Chapter 12
Physical Hilbert Spaces

The Wheeler–DeWitt equation or the difference equation of loop quantum cosmology present a constraint that states have to satisfy, analogous to the Friedmann equation which is a constraint in canonical relativity. Zero eigenvalues of the constraint operator are thus to be found. For the difference equation (4.15) encountered for isotropic models there is in fact a kinematically normalizable eigenstate of zero eigenvalue, given by $\psi_\mu = \delta_{\mu,0}$. This state, however, is supported only on the classical singularity and of no interest to describe an expanding universe.

Sometimes also interesting states can correspond to normalizable zero-eigenstates, for instance if they belong to a model in which all solutions recollapse, requiring exponentially decaying wave functions at large volume. But in general one has to deal with states that belong to a zero eigenvalue as part of the continuous spectrum, and which cannot be normalizable in the kinematical inner product as it is defined on the Bohr Hilbert space. Then, solutions to the constraint equation cannot form a subspace of the kinematical Hilbert space but constitute a new physical Hilbert space to be constructed by endowing the solution space with a physical inner product. Several techniques to do so exist, but explicit constructions are complicated in general models. For a reparameterization–invariant system as realized by models of general relativity, the physical Hilbert space issue is often related to the problem of time.

12.1 Group Averaging

One method to derive a physical Hilbert space for a given constrained system is group averaging [1]. It applies in particular if all the constraints to be solved generate a unitary group action on the kinematical Hilbert space. One can then integrate over the group to ensure that states considered are invariant under the action.

For a single self-adjoint constraint $\hat{C}$, the unitary group action is Abelian: $\exp(\mathrm{i}t\hat{C})$, $t \in \mathbb{R}$. Starting with an arbitrary kinematical state $|\psi\rangle$, an averaged state is obtained by integrating $\exp(\mathrm{i}t\hat{C})|\psi\rangle$ over all values of t. The integration does

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not necessarily exist for all states, and the result may not be another normalizable state. However, for states $|\psi\rangle$ in a suitable dense set $\mathscr{D} \subset \mathscr{H}$ the integration can be made sense of as a distribution: a linear functional $\langle\eta_\psi|: \mathscr{D} \to \mathbb{C}$ mapping every state $|\phi\rangle \in \mathscr{D}$ to a complex number. This number is defined by

$$\langle\eta_\psi|\phi\rangle = \int_{-\infty}^{\infty} \mathrm{d}t\, \langle\psi|\exp(\mathrm{i}t\hat{C})|\phi\rangle. \tag{12.1}$$

Heuristically, one can interpret $\int_{-\infty}^{\infty} \mathrm{d}t \exp(\mathrm{i}t\hat{C}) = \delta(\hat{C})$ as a delta-function whose insertion ensures that the action of $\hat{C}$ on states vanishes. The distributional, group-averaged state may then be written as $\eta_\psi = \delta(\hat{C})\psi$ which is not normalizable in the kinematical inner product because one would have to multiply two delta-functions before integrating. The group-averaging inner product makes sense of such an expression by elegantly removing one of the delta-functions, making the integral well-defined and invariant under the group action.

In a similar way, symmetric states, obtained not by implementing first-class constraints but by imposing symmetry, can be interpreted as ordinary states multiplied with delta-functions to make a state vanish on non-symmetric configurations. Distributional techniques similar to group averaging have been used to make sense of the reduced state spaces, giving rise to the models of loop quantum cosmology as in Sects. 8.2.5 and 10.1.2.2.

