Chapter 2
Approximately Globally Convergent Numerical Method

In this chapter, we present our approximately globally convergent numerical method for a multidimensional CIP for a hyperbolic PDE. This method also works for a similar CIP for a parabolic PDE. The numerical method of the current chapter addresses the first central question of this book (Sect. 1.1). The first publication about this method was [24] with follow-up works [25–29, 109, 114–117, 160]. We remind that only multidimensional CIPs with single measurement data are considered in this book. Recall that the term "single measurement" means that the boundary data are generated either by a single position of the point source or by a single direction of the initializing plane wave (Sect. 1.1.2). It will become clear from the material below that when approximately solving certain nonlinear integral differential equations with Volterra-like integrals, we use an analog of the well-known predictor-corrector approach.

We describe this numerical method and prove its approximate global convergence property. The development of approximately globally convergent numerical methods for multidimensional CIPs has started from the so-called *convexification* algorithm [100–102, 157–160], which we consider as the approximately globally convergent numerical method of the first generation. First, the convexification comes up with a nonlinear integral differential equation, which is the same as (2.20) in Sect. 2.3. The *key point* is that this equation does not contain the unknown coefficient, which is similar with one of the ideas of the Bukhgeim-Klibanov method; see Sects. 1.10 and 1.11. A numerical method for the solution of this equation represents the main difficulty of both the convexification and the approach of this book. To solve that equation, the convexification uses a layer stripping procedure with respect to a spatial variable z and the projection method with respect to the rest of spatial variables. In this case, both Dirichlet and Neumann boundary conditions at a part of a plane orthogonal to the z-axis are used. Also, z dependent CWFs are involved in the convexification. Because of this, the convexification can use boundary conditions only at one part of the boundary, i.e., at a side of a rectangular prism, which is orthogonal to z.

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The numerical method of this chapter is the approximately globally convergent numerical method of the second generation. Its radical difference with the convexification is in the solution of the abovementioned nonlinear integral differential equation. Unlike the convexification, the current method is not using neither the projection with respect to some spatial variables nor the layer stripping with respect to a spatial variable. In Chaps. 2–5, the current method uses the Dirichlet boundary condition at the entire boundary $\partial\Omega$ of a finite domain of interest Ω. The target coefficient is unknown in Ω and has a known constant value outside of Ω.

We use the layer stripping procedure with respect to the parameter $s > 0$, where s is the parameter of the Laplace transform of a hyperbolic PDE, for which the CIP is considered. We call s *pseudo frequency.* Since the differential operator with respect to s is not involved in the corresponding PDE, unlike the differential operator with respect to z in the convexification, then this procedure is more stable than the convexification. On each thin s layer, the Dirichlet boundary value problem for a nonlinear elliptic PDE of the second order is solved via the FEM. Dirichlet boundary conditions for these elliptic PDEs are originally generated by the data for the inverse problem. Also, s dependent CWFs are present in our numerical scheme. This presence is important, because it enables one to weaken the influence of the nonlinear term in each of those elliptic PDEs, thus solving a linear problem on each iteration.

Starting from the remarkable work of Carleman [50], weight functions carrying his name have been widely used for proofs of uniqueness and conditional stability results for ill-posed Cauchy problems for PDEs [102, 124], as well as for multidimensional CIPs with the single measurement data (see Sects. 1.10 and 1.11 above for the latter). In this capacity, CWFs were dependent on spatial variables since they have provided weighted estimates for differential operators. However, one of new points of our method is that CWFs are used for integral Volterra-like operators, they are involved in the numerical scheme, and depend on the pseudo frequency s, rather than on a spatial variable.

An important element of our technique is the procedure of working with the so-called *tail functions*. The tail function complements a certain truncated integral with respect to s. We refer to earlier works [73, 155] for similar treatments of tails for some other numerical methods for CIPs.

Theorems 2.8.2 and 2.9.4 ensure the approximate global convergence property of our technique within frameworks of two approximate mathematical models. It follows from these theorems that the accuracy of the solution mainly depends from the accuracy of the reconstruction of the tail functions. On the other hand, it follows from the second approximate mathematical model (Sect. 2.9.2) that the reconstruction of the first tail function can be done via solving the Dirichlet boundary value problem for the Laplace equation. Thus, if the noise in the boundary data is small, then the solution of the latter problem is accurate. The accuracy of the reconstruction of the rest of tail functions depends on the accuracy of the first tail. This indicates that it is because of the successful choice of our approximate mathematical models, a small noise in the boundary data is the main input for a good accuracy of our algorithm. In the theory of ill-posed problems, the small noise condition is a natural requirement.

A substantially different layer stripping procedure with respect to the frequency (rather than pseudo frequency) was previously developed in [55], where a convergence theorem was not proved (see Remark 1.1 in [55]). The paper [55] works with the Fourier transform of the hyperbolic equation $c(x) u_{tt} = \Delta u$ with the unknown coefficient $c(x)$. The iterative process of [55] starts from a low frequency value. Unlike this, we start from a high value of the pseudo frequency.

2.1 Statements of Forward and Inverse Problems

Everywhere in this book, the forward problem is the Cauchy problem for either a hyperbolic or a parabolic PDE. The case of a boundary value problem in a finite domain is not considered here only because an analogue of the asymptotic behavior (2.14) is not proved in this case, since (2.14) is actually derived from Theorem 4.1 of [144] as well as from [145]. That theorem establishes a certain asymptotic behavior of the fundamental solution of a hyperbolic equation near the characteristic cone. In our numerical experiments, we verify the asymptotic behavior (2.14) computationally; see Sect. 3.1.2. We also note that the existence of the fundamental solution of the hyperbolic equation (2.1) is currently proven only for the case when the coefficient $c \in C^k(\mathbb{R}^3)$ with $k \geq 2$ and the geodesic lines are regular [144, 145]. These justify the assumption (2.4) below.

Consider the Cauchy problem for the hyperbolic equation:

$$c(x) u_{tt} = \Delta u \text{ in } \mathbb{R}^3 \times (0, \infty), \tag{2.1}$$

$$u(x, 0) = 0, \; u_t(x, 0) = \delta(x - x_0). \tag{2.2}$$

Equation (2.1) governs a wide range of applications, including, for example, propagation of acoustic and EM waves. In the acoustical case, $1/\sqrt{c(x)}$ is the sound speed. In the 2D case of EM waves propagation in a non-magnetic medium, the dimensionless coefficient is $c(x) = \varepsilon_r(x)$, where $\varepsilon_r(x)$ is the spatially distributed dielectric constant of the medium, see, for example, [57], where (2.1) was derived from the Maxwell equations in the 2D case. Unlike the 2D case, (2.1) cannot be derived from the Maxwell equations in the 3D case if $c(x) = \varepsilon_r(x) \neq \text{const.}$ Nevertheless, this equation was successfully used to work with experimental data in [28, 109] in 3D; see Chap. 5.

Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$. Let $d = \text{const.} > 1$. We assume that the coefficient $c(x)$ of (2.1) is such that

$$c(x) \in [1, d], \;\; c(x) = 1 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega, \tag{2.3}$$

$$c \in C^3(\mathbb{R}^3). \tag{2.4}$$

In accordance with the second condition of the fundamental concept of Tikhonov (Sect. 1.4), we a priori assume knowledge of the constant d, which amounts to the knowledge of the correctness set. However, we do *not* assume that the number $d-1$ is small, i.e., we do not impose smallness assumptions on the unknown coefficient $c(x)$.

Coefficient Inverse Problem 2.1. *Suppose that the coefficient $c(x)$ satisfies (2.3) and (2.4). Assume that the function $c(x)$ is unknown in the domain Ω. Determine the function $c(x)$ for $x \in \Omega$, assuming that the following function $g(x,t)$ is known for a single source position $x_0 \notin \overline{\Omega}$:*

$$u(x,t) = g(x,t), \forall (x,t) \in \partial\Omega \times (0,\infty). \tag{2.5}$$

The reason why we assume here that the source $x_0 \notin \overline{\Omega}$ is that we do not want to deal with singularities near the source location. In applications, the assumption $c(x) = 1$ for $x \in \mathbb{R}^3 \diagdown \Omega$ means that the target coefficient $c(x)$ has a known constant value outside of the medium of interest Ω. Another argument here is that one should bound the coefficient $c(x)$ from the below by a positive number to ensure that the operator in (2.1) is a hyperbolic one on all iterations of our method. The function $g(x,t)$ models time-dependent measurements of the wave field at the boundary of the domain of interest. Practical measurements are calculated at a number of detectors, of course. In this case, the function $g(x,t)$ can be obtained via one of standard interpolation procedures.

Remarks 2.1. 1. As it was stated in Sect. 1.10.1, uniqueness theorem for this inverse problem is a long-standing and well-known open question because of the $\delta(x-x_0)$ function in the initial condition (2.2), although see (1.76). Thus, we assume everywhere below that uniqueness theorem is valid for this problem, as well as for all other CIPs considered in this book. It is an opinion of the authors that because of applications, it is worthy to study numerical methods for CIPs of this book, assuming that the uniqueness holds.
2. Our computational experience shows that the assumption of the infinite time interval in (2.5) is not a restrictive one. In the case of a finite time interval, on which measurements are performed, one should assume that this interval is large enough. Thus, the t-integral of the Laplace transform over this interval is approximately the same as the one over $(0,\infty)$. Our work with experimental data in Chaps. 5 and 6 verifies this point.

2.2 Parabolic Equation with Application in Medical Optics

In this section, we formulate both forward and inverse problems for a parabolic equation which governs applications particularly in medical optical imaging. The optical medical imaging consists of two stages. On the first stage, a device collects the light scattering data at the boundary of a human tissue (e.g., at the surface of a brain or a female breast); see, for example, [150]. On the second stage, a mathematical algorithm for a CIP for a diffusion PDE is applied to approximately calculate the spatially distributed absorption coefficient inside that tissue. The map of this coefficient produces the desired image. We are interested in the second stage. Because of the necessity to solve a CIP, this stage represents a major mathematical challenge.

It was shown experimentally that the diffusion coefficient of light changes slowly in human tissues [76]. Hence, we can assume that it is a known constant and consider the following parabolic equation governing light propagation in human tissues [8]:

$$\begin{aligned} U_t &= D\Delta U - a(x)U \ \text{ in } \mathbb{R}^3 \times (0,\infty), \\ U(x,0) &= \delta(x - x_0). \end{aligned} \tag{2.6}$$

Here, $\{x = x_0\}$ is the location of the light source, $U(x,t)$ is the light amplitude, $a(x) = \mu_{\mathrm{a}}(x) \geq \text{const.} > 0$ is the absorption coefficient, and $D = D_0 = \text{const.} > 0$ is the diffusion coefficient $D = 1/3\mu_{\mathrm{s}}'$, where μ_{s}' is the reduced scattering coefficient. We assume below that the diffusion coefficient D_0 is known.

Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$. Let $a_0, a_1 = \text{const.} > 0$, $a_0 < a_1$. We assume that the absorption coefficient $a(x)$ of (2.6) is such that

$$a(x) \in [a_0, a_1], \ \ a(x) = a_0 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega, \tag{2.7}$$

$$a(x) \in C^{\alpha}\left(\mathbb{R}^3\right), \alpha \in (0,1). \tag{2.8}$$

Let

$$U_0(x,t) = \frac{1}{\left(2\sqrt{\pi t}\right)^3} \exp\left(-\frac{|x - x_0|^2}{4t}\right)$$

be the solution of the problem (2.6) for $a \equiv 0$. It follows from (2.7) and (2.8) that there exists unique solution U of the forward problem (2.6) such that the function $(U - U_0) \in C^{2+\alpha,1+\alpha/2}\left(\mathbb{R}^3 \times [0,T]\right), \forall T > 0$ [120].

It was established experimentally [76, 156] that cancerous tumors absorb light more than the surrounding tissue. The tumor/background absorption contrast is between 2:1 and 3:1 [76]. Realistic value of optical coefficients of light propagation in human tissues are [156] $\mu_{\mathrm{a}} \in [0.004, 0.016]\,\text{mm}^{-1}$, $\mu_{\mathrm{s}}' \in [0.6, 1.2]\,\text{mm}^{-1}$, where μ_{a} and μ_{s}' are absorption and reduced scattering coefficients, respectively. The absorption coefficient characterizes blood oxygenation. Since malignant tumors are less oxygenated than healthy tissues, a hope of researchers is to detect these tumors on early stages using optical methods. Thus, our goal is to determine the absorption coefficient in (2.6). We now pose the following inverse problem for (2.6).

Coefficient Inverse Problem 2.2. Let $\Omega \subset R^3$ be a convex bounded domain with the piecewise smooth boundary $\partial\Omega$. Suppose that the coefficient $a(x)$ satisfies conditions (2.7) and (2.8) and is unknown in Ω. Suppose also that the diffusion coefficient $D = D_0 = \text{const.} > 0$ is known. Determine the coefficient $a(x)$ for

$x \in \Omega$, assuming that the following function $\varphi(x,t)$ is known for a fixed source position some position $x_0 \notin \overline{\Omega}$:

$$U(x,t) = \widetilde{g}(x,t), \forall (x,t) \in \partial\Omega \times (0,\infty). \tag{2.9}$$

2.3 The Transformation Procedure for the Hyperbolic Case

In this section, we reduce inverse problem 2.1 to the Dirichlet boundary value problem for a nonlinear integral differential equation. Consider the Laplace transform of the functions u in the hyperbolic equation (2.1):

$$w(x,s) = \int_0^\infty u(x,t)\mathrm{e}^{-st}\mathrm{d}t, \text{ for } s > \underline{s} = \text{const.} > 0, \tag{2.10}$$

where $\underline{s}$ is a certain number. It is sufficient to choose $\underline{s}$ such that the integral (2.10) would converge together with corresponding (x,t) derivatives. So, we can assume that the number $\underline{s}$ is sufficiently large. We call the parameter s *pseudo frequency*. Recall that $x_0 \notin \overline{\Omega}$. It follows from (2.1), (2.2), and (2.10) that the function w is the solution of the following problem:

$$\Delta w - s^2 c(x) w = -\delta(x - x_0), \ x \in \mathbb{R}^3, \tag{2.11}$$

$$\lim_{|x|\to\infty} w(x,s) = 0. \tag{2.12}$$

We prove (2.12) in Theorem 2.7.1 Likewise, we specify properties of the function $w(x,s)$ in Theorem 2.7.2 In particular, it follows from Theorems 2.7.1 and 2.7.2 that $w(x,s) \in C^3\left(\mathbb{R}^3 \diagdown \{|x - x_0| < \varepsilon\}\right), \forall \varepsilon > 0$. To justify the asymptotic behavior of the function $w(x,s)$ at $s \to \infty$, we need Lemma 2.3.

Lemma 2.3([102]). *Assume that conditions (2.3) and (2.4) are satisfied. Let the function $w(x,s) \in C^3\left(\mathbb{R}^3 \diagdown \{|x - x_0| < \varepsilon\}\right), \forall \varepsilon > 0$ be the solution of the problem (2.11) and (2.12). Assume that geodesic lines, generated by the eikonal equation corresponding to the function $c(x)$ are regular, i.e., any two points in $\mathbb{R}^3$ can be connected by a single geodesic line. Let $l(x,x_0)$ be the length of the geodesic line connecting points x and x_0. Then the following asymptotic behavior of the function w and its derivatives takes place for $|\beta| \le 3, k = 0,1, x \neq x_0$:*

$$D_x^\beta D_s^k w(x,s) = D_x^\beta D_s^k \left\{ \frac{\exp[-sl(x,x_0)]}{f(x,x_0)} \left[1 + O\left(\frac{1}{s}\right)\right]\right\}, s \to \infty, \tag{2.13}$$

where $f(x,x_0)$ is a certain function and $f(x,x_0) \neq 0$ for $x \neq x_0$. This behavior is uniform for $x \in \overline{\Omega}$.

The C^2-smoothness required by Lemma 2.1 is also because of Theorem 4.1 of [144], which implies the asymptotic behavior (2.13). Note that Theorem 4.1 of [144] actually requires a higher smoothness of coefficients. This is because it is concerned with many terms of the asymptotic behavior of the fundamental solution of the hyperbolic equation near the characteristic cone. However, since (2.13) is dealing only with the first term of this behavior, then it follows from the proof of that theorem that the C^2-smoothness is sufficient; also see [145] for the smoothness.

Remark 2.3.1. Actually, it follows from Theorem 4.1 of [144] that the asymptotic behavior (2.13) is valid for the Laplace transform for a general hyperbolic equation of the second order, as long as the condition of the regularity of geodesic lines is in place. This condition cannot be effectively verified, unless the coefficient $c(x)$ is close to a constant. The authors are unaware about any meaningful analytical results for multidimensional hyperbolic CIPs without either this or somewhat close condition imposed. For example, it was shown in [144] that condition (1.82) is close to the condition of the regularity of geodesic lines. On the other hand, conditions of this lemma are only sufficient, but not necessary ones for the asymptotic behavior (2.13). Therefore, we assume everywhere in this book that the asymptotic behavior (2.13) holds. We verify (2.13) computationally in some of our numerical studies; see Sect. 3.1.2 below.

We now work only with the function $w(x,s)$. It will be shown in Theorems 2.7.1 and 2.7.2 that $w(x,s) > 0$. Hence, we can consider functions $v(x,s)$ and $H(x,s)$ defined as

$$v(x,s) = \frac{\ln w(x,s)}{s^2}.$$

Assuming that the asymptotic behavior (2.13) holds (Remark 2.3.1), we obtain the following asymptotic behavior of the function v:

$$\left\| D_x^{\beta} D_s^k v(x,s) \right\|_{C^3(\overline{\Omega})} = O\left(\frac{1}{s^{k+1}}\right), \; s \to \infty, k = 0, 1. \tag{2.14}$$

Substituting $w = e^v$ in (2.11), keeping in mind that the source $x_0 \notin \overline{\Omega}$ and then dividing the resulting equation for v by s^2, we obtain

$$\Delta v + s^2 (\nabla v)^2 = c(x), x \in \Omega. \tag{2.15}$$

Denote

$$q(x,s) = \partial_s v(x,s). \tag{2.16}$$

By (2.14) and (2.16),

$$v(x,s) = -\int_s^{\infty} q(x,\tau)\, d\tau.$$

We rewrite this integral as

$$v(x,s) = -\int_{s}^{\overline{s}} q(x,\tau)\,d\tau + V(x,\overline{s}), \tag{2.17}$$

where the truncation pseudo frequency $\overline{s} > \underline{s}$ is a large number. It is important that in (2.17), $V(x,\overline{s})$ is not an arbitrary function, but rather

$$V(x,\overline{s}) = v(x,\overline{s}) = \frac{\ln w(x,\overline{s})}{\overline{s}^2}, \tag{2.18}$$

where $w(x,\overline{s})$ is the Laplace transform (2.10) of the solution of the forward problem (2.1) and (2.2) at $s := \overline{s}$, or, which is equivalent, the solution of the elliptic forward problem (2.10), (2.10) at $s := \overline{s}$. The number $\overline{s}$ should be chosen in numerical experiments. We call $V(x,\overline{s})$ the "tail," and this function is unknown. By (2.14) and (2.18),

$$\|V(x,\overline{s})\|_{C^3(\overline{\Omega})} = O\left(\frac{1}{\overline{s}}\right), \quad \|\partial_{\overline{s}} V(x,\overline{s})\|_{C^3(\overline{\Omega})} = O\left(\frac{1}{\overline{s}^2}\right). \tag{2.19}$$

In other words, the tail is small for large values of $\overline{s}$. In principle, therefore, one can set $V(x,\overline{s}) := 0$. However, our numerical experience shows that it would be better to update somehow the tail function in an iterative procedure. We call the updating procedure "iterations with respect to tails" and describe it in Sect. 2.7.

Remark 2.3.2. The integral in (2.17) is sort of truncated at a large value $\overline{s}$ of the pseudo frequency, which is similar with a routine truncation of high frequencies in science and engineering. We use words "sort of" because instead of just setting the tail function to zero, as it would be the case of a "straight" truncation, we iteratively update it in our algorithm. Hence, $\overline{s}$ is one of the regularization parameters of our numerical method. In the computational practice, this parameter is chosen in numerical experiments.

Thus, differentiating (2.15) with respect to s and using (2.16) and (2.17), we obtain the following integral nonlinear differential equation:

$$\Delta q - 2s^2 \nabla q \int_{s}^{\overline{s}} \nabla q(x,\tau)\,d\tau + 2s\left[\int_{s}^{\overline{s}} \nabla q(x,\tau)\,d\tau\right]^2$$
$$+2s^2 \nabla q \nabla V - 4s \nabla V \int_{s}^{\overline{s}} \nabla q(x,\tau)\,d\tau + 2s(\nabla V)^2 = 0, x \in \Omega. \tag{2.20}$$

In addition, (2.5) and (2.16) imply that the following Dirichlet boundary condition is given for the function q:

$$q(x,s) = \psi(x,s), \ \forall (x,s) \in \partial\Omega \times [\underline{s}, \overline{s}], \tag{2.21}$$

where:

$$\psi(x,s) = \frac{\partial_s \ln\varphi}{s^2} - \frac{2\ln\varphi}{s^3},$$

and $\varphi(x,s)$ is the Laplace transform (2.10) of the function $g(x,t)$ in (2.5).

Suppose for a moment that functions q and V are approximated in Ω together with their derivatives $D_x^\alpha q, D_x^\alpha V, |\alpha| \leq 2$. Then the corresponding approximation for the target coefficient can be found via (2.15) as

$$c(x) = \Delta v + \underline{s}^2 (\nabla v)^2, x \in \Omega, \tag{2.22}$$

where the function H is approximated via (2.17). Although any value of the pseudo frequency $s \in [\underline{s}, \overline{s}]$ can be used in (2.22), we found in our numerical experiments that the best value is $s := \underline{s}$.

If integrals would be absent in (2.20) and the tail function would be known, then (2.20) and (2.21) would be the classical Dirichlet boundary value problem for the Laplace equation. However, the presence of integrals implies the nonlinearity and represents the main difficulty here. Another obvious difficulty is that (2.20) has two unknown functions q and V. The reason why we can handle this difficulty is that we treat functions q and V differently: while we iteratively approximate the function q being sort of "restricted" only to (2.20), we find updates for V using solutions of forward problems (2.1) and (2.2), the Laplace transform (2.10), and the formula (2.18). In those forward problems, we use approximations for the unknown coefficient c obtained from (2.22). The algorithm of approximating both functions q and V is described in Sect. 2.6.

2.4 The Transformation Procedure for the Parabolic Case

The goal of this section is to show that the coefficient inverse problem 2.2 for the parabolic equation (2.6) can be solved numerically along the same lines as the coefficient inverse problem 2.1 for the hyperbolic equation (2.1). However, we do not study further the parabolic case in this book. In the case of parabolic equation (2.6), consider the Laplace transform of the solution of the parabolic Cauchy problem (2.6)

$$W(x,s) = \int_0^\infty U(x,t) \exp\left(-s^2 t\right) dt, \ s \geq \underline{s} = \text{const.} > 0. \tag{2.23}$$

For simplicity, let $D_0 = 1$. It follows from (2.6) and (2.23) that the function W is the solution of the following problem:

$$\Delta W - s^2 W - a(x) W = -\delta(x - x_0), \forall s \geq \underline{s} = \text{const.} > 0,$$
$$\lim_{|x| \to \infty} W(x,s) = 0. \tag{2.24}$$

The second condition (2.24) is valid for sufficiently large $\underline{s}$ and can be proved by the method, which is similar with the one of Sect. 2.5**.** Theorem 11 of Chap. 2 of [69] ensures that the fundamental solution of a general parabolic equation is positive for $t > 0$. This means that $U(x,t) > 0$ for $t > 0$.Hence, $W(x,s) > 0$. Hence, we can consider the function $\overline{P} = \ln W$. Since $x_0 \notin \overline{\Omega}$, we obtain from (2.24)

$$\Delta \overline{P} + \left|\nabla \overline{P}\right|^2 - s^2 = a(x), x \in \Omega. \tag{2.25}$$

Consider now the following hyperbolic Cauchy problem:

$$u_{tt} = \Delta u - a(x) u \text{ in } \mathbb{R}^3 \times (0, \infty),$$
$$u(x,0) = 0, u_t(x,0) = \delta(x - x_0).$$

Applying to the function u the Laplace transform (2.10), we obtain (2.24). Hence,

$$W(x,s) = \int_0^\infty u(x,t) \exp(-st)\,\mathrm{d}t, \; s \geq \underline{s} = \text{const.} > 0.$$

Geodesic lines are straight lines in this case. Therefore, the asymptotic behavior (2.13) holds (the first sentence of Remark 2.3.1),

$$D_x^\beta D_s^k W(x,s) = D_x^\beta D_s^k \left\{ \frac{\exp[-s|x - x_0|]}{4\pi |x - x_0|} \left[1 + O\left(\frac{1}{s}\right)\right]\right\}, s \to \infty,$$

where $|\beta| \leq 2, k = 0, 1$. Consider the function P:

$$P(x,s) = \overline{P}(x,s) - \ln\left(\frac{\exp[-s|x - x_0|]}{4\pi |x - x_0|}\right) := \overline{P}(x,s) - P_0(x,s).$$

Then the following asymptotic behavior is valid:

$$D_x^\beta D_s^k P(x,s) = O\left(\frac{1}{s^{k+1}}\right), \; s \to \infty, \; |\beta| \leq 2, \; k = 0, 1, \; x \in \overline{\Omega}. \tag{2.26}$$

Next, similarly with (2.16), we "eliminate" the unknown coefficient $a(x)$ from equation for the function P via the differentiation with respect to s. Denote

$$Q(x,s) = \partial_s P(x,s). \tag{2.27}$$

By (2.26) and (2.27),

$$P(x,s) = -\int_s^\infty Q(x,\tau)\,d\tau.$$

We represent this integral as

$$P(x,s) = -\int_s^{\overline{s}} Q(x,\tau)\,d\tau + V(x,\overline{s}), \tag{2.28}$$

where $\overline{s} > \underline{s}$ is a large number. Again, we call the function $\tilde{V}(x,\overline{s})$ in (2.28) as "the tail function" and it is defined as

$$V(x,\overline{s}) = P(x,\overline{s}) = \ln W(x,\overline{s}) - \ln\left(\frac{\exp\left[-\overline{s}\,|x-x_0|\right]}{4\pi\,|x-x_0|}\right). \tag{2.29}$$

Similarly with the hyperbolic case, differentiating equation for P with respect to s and using (2.27)–(2.29), we obtain the following integral nonlinear differential equation for the function Q:

$$\Delta Q - 2\frac{(s+1)}{|x-x_0|}(\nabla Q, x-x_0) - 2\nabla Q\int_s^{\overline{s}}\nabla Q(x,\tau)\,d\tau + 2\nabla Q\nabla V$$
$$+\frac{2}{|x-x_0|}\left(\int_s^{\overline{s}}\nabla Q(x,\tau)\,d\tau, x-x_0\right) - \frac{2}{|x-x_0|}(\nabla V, x-x_0) = 0. \tag{2.30}$$

Here, $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^3$. The boundary condition for the function Q is

$$Q\mid_\Omega = \bar{\psi}(x,s), (x,s) \in \partial\Omega \times [\underline{s},\overline{s}], \tag{2.31}$$

where the function $\bar{\psi}$ is defined as

$$\bar{\psi}(x,s) = \partial_s \ln\bar{\varphi}(x,s) + |x-x_0|,$$

where $\bar{\varphi}(x,s)$ is the Laplace transform of the function $\widetilde{g}(x,t)$ in (2.9).

