Chapter 9
Bundles and Connections

In previous chapters we have described how the notion of symmetry of a physical theory forced us to change the definition of derivative of a field so as to make the action principle invariant. The invariance of the action means that it must be the same for all observers (when considering coordinate transformations) and for the observable fields (when referring to gauge transformations). It has to be such invariant to fulfill the purposes of Maupertuis, Euler, Lagrange, and Hamilton. The modification of the derivative is quite intuitive when we are talking about coordinate transformations, but it is less intuitive when we are talking about field theory. In this chapter we will introduce the basic tools to the formal definition of gauge covariant derivatives.

Starting with the more intuitive coordinate transformations, we have a very naive notion of Euclidean 3-dimensional background Euclidean space where we make frequent use of Cartesian coordinates defined in an absolute (or canonical) reference frame. The necessity to change coordinates away from the Cartesian system was regarded as a way to simplify the equations and not as a fundamental issue. However, as we have already exemplified in the previous chapters, in doing mathematical analysis in a physical manifold we no longer have the Cartesian frames at our disposal. In order to make sense as an invariant property to the allowed observers, the notion of derivative needs to be modified by the introduction of a connection or covariant derivative. In the following we generalize the notions of vector bundles to *principal fiber bundles*, so that we may incorporate a symmetry group to our physical manifold structure [108–112].

9.1 Fiber Bundles

Definition 9.1 (Fiber Bundle) Consider an n-dimensional differentiable manifold $\mathscr{M}$, to each point p of which there is an attached another m-dimensional manifold $\mathscr{B}_p$. The word attachment means that in principle there is not any specific relationship between $\mathscr{M}$ and $\mathscr{B}_p$, besides the point of contact. Then we may collect all these manifolds $\mathscr{B}_p$ in a set $T\mathscr{B}$, in such a way that we may identify the contact point p in $\mathscr{M}$ by a map $\pi : T\mathscr{B} \to \mathscr{M}$. The differentiable *fiber bundle* with base $\mathscr{M}$ and total space $T\mathscr{B}$ is the triad

M.D. Maia, *Geometry of the Fundamental Interactions*,
DOI 10.1007/978-1-4419-8273-5_9,

$$(\mathscr{M},\ \pi,\ T\mathscr{B})$$

If $p \in \mathscr{M}$, then the map $s(p) \to T\mathscr{B}$ such that $s(p) \circ \pi = 1$ is called a section of the fiber bundle, all it does is to identify the fiber $\mathscr{B}_p \subset T\mathscr{B}$.

The tangent bundle is a particular fiber bundle. Higher order tangent bundles such as the osculating paraboloid to a surface are interesting non-trivial examples.

Other examples are given by Galilean and Newton's space–time in which the total space is the space–times and the fibers are of the three-dimensional simultaneity sections. In both cases the base space is the absolute time axis $I\!R$.

Given a fiber bundle $(\mathscr{M},\ \pi,\ T\mathscr{B})$ and neighborhood of a point x, $\cup_x \subset \mathscr{M}$, we may define a *local fiber bundle* as the restriction of a fiber bundle to the points of $\cup_x$

$$\text{Local fiber bundle} = (\cup_x,\ \pi,\ T\mathscr{B}\rfloor_{\cup_x})$$

As interesting examples consider the cylinder (Fig. 9.1) and Möbius strip (Fig. 9.2) as fiber bundles.