The space of distributions on $\mathscr{D}$, called $\mathscr{D}'$, does not carry a natural inner product. But on its subspace obtained by group averaging one can easily introduce one. Given two such distributions $\langle\eta_\psi|$ and $\langle\eta_\phi|$, we define the bilinear form

$$\langle\eta_\phi|\eta_\psi\rangle_{\mathrm{phys}} := \int_{-\infty}^{\infty} \mathrm{d}t\, \langle\phi|\exp(\mathrm{i}t\hat{C})|\psi\rangle. \tag{12.2}$$

On the right-hand side we use kinematical states $|\phi\rangle$ and $|\psi\rangle$ averaged to $|\eta_\phi\rangle$ and $|\eta_\phi\rangle$, respectively. Such states are not unique, but thanks to unitarity of the group action the integral in the definition of $\langle\eta_\phi|\eta_\psi\rangle_{\mathrm{phys}}$ does not depend on which representative is chosen: Any other states averaging to the same distributions must be of the form $|\phi'\rangle = \exp(\mathrm{i}u\hat{C})|\phi\rangle$ and $|\psi'\rangle = \exp(\mathrm{i}v\hat{C})|\psi\rangle$ with real u and v, such that

$$\int_{-\infty}^{\infty} \mathrm{d}t\, \langle\phi'|\exp(\mathrm{i}t\hat{C})|\psi'\rangle = \int_{-\infty}^{\infty} \mathrm{d}t\, \langle\phi|\exp(\mathrm{i}(t+v-u)\hat{C})|\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}t'\, \langle\phi|\exp(\mathrm{i}t'\hat{C})|\psi\rangle.$$

Factoring out a possible kernel of the bilinear form and Cauchy-completing the space provides the physical Hilbert space. Uniqueness properties and examples can be found in [2–4].

Alternatively, one can understand the procedure by first introducing a parameter λ along the unitary flow generated by the constraint operator on arbitrary states:

$$\hat{C}|\phi_\lambda\rangle = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\lambda}|\phi_\lambda\rangle. \tag{12.3}$$

One thus deals with the constraint in a way similar to a Hamiltonian providing a Schrödinger-type equation. A family of states solving the Schrödinger equation (12.3) is a solution to the constraint only if it is actually λ-independent such that $\hat{C}|\phi_\lambda\rangle = 0$ follows from $\mathrm{d}|\phi_\lambda\rangle/\mathrm{d}\lambda = 0$. In the solution space, this can be achieved by integrating over λ and defining $|\phi\rangle = \int_{-\infty}^{\infty} \mathrm{d}\lambda|\phi_\lambda\rangle$. The state $|\phi\rangle$ is then annihilated by the constraint, and noting that we can always write $|\phi_\lambda\rangle = \exp(-\mathrm{i}\lambda\hat{C})|\phi_0\rangle$ thanks to the Schrödinger equation $|\phi_\lambda\rangle$ satisfies, we recognize $|\phi\rangle$ as the group average of $|\phi_0\rangle$. In general one has to be careful with commuting the action of $\hat{C}$ and the λ-integration. This is taken care of in the detailed procedure of group averaging by selecting an appropriate dense set $\mathscr{D}$.

Example 12.1 Self-adjointness is not always required for group averaging integrals to exist, but violations can have undesired consequences. First taking the simple self-adjoint case $\hat{C} = \mathrm{i}\partial_x$, the flow equation $-\partial_x\phi_\lambda(x) = \partial_\lambda\phi_\lambda(x)$ has the general solution $\phi_\lambda(x) = \phi(\lambda + x)$ which easily integrates over λ to a constant independent of x. Now looking at the constraint $\hat{C} = \partial_x$ which is not self-adjoint, we have $\phi_\lambda(x) = \phi(\lambda + \mathrm{i}x)$ as the general solution of the flow equation. Interpreting this general solution as an arbitrary holomorphic function on the complex plane with coordinate $\lambda + \mathrm{i}x$, the λ-integration is performed along a line parallel to the real axis but shifted by an amount of x. Clearly, ϕ must fall off at infinity for the integration to exist. Since any bounded entire function is a constant according to Liouville's theorem, the only pole-free function allowed is the trivial zero function. Otherwise, ϕ must have poles on the complex plane. But then, changing x and shifting one of the integration contours past a pole with non-vanishing residue will change the value of the integration. Except for the zero solution, averaged solutions cannot be constant and independent of x, as one would expect it for the constraint ∂_x. They are only piecewise constant and include discontinuous steps at values of x corresponding to the imaginary parts of poles of ϕ.

