Chapter 1
Two Central Questions of This Book and an Introduction to the Theories of Ill-posed and Coefficient Inverse Problems

This is an introductory chapter. In Sect. 1.1, we outline two central questions discussed in this book. Sections 1.2–1.9 are introductory ones to the theory of ill-posed problems. In Sects. 1.10 and 1.11, we present main uniqueness results for coefficient inverse problems (CIPs) with the single measurement data. The material of this chapter might serve as an introductory course for theories of ill-posed and CIPs. We refer to books [7, 10, 41, 45, 48, 51, 54, 60, 65, 83, 84, 90, 93, 94, 102, 124, 138, 143, 144, 153, 154] where various Ill-Posed and CIPs were studied.

This book focuses on CIPs with single measurement time resolved data. "Single measurement" means that the data are generated by either a single location of the point source or a single direction of the incident plane wave. More generally, in the case of a CIP for a hyperbolic partial differential equation (PDE), "single measurement" means that only one pair of initial conditions is available, and in the case of a CIP for a parabolic PDE, only one initial condition is available. In other words, single measurement amounts to the minimal information content. The single measurement arrangement is the most suitable one for military applications. Indeed, because of various dangers on the battlefield, it is desirable to minimize the number of measurements in the military environment.

The single measurement case is the most economical way of data collection with the minimal available information. At the same time, because of the minimal information content, it is apparently more challenging than the multiple measurement case. At the time of the submission of this book the authors are unaware about other research groups working on non-local numerical methods for multidimensional CIPs with single measurement data.

CIPs with the data resulting from multiple measurements are also considered in the mathematical literature. These CIPs have applications in, for example, medical imaging and geophysics. In the case of multiple measurements, either the point source should run along a manifold or the direction of the incident plane wave should vary within a certain cone. We refer to, for example, [3, 35, 46, 63, 75, 82, 90, 92, 129–132] and references cited therein for some nonlocal algorithms for CIPs with multiple measurements.

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CIPs have many applications, for example, geophysics, detection of explosives (e.g., land mines), and medical imaging of malignant tumors. Because of these applications, we focus our numerical studies on imaging of small sharp abnormalities embedded in an otherwise slowly changing background medium. We image both locations of these inclusions and values of the unknown coefficient inside them. However, we are not interested in imaging of slowly changing backgrounds. We point out to an important point: our algorithms, which address the first central question of this book, as well as those of the two-stage numerical procedure (Sect. 1.1), do not use a priori knowledge of the background medium. An application to the detection of explosives is addressed in Sect. 6.9 of Chap. 6 for the case of blind experimental data collected by a radar in the field.

1.1 Two Central Questions of This Book

Since the field of inverse problems is an applied one, it is important to develop numerical methods for these problems. The following are the two central questions which inevitably surface in the computational treatment of any CIP for a PDE:

The First Central Question. Consider a CIP and suppose that this problem has unique exact solution for noiseless data. Assume that we have a small noise in the data. Then the question is, *how to develop such a numerical method for this CIP, which would provide an approximate solution located in a sufficiently small neighborhood of that exact solution without any a priori knowledge of this neighborhood?* The most important point here is that this method should not rely on the assumption of a priori knowledge of that neighborhood. The second very important point is that the property of obtaining such an approximation should be rigorously guaranteed. However, since CIPs are enormously challenging ones, then one has no choice but to "allow" this rigorous guarantee to be within the framework of a certain reasonable approximate mathematical model. Numerical studies should confirm this property. It is also desirable to provide an addition confirmation for the case of experimental data.

The most challenging case of **blind** experimental data is especially persuasive one. Indeed, since results are unbiased in this case, then the success in the blind data case would mean an ultimate verification of that approximate mathematical model. Similarly, the ultimate verification of any Partial Differential Equation of Mathematical Physics is in experiments. Results for blind experimental data are described in Chap. 5 and Sect. 6.9 of this book.

The Second Central Question. Suppose that the approximate solution mentioned in the first central question is computed. The second central question is, *how to refine this solution*? Indeed, since an approximate mathematical model is used, then the room might be left for a refinement.

Roughly speaking, any numerical method addressing the first central question is called *globally convergent*. However, because of the abovementioned approximation, we use in this book the term *approximate global convergence*. A rigorous definition of this term is presented in Sect. 1.1.2. Still, a short term for the latter is *global convergence*.

It is well known that there are a number of numerical methods for one-dimensional CIPs which do not require a priori knowledge of a small neighborhood of the exact solution; see, for example, [40, 47, 51, 56, 90] and references cited therein. At the same time, the latter is not the case for multidimensional CIPs. In this book, we consider only multidimensional CIPs with the only exception of Sect. 6.9. Thus, below, the abbreviation "CIP" always means an n-D CIP ($n = 2, 3$).

Conventional numerical methods for CIPs, such as, for example, various versions of Newton and gradient methods, converge locally, i.e., they need to use a good approximation for the exact solution to start from; see, for example, books [10, 93] for these methods for ill-posed problems. However, in the case of CIPs, such an approximation is rarely available in applications. Nevertheless, locally convergent methods can well be used to address the second central question. Indeed, the *main input* which any locally convergent algorithm needs is a good approximation for the exact solution. This approximation would be used as the starting point for iterations.

The above two questions were addressed in a series of recent publications of the authors for 2D and 3D CIPs for a hyperbolic PDE [9, 24–29, 109, 114–117, 160]. In particular, numerical methods addressing first and second central questions were synthesized in these publications in a *two-stage numerical procedure*. On the first stage, a good approximation for the exact solution is obtained for a CIP via our approximately globally convergent algorithm. Hence, the first central question is addressed on the first stage. On the second stage, this approximation is taken as the starting point for iterations of a locally convergent adaptive finite element method (adaptivity). In other words, the second central question is addressed on the second stage.

Unlike traditional numerical methods for CIPs, our technique, which addresses the first central question, does not use least squares functionals. Rather, only the structure of the underlying PDE operator is used. Also, it does not use a knowledge of the background values of the unknown coefficient. The goal of this book is to present results of above cited publications of the authors in a concise way. In addition, some previous results of the authors are presented as well.

An approximately globally convergent numerical method, which is similar to the one of this book, was developed in parallel for the case of a CIP for an elliptic PDE:

$$\Delta u - a(x) u = -\delta(x - x_0), x \in \mathbb{R}^2,$$

with the point source $\{x_0\}$ running along a straight line. This CIP has direct applications in medical optical imaging. That effort was undertaken by a team of researchers from the University of Texas at Arlington in collaboration with the second author of this book [110, 135, 147, 149, 150]. However, a description of this effort is outside of the scope of the current book.

1.1.1 *Why the Above Two Questions Are the Central Ones for Computations of CIPs*

Consider a radiation propagating through a medium. Some examples of the radiation are electromagnetic (EM), acoustical, thermo, light, and nuclear. Usually, the propagation of a radiation is governed by a PDE. Suppose that one needs to figure out the spatial dependence of one of properties of that medium. That property of interest is described by one of coefficients of the governing PDE. Some examples of such properties are the spatially distributed dielectric constant, electric conductivity, speed of sound, and absorption coefficient of light. If one would approximately calculate the spatial distribution of the property of interest, then one would create an image of the interior of that medium.

An attractive goal is to image that property of interest without placing detectors inside the medium. The latter is called *noninvasive* imaging. To obtain a noninvasive image, one can place detectors at some positions either at the entire boundary of the medium or at a part of it. In the first case, one would have *complete data,* and one would have *incomplete data* in the second case. Quite often, detectors can be placed only rather far from the medium. The latter is the case in, for example, imaging of explosives. Detectors would measure the output radiation. That output signal should have some trace of the property of interest. Suppose that readings of those detectors are interpolated in one of standard ways over the surface where those detectors are placed. Then the resulting function represents a boundary condition for that PDE. This is an extra boundary condition, the one which is given in addition to the original boundary condition for that equation. We call this boundary condition *the measured data* or shortly *the data*. For example, if originally one has the Neumann boundary condition, then the additional one is the Dirichlet boundary condition. The idea is to compute that unknown coefficient of the governing PDE (i.e., the unknown property of ones interest) using this additional boundary condition. Hence, we arrive at a CIP for that PDE.

Therefore, a CIP for a PDE is the problem of the reconstruction of an unknown spatially dependent coefficient of that PDE, given an additional boundary condition. This boundary condition can be given either at the entire boundary or at its part, and it models measurements of the corresponding output signal propagating through the medium of interest. Thus, to find a good approximation of the target property, one should solve numerically that CIP using the measured data. Clearly, these data contain a noisy component, since noise is inevitable in any measurement.

It is well known that it is extremely hard to solve a CIP. First, an important theoretical question is far not easy to address. Namely, this is the question about the uniqueness of the solution of a CIP. It will be clear from the material of this chapter that the uniqueness is one of the central questions to address in order to justify numerical methods for CIPs. This is why many mathematicians work on proofs of uniqueness theorems for CIPs. At the same time, since the discipline of inverse problems is an applied one, it is insufficient only to prove a uniqueness theorem. Along with proofs of uniqueness results, an important question is to construct reliable numerical methods. However, there are two main phenomena which cause huge challenges in the latter topic. These phenomena are the nonlinearity and the ill-posedness of CIPs combined. A problem is called *ill-posed* if small fluctuations of input data, which are inevitable in any experiment, can cause large fluctuations of resulting solutions. In other words this problem is unstable.

Here is a trivial example of the nonlinearity. Consider the Cauchy problem for the simplest ordinary differential equation:

$$y' = ay, y(0) = 1, \tag{1.1}$$

where $a = \text{const.} \neq 0$. The solution of the problem (1.1) is $y(t,a) = e^{at}$. Obviously, the function $y(t,a)$ depends nonlinearly on the coefficient a.

As to the numerical solution of a CIP, the first idea which naturally comes in mind is to construct a least squares cost functional and to minimize it then. It seems to be, on the first glance, that the point of the minimum of this functional should provide a good approximation for the exact solution. However, there are some serious problems associated with this idea. Indeed, because of the nonlinearity and the ill-posedness of CIPs, corresponding cost functionals usually suffer from the problem of multiple local minima and ravines; see, for example, [102] for some examples. Furthermore, there is no guarantee that a point of a global minimum is indeed close to the correct solution. Suppose, for example, that a cost functional has one hundred (100) points of local minima, one of them is a global one, the values of this functional at those points of local minima differ from each other by 0.5%, and the noise in the measured data is 5%. This might well happen when solving a 3D/2D CIP. Hence, there are no rigorous methods to decide which of these local minima is indeed close to the correct solution. Therefore, the idea of the minimization of the cost functional can work only in the case when a good first approximation for the exact solution is known in advance. However, the latter is a luxury in many applications.

A standard way to treat an ill-posed problem numerically is to minimize the Tikhonov regularization functional; see Sects. 1.7 and 1.8 below in this chapter for this functional. However, if the original problem is nonlinear, for example, a CIP, then this idea also cannot work in practical computations unless a good first approximation for the exact solution is available. In other words, one should know in advance such an approximation, which is located in the ε- neighborhood of the exact solution, where $\varepsilon > 0$ is sufficiently small. Indeed, the theory of the Tikhonov functional is based on the assumption that one can find a minimizing sequence, which ensures the convergence of the values of that functional to its infimum. However, the search of such a sequence can well face the abovementioned problem of local minima and ravines; see, for example, p. 3 of [93] for a similar observation.

Since the first central question is very challenging one to address, then it is hard to anticipate that it can be addressed without some approximations. In other words, a certain reasonable approximate mathematical model should likely be used. Because this model is not an exact one, it is likely that the above good approximation for the exact solution can be refined by one of locally convergent numerical methods. Thus, we arrive at the above second central question.

1.1.2 Approximate Global Convergence

Because of the approximate mathematical model mentioned both in the beginning of Sect. 1.1 and in the end of Sect. 1.1.1, we now discuss the notion of the global convergence. The common perception of the notion of a globally convergent numerical method is that this should be such an iterative algorithm which converges to the exact solution of a corresponding problem starting from an arbitrary point of a sufficiently large set. However, if thinking more carefully, all what one needs is to obtain a point in a sufficiently small neighborhood of the exact solution, provided that iterations would start not from an arbitrary point but rather from a prescribed and rather easily selected point. At the same time, the choice of that starting point should not be based on an a priori knowledge of a small neighborhood of the exact solution. In addition, one should have a rigorous guarantee of reaching that small neighborhood if starting from that selected point. Furthermore, it would be sufficient if that small neighborhood would be reached after a finite number of iterations. In other words, it is not necessary to consider infinitely many iterations, as it is usually done in the classical convergence analysis. On the other hand, since nonlinear problems are usually extremely challenging ones, some approximations should be allowed when developing such numerical methods. A valuable illustration of the idea of "allowed approximations" is the fifth Remark 1.1.2.1 below in this section. These thoughts have generated our definition of the approximate global convergence property.