Assume that we can approximate both functions Q and V in Ω together with their derivatives $D_x^\alpha Q, D_x^\alpha V, |\alpha| \leq 2$. Then the corresponding approximation for the absorption coefficient $a(x)$ can be found via (2.25) as

$$a(x) = \Delta(P+P_0) + |\nabla(P+P_0)|^2 - \underline{s}^2,\ x \in \Omega,$$

where the function P is approximated via (2.28). As it was mentioned above, the main difficulty of our method consists in the numerical solution of the nonlinear integral differential equation (2.30) with the boundary condition (2.31).

2.5 The Layer Stripping with Respect to the Pseudo Frequency *s*

In this section, we introduce the layer stripping procedure with respect to the pseudo-frequency for the solution of the integral-differential equation (2.20). Almost the same procedure can be applied for the solution of (2.30), although it is not presented here.

We approximate the function $q(x,s)$ in (2.20) as a piecewise constant function with respect to the pseudo frequency s. That is, we assume that there exists a partition

$$\underline{s} = s_N < s_{N-1} < ... < s_1 < s_0 = \overline{s}, s_{i-1} - s_i = h$$

of the interval $[\underline{s}, \overline{s}]$ with a sufficiently small grid step size h such that $q(x,s) = q_n(x)$ for $s \in (s_n, s_{n-1}]$. We set

$$q_0 \equiv 0. \tag{2.32}$$

Hence,

$$\int_s^{\overline{s}} \nabla q(x,\tau) d\tau = (s_{n-1} - s)\nabla q_n(x) + h \sum_{j=0}^{n-1} \nabla q_j(x), s \in (s_n, s_{n-1}). \tag{2.33}$$

We approximate the boundary condition (2.21) as a piecewise constant function:

$$q_n(x) = \overline{\psi}_n(x), x \in \partial\Omega, \tag{2.34}$$

$$\overline{\psi}_n(x) = \frac{1}{h} \int_{s_n}^{s_{n-1}} \psi(x,s) ds. \tag{2.35}$$

For each subinterval $(s_n, s_{n-1}], n \geq 1$, we assume that functions $q_j(x)$, $j = 1, ..., n-1$ for all previous subintervals are known. We obtain from (2.20) the following approximate equation for the function $q_n(x)$:

$$\begin{aligned}\widetilde{L}_n(q_n) := & \Delta q_n - 2\left(s^2 - 2s(s_{n-1} - s)\right)\left(h\sum_{j=1}^{n-1} \nabla q_j\right) \nabla q_n \\ & + 2\left(s^2 - 2s(s_{n-1} - s)\right) \nabla q_n \nabla V \\ = & \ 2(s_{n-1} - s)\left[s^2 - s(s_{n-1} - s)\right](\nabla q_n)^2 - 2sh^2 \left(\sum_{j=1}^{n-1} \nabla q_j\right)^2 \\ & + 4s\nabla V \left(h \sum_{j=1}^{n-1} \nabla q_j\right) - 2s|\nabla V|^2, s \in (s_{n-1}, s_n].\end{aligned} \tag{2.36}$$

Equation (2.36) is nonlinear, and it depends on the parameter s, whereas the function $q_n(x)$ is independent on s. This discrepancy is due to the approximation of the function $q(x,s)$ by a piecewise constant function. Although it seems that (2.36) is over-determined because the function $q_n(x)$ is not changing with the change of s, variations of s dependent coefficients of (2.36) are small over $s \in [s_n, s_{n-1})$ because this interval is small. This discrepancy is actually helpful for our method since it enables us to "mitigate" the influence of the nonlinear term $(\nabla q_n)^2$ in (2.36) via introducing the s dependent CWF.

In addition, we add the term $-\varepsilon q_n$ to the left-hand side of (2.36), where $\varepsilon > 0$ is a small parameter. We are doing so because, by the maximum principle, if a function $p(x,s)$ is the classical solution of the Dirichlet boundary value problem

$$\widetilde{L}_n(p) - \varepsilon p = f(x,s) \text{ in } \Omega, \; p\mid_{\partial\Omega} = p_b(x,s),$$

then [118] (Chap. 3, Sect. 1)

$$\max_{\overline{\Omega}} |p| \leq \max\left[\max_{\partial\Omega} |p_b|, \varepsilon^{-1} \max_{\overline{\Omega}} |f|\right], \forall s \in (s_{n-1}, s_n]. \tag{2.37}$$

On the other hand, if $\varepsilon = 0$, then an analogue of the estimate (2.37) would be worse because of the involvement of some other constants. Therefore, it is anticipated that the introduction of the term $-\varepsilon q_n$ should provide a better stability of our process, and we indeed observe this in our computations.

After adding the term $-\varepsilon q_n$ to the left-hand side of (2.36), multiply this equation by the CWF of the form:

$$\mathcal{C}_{n,\lambda}(s) = \exp\left[\lambda(s - s_{n-1})\right], \;\; s \in (s_n, s_{n-1}], \tag{2.38}$$

and integrate with respect to s over (s_n, s_{n-1}). In (2.38) $\lambda >> 1$ is a parameter, which should be chosen in numerical experiments. Theorem 2.8.2 provides a recipe for this choice. Taking into account (2.34), we obtain

$$\begin{aligned} L_n(q_n) &:= \Delta q_n - A_{1,n}\left(h\sum_{j=0}^{n-1} \nabla q_j\right)\nabla q_n + A_{1n}\nabla q_n \nabla V - \varepsilon q_n \\ &= 2\frac{I_{1,n}}{I_0}(\nabla q_n)^2 - A_{2,n}h^2\left(\sum_{j=0}^{n-1}\nabla q_j(x)\right)^2 \\ &\quad + 2A_{2,n}\nabla V\left(h\sum_{j=0}^{n-1}\nabla q_j\right) - A_{2,n}(\nabla V)^2, n = 1, ..., N, \\ &\quad q_n\mid_{x\in\partial\Omega} = \overline{\psi}_n(x). \end{aligned} \tag{2.39}$$

In (2.39),

$$I_0 := I_0(\lambda, h) = \int_{s_n}^{s_{n-1}} \mathcal{C}_{n,\lambda}(s)\,\mathrm{d}s = \frac{1 - \mathrm{e}^{-\lambda h}}{\lambda},$$

$$I_{1,n} := I_{1,n}(\lambda, h) = \int_{s_n}^{s_{n-1}} (s_{n-1} - s)\left[s^2 - s(s_{n-1} - s)\right]\mathcal{C}_{n,\lambda}(s)\,\mathrm{d}s,$$

$$A_{1,n} := A_{1,n}(\lambda, h) = \frac{2}{I_0}\int_{s_n}^{s_{n-1}} \left(s^2 - 2s(s_{n-1} - s)\right)\mathcal{C}_{n,\lambda}(s)\,\mathrm{d}s,$$

$$A_{2,n} := A_{2,n}(\lambda, h) = \frac{2}{I_0}\int_{s_n}^{s_{n-1}} s\mathcal{C}_{n,\lambda}(s)\,\mathrm{d}s.$$

Thus, we have obtained the Dirichlet boundary value problem (2.39) for a nonlinear elliptic PDE with the unknown function $q_n(x)$. In (2.39), the tail function V is also unknown. An important observation is that

$$\frac{|I_{1,n}(\lambda, h)|}{I_0(\lambda, h)} \leq \frac{4\overline{s}^2}{\lambda}, \text{ for } \lambda h \geq 1. \tag{2.40}$$

Therefore, by taking $\lambda >> 1$, we mitigate the influence of the nonlinear term with $(\nabla q_n)^2$ in (2.39). This enables us to solve each elliptic Dirichlet boundary value problem (2.34) and (2.39) iteratively at each n via solving a linear problem on each step.

Remarks 2.5. 1. It is clear from (2.40) that the nonlinear term $(\nabla q_n)^2$ in (2.39) can be ignored for large values of λ. This is done in Sect. 2.6. However, ignoring this term does not mean linearization of the original problem. Indeed, the nonlinearity actually surfaces in iterations with respect to n, because of the involvement of terms $\nabla q_j \nabla q_n, \left(\nabla q_j\right)^2, \nabla q_j \nabla q_i; i, j \in [1, n-1]$ in (2.39). In addition, the tail function V, which we will calculate iteratively, depends nonlinearly on q_j, q_n.

2. In principle, one can avoid using the CWF via decreasing the step size h, which would also result in a small influence of the term $(\nabla q_n)^2$ in (2.39). However, this would lead to an unnecessary increase of the number of equations N in (2.39). Hence, one would need to solve too many Dirichlet boundary problems (2.39), which is time-consuming. Thus, the introduction of the s dependent CWF (2.38) in the numerical scheme makes this scheme more flexible.

2.6 The Approximately Globally Convergent Algorithm

The above considerations lead to the algorithm described in this section. This is an algorithm with the approximate global convergence property for coefficient inverse problem 2.1. This property is established in Theorems 2.8.2 and 2.9.4 for two different approximate mathematical models (Definition 1.1.2.1). We present in this section two versions of the algorithm. The first version, which is described in Sect. 2.6.1, is verified computationally in our above cited works. However, a simplified version of the algorithm of Sect. 2.6.2 is not yet verified computationally.

Everywhere below,

$$|f|_{k+\alpha} = \|f\|_{C^{k+\alpha}\left(\overline{\Omega}\right)}, \ \forall f \in C^{k+\alpha}\left(\overline{\Omega}\right).$$

Our algorithm reconstructs iterative approximations $c_{n,i}(x) \in C^{\alpha}\left(\overline{\Omega}\right)$ of the function $c(x)$ only inside the domain Ω. To update tails, we should solve the forward problem (2.1) and (2.2). Hence, we should extend each function $c_{n,i}(x)$ outside of the domain Ω in such a way that the resulting function $\widehat{c}_{n,i} \in C^{\alpha}\left(\mathbb{R}^3\right)$, $\widehat{c}_{n,i} \geq 1$ in Ω and $\widehat{c}_{n,i} = 1$ outside of Ω. So, we first describe a rather standard procedure of such an extension. Choose a smaller subdomain $\Omega' \subset \Omega$. Choose a function $\chi(x)$ such that

$$\chi \in C^1\left(\mathbb{R}^3\right), \ \chi(x) = \begin{cases} 1 \text{ in } \Omega', \\ \in [0,1] \text{ in } \Omega \setminus \Omega', \\ 0 \text{ outside of } \Omega. \end{cases}$$

The existence of such functions $\chi(x)$ is well known from the real analysis course. Define the target extension of the function $c_{n,i}$ as

$$\widehat{c}_{n,i}(x) := (1 - \chi(x)) + \chi(x)\, c_{n,i}(x), \forall x \in \mathbb{R}^3.$$

Hence, $\widehat{c}_{n,i}(x) = 1$ outside of the domain Ω and $\widehat{c}_{n,i} \in C^{\alpha}\left(\mathbb{R}^3\right)$. Furthermore, since $c_{n,i}(x) \in [1, d+1]$ in Ω, then $\widehat{c}_{n,i}(x) \in [1, d+1]$ in Ω. Indeed,

$$\widehat{c}_{n,i}(x) - 1 = \chi(x)\left(c_{n,i}(x) - 1\right) \geq 0, \ x \in \Omega,$$

$$\widehat{c}_{n,i}(x) - (d+1) = 1 - \chi(x) + \chi(x)\, c_{n,i}(x) - \chi(x)(d+1) - (1 - \chi(x))$$

$$(d+1) = -(1-\chi(x))\, d + \chi(x)\left(c_{n,i}(x) - d - 1\right) \leq 0, \ x \in \Omega.$$

In accordance with (2.17), (2.22), and (2.33), denote

$$v_{n,i}(x) = -hq_{n,i}(x) - h\sum_{j=0}^{n-1} q_j(x) + V_{n,i}(x), x \in \Omega, \tag{2.41}$$

$$c_{n,i}(x) = \left[\Delta v_{n,i} + s_n^2 (\nabla v_{n,i})^2 \right](x), \ x \in \Omega, \tag{2.42}$$

where functions $q_j, q_{n,i}, V_{n,i}$ are defined in this section below. Here, $V_{n,i}(x)$ is a certain approximation for the tail function and m_n is the number of iterations with respect to tails for a given $n \geq 1$, where $k = 1, ..., m_n$. Recall that by (2.32) $q_0 \equiv 0$. Hence, we set

$$q_{1,1}^0 := 0, V_{1,1}(x) := V_{1,1}^0(x), \tag{2.43}$$

$$q_{n,1}^0 := q_{n-1}, V_{n,1} := V_{n-1,m_{n-1}}, \text{for } n \geq 2, \tag{2.44}$$

where $V_{1,1}^0(x)$ is a certain starting value for the tail function.

In our iterative algorithm below, iterations with respect to k in $q_{n,1}^k$ are conducted in order to take into account the nonlinear term $(\nabla q_n)^2$in (2.39). As a limiting case, we construct the function $q_{n,1}$ for each n. Next, we iterate with respect to the tail and construct functions $q_{n,i}, i = 2, ..., m_n$. However, we do not iterate with respect to the nonlinear term for functions $q_{n,i}$ with $i \geq 2$.

Remarks 2.6. We now need to comment on the choice of the function $V_{1,1}^0(x)$.

1. By (2.14) and (2.18), this function should be small for large $\overline{s}$. In our numerical studies, we work with the incident plane wave rather with the point source in (2.2). The reason is that it is more convenient to computationally implement the case of the plane wave. On the other hand, we have chosen the case of the point source in (2.2) because Lemma 2.3 is actually derived from Theorem 4.1 of [144]. And this theorem was proven for the case of the point source.
2. In the first work [24], we took $V_{1,1}^0(x) \equiv 0$, and this is the case of numerical studies in Chap. 3. In follow-up publications, we have taken $V_{1,1}^0(x) = V_{\text{uniform}}(x)$, where

$$V_{\text{uniform}}(x) = \frac{\ln(w_{\text{uniform}}(x, \overline{s}))}{\overline{s}^2}.$$

   Here, $w_{\text{uniform}}(x, \overline{s})$ is the solution of the problem (2.11) and (2.12) for $c(x) \equiv 1, s := \overline{s}$ in the case of the incident plane wave. The latter is the case of numerical studies in Chaps. 4–6. In other words, $V_{\text{uniform}}(x)$ corresponds to the solution of the problem (2.11) and (2.12) for the case of the uniform medium which surrounds our domain of interest Ω; see (2.3). Recall that we do not assume any knowledge of the function $c(x)$ inside the domain Ω. We have discovered in our computational experiments that both these choices provide about the same solutions. However, the second one leads to a faster numerical convergence.
3. The second approximate mathematical model leads to another choice for the initial tail function $V_{1,1}(x)$; see Sect. 2.9.2. We have tested numerically this choice as well. Our computations have shown that although this choice provides a little bit better accuracy than the above two, the difference is still insignificant.

2.6.1 The First Version of the Algorithm

Step $n_1, n \geq 1$. Suppose that functions

$$q_1, ..., q_{n-1}, q_{n,1}^0 := q_{n-1} \in C^{2+\alpha}(\overline{\Omega}), c_{n-1} \in C^{\alpha}(\overline{\Omega})$$

and the tail function $V_{n,1}(x, \overline{s}) \in C^{2+\alpha}(\overline{\Omega})$ are constructed; see (2.43) and (2.44). We now construct the function $q_{n,1}$. To do this, we solve iteratively the following Dirichlet boundary value problems:

$$\Delta q_{n,1}^k - A_{1,n} \left(h \sum_{j=0}^{n-1} \nabla q_j \right) \cdot \nabla q_{n,1}^k - \varepsilon q_{n,1}^k + A_{1,n} \nabla q_{n,1}^k \cdot \nabla V_{n,1}$$
$$= 2 \frac{I_{1n}}{I_0} \left(\nabla q_{n,1}^{k-1} \right)^2 - A_{2,n} h^2 \left(\sum_{j=0}^{n-1} \nabla q_j \right)^2 + 2 A_{2,n} \nabla V_{n,1} \cdot \left(h \sum_{j=0}^{n-1} \nabla q_j \right)$$
$$-A_{2n} \left(\nabla V_{n,1} \right)^2, \ x \in \Omega, \tag{2.45}$$

$$q_{n,1}^k (x) = \overline{\psi}_n (x), x \in \partial \Omega, \tag{2.46}$$

where $k = 1, 2, ...,$ functions $\overline{\psi}_n (x)$ are defined in (2.34) and (2.35) and functions $q_{n,1}^k \in C^{2+\alpha} \left(\overline{\Omega} \right)$. We call these "iterations with respect to the nonlinear term."It can be proven that this process converges; see Theorem 2.8.2 So, we set

$$q_{n,1} = \lim_{k \to \infty} q_{n,1}^k \text{ in the } C^{2+\alpha} \left(\overline{\Omega} \right) \text{ norm.} \tag{2.47}$$

Our numerical convergence criterion for the sequence $\{q_{n,1}^k\}_{k=1}^{\infty}$ is described in Chap. 3. Next, we reconstruct an approximation $c_{n,1}(x), x \in \Omega$ for the unknown function $c(x)$ using the resulting function $q_{n,1}(x)$ and formulas (2.41) and (2.42) at $i = 1$. Hence, $c_{n,1} \in C^{\alpha} \left(\overline{\Omega} \right)$. Construct the function $\widehat{c}_{n,1}(x) \in C^{\alpha} \left(\mathbb{R}^3 \right)$. Next, solve the forward problem (2.1) and (2.2) with $c(x) := \widehat{c}_{n,1}(x)$. We obtain the function $u_{n,1}(x,t)$. Calculate the Laplace transform (2.10) of this function and obtain the function $w_{n,1}(x, \overline{s})$ this way. Next, following (2.18), we set for $x \in \Omega$

$$V_{n,2}(x, \overline{s}) = \frac{\ln w_{n,1}(x, \overline{s})}{\overline{s}^2} \in C^{2+\alpha} \left(\overline{\Omega} \right). \tag{2.48}$$

Step n_i, $i \geq 2, n \geq 1$. We now iterate with respect to the tails. Suppose that functions $q_{n,i-1}, V_{n,i}(x, \overline{s}) \in C^{2+\alpha} \left(\overline{\Omega} \right)$ are constructed. Then we solve the following Dirichlet boundary value problem:

$$\Delta q_{n,i} - A_{1n} \left(h \sum_{j=0}^{n-1} \nabla q_j \right) \cdot \nabla q_{n,i} - \varepsilon q_{n,i} + A_{1n} \nabla q_{n,i} \cdot \nabla V_{n,i}$$

$$= 2\frac{I_{1n}}{I_0}\left(\nabla q_{n,i-1}\right)^2 - A_{2n}h^2\left(\sum_{j=0}^{n-1}\nabla q_j\right)^2 + 2A_{2n}\nabla V_{n,i}\cdot\left(h\sum_{j=0}^{n-1}\nabla q_j\right)$$

$$-A_{2n}\left(\nabla V_{n,i}\right)^2,\ x\in\Omega, \tag{2.49}$$

$$q_{n,i}\left(x\right)=\overline{\psi}_n\left(x\right), x\in\partial\Omega. \tag{2.50}$$

Having the function $q_{n,i}$, we reconstruct the next approximation $c_{n,i} \in C^{\alpha}(\overline{\Omega})$ for the target coefficient using (2.41) and (2.42). Next, we construct the function $\widehat{c}_{n,i} \in C^{\alpha}(\mathbb{R}^3)$. Next, we solve the forward problem (2.1) and (2.2) with $c(x) := \widehat{c}_{n,i}(x)$, calculate the Laplace transform (2.10), and update the tail as in (2.48), where $(w_{n,1}, V_{n,2})$ is replaced with $(w_{n,i}, V_{n,i+1})$. Alternatively to the solution of the problem (2.1) and (2.2), one can also solve the problem (2.11) and (2.12) at $s := \overline{s}$; see Theorem 2.7.2 for the justification. We iterate with respect to i until convergence occurs at the step $i := m_n$. Then we set

$$q_n := q_{n,m_n} \in C^{2+\alpha}\left(\overline{\Omega}\right), c_n := c_{n,m_n} \in C^{\alpha}\left(\overline{\Omega}\right), \tag{2.51}$$

$$V_{n+1,1}\left(x,\overline{s}\right) = \frac{1}{\overline{s}^2}\ln w_{n,m_n}(x,\overline{s}) \in C^{2+\alpha}\left(\overline{\Omega}\right). \tag{2.52}$$

While convergence of the sequence $\left\{q_{n,1}^k\right\}_{k=1}^{\infty}$, which is generated by iterations with respect to the nonlinear term (see Step n_1) can be proven (Theorem 2.8.2), convergence of the sequence $\{q_{n,i}\}$ (with respect to i) cannot be proven. Hence, we have established a stopping rule for the latter sequence numerically; see details in Chap. 3. So, if the stopping rule is not yet reached, then we proceed with Step $(n+1)$. Alternatively we stop.

The stopping rule is chosen in numerical experiments; see Chap. 3. In addition, Theorem 2.8.2 claims that a subsequence of the sequence $\{c_{n,i}\}_{i=1}^{\infty}$ converges in the $L_2(\Omega)$-norm. We use the discrete $L_2(\Omega)$ norm in our computations for the stopping rule. This norm is the most convenient one for the computational analysis. Therefore, the stopping rule of Chap. 3 is indeed a reasonable one.

2.6.2 A Simplified Version of the Algorithm

We now briefly present a simplified version of the algorithm, which is not yet computationally verified. The idea is generated by the standard way of solving Volterra integral equations. First, we present the latter idea in brief. Consider a Volterra-like integral equation

$$y\left(t\right) = \int_0^t f\left(t,\tau,y\left(\tau\right)\right)\mathrm{d}\tau + g\left(t\right), t>0, \tag{2.53}$$

where $f, \partial_y f, g$ are continuous functions of their variables. Equation (2.53) can be solved iteratively as

$$y_n(t) = \int_0^t f(t, \tau, y_{n-1}(\tau))\,d\tau + g(t), \; n \geq 1, \; y_0(t) = g(t). \tag{2.54}$$

It is proved in the standard ordinary differential equations course that this process converges as long as $t \in (0, \varepsilon)$, where $\varepsilon > 0$ is a sufficiently small number. Furthermore, solution of (2.53) is unique for all $t > 0$, as long as $f, \partial_y f$ are continuous functions of their variables for appropriate values of τ, t, y. At the same time, existence of the solution of (2.53) as well as convergence of the iterative process (2.54) can be proved only for small values $t \in (0, \varepsilon)$.

Equation (2.20) can be written in the form, which is similar with (2.53):

$$\begin{aligned}\Delta q &= 2s^2 \nabla q \int_s^{\bar{s}} \nabla q(x,\tau)\,d\tau - 2s \left[\int_s^{\bar{s}} \nabla q(x,\tau)\,d\tau \right]^2 \\ &\quad - 2s^2 \nabla q \nabla V + 4s \nabla V \int_s^{\bar{s}} \nabla q(x,\tau)\,d\tau - 2s(\nabla V)^2 = 0, \;\; x \in \Omega, s \in [\underline{s}, \bar{s}].\end{aligned} \tag{2.55}$$

The boundary condition (2.21) is

$$q(x,s) = \psi(x,s), \; \forall (x,s) \in \partial\Omega \times [\underline{s}, \bar{s}]. \tag{2.56}$$

Hence, the idea is to solve the problem (2.55) and (2.56) for each appropriate tail function V iteratively via the process, which is similar with (2.54). Next, the tail should be updated, and the process should be repeated.

We use (2.32) and (2.33). Similarly with (2.43), we set

$$q_0 := 0, V_0(x) := V_0^0(x) \in C^{2+\alpha}(\overline{\Omega}),$$

where $V_0^0(x)$ is the first guess for the tail; see Remarks 2.6.

Step n_0, $n \geq 1$. Assume that functions $q_j^0 \in C^{2+\alpha}(\overline{\Omega}), j \in [0, n-1]$ are constructed. To find the function q_n^0, solve the following Dirichlet boundary value problem:

$$\begin{aligned}&\Delta q_n^0 - A_{1n} \left(h \sum_{j=0}^{n-1} \nabla q_j^0 \right) \nabla q_n^0 - \varepsilon q_n^0 + A_{1n} \nabla q_n^0 \cdot \nabla V_0 \\ &\quad = -A_{2n} h^2 \left(\sum_{j=0}^{n-1} \nabla q_j \right)^2 + 2A_{2n} \nabla V_0 \left(h \sum_{j=0}^{n-1} \nabla q_j \right) - A_{2n} (\nabla V_0)^2, \; x \in \Omega,\end{aligned} \tag{2.57}$$

$$q_n^0(x) = \overline{\psi}_n(x), x \in \partial\Omega. \tag{2.58}$$

Continue until $n = N$. Next, reconstruct an approximation $c_1(x), x \in \Omega$ for the unknown function $c(x)$ using the resulting vector function $q^0(x) = \left(q_0^0, q_1^0, ..., q_N^0\right)$ and obvious analogs of formulas (2.41) and (2.42). Next, construct the function $\widehat{c}_1(x) \in C^\alpha\left(\mathbb{R}^3\right)$ and solve the problem (2.11) and (2.12) with the coefficient $\widehat{c}_1(x)$ at $s = \overline{s}$. We obtain the function $w(x, \overline{s}; \widehat{c}_1)$. Next, construct the function $V_1(x)$ as

$$V_1(x) = \frac{1}{\overline{s}^2} \ln w(x, \overline{s}; \widehat{c}_1) \in C^{2+\alpha}\left(\overline{\Omega}\right). \tag{2.59}$$

Set $q_0^1 := 0$.

