write

$$f(z) = \frac{\sigma_\alpha(z) g(z)}{(z-a_1)\cdots(z-a_{N+k-1})},$$

where g is an entire function. Since $F^p_\alpha \subset F^\infty_\alpha$, it follows from (5.5) and Cauchy's integral estimates that g is a polynomial with degree less than or equal to $N+k-1$. If the degree of g is $j > k-1$, then

$$\frac{g(z)}{(z-a_1)\cdots(z-a_{N+k-1})} \sim \frac{1}{z^{N+k-1-j}}, \qquad z \to \infty.$$

This together with $f \in F^p_\alpha$ shows that (5.6) still holds when N is replaced by $N+k-1-j$, which contradicts our minimality assumption on N. Thus, $j \le k-1$, which shows that I_Z is k dimensional. □

The following result describes the structure of I_Z when it is finite dimensional.

Theorem 5.10. *Suppose Z is a zero set for F^p_α and $\dim(I_Z) = k$ is a positive integer. Then there exists a function $g \in I_Z$ such that $I_Z = gP_{k-1}$, where P_{k-1} is the set of all polynomials of degree less than or equal to $k-1$.*

Proof. First, observe that if $\dim(I_Z) = k < \infty$, then $Z' = Z \cup \{a_1, \cdots, a_k\}$ is a uniqueness set for F^p_α for all $\{a_1, \cdots, a_k\}$. Here, the union in Z' should be understood in the sense of zero sequences, where multiplicities are taken into account. In fact, if there exists a function $f \in F^p_\alpha$, not identically zero, such that f vanishes on Z', then the functions

$$f(z), \quad \frac{f(z)}{z-a_1}, \quad \cdots, \quad \frac{f(z)}{z-a_k}$$

all belong to F^p_α and vanish on Z. Here again, if zeros of higher multiplicity are involved, then some obvious adjustments should be made. It is clear that the functions listed above are linearly independent, so the dimension of I_Z is at least $k+1$, a contradiction.

Next, observe that if $\dim(I_Z) > m$, then $Z' = Z \cup \{a_1, \cdots, a_m\}$ is not a uniqueness set for F^p_α for any collection $\{a_1, \cdots, a_m\}$. To see this, pick any $m+1$ linearly independent functions $f_1, \cdots, f_{m+1}$ from I_Z, let

$$f = c_1 f_1 + \cdots + c_{m+1} f_{m+1},$$

and consider the system of linear equations

$$c_1 f_1(a_j) + \cdots + c_{m+1} f_{m+1}(a_j) = 0, \quad 1 \le j \le m.$$

Once again, obvious adjustments should be made when there are zeros of higher multiplicity. The homogeneous system above has m equations but $m+1$ unknowns, so it always has nonzero solutions c_j, $1 \le j \le m+1$. With such a choice of c_j, the function f is not identically zero but vanishes on Z', so Z' is not a uniqueness set.

It follows that if $1 \le j < k$ and $Z' = Z \cup \{a_1, \cdots, a_j\}$, then Z' is not a uniqueness set for F^p_α. We can actually show that Z' is a zero set for F^p_α. In fact, if f is a function in F^p_α, not identically zero, such that f vanishes on Z' (but not necessarily exactly on Z'), then the conclusion of the previous paragraph shows that the number of zeros of f in addition to those in Z' cannot exceed $k-j$. If these additional zeros a are divided out of f by the appropriate powers of $z-a$, the resulting function is still in F^p_α and vanishes exactly on Z'. Thus, Z' is a zero set for F^p_α.

Fix a function $g \in I_Z$ that has exactly Z as its zero set. If f is any function in I_Z, not identically zero, then just as in the previous paragraph, we can show that the zeros of f must be of the form $Z' = Z \cup \{a_1, \cdots, a_j\}$, where $j \le k-1$. Thus, we can factor f as follows: $f = gP\mathrm{e}^h$, where $P \in P_{k-1}$ and h is entire. It is clear that dividing a polynomial out of f, whenever the division is possible, always results in a function in F^p_α. Therefore, the function $g\mathrm{e}^h$ belongs to I_K as well. It follows that the function $g\mathrm{e}^h - g = g(\mathrm{e}^h - 1)$ belongs to I_Z. If h is not constant, then by Picard's theorem, $\mathrm{e}^h - 1$ has infinitely many zeros, so $g\mathrm{e}^h - g$ is a function in I_Z that has infinitely many zeros in addition to those in Z, a contradiction. This shows that h is constant and $I_Z \subset gP_{k-1}$. A count of dimension then gives $I_Z = gP_{k-1}$. □

In the classical theories of Hardy and Bergman spaces, every interpolating sequence is necessarily a zero sequence. We now show that this is not true for Fock spaces.