Example 12.2 ([5]). Let the constraint be $\hat{C} = -a\hat{x}\hat{p} + b$ with two constants a and b which may be complex-valued. To implement group averaging we first solve the flow equation

$$(ax\mathrm{i}\partial_x + b)\phi_\lambda(x) = \mathrm{i}\partial_\lambda\phi_\lambda(x)$$

in the x-representation. The general solution is

$$\phi_\lambda(x) = \mathrm{e}^{\frac{1}{2}\mathrm{i}b(a^{-1}\log(x)-\lambda)} f(\log(x)/a + \lambda)$$

with an arbitrary differentiable function f. This solution realizes the flow of the constraint on the state space. To proceed further and integrate over λ, it is useful to Fourier transform f provided that a is real: $f(u) = (2\pi)^{-1}\int_{-\infty}^{\infty}\mathrm{d}\omega\mathrm{e}^{-\mathrm{i}\omega u}\tilde{f}(\omega)$. Then,

$$\int_{-\infty}^{\infty} \mathrm{d}\lambda \phi_\lambda(x) = e^{\frac{1}{2}\mathrm{i}b\log(x)/a} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} e^{-\mathrm{i}\omega\log(x)/a} \tilde{f}(\omega) \int_{-\infty}^{\infty} \mathrm{d}\lambda e^{-\mathrm{i}(\omega+b/2)\lambda}$$

if we are allowed to commute the integrations. If b is real, the λ-integration results in $\delta(\omega + b/2)$ and we have invariant solutions $\phi(x) \propto e^{\mathrm{i}b\log(x)/a}$. If b is not real, the integration would have to be done more carefully, and appropriate fall-off conditions for $\tilde{f}$ would be required for convergence.

12.2 Observables

Physical observables are self-adjoint operators on the physical Hilbert space. If one knows a kinematical operator $\hat{O}$ which commutes with the constraint $\hat{C}$, then called a Dirac observable, its action can directly be taken over to group-averaged states by the dual action:

$$\langle \hat{O}\eta_\psi|\phi\rangle := \langle \eta_\psi|\hat{O}^\dagger|\phi\rangle = \int_{-\infty}^{\infty} \mathrm{d}t \langle \psi| \exp(\mathrm{i}t\hat{C})\hat{O}^\dagger|\phi\rangle \tag{12.4}$$

provides an action on physical states. It is independent of the representative chosen: Another one is related to $|\phi\rangle$ by $|\phi'\rangle = \exp(\mathrm{i}u\hat{C})|\phi\rangle$ with a real number u that just adds to t in the integration without changing the result. If $\hat{O}$ does not commute with $\hat{C}$ one may still define its action on physical states in this way, but the notion is ambiguous: it depends on which representative $|\psi\rangle$ is chosen for the physical state $|\eta_\psi\rangle$ before averaging the right-hand side.

A natural action of operators not commuting with the constraint on physical states is obtained for evolving observables, a special class of relational observables [6–10], provided one manages to write the constraint as $\hat{C} = -\mathrm{i}\hbar\partial_\varphi + \hat{H}$ with a suitable phase–space variable φ (called internal time) and a φ-independent $\hat{H}$. Then,

$$\hat{\mathscr{O}}(\varphi) = \mathrm{e}^{-\mathrm{i}\varphi\hat{H}/\hbar}\hat{O}\mathrm{e}^{\mathrm{i}\varphi\hat{H}/\hbar} \tag{12.5}$$

provides parameterized families of operators commuting with the constraint $\hat{H} - \mathrm{i}\hbar\partial_\varphi$:

$$[\hat{\mathscr{O}}(\varphi), \hat{C}] = \mathrm{i}\hbar\partial_\varphi\hat{\mathscr{O}}(\varphi) + \exp(-\mathrm{i}\varphi\hat{H}/\hbar)[\hat{O}, \hat{H}]\exp(\mathrm{i}\varphi\hat{H}/\hbar) = 0.$$

Here, we crucially use the φ-independence of $\hat{H}$, which implies $\mathrm{i}\hbar\partial_\varphi\hat{\mathscr{O}}(\varphi) = \exp(-\mathrm{i}\varphi\hat{H}/\hbar)\hat{H}\hat{O}\exp(\mathrm{i}\varphi\hat{H}/\hbar) - \exp(-\mathrm{i}\varphi\hat{H}/\hbar)\hat{O}\hat{H}\exp(\mathrm{i}\varphi\hat{H}/\hbar)$. A physical state is thus mapped into a physical state by $\hat{\mathscr{O}}(\varphi)$.

