Chapter 7
Vector, Tensor, and Spinor Fields

7.1 Vector Fields

The prime example of a vector field is the electromagnetic field in Minkowski space–time. It is an essential component of the development of modern physics, including the emergence of relativity and the relevance of the concept of symmetry in physics. Due to the importance of this combination of theoretical and experimental results to the development of gauge theory let us briefly review the basics of the electromagnetic field (for more details, see, e.g., [93]) and some more advanced topics involving its interactions with scalar fields.

7.1.1 The Electromagnetic Field

The systematic observations of electricity is credited to Stephen Grey and François da Fay between 1736 and 1739. However, real progress was possible only after the Leyden (Holland) bottles were made in 1746. Quantitative results started to show up around 1777 with the invention of the torsion balance by Charles Augustin Coulomb, leading to the Coulomb law:

$$\mathbf{F} = K\frac{qq'}{r^2}\frac{\mathbf{r}}{r}, \quad K = constant$$

The systematic study of the electric current was possible only after 1794 with the invention of the electric battery by Allesandro Volta, allowing for the use of the electric current in a controlled way. The electromagnetism appeared around 1819, after Hans Christian Oersted observed that magnetic forces, originally observed only with permanent magnets, could also be *induced* by the presence of varying electric current. The relation between this magnetic field and the derivative of the current with respect to time was discovered by André Marie Ampére in 1819, leading to a differential expression for the Coulomb law.

Ampère imagined the electric current as formed by small cylindrical moving sections with length $d\ell$ and area A of the conductor, with charge density ρ, so that

M.D. Maia, *Geometry of the Fundamental Interactions*,
DOI 10.1007/978-1-4419-8273-5_7,

the charge in each section is $dq = \rho dV = \rho d\ell dA$. Replacing in Coulomb's law he obtained a differential expression for the electric force

$$|d\mathbf{F}| = K\frac{q\rho A d\ell}{r^2}$$

The experimental observation by Michael Faraday that this force induced a magnetic attraction between two close wires led to the concept of magnetic induction and magnetic field flux and eventually to the Faraday law of 1831 stating that, *The variation of the magnetic flux with time induces an electric current on a conductor which is proportional to that variation.*

The final step in this rather complex development was given by Jean-Baptiste Biot and Félix Savart in 1822, obtaining the expression

$$\mathbf{F}_{\text{mag}} = K'\frac{d\ell \wedge d\ell'}{r}, \quad K' = constant$$

where $d\ell$ and $d\ell'$ are the tangent vectors to two small cylindrical sections of two conductors separated by a distance r.

Thus the electric and magnetic fields which were originally thought to be two independent fields become related to each other. However, the differential second-order equations describing these two fields were not quite consistent. The completion of the consistency process was elaborated in 1861 by James Clark Maxwell [94].

Of course, the electric and magnetic field equations were originally written with the absolute time t and consequently with the idea of simultaneity sections Σ_t as in the Galilean space–time. They were expressed in terms of the scalar ϕ and vector potentials $\mathbf{A}$ as

$$\mathbf{B} = \nabla \wedge \mathbf{A}, \qquad \mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{7.1}$$

From these expressions we obtain immediately two homogeneous equations

$$< \nabla, \mathbf{B} > = 0, \qquad \nabla \wedge \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \tag{7.2}$$

The two remaining equations, the Coulomb and Ampère equations, involve electrical charges and current

$$\nabla^2\phi = -4\pi\rho, \qquad \nabla^2 A = -\frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

Originally these equations were inconsistent because the Faraday and Ampère equations hold under different conditions. This was fixed by Maxwell, and today they combine in the four Maxwell's equations

$$< \nabla, \mathbf{B} > = 0 \tag{7.3}$$

$$\nabla \wedge \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \tag{7.4}$$

$$< \nabla, \mathbf{E} > = 4\pi\rho \tag{7.5}$$

$$\nabla \wedge \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = -4\pi \mathbf{J} \tag{7.6}$$

Only the last two (non-homogeneous) equations are the Euler–Lagrange equations with respect to the vector potential **A** and the scalar potential ϕ in the Lagrangian

$$\mathscr{L} = \frac{< \mathbf{E}, \mathbf{E} > - < \mathbf{B}, \mathbf{B} >}{8\pi} - \rho\phi - < \mathbf{J}, \mathbf{A} > \tag{7.7}$$

This Lagrangian is invariant under a special local transformation of the potential functions given by

$$\mathbf{A}' = \mathbf{A} + \nabla\theta \tag{7.8}$$

$$\phi' = \phi - \frac{1}{c}\frac{\partial \theta}{\partial t} \tag{7.9}$$

where the parameter θ *is a function of the space–time coordinates*.

The invariance of the Lagrangian under the above transformations follows directly from the fact that in (7.1), the expressions of **E** and **B** are invariant under the above transformations. Indeed

$$\begin{aligned} \mathbf{E}' &= -\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\nabla\theta - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c}\frac{\partial}{\partial t}\nabla\theta = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \mathbf{E} \\ \mathbf{B}' &= \nabla \wedge (\mathbf{A} + \nabla\theta) = \nabla \wedge \mathbf{A} = \mathbf{B} \end{aligned}$$

Consequently, Maxwell's equations also do not change under the same transformations.