Step $n_k, n \geq 1, k \geq 1$. Assume that functions $V_k, q_j^k \in C^{2+\alpha}\left(\overline{\Omega}\right), j \in [0, n-1]$ are constructed and $q_0^k = 0$. To construct the function q_n^k, solve the following analog of the Dirichlet boundary value problem (2.57) and (2.58):

$$\Delta q_n^k - A_{1n}\left(h\sum_{j=0}^{n-1}\nabla q_j^k\right)\nabla q_n^k - \varepsilon q_n^0 + A_{1n}\nabla q_n^k \nabla V_k$$

$$= -A_{2n}h^2\left(\sum_{j=0}^{n-1}\nabla q_j^k\right)^2 + 2A_{2n}\nabla V_k\left(h\sum_{j=0}^{n-1}\nabla q_j^k\right) - A_{2n}\left(\nabla V_k\right)^2,\ x \in \Omega,$$

$$q_n^k(x) = \overline{\psi}_n(x), x \in \partial\Omega.$$

Continue until $n = N$. Next, reconstruct an approximation $c_k(x), x \in \Omega$ for the unknown function $c(x)$ using the resulting vector function $q^k(x) = \left(q_0^k, q_1^k, ..., q_N^k\right)$ and obvious analogs of formulas (2.41), (2.42). Next, construct the function $\widehat{c}_k(x) \in C^\alpha\left(\mathbb{R}^3\right)$ and solve the problem (2.11) and (2.12) with the coefficient $\widehat{c}_k(x)$ at $s = \overline{s}$. A "good" solution of this problem exists and is unique; see Theorem 2.7.2 We obtain the function $w(x, \overline{s}; \widehat{c}_k)$. Next, construct the function $V_{k+1}(x)$ similarly with (2.59):

$$V_{k+1}(x) = \frac{1}{\overline{s}^2} \ln w(x, \overline{s}; \widehat{c}_k) \in C^{2+\alpha}\left(\overline{\Omega}\right).$$

Continue above iterations with respect to k until a convergence criterion is met. That convergence criterion should be established computationally.

2.7 Some Properties of the Laplace Transform of the Solution of the Cauchy Problem (2.1) and (2.2)

We need the material of this section for our analysis of the approximate global convergence property of the algorithm of Sect. 2.6.1. Indeed, we have not proven the limit (2.12) in Sect. 2.3. This is done in Sect. 2.7.1. In Sect. 2.7.2, we establish some additional properties of the solution of the problem (2.11) and (2.12).

2.7.1 The Study of the Limit (2.12)

Theorem 2.7.1. *Let $x_0 \notin \overline{\Omega}$, the function $c(x)$ satisfies conditions (2.3) and also $c \in C^{k+\alpha}(\mathbb{R}^3)$, where $k \geq 0$ is an integer and the number $\alpha \in (0,1)$. Assume that there exist constants*

$$M_1 = M_1(c) > 0, M_2 = M_2(x,c) > 0, \underline{s}_1 = \underline{s}_1(c) > 1,$$

such that for $k = 0, 1, 2$ and $|\gamma| \leq 2$,

$$\left|D_t^k u(x,t)\right|, \left|D_x^\gamma u(x,t)\right| \leq M_1(c)\, e^{\underline{s}_1 t},\ t > M_2(x,c), \forall x \in \mathbb{R}^3, \tag{2.60}$$

where $u(x,t)$ is the solution of the problem (2.1) and (2.2). Then there exists a constant $\underline{s}_2 = \underline{s}_2(c) \geq \underline{s}_1(c) > 1$ such that for all $s > \underline{s}_2$, the function $w(x,s)$, which is the Laplace transform (2.10) of the function $u(x,t)$, satisfies the following conditions:

$$\Delta w - s^2 c(x)\, w = -\delta(x - x_0),\ \forall s > \underline{s}_2, \tag{2.61}$$

$$\lim_{|x| \to \infty} w(x,s) = 0,\ \forall s > \underline{s}_2, \tag{2.62}$$

$$w(x,s) > 0 \text{ for } x \neq x_0, \tag{2.63}$$

$$w(x,s) = \frac{\exp(-s|x - x_0|)}{4\pi |x - x_0|} + \overline{w}(x,s) := w_1(x,s) + \overline{w}(x,s),\ \forall s > \underline{s}_2(c), \tag{2.64}$$

$$\overline{w}(x,s) \in C^{k+2+\alpha}(\mathbb{R}^3),\ \forall s > \underline{s}_2. \tag{2.65}$$

Proof. The limit (2.12) can be proven as follows. First, apply to (2.1) and (2.2) the integral transformation (1.162), which is an analog of the Laplace transform:

$$v(x,t) = \frac{1}{2\sqrt{\pi}t^{3/2}} \int_0^\infty u(x,\tau)\,\tau \exp\left(-\frac{\tau^2}{4t}\right) d\tau := \mathcal{L}_1 u. \tag{2.66}$$

Let $f(t), t \in [0,\infty)$ be a piecewise continuous function such that the function $|f(t)|\, e^{-\underline{s}_1 t}$ is bounded for $t \in [0,\infty)$. Two other types of the Laplace transform which we use are

$$\mathcal{L}_2 f = \int_0^\infty f(t)\, e^{-s^2 t} dt, s > \underline{s}_1 > 1, \tag{2.67}$$

$$\mathcal{L} f = \int_0^\infty f(t)\, e^{-st} dt, s > \underline{s}_1 > 1. \tag{2.68}$$

One can easily verify that

$$(\mathcal{L} f)(s) = (\mathcal{L}_2 (\mathcal{L}_1 f))(s), \forall s > \underline{s}_1 > 1. \tag{2.69}$$

It follows from (2.66) that [102, 123, 124]

$$c(x)\, v_t = \Delta v, \tag{2.70}$$

$$v(x, 0) = \delta(x - x_0). \tag{2.71}$$

Hence, it follows from (2.1), (2.2), (2.60), and (2.67)–(2.71) that the function $w = \mathcal{L}u$ satisfies (2.61). We now need to establish (2.62)–(2.65). First, (2.63) follows from $w = \mathcal{L}_2 v$. Indeed, Theorem 11 of Chap. 2 of the book [69] claims that the fundamental solution of a general parabolic equation is positive for $t > 0$.

Detailed estimates of the fundamental solution of a general parabolic equation with variable coefficients can be found in Sects. 11–13 of Chap. 4 of the book [120]. In particular, it follows from the formula (13.1) of that chapter of [120] that the following estimate is valid for (x, t)-derivatives $D_t^r D_x^n v$ of the function v:

$$|D_t^r D_x^n v| \leq v_{r,k} := C_1 \frac{\exp\left(-C_2 \frac{|x - x_0|^2}{t}\right)}{t^p} \exp(C_3 t); \; p = \frac{3 + 2r + n}{2}, \tag{2.72}$$

where $2r + n \leq 2$ and C_1, C_2, C_3 are certain positive constants depending only on the upper estimate of the norm $\|c\|_{C^\alpha(\mathbb{R}^3)}$. Let $w_{r,n}(x, s) = \mathcal{L}_2 \left(D_t^r D_x^n v\right)$. Using estimate (2.72) as well as formula (29) in Sect. 4.5 in the table of the Laplace transform of the book [13], we obtain

$$|w_{r,n}(x, s)| \leq 2C_1 \left(\frac{C_2 |x - x_0|}{\sqrt{s^2 - C_3}}\right)^{(1-p)/2} K_{p-1}\left(2\sqrt{C_2 (s^2 - C_3)} \cdot |x - x_0|\right), \tag{2.73}$$

where $s \geq \underline{s} > \sqrt{C_3}$ and K_{p-1} is the McDonald function. Note that $K_{p-1} = K_{1-p}$ [1]. Since for $y \in \mathbb{R}$, the function $K_{p-1}(y) \in C^\infty(y \geq \varrho), \forall \varrho > 0$, then it follows from (2.73) that the function $w \in C^2(\{|x - x_0| \geq \vartheta\}), \forall \vartheta > 0$, for $s > \sqrt{C_3}$. Furthermore, since the function $K_{p-1}(y)$ decays exponentially when $y \to \infty$, $y \in$

$\mathbb{R}$, then we obtain from (2.73) that

$$\lim_{|x|\to\infty} D_x^n w(x,s) = 0 \text{ for } n = 0,1,2 \text{ and } s > \sqrt{C_3}, \tag{2.74}$$

from which (2.62) follows.

Consider now the fundamental solution v_0 of the heat equation:

$$v_{0t} = \Delta v_0, \text{ in } \mathbb{R}^3,$$
$$v_0(x,0) = \delta(x - x_0).$$

Hence,

$$v_0(x,t) = \frac{1}{\left(2\sqrt{\pi t}\right)^3} \exp\left(-\frac{|x-x_0|^2}{4t}\right). \tag{2.75}$$

Let $v_1 = v - v_0$. Then (2.70) and (2.71) imply that

$$c(x)\, v_{1t} = \Delta v_1 - (c(x) - 1)\, v_{0t}, \tag{2.76}$$

$$v_1(x,0) = 0. \tag{2.77}$$

Since the source $x_0 \notin \overline{\Omega}$, the function $c \in C^{k+\alpha}\left(\mathbb{R}^3\right)$, and by (2.3) $c(x) - 1 = 0$ outside of Ω, then it follows from (2.75) that

$$(c(x) - 1)\, v_{0t} \in C^{k+\alpha}\left(\mathbb{R}^3 \times [0,T]\right), \forall T > 0.$$

Consider the function $w^{(1)} := \mathcal{L}_2 v_1$ for $s \geq \underline{s} > \sqrt{C_3}$. Estimates for the solution of the Cauchy problem for a general parabolic equation with variable coefficients are obtained in Sect. 14 of Chap. 4 of [120] for the case when the right-hand side of this equation belongs to $C^{\alpha}\left(\mathbb{R}^3 \times [0,T]\right), \forall T > 0$. So, these estimates as well as (2.73), (2.75), (2.76), and (2.77) imply that $w^{(1)} \in C^2\left(\mathbb{R}^3\right)$.

Consider the function $\mathcal{L}_2 v_0$. Formula (28) of Sect. 4.5 of [13] implies that

$$\mathcal{L}_2 v_0 = \frac{\exp(-s|x - x_0|)}{4\pi |x - x_0|} = w_1(x,s). \tag{2.78}$$

Next, by (2.62) and (2.76)–(2.78) the function $\overline{w} = w - w_0$ satisfies the following conditions:

$$\Delta \overline{w} - s^2 c(x)\, \overline{w} = s^2 (c(x) - 1)\, w_1, \ s \geq \underline{s} > \sqrt{C_3}, \tag{2.79}$$

$$\lim_{|x|\to\infty} \overline{w}(x,s) = 0. \tag{2.80}$$

Now, since the function $(c(x) - 1)\, w_1 \in C^{k+\alpha}\left(\mathbb{R}^3\right)$, then Theorem 6.17 of [72] ensures that $\overline{w} \in C^{k+2+\alpha}\left(\mathbb{R}^3\right)$.

We now show that the solution $\overline{w} \in C^2(\mathbb{R}^3)$ of the problem (2.79) and (2.80) is unique. Suppose that there exists another function $\widetilde{w} \in C^2(\mathbb{R}^3)$ satisfying conditions (2.79) and (2.80). Let $w_2 = \overline{w} - \widetilde{w}$. Then

$$\Delta w_2 - s^2 c(x) w_2 = 0, \; s > \sqrt{C_3},$$

$$\lim_{|x|\to\infty} w_2(x,s) = 0.$$

Fix a pseudo frequency $s, s > \sqrt{C_3}$. Let $\varepsilon \in (0,1)$ be an arbitrary number. Choose a sufficiently large number $R(\varepsilon) > 0$ such that $|w_2(x,s)| < \varepsilon$ for $x \in \{|x| = R(\varepsilon)\}$. Then by the maximum principle (see Sect. 1 in Chap. 3 of [118])

$$\max_{|x|\le R(\varepsilon)} |w_2(x,s)| \le \max_{|x|=R(\varepsilon)} |w_2(x,s)| < \varepsilon.$$

Since $\kappa \in (0,1)$ is an arbitrary number, then $w_2(x,s) \equiv 0$. Hence, $\overline{w}(x,s) \equiv \widetilde{w}(x,s)$. Furthermore, since the above function $w^{(1)} \in C^2(\mathbb{R}^3)$ is $w^{(1)} := \mathcal{L}_2 v_1$ for $s \ge \underline{s} > \sqrt{C_3}$, then by (2.76), (2.77), and (2.74) imply that the function $w^{(1)}$ satisfies conditions (2.79) and (2.80). Hence, $w^{(1)} = \overline{w}$. Thus, conditions (2.61)–(2.65) are established. □

2.7.2 *Some Additional Properties of the Solution of the Problem (2.11) and (2.12)*

An inconvenient point of Theorem 2.7.1 is that it works only for $s > \underline{s}_2(c)$. The next natural question is whether its analog would be valid for values of s, which are independent on the function $c(x)$. In addition, the question about lower and upper bounds for the function w is important for our convergence analysis in Sect. 2.9. Thus, we need to prove Theorem 2.7.2 It should be noticed that this theorem does not follow from classical results of the theory of elliptic PDEs, since there results are known only for bounded domains. Unlike this, Theorem 2.7.2 is concerned with the elliptic problem (2.11) and (2.12) in the entire space $\mathbb{R}^3$.

First, we copy condition (2.3) for the convenience of the reader:

$$c(x) \in [1,d], \;\; c(x) = 1 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega. \tag{2.81}$$

Theorem 2.7.2. *Let $x_0 \notin \overline{\Omega}$ and the function $c(x)$ satisfies condition (2.81) as well as the following smoothness condition:*

$$c \in C^{k+\alpha}(\mathbb{R}^3), \; k \ge 0, \; \alpha \in (0,1). \tag{2.82}$$

Denote

$$w_1(x,s) = \frac{\exp(-s|x-x_0|)}{4\pi|x-x_0|} \text{ and } w_d(x,s) = \frac{\exp\left(-s\sqrt{d}\,|x-x_0|\right)}{4\pi|x-x_0|}, \tag{2.83}$$

the solutions of the problem (2.11) and (2.12) for $c(x) \equiv 1$ *and* $c(x) \equiv d$, *respectively. Then for any* $s > 0$, *there exists unique solution of the problem (2.11) and (2.12) of the form*

$$w(x,s) = w_1(x,s) + \overline{w}(x,s), \text{ where } \overline{w} \in C^{k+2+\alpha}\left(\mathbb{R}^3\right). \tag{2.84}$$

Furthermore,

$$w_d(x,s) < w(x,s) \le w_1(x,s), \ \forall x \neq x_0. \tag{2.85}$$

Proof. Consider the following parabolic Cauchy problem for $(x,t) \in \mathbb{R}^3 \times (0,\infty)$:

$$c(x)\,v_t = \Delta v, \ v(x,0) = \delta(x-x_0). \tag{2.86}$$

Let the function $v_0(x,t)$ in (2.75) be the solution of the problem (2.86) with $c \equiv 1$. Also, consider the function $\overline{v}(x,t)$:

$$\overline{v}(x,t) = \int_0^t (v - v_0)(x,\tau)\,d\tau. \tag{2.87}$$

Denote $b(x) = c(x) - 1$. By (2.81) and (2.82),

$$b(x) = 0 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega, \ b \in C^{k+\alpha}\left(\mathbb{R}^3\right). \tag{2.88}$$

We obtain from (2.86) and (2.87)

$$\Delta\overline{v} - c(x)\,\overline{v}_t = b(x)\,v_0, \ \overline{v}(x,0) = 0, \ (x,t) \in \mathbb{R}^3 \times (0,\infty). \tag{2.89}$$

Since $x_0 \notin \overline{\Omega}$, then it follows from (2.75) and (2.88) that the right hand of (2.89) does not have a singularity in $\mathbb{R}^3 \times [0,\infty)$. Let $T, R > 0$ be two arbitrary numbers and $B_R(T) = \{|x| < R\} \times (0,T)$. By (2.81) and (2.22), $b(x)\,v_0(x,t) \ge 0$ for $(x,t) \in \mathbb{R}^3 \times (0,\infty)$. Hence, applying to (2.89), the maximum principle of Theorem 1 of Chap. 2 of [69], we obtain $\max_{\overline{B}_R(T)} \overline{v}(x,t) \le 0$. Since $R, T > 0$ are arbitrary numbers, then

$$\overline{v}(x,t) \le 0 \text{ in } \mathbb{R}^3 \times [0,\infty). \tag{2.90}$$

On the other hand, Theorem 11 of Chap. 2 of [69] ensures that the fundamental solution of the parabolic equation is positive for $t > 0$. Hence, (2.87) and (2.90) imply that

$$0 < \int_0^t v(x,\tau)\,d\tau \le \int_0^t v_0(x,\tau)\,d\tau \text{ and } v(x,t) > 0 \text{ for } t > 0. \tag{2.91}$$

Next, we apply to the function v the operator $\mathcal{L}_2$ of the Laplace transform (2.67):

$$\mathcal{L}_2 v = \int_0^\infty v(x,t)e^{-s^2 t}dt. \tag{2.92}$$

By one of the well-known properties of the Laplace transform,

$$\mathcal{L}_2\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s^2}\mathcal{L}_2 f \tag{2.93}$$

for any appropriate function f. By (2.75), the integral

$$\mathcal{L}_2 v_0 = \int_0^\infty v_0(x,t)e^{-s^2 t}dt$$

converges for all $s > 0$. Formula (28) of Sect. 4.5 of Tables [13] gives $\mathcal{L}_2 v_0 = w_1, \forall s > 0$. Hence, (2.91)–(2.93), and Fubini theorem lead to

$$\mathcal{L}_2\left(\int_0^t v(x,\tau)\,d\tau\right) = \frac{1}{s^2}\mathcal{L}_2 v \le \frac{1}{s^2}\mathcal{L}_2(v_0) = \frac{1}{s^2}w_1(x,s),\ \forall s > 0. \tag{2.94}$$

Hence, the integral (2.92) converges absolutely. Next, by (2.89), for any $A > 0$,

$$\Delta \int_0^A \overline{v}(x,t)\,e^{-s^2 t}dt = \int_0^A \Delta\overline{v}(x,t)\,e^{-s^2 t}dt = \int_0^A [c\overline{v}_t + (c-1)v_0]\,e^{-s^2 t}dt.$$

Setting here $A \to \infty$ and using that by (2.87) $c\overline{v}_t + (c-1)v_0 = cv - v_0$, we obtain

$$\lim_{A\to\infty} \Delta \int_0^A \overline{v}(x,t)\,e^{-s^2 t}dt = \lim_{A\to\infty}\int_0^A \Delta\overline{v}(x,t)\,e^{-s^2 t}dt = c\mathcal{L}_2 v - \mathcal{L}_2 v_0. \tag{2.95}$$

Hence, it follows from (2.95) that $\Delta\mathcal{L}_2(\overline{v})$ and $\mathcal{L}_2(\Delta\overline{v})$ exist and $\Delta\mathcal{L}_2(\overline{v}) = \mathcal{L}_2(\Delta\overline{v})$. Furthermore, by (2.87) and (2.93)–(2.95):

$$\Delta\mathcal{L}_2(\overline{v}) = s^{-2}\Delta(\mathcal{L}_2 v - \mathcal{L}_2 v_0) = c\mathcal{L}_2 v - \mathcal{L}_2 v_0.$$

Hence, denoting $w := \mathcal{L}_2(v)$ and using $\mathcal{L}_2 v_0 = w_1$ as well as $\Delta w_1 - s^2 w_1 = -\delta(x - x_0)$, we obtain that the function w satisfies (2.11):

$$\Delta w - s^2 c(x) w = -\delta(x - x_0), \ x \in \mathbb{R}^3. \tag{2.96}$$

We now prove (2.12). Since $c\overline{v}_t = \overline{v}_t + b\overline{v}_t$, then using (2.87) and (2.89), we obtain

$$\overline{v}_t - \Delta\overline{v} = -b(x) v, \ \overline{v}(x, 0) = 0. \tag{2.97}$$

Since by (2.88) $b(x) = 0$ near x_0 and $b \in C^{k+\alpha}(\mathbb{R}^3)$, then at least

$$bv \in C^{\alpha,\alpha/2}(\mathbb{R}^3 \times [0, T]). \tag{2.98}$$

Hence, it follows from formula (13.2) of Chap. 4 of [120] that

$$\overline{v} \in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^3 \times [0, T]), \forall T > 0. \tag{2.99}$$

Consider (2.97) as the Cauchy problem for the heat equation with the right hand $(-b(x) v)$. It follows from (2.88), (2.98) and (2.99), and results of Sect. 1 of Chap. 4 of [120] that the solution of the problem (2.97) can be written in the following form:

$$\overline{v}(x, t) = -\int_0^t \int_\Omega v_0(x - \xi, t - \tau) b(\xi) v(\xi, \tau) d\xi d\tau. \tag{2.100}$$

By (2.87), (2.92), and (2.94), $\mathcal{L}_2\overline{v} = s^{-2}(\mathcal{L}_2 v - w_1) = s^{-2}(w - w_1)$. Hence, applying the Laplace transform $\mathcal{L}_2$ to both sides of (2.100) and using the convolution theorem, we obtain

$$w(x, s) = w_1(x, s) - s^2 \int_\Omega w_1(x - \xi, s) b(\xi) w(\xi, s) d\xi. \tag{2.101}$$

By (2.83) and (2.101), functions $w(x, s)$, $w_1(x, s)$ and $(w - w_1)(x, s)$ satisfy condition (2.12):

$$\lim_{|x|\to\infty} w(x, s) = \lim_{|x|\to\infty} w_1(x, s) = \lim_{|x|\to\infty} (w - w_1)(x, s) = 0. \tag{2.102}$$

We now establish (2.84). Let $\Omega_1 \subset \mathbb{R}^3$ be a bounded domain such that

$$\Omega \subset \Omega_1,\ \partial\Omega \cap \partial\Omega_1 = \varnothing,\ \partial\Omega_1 \in C^3,\ x_0 \notin \overline{\Omega}_1.$$

It follows from (2.101) that the function $w(x,s) \in C^1\left(\overline{\Omega}_1\right)$. Hence, by (2.88), the function $b(x)\,w(x,s) \in C^{\alpha}\left(\overline{\Omega}_1\right)$. Hence, Lemma 2.9.1.4 (Sect. 2.9.1) and (2.101) imply that the function $(w - w_0)(x,s) \in C^{2+\alpha}\left(\mathbb{R}^3\right)$. Finally, the abovementioned Theorem 6.17 of [72] combined with (2.96) ensures that (2.84) is true for any $k \geq 0$.

Thus, we have proven the existence of the solution of the problem (2.11) and (2.12) in the form (2.84). The proof of the uniqueness is the same as in the last part of the proof of Theorem 2.7.1 (after (2.80)).

Finally, we prove (2.85). We have established above in this proof that $w = \mathcal{L}_2(v)$ and $v \geq 0, b \geq 0$. Hence, the right inequality (2.85) follows from (2.101). Consider the function $\widetilde{w}(x,s) = w(x,s) - w_d(x,s)$. Then (2.96), (2.102), and (2.83) imply that

$$\Delta\widetilde{w} - s^2 c\widetilde{w} = s^2\left(c(x) - d\right)w_d,\quad \lim_{|x|\to\infty}\widetilde{w}(x,s) = 0. \tag{2.103}$$

By (2.83) and (2.84),

$$\frac{\widetilde{w}(x,s)}{w_d(x,s)} = \exp\left[s\left(\sqrt{d} - 1\right)|x - x_0|\right]\left[1 + O\left(|x - x_0|\right)\right] > 0, x \to x_0, x \neq x_0.$$

Hence, there exists a sufficiently small number $\varepsilon > 0$ such that

$$\widetilde{w}(x,s) > 0 \text{ for } x \in \left\{|x - x_0| \leq \varepsilon, x \neq x_0\right\}. \tag{2.104}$$

For $R > 0$, consider the domain $B_{R,\varepsilon} = \left\{|x| < R, |x - x_0| > \varepsilon\right\}$. Assuming that $B_{R,\varepsilon} \neq \varnothing$, which is true for sufficiently large R, we obtain $\widetilde{w} \in C^{2+\alpha}\left(\overline{B}_{R,\varepsilon}\right)$ and $s^2\left(c(x) - d\right)w_d \leq 0$ in $B_{R,\varepsilon}$. Hence, applying the maximum principle to (2.103), we obtain

$$\min_{\overline{B}_{R,\varepsilon}}\widetilde{w} \geq \min_{\partial B_{R,\varepsilon}}\widetilde{w}.$$

Setting $R \to \infty$ and using the second condition (2.103) as well as (2.104), we obtain

$$\min_{|x - x_0| \geq \varepsilon}\widetilde{w} \geq \min_{|x - x_0| = \varepsilon}\widetilde{w} > 0.$$

Thus, $w(x,s) > w_d(x,s)$ for $x \neq x_0$. □

2.8 The First Approximate Global Convergence Theorem

In this section, we present the first version of the proof of the approximate global convergence property of the algorithm of Sect. 2.6.1 for coefficient inverse problem 2.1. In other words, we show that this algorithm addresses the first central question of this book; see Sect. 1.1. Following the fundamental concept of Tikhonov (Sect. 1.4), we should assume first that there exists an "ideal" exact solution of an ill-posed problem with the exact data. Next, one should assume the presence of an error of the level σ in the data and construct the solution for each such σ. So constructed solution is called a "regularized solution", if it tends to the exact solution as $\sigma \to 0$.

2.8.1 Exact Solution

Following the fundamental concept of Tikhonov (Sect. 1.4), we introduce first the definition of the exact solution of coefficient inverse problem 2.1. We assume that there exists a coefficient $c^*(x)$ satisfying conditions (2.3) and (2.4), and this function is the exact solution of this CIP with the "ideal" exact data in $g^*(x,t)$ in (2.5). Recall that by Remark 2.1, we always assume that the uniqueness theorem is in place for each inverse problem considered in Chaps. 2–6. The Laplace transform (2.10) of the function $g^*(x,t)$ leads to the exact function $\varphi^*(x,s) = w^*(x,s), \forall (x,s) \in \partial\Omega \times [\underline{s}, \overline{s}]$.

Denote

$$v^*(x,s) = \frac{\ln[w^*(x,s)]}{s^2},\ q^*(x,s) = \frac{\partial v^*(x,s)}{\partial s},\ V^*(x,\overline{s}) = v^*(x,\overline{s}).$$

Recall that (2.20) for the exact function $q^*(x,s)$ is

$$\begin{gathered}\Delta q^* - 2s^2\nabla q^* \int\limits_s^{\overline{s}} \nabla q^*(x,\tau)\,\mathrm{d}\tau + 2s\left[\int\limits_s^{\overline{s}} \nabla q^*(x,\tau)\,\mathrm{d}\tau\right]^2 \\ + 2s^2\nabla q^*\nabla V^* - 4s\nabla V^* \int\limits_s^{\overline{s}} \nabla q^*(x,\tau)\,\mathrm{d}\tau + 2s\left(\nabla V^*\right)^2 = 0, \\ x \in \Omega, s \in [\underline{s}, \overline{s}]. \end{gathered} \tag{2.105}$$

In addition, by (2.21) and (2.15),

$$q^*(x,s) = \psi^*(x,s),\ \forall (x,s) \in \partial\Omega \times [\underline{s}, \overline{s}]. \tag{2.106}$$

$$c^*(x) = \left[\Delta v^* + s^2 |\nabla v^*|^2\right](x,s),\ (x,s) \in \Omega \times [\underline{s}, \overline{s}]. \tag{2.107}$$

In (2.106),

$$\psi^*(x,s) = \frac{1}{\varphi^* s^2} \cdot \frac{\partial \varphi^*}{\partial s} - \frac{2\ln \varphi^*}{s^3}.$$

The formula (2.107) is used to reconstruct the exact solution c^* from the function v^*.

