Chapter 2
The Physical Manifold

2.1 Manifolds

The basic concept of a physical space was formulated by Kant in his Critique of Pure reason 1781, where he used the word *mannigfaltigkeit* to describe the set of all *space and time perceptions* [42]. Except for the lack of specification of a geometry and of the measurement conditions, Kant's concept of physical space is very close to our present notion of space–time.

The same word *mannigfaltigkeit* was used by Riemann in 1854, with a slightly different meaning to define his metric geometry. Riemann was less emphatic on the observational detail and more concerned with the geometry itself, the idea of proximity of the objects, and with the notion of the shape or topological qualities. These concepts were introduced by Riemann in his original paper [5]. Since Riemann's paper used very little mathematical language and expressions, it led to different interpretations. The impact of that paper on essentially all modern physics, geometry, mathematical analysis, and the subsequent technology, we can hardly avoid commenting on some fundamental aspects of Riemann's geometry and how it is used today.

Riemann's paper was translated to English in 1871 by Clifford where the word *mannigfaltigkeit* was translated to "manifold," and this was subsequently adopted as the translation of *mannigfaltigkeit* in all current dictionaries. Inevitably, in the translation process, some of the original concepts of Kant, specially the perception aspect, was shaded by the concept of topological space, another invention of Riemann in the same paper [5, 43, 44].

The *topological space* of Riemann is the same as we understand today: Any set endowed with a collection of *open sets* such that their intersections and unions are also open sets and that such collection covers the whole manifold. With such topology we may define the notions of limits and derivatives of functions on manifolds [44].

Such topology is *primarily borrowed* from the metric topology of the parameter space $I\!R^n$, so that the standard mathematical analysis in Euclidean spaces can be readily used [43, 45–48]. Once this choice is made, then it is possible to define

other topological basis, although they are not always practical as the borrowed topology of $I\!R^n$. One drawback of the borrowed topology is that a manifold can be described as being locally equivalent to $I\!R^n$, leading to the wrong interpretation that the manifold is composed of dimensionless points, like those of the $I\!R^n$. This conflicts with the Kant description of manifolds as a set of perceptions, unless we understand that point particles are not really points but just a mathematical name, capable of carrying physical qualities such as mass, charge, energy, and momenta, thus occupying a non-zero volume. In this sense a point particle can be a galaxy, an elephant, a membrane, a string, or a quark, as long as it can be assigned a time and position (as if endowed with a global positioning system (GPS)). Thus, the local equivalence between a manifold and the parameter space $I\!R^n$ does not extend to the physical meaning of the manifold. Here and in the following we use the concept of manifold as a physical space (in the sense of Kant) and often refer to its objects as points, not to be confused with the points of the parameter space.

Another topic on manifolds which deserves a comment is the choice of $I\!R^n$ as the parameter space. For some, the physical space is composed primarily of *elementary particles* and as such they should be parameterized by a discrete set and not continuous because particles are of quantum nature, characterized by a discrete spectra of eigenvalues. It is also argued that the differentiable nature associated with Riemann's topology of open sets can be replaced by a discrete topology. Thus, the usual differential equations are replaced by finite difference equations. In this interpretation the continuum would be only a non-fundamental short sight view of a discrete physical space [49–52].

On the other hand, the choice of $I\!R^n$ as the parameter space makes sense when we consider that the observers, the observables, and the conditions of measurement are defined primarily by classical observers using classical physics based on the continuum. After all, it was the differentiable structure that allowed those classical observers and their instruments to construct quantum mechanics, the present notion of elementary particles and their observables, defined by the eigenvalues of the Casimir operators of the Poincaré group. One of the most complete discussions on this fundamental subject was presented by Weyl, when he combines the foundations of mathematics with that of physics [53, 54]. In this book we base our arguments on the type of spectra of the Casimir operators. We do not see why the discrete spin spectrum of eigenvalues should be favored in presence of the spectrum of the mass operator of the Poincaré group, which, unlike the spin spectrum, is continuous (although assuming only discrete values) [31, 33]. In this sense we agree with Weyl's conclusion that the parameter space is $I\!R^n$, where continuous fields gives the fundamental physical structures with the quantum masses, spins, color, strangeness, etc. as secondary characteristics.

After these considerations we may proceed with the standard definition and properties of manifolds as found in most textbooks:

Definition 2.1 (Manifold) A manifold $\mathscr{M}$ is a set of objects (generally called points and denoted by p) with the following properties:

(a) For each of these objects we may associate n coordinates in $\mathbb{R}^n$, by means of an 1:1 map $\sigma : \mathcal{M} \to \mathbb{R}^n$,

$$\sigma(p) = (x^1, x^2, \ldots, x^n)$$

with inverse $\sigma^{-1} : \mathbb{R}^n \to \mathcal{M}$ such that

$$\sigma^{-1}((x^1, x^2, \ldots, x^n)) = p$$

(b) Given another such map τ, associate with the same p another set of coordinates $\tau : \mathcal{M} \to \mathbb{R}^n$,

$$\tau(p) = (x'^1, x'^2, \ldots, x'^n)$$

with inverse

$$\tau^{-1}((x'^1, x'^2, \ldots, x'^n)) = p$$

Then the composition $\phi = \sigma^{-1} \circ \tau : \mathbb{R}^n \to \mathbb{R}^n$ is the same as a coordinate transformation in $\mathbb{R}^n : x'^i = \phi^i(x^j)$ (see Fig. 2.1).