Virtues and disadvantages of evolving observables have been discussed in [11]. Evolving observables satisfy the equation

$$\frac{\partial\langle\hat{\mathcal{O}}(\varphi)\rangle}{\partial\varphi} = -\frac{\langle[\hat{\mathcal{O}},\hat{H}]\rangle}{\mathrm{i}\hbar}. \tag{12.6}$$

Notice the minus sign, to be commented on at the end of Sect. 12.3.1.

12.3 Internal Time

Solving quantum constraints and computing observable quantities is often facilitated if global internal-time variables exist; most of the explicit techniques even rely on that feature: the existence of simple phase–space functions that can be used as global parameters in dynamical solutions. We have already seen this in the preceding section, where the existence of a phase–space variable φ allowing us to write the constraint as a linear momentum of time φ plus a time-independent Hamiltonian, providing general expressions for observables. This procedure is called deparameterization.

12.3.1 Non-Relativistic Parameterized Systems

Constraints are rather simple to solve if a phase–space variable φ exists such that they take the form $\hat{C} = \hat{p}_\varphi + \hat{H}$ where H is independent of φ and its momentum p_φ. One can then replace the constraint equation $\hat{C}|\psi\rangle = 0$ by a conventional Schrödinger flow in the internal time φ. If $\hat{H}$ in this decomposition is self-adjoint, a physical inner product can be defined using the kinematical one at any fixed value of φ.

Physical states in such a case are solutions to the Schrödinger equation

$$\mathrm{i}\hbar\partial_\varphi\psi(\varphi) = \hat{H}\psi(\varphi)$$

and thus clearly depend on the internal time φ. In contrast to the discussion of general constraints where we introduced an identical equation in terms of λ instead of φ, however, no further averaging is necessary: φ represents a variable on the phase space of the system, not an auxiliary quantity to reformulate the equations. The constraint equation $\hat{C}|\psi\rangle$ is solved by the Schrödinger flow, and we can easily define the physical inner product

$$\langle\psi_1(\varphi)|\psi_2(\varphi)\rangle_{\mathrm{phys}} = \langle\psi_1(\varphi_0)|\psi_2(\varphi_0)\rangle_{\varphi_0}$$

in terms of the kinematical inner product of unconstrained states, assumed to take the form $\langle\cdot|\cdot\rangle = \int \mathrm{d}\varphi\langle\cdot|\cdot\rangle_\varphi$. We are thus dropping one φ-integration and replace it with evaluation at some fixed φ_0. Since the inner product is preserved by any unitary φ-evolution, the physical inner product is independent of the choice of φ_0.

A complete set of observables can formally be constructed in the evolving-observable sense. First, $\hat{p}_\varphi$ is clearly an observable since we assumed $\hat{H}$ to be φ-independent: $[\hat{p}_\varphi,\hat{C}] = 0$. Any kinematical operator $\hat{O}$ other than $\hat{\varphi}$ can be made into an evolving observable

$$\hat{\mathscr{O}}(\varphi) = \exp(-\mathrm{i}(\varphi - \varphi_0)\hat{H}/\hbar)\hat{O}\exp(\mathrm{i}(\varphi - \varphi_0)\hat{H}/\hbar) \tag{12.7}$$

mapping physical solutions into other physical solutions. Properties such as expectation values in physical states can simply be computed by the restricted inner product $\langle\cdot|\cdot\rangle_{\varphi_0}$.