Definition 2.8.1. We call the function $q^{*}(x,s)$ the *exact solution* of the problem (2.20) and (2.21), or, equivalently, of the problem (2.105) and (2.106), with the *exact boundary* condition $\psi^{*}(x,s)$.

Hence,

$$q^{*}(x,s) \in C^{3+\alpha}\left(\overline{\Omega}\right) \times C^{1}\left[\underline{s},\overline{s}\right]. \tag{2.108}$$

We now follow (2.33)–(2.36), (2.38), and (2.39). First, we approximate functions $q^{*}(x,s)$ and $\psi^{*}(x,s)$ via piecewise constant functions with respect to $s \in [\underline{s},\overline{s}]$. For $n \in [1,N]$, let

$$q_{n}^{*}(x) = \frac{1}{h}\int\limits_{s_n}^{s_{n-1}} q^{*}(x,s)\,\mathrm{d}s,\ \overline{\psi}_{n}^{*}(x) = \frac{1}{h}\int\limits_{s_n}^{s_{n-1}} \psi^{*}(x,s)\,\mathrm{d}s,\ q_{0}^{*}(x) \equiv 0. \tag{2.109}$$

Hence,

$$\begin{aligned} q^{*}(x,s) &= q_{n}^{*}(x) + Q_{n}(x,s),\ \psi^{*}(x,s) \\ &= \overline{\psi}_{n}^{*}(x) + \Psi_{n}(x,s),n \in [1,N], \quad s \in [s_n, s_{n-1}], \end{aligned}$$

where by (2.108), functions Q_n, Ψ_n are such that

$$|Q_n(x,s)|_{2+\alpha} \leq C^{*}h,\ |\Psi_n(x,s)|_{2+\alpha} \leq C^{*}h,\ \text{for } s \in [s_n, s_{n-1}]. \tag{2.110}$$

Here, the constant $C^{*} = C^{*}(\|q^{*}\|_{C^{2+\alpha}(\overline{\Omega})\times C^{1}[\underline{s},\overline{s}]}) > 0$ depends only on the $C^{2+\alpha}\left(\overline{\Omega}\right) \times C^{1}[\underline{s},\overline{s}]$ norm of the function $q^{*}(x,s)$. Hence, we can assume that

$$\max_{1\leq n\leq N} |q_{n}^{*}|_{2+\alpha} \leq C^{*}. \tag{2.111}$$

Without any loss of generality, we assume that

$$C^{*} \geq 1. \tag{2.112}$$

By the fundamental concept of Tikhonov (Sect. 1.4), we assume that the constant C^{*} is known a priori. By (2.14), it is reasonable to assume that C^{*} is independent on $\overline{s}$, although we do not use this assumption. By (2.109),

$$q_{n}^{*}(x) = \overline{\psi}_{n}^{*}(x),\ x \in \partial\Omega. \tag{2.113}$$

Hence, we obtain from (2.105) the following analogue of (2.39):

$$\Delta q_{n}^{*} - A_{1,n}\left(h\sum_{j=0}^{n-1}\nabla q_{j}^{*}\right)\nabla q_{n}^{*} + A_{1,n}\nabla q_{n}^{*}\nabla V^{*}$$

$$= 2\frac{I_{1,n}}{I_0}\left(\nabla q_n^*\right)^2 - A_{2,n}h^2\left(\sum_{i=1}^{n-1}\nabla q_i^*\right)^2$$

$$+2A_{2,n}\nabla V^*\left(h\sum_{j=0}^{n-1}\nabla q_j^*\right) - A_{2,n}\left|\nabla V^*\right|^2 + F_n\left(x,h,\lambda\right), \quad (2.114)$$

where the function $F_n\left(x,h,\lambda\right) \in C^{\alpha}\left(\overline{\Omega}\right)$ and

$$\max_{\lambda h \geq 1}\left|F_n\left(x,h,\lambda\right)\right|_{\alpha} \leq C^* h,\ \lambda h \geq 1. \quad (2.115)$$

Let

$$v_n^*\left(x\right) = -hq_n^*\left(x\right) - h\sum_{j=0}^{n-1}q_j^*\left(x\right) + V^*\left(x\right),\ x \in \Omega,\ n \in [1, N]. \quad (2.116)$$

Then (2.107), (2.108), and (2.115) imply that

$$c^*\left(x\right) = \left[\Delta v_n^* + s_n^2\left|\nabla v_n^*\right|^2\right](x) + \overline{F}_n\left(x\right), \quad (2.117)$$

where $\left|\overline{F}_n\right|_{\alpha} \leq C^* h$. To simplify the presentation, we replace the latter inequality with

$$\left|\overline{F}_n\right|_{\alpha} \leq h. \quad (2.118)$$

This is not a severe restriction since a similar convergence analysis can be conducted for the case $\left|\overline{F}_n\right|_{\alpha} \leq C^* h$, although it would take more space.

We also assume that the function $g(x,t)$ in (2.5) is given with an error. This naturally produces an error in the function $\psi\left(x,s\right)$ in (2.21). An additional error is introduced due to the averaging in (2.35) and (2.109). Hence, we assume that in (2.34) functions $\overline{\psi}_n\left(x\right) \in C^{2+\alpha}\left(\partial\Omega\right)$ and

$$\left\|\overline{\psi}_n^*\left(x\right) - \overline{\psi}_n\left(x\right)\right\|_{C^{2+\alpha}(\partial\Omega)} \leq C^*\left(\sigma + h\right), \quad (2.119)$$

where $\sigma > 0$ is a small parameter characterizing the level of the error in the data $\psi\left(x,s\right)$. The parameter h can also be considered as a part of the error in the data.

2.8.2 The First Approximate Global Convergence Theorem

First, we reformulate the Schauder theorem in a simplified form, which is sufficient for our case; see Chap. 3, Sect. 1 in [118] for this theorem. Assuming that

$$\overline{s} > 1, \ \lambda h \geq 1, \tag{2.120}$$

and using (2.38) as well as formulas for numbers $A_{1,n}, A_{2,n}$, we obtain [24]

$$\max_{1 \leq n \leq N} \{|A_{1,n}| + |A_{2,n}|\} \leq 8\overline{s}^2. \tag{2.121}$$

Introduce the positive constant $M^* = M^* (C^* : \overline{s})$,

$$M^* = 16C^*\overline{s}^2 = 2C^* \max \left(8\overline{s}^2, \max_{1 \leq n \leq N} \{|A_{1,n}| + |A_{2,n}|\} \right) > 16. \tag{2.122}$$

The inequality $M^* > 16$ follows from (2.112) and (2.120). Consider the Dirichlet boundary value problem:

$$\Delta u + \sum_{j=1}^{3} b_j(x) u_{x_j} - b_0(x) u = f(x), x \in \Omega,$$
$$u \mid_{\partial\Omega} = g(x) \in C^{2+\alpha}(\partial\Omega). \tag{2.123}$$

Assume that the following conditions are satisfied:

$$b_j, b_0, f \in C^{\alpha}\left(\overline{\Omega}\right), b_0(x) \geq 0; \ \max_{j \in [0,n]} \left(\left|b_j\right|_{\alpha}\right) \leq 1. \tag{2.124}$$

This upper bound is chosen to simplify the presentation since this is sufficient for our goal. By the Schauder theorem, there exists unique solution $u \in C^{2+\alpha}\left(\overline{\Omega}\right)$ of the boundary value problem (2.123). Furthermore, with a constant $K = K(\Omega) > 1$, depending only on the domain Ω, the following estimate holds:

$$|u|_{2+\alpha} \leq K \left[\|g\|_{C^{2+\alpha}(\partial\Omega)} + |f|_{\alpha} \right]. \tag{2.125}$$

We point out that the constant K depends only on the domain Ω as long as estimate (2.124) for coefficients is in place. In general, however, K depends on both the domain Ω and the upper estimate of the $C^{\alpha}\left(\overline{\Omega}\right)$-norm of coefficients. Note that the definition of the $C^{\alpha}\left(\overline{\Omega}\right)$-norm implies that

$$|f_1 f_2|_{\alpha} \leq |f_1|_{\alpha} |f_2|_{\alpha}, \ \forall f_1, f_2 \in C^{\alpha}\left(\overline{\Omega}\right). \tag{2.126}$$

Theorem 2.8.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with the boundary $\partial\Omega \in C^3$. Consider the algorithm of Sect. 2.6.1, where $\overline{s} =$ const. > 1. Assume that all functions $c_{n,i}$ reconstructed in this algorithm are such that*

$$c_{n,i}(x) \geq 1, \ x \in \Omega. \tag{2.127}$$

Let the exact coefficient $c^*(x)$ *satisfies conditions (2.3) and (2.4), i.e.,*

$$c^*(x) \in [1, d], \;\; c^*(x) = 1 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega,$$

$$c^* \in C^3\left(\mathbb{R}^3\right),$$

where the number $d > 1$ *is given. Let* $C^* \geq 1$ *be the constant defined in (2.111) and (2.112). Let in (2.34) boundary functions* $\overline{\psi}_n \in C^{2+\alpha}(\partial\Omega)$. *Assume that (2.115), (2.119), and (2.120) hold. For any function* $c(x) \in C^\alpha\left(\mathbb{R}^3\right)$ *such that* $c(x) \in [1, d+1]$ *in* Ω *and* $c(x) = 1$ *in* $\mathbb{R}^3 \diagdown \Omega$, *consider the solution* $w_c(x, \overline{s})$ *of the problem (2.11) and (2.12),*

$$\Delta w_c - \overline{s}^2 c(x) w_c = -\delta(x - x_0), \; x \in \mathbb{R}^3, \tag{2.128}$$

$$\lim_{|x| \to \infty} w_c(x, \overline{s}) = 0, \tag{2.129}$$

satisfying condition (2.84) with $k = 0$. *Consider the corresponding tail functions,*

$$V^*(x) = \frac{\ln w^*(x, \overline{s})}{\overline{s}^2} \in C^{2+\alpha}\left(\overline{\Omega}\right), \; V_c(x) = \frac{\ln w_c(x, \overline{s})}{\overline{s}^2} \in C^{2+\alpha}\left(\overline{\Omega}\right), \tag{2.130}$$

where $w^*(x, \overline{s})$ *is the solution of the problem (2.128) and (2.129) of the form (2.84) with* $k = 3$ *for* $c(x) := c^*(x)$. *Suppose that the number* $\overline{s}$ *is so large that the following estimates hold*

$$|V^*|_{2+\alpha} \leq \xi, \; |V_c|_{2+\alpha} \leq \xi, \tag{2.131}$$

for all such functions c, *where* $\xi \in (0, 1)$ *is a sufficiently small number. Let* $V_{1,1}(x, \overline{s}) \in C^{2+\alpha}\left(\overline{\Omega}\right)$ *be the initial tail function and let*

$$|V_{1,1}|_{2+\alpha} \leq \xi. \tag{2.132}$$

Denote

$$\eta := 2(h + \sigma + \xi + \varepsilon). \tag{2.133}$$

Let $K = K(\Omega) > 1$ *be the constant of the Schauder theorem in (2.84) and* $\overline{N} \leq N$ *be the total number of functions* q_n *calculated by the algorithm of Sect. 2.6.1. Suppose that the number* $\overline{N} = \overline{N}(h)$ *is connected with the step size* h *via* $\overline{N}(h)h = \beta$, *where the constant* $\beta > 0$ *is independent on* h. *Let* β *be so small that*

$$\beta \leq \frac{1}{24KM^*} = \frac{1}{384K\overline{s}^2}, \tag{2.134}$$

where the number M^* was defined in (2.122). In addition, let the number η in (2.133) and the parameter λ of the CWF in (2.38) satisfy the following estimates:

$$\eta \leq \eta_0\left(K, C^*, \overline{s}\right) = \frac{1}{16KM^*} = \frac{1}{256KC^*\overline{s}^2}, \tag{2.135}$$

$$\lambda \geq \lambda_0\left(C^*, K, \overline{s}, \eta\right) = \max\left(\left(C^*\right)^2, 96KC^*\overline{s}^2, \frac{1}{\eta^2}\right). \tag{2.136}$$

Then for each $n \in \left[1, \overline{N}\right]$, the sequence $\left\{q_{n,1}^k\right\}_{k=1}^{\infty}$ converges in $C^{2+\alpha}\left(\overline{\Omega}\right)$. Furthermore, functions

$$c_{n,i} \in C^{\alpha}\left(\overline{\Omega}\right), \widehat{c}_{n,k} \in C^{\alpha}\left(\mathbb{R}^3\right),$$

$$c_{n,i}\left(x\right), \widehat{c}_{n,i}\left(x\right) \in \left[1, d+1\right] \ in\ \Omega.$$

In addition, the following estimates hold :

$$\left|q_n - q_n^*\right|_{2+\alpha} \leq 2KM^*\left(\frac{1}{\sqrt{\lambda}} + 3\eta\right),\ n \in \left[1, \overline{N}\right], \tag{2.137}$$

$$\left|q_n\right|_{2+\alpha} \leq 2C^*,\ n \in \left[1, \overline{N}\right], \tag{2.138}$$

$$\left|c_n - c^*\right|_{\alpha} \leq \frac{1}{2\cdot 9^{n-1}}\eta + \frac{23}{8}\eta,\ n \in \left[2, \overline{N}\right]. \tag{2.139}$$

Denote

$$\varepsilon = \left(\frac{1}{18} + \frac{23}{8}\right)\eta = \frac{211}{72}\eta.$$

By (2.135), $\varepsilon \in (0, 0.012)$. Therefore, estimate (2.139) implies the approximate global convergence property of the algorithm of Sect. 2.6.1 of the level ε within the framework of the first approximate mathematical model of Sect. 2.8.4 (below).

It is worthy to make some comments prior to the proof of this theorem. We formulate these comments as the following remarks:

Remarks 2.8.2. 1. The existence and uniqueness of the solution of the problem (2.128) and (2.129) is guaranteed by Theorem 2.7.2 This theorem also guarantees that $w_c\left(x, \overline{s}\right) > 0$ for $x \neq x_0$, which justifies the consideration of $\ln w_c\left(x, \overline{s}\right)$ in (2.130). We impose condition (2.127) because of Theorem 2.7.2

2. We have observed in our computations that the inequality (2.127) holds indeed for computed functions $c_{n,i}\left(x\right)$; see Sect. 3.1.2. In fact, if we would need to estimate norms $\left\|c_n - c^*\right\|_{L_2(\Omega)}$ instead of Hölder norms above, then we would ensure (2.127) via replacing (2.42) with

$$c_{n,i}\left(x\right) = \min\left\{1, \left[\Delta v_{n,i} + s_n^2\left(\nabla v_{n,i}\right)^2\right](x)\right\},\ x \in \Omega. \tag{2.140}$$

Clearly, this function belongs to $C^{\alpha}\left(\overline{\Omega}\right)$ if the function $\left[\Delta v_{n,i} + s_n^2 \left(\nabla v_{n,i}\right)^2\right] \in C^{\alpha}\left(\overline{\Omega}\right)$. An analog of (2.140) is used in Sect. 6.4.3, since the follow up Theorem 6.7 estimates the reconstruction accuracy in the L_2-norm.

3. In fact, it is established in the proof of this theorem that $|c_n - c^*|_{\alpha} \leq 8\eta/3$, which is stronger than estimate (2.139). Nevertheless, estimate (2.139) is interesting in its own right because it shows the dependence from the iteration number n. Indeed, it follows from (2.139) that initially, the reconstruction accuracy improves with iterations. However, for larger values of n, one should expect a stabilization of functions c_n, since $\eta/\left(2 \cdot 9^{n-1}\right) \approx 0$ for large n. This is exactly what we observe in our computations.
4. The number $\beta = \overline{N}(h)\,h$ is the length of the s interval, which is covered by the algorithm of Sect. 2.6.1. The smallness condition (2.134) imposed on β seems to be inevitable since (2.39) are actually generated by (2.20), which contains Volterra integrals in nonlinear terms.

Proof of Theorem 2.8.2. We obtain from (2.112), (2.135), and (2.136) that

$$\frac{C^*}{2\sqrt{\lambda}} \leq 1, \quad \frac{1}{\sqrt{\lambda}} + 3\eta \leq \frac{C^*}{2KM^*}, \quad \frac{KM^*}{\lambda} \leq \frac{1}{3}, \quad \frac{1}{\sqrt{\lambda}} \leq \eta. \tag{2.141}$$

Denote

$$\widetilde{q}_{n,1}^{k} = q_{n,1}^{k} - q_n^*, \ \ \widetilde{q}_{n,i} = q_{n,i} - q_n^*,$$
$$\widetilde{V}_{n,i} = V_{n,i} - V^*, \ \ \widetilde{c}_{n,i} = c_{n,i} - c^*, \widetilde{\psi}_n = \overline{\psi}_n - \overline{\psi}_n^*,$$
$$\widetilde{v}_{n,i}(x) = v_{n,i}(x) - v^*(x, s_n), \ \ \widetilde{v}_n(x) = v_n(x) - v^*(x, s_n),$$

where $H^*(x,s)$ is the function $H_n(x)$ in (2.41) in the case when functions q_j and V_n are replaced with q_j^* and V^*, respectively. Recall that by (2.40),

$$\frac{|I_{1,n}(\lambda,h)|}{I_0(\lambda,h)} \leq \frac{4\overline{s}^2}{\lambda}, \ \text{ for } \lambda h \geq 1. \tag{2.142}$$

The proof of Theorem 2.8.2 basically consists in estimating norms $|\widetilde{q}_{n,1}^{k}|_{2+\alpha}$, $|\widetilde{q}_{n,i}|_{2+\alpha}$ from the above. First, we estimate norms $|\widetilde{q}_{1,k}^{1}|_{2+\alpha}$. By (2.132) and (2.133),

$$\left|\widetilde{V}_{1,1}\right|_{2+\alpha} \leq 2\xi \leq \eta. \tag{2.143}$$

Substituting $n = 1$ in (2.114), subtracting it from (2.45), and subtracting (2.113) from (2.46), we obtain

$$\begin{aligned} \Delta\widetilde{q}_{1,1}^{k} - \varepsilon\widetilde{q}_{1,1}^{k} + A_{1,1}\nabla V_{1,1}\nabla\widetilde{q}_{1,1}^{k} &= 2\frac{I_{1,1}}{I_0}\nabla\widetilde{q}_{1,1}^{k-1}\left(\nabla q_{1,1}^{k-1} + \nabla q_1^*\right) \\ &\quad - A_{1,1}\nabla\widetilde{V}_{1,1}\nabla q_1^* - A_{2,1}\nabla\widetilde{V}_{1,1}\left(\nabla V_{1,1} + \nabla V^*\right) + \varepsilon q_1^* - F_1, \end{aligned} \tag{2.144}$$

$$\widetilde{q}_{1,1}^{1}(x) = \widetilde{\psi}_1(x), x \in \partial\Omega. \tag{2.145}$$

By (2.133) and (2.135), $\varepsilon \leq \eta/2 < 1$. Also, since $K, C^* > 1$, then by (2.122), (2.132), (2.133), and (2.135),

$$|A_{1,1}\nabla V_{1,1}| \leq 4\overline{s}^2\eta \leq \frac{1}{64KC^*} \leq \frac{1}{64K} < 1. \tag{2.146}$$

Hence, combining the Schauder theorem (2.125) with (2.115)–(2.122), (2.131), (2.135), and (2.143)–(2.146), we obtain

$$\left|\widetilde{q}_{1,1}^{k}\right|_{2+\alpha} \leq \frac{KM^*}{2C^*\lambda}\left|\widetilde{q}_{1,1}^{k-1}\right|_{1+\alpha}\left|q_{1,1}^{k-1} + q_1^*\right|_{1+\alpha} + 3KM^*\eta. \tag{2.147}$$

First, let $k = 1$. Since by (2.43) and (2.44), $q_{1,1}^{0} = 0$, then $\widetilde{q}_{1,1}^{0} = -q_1^*$. By (2.111) $\left|\nabla q_1^*\right|_\alpha^2 \leq (C^*)^2$. Hence, (2.147) implies that

$$\left|\widetilde{q}_{1,1}^{1}\right|_{2+\alpha} \leq KM^*\left[\frac{C^*}{2\lambda} + 3\eta\right].$$

Hence, using the first inequality (2.141), we obtain

$$\left|\widetilde{q}_{1,1}^{1}\right|_{2+\alpha} \leq KM^*\left(\frac{1}{\sqrt{\lambda}} + 3\eta\right) \leq 2KM^*\left(\frac{1}{\sqrt{\lambda}} + 3\eta\right).$$

Hence, the second inequality (2.141) and (2.111) imply that

$$\left|q_{1,1}^{1}\right|_{2+\alpha} \leq \left|\widetilde{q}_{1,1}^{1}\right|_{2+\alpha} + |q^*|_{2+\alpha} \leq 2C^*. \tag{2.148}$$

Assume now that

$$\left|\widetilde{q}_{1,1}^{k-1}\right|_{2+\alpha} \leq 2KM^*\left(\frac{1}{\sqrt{\lambda}} + 3\eta\right), k \geq 2. \tag{2.149}$$

Then similarly with (2.148), we obtain

$$\left|q_{1,1}^{k-1}\right|_{2+\alpha} \leq 2C^*. \tag{2.150}$$

We obtain from (2.147), (2.149), and (2.150)

$$\left|\widetilde{q}_{1,1}^{k}\right|_{2+\alpha} \leq \frac{3(KM^*)^2}{\lambda}\left(\frac{1}{\sqrt{\lambda}} + 3\eta\right) + 3KM^*\eta.$$

Hence, the third inequality (2.141) leads to

$$\left|\widetilde{q}_{1,1}^{k}\right|_{2+\alpha} \leq 2KM^{*}\left(\frac{1}{\sqrt{\lambda}}+3\eta\right), k \geq 1. \tag{2.151}$$

Hence, we obtain similarly with (2.148) that

$$\left|q_{1,1}^{k}\right|_{2+\alpha} \leq 2C^{*}, k \geq 1. \tag{2.152}$$

Estimates (2.151) and (2.152) enable us to prove convergence of functions $q_{1,1}^{k}$ for $k \to \infty$. Let $m, r > 2$ be two positive integers. Denote $a_{m,r} = q_{1,1}^{m} - q_{1,1}^{r}$. Then, $a_{m,r} = \widetilde{q}_{1,1}^{m} - \widetilde{q}_{1,1}^{r}$. First, set in (2.144) $k := m$ and then set $k := r$. Next, subtract two resulting equations and use the following:

$$\begin{aligned}
&\nabla\widetilde{q}_{1,1}^{m-1}\left(\nabla q_{1,1}^{m-1}+\nabla q_{1}^{*}\right)-\nabla\widetilde{q}_{1,1}^{r-1}\left(\nabla q_{1,1}^{r-1}+\nabla q_{1}^{*}\right) \\
&\quad = \nabla\widetilde{q}_{1,1}^{m-1}\left(\nabla q_{1,1}^{m-1}+\nabla q_{1}^{*}\right)-\nabla\widetilde{q}_{1,1}^{r-1}\left(\nabla q_{1,1}^{m-1}+\nabla q_{1}^{*}\right) \\
&\quad\quad + \nabla\widetilde{q}_{1,1}^{r-1}\left(\nabla q_{1,1}^{m-1}+\nabla q_{1}^{*}\right)-\nabla\widetilde{q}_{1,1}^{r-1}\left(\nabla q_{1,1}^{r-1}+\nabla q_{1}^{*}\right) \\
&\quad = \nabla a_{m-1,r-1}\left(\nabla q_{1,1}^{m-1}+\nabla q_{1}^{*}\right)+\nabla\widetilde{q}_{1,1}^{r-1}\cdot\nabla a_{m-1,r-1} \\
&\quad = \nabla a_{m-1,r-1}\left(\nabla q_{1,1}^{m-1}+\nabla\widetilde{q}_{1,1}^{r-1}+\nabla q_{1}^{*}\right).
\end{aligned}$$

We obtain

$$\Delta a_{m,r} - \varepsilon a_{m,r} + A_{1,1}\nabla V_{1,1}\nabla a_{mr} = 2\frac{I_{1,1}}{I_0}\nabla a_{m-1,r-1}\cdot\left(\nabla q_{1,1}^{m-1}+\nabla\widetilde{q}_{1,1}^{r-1}+\nabla q_{1}^{*}\right),$$

$$a_{m,r}\mid_{\partial\Omega} = 0.$$

Hence, by the Schauder theorem (2.125), second and third inequalities (2.141), (2.151), (2.152), and (2.142),

$$\left|a_{m,r}\right|_{2+\alpha} \leq \frac{2KM^{*}}{\lambda}\left|a_{m-1,r-1}\right|_{2+\alpha} \leq \frac{2}{3}\left|a_{m-1,r-1}\right|_{2+\alpha}. \tag{2.153}$$

It follows from (2.153) that the sequence $\{q_{1,1}^{k}\}_{k=1}^{\infty}$ satisfies the Cauchy convergence criterion. Convergence of other sequences $\{q_{n,1}^{k}\}_{k=1}^{\infty}$ can be proven similarly. Thus, these proofs are omitted below.

Since functions $\widetilde{q}_{1,1}$ and $q_{1,1}$ are estimated via (2.151) and (2.152), we now can estimate the norm $\left|\widetilde{c}_{1,1}\right|_{\alpha}$. First, we note that by (2.41), (2.42), and (2.116)–(2.118),

$$\left|\widetilde{c}_{1,1}\right|_{\alpha} \leq \left(\left|\widetilde{v}_{1,1}\right|_{2+\alpha}+\frac{\eta}{2}\right)\left[1+\overline{s}^{2}\left(\left|v_{1,1}\right|_{2+\alpha}+\left|v_{1}^{*}\right|_{2+\alpha}\right)\right].$$

As to the truncation in (2.42), it does not affect this estimate because $c^* \geq 1$. By (2.41), (2.116), (2.131)–(2.134) and the fourth inequality (2.141),

$$|\widetilde{c}_{1,1}|_{2+\alpha} + \frac{\eta}{2} \leq 8KM^*\beta\eta + \frac{3}{2}\eta \leq \frac{1}{3}\eta + \frac{3}{2}\eta = \frac{11}{6}\eta.$$

Next, (2.41), (2.116), (2.132), (2.134), (2.135), and (2.152) lead to

$$1 + \overline{s}^2\left(|v_{1,1}|_{2+\alpha} + |v_1^*|_{2+\alpha}\right) \leq 1 + 3\overline{s}^2 C^*\beta + \overline{s}^2\eta \leq 1 + \frac{M^*}{5}\beta + \frac{1}{256}$$
$$\leq 1 + \frac{1}{120} + \frac{1}{256} < \frac{16}{11}.$$

Thus, the last three inequalities combined with (2.135) imply that

$$|\widetilde{c}_{1,1}|_{\alpha} \leq \frac{8}{3}\eta < \frac{1}{2}. \tag{2.154}$$

Since the function c^* satisfies conditions (2.42), then it follows from (2.154) and (2.127) that functions $c_{1,1}, \widehat{c}_{1,1} \in [1, d + 1/2]$. This, along with one of conditions of Theorem 2.8.2, ensures that $|V_{1,2}|_{2+\alpha} \leq \xi$. Hence, similarly with the above, one can prove that estimates (2.151), (2.152), and (2.154) are valid for functions $\widetilde{q}_{1,2}, q_{1,2}$ and $\widetilde{c}_{1,2}$, respectively. To do this, one should use (2.45) and (2.46) at $n = 1$, $i = 2$. Repeating this process m_1 times, we obtain the same estimates for functions $\widetilde{q}_1, q_1, \widetilde{c}_1$. In addition, we also obtain that functions $c_1, \widehat{c}_1 \in [1, d + 1/2]$. Hence, one of conditions of this theorem implies that $|V_{2,1}|_{2+\alpha} \leq \xi$.