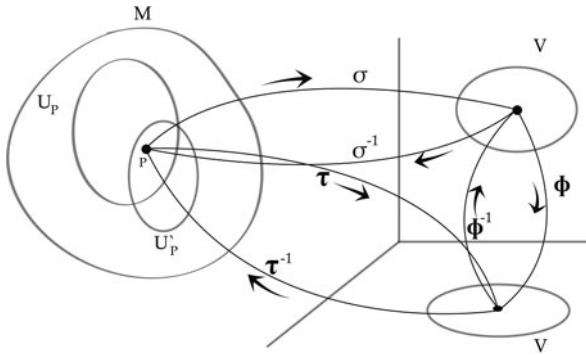(c) For all points of $\mathcal{M}$ we can define one such map and the set of such maps covers the whole $\mathcal{M}$.

Fig. 2.1 Manifold

The maps $\sigma, \tau, \ldots$ are called *charts* and the set of all *charts* is called an atlas of $\mathcal{M}$. A *differentiable manifold* is a manifold for which ϕ is a differentiable map in $\mathbb{R}^n$. In this case we say that the differentiable manifold $\mathcal{M}$ has a differentiable atlas. The smallest n required to form an atlas is called the *dimension of the manifold*.

From the inverse σ^{-1} of each chart we may obtain a topology in $\mathscr{M}$ in the following way: Denoting by $\vee_q$ an open set in $\mathbb{R}^n$, then all points in this open set are mapped by σ^{-1} in an open set $\cup_p$ in $\mathscr{M}$ (Fig. 2.1). Thus, we obtain the borrowed topology in $\mathscr{M}$, where all topological properties of $\mathbb{R}^n$ are transferred to $\mathscr{M}$, including the Hausdorff property meaning that for each object in $\mathscr{M}$ there is a neighborhood containing another object of $\mathscr{M}$.

The simplest examples of manifolds are the already known curves and surfaces of $\mathbb{R}^3$. The coordinate space $\mathbb{R}^n$ itself is a trivial manifold, whose charts are identity maps. Less trivial examples are the space–times as we shall see later.

A *differentiable map* between two arbitrary manifolds can be defined through the use of the borrowed topology as follows: Let $\mathscr{M}$ and $\mathscr{N}$ be manifolds with dimensions m and n, respectively. A map $F : \cup_p \to \cup_q$, with $\cup_p \in \mathscr{M}$ and $\cup_q \in \mathscr{N}$, is said to be differentiable if for any chart σ in $\mathscr{M}$, and any chart τ in $\mathscr{N}$, the composition

$$\tau \circ F \circ \sigma^{-1} : \vee \to \vee'$$

is a differentiable map from $\mathbb{R}^m$ to $\mathbb{R}^n$. A *homeomorphism* F between manifolds is an invertible map such that $\tau \circ F \circ \sigma^{-1}$ is continuous. If this map is also differentiable then F is called a *diffeomorphism* (Fig. 2.2).

As an example consider that $\mathscr{M} \equiv \mathbb{R}$ and $\mathscr{N}$ is an arbitrary manifold. Then it follows from the above definition that the map

$$\alpha : \cup_t \to \cup', \quad \cup' \in \mathscr{N}, \quad t \in \cup_t \subset \mathbb{R}$$
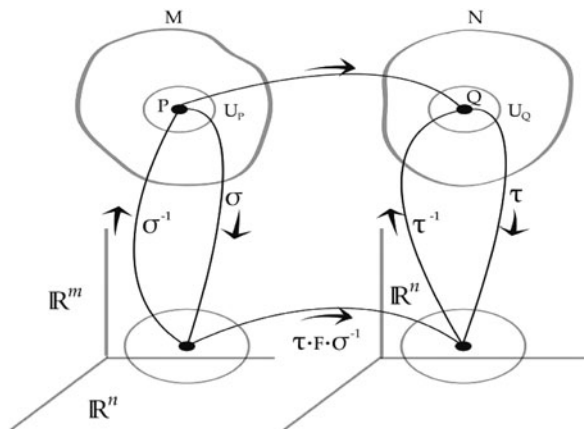
is differentiable when the composition

Fig. 2.2 Manifold mappings

$$I \circ \alpha \circ \sigma^{-1} = \alpha \circ \sigma^{-1} : I\!R \to I\!R^n$$

is differentiable (here the chart of $I\!R$ is the identity map I).