Definition 1.1.2.1 (Approximate global convergence). Consider a nonlinear ill-posed problem P. Suppose that this problem has a unique solution $x^* \in B$ for the noiseless data y^*, where B is a Banach space with the norm $\|\cdot\|_B$. We call x^* "exact solution" or "correct solution." Suppose that a certain approximate mathematical model M_1 is proposed to solve the problem P numerically. Assume that, within the framework of the model M_1, this problem has unique exact solution $x^*_{M_1}$. Also, let one of assumptions of the model M_1 be that $x^*_{M_1} = x^*$. Consider an iterative numerical method for solving the problem P. Suppose that this method produces a sequence of points $\{x_n\}_{n=1}^N \subset B$, where $N \in [1, \infty)$. Let the number $\varepsilon \in (0, 1)$. We call this numerical method *approximately globally convergent of the level* ε*,* or shortly *globally convergent*, if, within the framework of the approximate model M_1, a theorem is proven, which guarantees that, without any a priori knowledge of a sufficiently small neighborhood of x^*, there exists a number $\overline{N} \in [1, N)$ such that

$$\|x_n - x^*\|_B \leq \varepsilon, \forall n \geq \overline{N}. \tag{1.2}$$

Suppose that iterations are stopped at a certain number $k \geq \overline{N}$. Then the point x_k is denoted as $x_k := x_{\text{glob}}$ and is called "the approximate solution resulting from this method."

This is our formal mathematical definition of the approximate global convergence property. However, since the approximate mathematical model M_1 is involved in it, then a natural question can be raised about the validity of this model. This question can be addressed only via computational experiments. In fact, it is a success in computational experiments, which is the true key for the verification of the model M_1. In addition, it would be good to verify M_1 on experimental data. These thoughts lead to the following informal definition of the approximate global convergence property.

Definition 1.1.2.2 (informal definition of the approximate global convergence property). Consider a nonlinear ill-posed problem P. Suppose that this problem has a unique solution $x^* \in B$ for the noiseless data y^*, where B is a Banach space with the norm $\|\cdot\|_B$. Suppose that a certain approximate mathematical model M_1 is proposed to solve the problem P numerically. Assume that, within the framework of the model M_1, this problem has unique exact solution $x^*_{M_1}$. Also, let one of assumptions of the model M_1 be that $x^*_{M_1} = x^*$. Consider an iterative numerical method for solving the problem P. Suppose that this method produces a sequence of points $\{x_n\}_{n=1}^N \subset B$, where $N \in [1, \infty)$. Let the number $\varepsilon \in (0, 1)$. We call this numerical method *approximately globally convergent of the level* ε, or shortly *globally convergent*, if the following three conditions are satisfied:

1. Within the framework of the approximate model M_1, a theorem is proven, which claims that, without any knowledge of a sufficiently small neighborhood of x^*, there exists a number $\overline{N} \in [1, N)$ such that the inequality (1.2) is valid.
2. Numerical studies confirm that x_{glob} is indeed a sufficiently good approximation for the true exact solution x^*, where x_{glob} is introduced in Definition 1.1.2.1.
3. Testing of this numerical method on appropriate experimental data also demonstrates that iterative solutions provide a good approximation for the exact one (optional).

We consider the third condition as an optional one because it is sometimes both hard and expensive to obtain proper experimental data. Furthermore, these data might be suitable only for one version of that numerical method and not suitable for other versions. Nevertheless, we believe that good results obtained for experimental data provide an ultimate confirmation of the validity of the approximate mathematical model M_1.

Remarks 1.1.2.1. 1. We repeat that we have introduced these two definitions because of substantial challenges which one inevitably faces when attempting to construct reliable numerical methods for CIPs. Indeed, because of these challenges, it is unlikely that the desired good approximation for the exact solution would be obtained without a "price." This price is the approximate mathematical model M_1.

2. The *main requirement* of the above definitions is that this numerical method should provide a sufficiently good approximation for the exact solution x^* without any a priori knowledge of a sufficiently small neighborhood of x^*. Furthermore, it is important that one should have a rigorous guarantee of the latter, within the framework of the model M_1.

3. Unlike the classical convergence, these definitions only require that points $\{x_n\}_{n=1}^k$ belong to a small neighborhood of the exact solution x^*. However, the total number of iterations N can be finite in Definitions 1.1.2.1, 1.1.2.2. Such algorithms are not rare in the theory of Ill-Posed Problems. As two examples, we refer to Theorem 4.6 of [10] and Lemma 6.2 on page 156 of [65] for some other numerical methods with the property (1.2). Actually, (1.2) is sufficient, since one can apply a refinement procedure on the second stage, i.e. a procedure addressing The Second Central Question.
4. Therefore, the above definitions leave the room for a refinement of the approximate solution x_{glob} via a subsequent application of a locally convergent numerical method. The latter is exactly what the second central question is about.
5. As to the approximate mathematical model M_1, here is a good analogy. First of all, all equations of mathematical physics are approximate ones. More precisely, it is well known that the Huygens-Fresnel optics is not yet rigorously derived from the Maxwell equations. We now cite some relevant statements from Sect. 8.1 of the classical book of Born and Wolf [36]. "*Diffraction problems are amongst the most difficult ones encountered in optics. Solutions which, in some sense, can be regarded as rigorous are very rare in diffraction theory.*" Next, "*because of mathematical difficulties, approximate models must be used in most cases of practical interest. Of these the theory of Huygens and Fresnel is by far the most powerful and is adequate for the treatment of the majority of problems encountered in instrumental optics.*" It is well known that the entire optical industry nowadays is based on the Huygens-Fresnel theory. Analogously, although the numerical method of this book works only with approximate models, its accurate numerical performance has been consistently demonstrated in [24–29, 109, 114–116], including the most challenging case of *blind* experimental data; see [109], Chap. 5, and Sect. 6.9.

Based on Definitions 1.1.2.1, 1.1.2.2, we address The First Central Question of this book via six steps listed below.

Step 1. A reasonable approximate mathematical model is proposed. The accuracy of this model cannot be rigorously estimated.

Step 2. A numerical method is developed, which works within the framework of this model.

Step 3. A theorem is proven, which guarantees that, within the framework of this model, the numerical method of Step 2 indeed reaches a sufficiently small neighborhood of the exact solution, as long as the error, both in the data and in some additional approximations is sufficiently small. It is a **crucial requirement** of our approach that this theorem should not rely neither on the assumption about a knowledge of any point in a small neighborhood of the exact solution nor on the assumption of a knowledge of the background medium inside the domain of interest.

Step 4. Testing of the numerical method of Step 2 on computationally simulated data.

Step 5. Testing of the numerical method of Step 2 on experimental data (if available). To have a truly unbiased case, the most challenging case of **blind** experimental data is preferable.

Step 6. Finally, if results of Step 4 and Step 5 are good ones, then we conclude that our approximate mathematical model is a valid one. However, if experimental data are unavailable, while results of Step 4 are good ones, then we still conclude that our approximate mathematical model is a valid one.

Step 6 is logical, because its condition is that the resulting numerical method is proved to be effective. It is sufficient to achieve that small neighborhood of the exact solution after a finite (rather than infinite) number of iterations. Next, because of approximations in the mathematical model, the resulting solution can be refined via a locally convergent numerical method, i.e. the Second Central Question should be addressed.

Therefore, the key philosophical focus of Definitions 1.1.2.1 and 1.1.2.2 is the point about natural assumptions/approximations which make the technique numerically efficient and, at the same time, independent on the availability of a good first guess.

The next definition is about a locally convergent numerical method for a nonlinear ill-posed problem. In this definition, we consider the Tikhonov functional which is introduced in Sect. 1.7. While sometimes the existence of a minimizer of the Tikhonov functional can be proved in an infinitely dimensional space, in a generic case of a nonlinear ill-posed problem, for example, CIP, this existence cannot be guaranteed; see Sects. 1.7.1 and 1.7.2. On the other hand, the existence of a minimizer for the classical Tikhonov regularization functional is guaranteed only in the case of a finite dimensional space (Sect. 1.8). This minimizer is called a *regularized solution* (in principle, one might have many minimizers). A good example of such a finite dimensional space is the space of piecewise linear finite elements. Furthermore, this is a natural space to use in practical computations, and we use it throughout this book.

Still, the resulting finite dimensional problem inherits the ill-posed nature of the original ill-posed problem. Thus, the Tikhonov regularization functional should be used in that finite dimensional space. At the same time, since a finite dimensional space is taken instead of an infinitely dimensional one, then this can be considered as an approximate mathematical model of the original ill-posed problem. Thus, the *approximate mathematical model* M_2 for an ill-posed problem P means that P is considered in a finite dimensional space.

Definition 1.1.2.3. Consider a nonlinear ill-posed problem P. Suppose that this problem has a exact unique solution $x^* \in B$ for the noiseless data y^*, where B is a Banach space. Consider the approximate mathematical model M_2 for the problem P. The model M_2 means the replacement of the infinitely dimensional space B with a finite dimensional Banach space B_k, $\dim B_k = k$. Assume that, within the framework of the model M_2, the problem P has unique exact solution $x^*_{M_2} \in B_k$ and let one of assumptions of the model M_2 be that $x^*_{M_2} = x^*$. Let $G \subset B_k$ be an open bounded set. Let the small number $\delta > 0$ be the level of the error in the data and $\alpha = \alpha(\delta)$ be the regularization parameter depending on δ (Sect. 1.4). For the problem P, consider the Tikhonov functional defined in Sect. 1.7. Consider an iterative numerical method of the minimization of this functional on the set $\overline{G}$. Suppose that this method starts its iterations from the point x_0 and produces iterative solutions $\{x_n^\delta\}_{n=1}^{\infty} \subset \overline{G}$. Let $x_{\alpha(\delta)} \in \overline{G}$ be a minimizer of the Tikhonov functional with $\alpha = \alpha(\delta)$. Let $\delta_0, \rho \in (0,1)$ be two sufficiently small numbers. We call this method *locally convergent*, if the following two conditions are satisfied:

1. A theorem is proven, which ensures that if $\delta \in (0, \delta_0)$ and $\|x_0 - x^*\|_{B_k} \leq \rho$, then
$$\lim_{n\to\infty} \left\| x_n^\delta - x_{\alpha(\delta)} \right\|_{B_k} = 0, \ \forall \delta \in (0, \delta_0).$$
2. This theorem also claims that
$$\lim_{\delta\to 0} \left\| x_{\alpha(\delta)} - x^* \right\|_{B_k} = 0.$$

On the other hand, the global convergence in the classical sense intuitively means that, regardless on the absence of a good first approximation for the exact solution, the iterative solutions tend to the exact one, as long as certain parameters tend to their limiting values. This, as well as Definitions 1.1.2.1 and 1.1.2.3 lead to Definition 1.1.2.4. Prior this definition, we need to impose the Assumption 1.1.2. We impose this assumption only for the simplicity of the presentation. Note that Assumption 1.1.2 makes sense only if the two-stage numerical procedure mentioned in Sect. 1.1 is applied. However, if only the first stage is applied, then we do not need this assumption.

Assumption 1.1.2. Suppose that a nonlinear ill-posed problem P is the same in both Definitions 1.1.2.1 and 1.1.2.3. Suppose that the two-stage numerical procedure mentioned in Sect. 1.1 is applied. Then, we assume throughout the book that the finite dimensional space $B_k \subseteq B$ and that the exact solution x^* is the same for both mathematical models M_1, M_2 of these two stages.

Definition 1.1.2.4. Consider a nonlinear ill-posed problem. Let B and B_k be the Banach spaces, ε and ρ be the numbers of Definitions 1.1.2.1 and 1.1.2.3, respectively, and let $B_k \subseteq B$ and $\varepsilon \in (0, \rho]$. Consider a numerical procedure for this problem, which consists of the following two stages:

1. On the first stage, a numerical method satisfying conditions of Definitions 1.1.2.1 is applied, and it ends up with an element $x_{\text{glob}} \in B_k$ satisfying inequality (1.2).
2. On the second stage, a locally convergent numerical method satisfying conditions of Definition 1.1.2.3 is applied. This method takes $x_{\text{glob}} := x_0 \in B_k$ as the starting point for iterations.