Let now $n \geq 2$. Assume that

$$\left|\widetilde{q}_j\right|_{2+\alpha} \leq 2KM^*\left(\frac{1}{\sqrt{\lambda}} + 3\eta\right),\ j \in [1, n-1], \tag{2.155}$$

$$\left|q_j\right|_{2+\alpha} \leq 2C^*,\ j \in [1, n-1], \tag{2.156}$$

$$\left|\widetilde{c}_j\right|_{\alpha} \leq \frac{8}{3}\eta < \frac{1}{2},\ j \in [1, n-1], \tag{2.157}$$

$$c_j, \widehat{c}_j \in [1, d+1],\ \widehat{c}_j(x) = 1 \text{ in } \mathbb{R}^3 \diagdown \Omega,\ c_j, \widehat{c}_j \in C^\alpha\left(\mathbb{R}^3\right),\ j \in [1, n-1]. \tag{2.158}$$

We now obtain these estimates at $j = n$. It follows from (2.131), (2.133), and (2.158) that

$$|V_{n,1}|_{2+\alpha} \leq \xi \leq \frac{\eta}{2},\ \left|\widetilde{V}_{n,1}\right| \leq 2\xi \leq \eta. \tag{2.159}$$

For brevity, consider only functions $q_{n,i}$ with $i \geq 1$, since convergence of the sequence $\left\{q_{n,1}^k\right\}_{k=1}^\infty$ can be proved very similarly with the above case of $\left\{q_{1,1}^k\right\}_{k=1}^\infty$. Also, for brevity set,

$$q_{n,0} := q_{n-1}. \tag{2.160}$$

Recall that by (2.47), $\lim_{k\to\infty} q_{n,1}^k = q_{n,1}$ in the norm of the space $C^{2+\alpha}\left(\overline{\Omega}\right)$.

Subtracting (2.114) from (2.45) and (2.113) from (2.46), we obtain for $i \geq 1$

$$\Delta\widetilde{q}_{n,i} - A_{1,n}\left(h\sum_{j=0}^{n-1}\nabla q_j\left(x\right)\right)\nabla\widetilde{q}_{n,i} + A_{1,n}\nabla V_{n,i}\cdot\nabla\widetilde{q}_{n,i} - \varepsilon\widetilde{q}_{n,i}$$

$$= 2\frac{I_{1,n}}{I_0}\left[\nabla\widetilde{q}_{n,i-1}\left(\nabla q_{n,i-1} + \nabla q_n^*\right)\right]$$

$$+\left(A_{1,n}\nabla q_n^* - A_{2,n}h\sum_{j=0}^{n-1}\left(\nabla q_j + \nabla q_j^*\right) + 2A_{2,n}\nabla V_{n,i}\right)\left(h\sum_{j=0}^{n-1}\nabla\widetilde{q}_j\right)$$

$$+\left[2A_{2,n}h\sum_{j=0}^{n-1}\nabla q_j^* - A_{1,n}\nabla q_n^* - A_{2,n}\left(\nabla V_{n,i} + \nabla V^*\right)\right]\nabla\widetilde{V}_{n,i} + \varepsilon q_n^* - F_n, \tag{2.161}$$

$$\widetilde{q}_{n,i}\mid_{\partial\Omega} = \widetilde{\psi}_n(x). \tag{2.162}$$

We estimate the sum of 2nd, 3rd, 4th, and 5th terms in the right-hand side of (2.161). As to the second term, using (2.111), (2.122), (2.135), (2.156), and (2.159), we obtain

$$\left|A_{1,n}\nabla q_n^* - A_{2,n}h\sum_{j=1}^{n-1}\left(\nabla q_j + \nabla q_j^*\right) + 2A_{2,n}\nabla V_{n,1}\right|_\alpha$$

$$\leq \frac{M^*}{2} + \frac{3M^*\beta}{2} + \frac{M^*}{2} = M^*\left(1 + \frac{3}{2}\beta\right).$$

On the other hand, by (2.155),

$$h\sum_{j=0}^{n-1}\left|\nabla\widetilde{q}_j\right|_\alpha \leq 2KM^*\beta\left(\frac{1}{\sqrt{\lambda}} + 3\eta\right). \tag{2.163}$$

Hence, (2.131), (2.133), and (2.163) imply that

$$\left|A_{1,n}\nabla q_n^* - A_{2,n}h\sum_{j=0}^{n-1}\left(\nabla q_j + \nabla q_j^*\right) + 2A_{2,n}\nabla V_{n,1}\right|_\alpha \cdot \left|h\sum_{j=1}^{n-1}\nabla\widetilde{q}_j\left(x\right)\right|_\alpha$$

$$\leq 2K\left(M^*\right)^2\beta\left(1 + \frac{3}{2}\beta\right)\left(\frac{1}{\sqrt{\lambda}} + 3\eta\right). \tag{2.164}$$

Estimate now the sum of 3rd, 4th, and 5th terms in the right-hand side of (2.161). We obtain similarly with the above:

$$\left|\left(2A_{2,n}h\sum_{j=0}^{n-1}\nabla q_j^* - A_{1,n}\nabla q_n^* - A_{2,n}\left(\nabla V_{n,1}+\nabla V^*\right)\right)\nabla\widetilde{V}_{n,1}+\varepsilon q_n^* - F_1\right|_\alpha$$
$$\leq 2M^*\left(1+\frac{\beta}{2}\right)\eta. \tag{2.165}$$

Combining (2.165) with (2.163), we obtain the following estimate the sum of 3rd, 4th, and 5th terms in the right-hand side of (2.161):

$$\left|\left(A_{1,n}\nabla q_n^* - A_{2,n}h\sum_{j=0}^{n-1}\left(\nabla q_j+\nabla q_j^*\right)+2A_{2,n}\nabla V_{n,1}\right)\left(h\sum_{j=0}^{n-1}\nabla\widetilde{q}_j\right)\right|_\alpha$$
$$+\left|\left(2A_{2,n}h\sum_{j=0}^{n-1}\nabla q_j^* - A_{1,n}\nabla q_n^* - A_{2,n}\left(\nabla V_{n,1}+\nabla V^*\right)\right)\nabla\widetilde{V}_{n,1}+\varepsilon q_n^* - F_1\right|_\alpha$$
$$\leq 2K\left(M^*\right)^2\beta\left(1+\frac{3}{2}\beta\right)\left(\frac{1}{\sqrt{\lambda}}+3\eta\right)+2M^*\left(1+\frac{\beta}{2}\right)\eta. \tag{2.166}$$

Since $K, M^* > 1$, then (2.134) and the 4th inequality (2.141) imply that

$$2K\left(M^*\right)^2\beta\left(1+\frac{3}{2}\beta\right)\left(\frac{1}{\sqrt{\lambda}}+3\eta\right)\leq 8K\left(M^*\right)^2\beta\left(1+\frac{3}{2}\beta\right)\eta\leq\frac{1}{2}M^*\eta. \tag{2.167}$$

By (2.134),

$$2M^*\left(1+\frac{\beta}{2}\right)\eta\leq\frac{5}{2}M^*\eta. \tag{2.168}$$

Hence, we obtain from (2.166)–(2.168) that

$$\left|\left(A_{1,n}\nabla q_n^* - A_{2,n}h\sum_{j=0}^{n-1}\left(\nabla q_j+\nabla q_j^*\right)+2A_{2,n}\nabla V_{n,1}\right)\left(h\sum_{j=0}^{n-1}\nabla\widetilde{q}_j\right)\right|_\alpha$$
$$+\left|\left(2A_{2,n}h\sum_{j=0}^{n-1}\nabla q_j^* - A_{1,n}\nabla q_n^* - A_{2,n}\left(\nabla V_{n,1}+\nabla V^*\right)\right)\nabla\widetilde{V}_{n,1}+\varepsilon q_n^* - F_1\right|_\alpha$$
$$\leq\frac{1}{2}M^*\eta+\frac{5}{2}M^*\eta=3M^*\eta. \tag{2.169}$$

It follows from (2.134), (2.137), and (2.156) that $C^{\alpha}\left(\overline{\Omega}\right)$ norms of coefficients at $\nabla\widetilde{q}_{n,i}, \widetilde{q}_{n,i}$ in the left-hand side of (2.161) do not exceed 1. Hence, we can apply the estimate (2.125) of the Schauder theorem to the Dirichlet boundary value problem (2.161) and (2.162). Using that estimate and (2.142), we obtain

$$|\widetilde{q}_{n,i}|_{2+\alpha} \leq \frac{KM^*}{2C^*\lambda} |\nabla\widetilde{q}_{n,i-1}|_{\alpha} |\nabla q_{n,i-1} + \nabla q_n^*|_{\alpha} + 3KM^*\eta.$$

First, consider the case $i = 1$. By (2.160) $q_{n,0} = q_{n-1}$. Since estimates (2.155) and (2.156) hold true for functions $\widetilde{q}_{n-1}, q_{n-1}$, then (2.111), (2.136), (2.155), and (2.156) imply that

$$|\widetilde{q}_{n,1}|_{2+\alpha} \leq \frac{3KM^*}{\lambda} KM^* \left(\frac{1}{\sqrt{\lambda}} + 3\eta\right) + 3KM^*\eta \leq 2KM^* \left(\frac{1}{\sqrt{\lambda}} + 3\eta\right), \tag{2.170}$$

which establishes (2.155) for the function $\widetilde{q}_{n,1}$. Hence, similarly with (2.148), we obtain $|q_{n,1}|_{2+\alpha} \leq 2C^*$, which proves (2.156) for $q_{n,1}$. Using (2.41), (2.42), (2.116)–(2.118), (2.155), (2.156), (2.170), and the fourth inequality (2.141), we obtain similarly with (2.154) that

$$|\widetilde{c}_{n,1}|_{\alpha} \leq \frac{8}{3}\eta < \frac{1}{2},$$

which establishes (2.157) for $\widetilde{c}_{n,1}$. We obtain from (2.127) and (2.157) that functions

$$c_{n,1}, \widehat{c}_{n,1} \in [1, d+1], \ \widehat{c}_n(x) = 1 \text{ in } \mathbb{R}^3 \diagdown \Omega, \ c_{n,1}, \widehat{c}_{n,1} \in C^{\alpha}\left(\mathbb{R}^3\right).$$

This establishes (2.158) for functions $c_{n,1}, \widehat{c}_{n,1}(x)$. The latter, (2.48), and one of conditions of this theorem guarantee that $|V_{n,2}|_{2+\alpha} \leq \xi$. Recalling that $q_n = q_{n,m_n}$ and applying the mathematical induction principle, we obtain that estimates (2.155)–(2.159) are valid for $j = n$.

Having estimates (2.155)–(2.158) for $j = 1, ..., n$, we now obtain estimate (2.139). Denote

$$p_n := \sum_{j=0}^{n} \left|\widetilde{q}_j\right|_{2+\alpha}, g_n = hp_n, n \in \left[1, \overline{N}\right].$$

It follows from the above proof that

$$\left|\left(A_{1,n}\nabla q_n^* - A_{2,n}h\sum_{j=0}^{n-1}\left(\nabla q_j + \nabla q_j^*\right) + 2A_{2,n}\nabla V_{n,1}\right)\left(h\sum_{j=0}^{n-1}\nabla\widetilde{q}_j\right)\right|_{\alpha}$$
$$\leq M^*\left(1 + \frac{3}{2}\beta\right)hp_{n-1} \leq 2M^*hp_{n-1}.$$

Hence, it follows from (2.165) and (2.168) that the sum of all terms in the right-hand side of (2.161), excluding the first one, can be estimated from the above via $2M^*hp_{n-1} + 5/2 \cdot M^*\eta$. First, consider the case when in (2.161) $\widetilde{q}_{n,i}$ is replaced with $\widetilde{q}_{n,1}^k$ and $\widetilde{q}_{n,i-1}$ is replaced with $\widetilde{q}_{n,1}^{k-1}$, respectively. Since the sequence $\{q_{n,1}^k\}_{k=1}^{\infty}$ converges, we can replace in (2.161) the vector $(\widetilde{q}_{n,1}^k, q_{n,1}^{k-1})$ with the vector $(\widetilde{q}_{n,1}, q_{n,1})$. Hence, applying to the boundary value problem (2.161) and (2.162), the estimate (2.125) of the Schauder theorem as well as (2.135)–(2.137), and the fourth inequality (2.141), we obtain

$$|\widetilde{q}_{n,1}|_{2+\alpha} \leq \frac{|\widetilde{q}_{n,1}|_{2+\alpha}}{4} + 2KM^*hp_{n-1} + \frac{5}{2}KM^*\eta$$

or

$$|\widetilde{q}_{n,1}|_{2+\alpha} \leq \frac{8}{3}KM^*hp_{n-1} + \frac{10}{3}KM^*\eta. \tag{2.171}$$

Similarly, we obtain for $\widetilde{q}_{n,i}, i \in [2, m_n]$

$$\begin{aligned} |\widetilde{q}_{n,i}|_{2+\alpha} &\leq \frac{KM^*}{2C^*\lambda} |\widetilde{q}_{n,i-1}|_{2+\alpha} |\nabla q_{n,i-1} + \nabla q_n^*| + 2KM^*hp_{n-1} + \frac{5}{2}KM^*\eta \\ &\leq \frac{3KM^*}{\lambda} |\widetilde{q}_{n,i-1}|_{2+\alpha} + 2KM^*hp_{n-1} + \frac{5}{2}KM^*\eta \\ &\leq \frac{24(KM^*)^2}{\lambda}\eta + 2KM^*hp_{n-1} + \frac{5}{2}KM^*\eta \leq 2KM^*hp_{n-1} \\ &+ \frac{11}{4}KM^*\eta. \end{aligned} \tag{2.172}$$

Thus, it follows from (2.171) and (2.172) that

$$|\widetilde{q}_{n,i}|_{2+\alpha} \leq 2KM^*hp_{n-1} + \frac{10}{3}KM^*\eta, \ i \in [1, m_n].$$

Hence, recalling that $\widetilde{q}_n = \widetilde{q}_{n,m_n}$, we obtain

$$|\widetilde{q}_n|_{2+\alpha} \leq \frac{8}{3}KM^*hp_{n-1} + \frac{10}{3}KM^*\eta. \tag{2.173}$$

Substituting in (2.173) $\widetilde{q}_{n-k}$ for $\widetilde{q}_n$, we obtain the following sequence of estimates:

$$|\widetilde{q}_{n-k}|_{2+\alpha} \leq \frac{8}{3}KM^*hp_{n-k-1} + \frac{10}{3}KM^*\eta, \ 0 \leq k \leq n-2. \tag{2.174}$$

Summing up all estimates (2.174) for functions $\widetilde{q}_{n-k}$ with $0 \leq k \leq n-2$, we obtain

$$p_n - |\widetilde{q}_1|_{2+\alpha} \leq \frac{8}{3}KM^*h\sum_{i=1}^{n-1} p_i + \frac{10}{3}KM^*n\eta.$$

Since $|p_i|_{2+\alpha} \le |p_{i+1}|_{2+\alpha}$ and $h\overline{N} = \beta$, then

$$|p_n|_{2+\alpha} \le \frac{8}{3} K M^* \beta p_{n-1} + \frac{10}{3} K M^* \overline{N} \eta + |\widetilde{q}_1|_{2+\alpha}.$$

Hence, multiplying by h and using (2.155) and the fourth inequality (2.141), we obtain

$$g_n \le \frac{8}{3} K M^* \beta g_{n-1} + \frac{10}{3} K M^* \beta \eta + 4 K M^* \eta^2.$$

Hence, (2.134) and (2.135) imply that

$$g_n \le \frac{1}{9} g_{n-1} + \frac{7}{18} \eta, \ n \in \left[2, \overline{N}\right].$$

Iterating this inequality and using the formula for the sum of the geometrical progression, we obtain

$$g_n \le \frac{1}{9^{n-1}} g_1 + \frac{7}{16} \eta, \ n \in \left[2, \overline{N}\right].$$

Since $g_1 = h\,|\widetilde{q}_1|_{2+\alpha} \le |\widetilde{q}_1|_{2+\alpha}\,\eta/2$, then (2.135), (2.155), and the fourth inequality (2.141) imply that

$$g_n \le \frac{\eta}{4 \cdot 9^{n-1}} + \frac{7}{16} \eta, \ n \in \left[2, \overline{N}\right]. \tag{2.175}$$

We now prove (2.139). Repeating the above arguments, which were presented for $|\widetilde{c}_{1,1}|_\alpha$, we obtain

$$|\widetilde{c}_n|_\alpha \le |\widetilde{v}_n|_{2+\alpha} \left[1 + \overline{s}^2 \left(|v_n|_{2+\alpha} + |v_n^*|_{2+\alpha}\right)\right] \le 2\,|\widetilde{v}_n|_{2+\alpha}. \tag{2.176}$$

Also, by (2.41) and (2.131), $|\widetilde{v}_n|_{2+\alpha} \le g_n + \eta$. Hence, it follows from (2.176) that $|\widetilde{c}_n|_\alpha \le 2\,(g_n + \eta)$. Combining this with (2.175), we obtain

$$|\widetilde{c}_n|_\alpha \le \frac{1}{2 \cdot 9^{n-1}} \eta + \frac{23}{8} \eta, \ n \in \left[2, \overline{N}\right],$$

which is (2.139). □

2.8.3 Informal Discussion of Theorem 2.8.2

In this section, we informally discuss the meaning of the parameter ξ. In Sect. 2.8.4, we formalize this discussion via the introduction of the first approximate mathematical model; see Definition 1.1.2.1 in Sect. 1.1.2 for this notion. Theorem 2.8.2, which fits this model, was our first result about the approximate global convergence [24]. The second approximate mathematical model of Sect. 2.9.2 imposes less restrictive conditions than the first one. That model is free from the discrepancy mentioned below in the current section.

By (2.14), (2.18), (2.131), and (2.132) the parameter ξ is small as long as the truncated pseudo frequency $\overline{s}$ is large. This implies, of course that the parameter η is also small since other numbers in (2.133) are those occurred either in the approximation procedure or the noise level. And the latter parameters traditionally assumed to be small in the numerical analysis. There is nothing unusual in the smallness assumption imposed on ξ. Indeed, since by (2.19), (2.131), and (2.132)

$$\xi = O\left(\frac{1}{\overline{s}}\right), \ \overline{s} \to \infty, \tag{2.177}$$

then that smallness assumption is similar with the truncation of high frequencies. And the latter is routinely done in engineering. Nevertheless, Theorem 2.8.2 has a discrepancy related to the parameter ξ. Indeed, by (2.135) we should have

$$\eta \leq \frac{1}{256KC^*\overline{s}^2}. \tag{2.178}$$

Conditions (2.177) and (2.178) are incompatible. In addition, since by (2.122) $M^* = O\left(\overline{s}^2\right)$ as $\overline{s} \to \infty$, then there is no guarantee that the right-hand side of (2.137) is indeed small.

We explain the discrepancy between (2.177) and (2.178) the same way as we have explained Definition 1.1.2.1 of the approximate global convergence property. The problem of construction of globally convergent numerical methods for our CIP is obviously an extremely challenging one because of three factors combined: nonlinearity, ill-posedness, and single measurement data. Hence, we need to come up with a certain compromise. One version of such a compromise is outlined in the previous paragraph. In simple terms, not everything can be covered by the theory, while numerical results are fortunately more optimistic than theoretical ones. Analogously, see the fifth Remark 1.1.2.1 about the well-known discrepancy between the Huygens-Fresnel theory of optics and the Maxwell equations.

Likewise, if we would prove convergence of our method as $\overline{s} \to \infty$, then we would also prove uniqueness of the above formulated inverse problems, which is a long-standing and not yet addressed question; see Remark 2.1

2.8.4 The First Approximate Mathematical Model

We now introduce the first approximate mathematical model which ensures that, within the framework of this model, Theorem 2.8.2 claims the approximate global convergence property of the algorithm of Sect. 2.6.1. We follow Definition 1.1.2.1 in Sect. 1.1.2.

Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$. Let the exact coefficient $c^*(x)$ satisfies conditions (2.3) and (2.4):

$$c^*(x) \in [1, d], \quad c^*(x) = 1 \text{ for } x \in \mathbb{R}^3 \setminus \Omega,$$

$$c^* \in C^3(\mathbb{R}^3), |c^*|_\alpha \leq \overline{d},$$

where the numbers $d, \overline{d} > 1$ are given. Let the cut-off pseudo frequency $\overline{s} =$ const. > 1. For any function $c(x)$ such that

$$c \in [1, d+1] \text{ in } \Omega, \; c(x) = 1 \text{ for } x \in \mathbb{R}^3 \setminus \Omega, \tag{2.179}$$

$$c(x) \in C^\alpha(\mathbb{R}^3), |c|_\alpha \leq \overline{d}, \tag{2.180}$$

consider the solution $w_c(x, \overline{s})$ of the problem:

$$\Delta w_c - \overline{s}^2 c(x) w_c = -\delta(x - x_0), \; x \in \mathbb{R}^3,$$

$$\lim_{|x| \to \infty} w_c(x, \overline{s}) = 0.$$

We seek solution of this problem in the class represented as

$$w_c(x, s) = w_1(x, s) + \overline{w}_c(x, s),$$

where

$$w_1(x, s) = \frac{\exp(-s|x - x_0|)}{4\pi|x - x_0|}, \; \overline{w}_c \in C^{2+\alpha}(\mathbb{R}^3).$$

Consider the corresponding tail function $V_c(x)$:

$$V_c(x) = \frac{\ln w_c(x, \overline{s})}{\overline{s}^2} \in C^{2+\alpha}(\overline{\Omega}).$$

Suppose that the following inequality holds for all functions c satisfying (2.179) and (2.180):

$$|\nabla V_c|_{1+\alpha} \leq \xi, \tag{2.181}$$

where $\xi \in (0, 1)$ is a sufficiently small number. It follows from Theorem 2.9.1.2 that norms $|\nabla V_c|_{1+\alpha}$ are indeed uniformly bounded for all functions $c(x)$ satisfying conditions (2.179) and (2.180).

The First Approximate Mathematical Model for the Algorithm of Sect. 2.6.1 consists of the following two assumptions.

Assumptions:

1. We assume that the number ξ in (2.181) is a free parameter, which can be made infinitely small independently on the parameter $\overline{s}$.
2. We assume that the tail function $V^*(x)$ is unique.

Actually, the first assumption was realized numerically in our works with experimental data [28,109]; also see Sect. 5.7. Indeed, it is stated in Sect. 7.2 of [109] that we have used derivatives of tails $\partial_{\overline{s}} V_c(x,\overline{s})$ instead of tails $V_c(x,\overline{s})$ themselves. Assuming that conditions of Lemma 2.3 hold, it follows from this lemma and (2.19) that

$$|\partial_{\overline{s}} V_c(x,\overline{s})|_{2+\alpha} << |V_c(x,\overline{s})|_{2+\alpha}, \ \overline{s} \to \infty. \tag{2.182}$$

Hence, it is reasonable to assume that in the formulation of Theorem 2.8.2, tails V^* and V_c are replaced with $\partial_{\overline{s}} V^*$ and $\partial_{\overline{s}} V_c$, respectively. Theorem 2.8.2 is still valid in this case with an insignificant change of its proof.

The second above assumption is imposed to make sure that the solution of (2.105) with the boundary condition (2.106) and the smoothness condition (2.108) is unique. Recall that its existence is assumed a priori by the fundamental concept of Tikhonov (Sects. 1.4 and 2.8.1). Uniqueness can be proven similarly with the proof of Lemma 2.9.2.

Remarks 2.8.4. 1. As it is stated in Theorem 2.8.2, (2.139) implies the approximate global convergence property of the algorithm of Sect. 2.6.1 within the framework of the first approximate mathematical model.

2. The only way to justify assumption 1 is via numerical studies. Numerical experiments of Chaps. 3 and 4 demonstrate that this model is reasonable since computational results confirm the validity of Theorem 2.8.2. The same is true for the second approximate mathematical model of Sect. 2.9.2. It is an opinion of the authors that results of testing of experimental data in [109] and [28] completely justify both approximate mathematical models; see the informal Definition 1.1.2.2 of the approximate global convergence property. Indeed, in [109], very accurate images of both locations and refractive indices of dielectric abnormalities were obtained for the most challenging case of *blind* experimental data when answers were unknown in advance. The follow-up refinement stage of nonblind testing in [28] led to very accurate images of all three components: locations, shapes, and refractive indices of those dielectric abnormalities. These results are presented in Chap. 5.

2.9 The Second Approximate Global Convergence Theorem

In this section, we present the second version of the proof of the approximate global convergence property of the algorithm of Sect. 2.6.1. Unlike Sect. 2.8, we estimate tail functions first. Next, we present the second approximate mathematical model. This model sounds more convenient than the first one because it basically amounts to the truncation of all terms of the asymptotic series for the tail function $V(x,\overline{s})$ at $\overline{s} \to \infty$, except of the first one. Finally, based on this model as well as on estimates for tail functions, we prove the second approximate global convergence Theorem 2.9.4 This theorem claims that the algorithm of Sect. 2.6.1 has the approximate global convergence property within the framework of the second approximate mathematical model; see Definition 1.1.2.1 in Sect. 1.1.2 for this property.