Example 12.3 (*Linear Hamiltonian*) For $\hat{H} = \hat{p}$, the Schrödinger equation $\partial_\varphi\psi = -\partial_q\psi$ has the general solution $\psi(\varphi - q)$. We immediately see that the motion of the expectation value $\langle\hat{q}\rangle$ must be linear in internal time φ, and that the initial shape of the state is preserved during the motion. While $\hat{p}$ is already an observable, the position observable must be an evolving one:

$$\hat{\mathscr{Q}} = \exp(-\mathrm{i}(\varphi - \varphi_0)\hat{p}/\hbar)\hat{q}\exp(\mathrm{i}(\varphi - \varphi_0)\hat{p}/\hbar) = \hat{q} - (\varphi - \varphi_0) \tag{12.8}$$

(using $[\hat{q}, f(\hat{p})] = \mathrm{i}\hbar f'(\hat{p})$). Thus, $\partial\langle\hat{\mathscr{Q}}\rangle/\partial\varphi = -1$, in accordance with (12.6). At this stage, we can explain the possibly unexpected minus sign in (12.6), opposite of what one normally has for Schrödinger evolution, for instance in (5.9). We invert (12.8) to write $\hat{q}(\varphi) = \hat{\mathscr{Q}} + \varphi - \varphi_0$, where $\hat{\mathscr{Q}}$, as a Dirac observable, is a constant of motion. For the time-dependent expectation value $\langle\hat{q}\rangle(\varphi)$ we then have $\mathrm{d}\langle\hat{q}\rangle/\mathrm{d}\varphi = 1$, with the expected sign. In general, the inversion involved in the transition between observables and internal-time dependent operators is the origin of the sign difference between (12.6) and (5.9).

12.3.2 Relativistic Parameterized Systems

Relativistic systems have constraints of the form $\hat{C} = \hat{p}_\varphi^2 - \hat{H}^2$, and are deparameterizable if $\hat{H}$ does not depend on φ. They can thus be brought into the non-relativistic parameterized form only by taking a square root, which requires sign choices. To view the resulting equations as those corresponding to a system with definite frequency sign of φ-evolution, one factorizes

$$\hat{p}_\varphi^2 - \hat{H}^2 = (\hat{p}_\varphi + |\hat{H}|)(\hat{p}_\varphi - |\hat{H}|).$$

The two sectors of solutions annihilated by either of the two factors correspond to positive and negative frequency, respectively. Within these sectors, solutions can also be classified by the sign of $\hat{H}$, which are called right-moving and left-moving by analogy with the case $\hat{H} = \hat{p}$ for a free, massless relativistic Klein–Gordon particle.

For a simple form of $\hat{H}$, such as $\hat{p}^2 + m^2$ for a free relativistic particle of mass m, one can formulate a complete inner product, find all physical solutions and derive properties such as the φ-dependence of expectation values and moments. A useful formulation employs a Dirac-style procedure to take the square root in terms of matrices [12]. Others are based on Fourier transformation.

Example 12.4 (*Klein–Gordon inner product*) For a free, massless relativistic particle we consider the constraint $E^2 - p^2 = 0$, quantized to

$$\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} = 0$$

Solutions can easily be found as $\psi(t,x) = \psi_+(x+t) + \psi_-(x-t)$ with arbitrary ψ_+ and ψ_-. The inner product is easiest to discuss after a Fourier transformation, for which we consider wave functions satisfying $(\omega^2 - k^2)\tilde{\psi}(\omega,k) = 0$. Thus, $\tilde{\psi}$ is supported only on $\omega = |k|$ (positive frequency) or $\omega = -|k|$ (negative frequency).

The Klein–Gordon bilinear form

$$(\psi_1, \psi_2) = \mathrm{i}\int \mathrm{d}x \left(\psi_1^* \frac{\partial\psi_2}{\partial t} - \frac{\partial\psi_1^*}{\partial t}\psi_2\right)$$

is preserved in time for solutions ψ_1 and ψ_2 to the Klein–Gordon equation. However, it does not directly provide an inner product because it is not positive definite:

$$\begin{aligned}(\psi_1, \psi_2) &= \int \mathrm{d}x \left(\int \mathrm{d}k_1 \tilde{\psi}_1^* e^{\mathrm{i}(\omega_1 t - k_1 x)} \int \mathrm{d}k_2 \omega_2 \tilde{\psi}_2 \mathrm{e}^{-\mathrm{i}(\omega_2 t - k_2 x)}\right.\\ &\qquad \left. + \int \mathrm{d}k_1 \omega_1 \tilde{\psi}_1^* \mathrm{e}^{\mathrm{i}(\omega_1 t - k_1 x)} \int \mathrm{d}k_2 \tilde{\psi}_2 \mathrm{e}^{\mathrm{i}(\omega_2 t - k_2 x)}\right)\\ &= \int \mathrm{d}k (\omega_1(k) + \omega_2(k)) \tilde{\psi}_1^* \tilde{\psi}_2 \mathrm{e}^{-\mathrm{i}(\omega_1(k) - \omega_2(k))t}\end{aligned}$$

shows that there are three different cases:

- $\psi_1 = \psi_2$, positive frequency: $\tilde{\psi}_1$ supported on $\omega = |k|$. Then, $(\psi_1, \psi_1) = 2\int \mathrm{d}k |k| \tilde{\psi}_1^* \tilde{\psi}_1 > 0$ and we have positive Klein–Gordon norm.
- $\psi_1 = \psi_2$, negative frequency: $\tilde{\psi}_1$ supported on $\omega = -|k|$. Then, $(\psi_1, \psi_1) = -2\int \mathrm{d}k |k| \tilde{\psi}_1^* \tilde{\psi}_1 < 0$ and we have negative Klein–Gordon norm.
- ψ_1 and ψ_2 of opposite frequency signs, e.g. $\omega_1 = |k_1|$, $\omega_2 = -|k_2|$. Then, $(\psi_1, \psi_2) = 0$ and the solutions are orthogonal.

A Hilbert space with a definite inner product can be defined as $(\mathscr{H}, \langle\cdot,\cdot\rangle) = (\mathscr{H}_+, (\cdot,\cdot)) \oplus (\mathscr{H}_-, -(\cdot,\cdot))$ [13].

Example 12.5 (*Relativistic harmonic oscillator* [14]). We consider a system with two degrees of freedom q and φ with momenta p and p_φ, subject to the constraint

$$\hat{C} = \hat{p}_\varphi^2 - \hat{p}^2 - \hat{q}^2.$$

Physical states must solve the equation

$$\mathrm{i}\hbar \frac{\partial}{\partial\varphi}\psi(q,\varphi) = \pm\sqrt{\hat{p}^2 + \hat{q}^2}\,\psi(q,\varphi)$$

and can be written as

$$\psi_\pm(q,\varphi) = \sum_{n=0}^{\infty} c_n \phi_n(q) \exp(\mp \mathrm{i}\lambda_n \varphi/\hbar)$$

in terms of harmonic-oscillator eigenstates $\phi_n(q)$, $\lambda_n = \sqrt{(2n+1)\hbar}$. The constants c_n in this general solution are determined by "initial" values chosen at a fixed value of φ. For instance, using the well-known harmonic-oscillator coherent states at $\varphi = 0$ (in the present context playing the role of a kinematical coherent state), we have

$$c_n = \exp\left(-\frac{|z|^2}{2}\right)\frac{z^n}{\sqrt{n!}}, \quad z \in \mathbb{C}$$

such that

$$\psi(q,0) = \left(\frac{2}{\pi}\right)^{1/4} \exp\left(-\frac{1}{2}|z|^2 - z^2 + 2q^2 - 4\mathrm{i}zq\right)$$

The corresponding evolving state has coefficients

$$c_n \mathrm{e}^{-\mathrm{i}\lambda_n\varphi/\hbar} = \frac{1}{\sqrt{n!}}\mathrm{e}^{-|z|^2/2} z^n \exp(-\mathrm{i}\sqrt{2n+1}\varphi/\sqrt{\hbar})$$

and is non-coherent for $\varphi \neq 0$, but may be considered as a physical semiclassical state for some time of the evolution [14].