Then, we call this two-stage numerical procedure *globally convergent in the classical sense within frameworks of the pair of approximate mathematical models* (M_1, M_2). In short, we call this procedure *globally convergent in the classical sense.*

Remarks 1.1.2.2. 1. The single most important point of Definition 1.1.2.4 is that the two-stage numerical procedure converges *globally* in the *classical* sense to the exact solution within the frameworks of the pair (M_1, M_2). In other words, it converges *regardless* on the availability of a good first guess for the exact solution.
2. The two-stage numerical procedure for CIPs which is developed in this book satisfies conditions of Definition 1.1.2.4.

1.1.3 Some Notations and Definitions

The theory of ill-posed problems addresses the following fundamental question: *How to obtain a good approximation for the solution of an ill-posed problem in a stable way?* Roughly speaking, a numerical method, which provides a stable and accurate solution of an ill-posed problem, is called the *regularization* method for this problem; see Sect. 1.7 for a rigorous definition. Foundations of the theory of ill-posed problems were established by three Russian mathematicians: Tikhonov [152–154], Lavrent'ev [122, 124], and Ivanov [85, 86] in the 1960s. The first foundational work was published by Tikhonov in 1943 [152].

We now briefly introduce some common notations which will be used throughout this book. These notations can be found in, for example, the textbook [127]. We work in this book only with real valued functions. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We will always assume in our analytical derivations that its boundary $\partial\Omega \in C^3$, although we will work with piecewise smooth boundaries in numerical studies. This is one of natural discrepancies between the theory and its numerical implementation, which always exist in computations. Let $u(x)$, $x = (x_1, ..., x_n) \in \Omega$ be a k times continuously differentiable function defined in Ω. Denote

$$D^{\alpha}u = \frac{\partial^{|\alpha|}u}{\partial^{\alpha_1}x_1 \ldots \partial^{\alpha_n}x_n}, \quad |\alpha| = \alpha_1 + \ldots + \alpha_n,$$

the partial derivative of the order $|\alpha| \leq k$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index with integers $\alpha_i \geq 0$. Denote $C^k\left(\overline{\Omega}\right)$ the Banach space of functions $u(x)$ which are continuous in the closure $\overline{\Omega}$ of the domain Ω together with their derivatives $D^{\alpha}u, |\alpha| \leq m$. The norm in this space is defined as

$$\|u\|_{C^k(\overline{\Omega})} = \sum_{|\alpha| \leq m} \sup_{x \in \Omega} |D^{\alpha}u(x)| < \infty.$$

By definition, $C^0\left(\overline{\Omega}\right) = C\left(\overline{\Omega}\right)$ is the space of functions continuous in $\overline{\Omega}$ with the norm

$$\|u\|_{C\left(\overline{\Omega}\right)} = \sup_{x\in\Omega} |u(x)|.$$

We also introduce Hölder spaces $C^{k+\alpha}\left(\overline{\Omega}\right)$ for any number $\alpha \in (0,1)$. The norm in this space is defined as

$$\|u\|_{C^{k+\alpha}\left(\overline{\Omega}\right)} := |u|_{k+\alpha} := \|u\|_{C^k\left(\overline{\Omega}\right)} + \sup_{x,y\in\Omega,\, x\neq y} \frac{|u(x) - u(y)|}{|x-y|^{\alpha}},$$

provided that the last term is finite. It is clear that if the function $u \in C^{k+1}\left(\overline{\Omega}\right)$, then $u \in C^{k+\alpha}\left(\overline{\Omega}\right)$, $\forall \alpha \in (0,1)$, and:

$$|u|_{k+\alpha} \leq C\, \|u\|_{C^{k+1}\left(\overline{\Omega}\right)}, \ \forall u \in C^{k+1}\left(\overline{\Omega}\right),$$

where $C = C(\Omega, \alpha) > 0$ is a constant independent on the function u. Sometimes, we also use the notion of Hölder spaces for infinite domains. Let D be such a domain. It is convenient for us to say that the function $u \in C^{k+\alpha}(D)$ if $u \in C^{k+\alpha}\left(\overline{\Omega}\right)$ for every bounded subdomain $\Omega \subset D$. Although sometimes people say that $u \in C^{k+\alpha}\left(\overline{D}\right)$ if the above Hölder norm in $\overline{D}$ is finite.

Consider the Sobolev space $H^k(\Omega)$ of all functions with the norm defined as

$$\|u\|_{H^k(\Omega)}^2 = \sum_{|\alpha|\leq k} \int_{\Omega} |D^{\alpha} u|^2 \,\mathrm{d}x < \infty,$$

where $D^{\alpha}u$ are weak derivatives of the function u. By the definition, $H^0(\Omega) = L_2(\Omega)$. It is well known that $H^k(\Omega)$ is a Hilbert space with the inner product defined as

$$(u,v)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} \int_{\Omega} D^{\alpha} u D^{\alpha} v \mathrm{d}x.$$

Let $T > 0$ and $\Gamma \subseteq \partial\Omega$ be a part of the boundary $\partial\Omega$ of the domain Ω. We will use the following notations throughout this book:

$$Q_T = \Omega \times (0,T),\, S_T = \partial\Omega \times (0,T),\, \Gamma_T = \Gamma \times (0,T),\, D_T^{n+1} = \mathbb{R}^n \times (0,T).$$

The space $C^{2k,k}\left(\overline{Q}_T\right)$ is defined as the set of all functions $u(x,t)$ having derivatives $D_x^{\alpha} D_t^{\beta} u \in C\left(\overline{Q}_T\right)$ with $|\alpha| + 2\beta \leq 2k$ and with the following norm:

$$\|u\|_{C^{2k,k}\left(\overline{Q}_T\right)} = \sum_{|\alpha|+2\beta\leq 2k} \max_{\overline{Q}_T} \left| D_x^{\alpha} D_t^{\beta} u(x,t) \right|.$$

The Hölder space $C^{2k+\alpha,k+\alpha/2}\left(\overline{Q}_T\right)$, $\alpha \in (0,1)$ is defined similarly [120].

We now remind some definitions from the standard course of functional analysis.

Definition 1.1.3.1. Let B be a Banach space. The set $V \subset B$ is called *precompact* set if every sequence $\{x_n\}_{n=1}^{\infty} \subseteq V$ contains a fundamental subsequence (i.e., the Cauchy subsequence).

Although by the Cauchy criterion the subsequence of Definition 1.1.3.1 converges to a certain point, there is no guarantee that this point belongs to the set V. If we consider the closure of V, i.e., the set $\overline{V}$, then all limiting points of all convergent sequences in V would belong to V. Therefore, we arrive at Definition 1.1.3.2.

Definition 1.1.3.2. Let B be a Banach space. The set $V \subset B$ is called *compact* set if V is a closed set, $V = \overline{V}$, every sequence $\{x_n\}_{n=1}^{\infty} \subseteq V$ contains a fundamental subsequence, and the limiting point of this subsequence belongs to the set V.

Definition 1.1.3.3. Let B_1 and B_2 be two Banach spaces, $U \subseteq B_1$ be a set and $A : U \to B_2$ be a continuous operator. The operator A is called a *compact operator* or *completely continuous* operator if it maps any bounded subset $U' \subseteq U$ in a precompact set in B_2. Clearly, if U' is a closed set, then $A(U')$ is a compact set.

The following theorem is well known under the name of Ascoli-Archela theorem (More general formulations of this theorem can also be found).

Theorem 1.1.3.1. *The set of functions $\mathcal{M} \subset C\left(\overline{\Omega}\right)$ is a compact set if and only if it is uniformly bounded and equicontinuous. In other words, if the following two conditions are satisfied:*

1. There exists a constant $M > 0$ such that

$$\|f\|_{C(\overline{\Omega})} \leq M, \ \forall f \in \mathcal{M}.$$

2. For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \quad \forall x, y \in \{|x - y| < \delta\} \cap \overline{\Omega}, \ \forall f \in \mathcal{M}.$$

In particular, because of some generalizations of this theorem, any bounded set in $C^k\left(\overline{\Omega}\right)$ (or $H^k(\Omega)$), $k \geq 1$ is a compact set in $C^p\left(\overline{\Omega}\right)$ (respectively $H^p(\Omega)$) for $p \in [0, k-1]$. We also remind one of the Sobolev embedding theorems for spaces $H^k(\Omega)$. Let $[n/2]$ be the least integer which does not exceed $n/2$.

Theorem 1.1.3.2 ([127]). *Suppose that $k > [n/2] + m$, the domain Ω is bounded and $\partial\Omega \in C^k$. Then $H^k(\Omega) \subset C^m\left(\overline{\Omega}\right)$ and $\|f\|_{C^m(\overline{\Omega})} \leq C\|f\|_{H^k(\Omega)}, \forall f \in H^k(\Omega)$, where the constant $C = C(\Omega, k, m) > 0$ depends only on Ω, k, m. In addition, any bounded set in $H^k(\Omega)$ is a precompact set in $C^m\left(\overline{\Omega}\right)$.*

Theorem 1.1.3.2 actually claims that the space $H^k(\Omega)$ is compactly embedded in the space $C^m\left(\overline{\Omega}\right)$. "Compactly embedded" means that $\|f\|_{C^m(\overline{\Omega})} \leq C\|f\|_{H^k(\Omega)}, \forall f \in H^k(\Omega)$, and any bounded set in $H^k(\Omega)$ is a precompact set in $C^m\left(\overline{\Omega}\right)$. In other words, any sequence bounded in $H^k(\Omega)$ contains a subsequence, which converges in $C^m\left(\overline{\Omega}\right)$, although the limit of this subsequence does not necessarily belong to $H^k(\Omega)$.

1.2 Some Examples of Ill-posed Problems

Example 1 (J. Hadamard). We now describe the classical example of Hadamard; see, for example, [124]. Consider the Cauchy problem for the Laplace equation for the function $u(x, y)$:

$$\Delta u = 0, \quad x \in (0, \pi), \quad y > 0, \tag{1.3}$$

$$u(x, 0) = 0, u_y(x, 0) = \alpha \sin(nx), \tag{1.4}$$

where $n > 0$ is an integer. It is well known that the Cauchy problem for a general elliptic equation with "good" variable coefficients has at most one solution [102, 124] (although it might not have solutions at all). The unique solution of the problems (1.3) and (1.4) is

$$u(x, y) = \frac{\alpha}{n} \sinh(ny) \sin(nx). \tag{1.5}$$

Choose sufficiently small numbers $\varepsilon > 0, \alpha = \alpha(\varepsilon) > 0$ and a number $y := y_0 > 0$. Let in (1.5) $x \in (0, \pi)$. Since the function

$$\sinh(ny_0) = \frac{e^{ny_0}\left(1 + e^{-2ny_0}\right)}{2}$$

grows exponentially as $n \to \infty$, then it is clear from (1.5) that for any pair of reasonable functional spaces $C^k[0, \pi], L_2[0, \pi], H^k[0, \pi]$, etc., one can choose such two numbers $c > 0, n_0 > 0$ depending only on numbers ε, α, y_0 that

$$||\alpha \sin(nx)||_1 < \varepsilon, \quad \forall n \geq n_0,$$

$$\|u(x, y_0)\|_2 = \left\|\frac{\alpha}{n} \sinh(ny_0) \sin(nx)\right\|_2 > c, \quad \forall n \geq n_0,$$

where $\|\cdot\|_1$ is the norm in one of those spaces and $\|\cdot\|_2$ is the norm in another one.

The above example demonstrates that although both the Dirichlet and Neumann boundary data are small, any reasonable norm of the solution is still large. In other words, this is a manifestation of a high instability of this problem. Based on this example, Hadamard has concluded that it makes no sense to consider unstable problems. However, his conclusion was an exaggeration. Indeed, unstable problems arise in many applications. Being inspired by applications to geophysics, Tikhonov has proposed in 1943 [152] the fundamental concept for solving unstable problems; see Sect. 1.3.

