Chapter 7
Deterministic Models

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7.1 The Basic EOQ Model with Perishability

As we have seen, managing perishables when demand is uncertain is a challenging problem. When demand is known, one can always find an ordering rule that guarantees no outdating. Hence, at first glance, it would appear that the deterministic demand case is trivial. However, this is not the true when demand is nonstationary.

Consider first the stationary demand case (namely, the basic EOQ model). In this case, incorporating perishability is straightforward. We follow the notation from Nahmias (2009). It is well known that in the EOQ setting, the optimal number of units to order each cycle is given by the formula

$$Q^* = \sqrt{\frac{2K\lambda}{h}},$$

where K is the fixed order cost, h is the holding cost measured on a unit per unit time basis, and λ is the fixed rate of consumption also in units per unit time. The cycle time, T, is the time between placement of orders, given by

$$T = Q^{*/\lambda} = \sqrt{\frac{2K}{\lambda h}}.$$

The model is based on the assumption that goods are durable (i.e., have an infinite lifetime). Now, let us suppose that new orders have a lifetime of m units of time. If $T \le m$, there is no modification of the policy required. All units are consumed by demand prior to outdating. Consider the case $T > m$. Here, all units on hand at time

S. Nahmias, *Perishable Inventory Systems*, International Series in Operations Research & Management Science 160, DOI 10.1007/978-1-4419-7999-5_7,

m (which will be $Q^* - \lambda m$) expire, and one must immediately reorder to avoid shortages. It is obvious that in this situation, reducing the order size from Q^* to λm eliminates outdating and reduces the holding cost and marginal order cost and has no effect on fixed costs. Hence, the optimal policy is to order $\min(Q^*, \lambda m)$.

It follows that in the EOQ environment one orders so that outdating never occurs. It appears safe to assume that this property of an optimal solution carries over to *all* deterministic perishable inventory problems. This includes problems with nonstationary costs, capacity restrictions, etc.

7.2 Dynamic Deterministic Model with Perishability

In what has become a seminal paper, Wagner and Whitin (1958) provided the first analysis of the economic lot scheduling (ELS) problem. Consider a set of known requirements over N planning periods, say $\mathbf{r} = (r_1, r_2, ...r_N)$. They assumed stationary only fixed order costs and holding costs, and furthermore that these costs are not changing with time. The key result which allowed them to construct an efficient solution algorithm was that an optimal policy only ordered in periods where starting inventory was zero. (We refer to this as the zero inventory property.) This means that if starting inventory at the beginning of the planning horizon is assumed to be zero, an optimal policy is completely specified by knowing the periods in which ordering occurs. It also means that every order quantity is the sum of requirements for some set of future periods (this is known as an exact requirements policy). Because of this result, the problem can be formulated as a shortest path through an N node network and solved efficiently either by forward or backward dynamic programing. Their paper sparked a great deal of interest among researchers, and led to several generalizations. It has become known as the ELS problem.

Consider the extension of the Wagner Whitin ELS problem, but assume that the usable lifetime of the product is m periods. Clearly, it is now optimal to restrict attention to policies that allow no outdating. Suppose that $\mathbf{y} = (y_1, y_2, ..., y_N)$are the production (or purchase) quantities over the N period planning horizon, and further suppose that the order quantity y_i results in outdating of k units in period $i + m$. Then, replacing y_i with $y_i{}' = y_i - k$ must result in costs less than or equal to those incurred by ordering y_i, since (as with the EOQ model) fixed order costs are unaffected, but both holding costs and marginal order costs as well as outdate costs, are reduced. Hence, one might think that this implies that the ELS problem with perishability is uninteresting. This is certainly not the case, however.

The first to consider the extension of the ELS problem to the case of perishable inventory appears to have been Smith (1975). Smith assumed that the zero inventory property carried over to the perishable case (which is not true). He also seems to have ignored the property that (in the fixed life case) an optimal policy always has zero outdating.