In general, the construction of physical semiclassical states via wave functions can be surprisingly subtle. For instance, in the physical Hilbert space underlying loop quantum cosmology sourced be a free massless scalar [15], one may choose Gaussian states at a fixed value of φ and evolve with φ as internal time. If the Gaussian is sharp enough, one expects to have a good semiclassical state with small volume and curvature fluctuations. However, an actual computation of the fluctuations reveals that in this innocent-looking state they are infinite [16]. One can define semiclassical states via the moments, as already used in Chap. 5, but reconstructing the form of a wave function from given fluctuations or higher moments is highly non-trivial.

12.4 Examples in Quantum Cosmology

If φ is taken as the free massless scalar field in a model of quantum cosmology, one can use the procedures seen here to derive physical Hilbert spaces and their properties. As already used for the solvable models of Chap. 5, one can rewrite the Friedmann equation as a relativistically deparameterizable constraint

$$p_\varphi^2 - \frac{16\pi G}{3}(1-x)V^2P^2 = 0 \tag{12.9}$$

in the canonical variables (5.3). ; see (5.4). In a similar form, physical Hilbert spaces for the Wheeler–DeWitt quantization of this constraint have been introduced in [17]. The φ-Hamiltonian is then the one used in the discussion of solvable models. Analogously, holonomy corrections and inverse-triad corrections of loop quantum cosmology can be included in the Hamiltonian before applying the same techniques to

compute the physical Hilbert space [15, 18]. In addition to this extension of the methods of [17] to loop quantum cosmology, [15, 18] have provided detailed numerical tools to compute physical states and extract observables. An alternative derivation of the physical Hilbert space for such systems, based on a Dirac-style first-order formulation of the constraint operator quadratic in $\hat{p}_\varphi$ can be found in [12].

For meaningful physical evolution in internal time it is important that the φ-Hamiltonian is self-adjoint. As shown by [19] and [20], the φ-Hamiltonian of isotropic loop quantum cosmology is essentially self-adjoint for the flat and closed models with vanishing (or negative) cosmological constant. With a positive cosmological constant, however, the Hamiltonian is not essentially self-adjoint [21]. This property is related to the fact, mentioned in Sect. 5.4.1.3, that the classical volume diverges at a finite φ. A semiclassical state thus reaches the boundary at infinity in a finite amount of internal time, and must be reflected back in order to preserve probability. Reflection conditions for wave functions are not unique and do not follow from classical physics; thus, there is no unique self-adjoint extension of the quantum Hamiltonian. Physical interpretations of the reflection and possible extensions of the dynamics are questionable, however, because infinite volume requires an infinite amount of proper time to be reached.

Another approach to construct a physical inner product is based on spin–foam models [23], a suggested covariant, path-integral like version of loop quantum gravity which has branched out into quantum cosmology. The first construction of spin–foam motivated physical inner products in quantum cosmology was done for a model without matter but a positive cosmological constant [22]. More recently, a more general formulation of spin–foam cosmology has been attempted [24, 25] and is still being developed. Well-known results of loop quantum cosmology regarding physical Hilbert spaces can then be used to shed light on more complicated constructions for general spin foams [26–31].

Constructions of physical Hilbert spaces in quantum cosmology are straightforward applications of standard methods if there is no non-trivial matter potential, such that the resulting Hamiltonians are indeed φ-independent. Otherwise the whole procedure complicates enormously to the degree of being intractable. If deparameterization is formally applied to models without good internal time, instead using a variable with turning point (a zero of its momentum along a physical solution) for a description local in time, the system freezes at the turning point [32, 33]. At this point, or even before the freeze-out [34], it is difficult to decide which aspects of the physical state can still be trusted. Transformations to a new time variable which remains valid around the turning point of the old time would have to change the physical Hilbert space, whose construction is based on the choice of time for deparameterization. Even if the physical Hilbert space could be derived completely, such transformations would be difficult to perform. Moreover, time must be chosen before quantization in such a setting, and different choices of time give rise to different quantum representations with possibly differing physics. Here, effective tools to address physical Hilbert space issues become essential to understand physical properties of generic models in loop quantum cosmology, as introduced in the next and final chapter.