A *continuous curve* in $\mathcal{N}$ is a simple continuous map $\alpha(t) : I\!R \to \mathcal{N}$. A *differentiable curve* in $\mathcal{N}$ occurs when the map α is differentiable. If in addition the derivative $d\alpha/dt$ does not vanish, we have a *regular curve* in $\mathcal{N}$. From Fig. 2.3 we see that the curve in $\mathcal{N}$ is the image of a curve in $I\!R^n$ by the inverse chart. In particular, when $I\!R$ is replaced by one of the coordinate axis x^α of the $I\!R^n$, the curve $\alpha(x^\alpha)$ is called the *coordinate curve* in the manifold, whose parameter is the coordinate itself x^α.

From the definition it follows that in general a manifold is not a vector space. Therefore the notions of force, pressure, momenta, and other physical fields that depend on the specification of a direction on different points of a manifold are not defined. This may seem conflicting with the concept of a manifold as a set of observations because these observations involve interactions or forces. Vectors and vector fields are implemented in the differentiable structure of manifolds in the form of tangent vectors.

Definition 2.2 (The Tangent Bundle) A *tangent vector* to a manifold $\mathcal{M}$ at a point p is a tangent vector to a curve on $\mathcal{M}$ passing through p. To define a tangent vector to a curve on $\mathcal{M}$, consider the set of all differentiable functions defined in $\mathcal{M}$, $\mathcal{F}(\mathcal{M})$, and $f \in \mathcal{F}(\mathcal{M})$. The tangent vector field to the curve $\alpha(t)$ at the point $p = \alpha(t_0)$ can be defined by the operation

$$\frac{d}{dt} f(\alpha(t))\rfloor_{t_0} = \frac{\partial f}{\partial x^\beta} \frac{d\alpha^\beta}{dt}\rfloor_p$$
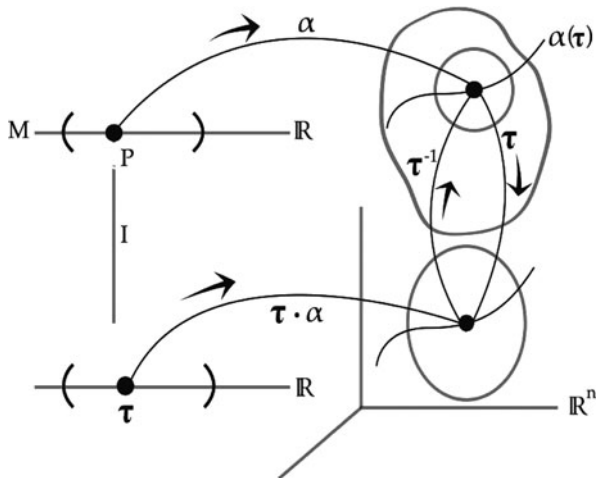

Fig. 2.3 Curve on a manifold

The derivative $d/dt\, f(\alpha(t))\rfloor_{t_0}$ is called the *directional derivative* of f with respect to the vector $\alpha'(t_0) = v_p$. It is also denoted by

$$\alpha'(t_0)[f] = v_p[f] = \frac{df}{dt}\rfloor_p$$

The set of all tangent vectors to $\mathscr{M}$ at p generates a *tangent space*, denoted by $T_p\mathscr{M}$, with respect to the vector addition rule at p: if $v_p = \alpha'(t_0)$ and $w_p = \beta'(t_0)$ are tangent vectors to two curves passing through p, then the linear combination $mv_p + nw_p = u_p$ defines another curve $\gamma(t)$ in $\mathscr{M}$ with tangent $\gamma'(t_0) = u_p$ passing through the same point $\gamma(t_0) = p$. Clearly such rule does not apply to tangent vectors in different points of $\mathscr{M}$, so that tangent vectors and tangent spaces to a manifold are only locally defined. In some textbooks a tangent vector at p is called a *vector applied to a point*.

Since $\mathscr{M}$ has dimension n, $T_p\mathscr{M}$ has dimension n and a basis of $T_p\mathscr{M}$ is composed of n linearly independent vectors, tangent to n curves in $\mathscr{M}$. In particular, these curves can be taken to be the curves defined by the coordinates x^α with tangent vectors

$$e_\alpha[f] = \alpha'(x^\alpha)[f]\rfloor_p = \frac{\partial f}{\partial x^\beta}\frac{dx^\beta}{dx^\alpha}\rfloor_p = \frac{\partial f}{\partial x^\alpha}\rfloor_p$$

Since this applies to all differentiable functions we may omit f and write the tangent basis as an operator

$$e_\alpha = \frac{\partial}{\partial x^\alpha}$$

Such basis is naturally called the *coordinate basis* of $T_p\mathscr{M}$.

The collection $T\mathscr{M}$ of all tangent spaces to $\mathscr{M}$ in all points of $\mathscr{M}$, endowed with a diffeomorphism $\pi : T\mathscr{M} \to \mathbb{R}$, is called the total tangent space (or simply the *total space*). The *tangent bundle* of $\mathscr{M}$ is the triad

$$(\mathscr{M}, \pi, T\mathscr{M})$$

where the manifold $\mathscr{M}$ is called *the base manifold* and π is called *the projection map*. Each tangent space $T_p\mathscr{M} \in T\mathscr{M}$ is called a *fiber over p*.