The set of transformations (7.8) and (7.9) constitute a group with respect to the composition

$$\begin{aligned} &\mathbf{A}' = \mathbf{A} + \nabla\theta, \qquad \mathbf{A}'' = \mathbf{A}' + \nabla\theta' \\ &\phi' = \phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \quad \phi'' = \phi' - \frac{1}{c}\frac{\partial \theta'}{\partial t} \end{aligned}$$

which combine into transformations of the same kind

$$\mathbf{A}'' = \mathbf{A} + \nabla(\theta + \theta') = \mathbf{A} + \nabla\theta''$$
$$\phi'' = \phi - \frac{1}{c}\frac{\partial}{\partial t}(\theta + \theta') = \phi - \frac{1}{c}\frac{\partial \theta''}{\partial t}$$

The identity transformation corresponds to $\theta =$ constant. Choosing this constant to be zero, the inverse of a transformation corresponds naturally to $-\theta$. We may easily check that the above composition is associative. Since the order of composition does not affect the result we have an Abelian group, where the only parameter θ is a function of the coordinates. This is the *electromagnetic gauge group*. This group is a Lie group with one coordinate-dependent parameter. Therefore, this group is isomorphic to the local group of rotations $SO(2)$ and as we have seen also to the local unitary group $U(1)$.

With the appropriate choice of conditions imposed on θ, we obtain different solutions of Maxwell's equations. The two most common choices are as follows:

(a) The Lorentz gauge
From (7.8) and (7.9) we may write

$$< \nabla, \mathbf{A}' > = < \nabla, \mathbf{A} > + \nabla^2\theta$$
$$\frac{1}{c}\frac{\partial \phi}{\partial t} = \frac{1}{c}\frac{\partial \phi}{\partial t} - \frac{1}{c}\frac{\partial^2 \theta}{\partial t^2}$$

so that

$$\left(< \nabla, \mathbf{A}' > + \frac{1}{c}\frac{\partial \phi'}{\partial t}\right) = \left(< \nabla, \mathbf{A} > + \frac{1}{c}\frac{\partial \phi}{\partial t}\right) + \nabla^2\theta - \frac{1}{c^2}\frac{\partial^2 \theta}{\partial t^2}$$

Therefore, assuming that θ is such that

$$\nabla^2\theta - \frac{1}{c}\frac{\partial^2 \theta}{\partial t^2} = 0 \tag{7.10}$$

it follows that

$$< \nabla, \mathbf{A}' > + \frac{1}{c}\frac{\partial \phi'}{\partial t} = < \nabla, \mathbf{A} > - \frac{1}{c}\frac{\partial \phi}{\partial t} = C$$

where C is a constant. In particular choosing this constant to be $C = 0$, we obtain the Lorentz gauge condition

$$< \nabla, \mathbf{A} > - \frac{1}{c}\frac{\partial \phi}{\partial t} = 0$$

which is compatible with the electromagnetic wave solution of Maxwell's equations.

(b) The Coulomb gauge
Here we consider a more restrictive condition where the scalar potential does not depend on time. Then we obtain

$$< \nabla, \mathbf{A} > = 0$$

Replacing this in Maxwell's equations we obtain

$$\nabla^2 \phi = 4\pi\rho$$

which is Poisson's equation for a charge density $\rho(x, t)$ and whose solution describes the Coulomb potential for the electrostatic field of an isolated particle.

7.1.2 The Maxwell Tensor

The development of the electromagnetic theory in the beginning of the 20th century led to the conclusion that Maxwell's theory written in the Galilean space–time with its simultaneous sections was not compatible with the explanations of the negative results of the *Michelson–Morley experiment* on the propagation of light within the context of the Galilean space–time. This resulted in the theory of special relativity based on the Minkowski space–time previously described. Then Maxwell's equations are more appropriately written in the Minkowski space–time, using the concept of proper time denoted by τ. Correspondingly, the Galilean group was replaced by the Poincaré group and the light speed was assumed to be a fundamental constant of nature.