Example 2 (Differentiation of a Function Given with a Noise). The differentiation of functions given by analytic formulas is a trivial exercise. In the reality, however, functions are often measured in experiments. Since experimental data always contain noise, then measured functions are given with a noise. Quite often, it is necessary to differentiate these noisy functions. We demonstrate now that the problem of the differentiation of noisy functions is unstable. Suppose that the function $f(x), x \in [0,1]$ is given with a noise. In other words, suppose that instead of $f(x) \in C^1[0,1]$ the following function $f_\delta(x)$ is given:

$$f_\delta(x) = f(x) + \delta f(x), x \in [0,1],$$

where $\delta f(x)$ is the noisy component. Let $\delta > 0$ be a small parameter characterizing the level of noise. We assume that the noisy component is small, $\|\delta f\|_{C[0,1]} \leq \delta$. The problem of calculating the derivative $f'_\delta(x)$ is unstable. Indeed, let, for example,

$$\delta f(x) = \frac{\sin(n^2 x)}{n},$$

where $n > 0$ is a large integer. Then the $C[0,1]$-norm of the noisy component is small:

$$\|\delta f\|_{C[0,1]} \leq \frac{1}{n}.$$

However, the difference between derivatives of noisy and exact functions

$$f'_\delta(x) - f'(x) = n \cos n^2 x$$

is not small in any reasonable norm.

We now describe a simple regularization method of stable calculation of derivatives. The idea is that the step size h in the corresponding finite difference should be connected with the level of noise δ. Thus, h cannot be made arbitrary small, as it is the case of the classic definition of the derivative. We obviously have

$$f'_\delta(x) \approx \frac{f(x+h) - f(x)}{h} + \frac{\delta f(x+h) - \delta f(x)}{h}. \tag{1.6}$$

The first term in the right-hand side of (1.6) is close to the exact derivative $f'(x)$, if h is small enough. The second term, however, comes from the noise. Hence, we need to balance these two terms via an appropriate choice of $h = h(\delta)$. Obviously:

$$\left| f'_\delta(x) - \frac{f(x+h) - f(x)}{h} \right| \leq \frac{2\delta}{h}.$$

Hence, we should choose $h = h(\delta)$ such that

$$\lim_{\delta \to 0} \frac{2\delta}{h(\delta)} = 0.$$

For example, let $h(\delta) = \delta^{\mu}$, where $\mu \in (0, 1)$. Then

$$\lim_{\delta \to 0} \left| f_\delta'(x) - \frac{f(x+h) - f(x)}{h} \right| \leq \lim_{\delta \to 0} \left(2\delta^{1-\mu} \right) = 0.$$

Hence, the problem becomes stable for this choice of the grid step size $h(\delta) = \delta^{\mu}$. This means that $h(\delta)$ is the regularization parameter here. There are many practical methods in the literature designed for stable differentiation. For example, one can approximate the function $f_\delta(x)$ via cubic B splines and differentiate this approximation then; see, for example, [73]. However, the number of these splines should not be too large; otherwise, the problem would become unstable. So the number of cubic B splines is the regularization parameter in this case, and its intuitive meaning is the same as the meaning of the number $1/h(\delta)$. A more detailed description of regularization methods for the differentiation procedure is outside of the scope of this book.

Let $\Omega \subset \mathbb{R}^n$ is a bounded domain and the function $K(x, y) \in C\left(\overline{\Omega} \times \overline{\Omega}\right)$. Recall that the equation

$$g(x) + \int_{\Omega} K(x, y)\, g(y)\, \mathrm{d}y = p(x), x \in \Omega, \tag{1.7}$$

where $p(x)$ is a bounded function, is called *integral equation of the second kind*. These equations are considered quite often in the classic theory of PDEs. The classical Fredholm theory works for these equations; see, for example, the textbook [127]. Next, let $\Omega' \subset \mathbb{R}^n$ be a bounded domain and the function $K(x, y) \in C\left(\overline{\Omega} \times \overline{\Omega}\right)$. Unlike (1.7), the equation

$$\int_{\Omega} K(x, y)\, g(y)\, \mathrm{d}y = p(x), x \in \Omega' \tag{1.8}$$

is called the integral equation of the first kind. The Fredholm theory does not work for such equations. The problem of solution of (1.8) is an ill-posed problem; see Example 3.

Example 3 (Integral Equation of the First Kind). Consider (1.8). The function $K(x, y)$ is called *kernel* of the integral operator. Equation (1.8) can be rewritten in the form

$$Kf = p, \tag{1.9}$$

where $K : C\left(\overline{\Omega}\right) \to C\left(\overline{\Omega}'\right)$ is the integral operator in (1.8). It is well known from the standard functional analysis course that K is a compact operator. We now show that the problem (1.9) is ill-posed. Let $\Omega = (0,1)$, $\Omega' = (a,b)$. Replace the function f with the function $f_n(x) = f(x) + \sin nx$. Then

$$\int_0^1 K(x,y) f_n(y)\,\mathrm{d}y = g_n(x), \; x \in (0,1), \tag{1.10}$$

where $g_n(x) = p(x) + p_n(x)$ and

$$p_n(x) = \int_0^1 K(x,y) \sin ny \,\mathrm{d}y.$$

By the Lebesque lemma,

$$\lim_{n\to\infty} \|p_n\|_{C[a,b]} = 0.$$

However, it is clear that

$$\|f_n(x) - f(x)\|_{C[0,1]} = \|\sin nx\|_{C[0,1]}$$

is not small for large n.

Example 4 (The Case of a General Compact Operator). We now describe an example of a general ill-posed problem. Let H_1 and H_2 be two Hilbert spaces with $\dim H_1 = \dim H_2 = \infty$. We remind that a sphere in an infinitely dimensional Hilbert space is not a compact set. Indeed, although the orthonormal basis in this space belongs to the unit sphere, it does not contain a fundamental subsequence.

Theorem 1.2. *Let* $G = \left\{\|x\|_{H_1} \le 1\right\} \subset H_1$. *Let* $A : G \to H_2$ *be a compact operator and let* $R(A) := A(G)$ *be its range. Consider an arbitrary point* $y_0 \in R(A)$. *Let* $\varepsilon > 0$ *be a number and* $U_\varepsilon(y_0) = \left\{y \in H_2 : \|y - y_0\|_{H_2} < \varepsilon\right\}$. *Then there exists a point* $y \in U_\varepsilon(y_0) \diagdown R(A)$. *If, in addition, the operator* A *is one-to-one, then the inverse operator* $A^{-1} : R(A) \to G$ *is not continuous. Hence, the problem of the solution of the equation*

$$A(x) = z, x \in G, z \in R(A) \tag{1.11}$$

is unstable, i.e., this is an ill-posed problem.

Proof. First, we prove the existence of a point $y \in U_\varepsilon(y_0) \diagdown R(A)$. Assume to the contrary, i.e., assume that $U_\varepsilon(y_0) \subset R(A)$. Let $\{y_n\}_{n=1}^{\infty} \subset H_2$ be an orthonormal basis in H_2. Then the sequence

$$\left\{y_0 + \frac{\varepsilon}{2} y_n\right\}_{n=1}^{\infty} := \{z_n\}_{n=1}^{\infty} \subset \left\{\|y - y_0\| = \frac{\varepsilon}{2}\right\} \subset U_\varepsilon(y_0).$$

We have

$$\|z_n - z_m\|_{H_2} = \frac{\varepsilon}{\sqrt{2}}.$$

Hence, the sequence $\{z_n\}_{n=1}^{\infty}$ does not contain a fundamental subsequence. Therefore, $U_\varepsilon(y_0)$ is not a precompact set in H_2. On the other hand, since G is a closed bounded set and A is a compact operator, then $R(A)$ is a compact set. Hence, $U_\varepsilon(y_0)$ is a precompact set. We got a contradiction, which proves the first assertion of this lemma.

We now prove the second assertion. Assume to the contrary that the operator $A^{-1} : R(A) \to G$ is continuous. By the definition of the operator A, we have $A^{-1}(R(A)) = G$. Since $R(A)$ is a compact set in H_2, then the continuity of A^{-1} implies that G is a compact set in H_1, which is not true.

We now summarize some conclusions which follow from Theorem 1.2. By this theorem, the set $R(A)$ is not dense everywhere. Therefore, the question about the existence of the solution of either of (1.9) or (1.11) does not make an applied sense. Indeed, since the set $R(A)$ is not dense everywhere, then it is very hard to describe a set of values y belonging to this set. As an example, consider the case when the kernel $K(x, y) \in C([a, b] \times [0, 1])$ in (1.10) is an analytic function of the real variable $x \in (a, b)$. Then the right hand side $p(x)$ of (1.8) should also be analytic with respect to $x \in (a, b)$. However, in applications, the function $p(x)$ is a result of measurements, it is given only at a number of discrete points and contains noise. Clearly, it is impossible to determine from this information whether the function $p(x)$ is analytic or not. Hence, we got the following important conclusion.

Conclusion. Assuming that conditions of Theorem 1.2 are satisfied, the problem of solving (1.11) is ill-posed in the following terms: (a) the proof of an existence theorem makes no applied sense, and (b) small fluctuations of the right hand side y can lead to large fluctuations of the solution x, i.e., the problem is unstable.

Example 5 (A Coefficient Inverse Problem (CIP)). Let the functions $a(x) \in C^{\alpha}(\mathbb{R}^n), \alpha \in (0, 1)$, and $a(x) = 0$ outside of the bounded domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^3$. Consider the following Cauchy problem:

$$u_t = \Delta u + a(x)u, \quad (x, t) \in D_T^{n+1}, \tag{1.12}$$

$$u(x, 0) = f(x). \tag{1.13}$$

Here, the function $f(x) \in C^{2+\alpha}(\mathbb{R}^n)$ has a finite support in $\mathbb{R}^n$. Although less restrictive conditions on f can also be imposed, we are not doing this here for brevity; see details in the book [120]. Another option for the initial condition is

$$f(x) = \delta(x - x_0), \tag{1.14}$$

where the source position $x_0 \notin \overline{\Omega}$. Throughout the book, we will always assume that the source is located outside of the domain of interest Ω. The reason of doing this is that we do not want to work with singularities since CIPs are very complicated even without singularities. The second reason is that in the majority of applications, sources are indeed located outside of domains of interest; see Chaps. 5 and 6 for experimental data.

Statement of a Coefficient Inverse Problem. Assume that the function $a(x)$ is unknown inside the domain Ω. Determine this function for $x \in \Omega$ assuming that the following function $g(x,t)$ is known:

$$u\mid_{S_T} = g(x,t). \tag{1.15}$$

The function $g(x,t)$ is an additional boundary condition. This function can be interpreted as a result of measurements: One is measuring the function $u(x,t)$ at the boundary of the domain Ω in order to reconstruct the function $a(x)$ inside Ω. Indeed, if the coefficient $a(x)$ would be known in the entire space $\mathbb{R}^n$, then one would uniquely determine the function $u(x,t)$ in D_T^{n+1}. But since $a(x)$ is unknown, then the function $u\mid_{S_T}$ can be determined only via measurements. Note that since $a(x) = 0$ outside of Ω, then one can uniquely solve the following initial boundary value problem outside of Ω :

$$u_t = \Delta u, (x,t) \in (\mathbb{R}^n \diagdown \Omega) \times (0,T),$$

$$u(x,0) = f(x), x \in \mathbb{R}^n \diagdown \Omega,$$

$$u\mid_{S_T} = g(x,t).$$

Hence, one can uniquely determine the Neumann boundary condition for the function u at the boundary $\partial\Omega$, and we will use this consideration throughout this book. Thus, the following function $g_1(x,t)$ is known along with the function $g(x,t)$ in (1.15):

$$\partial_n u\mid_{S_T} = g_1(x,t).$$

This CIP has direct applications in imaging of the *turbid media* using light propagation [8, 76, 156]. In a turbid medium, photons of light, originated by a laser, propagate randomly in the diffuse manner. In other words, they experience many random scattering events. Two examples of turbid media are smog and flames in the air. The most popular example is the biological tissue, including human organs. Assuming that the diffusion coefficient $D = 1$, we obtain that in (1.12) the coefficient $a(x) = -\mu_a(x) \leq 0$, where $\mu_a(x)$ is the absorption coefficient of the medium. The case of smog and flames has military applications. Since $\mu_a(x) = \infty$ for any metallic target, then imaging small inhomogeneities with large values of the absorption coefficient might lead to detection of those targets. In the case of medical applications, high values of $\mu_a(x)$ usually correspond to malignant legions. Naturally, one is interested to image those legions noninvasively via solving a CIP.

Thus, in both applications, the main interest is in imaging of small sharp abnormalities, rather than in imaging of a slowly changing background function. Furthermore, to correctly identify those abnormalities, one needs to image with a good accuracy the value of the coefficient $\mu_a(x)$ within them. Naturally, in both applications, one should use the function (1.14) as the initial condition. In this case, x_0 is the location of the light source.