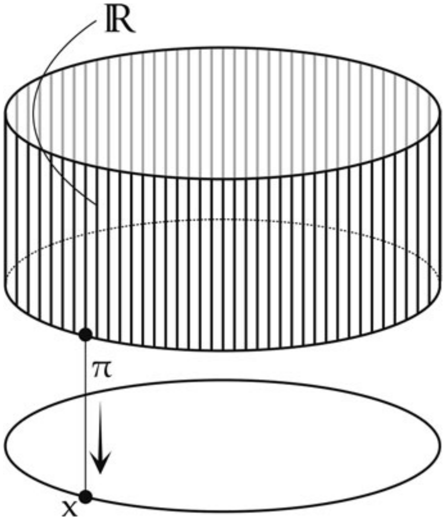


Fig. 9.1 Cylinder fiber bundle

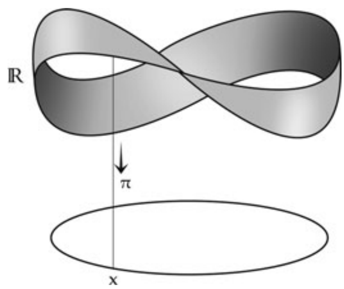


Fig. 9.2 The Möbius fiber bundle

The cylinder can be described as a fiber bundle with a circle S^1 as the base space and where each fiber is a segment of line (a compact manifold) with fixed length ℓ. The total space is then the set of lines in the rectangle $I\!R^2$, with identified lateral sides forming a cylinder, where the fibers are the line segments at each point, and the total space is the cylinder itself

$$(S^1, \pi, S^1 \times \ell)$$

On the other hand, by giving a rotation of each fiber around the middle line, we obtain as the total space the Möbius strip itself

$$(S^1, \pi, \text{ Möbius})$$

The helicity of the fibers prevents the identification of the total space Möbius with $S^1 \times \ell$. However, locally they can be identified.

9.2 Base Morphisms

Consider two fiber bundles $(\mathscr{M}, \pi, T\mathscr{B})$ and $(\mathscr{M}', \pi, T\mathscr{B}')$. A *morphism* between them is a differentiable map

$$\varphi : T\mathscr{B} \rightarrow T\mathscr{B}'$$

such that it takes a fiber of $T\mathscr{B}$ into a fiber of $T\mathscr{B}'$. In particular, two differentiable fiber bundles are isomorphic when φ is 1:1. If we take the bases $\mathscr{M}$ and $\mathscr{M}'$ to be the same, then we have a *base-isomorphism*.

Definition 9.2 (Trivial Fiber Bundle) A fiber bundle with base $\mathscr{M}$ and fibers $\mathscr{B}_p$ is said to be a *trivial fiber bundle* when the total space is the Cartesian product of the base $\mathscr{M}$ with a manifold Σ that is isomorphic to all fibers $\mathscr{B}_p$

$$(\mathscr{M}, \pi, \mathscr{M} \times \Sigma)$$

Σ is called the *typical fiber*.

The designation trivial fiber bundle results from the fact that the total space can be represented by a box where its elements (x, v) with $x \in \mathscr{M}$ and $v \in \Sigma$ are the sides of the box: In a trivial fiber bundle the fiber over p is

$$\mathscr{B}_p = (p, \Sigma) = \{(p, y), \ \forall\ y \in \Sigma\}$$

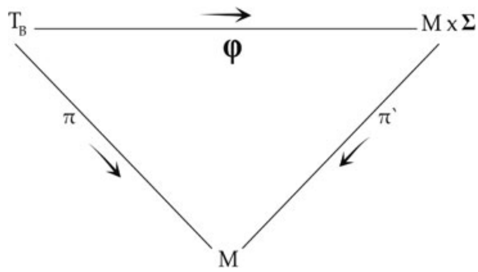
A simple example of trivial fiber bundle is the cylinder bundle shown in Fig. 9.1. In the particular case where the typical fiber Σ is a vector space we have a *trivial vector bundle*.

Definition 9.3 (Trivialization of Fiber Bundles) A fiber bundle is said to be trivializable when there is a base morphism

$$(\mathscr{M},\ \pi,\ T\mathscr{B})\rightarrow(\mathscr{M},\ \pi,\ \mathscr{M}\times\Sigma)$$

Note from this definition that for a given fiber bundle we may have several different trivializations, even considering those with the same typical fiber Σ, but with different base morphisms.

Fig. 9.3 Trivialization of a fiber bundle



Example 9.1 (Galilean Space–Time) As we have seen, the Galilean space–time $\mathscr{G}_4$ is the total space of a fiber bundle where the fibers are the simultaneity sections Σ_t and the base space is the absolute time axis $I\!R$.

$$(I\!R,\ \pi,\ \mathscr{G}_4)$$

In that space–time each simultaneous sections Σ_t is isomorphic to $I\!R^3$, so that $\mathscr{G}_4 = I\!R \times I\!R^3$. Therefore this fiber bundle is trivialized by the existence and the properties of the absolute time:

$$(I\!R,\ \pi,\ \mathscr{G}_4)\longrightarrow(I\!R,\ \pi,\ I\!R\times I\!R^3)$$

Since $\mathscr{G}_4$ is parameterized by $I\!R^4$, it follows that the trivialization can be defined by a specific choice of coordinate chart of $\mathscr{G}_4$ which are compatible with the Galilean group. For each Galilean transformation we have a trivialization, with the same typical fiber $I\!R^3$.