The electromagnetic field which was written as a pair of vectors ($\mathbf{E}$, $\mathbf{B}$) can now be more appropriately written as an anti-symmetric rank two tensor field composed of the components of $\mathbf{E}$ and $\mathbf{B}$ given by (7.1):

$$E_i = -\partial_i \phi - \frac{1}{c}\frac{\partial}{\partial \tau} A_i \tag{7.11}$$

$$B_i = \partial_j A_k - \nabla_k A_j \qquad (i, j, k \text{ cyclic} = 1, 2, 3) \tag{7.12}$$

Define the tensor $F = (F_{\mu\nu})$ by its components

$$F_{ij} = \partial_i A_j - \partial_j A_i, \quad F_{ii} = 0$$

$$F_{i4} = \partial_i A_4 - \partial_4 A_i, \quad F_{44} = 0$$

where we have denoted $\partial_4 = \frac{1}{c}\frac{\partial}{\partial \tau}$. Introducing the potential four-vector

$$A = (\mathbf{A}, -\phi) = (A_1,\ A_2,\ A_3, A_4)$$

and denoting its individual components by A_μ, the expressions of $F_{\mu\nu}$ can be summarized as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \mu; \nu = 1, \ldots, 4$$

Therefore, comparing with (7.11) and (7.12) we obtain

$$F_{12}=\partial_1 A_2-\partial_2 A_1=B_3,\ F_{13}=\partial_1 A_3-\partial_3 A_1=-B_2,\ F_{23}=\partial_2 A_3-\partial_3 A_2=B_1,$$
$$F_{i4} = \partial_i A_4 - \partial_4 A_i = -\partial_i \phi - \frac{\partial}{\partial \tau} A_i = E_i, \quad F_{44} = 0$$

In a shorter notation we may write

$$F_{i4} = E_i, \qquad F_{ij} = \varepsilon_{ijk} B_k \tag{7.13}$$

where ε_{ijk} is the standard Levi-Civita permutation symbol for $i, j, k = 1, \ldots, 3$.

Explicitly, we obtain an array (it is not a matrix)

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_3 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & B_2 \\ E_2 & -B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

which is known as the covariant Maxwell tensor. The corresponding contravariant tensor has components

$$F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\beta} F_{\rho\beta}$$

In order to write Maxwell's equations in terms of the Maxwell tensor, we start with the components of the two non-homogeneous equations (7.5) and (7.6)

$$\sum \partial_i E_i = 4\pi\rho$$
$$\sum \varepsilon_{ijk}\partial_j B_k - \frac{1}{c}\frac{\partial E_i}{\partial \tau} = -4\pi J_i$$

Using (7.13), the two non-homogeneous equations correspond to

$$\sum \partial_i F_{i4} = 4\pi\rho$$
$$\sum \partial_j F_{ij} - \frac{1}{c}\frac{\partial F_{i4}}{\partial \tau} = -4\pi J_i$$

However, $F^{j4} = -F_{j4}$ and $F^{ij} = F_{ij}$ so that

$$-\sum \partial_j F^{j4} = 4\pi\rho \quad \text{(Coulomb)}$$
$$\sum \partial_j F^{ij} + \partial_4 F^{j4} = J \quad \text{(Ampère)}$$

Defining the four-dimensional current $(\mathbf{J}, -c\rho) = (J_1, J_2, J_3, J_4)$ with components J_μ and $J^\mu = \eta^{\mu\rho} J_\rho$, it follows that the above two equations can be summarized as

$$F^{\mu\nu}{}_{,\nu} = \frac{4\pi}{c} J^\mu$$

On the other hand, the two homogeneous Maxwell's equations are

$$\sum \partial_i B_i = 0, \quad \sum \varepsilon_{ijk} \partial_j E_k + \frac{1}{c} \frac{\partial B_i}{\partial \tau} = 0$$

which can also be written in terms of $F_{\mu\nu}$ as

$$\partial_i \varepsilon_{ijk} F_{jk} = 0$$

$$\varepsilon_{ijk} \partial_j F_{k4} + \frac{1}{c} \frac{\partial}{\partial \tau} \varepsilon_{ijk} F_{jk} = 0$$

or, using the four-dimensional Levi-Civita permutation symbol

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of 1234} \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of 1234} \\ 0 & \text{in any other case} \end{cases}$$

The last two equations can be summarized as

$$\varepsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0$$

Therefore the four Maxwell's equations are equivalent to

$$F^{\mu\nu}_{\ \ ,\nu} = 4\pi J^\mu \tag{7.14}$$

$$\varepsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0 \tag{7.15}$$

which are known as the manifestly covariant Maxwell's equations, having the same shape in any Lorentz frame in Minkowski's space–time.

7.1.2.1 The Lagrangian of the Electromagnetic Field

The Lagrangian of the electromagnetic field (7.7) can be written in terms of the Maxwell tensor $F_{\mu\nu}$. For that purpose consider the four-vectors J_μ and A_μ defined previously and define the Lorentz-invariant quantity

$$\eta^{\mu\nu} A_\mu J_\nu = < \mathbf{J}, \mathbf{A} > + \rho\phi$$

On the other hand, from the components of **E** and **B** written in terms of $F_{\mu\nu}$ we obtain

$$< \mathbf{E}, \mathbf{E} > = \sum E_i E_i = -\sum F_{14} F_{14} - \sum F_{24} F_{24} - \sum F_{34} F_{34}$$

and

$$< \mathbf{B}, \mathbf{B} > = \sum B_i B_i = F_{12} F^{12} + F_{13} F^{13} + F_{23} F^{23}$$