For reader's convenience, we remind here some facts from previous sections of this chapter. Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$. Let $c^*(x)$ be the exact solution of Inverse Problem 2.1. Just as above, we assume that the exact coefficient $c^*(x)$ satisfies the following conditions:

$$c^*(x) \in [1, d], \quad c^*(x) = 1 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega, \tag{2.183}$$

$$c^* \in C^3\left(\mathbb{R}^3\right), |c^*|_\alpha \leq \overline{d}, \tag{2.184}$$

where the numbers $d, \overline{d} > 1$ are given. In addition, we consider functions $c(x)$ satisfying conditions (2.179) and (2.180):

$$c \in [1, d+1] \text{ in } \Omega, \; c(x) = 1 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega, \tag{2.185}$$

$$c(x) \in C^\alpha\left(\mathbb{R}^3\right), |c|_\alpha \leq \overline{d}. \tag{2.186}$$

For each such function c and for each $s > 0$, we consider the solution $w_c(x, s)$ of the following problem

$$\Delta w_c - \overline{s}^2 c(x) w_c = -\delta(x - x_0), \; x \in \mathbb{R}^3, \tag{2.187}$$

$$\lim_{|x| \to \infty} w_c(x, \overline{s}) = 0, \tag{2.188}$$

such that

$$w_c(x, s) = w_1(x, s) + \overline{w}_c(x, s), \text{ where } \overline{w}_c \in C^{2+\alpha}\left(\mathbb{R}^3\right), \tag{2.189}$$

$$w_1(x, s) = \frac{\exp(-s|x - x_0|)}{4\pi |x - x_0|}. \tag{2.190}$$

The existence and uniqueness of the solution w_c of the problem (2.187)–(2.190) is guaranteed by Theorem 2.7.2. Let the function $w_{d+1}(x, s)$ be the solution of the problem (2.187) and (2.188) for the case $c(x) \equiv d+1$:

$$w_{d+1}(x, s) = \frac{\exp\left(-s\sqrt{d+1}\,|x - x_0|\right)}{4\pi |x - x_0|}. \tag{2.191}$$

By Theorem 2.7.2,

$$w_{d+1}(x, s) < w_c(x, s) \leq w_1(x, s), \forall s > 0, \tag{2.192}$$

for all functions $c(x)$ satisfying conditions (2.179) and (2.180). Also, we define tail functions as

$$V^*(x) := V^*(x,\overline{s}) = \frac{\ln w^*(x,\overline{s})}{\overline{s}^2}, \tag{2.193}$$

$$V_c(x) := V_c(x,\overline{s}) = \frac{\ln w_c(x,\overline{s})}{\overline{s}^2}, \tag{2.194}$$

where $w^*(x,\overline{s})$ is the solution of the problem (2.187)–(2.190) with the function $c(x) := c^*(x)$ satisfying conditions (2.183) and (2.184).

2.9.1 Estimates of the Tail Function

In Theorem 2.9.1.1 of this section, we estimate tails in non-Hölder norms. We will need these estimates in Chap. 6. And in Theorem 2.9.1.2, we estimate tails in Hölder norms. We will use Theorem 2.9.1.2 in Sect. 2.9.4.

Theorem 2.9.1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Let the source $x_0 \notin \overline{\Omega}$. Let the function $c^*(x)$ satisfying (2.183) and (2.184) be the exact solution of Inverse Problem 2.1 and the parameter $\overline{s} > 1$ and $V^*(x)$ be the exact tail function as in (2.193). For each function $c(x)$ satisfying condition (2.185) and (2.186), let $w(x,\overline{s}) := w_c(x,\overline{s})$ be the solution of the problems (2.187)–(2.190) (Theorem 2.7.2) and $V_c(x)$ be the corresponding tail function as defined in (2.194). Then there exists a constant $B = B(\Omega,\overline{s},d,x_0) > 2$ depending only on listed parameters such that for all functions $c(x)$ satisfying (2.185) and (2.186) the following inequalities hold:*

$$\|\nabla V_c\|_{C(\overline{\Omega})}, \|\nabla V^*\|_{C(\overline{\Omega})} \leq B, \tag{2.195}$$

$$\|\nabla V_c - \nabla V^*\|_{L_2(\Omega)} + \|\Delta V_c - \Delta V^*\|_{L_2(\Omega)} \leq B\|c - c^*\|_{L_2(\Omega)}. \tag{2.196}$$

Proof. In this proof, $B = B(\Omega,\overline{s},d,x_0) > 1$ denotes different constants depending on listed parameters. Temporary denote in this proof only $\mathbf{x} = (x,y,z)$. For brevity, we estimate only $\|V_x\|_{C(\overline{\Omega})}$. Estimates of two other first derivatives are similar. By (2.192)–(2.194)

$$|\partial_x V_c| = \left|\frac{w_x}{w}(\mathbf{x},\overline{s})\right| \leq B\,|w_x(\mathbf{x},\overline{s})|, \quad |\partial_x V^*| = \left|\frac{w_x^*}{w^*}(\mathbf{x},\overline{s})\right| \leq B\,|w_x^*(\mathbf{x},\overline{s})|. \tag{2.197}$$

Theorem 2.7.2, (2.190), and (2.101) imply that for $\boldsymbol{\xi} = (\xi_1,\xi_2,\xi_3), \mathbf{x} \in \Omega, b(\mathbf{x}) = c(\mathbf{x}) - 1$,

$$w_x(\mathbf{x},\overline{s}) = w_{1x}(\mathbf{x},\overline{s}) + \frac{s^2}{4\pi}\int_{\Omega}\Bigg[\left(s\frac{x-\xi_1}{|\mathbf{x}-\boldsymbol{\xi}|^2}+\frac{x-\xi_1}{|\mathbf{x}-\boldsymbol{\xi}|^3}\right) \times \exp(-\overline{s}|\mathbf{x}-\boldsymbol{\xi}|)\, b(\boldsymbol{\xi})\, w(\boldsymbol{\xi},\overline{s})\Bigg]\mathrm{d}\xi. \tag{2.198}$$

Since $x_0 \notin \overline{\Omega}$, then functions w_0, w_{0x} do not have a singularity for $\mathbf{x} \in \overline{\Omega}$. Hence, (2.192) and (2.198) imply that

$$|w_x(\mathbf{x},\overline{s})| \leq B + B\int_{\Omega}\left(\frac{1}{|\mathbf{x}-\boldsymbol{\xi}|}+\frac{1}{|\mathbf{x}-\boldsymbol{\xi}|^2}\right)\exp(-\overline{s}|\mathbf{x}-\boldsymbol{\xi}|)\,\mathrm{d}\boldsymbol{\xi} \leq B,\ \mathbf{x}\in\Omega. \tag{2.199}$$

Hence, (2.196) follows from (2.197) and (2.199). Denote $\widetilde{w} := w - w^*$. Then (2.193) and (2.194) imply that

$$\partial_x V_c - \partial_x V^* = \left(\frac{\widetilde{w}_x}{w} - \frac{w_x^*}{ww^*}\widetilde{w}\right)(\mathbf{x},\overline{s}),\ \mathbf{x}\in\Omega.$$

Hence, by (2.192) and (2.199),

$$\|\nabla V_c - \nabla V^*\|_{L_2(\Omega)} \leq B\left(\|\nabla\widetilde{w}\|_{L_2(\Omega)} + \|\widetilde{w}\|_{L_2(\Omega)}\right) \leq B\left(\|\nabla\widetilde{w}\|_{L_2(\mathbb{R}^3)} + \|\widetilde{w}\|_{L_2(\mathbb{R}^3)}\right). \tag{2.200}$$

Let $\widetilde{c}(\mathbf{x}) = c(\mathbf{x}) - c^*(\mathbf{x})$. Since

$$c(\mathbf{x})\, w(\mathbf{x},\overline{s}) - c^*(\mathbf{x})\, w^*(\mathbf{x},\overline{s}) = c(\mathbf{x})\,\widetilde{w}(\mathbf{x},\overline{s}) + \widetilde{c}(\mathbf{x})\, w^*(\mathbf{x},\overline{s}),$$

we obtain from (2.187)

$$\Delta\widetilde{w}(\mathbf{x},\overline{s}) - \overline{s}^2 c(\mathbf{x})\,\widetilde{w}(\mathbf{x},\overline{s}) = \overline{s}^2\widetilde{c}(\mathbf{x})\, w^*(\mathbf{x},\overline{s}),\ \mathbf{x}\in\mathbb{R}^3. \tag{2.201}$$

Since $\widetilde{c}(\mathbf{x}) = 0$ outside of Ω and $\mathbf{x}_0 \notin \overline{\Omega}$, then $\overline{s}^2\widetilde{c}(\mathbf{x})\, w^*(\mathbf{x},\overline{s}) = 0$ near x_0. In particular, $\overline{s}^2\widetilde{c}(\mathbf{x})\, w^*(\mathbf{x},\overline{s}) = 0$ for $\mathbf{x}\in\mathbb{R}^3 \diagdown \Omega$. Let the number $R > 0$ be so large that $\Omega \subset B_R = \{|\mathbf{x}| < R\}$. Multiply both sides of (2.201) by $(-\widetilde{w})$ and integrate over B_R. We obtain

$$\int_{B_R}\left(|\nabla\widetilde{w}(\mathbf{x},\overline{s})|^2 + \overline{s}^2 c(\mathbf{x})\,\widetilde{w}^2(\mathbf{x},\overline{s})\right)\mathrm{d}\mathbf{x} - \int_{\partial B_R}\left(\widetilde{w}\frac{\partial\widetilde{w}}{\partial n}\right)(\mathbf{x},\overline{s})\,\mathrm{d}S = -\overline{s}^2\int_{\Omega}\widetilde{c}(\mathbf{x})\left[w^*\widetilde{w}\right](\mathbf{x},\overline{s})\,\mathrm{d}\mathbf{x}. \tag{2.202}$$

It follows from (2.101) and (2.198) that $\nabla\widetilde{w}(\mathbf{x},\overline{s}), \widetilde{w}(\mathbf{x},\overline{s}) \in L_2(\mathbb{R}^3)$ and the second term in the left-hand side of (2.202) tends to zero as $R \to \infty$. Hence, setting in (2.202) $R \to \infty$, we obtain

$$\int_{\mathbb{R}^3} \left(|\nabla\widetilde{w}(\mathbf{x},\overline{s})|^2 + \overline{s}^2 c(\mathbf{x})\,\widetilde{w}^2(\mathbf{x},\overline{s})\right) d\mathbf{x} = -\overline{s}^2 \int_{\Omega} \widetilde{c}(\mathbf{x})\left(w^*\widetilde{w}\right)(\mathbf{x},\overline{s})\, d\mathbf{x}.$$

Since $c \geq 1$, then $\overline{s}^2 c(\mathbf{x})\,\widetilde{w}^2(\mathbf{x},\overline{s}) \geq \overline{s}^2\widetilde{w}^2(\mathbf{x},\overline{s})$. Hence, using (2.192) and the Cauchy-Schwarz inequality, we obtain

$$\|\widetilde{w}(\mathbf{x},\overline{s})\|_{H^1(\mathbb{R}^3)} \leq B\,\|\widetilde{c}\|_{L_2(\Omega)}. \tag{2.203}$$

Next,

$$\Delta V_c - \Delta V^* = \left[\frac{\Delta\widetilde{w}}{w} - \frac{\nabla(w+w^*)}{w^2}\nabla\widetilde{w} - \left(\frac{\Delta w^*}{ww^*} - \frac{(\nabla w^*)^2(w+w^*)}{(ww^*)^2}\right)\widetilde{w}\right](\mathbf{x},\overline{s}). \tag{2.204}$$

Since $\Delta w^*(\mathbf{x},\overline{s}) = \overline{s}^2 c^*(\mathbf{x})\, w^*(\mathbf{x},\overline{s})$ for $\mathbf{x} \in \overline{\Omega}$, then (2.192), (2.199), and (2.204) imply that

$$|\Delta V_c - \Delta V^*| \leq B\left(|\Delta\widetilde{w}| + |\nabla\widetilde{w}| + |\widetilde{w}|\right), \mathbf{x} \in \overline{\Omega}. \tag{2.205}$$

By (2.201),

$$\|\Delta\widetilde{w}\|_{L_2(\mathbb{R}^3)} \leq B\left(\|\widetilde{w}\|_{L_2(\mathbb{R}^3)} + \|\widetilde{c}\|_{L_2(\Omega)}\right).$$

Hence, (2.200), (2.203), and (2.205) imply (2.196). □

We now want to prove an analog of Theorem 2.9.1.1 for the Hölder norms. Let $c^*(x)$ be the exact solution of Inverse Problem 2.1. In applications, the domain of interest Ω can often be increased if necessary. In terms of Inverse Problem 2.1, this means that one can have measured data $g(x,t)$ in (2.5) at the boundary of a domain which is a little bit larger than the original domain of interest. Hence, let $\Omega' \subset \Omega$ be a subdomain of the domain Ω and $\partial\Omega' \cap \partial\Omega = \varnothing$. We replace condition (2.183) by a little bit different one:

$$c^*(x) \in [1,d],\quad c^*(x) = 1 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega'. \tag{2.206}$$

Recall that in Sect. 2.6 we have introduced the following cut-off function $\chi(x)$:

$$\chi \in C^3\left(\mathbb{R}^3\right),\ \chi(x) = \begin{cases} 1 \text{ in } \Omega', \\ \text{between 0 and 1 in } \Omega \diagdown \Omega', \\ 0 \text{ outside of } \Omega. \end{cases} \tag{2.207}$$

Consider the set of functions $P\left(d,\overline{d}\right)$ defined as

$$P\left(d,\overline{d}\right)=\left\{c\in C^{\alpha}\left(\overline{\Omega}\right):|c|_{\alpha}\leq\overline{d}+1,c\in[1,d+1]\right\}. \tag{2.208}$$

Hence, (2.184) and (2.206) imply that

$$c^{*}\in P\left(d,\overline{d}\right). \tag{2.209}$$

For each function $c\in P\left(d,\overline{d}\right)$, consider the function $\widehat{c}\left(x\right)$:

$$\widehat{c}\left(x\right)=\left(1-\chi\left(x\right)\right)+\chi\left(x\right)c\left(x\right), \tag{2.210}$$

where the function $\chi\left(x\right)$ is defined in (2.207). Then (Sect. 6.1),

$$\widehat{c}\in C^{\alpha}\left(\mathbb{R}^{3}\right),\ \widehat{c}\in[1,d+1]\ \text{ in }\Omega,\ \widehat{c}\left(x\right)=1\text{ for }x\in\mathbb{R}^{3}\diagdown\Omega. \tag{2.211}$$

Next, consider the solution $w_{\widehat{c}}(x,\overline{s})$ of the problem (2.187)–(2.190) with $c\left(x\right):=\widehat{c}\left(x\right)$. The existence and uniqueness of this solution is guaranteed by Theorem 2.7.2 Hence, by (2.192),

$$w_{d+1}\left(x,\overline{s}\right)<w_{\widehat{c}}(x,\overline{s})\leq w_{1}\left(x,\overline{s}\right),\ \forall x\neq x_{0},\ \forall c\in P\left(d,\overline{d}\right). \tag{2.212}$$

Lemma 2.9.1.1. *Let functions $c,c^{*}\in P\left(d,\overline{d}\right)$ (see (2.209) for c^{*}). Consider the function $\widehat{c}\left(x\right)$ defined in (2.210). Then*

$$\left|\widehat{c}-c^{*}\right|_{\alpha}\leq|\chi|_{\alpha}\left|c-c^{*}\right|_{\alpha}.$$

Proof. By (2.210),

$$\widehat{c}\left(x\right)-c^{*}\left(x\right)=\chi\left(x\right)\left(c\left(x\right)-c^{*}\left(x\right)\right)+\left(1-\chi\left(x\right)\right)\left(1-c^{*}\left(x\right)\right).$$

Since $1-c^{*}\left(x\right)=0$ for $x\in\Omega\diagdown\Omega'$ and $1-\chi\left(x\right)=0$ in Ω', then $\left(1-\chi\left(x\right)\right)\left(1-c^{*}\left(x\right)\right)\equiv0$. Hence, $\widehat{c}\left(x\right)-c^{*}\left(x\right)=\chi\left(x\right)\left(c\left(x\right)-c^{*}\left(x\right)\right)$, which implies the assertion of this lemma. □

Note that there exists a constant $C=C\left(\Omega,\alpha\right)>0$ depending only on the domain Ω and the parameter $\alpha\in(0,1)$ such that

$$|f|_{\alpha}\leq C\left\|f\right\|_{C^{1}\left(\overline{\Omega}\right)},\ \forall f\in C^{1}\left(\overline{\Omega}\right). \tag{2.213}$$

Lemma 2.9.1.2. *Let the source $x_{0}\notin\overline{\Omega}$. Let the function $c\in P\left(d,\overline{d}\right)$. Consider the function $\widehat{c}(x)$ defined in (2.210). Let $w_{\widehat{c}}(x,\overline{s})$ be the solution of the* *problem (2.187)–(2.190) with* $c(x) := \widehat{c}(x)$. *Then* $w_{\widehat{c}}(x,\overline{s}) \in C^{2+\alpha}\left(\overline{\Omega}\right)$. *Also, there exists a constant* $Y = Y\left(\Omega, \overline{s}, d, \overline{d}, \chi, x_0\right) > 0$ *depending on listed parameters such that*

$$\left|w_{\widehat{c}}(x,\overline{s})\right|_{\alpha} \leq Y,\ \forall c \in P\left(d, \overline{d}\right).$$

Proof. Below in this proof, $Y = Y\left(\Omega, \overline{s}, d, \overline{d}, \chi, x_0\right) > 0$ denotes different constants depending on listed parameters. Denote $b(x) = \widehat{c}(x) - 1$. Recall that by (2.101),

$$w_{\widehat{c}}(x,\overline{s}) = w_1(x,\overline{s}) - \overline{s}^2 \int_{\Omega} w_1(x-\xi,\overline{s})\, b(\xi)\, w_{\widehat{c}}(\xi,\overline{s})\, d\xi. \tag{2.214}$$

Since $x_0 \notin \overline{\Omega}$, then by (2.189), $w_{\widehat{c}}(x,\overline{s}) \in C^{2+\alpha}\left(\overline{\Omega}\right)$. Next, (2.211), (2.212), and (2.214) imply

$$\left|w_{\widehat{c}}(x,\overline{s})\right| \leq Y + Y\,\|b\|_{C(\overline{\Omega})} \int_{\Omega} w_1(x-\xi,\overline{s})\, d\xi \leq Y,\ x \in \Omega.$$

In addition, by (2.211) and (2.199), $\left|\nabla w_{\widehat{c}}(x,\overline{s})\right| \leq Y,\ x \in \Omega$. Hence, $\left\|w_{\widehat{c}}(x,\overline{s})\right\|_{C^1(\overline{\Omega})} \leq Y$. The rest of the proof follows from (2.213). □

Consider a bounded domain $\Omega_1 \subset \mathbb{R}^3$ such that

$$\Omega \subset \Omega_1,\ \partial\Omega \cap \partial\Omega_1 = \varnothing,\ \partial\Omega_1 \in C^3,\ x_0 \notin \overline{\Omega}_1. \tag{2.215}$$

Lemma 2.9.1.3. *Let* $\Omega, \Omega_1 \subset \mathbb{R}^3$ *be two bounded domains satisfying conditions (2.215). Let the function* $c \in P\left(d, \overline{d}\right)$. *Consider the function* $\widehat{c}(x)$ *defined in (2.210). Let* $w_{\widehat{c}}(x,\overline{s})$ *be the solution of the problem (2.187)–(2.190) with* $c(x) := \widehat{c}(x)$. *Then the function* $w_{\widehat{c}}(x,\overline{s}) \in C^3(\partial\Omega_1)$. *Furthermore, there exists a constant* $B = B\left(\Omega, \Omega_1, \overline{s}, d, \overline{d}, \chi, x_0\right) > 2$ *depending only on listed parameters such that*

$$\left\|w_{\widehat{c}}(x,\overline{s})\right\|_{C^3(\partial\Omega_1)} \leq B,\ \forall c \in P\left(d, \overline{d}\right). \tag{2.216}$$

Let two functions $c_1, c_2 \in P\left(d, \overline{d}\right)$. *Denote* $\widetilde{w}(x) = w_{\widehat{c}_1}(x,\overline{s}) - w_{\widehat{c}_2}(x,\overline{s})$. *Then*

$$\|\widetilde{w}\|_{C^3(\partial\Omega_1)} \leq B\,|c_1 - c_2|_{\alpha},\ \forall c_1, c_2 \in P\left(d, \overline{d}\right).$$

Proof. In this proof, B denotes different positive constant depending on above parameters. The integrand of the formula (2.214) does not have a singularity for $x \in \Omega_1 \diagdown \overline{\Omega}$. Hence, it follows from (2.214) that $w_{\widetilde{c}}(x,\overline{s}) \in C^3(\partial\Omega_1)$. Next, (2.216) follows from (2.192) and (2.214).

Denote $\widetilde{c}(x) = c_1(x) - c_2(x)$. By (2.210) $\widehat{c}_1(x) - \widehat{c}_2(x) = \chi(x)\widetilde{c}(x)$. First, substitute in (2.214) $(b_1, w_{\widehat{c}_1}) = (\widehat{c}_1 - 1, w_{\widehat{c}_1})$. Next, substitute $(b_2, w_{c_2}) = (\widehat{c}_2 - 1, w_{\widehat{c}_2})$ and subtract the second equation from the first one. We obtain

$$\widetilde{w}(x) = -\overline{s}^2 \int\limits_{\Omega} w_1(x-\xi,\overline{s})\,\chi(\xi)\,\widetilde{c}(\xi)\,w_{\widehat{c}_1}(\xi,\overline{s})\,d\xi$$
$$-\overline{s}^2 \int\limits_{\Omega} w_1(x-\xi,\overline{s})\,b_2(\xi)\,\widetilde{w}(\xi)\,d\xi.$$

Let

$$I_1(x) = -\overline{s}^2 \int\limits_{\Omega} w_1(x-\xi,\overline{s})\,\chi(\xi)\,\widetilde{c}(\xi)\,w_{\widehat{c}_1}(\xi,\overline{s})\,d\xi,$$

$$I_2(x) = -\overline{s}^2 \int\limits_{\Omega} w_1(x-\xi,\overline{s})\,b_2(\xi)\,\widetilde{w}(\xi)\,d\xi.$$

Using the same arguments as ones in the proof of Lemma 2.9.1.2 as well the assertion of this lemma, we obtain

$$\|I_1\|_{C^3(\partial\Omega_1)} \leq B\,\|\widetilde{c}\|_{L_2(\Omega)} \leq B\,|\widetilde{c}|_{\alpha}.$$

Next, by (2.203),

$$\|\widetilde{w}\|_{L_2(\Omega)} \leq B\,\|\widetilde{c}\|_{L_2(\Omega)} \leq B\,|\widetilde{c}|_{\alpha}.$$

The latter estimate implies that $\|I_2\|_{C^3(\partial\Omega_1)} \leq B\,|\widetilde{c}|_{\alpha}$. □

We need Lemma 2.9.1.4 since we have referred to this lemma in the course of the proof of Theorem 2.7.2**.**

Lemma 2.9.1.4. *Let $\Omega, \Omega_1 \subset \mathbb{R}^3$ be two convex bounded domains satisfying conditions (2.215) and let $\partial\Omega \in C^3$. Let the function $f \in C^{\alpha}(\overline{\Omega}_1)$ and $f(x) = 0$ outside of the domain Ω. For a number $s > 0$ consider the function $u(x)$:*

$$u(x) = \int\limits_{\Omega_1} w_1(x-\xi,s)\,f(\xi)\,d\xi = \int\limits_{\Omega_1} \frac{\exp(-s|x-\xi|)}{4\pi|x-\xi|} f(\xi)\,d\xi. \tag{2.217}$$

Then,

$$u \in C^{2+\alpha}(\mathbb{R}^3), \quad \lim_{|x|\to\infty} u(x) = 0, \tag{2.218}$$

$$|u|_{2+\alpha} \leq C \, |f|_{\alpha}\,, \tag{2.219}$$

$$\Delta u - s^2 u = -f \ \ in \ \mathbb{R}^3, \tag{2.220}$$

where the constant $C > 0$ is independent on the function f.

Proof. In this proof, $C > 0$ denotes different constants independent on the function f. Since the function $w_1\,(x-\xi, s)$ does not have a singularity for $x \in \partial\Omega_1, \xi \in \overline{\Omega}$, then by (2.217), $u \in C^3\,(\partial\Omega_1)$ and

$$\|u\|_{C^{2+\alpha}(\partial\Omega_1)} \leq C \, |f|_{\alpha}\,. \tag{2.221}$$

First, consider the case when the function $f \in C^1\left(\overline{\Omega}_1\right)$. We have

$$\Delta w_1\,(x-\xi) - s^2 w_1\,(x-\xi) = -\delta\,(x-\xi)\,.$$

Hence, using the same method as the one used in the standard PDE course for the Poisson equation,

$$\Delta v = -g\,(x)\,, g \in C^1\left(\mathbb{R}^3\right),\ g\,(x) = 0 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega,$$

$$\lim_{|x|\to\infty} v\,(x) = 0,$$

one can prove that the function $u \in C^2\left(\overline{\Omega}_1\right)$ and satisfies (2.220). Hence, by the Schauder theorem and (2.221), $u \in C^{2+\alpha}\left(\overline{\Omega}_1\right)$ and

$$\|u\|_{C^{2+\alpha}\left(\overline{\Omega}_1\right)} \leq C\,\|u\|_{C^{2+\alpha}(\partial\Omega_1)} \leq C\,|f|_{\alpha}\,, \forall f \in C^1\left(\overline{\Omega}_1\right),$$
$$f\,(x) = 0 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega. \tag{2.222}$$

Consider now a function f such that $f \in C^{\alpha}\left(\overline{\Omega}_1\right)$ and $f\,(x) = 0$ for $x \in \Omega_1 \diagdown \Omega$. Consider a sequence of functions $\{f_n\,(x)\}_{n=1}^{\infty} \subset C^1\left(\overline{\Omega}_1\right)$ such that $f_n\,(x) = 0$ for $x \in \Omega_1 \diagdown \Omega$ and:

$$\lim_{n\to\infty} |f_n - f|_{\alpha} = 0.$$

Let $\{u_n\,(x)\}_{n=1}^{\infty}$ be the corresponding sequence of functions defined via (2.217), where f is replaced with f_n. Then $u_n \in C^{2+\alpha}\left(\overline{\Omega}_1\right)$ and estimate (2.222) is valid for each n with the replacement of the vector (u, f) with the vector (u_n, f_n) is valid. Hence, $\{u_n\,(x)\}_{n=1}^{\infty}$ is the Cauchy sequence in the space $C^{2+\alpha}\left(\overline{\Omega}_1\right)$. Hence, this is a convergent sequence. On the other hand, (2.217) implies that

$$\lim_{n\to\infty} \|u_n - u\|_{C^1\left(\overline{\Omega}_1\right)} = 0.$$

Hence, it is the function u which is the limit of the sequence $\{u_n(x)\}_{n=1}^{\infty}$ in the space $C^{2+\alpha}\left(\overline{\Omega}_1\right)$. Hence, (2.218) and (2.220) are valid. Also, it follows from the above that in (2.222), "$\forall f \in C^1\left(\overline{\Omega}_1\right)$" can be replaced with "$\forall f \in C^{\alpha}\left(\overline{\Omega}_1\right)$". The latter implies (2.219). □

Theorem 2.9.1.2 provides estimates of tails in Hölder norms.