Friedman and Hoch (1978) noted that Smith's algorithm was flawed, since it was based on incorrect assumptions. They provided the following example to show that when fixed life perishability is included, the zero inventory property does not always hold. Consider a problem with a three period planning horizon, and nonstationary costs. Suppose that requirements in each period are for one unit, set-up costs are 0.5 each period, holding costs are \$1 per unit held each period, and marginal production (or order) costs are respectively 8, 10, and 12. Furthermore, suppose that the product lifetime is two periods. The optimal solution to this problem is to order two units in period 1 and one unit in period 2. The cost of this policy is the following: \$1 for setting up in two periods, \$2 for holding since there is one unit on hand at the end of periods 1 and 2, and marginal production costs of \$16 + \$10, for a total cost of \$29 over the three period planning horizon. Since this policy never orders more than two periods of demand, no units outdate. The key point is that this policy results in placing an order in period 2 when the entering inventory is one unit. Hence, the zero inventory property is violated. (Notice, that if we removed the two period lifetime restriction, the optimal policy would call for ordering 3 units in period 1.)

The stochastic problem is difficult for several reasons. One is that it is necessary to define a multidimensional state variable to keep track of the on hand inventories of each age level. This is not necessary in the deterministic case, however. Friedman and Hoch discovered a very clever way to track the age of the inventory without having to employ a multidimensional state variable. If one keeps track of both the period in which production (or ordering) occurs for a unit, and the period in which that unit is used to satisfy demand, then one can easily find the age of the unit in every period it is stored in the system. In this way, one can compute the total cost associated with that unit during its lifetime in the system. This cost matrix (which excludes fixed costs) is the main driver of the computational algorithm.

They assume on hand inventory of each age level is subject to decay at the end of each period. That is, if there are I units on hand of age i at the end of a period (after satisfying demand in that period), then only $r_i I$ units will be available at the start of the next period, where $0 \le r_i \le 1$. On the surface, this looks like simple exponential decay. However, because decay constants are age dependent, it is in fact much more general. As they note, their model includes fixed life perishability as a special case by defining $r_i = 1$ for $1 \le i \le m$, and $r_i = 0$ for $i > m$.

The input to the algorithm is the cost matrix having elements, a_{jt} defined as the holding and marginal production cost of meeting one unit of demand in period t from production in period j, where $1 \le j \le t$. It follows that $a_{jt} = c_j \left(\prod_{k=1}^{t-j} r_k \right)^{-1} + \sum_{i=j}^{t-1} h_t \left(\prod_{k=i-j+1}^{t-j} r_k \right)^{-1}$ for $j < t$. The idea behind the first term is the following. If a unit is produced in period j and held until period t, it will have decayed in the intervening periods to $\prod_{k=1}^{t-j} r_k$, which means one will have had to have produced the inverse of this quantity in period j. By applying the decay factor in each of the intervening periods between periods j and t, one determines the holding cost in those intervening periods, thus accounting for the second term. Based on this cost matrix, the authors develop a dynamic programing algorithm for solving the problem. The cost matrix only includes marginal production costs and holding costs. Fixed production costs are treated separately.

While the authors show that it is not necessarily true that ordering only occurs in periods in which starting inventory is zero, they did correctly state that all of the demand in a period is completely satisfied by production in a single prior (or current) period only. This observation is an important feature of their solution algorithm.

Friedman and Hoch's results were extended by Hsu (2000). Hsu defined the following:

z_{it} = the amount of the demand from period t to be satisfied from production in period i.
y_{it} = the amount produced in period i and held at the beginning of period t which excludes the amount z_{it} used to satisfy the demand in period t.
$H_{it}(y_{it})$ = the cost of holding y_{it} units of inventory in period t, which are produced in period i.
α_{it} = the fraction of y_{it} which is lost during period t.

Hsu's development is easier to follow, as he clearly defines the state variables. He generalizes the aging mechanism defined by Friedman and Hoch. Note the variable α_{it} depends on both the age of the inventory and the period in which aging occurs. This allows for different rates of aging based not only on the age of the inventory, but also based on the time of year. For example, one might expect food products to age more quickly in hot weather than in cool weather. This would be reflected by larger values of α_{it} in summer months.

Hsu assumes a more general cost structure than Friedman and Hoch. In particular, both the production and holding cost functions are assumed to be nondecreasing concave functions. He makes the following assumptions:

For $1 \leq i \leq j \leq t \leq n$:

Assumption 1. $\alpha_{it} \geq \alpha_{jt}$
Assumption 2. $H_{it}(y) \geq H_{jt}(y)$ for $y \geq 0$.

Note that by the way the aging mechanism is defined, the age of an item in period t that is produced in period i is the difference $t - i$. It follows that since $j \geq i$, $t - j \leq t - i$. Therefore, Assumption 1 says that older items age deteriorate at least as fast as younger items. Assumption 2 says that as items age, the cost of holding older items is at least as high as the cost of holding younger items. Part of the justification for Assumption 2 is that the holding cost term may also include a disposal cost for perishable items. Note that Hsu's formulation uses exactly the same mechanism for tracking the age of items as that developed by Friedman and Hoch.