Therefore,

$$< \mathbf{E}, \mathbf{E} > - < \mathbf{B}, \mathbf{B} > = \frac{1}{2} F^{\mu\nu} F_{\mu\nu}$$

where the factor 1/2 was included to compensate for the repeated terms in the sum of the right-hand side. Therefore, the Lagrangian of the electromagnetic field (multiplied by 4π) is

$$\mathscr{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - 4\pi J^{\mu} A_{\mu} \tag{7.16}$$

To see that this form of the Lagrangian leads directly to the covariant equations let us calculate the variation with respect to A_μ

$$\frac{\partial \mathscr{L}}{\partial A_\mu} = 4\pi J^{\mu} \quad \text{and} \quad \frac{\partial \mathscr{L}}{\partial A_{\mu,\nu}} = \frac{1}{2} F^{\rho\sigma} \frac{\partial F_{\rho\sigma}}{\partial A_{\mu,\nu}} = \frac{1}{2} \left(\delta^{\alpha}_{\rho} \delta^{\nu}_{\sigma} - \delta^{\nu}_{\rho} \delta^{\mu}_{\sigma} \right) F^{\rho\sigma} = F^{\mu\nu}$$

Consequently, the electromagnetic Euler–Lagrange equations are

$$F^{\mu\nu}{}_{,\nu} = 4\pi J^{\mu} \tag{7.17}$$

The other two equations (the homogeneous equations) are obtained directly from the expressions of **E** and **B** in terms of A and ϕ. For this reason they are often referred to as non-dynamical equations. Indeed, they are part of an identity satisfied by $F_{\mu\nu}$ as we shall see later.

7.1.3 The Nielsen–Olesen Model

The Nielsen–Olesen model arose originally from an attempt to describe a quantized magnetic flux [95]. Consider that we have a scalar field φ and the electromagnetic field $F_{\mu\nu}$, as if they are non-interacting, given by the Lagrangian

$$\mathscr{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \eta^{\mu\nu} \varphi^{*}_{,\mu} \varphi_{,\nu} - U(\varphi) \tag{7.18}$$

The first term is just the electromagnetic Lagrangian, the second term is the kinetic term of the scalar field φ, and the third term is the potential energy of φ chosen to be a generalization of the quartic scalar potential seen in the previous chapter

$$U(\varphi) = -2\alpha\beta\varphi^*\varphi + \alpha^2(\varphi^*\varphi)^2$$

where α, β are constants. Notice that there is not an explicit interaction term involving the two fields, so they behave as if they were two independent fields.

As we have seen in the last section, the electromagnetic field is invariant under the gauge transformation (7.8) and (7.9), which can now be written in terms of the components of the four-vector potential as

$$A'_\mu = A_\mu + \theta(x)_{,\mu}$$

On the other hand, from the same arguments seen in the study of the quartic potential in the previous chapter, the scalar field component in the Lagrangian is invariant under the global $U(1)$ transformations, but not under the local $U(1)$ group given by the transformations

$$\varphi' = e^{i\theta'(x)}\varphi$$

Therefore, the Nielsen–Olesen Lagrangian also has two independent local gauge groups: the gauge group of the electromagnetic field with parameter $\theta(x)$ and the unitary gauge group $U(1)$ of the scalar field with parameter $\theta'(x)$. As we have seen, the latter gauge transformation is not a symmetry, unless we take infinitesimal transformations like in (6.12), and replacing the partial derivatives in the Lagrangian by the more general covariant derivative (6.13) $D_\mu = I\partial_\mu + i\theta'_{,\mu}$.

To solve the problem of handling two independent gauge transformations Nielsen and Olesen proposed that they are different manifestations of the same group, by assuming that

$$i\theta'_{,\mu} = gA_\mu, \quad g = \text{constant} \tag{7.19}$$

With such condition, the Lorentz gauge implies that $\partial^\mu\theta'_{,\mu} \equiv g\partial^\mu A_\mu = 0$. Therefore, using the Lorentz gauge, the two gauge symmetries become just one, namely $U(1)$, and the gauge covariant derivative becomes

$$D_\mu = \partial_\mu + i\theta'_{,\mu} = \partial_\mu + gA_\mu \tag{7.20}$$

Then the original Lagrangian can be rewritten with D_μ in place of the partial derivative ∂_μ:

$$\mathscr{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \eta^{\mu\nu}(D_\mu\varphi)^*(D_\nu\varphi) - U(\varphi) \tag{7.21}$$

With this covariant derivative the Lagrangian becomes constant under the local gauge $U(1)$.

Since D_μ depends on A_μ, the original Nielsen–Olesen Lagrangian acquired an interaction term that did not exist before. To see this term explicitly, let us expand the covariant derivatives

$$\mathscr{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \eta^{\mu\nu}\left(\varphi^*_{,\mu}\varphi_{,\nu} + gA_\mu\varphi^*\varphi_{,\nu} + gA_\nu\varphi^*_{,\mu}\varphi + g^2A_\mu A_\nu\varphi^*\varphi\right) - U(\phi)$$

where the interaction terms are those involving products of both fields and their derivatives. The emergence of the interaction has some interesting physical consequence.