We now show that this CIP is an ill-posed problem. Let the function u_0 be the fundamental solution of the heat equation $u_{0t} = \Delta u_0$:

$$u_0(x,t) = \frac{1}{\left(2\sqrt{\pi t}\right)^n} \exp\left(-\frac{|x|^2}{4t}\right).$$

It is well known that the function u has the following integral representation [120]:

$$u(x,t) = \int\limits_{\mathbb{R}^n} u_0(x-\xi,t) f(\xi) d\xi + \int\limits_0^t \int\limits_{\Omega} u_0(x-\xi,t-\tau) a(\xi) u(\xi,\tau) d\tau. \tag{1.16}$$

Because of the presence of the integral

$$\int\limits_0^t (\cdot) d\tau,$$

(1.16) is a Volterra-like integral equation of the second kind. Hence, it can be solved as [120]:

$$u(x,t) = \int\limits_{\mathbb{R}^n} u_0(x-\xi,t) f(\xi) d\xi + \sum_{n=1}^{\infty} u_n(x,t), \tag{1.17}$$

$$u_n(x,t) = \int\limits_0^t \int\limits_{\Omega} u_0(x-\xi,t-\tau) a(\xi) u_{n-1}(\xi,\tau) d\tau.$$

One can prove that each function $u_n \in C^{2+\alpha,1+\alpha/2}\left(\overline{D}_T^{n+1}\right)$ and [120]

$$\left|D_x^{\beta} D_t^k u_n(x,t)\right| \leq \frac{(Mt)^n}{n!}, \quad |\beta| + 2k \leq 2, \tag{1.18}$$

where $M = \|a\|_{C^{\alpha}(\overline{\Omega})}$. In the case when $f = \delta(x - x_0)$, the first term in the right-hand side of (1.17) should be replaced with $u_0(x - x_0, t)$. Let $u_0^f(x,t)$ be the first term of the right-hand side of (1.17) and $v(x,t) = u(x,t) - u_0^f(x,t)$. Using (1.18), one can rewrite (1.17) as

$$v(x,t) = \int_0^t \int_\Omega u_0(x-\xi, t-\tau)\left(a(\xi)\, u_0^f(\xi,\tau) + P(a)(\xi,\tau)\right) d\xi d\tau, \quad (1.19)$$

where $P(a)$ is a nonlinear operator applied to the function a. It is clear from (1.17)–(1.19) that the operator $P : C^\alpha\left(\overline{\Omega}\right) \to C^{2+\alpha,1+\alpha/2}\left(\overline{Q}_T\right)$ is continuous. Setting in (1.19) $(x,t) \in S_T$, recalling (1.15), and denoting $\overline{g}(x,t) = g(x,t) - u_0^f(x,t)$, we obtain a nonlinear integral equation of the first kind with respect to the unknown coefficient $a(x)$:

$$\int_0^t \int_\Omega u_0(x-\xi, t-\tau)\left(u_0^f(\xi,\tau)\, a(\xi) + P(a)(\xi,\tau)\right) d\xi d\tau = \overline{g}(x,t), (x,t) \in S_T. \quad (1.20)$$

Let $A(a)$ be the operator in the left-hand side of (1.20). Let $H_1 = L_2(\Omega)$ and $H_2 = L_2(S_T)$. Consider now the set U of functions defined as

$$U = \left\{a : a \in C^\alpha\left(\overline{\Omega}\right), \|a\|_{C^\alpha(\overline{\Omega})} \leq M\right\} \subset H_1.$$

Since the $L_2(\Omega)$ norm is weaker than the $C^\alpha\left(\overline{\Omega}\right)$-norm, then U is a bounded set in H_1. Using (1.18) and Theorem 1.1, one can prove that $A : U \to C(S_T)$ is a compact operator. Since the norm in $L_2(S_T)$ is weaker than the norm in $C(S_T)$, then $A : U \to H_2$ is also a compact operator. Hence, Theorem 1.2 implies that the problem of solution of the equation

$$A(a) = g, a \in U \subset H_1, g \in H_2$$

is ill-posed in terms of the above conclusion.

1.3 The Foundational Theorem of A.N. Tikhonov

This theorem "restores" stability of unstable problems, provided that uniqueness theorems hold for such problems. The original motivation for this theorem came from the collaboration of Tikhonov with geophysicists. To his surprise, Tikhonov has learned that geophysicists successfully solve problems which are unstable from the mathematical standpoint. Naturally, Tikhonov was puzzled by this. This puzzle has prompted him to explain that "matter of fact" stability of unstable problems from the mathematical standpoint. He has observed that geophysicists have worked with rather simple models, which included only a few abnormalities. In addition, they knew very well ranges of parameters they have worked with. Also, they knew that the functions, which they have reconstructed from measured date, had only very few oscillations. In other words, they have reconstructed only rather simple media structures. On the other hand, the Ascoli-Archela Theorem 1.1.3.1 basically requires a priori known upper bounds of both the function and its first derivatives. Clearly, there is a connection between the number of oscillations per a bounded set in $\mathbb{R}^n$ and the upper bound of the modulus of the gradient of the corresponding function. These observations have made Tikhonov to believe that actually geophysicists have worked with compact sets. This was the starting point for the formulation of the foundational Tikhonov theorem (below). In particular, this means that in an ill-posed problem, one should not expect to reconstruct a complicated fine structure of the medium of interest. Rather, one should expect to reconstruct rather simple features of this medium.

The key idea of Tikhonov was that to restore stability of an unstable problem, one should solve this problem on a compact set. The question is then whether it is reasonable to assume that the solution belongs to a specific compact set. The answer on this question lies in applications. Indeed, by, for example, Theorem 1.1.3.1, an example of a compact set in the space $C\left(\overline{\Omega}\right)$ is the set of all functions from $C^1\left(\overline{\Omega}\right)$ which are bounded together with the absolute values of their first derivatives by an a priori chosen constant. On the other hand, it is very often known in any specific application that functions of ones interest are bounded by a certain known constant. In addition, it is also known that those functions do not have too many oscillations, which is guaranteed by an a priori bound imposed on absolute values of their first derivatives. These bounds should be uniform for all functions under consideration. Similar arguments can be brought up in the case of other conventional functional spaces, like, for example, $C^k\left(\overline{\Omega}\right), H^k(\Omega)$. Another expression of these thoughts, which is often used in applications, is that the admissible range of parameters is known in advance. On the other hand, because of the compact set requirement of Theorem 1.3, the foundational Tikhonov theorem essentially requires a higher smoothness of sought for functions than one would originally expect. The latter is the true underlying reason why computed solutions of ill-posed problems usually look smoother than the original ones. In particular, sharp boundaries usually look as smooth ones.

Although the proof of Theorem 1.3 is short and simple, this result is one of only a few backbones of the entire theory of ill-posed problems.

Theorem 1.3 (Tikhonov [152], 1943). *Let B_1 and B_2 be two Banach spaces. Let $U \subset B_1$ be a compact set and $F : U \to B_2$ be a continuous operator. Assume that the operator F is one-to-one. Let $V = F(U)$. Then the inverse operator $F^{-1} : V \to U$ is continuous.*

Proof. Assume the opposite: that the operator F^{-1} is not continuous on the set V. Then, there exists a point $y_0 \in V$ and a number $\varepsilon > 0$ such that for any $\delta > 0$, there exists a point y_δ such that although $\|y_\delta - y_0\|_{B_2} < \delta$, still $\left\|F^{-1}(y_\delta) - F^{-1}(y_0)\right\|_{B_1} \geq \varepsilon$. Hence, there exists a sequence $\{\delta_n\}_{n=1}^{\infty}$, $\lim_{n\to\infty}\delta_n = 0^+$ and the corresponding sequence $\{y_n\}_{n=1}^{\infty} \subset V$ such that

$$\|y_{\delta_n} - y_0\|_{B_2} < \delta_n, \left\|F^{-1}(y_n) - F^{-1}(y_0)\right\|_{B_1} \geq \varepsilon. \tag{1.21}$$

Denote

$$x_n = F^{-1}(y_n), x_0 = F^{-1}(y_0). \tag{1.22}$$

Then

$$\|x_n - x_0\|_{B_1} \geq \varepsilon. \tag{1.23}$$

Since U is a compact set and all points $x_n \in U$, then one can extract a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subseteq \{x_n\}_{n=1}^{\infty}$ from the sequence $\{x_n\}_{n=1}^{\infty}$. Let $\lim_{k\to\infty} x_{n_k} = \overline{x}$. Then $\overline{x} \in U$. Since $F\left(x_{n_k}\right) = y_{n_k}$ and the operator F is continuous, then by (1.21) and (1.22), $F(\overline{x}) = y_0 = F(x_0)$. Since the operator F is one-to one, we should have $\overline{x} = x_0$. However, by (1.23), $\|\overline{x} - x_0\|_{B_1} \geq \varepsilon$. We got a contradiction. □

1.4 Classical Correctness and Conditional Correctness

The notion of the classical correctness is called sometimes *correctness by Hadamard*.

Definition 1.4.1. Let B_1 and B_2 be two Banach spaces. Let $G \subseteq B_1$ be an open set and $F : G \to B_2$ be an operator. Consider the equation

$$F(x) = y, \ x \in G. \tag{1.24}$$

The problem of solution of (1.24) is called *well-posed by Hadamard,* or simply *well-posed,* or *classically well-posed* if the following three conditions are satisfied:

1. For any $y \in B_2$, there exists a solution $x = x(y)$ of (1.24) (existence theorem).
2. This solution is unique (uniqueness theorem).
3. The solution $x(y)$ depends continuously on y. In other words, the operator $F^{-1} : B_2 \to B_1$ is continuous.

Thus, the well-posedness by Hadamard means the existence of the solution of the operator equation (1.24) for any right-hand side y. This solution should be unique. In addition, it should depend on the data y continuously. All classical boundary value problems for PDEs, which are studied in the standard PDE course, satisfy these criteria and are, therefore, well-posed by Hadamard.

If (1.24) does not satisfy to at least one these three conditions, then the problem (1.24) is called *ill-posed*. The most pronounced feature of an ill-posed problem is its instability, i.e., small fluctuations of y can lead to large fluctuations of the solution x. The definition of the correctness by Tikhonov, or conditional correctness, reflects the above Theorems 1.2 and 1.3.

Since the experimental data are always given with a random noise, we need to introduce the notion of the error in the data. In practice, this error is always due to that random noise as well as due to an inevitable discrepancy between the mathematical model and the reality. However, we do not assume the randomness of

y in (1.24). Let $\delta > 0$ be a small number. We say that the right-hand side of (1.24) is given with an error of the level δ if $\|y^* - y\|_{B_2} \leq \delta$, where y^* is the exact value of y, which has no error.

Definition 1.4.2. Let B_1 and B_2 be two Banach spaces. Let $G \subset B_1$ be an a priori chosen set of the form $G = \overline{G}_1$, where G_1 is an open set in B_1. Let $F : G \to B_2$ be a continuous operator. Suppose that the right-hand side of (1.24) $y := y_\delta$ is given with an error of the level $\delta > 0$, where δ is a small number, $\|y^* - y_\delta\|_{B_2} \leq \delta$. Here, y^* is the ideal noiseless data y^*. The problem (1.24) is called *conditionally well-posed on the set* G, or *well-posed by Tikhonov* on the set G, if the following three conditions are satisfied:

1. It is a priori known that there exists an ideal solution $x^* = x^*(y^*) \in G$ of this problem for the ideal noiseless data y^*.
2. The operator $F : G \to B_2$ is one-to-one.
3. The inverse operator F^{-1} is continuous on the set $F(G)$.

Definition 1.4.3. The set G of Definition 1.4.2 is called *correctness set* for the problem (1.24).

We point out that, unlike the classical well-posedness, the conditional well-posedness, does not require the correctness set G to coincide with the entire Banach space B_1. Likewise, Definition 1.4.2 does not require a proof of an existence theorem, unlike the classical case. Indeed, it follows from Theorem 1.2 that it is hopeless to prove such a theorem for (1.11). In addition, such a result would not have a practical meaning. For comparison, recall that a significant part of the classical PDE theory is devoted to proofs of existence theorems, as it is required by the definition of the classical well-posedness. On the other hand, in the definition of the conditional well-posedness the existence is assumed a priori. Still, the existence is assumed not for every y in (1.24) but only for an *ideal*, noiseless $y := y^*$. The assumption of the existence of the ideal solution x^* is a very important notion of the theory of ill-posed problems. Neither the ideal right-hand side y^* nor the ideal solution x^* are never known in applications. This is because of the presence of the noise in any experiment. Still, this assumption is a quite reasonable one because actually, it tells one that the physical process is indeed in place and that the mathematical model, which is described by the operator F, governs this process accurately.