In general when a symmetry is not specified then the coordinates are mapped onto one another by the diffeomorphism group of the manifold. In this case, two trivializations of the same fiber bundle are said to be compatible when there is a diffeomorphism between the two typical fibers. Given a fiber bundle $(\mathscr{M},\ \pi,\ T\mathscr{B})$, and two trivializations of it given by

$$\phi : T\mathscr{B}\rightarrow\mathscr{M}\times\Sigma \quad\text{and}\quad \phi' : T\mathscr{B}\rightarrow\mathscr{M}\times\Sigma$$

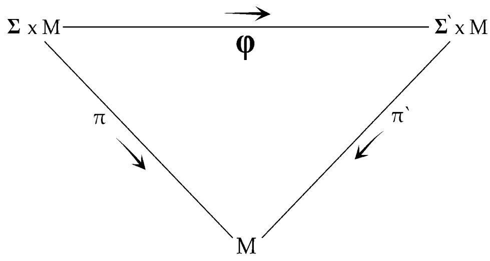


Fig. 9.4 Compatible trivializations

then the two trivializations are said to be equivalent when there is a base morphism φ such that the typical fiber of one is mapped in the typical fiber of the other, that is,

$$\varphi : \Sigma \to \Sigma'$$

Since $\pi = \pi' \circ \varphi$ we obtain

$$\varphi = \pi'^{-1} \circ \pi$$

which is the compatibility condition for equivalence. Two trivializations of the same fiber bundle are said to be equivalent when there is a base morphism between them.

Definition 9.4 (Local Trivialization) A given fiber bundle $(\mathscr{M}, \pi, T\mathscr{B})$ is *locally trivializable* when

a) $\forall x \in \mathscr{M}$, there is a neighborhood $\cup_x$ and manifold Σ such that

$$T\mathscr{B}\rfloor_{\cup_x} = \cup_x \times \Sigma$$

b) The manifold Σ is diffeomorphic to all fibers.

Example 9.2 (The Möbius Strip) An example of locally trivial fiber bundle is given by the Möbius strip. All fibers are equal segments of lines but they have different orientations (a helicity), which are diffeomorphic to a single segment.

Example 9.3 (The Newtonian Space–Time) Another example is given by Newton's space–time $\mathscr{N}_4$ associated with the fiber bundle

$$(I\!R,\ \pi, \mathscr{N}_4)$$

As in the Galilean case, the base of the fiber bundle is the absolute time axis. However, the simultaneity sections are only locally defined in the neighborhood $\cup_t$ of the time interval, isomorphic to $I\!R^3$. Therefore, Newton's space–time is *only locally trivialized* by the absolute time and its properties

$$(I\!R,\ \pi, \mathscr{N}_4) \to (\cup_t,\ \pi, \cup_t \times I\!R^3)$$

Trivializations of a fiber bundle can be associated with a symmetry group. For example, the transformations of coordinates of the Galilean group with a fixed origin (that is, not considering translations) are orthogonal transformations belonging to the group $SO(3)$. Since this is a three-parameter Lie group it is isomorphic to $I\!R^3$. In other words, the trivialization of the Galileo space–time can be written as

$$(I\!R,\ \pi,\ \mathscr{G}_4) \rightarrow (I\!R,\ \pi,\ I\!R \times I\!R^3) \rightarrow (I\!R,\ \pi,\ I\!R \times SO(3))$$

Similarly, for the Newtonian space–time, the generalized Galilean group

$$\begin{cases} x'^i = A^{\mu}_{j} x_j + c^i(t) \\ t' = at + b \\ \phi' = \phi + \dfrac{\partial^2 c^i(t)}{\partial t^2} x^i \end{cases}$$

and all simultaneity sections are locally isomorphic to $I\!R^3$. Fixing $c(x) = 0$ the group resumes to $S0(3)$ and we obtain the local trivialization with the same group

$$(I\!R, \pi,\ \mathscr{N}_4) \rightarrow (I\!R, \pi,\ \cup_t \times I\!R^3) \rightarrow (\cup_t,\ \pi,\ \cup_t \times SO(3))$$

These particular examples suggest the emergence of another type of fiber bundle which is associated with a Lie symmetry group G, called the principal fiber bundle of G.