7.1.3.1 The Meissner Effect

The Euler–Lagrange equations obtained from (7.21) with respect to A_μ are

$$F^{\mu\nu}{}_{,\mu} - g\,\eta^{\mu\nu}[\varphi^*_{,\nu}\varphi + \varphi_{,\nu}\varphi^* + 2gA_\nu\varphi^*\varphi] = 0 \tag{7.22}$$

and with respect to φ they are

$$g\eta^{\mu\nu}\left[A_\mu(D_\nu\varphi)^* - \frac{\partial U(\varphi)}{\partial\varphi} - g\eta^{\mu\nu}D_\nu(\varphi\varphi^*)\right] = 0 \tag{7.23}$$

(here we have written these equations using D_μ just for convenience. The Euler–Lagrange equations are usually written with the ordinary derivatives.) Since A_μ are the components of the electromagnetic potential, it must also satisfy the dynamical Maxwell's equations (7.17). Therefore, replacing the Maxwell tensor $F_{\mu\nu}$ in (7.22), we obtain a total of eight equations and only five unknowns A_μ and φ, so that the system is over-determined. This means that we cannot guarantee that the system remains consistent in its evolution.

The excess of equations can be lessened by reducing the number of dimensions from 4 to 3 = 2 + 1 (with coordinates x,y,t). There is no fundamental implication in this, as it means only that the sought solution is valid only in a three-dimensional subspace-time of space–time. In this case we obtain a compatible system with five equations. From (7.22) a solution of this system on empty space ($J^\mu = 0$), is given by

$$A_\mu = -\frac{(\varphi\varphi^*_{,\mu} - \varphi^*\varphi_{,\mu})}{2g\,\varphi\varphi^*} \tag{7.24}$$

Replacing this in (7.23) we obtain an equation involving only $\varphi(x, y, t)$.

In a practical application of this solution, consider that S is a flat surface limited by a circle c with radius r, in a region where there is no electrical current. Using the center of the circle as the center of a polar system of coordinates (r, θ, t) we may express the solution $\varphi = \sqrt{f(r)}e^{i\theta}$. Replacing this solution in (7.22) we obtain the electromagnetic potential A_μ in terms of $f(r)$ and θ.

We may choose the radius of the circle such that $\varphi^*\varphi = 1$. In this normalization, the magnetic field generated by this (2+1)-potential vector, as always, given by $B = \nabla \wedge A$, produces a *magnetic flux* across an arbitrary surface S in the (x, y) plane, limited by a closed curve c, given by

$$\Phi = \int\int < B, dS_n > = \int\int < \nabla \wedge A, dS_n > = \oint_c A_\mu dx^\mu$$

In the traditional electromagnetic field, this flux would be given by a current in c. Since here we have a vacuum solution ($J^\mu = 0$), then the flux should also be zero. However, using the above solution we find that

$$\Phi = \oint A_\mu dx^\mu = \frac{n}{g} \oint \theta_{,\mu} dx^\mu = \frac{n\pi}{g}$$

where n is the number of times in which the circle is run. *Contrary to the expectations it is not zero,* but it is discrete, depending on this integer n.

This result was confirmed by an experiment by Walther Meissner and Robert Ochsenfeld in 1933 and is known as the *Meissner effect* [96]. The circle c was drawn in a neutral metal plate (without any electrical current). A coil with n turns (*called the winding number*) with the same diameter as the circle was placed orthogonally to the plate. When a current flows in the coil, a magnetic flux should be produced on the disk drawn in the plate, but classically and at room temperature that flux is shielded by the plate itself. Nonetheless, at the critical temperature, they observed a flux distribution on the opposite side of the plate. The only possible interpretation of this somewhat strange result is that of a tunneling effect of a quantum *magnetic flux*. Within the assumptions made, the quantum effect on the flux appears under extremely low temperatures. When the temperature rises the quantum flux disappears.

When the plate is kept at room temperature and the magnetic field is produced by a cooled permanent magnet, then the flux causes a levitating effect on the magnet. The Meissner effect is thus responsible for the ongoing experiments on magnetic levitation and applications in public transportation.

The existence of a quantized flux only on one side of the plane may be also interpreted as the result of quantum *magnetic monopole* called the 'tHooft-Polyakov monopole [97, 98]. As in the example given by (6.1), the corresponding magnetic charge can be obtained by a symmetry breaking mechanism. More specifically, consider (7.21) where the parameters are chosen to be $\alpha^2 = \lambda/3! > 0$ and $2\alpha\beta = \mu^2$. Then the minimal energy condition $\partial U/\partial\varphi = 0$ gives

$$\varphi^* \left(\mu^2 + \frac{\lambda}{3!}(\varphi^*\varphi)\right) = 0$$

Therefore, if $\mu^2 > 0$, the only solution $\varphi = 0$. On the other hand, if $\mu^2 = -m^2 < 0$, then we have an infinite number of non-trivial vacuum states given by

$$\varphi_0 = \pm\sqrt{\frac{6m^2}{\lambda}} e^{i\theta}$$

When any of these infinite vacua states is chosen for a value of θ, the gauge symmetry of the field φ is broken and the Lagrangian acquires a mass term m^2 (proportional to $\varphi^*\varphi$), which can be interpreted as the *magnetic mass of the monopole*.