The projection π identifies on $\mathscr{M}$ the tangency point of $T_p\mathscr{M}$. Each tangent vector can be written as a pair $v_p = (p, v)$ while v is the vector properly. The projection of the pair gives $\pi(p, v_p) = p$. On the other hand, its inverse π^{-1} gives the whole tangent space at p:

$$\pi^{-1}(p) = T_p(\mathscr{M}) \in T\mathscr{M}$$

The *total space* $T\mathscr{M}$ contains all tangent spaces in all points of $\mathscr{M}$, so that it is composed of ordered pairs like (p, v), where $p \in \mathscr{M}$ and $v \in T_p\mathscr{M}$ (Fig. 2.4).
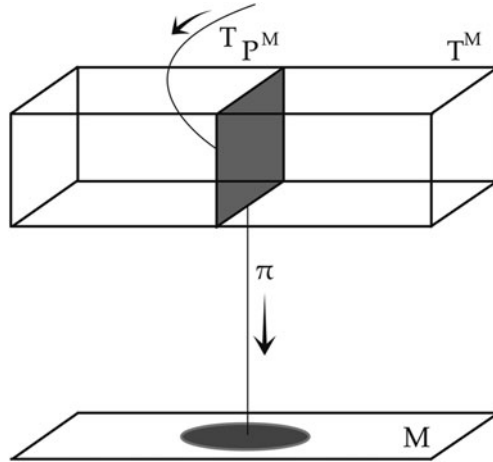
Fig. 2.4 The tangent bundle

Since $\mathcal{M}$ is a manifold with n dimensions, it follows that $T_p\mathcal{M}$ is also n-dimensional. Consequently, the set of all pairs $(p, v) \in \mathcal{M} \times T_p(\mathcal{M})$ is a manifold with dimension $2n$.

A well-known example is given by the *configuration space* of a mechanical system of idealized point particles defined in a region of a space–time $\mathcal{M}$. Supposing that all constraints to the motion are removed, we obtain a reduced representation space in which we mark ordered pairs $(x^i, \dot{x}^i)$, $i = 1..N$, where x^i denotes the coordinate of the system and $\dot{x}^i$ denotes the components of its velocity vector. This set of ordered pairs is the total space $T\mathcal{M}$ of the tangent bundle called the *representation space*.

The equations of motion of a mechanical system described in the configuration space are derived from a *Lagrangian* $\mathscr{L}(x^i, \dot{x}^i)$, which is a *differentiable function defined on the total space* $\mathscr{L} : T\mathcal{M} \rightarrow I\!R$ [55]. Classical mechanical systems evolved somewhat independently of the concept of manifold and the coordinates x^i were once called *generalized coordinates* [56].

Definition 2.3 (Tangent Vector Fields) The concept of *tangent vector field* arises naturally after the definition of the tangent bundle as a map $V : \mathcal{M} \rightarrow T\mathcal{M}$ such that it associates with each element $p \in \mathcal{M}$ a tangent vector $V(p) \in T_p\mathcal{M}$.

A *cross section* of the tangent bundle is a map $S : \mathcal{M} \rightarrow T\mathcal{M}$ such that $\pi \circ S = I$. It follows that a vector field is a particular cross section such that it specifies a vector $V(p) = v_p \in T_p(\mathcal{M})$.

Clearly, the set of vector fields on a manifold does not generate a vector space because we cannot sum vectors belonging to different tangent spaces.

The concept of *directional derivative* of a function with respect to a tangent vector can be easily extended to the directional derivative of a function with respect to a vector field: Consider a vector $v_p = V(p)$ and a curve $\alpha(t)$ such that $p = \alpha(t_0)$

and $\alpha'(t_0) = v_p$. Let f be a differentiable function on $\mathcal{M}$. Then we may calculate the directional derivative

$$v_p[f] = \frac{d}{dt} f(\alpha(t))\big\rfloor_{t=t_0}$$

where we have denoted $p = \alpha(t_0)$ and $\alpha'(t_0)_i = v_i(p)$. In local coordinates $\{x^\alpha\}$, the above expression is equivalent to

$$v_p[f] = \sum \alpha'^\alpha(t)\big\rfloor_{t=t_0} \frac{\partial f}{\partial x^\alpha}\big\rfloor_p$$

Thus, replacing $v_p = V(p)$ and $\alpha'^\alpha(t)|_{t=t_0} = V^\alpha(p)$ we obtain

$$V(p)[f] = \sum V^\alpha(p) \frac{\partial f}{\partial x^\alpha}(p)$$

Supposing that this holds true for all p belonging to the region of $\mathcal{M}$, we may simply suppress the point p, thus producing the directional derivative of f with respect to the vector field V in a coordinate basis:

$$V[f] = \sum V^\alpha \frac{\partial f}{\partial x^\alpha}$$

where V^α denotes the components of the vector field V in the chosen coordinates $\{x^\alpha\}$.