The second condition in Definition 1.4.2 means uniqueness theorem. Combined with Theorem 1.3, this condition emphasizes the importance of uniqueness theorems for the theory of ill-posed problems.

The third condition in Definition 1.4.2 means that the solution of the problem (1.24) is stable with respect to small fluctuations of the right-hand side y, as long as $x \in G$. This goes along well with Theorem 1.3. In other words, the third condition restores the most important feature: stability. The requirement that the correctness set $G \subset B_1$ is not conventionally used in the classical theory of PDEs. In other words, the requirement of x belonging to a "special" subset of B_1 is not imposed in classically well-posed problems.

Motivated by the above arguments, Tikhonov has introduced the Fundamental Concept of Tikhonov.

The Fundamental Concept of Tikhonov. This concept consists of the following three conditions which should be in place when solving the ill-posed problem (1.24):

1. One should a priori assume that there exists an ideal exact solution x^* of (1.24) for an ideal noiseless data y^*.
2. The correctness set G should be chosen a priori, meaning that some a priori bounds imposed on the solution x of (1.24) should be imposed.
3. To construct a stable numerical method for the problem (1.24), one should assume that there exists a family $\{y_\delta\}$ of right-hand sides of (1.24), where $\delta > 0$ is the level of the error in the data with $\|y^* - y_\delta\|_{B_2} \leq \delta$. Next, one should construct a family of approximate solutions $\{x_\delta\}$ of (1.24), where x_δ corresponds to y_δ. The family $\{x_\delta\}$ should be such that

$$\lim_{\delta \to 0+} \|x_\delta - x^*\| = 0.$$

1.5 Quasi-solution

The concept of quasi-solutions was originally proposed by Ivanov [85]. It is designed to provide a rather general method for solving the ill-posed problem (1.24). This concept is actually a quite useful, as long as one is seeking a solution on a compact set. An example is when the solution is parametrized, i.e.,

$$x = \sum_{i=1}^{N} a_i \varphi_i,$$

where elements $\{\varphi_i\}$ are a part of an orthonormal basis in a Hilbert space, the number N is fixed, and coefficients $\{a_i\}_{n=1}^{N}$ are unknown. So, one is seeking numbers $\{a_i\}_{n=1}^{N} \subset G$, where $G \subset \mathbb{R}^N$ is a priori chosen closed bounded set. This set is called sometimes "the set of admissible parameters."

Since the right-hand side y of (1.24) is given with an error, Theorem 1.2 implies that it is unlikely that y belongs to the range of the operator F. Therefore, the following natural question can be raised about the usefulness of Theorem 1.3: *Since the right-hand side y of (1.24) most likely does not belong to the range $F(G)$ of the operator F, then what is the practical meaning of solving this equation on the compact set G, as required by Theorem 1.3?* The importance of the notion of quasi-solutions is that it addresses this question in a natural way.

Suppose that the problem (1.24) is conditionally well-posed and let $G \subset B_1$ be a compact set. Then, the set $F(G) \subset B_2$ is also a compact set. We have $\|y - y^*\|_{B_2} \leq \delta$. Consider the minimization problem

$$\min_G J(x), \text{ where } J(x) = ||F(x) - y||_{B_2}^2. \tag{1.25}$$

Since G is a compact set, then there exists a point $x = x(y_\delta) \in G$ at which the minimum in (1.25) is achieved. In fact, one can have many points $x(y_\delta)$. Nevertheless, it follows from Theorem 1.5 that they are located close to each other, as long as the number δ is sufficiently small.

Definition 1.5. Any point $x = x(y) \in G$ of the minimum of the functional $J(x)$ in (1.25) is called *quasi-solution* of equation in (1.24) on the compact set G.

A natural question is, *how far is the quasi-solution from the exact solution x^*?* Since by Theorem 1.3 the operator $F^{-1} : F(G) \to G$ is continuous and the set $F(G)$ is compact, then one of classical results of real analysis implies that there exists the modulus of the continuity $\omega_F(z)$ of the operator F^{-1} on the set $F(G)$. The function $\omega_F(z)$ satisfies the following four conditions:

1. $\omega_F(z)$ is defined for $z \geq 0$.
2. $\omega_F(z) > 0$ for $z > 0$, $\omega_F(0) = 0$, and $\lim_{z\to 0^+} \omega_F(z) = 0$.
3. The function $\omega_F(z)$ is monotonically increasing for $z > 0$.
4. For any two points $y_1, y_2 \in F(G)$, the following estimate holds:

$$||F^{-1}(y_1) - F^{-1}(y_2)||_{B_1} \leq \omega_F(||y_1 - y_2)||_{B_2}).$$

The following theorem characterizes the accuracy of the quasi-solution:

Theorem 1.5. *Let B_1 and B_2 be two Banach spaces, $G \subset B_1$ be a compact set, and $F : G \to B_2$ be a continuous one-to-one operator. Consider (1.24). Suppose that its right-hand side $y := y_\delta$ is given with an error of the level $\delta > 0$, where δ is a small number, $\|y^* - y_\delta\|_{B_2} \leq \delta$. Here, y^* is the ideal noiseless data y^*. Let $x^* \in G$ be the ideal exact solution of (1.24) corresponding to the ideal data y^*, i.e., $F(x^*) = y^*$. Let x_δ^q be a quasi-solution of (1.24), i.e.,*

$$J(x_\delta^q) = \min_G ||F(x) - y_\delta||_{B_2}^2. \tag{1.26}$$

Let $\omega_F(z), z \geq 0$ be the modulus of the continuity of the operator $F^{-1} : F(G) \to G$ which exists by Theorem 1.3. Then the following error estimate holds

$$\left\|x_\delta^q - x^*\right\|_{B_1} \leq \omega_F(2\delta). \tag{1.27}$$

In other words, the problem of finding a quasi-solution is stable, and two quasi-solutions are close to each other as long as the error in the data is small.

Proof. Since $\|y^* - y_\delta\|_{B_2} \leq \delta$, then

$$J\left(x^*\right) = ||F(x^*) - y_\delta||_{B_2}^2 = \|y^* - y_\delta\|_{B_2}^2 \leq \delta^2.$$

Since the minimal value of the functional $J(x^*)$ is achieved at the point x_δ^q, then

$$J(x_\delta^q) \leq J\left(x^*\right) \leq \delta^2.$$

Hence, $||F(x_\delta^q) - y_\delta||_{B_2} \leq \delta$. Hence,

$$||F(x_\delta^q) - F(x^*)||_{B_2} \leq ||F(x_\delta^q) - y_\delta||_{B_2} + \|y_\delta - F(x^*)\|_{B_2}$$
$$= ||F(x_\delta^q) - y_\delta||_{B_2} + \|y_\delta - y^*\|_{B_2} \leq 2\delta.$$

Thus, we have obtained that $||F(x_\delta^q) - F(x^*)||_{B_2} \leq 2\delta$. Therefore, the definition of the modulus of the continuity of the operator F^{-1} implies (1.27). □

This theorem is very important for justifying the practical value of Theorem 1.3. Still, the notion of the quasi-solution has a drawback. This is because it is unclear how to actually find the target minimizer in practical computations. Indeed, to find it, one should minimize the functional $J(x)$ on the compact set G. The commonly acceptable minimization technique for any least squares functional is via searching points where the Frechét derivative of that functional equals zero. However, the well-known obstacle on this path is that this functional might have multiple local minima and ravines. Therefore, most likely, the norm of the Frechét derivative is sufficiently small at many points of, for example, a ravine. Thus, it is unclear how to practically select a quasi-solution. In other words, we come back again to the first central question of this book: *How to find a good approximation for the exact solution without an advanced knowledge of a small neighborhood of this solution?*

1.6 Regularization

To solve ill-posed problems, regularization methods should be used. In this section, we present main ideas of the regularization. Note that we do not assume in Definition 1.6 that the operator F is defined on a compact set.

Definition 1.6. Let B_1 and B_2 be two Banach spaces and $G \subset B_1$ be a set. Let the operator $F : G \to B_2$ be one-to-one. Consider the equation

$$F(x) = y. \tag{1.28}$$

Let y^* be the ideal noiseless right-hand side of (1.28) and x^* be the ideal noiseless solution corresponding to y^*, $F(x^*) = y^*$. Let $\delta_0 \in (0, 1)$ be a sufficiently small number. For every $\delta \in (0, \delta_0)$ denote

$$K_\delta\left(y^*\right) = \left\{z \in B_2 : \|z - y^*\|_{B_2} \leq \delta\right\}.$$

Let $\alpha > 0$ be a parameter and $R_\alpha : K_{\delta_0}(y^*) \to G$ be a continuous operator depending on the parameter α. The operator R_α is called the *regularization operator* for (1.28) if there exists a function $\alpha(\delta)$ defined for $\delta \in (0, \delta_0)$ such that

$$\lim_{\delta \to 0} \left\| R_{\alpha(\delta)}(y_\delta) - x^* \right\|_{B_1} = 0.$$

The parameter α is called the regularization parameter. The procedure of constructing the approximate solution $x_{\alpha(\delta)} = R_{\alpha(\delta)}(y_\delta)$ is called the *regularization procedure*, or simply *regularization*.

There might be several regularization procedures for the same problem. This is a simplified notion of the regularization. In our experience, in the case of CIPs, usually $\alpha(\delta)$ is a vector of regularization parameters, for example, the number of iterations, the truncation value of the parameter of the Laplace transform, and the number of finite elements. Since this vector has many coordinates, then its practical choice is usually quite time-consuming. This is because one should choose a proper combination of several components of the vector $\alpha(\delta)$.

The first example of the regularization was Example 2 of Sect. 1.6. We now present the second example. Consider the problem of the solution of the heat equation with the reversed time. Let the function $u(x,t)$ be the solution of the following problem:

$$u_t = u_{xx}, \; x \in (0,\pi), \; t \in (0,T),$$
$$u(x,T) = y(x) \in L_2(0,\pi),$$
$$u(0,t) = u(\pi,t) = 0.$$

Uniqueness theorem for this and a more general problem is well known and can be found in, for example, the book [124]. Obviously, the solution of this problem, if it exists, is

$$u(x,t) = \sum_{n=1}^{\infty} y_n e^{n^2(T-t)} \sin nx, \tag{1.29}$$

$$y_n = \sqrt{\frac{2}{\pi}} \int_0^{\pi} y(x) \sin nx \mathrm{d}x.$$

It is clear, however, that the Fourier series (1.29) converges a narrow class of functions $y(x)$. This is because the numbers $\{e^{n^2(T-t)}\}_{n=1}^{\infty}$ grow exponentially with n.

To regularize this problem, consider the following approximation for the function $u(x,t)$:

$$u_N(x,t) = \sum_{n=1}^{N} y_n e^{n^2(T-t)} \sin nx.$$

Here, $\alpha = 1/N$ is the regularization parameter. To show that this is indeed a regularization procedure in terms of Definition 1.6, we need to consider the following:

Inverse Problem. For each function $f \in L_2(0,\pi)$, consider the solution of the following initial boundary value problem

$$
\begin{aligned}
v_t &= v_{xx},\ x \in (0,\pi)\,,\ t \in (0,T),\\
v(x,0) &= f(x),\\
v(0,t) &= v(\pi,t) = 0.
\end{aligned}
\tag{1.30}
$$

Given the function $y(x) = v(x,T)$, determine the initial condition $f(x)$ in (1.30).

Define the operator $F : L_2(0,\pi) \to L_2(0,\pi)$ as $F(f) = v(x,T)$. It is known from the standard PDEs course that

$$
F(f) = v(x,T) = \int\limits_0^{\pi} G(x,\xi,T)\, f(\xi)\,\mathrm{d}\xi, \tag{1.31}
$$

$$
G(x,\xi,t) = \sum_{n=1}^{\infty} \mathrm{e}^{-n^2 t} \sin nx \sin n\xi,
$$

where G is the Green's function for the problem (1.30). In other words, we have obtained the integral equation (1.31) of the first kind. Hence, Theorem 1.2 implies that the operator F^{-1} cannot be continuous.