7.2 Spinor Fields

The best way to define *spinor fields* is through a particular tensor structure called a Clifford algebra defined on space–time.

Definition 7.1 (Clifford Algebras) The Clifford algebra $\mathbb{C}_n$ generated by an n-dimensional vector space V is the quotient of the tensor algebra $V \otimes V$ by the bilateral ideal I, defined by a bilinear form B in V and denoted by [99]

$$\mathbb{C}_n = (V \otimes V)/I$$

A bilinear form is a map $B : V \times V \rightarrow I\!R$, which is linear in both arguments $B(\mathbf{v}, \mathbf{w}) \in I\!R$. The above expression defines a subspace of the tensor algebra $V \otimes V$ given by the condition

$$\mathbf{v} \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{v} = B(\mathbf{v}, \mathbf{w})$$

This specifies that the rank-2 tensors in $\mathbb{C}_n$ are symmetric tensors $(V \otimes V)$ and that they are proportional to $B(\mathbf{v}, \mathbf{w})$. In terms of a basis $\{e_\alpha\}$ of V the bilateral ideal corresponds to imposing to the tensor algebra the condition

$$\mathbf{e}_\alpha \otimes \mathbf{e}_\beta + \mathbf{e}_\beta \otimes \mathbf{e}_\alpha = B(\mathbf{e}_\alpha, \mathbf{e}_\beta)$$

In general the tensor product notation in $\mathbb{C}_n$ is simplified to $e_\alpha \otimes e_\beta + e_\beta \otimes e_\alpha = e_\alpha e_\beta + e_\beta e_\alpha$. Denoting the coefficients of the bilinear form by $B(\mathbf{e}_\alpha, \mathbf{e}_\beta) = 2g_{\alpha\beta}$, we may write the Clifford algebra as

$$e_\alpha e_\beta + e_\beta e_\alpha = 2g_{\alpha\beta} e_0 \tag{7.25}$$

where e_0 denotes the identity element of the algebra:

$$e_\alpha e_0 = e_0 e_\alpha$$

The dimension of $\mathbb{C}_n$ is given by the maximum number 2^n of linearly independent elements of the algebra obtained with the independent products of the generators. Therefore, a generic element of $\mathbb{C}_n$ is given by the linear combination of the generators and their independent products:

$$X = X^0 e_0 + X^\alpha e_\alpha + X^{\alpha\beta} e_\alpha e_\beta + \cdots + X^{\alpha\beta\ldots\gamma} e_\alpha e_\beta, \cdots, e_\gamma$$

Example 7.1 (Complex Algebra) The complex algebra is the simplest Clifford algebra $\mathbb{C}_1 = \mathbb{C}$, with just one generator $e_1 = i$ plus the identity element $e_0 = 1$. The dimension of the algebra is $2^1 = 2$, and its generic elements are like

$$X = X_0 e_0 + X_1 e_1 = X_0 1 + X_1 i$$

It is usual to consider the real $I\!R$ as a Clifford algebra with just the identity element, denoted by $\mathbb{C}_0 = I\!R$.

After the complex algebra, the better known Clifford algebra is the quaternion algebra (or hypercomplex algebra) defined by William Hamilton in 1843 [100].

Example 7.2 (Quaternions) The quaternion algebra is the Clifford algebra $\mathbb{C}_2$ generated by a two-dimensional vector space.

Denoting by $\{e_\alpha\}$ an orthonormal basis of the three-dimensional space, with metric coefficients $\delta_{\alpha\beta}$, the quaternion algebra is given by the multiplication table

$$\begin{aligned} &e_1 e_2 + e_2 e_1 = 0 \\ &e_1 e_0 = e_1, \quad e_2 e_0 = e_2 \\ &e_1 e_1 = e_2 e_2 = -e_0 \end{aligned}$$

Denoting $e_3 = e_1 e_2$ and $X_{12} = X_3$, the quaternion can be written as

$$X = X^0 e_o + X^1 e_1 + X^2 e_2 + X^3 e_3$$

and the multiplication table can be simplified to

$$e_\alpha e_\beta + e_\beta e_\alpha = -2\eta_{\alpha\beta} e_0, \quad e_0 e_\alpha = e_\alpha e_0 = e_\alpha, \quad \alpha, \beta, \ldots 1..3 \tag{7.26}$$