Consider two manifolds $\mathcal{M}$ and $\mathcal{N}$ and a differentiable map $F : \mathcal{M} \to \mathcal{N}$. The *derivative map* of F, denoted by F_*, is a *linear* map between the respective total spaces,

$$F_* : T\mathcal{M} \to T\mathcal{N}$$

such that for a differentiable function $f : \mathcal{N} \to \mathbb{R}$ and $v_p \in T_p\mathcal{M}$, the result $F_*(v_p)[f]$ is the same as the directional derivative of $f \circ F$ with respect to v_p:

$$F_*(v_p)[f] = v_p[f \circ F]$$

The linearity of F_* is a consequence of the properties of the directional derivative:

$$F_*(av_p + bw_p)[f] = (av_p + bw_p)[f \circ F] = aF_*(v_p)[f] + bF_*(w_p)[f]$$

As an example consider that $v_p = \alpha'(t_0)$ is a tangent vector to a curve $\alpha(t)$ at a point $p = \alpha(t_0) \in \mathcal{M}$. The curve $\alpha(t)$ is mapped by F to a curve of $\mathcal{N}$ given by

$$\beta(t) = F(\alpha(t))$$

By the above definition it follows that

$$F_*(\alpha'(t_0))[f] = F_*(v_p)[f] = v_p[f \circ F(\alpha(t))] = v_p[f(F(\alpha))] = v_p[f(\beta)]$$

and from the definition of the directional derivative we obtain

$$v_p[f(\beta(t))] = \frac{d}{dt}[f(\beta(t))]|_{t=t_0} = \beta'(t_0)[f]$$

so that

$$F_*(\alpha'(t_0))[f] = \beta'(t_0)[f]$$

In other words, if $\beta = F(\alpha)$, then the tangent vector to β at the point $F(p) \in \mathcal{N}$ is $\beta' = F_*(\alpha'(t_0))$.

Definition 2.4 (Vector Bundle) Quite intuitively the definition of tangent bundle can be extended to the more general notion of *vector bundle* as follows: Given a manifold $\mathcal{M}$, we may attach to each point p a *local vector space* $\mathcal{V}_p$, not necessarily tangent to a curve in $\mathcal{M}$. Then we may collect these vector spaces in a total space $\mathcal{V}$, so that we can identify the point $p \in \mathcal{M}$ where $\mathcal{V}_p$ is defined, called the *fiber over p*, defined by a projection map $\pi : \mathcal{V} \to \mathcal{M}$. This *vector bundle* is represented by the triad

$$(\mathcal{M}, \pi, \mathcal{V})$$

Clearly, the tangent bundle is a particular example of vector bundle. A less trivial example is given by the *normal bundle* where the fiber over p is a vector space N_p orthogonal to the tangent spaces $T_p\mathcal{M}$. Another example of vector bundle is given by the space of matrices attached at each point of $\mathcal{M}$.

When all fibers $\mathcal{V}_p$ of a vector bundle have the same dimension, they are all isomorphic to a single vector space Σ, called the *typical fiber*. A particularly interesting case occurs when the total space is the Cartesian product $\mathcal{V} = \mathcal{M} \times \Sigma$, the vector bundle is called a product vector bundle, or simple *product bundle*, written as

$$(\mathcal{M}, \pi, \mathcal{M} \times \Sigma)$$

In this case, the total space $\mathcal{M} \times \Sigma$ can be graphically represented by a box, which represents the fiber bundle, with $\mathcal{M}$ in the base and Σ in the vertical side. Each element of this total space is just the pair (p, v) where the vector v represents any vector in each fiber. Because of this, these vector bundles are sometimes referred to as *trivial vector bundles*.

2.2 Geometry of Manifolds

A manifold does not necessarily come with a geometry, that is, with a measure of distances or of angles, so that we may draw parallel lines satisfying Euclid's axioms. A geometry can be implemented on a manifold as follows[1]:

Definition 2.5 (Metric Geometry on a Manifold) The most intuitive way to construct parallel lines in the Euclidean space is to use a graduated rule or metric geometry. This intuitiveness is a consequence of the fact that $I\!R^3$ is a manifold and also a vector space in which a scalar product is globally defined.

To define the same notion of parallels in a manifold $\mathcal{M}$ is a little more complicated. First, we need to define the metric by the introduction of a scalar product of vectors on the manifold. Since manifolds do not have vectors, we may locally define the metric in each tangent space as a map

$$< \ , \ >: T_p\mathcal{M} \times T_p\mathcal{M} \to I\!R$$

such that it is (a) bilinear and (b) symmetric. There is a third condition in Euclidean geometry which says that it should be positive definite: Given a vector v, then (c) $||v||^2 \ = < \ v, v \ > \geq 0$, and $||v||^2 \ = < \ v, v \ > = 0 \Longleftrightarrow v = 0$. This condition is omitted when we consider that geometry is an experimental science, whose results depend on the definition of the observers, of the observed object, and of the methods of observations. Thus the condition (c) may hold under certain measurements and not in others.[2]

Since the scalar product is locally defined, the metric components in an arbitrary basis

$$g_{\mu\nu} \ = < e_\mu, e_\nu >$$

are also locally defined. This makes it difficult to define distances between two distinct points of the manifold connected by a curve $\alpha(t)$, for in principle the metric varies from point to point. Therefore, the comparison of distances in different points requires an additional condition that the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

remains the same. Such *isometry* exists naturally in Galilean, Newtonian, and Minkowski's space–times, but there is no preliminary provision for it in general relativity. In this case (as in arbitrary metric manifolds), the metric components vary

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[1] A geometry can be of two basic kinds: The *metric geometries* based on the notion of distance or a graduated rule; the other is the *affine geometries* based on the notion of parallel transport of a vector field along a curve, keeping a constant angle with the tangent vector to that curve [5, 57].