9.3 Principal Fiber Bundles

Definition 9.5 (Principal Fiber Bundle) The *principal fiber bundle* or simply the principal bundle of a Lie group G is a fiber bundle with base $\mathscr{M}$ and where G *acts on the total space* $\mathscr{B}$ *as a map between fibers.* To make explicit the presence of G, a principal fiber bundle is usually denoted by a ordered tetrad

$$(G,\ \mathscr{M},\ \pi,\ T\mathscr{B})$$

The requirement that G is a Lie group will become clear in the next chapter. For now, it is sufficient to remind the fact that for a Lie group we can always deal with infinitesimal transformations, leading to its Lie algebra $\mathscr{G}$. Therefore, the principal fiber bundle of a Lie group can be always written in terms of its Lie algebra as

$$(\mathscr{G},\ \mathscr{M},\ \pi,\ T\mathscr{B})$$

From now on we shall refer to principal fiber bundles using *only its Lie algebra.*

Trivializations of principal fiber bundles are defined as before, given by a base morphism $\varphi : T\mathscr{B} \to \Sigma \times \mathscr{M}$, with the additional condition that *the Lie algebra* $\mathscr{G}$ *acts linearly upon the typical fiber* Σ

$$\mathscr{G} : \Sigma \to \Sigma$$

leading to the trivial principal bundle

$$(\mathscr{G},\ \mathscr{M},\ \pi,\ \mathscr{M} \times \Sigma)$$

where, as we said, the operators of $\mathscr{G}$ act as linear operators on the typical fiber Σ.

A particularly interesting trivialization is that defined by the space of the Lie algebra itself: *A trivialization of the principal bundle induces a particular representation of the Lie algebra where the representation space, is the space of the Lie algebra.* This particular representation was defined in Chapter 3 as the *adjoint representation* of $\mathscr{G}$. As we recall, this representation is defined by the structure constants of the group.

Reciprocally, *the adjoint representation of the Lie algebra of a group* G *induces a trivialization of the principal fiber bundle of* G.

The adjoint representation of a Lie algebra was defined as operators acting on the space of the Lie algebra, acting on its basis $\{X_a\}$ as

$$\tilde{\mathscr{G}}(X_a)X_b \overset{\text{def}}{=} [X_a, X_b] = f^c{}_{ab}X_c$$

Therefore, the adjoint representation is *unique as it is completely determined by the structure constants of the group* $f^c{}_{ab}$. This uniqueness is relevant because as we remember, in general, representations of a group are arbitrarily chosen. This is not the case of the adjoint representation which is self-contained in the Lie algebra structure.

From now on, we will denote the adjoint representation of a Lie algebra $\mathscr{G}$ by $\tilde{\mathscr{G}}$. It is the same algebra whose space is acted upon by the group (or algebra). Therefore, the trivialization of the principal bundle of a Lie group G by the adjoint representation of its Lie algebra is

$$(\tilde{\mathscr{G}},\ \mathscr{M},\ \pi,\ \mathscr{M} \times \tilde{\mathscr{G}})$$

As we see there is a notational redundancy, where $\tilde{\mathscr{G}}$ shows repeatedly. It is no longer necessary to specify the Lie algebra twice, and it has become common practice to denote the trivialized principal fiber bundle by the adjoint representation simply by the usual ordered triad

$$(\mathscr{M},\ \pi,\ \mathscr{M} \times \tilde{\mathscr{G}})$$

Using the adjoint representation we may reexamine the two previous examples.

Example 9.4 (Trivialization of the Galilean Space–Time) Taking the Lie algebra $\mathscr{G}_{SO(3)}$ as a subalgebra of the Galilean group defined in the Galilean space–time, the principal fiber bundle of this group is

$$(\mathscr{G}_{SO(3),I\!R},),\ \pi,\ \mathscr{G}_4)$$

which is trivialized to

$$(\mathscr{G}_{SO(3),I\!R},),\ \pi,\ I\!R \times \Sigma)$$

where the typical fiber Σ is isomorphic to all simultaneity sections and isomorphic to $I\!R^3$, upon which the $\mathscr{G}_{SO(3)}$ group acts. In particular taking Σ to be the space of the Lie algebra $\tilde{\mathscr{G}}_{SO(3)}$, we obtain the adjoint representation $\tilde{\mathscr{G}}_{SO(3)}$ and therefore the trivialization.