The conjugate of a quaternion is defined by

$$\bar{e}_\alpha = -e_\alpha, \quad \bar{e}_0 = e_0$$

and the norm of a quaternion is

$$||X||^2 = X\bar{X} = X_0^2 + X_1^2 + X_2^2 + X_3^2$$

It should be mentioned that the complex and the quaternion algebras are the only associative normed division algebras, that is, such that $||AB|| = ||A||||B||$ and $(AB)C = A(BC)$ (by extension the set of real numbers is considered as a Clifford algebra generated by the identity only). The division algebra property is relevant to the construction of the standard mathematical analysis based on the properties of limits and derivatives, allowing us to write

$$\lim_{\Delta x \to 0} ||\frac{\Delta F(x)}{\Delta x}|| = \lim_{\Delta x \to 0} \frac{||\Delta F(x)||}{||\Delta x||}$$

The set of real numbers $I\!R$ is a division algebra because we have the same division property, where in fact the concept appeared in the first place. There is a fourth division algebra called the octonion algebra with seven generators, although it is not associative. We shall return to it at the end in connection with the $SU(3)$ gauge theory.

Definition 7.2 (Spinors) Spinors are vectors of a representation of the group of automorphisms of a Clifford algebra defined on space–time, satisfying a given variational principle:

Given an algebra $\mathscr{A}$, an n-dimensional matrix representation of it is a homomorphism

$$\mathscr{R} : \mathscr{A} \to M_{n\times n}$$

where $M_{n\times n}$ denotes the $n \times n$ matrix algebra. Denoting by $\mathscr{R}(X)$ and $\mathscr{R}(Y)$ the matrix representing $X, Y \in \mathscr{A}$, the homomorphism condition says that the product of the algebra goes into the product of matrices $\mathscr{R}(XY) = \mathscr{R}(X)\mathscr{R}(Y)$.

Any matrix representation of an algebra can be seen as linear operators on some vector space $\mathscr{S}$, whose vectors are represented by a column

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{pmatrix} \tag{7.27}$$

In particular, we may construct spinor representations of Clifford algebras defined on a space–time. The basic example is the representations of the quaternion algebra given by the Pauli matrices associated with the spin properties of particles in quantum mechanics [101]. The Pauli matrices can be written in a variety of ways, corresponding to equivalent representations. Here we use the following:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0. \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.28}$$

such that they satisfy the multiplication table

$$\begin{cases} \sigma_i\sigma_j + \sigma_j\sigma_i = -2\delta_{ij}\sigma_0 \\ \sigma_0\sigma_i = \sigma_i\sigma_0 \\ \sigma_0\sigma_0 = \sigma_0 \end{cases}$$

which is the same multiplication table of the quaternion algebra.

The above representation can be used as the basis to construct a matrix representation of any Clifford by tensor products of matrices, known as the Brauer–Weyl representation or simply as the Weyl representation [102]:

$$\begin{cases} P_\alpha = \sigma_2 \otimes \sigma_2 \otimes \cdots \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0 \otimes \sigma_0 \\ P_{n+1} = \sigma_2 \otimes \sigma_2 \otimes \cdots\cdots\cdots\cdots\cdots\cdots \otimes \sigma_2 \otimes \sigma_2 \\ Q_\alpha = \sigma_2 \otimes \sigma_2 \otimes \cdots \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0 \otimes \sigma_0 \end{cases}$$

where the matrices σ_1 and σ_3 occupy the α position. The tensor product $\otimes$ is taken to be from left to right (that is, each entry of the left matrix is multiplied by the whole right matrix).[1]

The column vectors (7.27) of the representation space $\mathscr{S}$ of a matrix representation of a Clifford algebra are called *spinors*. From the above Brauer–Weyl representations we may conclude that the spinors of a representation of $\mathbb{C}_n$ with n generators $\{\mathbf{e}_\alpha\}$ have $N = 2^{[n]/2}$ independent components, where $[n] = n$ for even n and $[n] = n - 1$ for odd n.

An important result shows that $\mathbb{C}_{2n+1} \approx \mathbb{C}_{2n}/\mathbb{C}$, where the right-hand side denotes the Clifford algebra on the complex field (with complex coefficients). Thus, the Dirac matrices in five dimensions are essentially the same as Dirac matrices in four dimensions.

An interesting case occurs in eight dimensions, where the spinors have $2^{8/2} = 16$ components, but they split in two equivalent halves with eight components each [103]. If in addition these spinors are real, then each half spinor space is isomorphic to the generator space of the Clifford algebra.

Example 7.3 (Pauli Spinors) The Pauli matrices (7.28) define two-component spinor representation of the quaternion algebra. Indeed, the quaternion algebra is the Clifford algebra $\mathbb{C}_2$ with two generators in the case of a two-dimensional (complex) spinor representation $\mathscr{S}_2$. Thus, we obtain a two-dimensional spinor field in $\mathscr{M}$, defined by

$$\Psi : \mathscr{M} \to \mathscr{S}_2$$

which gives a two-component spinor at each point of the space–time

$$\Psi(p) = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}_p$$

[1] Tensor products in general are non-commutative. Here, Brauer and Weyl prescribed a specific way to do it. It is possible to reverse the order, obtaining a different representation. Other spinor representations, such as the Majorana and Majorana–Weyl, different from the one above are also used in field theory.