Following the fundamental concept of Tikhonov, let $y^* \in L_2(0,\pi)$ be the "ideal" noiseless function y, which corresponds to the function f^* in (1.30). Let the function $y_\delta \in L_2(0,\pi)$ be such that $\|y_\delta - y^*\|_{L_2(0,\pi)} \le \delta$. Define the regularization parameter $\alpha := 1/N$ and the regularization operator $R_\alpha(y)$ as

$$
R_\alpha(y_\delta)(x) = \sum_{n=1}^{N} y_n \mathrm{e}^{n^2(T-t)} \sin nx, \tag{1.32}
$$

$$
y_n = \sqrt{\frac{2}{\pi}} \int\limits_0^{\pi} y_\delta(x) \sin nx\mathrm{d}x.
$$

Let $f^* \in C^1[0,\pi]$ and $f^*(0) = f^*(\pi) = 0$. The integration by parts leads to

$$
f_n^* = \sqrt{\frac{2}{\pi}} \int\limits_0^{\pi} f^*(x) \sin nx\mathrm{d}x = \frac{1}{n}\sqrt{\frac{2}{\pi}} \int\limits_0^{\pi} \left(f^*(x)\right)' \cos nx\mathrm{d}x.
$$

Hence,

$$
\left(f_n^*\right)^2 \le \frac{\left\|(f^*(x))'\right\|^2}{n^2}.
$$

Hence,

$$
\sum_{n=N+1}^{\infty} \left(f_n^*\right)^2 \le \frac{C\left\|(f^*(x))'\right\|^2_{L_2(0,\pi)}}{N}, \tag{1.33}
$$

where $C > 0$ is a constant independent on the function f^*. Consider now the function $R_\alpha(y) - f^*$:

$$R_\alpha(y_\delta) - f^* = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{N} \left(y_n - y_n^*\right) e^{n^2 T} \sin nx - \sqrt{\frac{2}{\pi}} \sum_{n=N+1}^{\infty} f_n^* \sin nx.$$

Since functions $\left\{(2/\pi)^{1/2} \sin nx\right\}_{n=1}^{\infty}$ form an orthonormal basis in $L_2(0,\pi)$, then

$$\|R_\alpha(y) - f^*\|_{L_2(0,\pi)}^2 \le e^{2N^2 T} \sum_{n=1}^{N} \left(y_n - y_n^*\right)^2 + \sum_{n=N+1}^{\infty} \left(f_n^*\right)^2.$$

This implies that

$$\|R_\alpha(y) - f^*\|_{L_2(0,\pi)}^2 \le e^{2N^2 T} \delta^2 + \sum_{n=N+1}^{\infty} \left(f_n^*\right)^2. \tag{1.34}$$

The second term in the right-hand side of (1.34) is independent on the level of error δ. However, it depends on the exact solution as well as on the regularization parameter $\alpha = 1/N$. So, the idea of obtaining the error estimate here is to balance these two terms via equalizing them. To do this, we need to impose an a priori assumption first about the maximum of a certain norm of the exact solution f^*. Hence, we assume that $\left\|(f^*)'\right\|_{L_2(0,\pi)}^2 \le M^2$, where M is a priori given positive constant. This means, in particular, that the resulting estimate of the accuracy of the regularized solution will hold uniformly for all functions f^* satisfying this condition. This is a typical scenario in the theory of ill-posed problems and it goes along well with Theorem 1.3.

Using (1.33), we obtain from (1.34)

$$\|R_\alpha(y_\delta) - f^*\|_{L_2(0,\pi)}^2 \le e^{2N^2 T} \delta^2 + \frac{CM^2}{N}. \tag{1.35}$$

The right-hand side of (1.35) contains two terms, which we need to balance by equalizing them:

$$e^{2N^2 T} \delta^2 = \frac{CM^2}{N}.$$

Since $e^{2N^2 T} N \left(CM^2\right)^{-1} < e^{3N^2 T}$ for sufficiently large N, we set

$$e^{3N^2 T} = \frac{1}{\delta^2}.$$

Hence, the regularization parameter is

$$\alpha(\delta) := \frac{1}{N(\delta)} := \left\{ \left[\ln\left(\delta^{-2/3T}\right) \right]^{1/2} \right\}^{-1}.$$

Here, $\{a\}$ denotes the least integer for a number $a > 0$. Thus, (1.35) implies that

$$\|R_\alpha(y_\delta) - f^*\|^2_{L_2(0,\pi)} \leq \delta^{2/3} + \frac{CM^2}{\left[\ln\left(\delta^{-2/3T}\right)\right]^{1/2}}.$$

It is clear that the right-hand side of this inequality tends to zero as $\delta \to 0$. Hence, $R_{\alpha(\delta)}$ is indeed a regularization operator for the above inverse problem.

In simpler terms, the number N of terms of the Fourier series (1.32) rather than $1/N$ is the regularization parameter here. It is also well known from the literature that the number of iterations in an iterative algorithm can serve as a regularization parameter. Since in this chapter we want to outline only main principles of the theory of ill-posed problems rather than working with advanced topics of this theory, we now derive from the above a simple example illustrating that the iteration number can indeed be used as a regularization parameter; see [65, 93, 124, 153] for more advanced examples. Indeed, in principle, we can construct the regularized solution (1.32) iteratively via

$$\begin{aligned} f_1 &= y_1 e^{n^2(T-t)} \sin x, \; f_2 = f_1 + y_2 e^{2^2(T-t)} \sin 2x, ..., \\ f_N &= f_{N-1} + y_N e^{N^2(T-t)} \sin Nx. \end{aligned} \tag{1.36}$$

It is clear from (1.36) that the number of iterations $N = N(\delta)$ can be considered as a regularization parameter here.

1.7 The Tikhonov Regularization Functional

Tikhonov has constructed a general regularization functional which works for a broad class of ill-posed problems [153, 154]. That functional carries his name in the literature. In the current section, we construct this functional and study its properties. We point out that the first stage of the two-stage numerical procedure of this book does not use this functional. The Tikhonov functional has proven to be a very powerful tool for solving ill-posed problems.

1.7.1 The Tikhonov Functional

Let B_1 and B_2 be two Banach spaces. Let Q be another space, $Q \subset B_1$ as a set, and $\overline{Q} = B_1$, where the closure is understood in the norm of the space B_1. In addition, we assume that Q is compactly embedded in B_1. It follows from Theorems 1.1.3.1 and 1.1.3.2 that Q and B_1 are:

(a) $B_1 = L_2(\Omega), Q = H^k(\Omega), \forall k \geq 1$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain.
(b) $B_1 = C^m\left(\overline{\Omega}\right), Q = C^{m+k}\left(\overline{\Omega}\right), \forall m \geq 0, \forall k \geq 1$, where m and k are integers.
(c) $B_1 = C^m\left(\overline{\Omega}\right), Q = H^k(\Omega), k > [n/2] + m$, assuming that $\partial\Omega \in C^k$.

Let $G \subset B_1$ be the closure of an open set. Consider a continuous one-to-one operator $F : G \to B_2$. The continuity here is in terms of the pair of spaces B_1, B_2, rather that in terms of the pair Q, B_2. We are again interested in solving the equation

$$F(x) = y, x \in G. \tag{1.37}$$

Just as above, we assume that the right-hand side of this equation is given with an error of the level δ. Let y^* be the ideal noiseless right-hand side corresponding to the ideal exact solution x^* :

$$F\left(x^*\right) = y^*, \quad \|y - y^*\|_{B_2} < \delta. \tag{1.38}$$

To find an approximate solution of (1.37), we minimize the Tikhonov regularization functional $J_\alpha(x)$:

$$J_\alpha(x) = \frac{1}{2}\|F(x) - y\|_{B_2}^2 + \frac{\alpha}{2}\|x - x_0\|_Q^2, \tag{1.39}$$

$$J_\alpha : G \to \mathbb{R}, \quad x_0 \in G,$$

where $\alpha = \alpha(\delta) > 0$ is a small regularization parameter and the point $x_0 \in Q$. In general, the choice of the point x_0 depends on the problem at hands. Usually, x_0 is a good first approximation for the exact solution x^*. Because of this, x_0 is sometimes called the *first guess* or the *first approximation*. The dependence $\alpha = \alpha(\delta)$ will be specified later. The term $\alpha\|x - x_0\|_Q^2$ is called the *Tikhonov regularization term* or simply the *regularization term*. Consider a sequence $\{\delta_k\}_{k=1}^{\infty}$ such that $\delta_k > 0, \lim_{k\to\infty}\delta_k = 0$. We want to construct sequences $\{\alpha(\delta_k)\}, \left\{x_{\alpha(\delta_k)}\right\}$ such that

$$\lim_{k\to\infty}\left\|x_{\alpha(\delta_k)} - x^*\right\|_{B_1} = 0.$$

Hence, if such a sequence will be constructed, then we will approximate the exact solution x^* in a stable way, and this would correspond well with the second condition of the fundamental concept of Tikhonov.

Using (1.38) and (1.39), we obtain

$$J_\alpha\left(x^*\right) \le \frac{\delta^2}{2} + \frac{\alpha}{2}\|x^* - x_0\|_Q^2 \le \frac{\delta^2}{2} + \frac{\alpha}{2}\|x^* - x_0\|_Q^2. \tag{1.40}$$

Let

$$m_{\alpha(\delta_k)} = \inf_G J_{\alpha(\delta_k)}(x).$$

By (1.40),

$$m_{\alpha(\delta_k)} \le \frac{\delta_k^2}{2} + \frac{\alpha(\delta_k)}{2}\|x^* - x_0\|_Q^2.$$

Hence, there exists a point $x_{\alpha(\delta_k)} \in G$ such that

$$m_{\alpha(\delta_k)} \le J_{\alpha(\delta_k)}\left(x_{\alpha(\delta_k)}\right) \le \frac{\delta_k^2}{2} + \frac{\alpha(\delta_k)}{2}\|x^* - x_0\|_Q^2. \tag{1.41}$$

Hence, by (1.39) and (1.41),

$$\left\|x_{\alpha(\delta_k)} - x_0\right\|_Q^2 \le \frac{\delta_k^2}{\alpha(\delta_k)} + \|x^* - x_0\|_Q^2. \tag{1.42}$$

Suppose that

$$\lim_{k\to\infty}\alpha(\delta_k) = 0 \text{ and } \lim_{k\to\infty}\frac{\delta_k^2}{\alpha(\delta_k)} = 0. \tag{1.43}$$

Then (1.42) implies that the sequence $\left\{x_{\alpha(\delta_k)}\right\} \subset G \subseteq Q$ is bounded in the norm of the space Q. Since Q is compactly embedded in B_1, then there exists a subsequence of the sequence $\left\{x_{\alpha(\delta_k)}\right\}$ which converges in the norm of the space B_1. For brevity and without any loss of generality, we assume that the sequence $\left\{x_{\alpha(\delta_k)}\right\}$ itself converges to a point $\overline{x} \in B_1$:

$$\lim_{k\to\infty}\left\|x_{\alpha(\delta_k)} - \overline{x}\right\|_{B_1} = 0.$$

Then (1.41) and (1.43) imply that $\lim_{k\to\infty} J_{\alpha(\delta_k)}\left(x_{\alpha(\delta_k)}\right) = 0$. On the other hand,

$$\begin{aligned}\lim_{k\to\infty} J_{\alpha(\delta_k)}\left(x_{\alpha(\delta_k)}\right) &= \frac{1}{2}\lim_{k\to\infty}\left[\|F(x_k) - y_{\delta_k}\|_{B_2}^2 + \alpha(\delta_k)\left\|x_{\alpha(\delta_k)} - x_0\right\|_Q^2\right] \\ &= \frac{1}{2}\|F(\overline{x}) - y^*\|_{B_2}^2.\end{aligned}$$

Hence, $\|F(\overline{x}) - y^*\|_{B_2} = 0$, which means that $F(\overline{x}) = y^*$. Since the operator F is one-to-one, then $\overline{x} = x^*$. Thus, we have constructed the sequence of regularization parameters $\{\alpha(\delta_k)\}_{k=1}^\infty$ and the sequence $\left\{x_{\alpha(\delta_k)}\right\}_{k=1}^\infty$ such that

$\lim_{k\to\infty} \left\| x_{\alpha(\delta_k)} - x^* \right\|_{B_1} = 0$. To ensure (1.43), one can choose, for example, $\alpha(\delta_k) = C\delta_k^{\mu}, \mu \in (0,2)$. It is reasonable to call $\left\{ x_{\alpha(\delta_k)} \right\}_{k=1}^{\infty}$ *regularizing sequence*.