[2] In mathematical analysis when the condition (c) is omitted the analysis is referred to as analysis in Lorentzian manifolds.

from point to point, *so that the measurements of distances between lines depend on the existence of an affine connection which is compatible with the metric geometry.* This affine connection was defined by Levi-Civita, using the Christoffel symbols (see below).

Definition 2.6 (Affine Geometry) An affine geometry on a manifold $\mathcal{M}$ is defined by the existence of parallel transport of a vector field W along a curve $\alpha(t)$ on $\mathcal{M}$, such that the angle between W and the tangent vector to $\alpha(t)$ remains constant. Therefore, it offers an alternative but essential way to trace parallel lines in a manifold prior to the definition of a metric. Let us detail how this works.

Given a vector field W on a manifold $\mathcal{M}$, its *covariant derivative* with respect to the vector field $V = \alpha'(t)$ tangent to a curve $\alpha(t)$ at a point $p = \alpha(t_0)$ is the measure of the variation of W along α:

$$\nabla_V W(p) = \frac{d}{dt} W(\alpha) \Big|_{t=t_0} \tag{2.1}$$

satisfying the following properties (a and b are numbers and f is a real function defined on $\mathcal{M}$):

(a) $\nabla_V (aW + bW') = a\nabla_V W + b\nabla_V W'$
(b) $\nabla_{aV+bV'}(W) = a\nabla_V W + b\nabla_{V'} W$
(c) $\nabla_V f = V[f]$
(d) $\nabla_V (fW) = V[f]W + f\nabla_V W.$

These properties correspond to similar properties that hold in the particular case of $I\!R^n$, when we use arbitrary base vectors [58]. It is clear from the above definition that the covariant derivative of a vector field in $\mathcal{M}$ with respect to a tangent vector of $T_p\mathcal{M}$ is again a tangent vector field of the same space. It is also clear that it does not depend on the previous existence of a metric.

The above definition of covariant derivative can be easily extended to the region of definition of the involved vector fields, without specifying the point $p = \alpha(t_0)$. Denoting by $V = \alpha'(t)$ the tangent vector field to a curve $\alpha(t)$, then (2.1) gives

$$\nabla_{\alpha'} W = \frac{d}{dt} W(\alpha)$$

providing a measure of how the vector field W varies along the curve $\alpha(t)$.

Definition 2.7 (Parallel Transport) In the case when

$$\nabla_{\alpha'} W = \frac{d}{dt} W(\alpha) = 0$$

we say that the field W is *parallel transported* along $\alpha(t)$.

Thus, the existence of a *covariant derivative* is intimately associated with the existence of an affine geometry, and the covariant derivative operator ∇ is also referred to as the *affine connection operator*.

Let $\{e_\alpha\}$ be a set of n tangent vector fields to M, such that at each point p, $\{e_\alpha(p)\}$ is a basis of $T_p(M)$. Such basis is sometimes referred to as a *field basis*. Then the covariant derivative of e_α with respect to another field basis e_β is a linear combination of the same field basis:

$$\nabla_{e_\alpha} e_\beta = \Gamma^\gamma_{\alpha\beta} e_\gamma \tag{2.2}$$

where the coefficients $\Gamma^\gamma_{\alpha\beta}$ are called the connection coefficients or the *Christoffel symbols*. By different choices of the way in which the covariant derivative acts on the basis, we obtain different geometries. Thus, for example we can have Riemann, Weyl, Cartan, Einstein–Cartan, and Weitzenbock geometries, depending on the properties of these coefficients.

In the case of the Riemann geometry, the connection coefficients $\Gamma^\gamma_{\alpha\beta}$ are symmetric in the sense that

$$\nabla_{e_\alpha} e_\beta = \nabla_{e_\beta} e_\alpha$$

or equivalently, the symmetry is explicit in the two lower indices of the Christoffel symbols:

$$\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$$

Here and in the following we use the choice of Riemann and Einstein, with a symmetric connection.

In order to write the *components of the covariant derivative*, let us write the vector fields in an arbitrary field basis: $W = W^\alpha e_\alpha$ and $V = V^\beta e_\beta$. From the above properties of the covariant derivatives, we obtain

$$\nabla_V W = \nabla_V(W^\alpha e_\alpha) = V[W^\alpha]e_\alpha + W^\alpha \nabla_V e_\alpha = \left(V^\beta \frac{\partial W^\gamma}{\partial x_\beta} + W^\alpha V^\beta \Gamma^\gamma_{\alpha\beta} \right) e_\gamma$$

where in the last expression we have made a convenient change in the summing indices.