Theorem 2.9.1.2. *Let $\Omega, \Omega', \Omega_1 \subset \mathbb{R}^3$ be bounded domain with the boundaries $\partial\Omega, \partial\Omega_1 \in C^3$. Let condition (2.215) be satisfied and also let $\Omega' \subset \Omega, \partial\Omega' \cap \partial\Omega = \varnothing$. Let the function $c^*(x)$ satisfying conditions (2.206) and (2.209) be the exact solution of Inverse Problem 2.1, where constants $d, \overline{d} > 1$ are given. Let the parameter $\overline{s} > 1$ and $V^*(x)$ be the exact tail function as in (2.193). For each function ,$c \in P\left(d, \overline{d}\right)$ construct the function $\widehat{c}(x)$ by the formula (2.210), where the function $\chi(x)$ is defined in (2.207). Let $w_{\widehat{c}}(x, \overline{s})$ be the solution of the problems (2.187)–(2.190) with $c(x) := \widehat{c}(x)$ (Theorem 2.7.2). Let $V(x)$ be the corresponding tail function as defined in (2.194), where $c(x) := \widehat{c}(x)$. Then, there exists a constant $B = B\left(\Omega, \Omega_1, \overline{s}, d, \overline{d}, \chi, x_0\right) > 2$ depending only on listed parameters such that for all functions $c \in P(d, d^*)$ the following inequalities hold:*

$$|\nabla V^*|_{1+\alpha} \leq B, \tag{2.223}$$

$$|\nabla V_c|_{1+\alpha} \leq B, \tag{2.224}$$

$$|\nabla V_c - \nabla V^*|_{1+\alpha} \leq B\,|c - c^*|_{\alpha}. \tag{2.225}$$

Proof. In this proof, $B = B\left(\Omega, \Omega_1, \overline{s}, d, \overline{d}, \chi, x_0\right) > 2$ denotes different constants depending on listed parameters. It follows from (2.197), (2.204), and (2.212) that in order to prove (2.223), (2.224), and (2.225), it is sufficient to prove that

$$|w^*|_{2+\alpha}, \left|w_{\widehat{c}}\right|_{2+\alpha} \leq B, \tag{2.226}$$

$$\left|w_{\widehat{c}} - w^*\right|_{2+\alpha} \leq B\,|c - c^*|_{\alpha}. \tag{2.227}$$

For $x \in \partial\Omega_1$, denote $f_{\widehat{c}}(x) = w_{\widehat{c}}(x, \overline{s})\,|_{\partial\Omega_1}$. By (2.187) and (2.188), we have the following Dirichlet boundary value problem in the domain Ω_1:

$$\Delta w_{\widehat{c}} - \overline{s}^2 \widehat{c}(x)\, w_{\widehat{c}} = 0, x \in \Omega_1, \tag{2.228}$$

$$w_{\widehat{c}}(x, \overline{s})\,|_{\partial\Omega_1} = f_{\widehat{c}}(x). \tag{2.229}$$

By Lemma 2.9.1.3,

$$f_{\widehat{c}} \in C^3(\partial\Omega_1), \quad \|f_{\widehat{c}}\|_{C^3(\partial\Omega_1)} \leq B. \tag{2.230}$$

In addition, by (2.189),

$$w_{\widehat{c}}(x,\overline{s}) = w_1(x,\overline{s}) + \overline{w}_{\widehat{c}}(x,\overline{s}), \text{ where } \overline{w}_{\widehat{c}} \in C^{2+\alpha}\left(\mathbb{R}^3\right).$$

Since $x_0 \notin \overline{\Omega}_1$, then it follows from (2.190) that $w_1(x,\overline{s}) \in C^{\infty}\left(\overline{\Omega}_1\right)$. Hence, $w_{\widehat{c}}(x,\overline{s}) \in C^{2+\alpha}\left(\overline{\Omega}_1\right)$. Hence, Schauder theorem, (2.228)–(2.230) imply that

$$\left|w_{\widehat{c}}\right|_{2+\alpha} \leq \left\|w_{\widehat{c}}\right\|_{C^{2+\alpha}(\overline{\Omega}_1)} \leq B\left\|f_{\widehat{c}}\right\|_{C^{2+\alpha}(\partial\Omega_1)} \leq B\left\|f_{\widehat{c}}\right\|_{C^3(\partial\Omega_1)} \leq B,$$

which establishes (2.226) for the function $w_{\widehat{c}}(x,\overline{s})$. The proof for case of the function $w^*(x,\overline{s})$ is almost identical.

We now prove (2.227). Denote

$$\widetilde{w}(x) = w^*(x,\overline{s}) - w_{\widehat{c}}(x,\overline{s}), \; \widetilde{c}(x) = \widehat{c}(x) - c^*(x).$$

By Lemma 2.9.1.1,

$$|\widetilde{c}|_{\alpha} \leq |\chi|_{\alpha} |c - c^*|_{\alpha}. \tag{2.231}$$

Rewriting (2.228) for the function w^* and subtracting the resulting equation from (2.228), we obtain

$$\Delta\widetilde{w} - \overline{s}^2 c^*(x)\widetilde{w} = -\overline{s}^2\widetilde{c}(x)\, w_{\widehat{c}}(x,\overline{s}) \text{ in } \Omega_1, \tag{2.232}$$

$$\widetilde{w}\mid_{\partial\Omega_1} = f^*(x) - f_{\widehat{c}}(x), \tag{2.233}$$

where $f^*(x) = w^*\mid_{\partial\Omega_1}$. Using Lemma 2.9.1.3 and (2.231), we obtain

$$\left\|f^* - f_{\widehat{c}}\right\|_{C^3(\partial\Omega_1)} \leq B\,|c - c^*|_{\alpha}. \tag{2.234}$$

Next, since $\widetilde{c}(x) = 0$ outside of the domain Ω, then, using the second inequality (2.226) as well as (2.231), we obtain

$$\left\|\overline{s}^2\widetilde{c}(x)\, w_{\widehat{c}}(x,\overline{s})\right\|_{C^{\alpha}(\overline{\Omega}_1)} = \left\|\overline{s}^2\widetilde{c}(x)\, w_{\widehat{c}}(x,\overline{s})\right\|_{C^{\alpha}(\Omega)} \leq B\,|c - c^*|_{\alpha}. \tag{2.235}$$

Hence, applying the Schauder theorem to the Dirichlet boundary value problem (2.232), (2.233) and using (2.234) and (2.235), we obtain

$$|\widetilde{w}|_{2+\alpha} \leq \|\widetilde{w}\|_{C^{2+\alpha}(\overline{\Omega}_1)} \leq B\,|c - c^*|_{\alpha}. \qquad \square$$

2.9.2 The Second Approximate Mathematical Model

Assuming that the asymptotic behavior (2.13) holds, assumption below basically means that we take into account only the first term of the asymptotic behavior of each of functions $V^*(x,\overline{s}), q^*(x,\overline{s})$ at $\overline{s}\to\infty$ and ignore the rest:

$$V^*(x,\overline{s}) = \frac{p^*(x)}{\overline{s}} + O\left(\frac{1}{\overline{s}^2}\right) \approx \frac{p^*(x)}{\overline{s}},\ \overline{s}\to\infty,$$

$$q^*(x,\overline{s}) = -\frac{p^*(x)}{\overline{s}^2} + O\left(\frac{1}{\overline{s}^3}\right) \approx -\frac{p^*(x)}{\overline{s}^2},\ \overline{s}\to\infty.$$

Such assumptions are quite common in science. As an example, we refer to the geometrical optics assumption. Still, our technique is not just geometrical optics since we take into account not only the information at $s := \overline{s}$ but also the lower values of $s \in [\underline{s},\overline{s}]$. In addition, we update tails in the "corrector" procedure, via solving the problems (2.187)–(2.190), which is not the geometrical optics. Just as above, we assume that $\Omega \subset \mathbb{R}^3$ is a convex bounded domain with the boundary $\partial\Omega \in C^3$ and the source $x_0 \notin \overline{\Omega}$.

Recall that (2.105) for the exact function $q^*(x,s)$ is

$$\begin{aligned}\Delta q^* \ &-2s^2\nabla q^* \int_s^{\overline{s}} \nabla q^*(x,\tau)\,d\tau + 2s\left[\int_s^{\overline{s}} \nabla q^*(x,\tau)\,d\tau\right]^2 \\ &+2s^2\nabla q^*\nabla V^* - 4s\nabla V^* \int_s^{\overline{s}} \nabla q^*(x,\tau)\,d\tau + 2s\left(\nabla V^*\right)^2 = 0, \\ x \ &\in \Omega, s\in[\underline{s},\overline{s}].\end{aligned} \tag{2.236}$$

In addition, by (2.106) and (2.108),

$$q^*(x,s) = \psi^*(x,s),\ \forall (x,s) \in \partial\Omega\times[\underline{s},\overline{s}], \tag{2.237}$$

$$q^*(x,s) \in C^{3+\alpha}\left(\overline{\Omega}\right)\times C^1[\underline{s},\overline{s}]. \tag{2.238}$$

The second approximate mathematical model for the algorithm of Sect. 2.6.1 consists of the following:

Assumption. There exists a function $p^*(x) \in C^{2+\alpha}\left(\overline{\Omega}\right)$ such that the exact tail function $V^*(x)$ has the form:

$$V^*(x,s) := \frac{p^*(x)}{s},\ \forall s \geq \overline{s}. \tag{2.239}$$

And also (see (2.193)),

$$\frac{p^*(x)}{\overline{s}} = \frac{\ln w^*(x,\overline{s})}{\overline{s}^2}. \tag{2.240}$$

Since $q^*(x,s) = \partial_s V^*(x,s)$ for $s \geq \overline{s}$, we derive from (2.239) that

$$q^*(x,\overline{s}) = -\frac{p^*(x)}{\overline{s}^2}. \tag{2.241}$$

Set in (2.236), $s = \overline{s}$. Then, using (2.237), (2.239), and (2.241), we obtain the following *approximate* Dirichlet boundary value problem for the function $p^*(x)$:

$$\Delta p^* = 0 \text{ in } \Omega,\ p^* \in C^{2+\alpha}\left(\overline{\Omega}\right), \tag{2.242}$$

$$p^*|_{\partial\Omega} = -\overline{s}^2\psi^*(x,\overline{s}). \tag{2.243}$$

The approximate (2.242) is valid only within the framework of the above assumption. Although (2.242) is linear, formulas (2.107) and (2.117) of the reconstruction of the target coefficient c^* are nonlinear.

Recall that by (2.21),

$$q(x,s) = \psi(x,s),\ \forall (x,s) \in \partial\Omega \times [\underline{s},\overline{s}].$$

Assume that

$$\psi(x,\overline{s}) \in C^{2+\alpha}\left(\overline{\Omega}\right). \tag{2.244}$$

Consider the solution $p(x)$ of the following boundary value problem:

$$\Delta p = 0 \text{ in } \Omega,\ p \in C^{2+\alpha}\left(\overline{\Omega}\right), \tag{2.245}$$

$$p|_{\partial\Omega} = -\overline{s}^2\psi(x,\overline{s}). \tag{2.246}$$

By the Schauder theorem, there exists unique solution p of the problem (2.245) and (2.246). Furthermore, it follows from (2.242)–(2.246) that

$$|p - p^*|_{2+\alpha} \leq K\overline{s}^2 \left\|\psi(x,\overline{s}) - \psi^*(x,\overline{s})\right\|_{C^{2+\alpha}(\partial\Omega)}, \tag{2.247}$$

where $K = K(\Omega) > 1$ is the constant defined in formula (2.125) of Sect. 2.8.2**.** As the first guess for the tail function in the formula (2.43) of Sect. 2.6, we take

$$V_{1,1}(x) := \frac{p(x)}{\overline{s}}. \tag{2.248}$$

Remarks 2.9.2. 1. Let $p(x)$ be the solution of the problem (2.245), (2.246). Substituting (2.248) in (2.41), (2.42) at $n = i = 1$ and setting temporary $q_{1,1} := 0$, one obtains a good approximation for the exact coefficient $c^*(x)$.

Furthermore, Theorem 2.9.4 guarantees that all functions $c_{n,k}$ are good approximations for c^*, as long as the total number of iterations is not too large. Since we find the function $p(x)$ only using the boundary data, then this means that our approximate mathematical model is indeed a good one. Hence, we can stop iterations on any function $c_{n,k}$ for those indices (n,k), which are "allowed" by Theorem 2.9.4. Next, one can use the adaptivity procedure to refine the solution. However, if not using the adaptivity for refinement, then, quite naturally, one needs to find an optimal iteration number to stop. These considerations correspond well with Definitions 1.1.2.1, 1.1.2.2, and they are confirmed numerically in Chaps. 3–6.

2. Because of the approximate nature of our mathematical model, equation (2.242) does not match the asymptotic behavior (2.13). Indeed, actually one should have $|\nabla p^*(x)|^2 = c(x)$. The same can be stated about the Third Approximate Mathematical Model of Chap. 6. Nevertheless, it has been consistently demonstrated that our numerical method works well for both computationally simulated and experimental data, see Chaps. 3–6. Based on our numerical experience, we believe that this is because of two factors: (1) The truncation of the asymptotic series with respect to $1/\overline{s}$ at $\overline{s} \to \infty$ is reasonable, and (2) The procedure of updating tails via solutions of forward problems.

We now establish uniqueness within the framework of our approximate mathematical model.

Lemma 2.9.2. *Let assumption of this section holds. Then for* $(x,s) \in \Omega \times [\underline{s}, \overline{s}]$, *there exists at most one function* $q^*(x,s)$ *satisfying conditions (2.236)–(2.238). In addition, let (2.107) be true, i.e.,*

$$c^*(x) = \left[\Delta v^* + s^2 |\nabla v^*|^2\right](x,s), (x,s) \in \Omega \times [\underline{s}, \overline{s}], \tag{2.249}$$

where

$$v^*(x,s) = -\int_s^{\overline{s}} q^*(x,\tau)\, d\tau + V^*(x,\overline{s}), \tag{2.250}$$

with the tail function $V^*(x,s)$ *satisfying conditions (2.239) and (2.240). Then there exists at most one function* $c^*(x)$.

Proof. It follows from (2.242) and (2.243) that there exists unique function $p^*(x)$ satisfying these conditions. Hence, (2.239) implies uniqueness of the function $V^*(x,\overline{s})$. Below in this proof, $V^* := V^*(x,\overline{s})$. Assume that there exist two functions q_1^* and q_2^*. Let $\widetilde{q} = q_1^* - q_2^*$. Use the formulas

$$a_1 b_1 - a_2 b_2 = \widetilde{a} b_1 + a_2 \widetilde{b},\ \forall a_1, b_1, a_2, b_2 \in \mathbb{R},$$

$$\widetilde{a} = a_1 - a_2,\ \widetilde{b} = b_1 - b_2.$$

Hence, (2.236) and (2.237) lead to

$$\Delta\widetilde{q} - 2s^2\left(\int_s^{\overline{s}} \nabla q_1^*(x,\tau)\,d\tau\right)\nabla\widetilde{q} + 2s^2\nabla V^*\nabla\widetilde{q}$$

$$= -2s\left[\int_s^{\overline{s}} \nabla\left(q_1^* + q_2^*\right)(x,\tau)\,d\tau\right]\int_s^{\overline{s}} \nabla\widetilde{q}(x,\tau)\,d\tau + 4s\nabla V^*\int_s^{\overline{s}} \nabla\widetilde{q}(x,\tau)\,d\tau,$$

$$(x,s) \in \Omega \times [\underline{s},\overline{s}], \quad \widetilde{q}(x,s)\mid_{\partial\Omega} = 0. \tag{2.251}$$

Let

$$M_1 = 2\overline{s}^2 \max_{(x,s)\in\overline{\Omega}\times[\underline{s},\overline{s}]}\left\{\int_s^{\overline{s}}\left[|\nabla q_1^*| + |\nabla q_2^*|\right](x,\tau)\,d\tau\right\},$$

$$M_2 = \max\left(2\overline{s}^2, 4\overline{s}\right)\|V^*\|_{C^1(\overline{\Omega})},$$

$$M_3 = \max\left(M_1, M_2\right),$$

$$M = \max_{s\in[\underline{s},\overline{s}]}\|\widetilde{q}(x,s)\|_{C^2(\overline{\Omega})}.$$

For each fixed value of the parameter $s \in [\underline{s},\overline{s}]$, we consider (2.251) as the Dirichlet boundary value problem for the linear elliptic equation with the same right-hand side as one in (2.251). Then, Schauder theorem and implies that there exists a constant $K_1 = K_1(\Omega, M_3) > 0$ such that

$$\max_{s\in[\underline{s},\overline{s}]}\|\widetilde{q}(x,s)\|_{C^2(\overline{\Omega})} = M \le K_1(\overline{s} - s), \ \forall s \in [\underline{s},\overline{s}].$$

Substituting this in (2.251), we obtain

$$\max_{s\in[\underline{s},\overline{s}]}\|\widetilde{q}(x,s)\|_{C^2(\overline{\Omega})} = M \le K_1^2\int_s^{\overline{s}}(\overline{s} - \tau)\,d\tau = K_1^2\frac{(\overline{s} - s)^2}{2}.$$

Substituting this again in (2.251), we obtain

$$\max_{s\in[\underline{s},\overline{s}]}\|\widetilde{q}(x,s)\|_{C^2(\overline{\Omega})} = M \le \frac{1}{2}K_1^2\int_s^{\overline{s}}(\overline{s} - \tau)^2\,d\tau = K_1^3\frac{(\overline{s} - s)^2}{3!}.$$

Continuing this process, we obtain

$$\max_{s\in[\underline{s},\overline{s}]} \|\widetilde{q}(x,s)\|_{C^2(\overline{\Omega})} = M \leq K_1^n \frac{(\overline{s}-s)^n}{n!}.$$

Setting here, $n \to \infty$ leads to $M = 0$. Hence, (2.249) and (2.250) imply that the function $c^*(x)$ is also unique. □

2.9.3 Preliminaries

The goal of this and next sections is to prove the theorem about the approximate global convergence property within the framework of the second approximate mathematical model of Sect. 2.9.2. We assume that in (2.39) and (2.256),

$$2\frac{I_{1,n}}{I_0}(\nabla q_n)^2 := 0. \tag{2.252}$$

Therefore, we set

$$2\frac{I_{1,n}}{I_0}(\nabla q_{n,k})^2 := 0. \tag{2.253}$$

The Assumption (2.252) can be justified by (2.40) via choosing the parameter $\lambda >> 1$, which we do in our computations. We point out that an analog of Theorem 2.9.4 can be proven similarly even without (2.252). We are not doing so here only because we want to simplify the presentation. Assumptions (2.252) and (2.253) do not mean a linearization of the original problem, since the nonlinearity surfaces in terms $\nabla q_j \nabla q_{n,i}$ in (2.49). Also, tails $V_{n,i}$ in (2.49) depend nonlinearly on functions $q_j,\ j \in [0, n-1]$.

Assume that in (2.34), functions $\overline{\psi}_n \in C^{2+\alpha}(\partial\Omega)$. Then by (2.119),

$$\left\|\overline{\psi}_n(x) - \overline{\psi}_n^*(x)\right\|_{C^{2+\alpha}(\partial\Omega)} \leq C^*(h+\sigma). \tag{2.254}$$

Recall that by (2.114)–(2.118) we have for $x \in \Omega$

$$\Delta q_n^* - A_{1,n}\left(h\sum_{j=0}^{n-1}\nabla q_j^*\right)\nabla q_n^* + A_{1,n}\nabla q_n^*\nabla V^* = 2\frac{I_{1,n}}{I_0}\left(\nabla q_n^*\right)^2$$
$$- A_{2,n}h^2\left(\sum_{i=1}^{n-1}\nabla q_i^*\right)^2 + 2A_{2,n}\nabla V^*\left(h\sum_{j=0}^{n-1}\nabla q_j^*\right) - A_{2,n}\left|\nabla V^*\right|^2 + F_n(x,h,\lambda), \tag{2.255}$$

$$F_n(x,h,\lambda) \in C^{\alpha}\left(\overline{\Omega}\right), \ \max_{\lambda h \geq 1} |F_n(x,h,\lambda)|_{\alpha} \leq C^* h, \tag{2.256}$$

$$v_n^*(x) = -hq_n^*(x) - h\sum_{j=0}^{n-1} q_j^*(x) + V^*(x), \ x \in \Omega, \ n \in [1,N], \tag{2.257}$$

$$c^*(x) = \left[\Delta v_n^* + s_n^2 |\nabla v_n^*|^2\right](x) + \overline{F}_n(x), \ n \in [1,N], \tag{2.258}$$

$$\left|\overline{F}_n\right|_{\alpha} \leq C^* h. \tag{2.259}$$

By (2.254), Eq. (2.49) and the boundary condition (2.50) become:

$$\Delta q_{n,k} - A_{1n}\left(h\sum_{j=0}^{n-1}\nabla q_j\right)\nabla q_{n,k} - \varkappa q_{n,k} + A_{1n}\nabla V_{n,k}\nabla q_{n,k}$$
$$= -A_{2n}h^2\left(\sum_{j=0}^{n-1}\nabla q_j\right)^2 + 2A_{2n}\nabla V_{n,k}\left(h\sum_{j=0}^{n-1}\nabla q_j\right) - A_{2n}\left(\nabla V_{n,k}\right)^2, \ x \in \Omega, \tag{2.260}$$

$$q_{n,k}(x) = \overline{\psi}_n(x), x \in \partial\Omega, \tag{2.261}$$

where $\varkappa \in (0,1)$ is a small parameter of ones choice. Recall that by (2.41) and (2.42), $q_0(x) \equiv q_0^*(x) \equiv 0$,

$$v_{n,k}(x) = -hq_{n,k}(x) - h\sum_{j=0}^{n-1} q_j(x) + V_{n,k}(x), \ x \in \Omega, \ n \in [1,N], \tag{2.262}$$

$$c_{n,k}(x) = \left[\Delta v_{n,k} + s_n^2\left(\nabla v_{n,k}\right)^2\right](x), \ x \in \Omega, \ n \in [1,N]. \tag{2.263}$$

We now reformulate the estimate of the Schauder theorem of Sect. 2.8.2 since we impose now upper estimates on the coefficients of the elliptic equation, which are different from ones imposed in Sect. 2.8.2. Just as in Sect. 2.8.2, consider the Dirichlet boundary value problem

$$\Delta u + \sum_{j=1}^{3} b_j(x) u_{x_j} - b_0(x)u = f(x), x \in \Omega, u\mid_{\partial\Omega} = g(x) \in C^{2+\alpha}(\partial\Omega). \tag{2.264}$$

Assume that the following conditions are satisfied,

$$b_j, b_0, f \in C^{\alpha}\left(\overline{\Omega}\right), \ b_0(x) \geq 0, \ \max_{j \in [0,n]}\left(\left|b_j\right|_{\alpha}\right) \leq Q, \tag{2.265}$$

where $Q > 0$ is a certain constant. Then by the Schauder theorem (see Chap. 3, Sect. 1 in [118]), there exists unique solution $u \in C^{2+\alpha}\left(\overline{\Omega}\right)$ of the boundary value problem (2.264). Furthermore, there exists a constant $\overline{K} = \overline{K}(\Omega, Q) > 2$, depending only on the domain Ω and the constant Q such that the following estimate holds:

$$|u|_{2+\alpha} \leq \overline{K}\left[\|g\|_{C^{2+\alpha}(\partial\Omega)} + |f|_{\alpha}\right]. \tag{2.266}$$

2.9.4 The Second Approximate Global Convergence Theorem

Let N be the total number of functions q_n computed in the algorithm of Sect. 2.6.1. In principle, $\widetilde{N} \in (1, N]$. However, to avoid new notations, we denote for brevity $\widetilde{N} := N$. Keeping this in mind, we assume in Theorem 2.9.4 that the total number N of functions q_n of the algorithm of 2.6.1 is independent on the grid step size h in the s-direction. In addition, the number m_n of functions $\{q_{n,k}\}_{k=1}^{m_n}$ is bounded from the above:

$$\max_{n \in [1,N]} m_n = m. \tag{2.267}$$

Condition (2.280) of Theorem 2.9.4 provides a linkage between the level of the error η in the data and the total "allowable" number of iterations Nm, i.e., the allowable number of functions $\{c_{n,k}\}_{(n,k)=(1,1)}^{(N,m)}$. This is going along well with the theory of ill-posed problems. Indeed, it is well known that the maximal number of iterations and the error in the data are often connected with each other. So that the maximal number of iterations is a regularization parameter in this case, see pp. 156 and 157 of [65] as well as Sect. 1.6. Hence, Theorem 2.9.4 provides another example of such a connection, in addition to those of [65] and Sect. 1.6.

Theorem 2.9.4. *Consider the algorithm of Sect. 2.6.1. As in Theorem 2.9.1.2, let* $\Omega, \Omega_1 \subset \mathbb{R}^3$ *be two convex bounded domain with the boundaries* $\partial\Omega, \partial\Omega_1 \in C^3$ *and let condition (2.215) hold. Let the maximal pseudo frequency* $\overline{s} = \text{const.} > 1$*:*

- *Let assumptions of Sect. 2.9.2 be valid, the number* N *of functions* $\{q_n\}_{n=1}^{N}$ *be independent on the grid step size* h *of the partition of the* s *interval, and (2.267) holds.*
- *In addition, assume that all functions* $c_{n,k}(x)$ *in (2.263) are such that*

$$c_{n,k}(x) \geq 1, x \in \Omega. \tag{2.268}$$

- *Let in (2.34) and (2.35) functions* $\overline{\psi}_n \in C^{2+\alpha}(\partial\Omega)$.
- *Let the function* $c^*(x)$ *satisfying conditions (2.206) and (2.209) be the exact solution of Inverse Problem 2.1, where constants* $d, \overline{d} > 1$ *are given, where* $\Omega' \subset \Omega$ *is a subdomain of the domain* Ω, $\partial\Omega' \cap \partial\Omega = \varnothing$, *and let* $\chi(x)$ *be the cut-off function defined in (2.207).*

- *Assume that conditions (2.120), (2.244), (2.252)–(2.254), (2.259), and (2.267) hold, where the constant $C^* \geq 1$ is defined in (2.112).*
- *Let the first tail function $V_{1,1}(x)$ be constructed via (2.245), (2.246), and (2.248).*
- *Let h be the grid step size in the layer-stripping procedure with respect to s, σ be level of the error in the data, and $\varkappa \in (0,1)$ be a small parameter in (2.260). Denote*

$$\eta = 2(h + \sigma + \varkappa). \tag{2.269}$$

- *Choose the parameter λ of the CWF (2.38) so large that*

$$\lambda \geq \frac{8(\overline{s}C^*)^2}{\eta}. \tag{2.270}$$

- *Let $B = B\left(\Omega, \Omega_1, \overline{s}, d, \overline{d}, \chi, x_0\right) > 2$ be the constant of Theorem 2.9.1.2*

Then there exists a constant $B_1 = B_1\left(\Omega, \Omega_1, \overline{s}, d, \overline{d}, C^*, \chi, x_0\right) \geq B > 2$ such that if $\overline{K} = \overline{K}\left(\overline{s}^2 B_1\right) > 2$ is the constant in (2.266) and the parameter η is so small that

$$\eta \in (0, \eta_0), \ \eta_0 = \frac{1}{\overline{K} N B_1^{3Nm}}, \tag{2.271}$$

then functions

$$c_{n,k} \in C^\alpha\left(\overline{\Omega}\right), \widehat{c}_{n,k} \in C^\alpha\left(\mathbb{R}^3\right), \ (n,k) \in [1,N] \times [1,m], \tag{2.272}$$

$$c_{n,k}(x), \widehat{c}_{n,k}(x) \in [1, d+1] \ \text{in } \Omega, \ (n,k) \in [1,N] \times [1,m]. \tag{2.273}$$

In particular, all functions $c_{n,k} \in P\left(d, \overline{d}\right)$, where the set of functions $P\left(d, \overline{d}\right)$ is defined in (2.208). In addition, the following estimates hold for $(n,k) \in [1,N] \times [1,m]$:

$$|\nabla V_{n,k}|_{1+\alpha}, |\Delta V_{n,k}|_\alpha \leq B_1, \tag{2.274}$$

$$|\nabla V_{n,k} - \nabla V^*|_{1+\alpha} \leq B_1^{3[k-1+(n-1)m]+1} \cdot \eta, \tag{2.275}$$

$$|\Delta V_{n,k} - \Delta V^*|_\alpha \leq B_1^{3[k-1+(n-1)m]+1} \cdot \eta, \tag{2.276}$$

$$|q_{n,k} - q_n^*|_{2+\alpha} \leq \overline{K} B_1^{3[k+(n-1)m]} \cdot \eta, \tag{2.277}$$

$$|q_{n,k}|_{2+\alpha} \leq 2C^*, \ n \in [1,N], \tag{2.278}$$

$$|c_{n,k} - c^*|_\alpha \leq B_1^{3[k+(n-1)m]} \cdot \eta. \tag{2.279}$$

Denote

$$\omega = \frac{\ln\left(\overline{K}N\right)}{3Nm\ln B_1 + \ln\left(\overline{K}N\right)} \in (0,1)\,. \tag{2.280}$$

Then (2.279) becomes

$$|c_{n,k} - c^*|_\alpha \le \eta^\omega := \varepsilon, \tag{2.281}$$

where the number $\varepsilon \in (0,1)$. Hence, the algorithm of Sect. 2.6.1 possesses the approximate globally convergent property of the level ε in the framework of the second approximate mathematical model of Sect. 2.9.2.