9.4 Connections

The purpose of the trivialization of a principal bundle by the adjoint representation is to obtain a connection.

As we have seen, in the adjoint representation the Lie algebra acts on itself, inducing a trivialization of the principal fiber bundle of G with base $\mathscr{M}$ to

$$(\mathscr{M},\ \pi,\ \mathscr{M} \times \tilde{\mathscr{G}})$$

In this trivial fiber bundle the fibers are all isomorphic to the Lie algebra space $\mathscr{G}$.

Now, suppose that our field Ψ defined on $\mathscr{M}$ by a Lagrangian $\mathscr{L}(\Psi, \Psi_{,\mu})$ has a Lie symmetry group G. This means that G and hence its Lie algebra $\mathscr{G}$ act on Ψ, keeping the Lagrangian invariant. Since in the adjoint representation $\tilde{\mathscr{G}}$ acts on the algebra space $\mathscr{G}$, then in this representation Ψ corresponds to an element of $\mathscr{G}$. Using this double role of the adjoint representation, we may express the field Ψ as linear combination of the Lie algebra basis $\{X_a\}$

$$\Psi = \sum \Psi^a X_a$$

or equivalently, the field can also be written in terms of the dual of the Lie algebra $\tilde{\mathscr{G}}^*$. This provides another equivalent trivialization of the principal fiber bundle of G, the *dual adjoint trivialization*

$$(\mathscr{M},\ \pi,\ \mathscr{M} \times \tilde{\mathscr{G}}^*)$$

Using this representation, the same field Ψ is written in terms of the dual basis $\{X^a\}$ of the Lie algebra (defined by $X^a(X_b) = \delta^a_b$), so that Ψ is now regarded as a one-form field:

$$\Psi = \sum \Psi_a X^a$$

We may now consider the three types of symmetry:

(a) When G is a *coordinate symmetry*, such as the Poincaré group, we may express the coordinate transformation generally as

$$x'^{\mu} = f^{\mu}(x^{\nu}, \theta_a)$$

Then by its definition $\{X^a\}$ can be expressed directly in terms of the coordinate basis as (from (3.7))

$$X_a = \sum a_a^{\mu}(x)\frac{\partial}{\partial x^{\mu}}$$

and its dual

$$X^a = \sum a_{\mu}^{a}(x)dx^{\mu}$$

Therefore, in the dual adjoint representation of a coordinate transformation we may express the field as a one-form field

$$\Psi = \sum \Psi_a X^a = \sum \Psi_a a^a_{\mu}(x)dx^{\mu} = \sum \Psi_{\mu} dx^{\mu}$$

where we have denoted $\Psi_{\mu} = \Psi_a a^a_{\mu}(x)$. Therefore the exterior derivative of the field Ψ gives a two-form field

$$d \wedge \Psi = \sum d\Psi_{\mu} \wedge dx^{\mu} = \sum \frac{\partial \Psi_{\mu}}{\partial x^{\nu}} dx^{\nu} \wedge dx^{\mu}$$

(b) In the case of a *global gauge symmetry* of a field defined on a space–time, the gauge transformation is

$$\Psi'^{\mu} = f^{\mu}(\Psi, \theta)$$

where in the global case, θ does not depend on the coordinates of the space–time. Therefore, given the adjoint representation of the Lie algebra of the gauge group with generators X_a, by the same token we may express the field Ψ in terms of the Lie algebra basis $\{X_a\}$, or of its dual $\{X^a\}$ as

$$\Psi = \sum \Psi^a X_a = \sum \Psi_a X^a \tag{9.1}$$

Since in the global case X^a do not depend on the coordinates x^μ, the exterior derivative of Ψ is a two-form field given by

$$d \wedge \Psi = \sum d\Psi_a \wedge X^a$$

However, the components Ψ_a of the field are functions of the space–time coordinates, so that we may write $d\Psi_a = \Psi_{a,\mu} dx^\mu$. Consequently, as in the previous case, the exterior derivative of Ψ can also be written in terms of the dual basis of the tangent bundle of $\mathscr{M}$ as

$$d \wedge \Psi = \sum d\Psi_a \wedge X^a = -\sum \Psi_{a,\mu} X^a \wedge dx^\mu \tag{9.2}$$