Taking in particular $V = e_\alpha$, and using the semicolon to denote the components of the covariant derivative, it follows that

$$\nabla_{e_\mu} W = W^\beta{}_{;\mu} e_\beta$$

where we have denoted the components of the covariant derivative of W as

$$W^\beta{}_{;\mu} = \left(\frac{\partial W^\beta}{\partial x^\mu} + W^\gamma \Gamma^\beta_{\gamma\mu} \right) \tag{2.3}$$

The affine geometry can be made compatible with the metric geometry under the condition that the metric behaves as a constant with respect to the covariant derivative. This is what Riemann did when he postulated that the covariant derivative of the metric tensor g is zero:

$$(\nabla_{e_\rho} g)_{\mu\nu} = 0 \tag{2.4}$$

This is called the *metricity condition* of the affine connection, and it is often written in terms of the components as $g_{\mu\nu;\rho} = 0$. As we recall from the introduction, this condition was tentatively modified by Weyl in his 1919 theory.

2.3 The Riemann Curvature

The geometry of surfaces of $I\!R^3$ tells us *that the shape of a surface depends on how it deviates from the local tangent plane*. This characterizes a topological property of the surface, allowing to distinguish, for example, a plane from a cylinder. This variation of the local tangent plane can be studied alternatively by the variation of the normal vector field to the surface, and it is called the *extrinsic curvature* of the surface. It is extrinsic because it depends on a property that lies outside the surface.

Definition 2.8 (The Riemann Tensor) Consider two curves in a manifold $\mathcal{M}$, α and β intersecting at a point A, with unit *tangent independent vectors* U and V respectively. Then make a parallel displacement of V and U along the curves α and β, respectively, as indicated in Fig. 2.5. At the points B and C draw the curves α_1 and β_1 with tangent vectors parallel to U and V, respectively, obtaining the parallelogram. Next, consider a third vector field W, linearly independent from U and V, at the
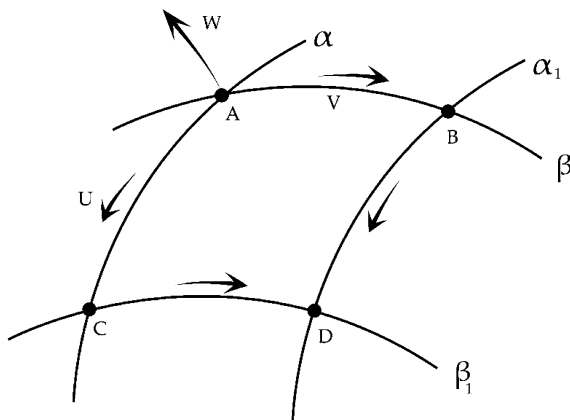

Fig. 2.5 The Riemann curvature

point A, and drag it along the curve β from A to B. Then drag it from B to D along the curve α_1. The result of such operation is the vector field

$$W' = \nabla_U \nabla_V W$$

On the other hand, dragging W from A to C and from C to D we obtain another vector

$$W'' = \nabla_V \nabla_U W$$

The difference $W' - W''$ gives the Riemann curvature tensor[3] of $\mathcal{M}$ [43, 47]

$$R(U, V)W = (\nabla_U \nabla_V - \nabla_V \nabla_U)W = [\nabla_U, \nabla_V]W \qquad (2.5)$$

As we see, this result does not depend on a metric, and from our previous comment, it is actually necessary to be so before any notion of constant distance is defined.

In the particular case of a flat plane of $\mathbb{R}^3$ the Riemann tensor vanishes. Therefore, Riemann's idea of curvature is compatible with the geometry of surfaces in $\mathbb{R}^3$, at least for some basic figures. *However, it is not sufficient to distinguish a plane from a cylinder or, in fact, from an infinite variety of ruled surfaces.* It is also interesting to note that for surfaces of $\mathbb{R}^3$ the Riemann tensor coincides with the *Gaussian curvature* $K = k_1 k_2$, where k_1 and k_2 are the *principal curvatures* measured by the maximum and minimum deviations of the normal vector field (see, e.g., [48, 58]). The Egregium theorem of Gauss shows that indeed K can be defined entirely as an intrinsic property of the surface.

The components of the Riemann tensor of a manifold $\mathcal{M}$ in an arbitrary tangent basis $\{e_\mu\}$ can be obtained from (2.5) when the operator is applied to the basis vectors, reproducing another vector

$$R(e_\alpha, e_\beta)e_\gamma = \nabla_{e_\alpha} \nabla_{e_\beta} e_\gamma - \nabla_{e_\beta} \nabla_{e_\alpha} e_\gamma = R_{\alpha\beta\gamma}{}^\delta e_\delta \qquad (2.6)$$

Using the metric we may also write $R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma}{}^\delta g_{\delta\varepsilon}$.