References

1. Marolf, D. (1995). gr-qc/9508015
2. Giulini, D., Marolf, D.: Class. Quantum Grav. **16**, 2479 (1999). gr-qc/9812024
3. Giulini, D., Marolf, D.: Class. Quantum Grav. **16**, 2489 (1999). gr-qc/9902045
4. Gomberoff, A., Marolf, D.: Int. J. Mod. Phys. D **8**, 519 (1999). gr-qc/9902069
5. Bojowald, M., Singh, P., Skirzewski, A.: Phys. Rev. D **70**, 124022 (2004). gr-qc/0408094
6. Bergmann, P.G.: Rev. Mod. Phys. **33**, 510 (1961)
7. Rovelli, C.: Class. Quantum Grav. **8**, 297 (1991)
8. Rovelli, C.: Class. Quantum Grav. **8**, 317 (1991)
9. Dittrich, B.: Gen. Rel. Grav. **39**, 1891 (2007). gr-qc/0411013
10. Dittrich, B.: Class. Quant. Grav. **23**, 6155 (2006). gr-qc/0507106
11. Lawrie, I.D.: Phys. Rev. D **83**, 043503 (2011). arXiv:1011.4444
12. Kreienbühl, A.: Phys. Rev. D **79**, 123509 (2009). arXiv:0901.4730
13. Hartle, J.B., Marolf, D.: Phys. Rev. D **56**, 6247 (1997). gr-qc/9703021
14. Bojowald, M., Tsobanjan, A.: Class. Quantum Grav. **27**, 145004 (2010). arXiv:0911.4950
15. Ashtekar, A., Pawlowski, T., Singh, P.: Phys. Rev. D **73**, 124038 (2006). gr-qc/0604013
16. Varadarajan, M.: Class. Quantum Grav. **26**, 085006 (2009). arXiv:0812.0272
17. Blyth, W.F., Isham, C.J.: Phys. Rev. D **11**, 768 (1975)
18. Ashtekar, A., Pawlowski, T., Singh, P.: Phys. Rev. D **74**, 084003 (2006). gr-qc/0607039
19. Kaminski, W., Lewandowski, J.: Class. Quant. Grav. **25**, 035001 (2008). arXiv:0709.3120
20. Szulc, L., Kaminski, W., Lewandowski, J.: Class. Quantum Grav. **24**, 2621 (2007). gr-qc/0612101
21. Kaminski, W., Pawlowski, T.: Phys. Rev. D **81**, 024014 (2010). arXiv:0912.0612
22. Perez, A.: Class. Quantum Grav. **20**, R43 (2003). gr-qc/0301113
23. Noui, K., Perez, A., Vandersloot, K.: Phys. Rev. D **71**, 044025 (2005). gr-qc/0411039
24. Rovelli, C., Vidotto, F.: Class. Quantum Grav. **25**, 225024 (2008). arXiv:0805.4585
25. Bianchi, E., Rovelli, C., Vidotto, F.: Phys. Rev. D **82**, 084035 (2010). arXiv:1003.3483
26. Ashtekar, A., Campiglia, M., Henderson, A.: Phys. Lett. B **681**, 347 (2009). arXiv:0909.4221
27. Ashtekar, A., Campiglia, M., Henderson, A.: Class. Quantum Grav. **27**, 135020 (2010). arXiv:1001.5147
28. Rovelli, C., Vidotto, F.: arXiv:0911.3097
29. Campiglia, M., Henderson, A., Nelson, W.: Phys. Rev. D **82**, 064036 (2010). arXiv:1007.3723
30. Henderson, A., Rovelli, C., Vidotto, F., Wilson-Ewing, E.: Class. Quantum Grav. **28**, 025003 (2011). arXiv:1010.0502
31. Calcagni, G., Gielen, S., Oriti, D.: Class. Quantum Grav. **28**, 125014 (2011). arXiv:1011.4290
32. Wald, R.M.: Phys. Rev. D **48**, 2377 (1993). gr-qc/9305024
33. Higuchi, A., Wald, R.M.: Phys. Rev. D **51**, 544 (1995). gr-qc/9407038
34. Rovelli, C.: Phys. Rev. D **43**, 442 (1991)