Example 7.4 (Dirac Spinors) The Dirac spinors are vectors of the four-dimensional spinor representation of the Clifford algebra $\mathbb{C}_4$, generated by a four-dimensional space.

Taking the generating space to be Minkowski's space–time $\{e_\mu\}$, together with the identity element e_0, we obtain an algebra with 16 components, whose general element is written as

$$X = X^0 e0 + X^\alpha e_\alpha + X^{\alpha\beta} e_\alpha e_\beta + X^{\alpha\beta\gamma} e_\alpha e_\beta e_\gamma + \cdots + X^{1234} e_1 e_2 e_3 e_4$$

The Brauer–Weyl matrix representation (simply known as the Weyl representation) of this algebra gives the $2^{[4]/2} \times 2^{[4]/2}$ matrices which are the Dirac matrices

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

These matrices act as linear operators on a four-dimensional complex space V_4, which is the Dirac spinor space in the Minkowski space–time

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

satisfying Dirac's equation for a relativistic charged particle with spin $1/2$ and mass m

$$(\gamma^\mu \partial_\mu - m)\psi = 0 \tag{7.29}$$

This equation can be derived from the Dirac Lagrangian [104]

$$\mathscr{L} = \bar{\psi}(\gamma^\mu \partial_\mu + m)\psi \tag{7.30}$$

where we have denoted $\bar{\psi} = \psi^\dagger \gamma^5$, and where $\psi^\dagger = (\psi^T)^*$ and $\gamma^5 = \gamma_1\gamma_2\gamma_3\gamma_4$.

7.2.1 Spinor Transformations

Since spinor fields are derived from representations of Clifford algebras, the (internal) automorphisms of these algebra correspond to a spinor transformation, that is, given a map $\tau : \mathbb{C}_n \to \mathbb{C}_n$ defined by $e'_\mu = \tau e_\mu \tau^{-1}$, such that it maintains invariant the multiplication table

$$\tau e_\mu e_\nu \tau^{-1} + \tau e_\nu e_\mu \tau^{-1} = g_{\mu\nu} e_0$$

It follows that these automorphisms necessarily correspond to isometries of the metric in the generating space. In the case of a matrix representation, the operators τ correspond to a matrix acting on the spinor space as

$$\Psi' = \mathscr{R}(\tau)\Psi$$

In quantum mechanics, spinors represent quantum states and therefore these matrix transformations $\mathscr{R}(\tau)$ correspond to unitary matrix operators denoted by u, such that $uu^\dagger = 1$ and acting on the spinors as $\Psi' = u\Psi$.

Example 7.5 (Transformations of Dirac Spinors) Let us detail the transformation of the Lagrangian of the Dirac spinor field under a unitary gauge transformation $\Psi' = u\Psi$, of the local group $U(1)$: $u = e^{i\theta(x)} e_0$. The derivative of the transformed spinor gives

$$\Psi'_{,\mu} = e^{i\theta(x)} \Psi_{,\mu} + i\theta_{,\mu} e^{i\theta(x)} \Psi$$

and similarly for $\bar{\Psi}'$. Replacing these transformations in the Dirac Lagrangian (7.30) we find that

$$\begin{aligned}\mathscr{L}(\Psi') &= e^{-i\theta(x)} \bar{\Psi}[\gamma^\mu (e^{i\theta} \Psi_{,\mu} + i\theta_{,\mu} e^{i\theta} \Psi)] - m\bar{\Psi}\Psi \\ &= \bar{\Psi}(\gamma^\mu \partial_\mu - m)\Psi + i\theta_{,\mu} \bar{\Psi}\Psi\end{aligned}$$

We see clearly that the Lagrangian is not invariant due to the presence of the derivative of the parameter θ. However, as it happened in the case of the complex scalar field, taking an infinitesimal transformation and defining the covariant derivative

$$\gamma^\mu D_\mu = \gamma^\mu \partial_\mu + i\theta_{,\mu}$$

Then the Lagrangian becomes invariant:

$$\mathscr{L}(\psi') = \bar{\psi}(i\gamma^\mu D_\mu - m)\Psi = \mathscr{L}(\psi)$$

Example 7.6 (Isospin) Returning to the quaternion algebra $\mathbb{C}_2$ satisfying the multiplication table (7.26), we have an algebra that is invariant under the group of rotations $SO(3)$ (that can be seen as a subgroup of the Galilei group in $\mathscr{G}_4$). When this algebra is represented by the Pauli matrices (7.28), the corresponding quantum states describe the orbital spin states.

On the other hand, we also have an internal action of the same algebra, but which has nothing to do with the rotations in space–time. The matrix representations of this global automorphism produce two-component spinors called isospin, which transforms as

$$\begin{pmatrix} \Psi_1' \\ \Psi_2' \end{pmatrix} = e^{i\Theta} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

where now θ is the parameter of the global gauge symmetry. The matrix representation of the automorphism is the same as Pauli matrices, but to avoid confusion we use a different notation

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$