Proposition 5.11. *There exists an interpolating sequence for F^p_α that is not a zero set for F^p_α.*

Proof. Fix some $\delta > 2/\sqrt{\alpha}$. For any positive integer k, let Z_k denote the set of $k+1$ points evenly spaced in the first quadrant on the circle $|z| = k\delta$, including the end-points $k\delta$ and $k\delta\mathrm{i}$. Let

$$Z = \bigcup_{k=1}^{\infty} Z_k = \{z_1, z_2, \cdots, z_n, \cdots\}.$$

Since the distance between any two neighboring points in Z_k is

$$2k\delta \sin\frac{\pi}{4k} > \delta,$$

the sequence Z is separated with a separation constant greater than $2/\sqrt{\alpha}$. This implies that Z is an interpolating sequence for F^p_α; see Exercise 16 in Chap. 4.

If Z is the zero sequence of some function $f \in F^p_\alpha$, then by Theorem 5.1, the order ρ of f is less than or equal to 2. On the other hand, for the sequence Z, we have $m = \rho_1 = 2$; see Sect. 1.1 for the definition of these constants. By Theorem 1.10, we have $\rho \geq m = 2$. Thus, $\rho = 2$, and Lindelöf's theorem (Theorem 1.11) applies.

For $r \in (m\delta, (m+1)\delta)$, we have

$$\begin{aligned} S(r) &= \sum_{|z_k|<r} \frac{1}{z_k^2} = \sum_{k=1}^{m} \frac{1}{(k\delta)^2} \sum_{j=0}^{k} \mathrm{e}^{-\mathrm{i}\pi j/k} \\ &= \sum_{k=1}^{m} \frac{1}{(k\delta)^2} \frac{1+\mathrm{e}^{-\mathrm{i}\pi/k}}{1-\mathrm{e}^{-\mathrm{i}\pi/k}} = \sum_{k=1}^{m} \frac{1}{(k\delta)^2} \frac{\cos(\pi/2k)}{\sin(\pi/2k)} \\ &\sim -\frac{2\mathrm{i}}{\pi\delta^2} \sum_{k=1}^{m} \frac{1}{k} \sim -\frac{2\mathrm{i}}{\pi\delta^2} \log m \sim -\frac{2\mathrm{i}}{\pi\delta^2} \log r \end{aligned}$$

as $r \to \infty$. This shows that $S(r)$ is not bounded in r. By Lindelöf's theorem, f has infinite type. This contradicts with Theorem 2.12, which asserts that f must have type less than or equal to $\alpha/2$ when f is of order 2. Therefore, Z cannot be a zero sequence for F^p_α. □

5.4 Notes

Theorem 5.1, the necessary condition for zero sets of Fock spaces, was obtained in [249]. Theorem 5.3, the sufficient condition for zero sets of Fock spaces, is classical and follows from the general theory of entire functions. The proof of Theorem 5.3 here is basically from [67].

The results in Sect. 5.3 were mostly from [249, 258]. The motivation for [249] was Horowitz's study of zero sets for Bergman spaces; see [127–129]. The most intriguing results concerning zero sequences for Fock spaces are probably Theorems 5.9 and 5.10, which were proved in [258]. One interesting problem that remains open is the following: if $p \neq q$, do F^p_α and F^q_α always have different zero sequences?

Lemma 5.7 shows that Λ_α is a set of uniqueness for F^p_α when $0 < p < \infty$. This result as well as its proof are from [209]. Proposition 5.11, which is a little surprising when compared to the corresponding questions in the Hardy and Bergman space settings, is from [225].