The optimization problem is then stated as

$$\text{Minimize} \sum_{t=1}^{n} [C_t(x_t) + \sum_{i=1}^{t} H_{it}(y_{it})]$$

$$\text{subject to: } x_t - z_{tt} = y_{tt} \qquad 1 \le t \le n$$

$$(1 - \alpha_{i,t-1})y_{i,t-1} - z_{it} = y_{it} \qquad 1 \le i < t \le n$$

$$\sum_{i=1}^{t} z_{it} = d_t, \qquad 1 \le t \le n$$

$$x_t, y_{it}, z_{it} \ge 0 \qquad 1 \le i \le t \le n$$

He assumes that both the order cost functions $C_t(x_t)$ and the holding cost functions $H_{it}(y_{it})$ are concave nondecreasing functions, generalizing Friedman and Hoch's assumptions that these were linear. Hsu shows that the solution of this generalized version of Friedman and Hoch's model is equivalent to a minimum cost network flow problem on a specially constructed network with flow loss.

Hsu shows that for every t, $1 \le t \le n$, there is a unique i, $1 \le i \le t$, such that $z^*_{it} = d_t$. This means that at an optimal solution, every demand is satisfied completely by production in a single prior period. Note that the production quantity must be inflated by the decay losses as was noted by Friedman and Hoch. A second structural result (which we do not quote here) is then used to construct the solution algorithm. Hsu notes the result obtained by Friedman and Hoch that the zero inventory property (namely that an optimal solution only produces when starting inventory is zero) does not necessarily hold when perishability is present. In fact, even when perishability is not present, nonstationary costs could result in this property failing to hold. The solution algorithm developed for solving the problem is similar to the one developed by Friedman and Hoch. However, because Hsu allows for nonlinear costs, the cost matrix approach of Friedman and Hoch is no longer possible. In summary, Hsu generalizes Friedman and Hoch's results in two ways: one is allowing for the decay variables to depend on both the age of the inventory and the planning period, and the other is allowing for more general holding and ordering cost functions.

An extension of these results to allow for backorders is considered in Hsu (2003). In this paper, Hsu essentially combines the results for Hsu (2000) and Hsu and Lowe (2001), which did not deal with perishables, but considered the ELS problem with backorders and age-dependent backorder costs. The idea is to allow for backorder costs to depend on both the period in which the backorder occurs and the period in which an item is produced to meet that backorder. We do not review these papers in detail, but note that the earlier Hsu and Lowe paper built a solution algorithm based on properties similar to those discovered by Hsu (2000) when generalizing Friedman and Hoch's model.

An extension of Hsu (2000) was considered by Chu et al. (2005). The model is identical to Hsu (2000), except that the order cost function, $C_t(^*)$ is generalized to the so-called *economies of scale function*.

A cost function, $F(X)$, defined on $[0, \infty]$ is called an *economies of scale function* if:

1. $F(0) = 0$.
2. $F(X)$ is nondecreasing on $[0, \infty]$.
3. The average cost function defined as $\bar{F}(X) = F(X)/X$ for $X>0$, is a nonincreasing function on $(0, \infty)$.

The idea behind the economies of scale function is that it includes many types of discount schedules not captured by the simpler concave nonincreasing function assumed in Hsu (2000). Because there is no requirement that the economies of scale function be continuous or have continuous derivatives, it includes incremental, all units, and carload discount schedules. Except for that, the assumptions and structure of the model are the same as Hsu (2000). The problem is that the important property of an optimal solution under the simpler cost structure that demand in a period is completely filled by production in a single prior period no longer holds. Hence, the algorithm developed in Hsu (2000) no longer holds. It turns out that finding an optimal solution to this problem is NP hard, so the authors consider approximations.

They suggest an approximation which has properties similar to the optimal solution structure in Hsu (2000). Their main result is that this approximation results in a cost that is no higher than $(4\sqrt{2} + 5)/7$ (1.5224) times the optimal solution, and this bound is tight. Unfortunately, this means the cost error can be more than 50%, making the value of this approximation questionable. Note that even though the general problem is NP hard, it does not mean that most reasonable sized problems cannot be solved for an optimal solution by dynamic programing.