From (2.2), the *Christoffel symbols of the first kind* are defined as

$$\Gamma_{\alpha\beta\gamma} = g_{\gamma\delta}\Gamma_{\alpha\beta}{}^\delta$$

and using Riemann's metricity condition (2.4) we find the expression of the Christoffel symbols of the first kind in terms of the derivatives of the metric

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha} - g_{\alpha\beta,\gamma})$$

which is *symmetric in the first two indices* $\Gamma_{\alpha\beta\gamma} = \Gamma_{\beta\alpha\gamma}$.

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[3] In general U, V, W need not be linearly independent, but in this case we need to add the term $\nabla_{[U,V]}W$ to compensate for the linear dependency in the construction of the parallelogram.

Replacing these components in (2.5) we obtain the components of the Riemann tensor:

$$R_{\alpha\beta\gamma\varepsilon} = \Gamma_{\beta\varepsilon\alpha;\gamma} - \Gamma_{\beta\varepsilon\gamma;\alpha} + \Gamma^{\mu}_{\beta\gamma}\Gamma_{\alpha\varepsilon\mu} - \Gamma^{\mu}_{\beta\alpha}\Gamma_{\gamma\varepsilon\mu} \tag{2.7}$$

From this expression we derive the following properties:

$$R_{\alpha\beta\gamma\varepsilon} = -R_{\beta\alpha\gamma\varepsilon} \tag{2.8}$$
$$R_{\alpha\beta\gamma\varepsilon} = -R_{\alpha\beta\varepsilon\gamma} \tag{2.9}$$
$$R_{\alpha\beta\gamma\varepsilon} = R_{\gamma\varepsilon\alpha\beta} \tag{2.10}$$
$$R_{\alpha\beta\gamma\varepsilon} + R_{\alpha\varepsilon\beta\gamma} + R_{\alpha\gamma\varepsilon\beta} = 0 \tag{2.11}$$

Finally the covariant derivative of Riemann's tensor gives the Bianchi's identities

$$R_{\alpha\beta\gamma\varepsilon;\mu} + R_{\alpha\beta\varepsilon\mu;\gamma} + R_{\alpha\beta\mu\gamma;\varepsilon} = 0 \tag{2.12}$$

Ricci's curvature tensor is derived from Riemann's tensor by a contraction

$$R_{\alpha\varepsilon} = g^{\beta\gamma} R_{\alpha\beta\gamma\varepsilon} \tag{2.13}$$

On the other hand, the contraction of Ricci's tensor gives the scalar curvature (or the Ricci scalar curvature).

$$R = g^{\alpha\beta} R_{\alpha\beta} \tag{2.14}$$

We shall return to the Riemann tensor in the latter sections, showing that it has the same structure for gravitation and for the gauge field strengths.

Example 2.1 (Geodesic) A *geodesic* in a manifold $\mathcal{M}$ is a curve such that its tangent vector is transported *parallel to itself*:

$$\nabla_{\alpha'}\alpha' = 0$$

From (2.3) we may derive the equation of a geodesic $\alpha(t)$, with parameter t, in coordinate basis. Taking $V = W = \alpha' = \sum x^{\mu} e_{\mu}$, and using the geodesic definition, we obtain

$$\frac{d^2 x^{\mu}}{dt^2} + \Gamma^{\mu}_{\alpha\beta} \frac{dx^{\alpha}}{dt} \frac{dx^{\beta}}{dt} = 0 \tag{2.15}$$

In particular, for $\mathcal{M} = I\!R^n$ this is the equation for a straight line in arbitrary coordinates.

As an exercise on the equivalence between metric and affine geometries under (2.4), let us show that geodesics generalize the concept of straight lines in the sense that describe the smallest distance between two points of $\mathcal{M}$, measured by a metric associated with a scalar product $<, >$.

Consider a family of curves passing through two arbitrary points p and q in $\mathcal{M}$, defined by the displacement of a vector field W over the geodesic α:

$$\gamma(t, u) = \alpha(t) + uW(\alpha(t))$$

It follows that $\gamma' = d\gamma/dt = \alpha' + u\,dW/dt$ and $d\gamma/du = W$. The arc-length between p and q along any curve of the family is given by

$$S(u) = \int_o^t \sqrt{<\gamma', \gamma'>}\,dt$$

The variation of this arc-length with respect to the family parameter u is

$$\frac{dS}{du} = \int_0^t \frac{<\frac{d\gamma'}{du}, \gamma'>}{\sqrt{<\gamma', \gamma'>}}\,dt$$

Since W is an arbitrary vector field we may take in particular $W = \alpha'$, so that $d\gamma/du = W = \alpha'$. Using the fact that the two parameters are independent we obtain

$$\frac{d\gamma'}{du} = \frac{d}{dt}\frac{d\gamma}{du} = \frac{d\alpha'}{dt} = \nabla_{\alpha'}\alpha'$$

Since α is a geodesic, we necessarily have

$$\frac{dS}{du} = \int_0^t \frac{<\nabla_{\alpha'}\alpha', \gamma'>}{\sqrt{<\gamma', \gamma'>}}\,dt = 0$$

showing that S is a maximum or a minimum. The maximum is infinity and therefore it is not interesting. The minimum occurs in the geodesic.