Remarks 2.9.4. 1. Since $\omega \in (0,1)$, then (2.281) is a Hölder-like estimate. We impose condition (2.268) to ensure that the right inequality (2.192) holds for all functions $c_{n,k}$. Indeed, we use the latter inequality quite extensively in Sect. 2.9.4. We have observed computationally that (2.268) holds; see Sect. 3.1.2.

2. The fact that the constant B_1 depends not only on the domain Ω but also on the domain Ω_1 as well does not affect the approximate global convergence property. It follows from (2.281) and (2.280) that as long as total iteration number Nm is not too large, conditions of the approximate global convergence of Definition 1.1.2.1 are satisfied. Hence, one can take any function $c_{n,k}(x)$ as $c_{\text{glob}}(x)$. The question of an optimal choice of the pair (n,k) should be decided in numerical experiments.
3. Theorem 2.9.4 implies that $P\left(d,\overline{d}\right)$ is our correctness set for the second approximate mathematical model, see Definitions 1.4.2 and 1.4.2 for the correctness set.
4. It is hard to establish a priori the upper limit for the number N in practical computations. This is the reason why we have consistently observed in our numerical tests that certain numbers indicating convergence grow steeply for $n \ge \overline{N}$ with a number $\overline{N} < N$, while they stabilize a few iterations before $\overline{N}$, i.e., at $n = \widetilde{N} < \overline{N}$. This phenomenon means that the process should be stopped at $n = \widetilde{N}$. The third Remark 1.1.2.1 is relevant here.

Proof of Theorem 2.9.4. The estimate (2.281) follows from (2.271) and (2.280). Thus, we focus below on the proof of estimates (2.272)–(2.279). Denote

$$\widetilde{V}_{n,k} = V_{n,k} - V^*,\ \widetilde{q}_{n,k} = q_{n,k} - q_n^*,$$
$$\widetilde{v}_{n,k} = v_{n,k} - v_{n,k}^*,\ \widetilde{c}_{n,k} = c_{n,k} - c^*,\ \widetilde{\psi}_n = \psi_n - \psi_n^*.$$

Estimates (2.274)–(2.276) for functions $V_{1,1}, \widetilde{V}_{1,1}$ follow from (2.111), (2.239)–(2.241) and (2.244)–(2.247).

Assume for a moment that the estimate (2.279) is valid. Then the function $c_{n,k} \in P(d,d^*)$. Indeed, by (2.209), (2.271), and (2.279),

$$|c_{n,k}|_\alpha = |c_{n,k} - c^* + c^*|_\alpha \le |c_{n,k} - c^*|_\alpha + |c^*|_\alpha$$
$$\le B_1^{3[k+(n-1)m]}\eta + \overline{d} < \overline{d} + 1.$$

Similarly, $c_{n,k} \leq d + 1$. These two estimates combined with (2.208) and (2.268) imply that $c_{n,k} \in P\left(d, \overline{d}\right)$. Next, since the function $c_{n,k} \in P\left(d, \overline{d}\right)$, then Theorem 2.9.1.2 implies (2.274). Also, since the function $c_{n,k} \in [1, d+1]$, then the function $\widehat{c}_{n,k} \in [1, d+1]$ as well; see Sect. 2.6. Thus, if (2.279) is valid, then (2.273) is valid as well.

We now prove (2.275)–(2.279) for $(n,k) = (1,1)$. Set in (2.255) and (2.260) $(n,k) = (1,1)$. Subtracting (2.255) from (2.260), we obtain

$$\Delta\widetilde{q}_{1,1} + A_{1,1}\nabla V_{1,1}\nabla\widetilde{q}_{1,1} - \varkappa\widetilde{q}_{1,1} = -A_{1,1}\nabla\widetilde{V}_{1,1}\nabla q_1^* \\ -A_{2,1}\nabla\widetilde{V}_{1,1}\left(\nabla V_{1,1} + \nabla V^*\right) + \varkappa q_1^* - \widehat{F}_1, \tag{2.282}$$

$$\widetilde{q}_{1,1}(x) = \widetilde{\psi}_1(x), x \in \partial\Omega, \tag{2.283}$$

$$\widehat{F}_n = F_n - \frac{2I_{1,n}}{I_0}\left(\nabla q_n^*\right)^2, \; n \in [1, N]. \tag{2.284}$$

Recall that by (2.111) and (2.112),

$$\max_{n\in[1,N]} |q_n^*|_{2+\alpha} \leq C^*, \; C^* \geq 1. \tag{2.285}$$

Hence, (2.142), (2.256), (2.269), (2.270), (2.284), and (2.285) imply that

$$\left|\widehat{F}_n\right|_\alpha \leq C^*\eta. \tag{2.286}$$

Estimate now the right-hand side of (2.282). Using (2.121), (2.223), (2.224), (2.269), (2.286) as well as (2.274) and (2.275) at $(n,k) = (1,1)$, we obtain

$$\left|A_{1,1}\nabla\widetilde{V}_{1,1}\nabla q_1^* + A_{2,1}\nabla\widetilde{V}_{1,1}\left(\nabla V_{1,1} + \nabla V^*\right) + \varkappa q_1^* - \widehat{F}_1\right|_\alpha \\ \leq 8\overline{s}^2 BC^*\eta + 16\overline{s}^2 B^2\eta + 2C^*\eta = 8\overline{s}^2 B\left(2B + C^* + \frac{C^*}{4\overline{s}^2}\right)\eta.$$

We choose the constant $B_1 = B_1\left(\Omega, \Omega_1, \overline{s}, d, \overline{d}, C^*, \chi, x_0\right) \geq B > 2$ such that

$$C^* + 2B + \frac{C^*}{4\overline{s}^2} \leq 2B\left(1 + \frac{C^*}{B}\right) \leq 3B_1. \tag{2.287}$$

By (2.287),

$$C^* < \frac{B_1}{2}. \tag{2.288}$$

Hence, it follows from (2.287) that

$$\left|A_{1,1}\nabla\widetilde{V}_{1,1}\nabla q_1^* + A_{2,1}\nabla\widetilde{V}_{1,1}\left(\nabla V_{1,1} + \nabla V^*\right) + \varkappa q_1^* - \widehat{F}_1\right|_\alpha \leq 24\overline{s}^2 B_1^2\eta. \tag{2.289}$$

Next, consider coefficients in the left-hand side of (2.282). We have

$$|A_{1,1}\nabla V_{1,1}| \leq 8\overline{s}^2 B_1,\ \varkappa \in (0,1).$$

Hence, conditions (2.265) are satisfied. Hence, it follows from (2.266) and (2.289) that the solution of the Dirichlet boundary value problem (2.282), (2.283) can be estimated as

$$|\widetilde{q}_{1,1}|_{2+\alpha} \leq 24\overline{s}^2\overline{K}B_1^2\eta + \overline{K}\left\|\widetilde{\psi}_1\right\|_{C^{2+\alpha}(\partial\Omega)}.$$

Using (2.254) and (2.269), we obtain from this inequality and (2.288)

$$|\widetilde{q}_{1,1}|_{2+\alpha} \leq \overline{K}B^2\left(24\overline{s}^2 + \frac{C^*}{2B_1^2}\right)\eta \leq \overline{K}B_1^2\left(24\overline{s}^2 + \frac{1}{8}\right)\eta \leq 25\overline{s}^2\overline{K}B_1^2\eta.$$

In addition to (2.287), we can assume without any loss of generality that

$$40\overline{s}^2 \leq B_1. \tag{2.290}$$

Hence,

$$|\widetilde{q}_{1,1}|_{2+\alpha} \leq \overline{K}B_1^3\eta. \tag{2.291}$$

Estimate (2.291) establishes (2.277) for the function $\widetilde{q}_{1,1}$. Next, using (2.271), (2.285), and (2.291), we obtain

$$|q_{1,1}|_{2+\alpha} \leq |\widetilde{q}_{1,1}|_{2+\alpha} + |q_1^*|_{2+\alpha} \leq \overline{K}B_1^3\eta + C^* \leq 2C^*. \tag{2.292}$$

This establishes (2.278) for $|q_{1,1}|_{2+\alpha}$.

Now, we estimate the norm $|\widetilde{c}_{1,1}|_\alpha$. Subtracting (2.258) from (2.263) for $(n,k) = (1,1)$, we obtain

$$\widetilde{c}_{1,1} = \Delta\widetilde{v}_{1,1} + s_n^2\nabla\widetilde{v}_{1,1}\left(\nabla v_{1,1} + \nabla v_1^*\right) - \overline{F}_1. \tag{2.293}$$

Since by (2.257) and (2.262), the function $\widetilde{v}_{1,1} \in C^{2+\alpha}\left(\overline{\Omega}\right)$, then it follows from (2.293) that the function $\widetilde{c}_{1,1} \in C^\alpha\left(\overline{\Omega}\right)$. Since $c^* \in C^\alpha\left(\overline{\Omega}\right)$ as well, then also $c_{1,1} \in C^\alpha\left(\overline{\Omega}\right)$, which establishes (2.272) for $(n,k) = (1,1)$. Hence, taking into account the estimate (2.259) for the function $\overline{F}_1$, we obtain from (2.293)

$$|\widetilde{c}_{1,1}|_\alpha \leq \max\left(|\Delta\widetilde{v}_{1,1}|_\alpha, |\nabla\widetilde{v}_{1,1}|_\alpha\right)\left[1 + \overline{s}^2\left(|\nabla v_{1,1}|_\alpha + |\nabla v_1^*|_\alpha\right)\right] + \frac{C^*}{2}\eta. \tag{2.294}$$

By (2.257) and (2.262),

$$\widetilde{v}_{1,1} = -h\widetilde{q}_{1,1} + \widetilde{V}_{1,1}.$$

Hence, it follows from (2.269), (2.271), (2.287), (2.291) as well as from (2.275) and (2.276) at $(n,k) = (1,1)$ that

$$|\Delta\widetilde{v}_{1,1}|_{\alpha}, |\nabla\widetilde{v}_{1,1}|_{\alpha} \leq \frac{1}{2}\overline{K} B_1^3 \eta^2 + B_1\eta \leq 2B_1\eta. \tag{2.295}$$

Next, using (2.223), (2.257), (2.262), (2.269) and (2.274) at $(n,k) = (1,1)$ and (2.292), we obtain

$$1 + \overline{s}^2 \left(|\nabla v_{1,1}|_{\alpha} + |\nabla v_1^*|_{\alpha} \right) \leq 1 + \overline{s}^2 \left(2C^*\eta + 2B_1 \right) \leq 4\overline{s}^2 B_1. \tag{2.296}$$

Hence, comparing this with (2.288), (2.290), (2.294), and (2.295), we obtain

$$|\widetilde{c}_{1,1}|_{\alpha} \leq 9\overline{s}^2 B_1^2 \eta \leq B_1^3 \eta. \tag{2.297}$$

This establishes (2.279) for $(n,k) = (1,1)$. Hence, using Theorem 2.9.1.2 and (2.297), we obtain estimates (2.274)–(2.276) for the tail function at $(n,k) = (1,2)$:

$$|\nabla V_{1,2}|_{1+\alpha}, |\Delta V_{1,2}|_{\alpha} \leq B_1,$$

$$|\nabla V_{1,2} - \nabla V^*|_{1+\alpha} \leq B_1^4 \eta, \ |\Delta V_{1,2} - \Delta V^*|_{\alpha} \leq B_1^4 \eta.$$

Recall that by the algorithm of Sect. 2.6.1,

$$q_n := q_{n,m_n}, c_n := c_{n,m_n},$$

$$V_{n+1,1}(x,\overline{s}) = \frac{1}{\overline{s}^2} \ln w_{n,m_n}(x,\overline{s}).$$

Also, recall that by (2.267) $m_n \in [1,m]$. Having functions q_n and $V_{n+1,1}(x,\overline{s})$, we calculate next the function $q_{n+1,1}$. Also, recall that $q_0 = q_0^* = 0$. Thus, for the convenience of the mathematical induction, we temporary set $q_{n,0} := q_{n-1}$ for $n \geq 1$ and also $c_0 := c^*$, $V_{0,0} := V_{1,1}$. Hence, (2.272)–(2.279) are valid for $(n,k) = (0,0)$. In addition, since we have established (2.272)–(2.279) for $(n,k) = (1,1)$, we can assume now that we have proved (2.272)–(2.279) for $(n',k') \in [0,n] \times [0,k-1]$, where $k \geq 2$. We now want to prove (2.272)–(2.279) for $(n',k') = (n,k)$.

Subtracting (2.255) from (2.260), we obtain

$$\Delta\widetilde{q}_{n,k} - A_{1,n}\left(h\sum_{j=0}^{n-1}\nabla q_j\,(x)\right)\nabla\widetilde{q}_{n,k} + A_{1,n}\nabla V_{n,k}\cdot\nabla\widetilde{q}_{n,k} - \varkappa\widetilde{q}_{n,k}$$

$$= \left(A_{1,n}\nabla q_n^* - A_{2,n}h\sum_{j=0}^{n-1}\left(\nabla q_j + \nabla q_j^*\right) + 2A_{2,n}\nabla V_{n,k}\right)\left(h\sum_{j=0}^{n-1}\nabla\widetilde{q}_j\right)$$

$$+\left[2A_{2,n}h\sum_{j=0}^{n-1}\nabla q_j^* - A_{1,n}\nabla q_n^* - A_{2,n}\left(\nabla V_{n,k} + \nabla V^*\right)\right]\nabla\widetilde{V}_{n,k} + \varkappa q_n^* - \widehat{F}_n, \tag{2.298}$$

$$\widetilde{q}_{n,i}\;|_{\partial\Omega} = \widetilde{\psi}_n(x). \tag{2.299}$$

The function $\widehat{F}_n$ is defined in (2.284), and the estimate (2.286) is valid. First, we estimate the difference of tails $\widetilde{V}_{n,k}$. Since estimates (2.272)–(2.279) are valid for $(n',k') \in [0,n]\times[0,k-1]$, then by Theorem 2.9.1.2,

$$|\nabla V_{n,k}|_{1+\alpha}, |\Delta V_{n,k}|_{\alpha} \le B_1,$$

$$\left|\nabla\widetilde{V}_{n,k}\right|_{1+\alpha} \le B\,|\widetilde{c}_{n,k-1}|_{\alpha} \le B_1 B_1^{3[k-1+(n-1)m]}\cdot\eta = B_1^{3[k-1+(n-1)m]+1}\cdot\eta,$$

$$\left|\Delta\widetilde{V}_{n,k}\right|_{\alpha} \le B^{3[k+(n-1)m]+1}\cdot\eta.$$

The last three estimates establish (2.274)–(2.276) for $(n',k') = (n,k)$.

We now need to estimate the right-hand side of (2.298) using (2.121) as well as above established estimates. We have

$$\left|A_{1,n}\nabla q_n^* - A_{2,n}h\sum_{j=0}^{n-1}\left(\nabla q_j + \nabla q_j^*\right) + 2A_{2,n}\nabla V_{n,k}\right|_{\alpha}$$

$$\le 8\overline{s}^2\left(C^* + 3C^*Nh + 2B_1\right) \le 8\overline{s}^2\left(C^* + 1 + 2B_1\right).$$

Since $B_1 > 2$, then this inequality and (2.288) imply that

$$\left|A_{1,n}\nabla q_n^* - A_{2,n}h\sum_{j=0}^{n-1}\left(\nabla q_j + \nabla q_j^*\right) + 2A_{2,n}\nabla V_{n,k}\right|_{\alpha} \le 25\overline{s}^2 B_1. \tag{2.300}$$

Next, since estimates (2.277) are valid for functions $\widetilde{q}_j = q_j - q_j^*$, $j \in [0, n-1]$, then using (2.271), we obtain

$$\left| h \sum_{j=0}^{n-1} \nabla \widetilde{q}_j \right|_\alpha \leq \frac{1}{2} \overline{K} B_1^{3Nm} N \eta^2 \leq \frac{\eta}{2}.$$

Hence, using (2.300), we obtain the following estimate for the first term in the right-hand side of (2.298):

$$\left| A_{1,n} \nabla q_n^* - A_{2,n} h \sum_{j=0}^{n-1} \left(\nabla q_j + \nabla q_j^* \right) + 2 A_{2,n} \nabla V_{n,k} \right|_\alpha \left| h \sum_{j=0}^{n-1} \nabla \widetilde{q}_j \right|_\alpha \leq 14 \overline{s}^2 B_1 \eta. \tag{2.301}$$

Next, using (2.121), (2.223), (2.271), and (2.274), we obtain

$$\left| 2 A_{2,n} h \sum_{j=0}^{n-1} \nabla q_j^* - A_{1,n} \nabla q_n^* - A_{2,n} \left(\nabla V_{n,k} + \nabla V^* \right) \right|_\alpha$$
$$\leq 4 \overline{s}^2 C^* N \eta + 8 \overline{s}^2 C^* + 16 \overline{s}^2 B_1 \leq \frac{1}{4} \overline{s}^2 C^* + 4 \overline{s}^2 B_1 + 16 \overline{s}^2 B_1 \leq 21 \overline{s}^2 B_1.$$

Hence,

$$\left| 2 A_{2,n} h \sum_{j=0}^{n-1} \nabla q_j^* - A_{1,n} \nabla q_n^* - A_{2,n} \left(\nabla V_{n,k} + \nabla V^* \right) \right|_\alpha \left| \nabla \widetilde{V}_{n,k} \right|_\alpha + \left| \varkappa q_n^* - \widehat{F}_n \right|_\alpha$$
$$\leq 21 \overline{s}^2 B_1 B_1^{3[k-1+(n-1)m]+1} \cdot \eta + \frac{3}{2} C^* \eta \leq 21 \overline{s}^2 B_1 B_1^{3[k-1+(n-1)m]+1} \eta + B_1 \eta.$$

Combining this with (2.301), we obtain

$$\begin{aligned} |rhs|_\alpha &\leq 21 \overline{s}^2 B_1 B_1^{3[k-1+(n-1)m]+1} \cdot \eta + 15 \overline{s}^2 B_1 \eta \\ &= 21 \overline{s}^2 B_1 B_1^{3[k-1+(n-1)m]+1} \left(1 + \frac{15}{22 B_1} \right) \eta \\ &\leq 21 \overline{s}^2 B_1 B_1^{3[k-1+(n-1)m]+1} \left(1 + \frac{1}{2} \right) \eta \\ &= 32 \overline{s}^2 B_1 B_1^{3[k-1+(n-1)m]+1} \cdot \eta, \end{aligned}$$

where rhs is the right-hand side of (2.298). Thus,

$$|rhs|_\alpha \leq 32 \overline{s}^2 B_1 B_1^{3[k-1+(n-1)m]+1} \cdot \eta. \tag{2.302}$$

We now estimate coefficients in the left-hand side of (2.298) using (2.271), (2.288), as well as (2.278) for functions q_j with $j \in [0, n-1]$. The first resulting estimate is

$$\left| A_{1,n} \left(h \sum_{j=0}^{n-1} \nabla q_j (x) \right) \right|_{\alpha} \leq 12\overline{s}^2 C^* N \eta \leq \frac{6\overline{s}^2}{\overline{K} B_1^{3Nm-1}} \leq \frac{3}{4} \overline{s}^2. \tag{2.303}$$

Next, by (2.121) and (2.274),

$$|A_{1,n} \nabla V_{n,k}|_{\alpha} \leq 8\overline{s}^2 B_1. \tag{2.304}$$

Hence, it follows from (2.303) and (2.304) that condition (2.265) is satisfied for (2.298). Hence, (2.254), (2.266), (2.288), (2.299), and (2.302) imply that

$$\begin{aligned} |\widetilde{q}_{n,k}|_{2+\alpha} &\leq \overline{K} \left[32\overline{s}^2 B_1 B_1^{3[k-1+(n-1)m]+1} + \frac{C^*}{2} \right] \eta \\ &\leq \overline{K} \cdot 40\overline{s}^2 B B^{3[k-1+(n-1)m]+1} \cdot \eta. \end{aligned}$$

Since by (2.290), $40\overline{s}^2 \leq B$, then the last estimate implies that

$$|\widetilde{q}_{n,k}|_{2+\alpha} \leq \overline{K} B^{3[k+(n-1)m]} \cdot \eta,$$

which proves (2.277). The inequality (2.278) can be derived from (2.277) similarly with the derivation of (2.292).

Estimate now the norm $|\widetilde{c}_{n,k}|_{\alpha}$. Using (2.288), we obtain similarly with (2.294)

$$|\widetilde{c}_{n,k}|_{\alpha} \leq \max \left(|\Delta \widetilde{v}_{n,k}|_{\alpha}, |\nabla \widetilde{v}_{n,k}|_{\alpha} \right) \left[1 + \overline{s}^2 \left(|\nabla v_{n,k}|_{\alpha} + |\nabla v_n^*|_{\alpha} \right) \right] + \frac{B_1}{4} \eta. \tag{2.305}$$

We have

$$\widetilde{v}_{n,k}(x) = -h\widetilde{q}_{n,k}(x) - h \sum_{j=0}^{n-1} \widetilde{q}_j(x) + \widetilde{V}_{n,k}(x), \ x \in \Omega.$$

Hence, by (2.271), (2.275), and (2.277)

$$\begin{aligned} |\Delta \widetilde{v}_{n,k}|_{\alpha}, |\nabla \widetilde{v}_{n,k}|_{\alpha} &\leq \frac{1}{2} \overline{K} N B_1^{3[k+(n-1)m]} \eta^2 + B_1^{3[k-1+(n-1)m]+1} \eta \\ &\leq \frac{5}{4} B_1^{3[k-1+(n-1)m]+1} \cdot \eta. \end{aligned} \tag{2.306}$$

Next, using expressions (2.257) and (2.262) for functions $v_{n,k}$ and v_n^*, we obtain

$$1+\overline{s}^2\left(|\nabla v_{n,k}|_\alpha+|\nabla v_n^*|_\alpha\right)\leq 1+\frac{3}{2}\overline{s}^2C^*N\eta+2\overline{s}^2B_1\leq 3\overline{s}^2B_1.$$

Combining this with (2.305) and (2.306) and taking into account (2.290), we obtain

$$\begin{aligned}|\widetilde{c}_{n,k}|_\alpha &\leq 4\overline{s}^2B_1B_1^{3[k-1+(n-1)m]+1}\cdot\eta+\frac{B_1}{4}\eta\leq 5\overline{s}^2B_1B_1^{3[k-1+(n-1)m]+1}\cdot\eta\\ &\leq B_1^2B_1^{3[k-1+(n-1)m]+1}\cdot\eta=B_1^{3[k+(n-1)m]}\cdot\eta.\end{aligned}$$

Thus, $|\widetilde{c}_{n,k}|_\alpha\leq B^{3[k+(n-1)m]}\eta$. This establishes (2.279). □

2.10 Summary

One can see from Theorem 2.9.4 that the accuracy of the reconstruction strongly depends from the accuracy of the reconstruction of the tail functions. On the other hand, it follows from the second approximate mathematical model that the first tail function $V_{1,1}(x)$ is proportional to the solution of the Dirichlet boundary value problem for the Laplace equation; see (2.245) and (2.246) in Sect. 2.9.2. Therefore, it follows from estimate (2.247) that as long as the noise in the boundary data is small, the function $V_{1,1}(x)$ is reconstructed accurately. On the other hand, the accuracy of the reconstruction of other tail functions $V_{n,k}(x)$ depends on the accuracy of the reconstruction of the function $V_{1,1}(x)$. This explains why the approximately globally convergent algorithm of Sect. 2.9.4 works well numerically; see Chaps. 3–5 for computational studies. The "small noise" assumption is a natural one which is used in almost all numerical methods.

Thus, all what our approximately globally convergent numerical method requires is that the noise in the boundary data should be small. Under this assumption, we have a rigorous guarantee, within the framework of the second approximate mathematical model, that our resulting solution will be located in a small neighborhood of the exact solution. The size of this neighborhood is completely defined by the "noise" parameter η in (2.269), as it is conventionally done in standard convergence theorems. It is important that no a priori knowledge of any point in a small neighborhood of the exact solution is required. Therefore, the approximately globally convergent numerical method of this chapter indeed addresses the first central question of this book (Sect. 1.1).

Now, about the constants in convergence estimates of Theorems 2.8.2 and 2.9.4. They are probably large. However, this is not a discouraging factor. Indeed, it is well known that constants in almost all convergence estimates of numerical analysis are largely over-estimated for both well-posed and ill-posed problems. Consider, for example, standard energy estimates for classical initial boundary value problems for hyperbolic and parabolic PDEs with variable coefficients and non-self-adjoint elliptic operators [119, 120]. The final step of these estimates usually consists in the application of the Gronwall's theorem. It is well known that this theorem implies that constants in those estimates are bounded from the above by $C_1 := \exp(CT)$, where T is the final time and $C > 0$ is a constant depending on coefficients of the corresponding PDE as well as on the spatial domain. Thus, the number C_1 is expected to be sufficiently large. On the other hand, it is well known that convergence estimates for both finite difference and FEMs are based on those energy estimates.