(c) For *local gauge symmetries*, the group and its Lie algebra are locally defined. Therefore its base $\{X_a\}$ and the dual $\{X^a\}$ also depend on the coordinates of $\mathscr{M}$. In this case, the exterior derivative acts on both factors of the one-form field

$$\Psi = \sum \Psi_a X^a \tag{9.3}$$

as

$$d \wedge \Psi = \sum d\Psi_a \wedge X^a + \Psi_a \, d \wedge X^a \tag{9.4}$$

where $d \wedge X^a$ is a two-form. As such it can be expressed as an exterior product of X^a with another one-form ω^a_b belonging to the same space:

$$d \wedge X^a = \sum \omega^a{}_b \wedge X^b$$

Now, we have a more complicated situation as compared with (9.2), because we need to relate also the one-form X^b to the cotangent coordinate basis dx^μ (Fig. 9.5). This relation is formally done by the derivative map of a base morphism between the cotangent bundle $(\mathscr{M}, \pi, T\mathscr{M}^*)$ and the trivialized dual principal fiber bundle $(\mathscr{M}, \pi, \tilde{\mathscr{G}}^*)$ which is defined by the Jacobian matrix $\left(\dfrac{\partial f^a}{\partial x^\mu}\right)$ of the transformation between basis

$$X^a = \frac{\partial f^a}{\partial x^\mu} dx^\mu \tag{9.5}$$

With this transformation we may express

$$d \wedge X^a = \omega^a_b \wedge \frac{\partial f^b}{\partial x^\mu} dx^\mu = \omega^a_\mu \wedge dx^\mu$$

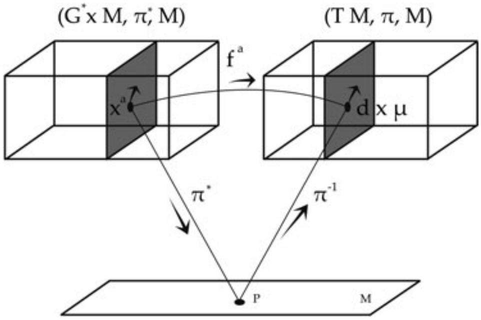


Fig. 9.5 Correspondence between X^a and dx^μ

where we have denoted

$$\omega^a_\mu = \omega^a_b \frac{\partial f^b}{\partial x^\mu}$$

Since ω^a_μ is a one-form with components in the dual adjoint representation $\tilde{\mathscr{G}}^*$, it may be expressed in terms of the dual basis $\{X^a\}$ as

$$\omega^a_\mu = -A^a_{b\,\mu} X^b$$

After inverting the order of the last exterior product and replacing in (9.4), we obtain

$$d \wedge \Psi = \Psi_{a,\mu} dx^\mu \wedge X^a - \Psi_a A^a_{\mu b} X^b \wedge = (\delta^a_b \partial_\mu + A^a_{\mu b})\, \Psi_a \; dx^\mu \wedge X^b$$

or, denoting

$$D^a_{\mu b} = \delta^a_b \partial_\mu + A^a_{\mu b} \tag{9.6}$$

we may write the exterior derivative of the field for a local gauge symmetry as

$$d \wedge \Psi = \sum D^a_{\mu b}\, \Psi_a \; dx^\mu \wedge X^b$$

The expression $D^a_{\mu b}\, \Psi_a$ extends the exterior derivative by the inclusion of the coefficients $A^a_{\mu b}$ defined in the Lie algebra of the local symmetry group.

Definition 9.6 (Gauge Connection) Given a Lie group G acting as symmetry of a physical field Ψ the vector-matrix derivative operator

$$D = Id + A \tag{9.7}$$

is called the *covariant exterior derivative operator* relative to the gauge connection matrix-vector A. The vector components in space-time are

$$D_\mu = I\partial_\mu + A_\mu \tag{9.8}$$

with matrix components $D^a_{\mu b}$ defined in the Lie algebra of the symmetry group.