5.5 Exercises

1. We say that an entire function $f(z)$ belongs to the Nevanlinna–Fock class F_α^* if

$$\int_{\mathbb{C}} \log^+ |f(z)|\, d\lambda_\alpha(z) < \infty.$$

Show that the zero sequence $\{z_n\}$ of any function f in F_α^* with $f(0) \neq 0$ satisfies the following condition:

$$\sum_{n=1}^{\infty} \frac{e^{-\alpha|z_n|^2}}{|z_n|^2} < \infty.$$

2. Let a be a nonzero complex number. Solve the extremal problem

$$\sup\{\operatorname{Re} f(0) : \|f\|_{2,\alpha} \leq 1, f(a) = 0\}.$$

3. Suppose Z is a zero set for F_α^p and k is a positive integer. Show that the following conditions are equivalent:
   (a) $\dim(I_Z) \leq k$.
   (b) $Z \cup \{a_1, \cdots, a_k\}$ is a uniqueness set for F_α^p for *all* $\{a_1, \cdots, a_k\}$.
   (c) $Z \cup \{a_1, \cdots, a_k\}$ is a uniqueness set for F_α^p for *some* $\{a_1, \cdots, a_k\}$.
4. Suppose Z is a zero set for F_α^p and k is a positive integer. Show that the following conditions are equivalent:
   (a) $\dim(I_Z) = k$.
   (b) For *any* $\{a_1, \cdots, a_k\}$, the sequence $Z \cup \{a_1, \cdots, a_{k-1}\}$ is not a uniqueness set for F_α^p but $Z \cup \{a_1, \cdots, a_k\}$ is.
   (c) For *some* $\{a_1, \cdots, a_{k-1}\}$, the sequence $Z \cup \{a_1, \cdots, a_{k-1}\}$ is not a uniqueness set for F_α^p, but for *some* $\{b_1, \cdots, b_k\}$, the sequence $Z \cup \{b_1, \cdots, b_k\}$ is a uniqueness set for F_α^p.
5. If Z is a zero set for F_α^p, then the sequence remains a zero set for F_α^p after any finite number of points are removed from it.
6. Suppose $0 < p < \infty$ and Z is uniformly close to Λ_α. Show that Z is a uniqueness set for F_α^p.
7. If $Z = \{z_n\}$ is a zero set for F_α^p, then

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^2 \log^{1+\varepsilon} |z_n|} < \infty$$

for all $\varepsilon > 0$, provided that $|z_n| \neq 0, 1$. Show that this is false in general if $\varepsilon = 0$.

8. Suppose $\{z_n\}$ is the zero sequence of a function $f \in F^p_\alpha$, where $f(0) = 1$, $0 < p < \infty$, and $\{|z_n|\}$ is nondecreasing. Show that

$$\prod_{k=1}^{n} \frac{1}{|z_n|^p} \le \frac{C}{\sqrt{n}} \left(\frac{\alpha e}{n}\right)^{\frac{np}{2}} \|f\|^p_{p,\alpha}$$

for all $n \ge 1$, where C is a positive constant independent of n and f.
9. Suppose $0 < p < q \le \infty$, Z is a zero set for F^q_α, and N is a positive integer with $Np > 2$. Show that if any N points are removed from Z, the remaining sequence becomes a zero set for F^p_α.
10. Let Z be a zero set for F^2_α with $0 \notin Z$. Show that there is no function $G_Z \in F^2_\alpha$ such that $G_Z(0) > 0$, $\|G_Z\|_{2,\alpha} = 1$, $Z(G_Z) = Z$, and $\|f/G_Z\|_{2,\alpha} \le \|f\|_{2,\alpha}$ for all $f \in F^2_\alpha$ with $f|Z = 0$. See [119] for information about the corresponding problem in the Bergman space setting.
11. Suppose $f \in F^p_\alpha$ has order 2 and type $\alpha/2$. Then f must have infinitely many zeros. See [22].
12. Show that the function

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n!}} z^n$$

belongs to F^2_1 but the function $zf(z)$ is no longer in F^2_1. See [22].
13. If Z is a zero set for F^p_α and $\dim(I_Z) < \infty$, then every function in I_Z has order 2 and type $\alpha/2$.
14. If Z is a zero set for F^p_α and $\dim(I_Z) < \infty$, then any two functions in I_Z whose zeros are exactly those in Z can only differ by a constant multiple. Thus, there is essentially just one function that vanishes exactly on Z.