However, to be consistent with the usual derivative operator and the covariant derivative defined in the geometry of manifolds of Chapter 2 and the exterior derivative defined in Chapter 4, the operator (9.8) must satisfy the formal conditions (which are typical of derivatives):

1) $D \wedge (\alpha\Psi + \beta\Phi) = \alpha D(\Psi) + \beta D(\Phi)$
2) $D \wedge f = df(x)$
3) $D \wedge (f(x)\Psi) = df(x) \wedge \Psi + f(x)D\Psi$
4) $D \wedge (\Psi \wedge \Phi) = (D\Psi) \wedge \Phi + \Psi \wedge (D\Phi)$

where $f(x)$ is a scalar function defined in $\mathscr{M}$.

It is important to observe that A_μ was not really postulated or defined. They are just coefficients of the variation of the Lie algebra basis in terms of the covariant coordinate basis $\{dx^\mu\}$. On the other hand, it was derived from a symmetry group of a Lagrangian, so that it must coincide with the same coefficients of the derivative operator (8.22) defined in Noether's theorem for local gauge transformations, which has also components in the Lie algebra. Therefore, the above connection components $A^a_{\mu b}$ are the same as those introduced by Noether, satisfying Noether's condition (8.21)

$$\sum_a \mathscr{F}_a \frac{\partial\theta^a}{\partial x^\mu} = \sum_a \mathscr{F}_a g A^a{}_{\mu b}\,\theta^b$$

Consequently the local gauge derivative used in Noether's theorem is the same covariant derivative obtained above from the Lie algebra trivialization of the symmetry group.

Summarizing, we have seen that a principal fiber bundle of a Lie symmetry group G, defined on a space–time $\mathscr{M}$ of some field Ψ, can be trivialized by taking the dual adjoint representation of the Lie algebra of G, $\tilde{\mathscr{G}}^*$, when the total space becomes a product of the base by a typical fiber

$$(\mathscr{M},\ G,\ \pi,\ T\mathscr{B}) \to (\mathscr{M},\ \pi,\ \mathscr{M} \times \tilde{\mathscr{G}}^*)$$

Here $T\mathscr{B}$ is the total space where the fields are defined as maps $\Psi : \mathscr{M} \to T\mathscr{B}$.

The trivialization is a direct consequence of the fact that the symmetry group G acts on the field manifold $T\mathscr{B}$ and also on any of its representation space. Such scheme is very general, but in the particular case of the adjoint representation of the Lie algebra of G, the field becomes a vector in the Lie algebra or in its dual $\tilde{\mathscr{G}}^*$.

In the case where the Lie groups are locally defined, the dependence of the parameters on the coordinates implies that the basis of its Lie algebra $\{X_a\}$ defined in (3.7) also depends on the coordinates of $\mathscr{M}$. In such cases the trivialization of the principal bundle by the adjoint representation implies that the field is expressed as

$$\Psi(x^\mu) = \Psi^a(x^\mu) X_a(x^\mu) = \Psi_a(x^\mu)\, X^a(x^\mu)$$

where both the components and the basis vectors depend on the coordinates x^μ. The consequence of this double dependence on the coordinates (of the components and of the basis vectors) is the emergence of an affine connection and of the exterior covariant derivative in local gauge fields defined by

$$D \wedge \Psi = d\Psi_a X^a + \Psi_a \wedge dX^a$$

or

$$D \wedge \Psi = D^a_{\mu b} \Psi_a \; dx^\mu \wedge X^b$$

where $D^a_{\mu b}$ are components of the *exterior covariant derivative* matrix operator

$$D_\mu = I\partial_\mu + A_\mu$$

with entries defined in the adjoint representation of the Lie algebra. Thus, in matrix notation we may write the gauge covariant derivative operator as matrix-vector operator D:

$$D = D_\mu dx^\mu \tag{9.9}$$

We say that the matrix-vector A_μ are the components of the connection associated with the Lie group G.

Since in Noether's theorem $A^b_{\mu a}$ were not really defined we still do not know what the connection is. We only know that it is required to define the conserved quantities for local gauge symmetries. The Yang–Mills theory described in the next section defines such connection.