Note that although both points $x_{\alpha(\delta_k)}$ and x^* belong to the space Q, convergence is proven in a weaker norm of the space B_1, which is typical for ill-posed problems. We point out that the original idea of Theorem 1.3 about compact sets plays a very important role in the above construction. The sequence $\left\{ x_{\alpha(\delta_k)} \right\}_{k=1}^{\infty}$ is called *minimizing sequence*. There are two inconveniences in the above construction. First, it is unclear how to find the minimizing sequence computationally. Second, the problem of multiple local minima and ravines of the functional (1.39) presents a significant complicating factor in the goal of the construction of such a sequence.

1.7.2 Regularized Solution

The construction of Sect. 1.7.1 does not guarantee that the functional $J_\alpha(x)$ indeed achieves it minimal value. Suppose now that the functional $J_\alpha(x)$ does achieve its minimal value, $J_\alpha(x_\alpha) = \min_G J_\alpha(x), \alpha = \alpha(\delta)$. Then $x_{\alpha(\delta)}$ is called a *regularized solution* of (1.37) for this specific value $\alpha = \alpha(\delta)$ of the regularization parameter. Let $\delta_0 > 0$ be a sufficiently small number. Suppose that for each $\delta \in (0, \delta_0)$, there exists an $x_{\alpha(\delta)}$ such that $J_{\alpha(\delta)}\left(x_{\alpha(\delta)}\right) = \min_G J_{\alpha(\delta)}(x)$. Even though one might have several points $x_{\alpha(\delta)}$, we select a single one of them for each $\alpha = \alpha(\delta)$. Indeed, it follows from the construction of Sect. 1.7.1 that all points $x_{\alpha(\delta)}$ are close to the exact solution x^*, as long as δ is sufficiently small. It makes sense now to relax a little bit the definition of Sect. 1.6 of the regularization operator. Namely, instead of the existence of a function $\alpha(\delta)$, we now require the existence of a sequence $\{\delta_k\}_{k=1}^{\infty} \subset (0,1)$ such that

$$\lim_{k\to\infty} \delta_k = 0 \text{ and } \lim_{k\to\infty} \left\| R_{\alpha(\delta_k)}\left(y_{\delta_k}\right) - x^* \right\|_{B_1} = 0.$$

For each $\delta \in (0, \delta_0)$ and for each y_δ such that $\|y_\delta - y^*\|_{B_2} \le \delta$, we define the operator $R_{\alpha(\delta)}(y) = x_{\alpha(\delta)}$, where $x_{\alpha(\delta)}$ is a regularized solution. Then it follows from the construction of Sect. 1.7.1 that $R_{\alpha(\delta)}(y)$ is a regularization operator. Hence, the parameter $\alpha(\delta)$ in (1.39) is a regularization parameter for the problem (1.37).

Consider now the case when the space B_1 is a finite dimensional space. Since all norms in finite dimensional spaces are equivalent, we can set $Q = B_1 = \mathbb{R}^n$. We denote the standard euclidean norm in $\mathbb{R}^n$ as $\|\cdot\|$. Hence, we assume now that $G \subset \mathbb{R}^n$ is the closure of an open bounded domain. Hence, G is a compact set. Let $x^* \in G$ and $\alpha = \alpha(\delta)$. We have

$$J_{\alpha(\delta)}(x) = \frac{1}{2}\|F(x) - y\|_{B_2}^2 + \frac{\alpha(\delta)}{2}\|x - x_0\|^2,$$

$$J_{\alpha(\delta)} : G \to \mathbb{R},\ x_0 \in G.$$

By the Weierstrass' theorem, the functional $J_{\alpha(\delta)}(x)$ achieves its minimal value on the set G. Let $x_{\alpha(\delta)}$ be a minimizer of the functional $J_{\alpha(\delta)}(x)$ on G (there might be several minimizers). Then

$$J_{\alpha(\delta)}\left(x_{\alpha(\delta)}\right) \leq J_{\alpha(\delta)}\left(x^*\right) = \frac{1}{2}\left\|F(x^*) - y\right\|_{B_2}^2 + \frac{\alpha}{2}\left\|x^* - x_0\right\|^2$$
$$\leq \frac{\delta^2}{2} + \frac{\alpha(\delta)}{2}\left\|x^* - x_0\right\|^2.$$

Hence,

$$\left\|x_{\alpha(\delta)} - x_0\right\| \leq \sqrt{\frac{\delta^2}{\alpha} + \left\|x^* - x_0\right\|^2} \leq \frac{\delta}{\sqrt{\alpha}} + \left\|x^* - x_0\right\|. \tag{1.44}$$

Since $\left\|x_{\alpha(\delta)} - x_0\right\| \geq \left\|x_{\alpha(\delta)} - x^*\right\| - \left\|x^* - x_0\right\|$, then we obtain from (1.44)

$$\left\|x_{\alpha(\delta)} - x^*\right\| \leq \frac{\delta}{\sqrt{\alpha}} + 2\left\|x^* - x_0\right\|. \tag{1.45}$$

An important conclusion from (1.45) is that for a given pair $(\delta, \alpha(\delta))$, the accuracy of the regularized solution is determined by the accuracy of the first guess x_0. This becomes even more clear when we recall that by (1.43), we should have $\lim_{\delta\to 0}\left(\delta/\sqrt{\alpha}\right) = 0$. This once again points toward the importance of the first central question of this book.

1.8 The Accuracy of the Regularized Solution for a Single Value of α

It was proven in Sect. 1.7.1 that the regularizing sequence $\left\{x_{\alpha(\delta_k)}\right\}_{k=1}^{\infty}$ converges to the exact solution x^* provided that $\lim_{k\to\infty}\delta_k = 0$. However, $\left\{x_{\alpha(\delta_k)}\right\}_{k=1}^{\infty}$ is only a subsequence of a certain sequence, which is inconvenient for computations. In addition, in practical computations, one always works only with a single value of the noise level δ and with a single value of the regularization parameter $\alpha(\delta)$. In these computations, people naturally work with finite dimensional spaces, in which the existence of a regularized solution is guaranteed; see Sect. 1.7.2. Naturally, one would want the regularized solution to be more accurate than the first guess for a single pair $(\delta, \alpha(\delta))$. It has been often observed in numerical studies of many researchers that even though parameters δ and $\alpha(\delta)$ are fixed, the regularized solution $x_{\alpha(\delta)}$ is indeed closer to the exact solution x^* than the first approximation x_0. The first analytical proof of this phenomenon was presented in the work [111]. Basically, Theorem 2 of [111] states that the regularized solution is indeed closer to the exact one than the first approximation in the case when uniqueness theorem holds for the original ill-posed problem. In this section, we present the main idea of [111]. An application of this idea to specific CIPs can be found in [111].

We assume that conditions of Sect. 1.7.1 which were imposed there on spaces and the operator F hold. Consider again the equation

$$F(x) = y, \ x \in G. \tag{1.46}$$

Just as above, we assume that the right-hand side of this equation is given with an error of the level δ. Let y^* be the ideal noiseless data corresponding to the ideal solution x^*:

$$F\left(x^*\right) = y^*, \quad \|y - y^*\|_{B_2} \leq \delta. \tag{1.47}$$

To find an approximate solution of (1.46), we minimize the Tikhonov regularization functional $J_\alpha(x)$:

$$J_\alpha(x) = \frac{1}{2}\|F(x) - y\|_{B_2}^2 + \frac{\alpha}{2}\|x - x_0\|_Q^2, \tag{1.48}$$

$$J_\alpha : G \to \mathbb{R}, x_0 \in G.$$

Since it is unlikely that one can get a better accuracy of the solution than δ, then it is usually acceptable that all other parameters involved in the regularization process are much larger than δ. For example, let the number $\mu \in (0,1)$. Since $\lim_{\delta \to 0}\left(\delta^{2\mu}/\delta^2\right) = \infty$, then there exists a sufficiently small number $\delta_0(\mu) \in (0,1)$ such that $\delta^{2\mu} > \delta^2, \forall \delta \in (0, \delta_0(\mu))$. Hence, we choose below in this section

$$\alpha(\delta) = \delta^{2\mu}, \mu \in (0,1). \tag{1.49}$$

We introduce the dependence (1.49) for the sake of definiteness only. In fact, other dependencies $\alpha(\delta)$ are also possible. Let $m_{\alpha(\delta)} = \inf_G J_{\alpha(\delta)}(x)$. Then

$$m_{\alpha(\delta)} \leq J_{\alpha(\delta)}\left(x^*\right). \tag{1.50}$$

Let $\dim B_1 = \infty$. As it was noticed in the beginning of Sect. 1.7.2, we cannot prove the existence of a minimizer of the functional J_α in this case. Hence, we work now with the minimizing sequence. It follows from (1.48) and (1.50) that there exists a sequence $\{x_n\}_{n=1}^\infty \subset G$ such that

$$m_{\alpha(\delta)} \leq J_{\alpha(\delta)}(x_n) \leq \frac{\delta^2}{2} + \frac{\alpha}{2}\|x^* - x_0\|_Q^2 \text{ and } \lim_{n \to \infty} J_{\alpha(\delta)}(x_n) = m(\delta). \tag{1.51}$$

By (1.42) and (1.51),

$$\|x_n\|_Q \leq \left(\frac{\delta^2}{\alpha} + \|x^* - x_0\|_Q^2\right)^{1/2} + \|x_0\|_Q. \tag{1.52}$$

Hence, it follows from (1.49) and (1.52) that $\{x_n\}_{n=1}^{\infty} \subset K(\delta, x_0)$, where $K(\delta, x_0) \subset Q$ is a precompact set in B_1 defined as

$$K(\delta, x_0) = \left\{ x \in Q : \|x\|_Q \leq \sqrt{\delta^{2(1-\mu)} + \|x^* - x_0\|_Q^2} + \|x_0\|_Q \right\}. \tag{1.53}$$

Note that the sequence $\{x_n\}_{n=1}^{\infty}$ depends on δ. Let $\overline{K}(\delta, x_0)$ be the closure of the set $K(\delta, x_0)$ in the norm of the space B_1. Hence, $\overline{K}(\delta, x_0)$ is a closed compact set in B_1.

Theorem 1.8 ([111]). *Let B_1 and B_2 be two Banach spaces. Let Q be another Banach space and $Q \subset B_1$ as a set. Assume that $\overline{Q} = B_1$and Q is compactly embedded in B_1. Let $G \subseteq Q$ be a convex set and $F : G \to B_2$ be a one-to-one operator, continuous in terms of norms $\|\cdot\|_{B_1}, \|\cdot\|_{B_2}$. Consider the problem of solution of (1.46). Let y^* be the ideal noiseless right-hand side of (1.46) and x^* be the corresponding exact solution of (1.46), $F(x^*) = y^*$. Let $\|y - y^*\|_{B_2} \leq \delta$. Consider the Tikhonov functional (1.48), assume that (1.49) holds and that $x_0 \neq x^*$. Let $\{x_n\}_{n=1}^{\infty} \subset K(\delta, x_0) \subseteq \overline{K}(\delta, x_0)$ be a minimizing sequence of the functional (1.48) satisfying (1.52). Let $\xi \in (0, 1)$ be an arbitrary number. Then there exists a sufficiently small number $\delta_0 = \delta_0(\xi) \in (0, 1)$ such that for all $\delta \in (0, \delta_0)$, the following inequality holds:*

$$\|x_n - x^*\|_{B_1} \leq \xi \|x_0 - x^*\|_Q, \forall n. \tag{1.54}$$

In particular, if $\dim B_1 < \infty$, then all norms in B_1 are equivalent. In this case, we set $Q = B_1$. Then a regularized solution $x_{\alpha(\delta)}$ exists (Sect. 1.7.2) and (1.54) becomes

$$\left\| x_{\alpha(\delta)} - x^* \right\|_{B_1} \leq \xi \|x_0 - x^*\|_{B_1}. \tag{1.55}$$

In the case of noiseless data with $\delta = 0$, the assertion of this theorem remains true if one replaces above $\delta \in (0, \delta_0)$ with $\alpha \in (0, \alpha_0)$, where $\alpha_0 = \alpha_0(\xi) \in (0, 1)$ is sufficiently small.

Proof. Note that if $x_0 = x^*$, then the exact solution is found, and all $x_n = x^*$. So this is not an interesting case to consider. By (1.47), (1.48), and (1.50),

$$\|F(x_n) - y\|_{B_2} \leq \sqrt{\delta^2 + \alpha \|x_0 - x^*\|_Q^2} = \sqrt{\delta^2 + \delta^{2\mu} \|x_0 - x^*\|_Q^2}.$$

Hence,

$$\begin{aligned} \left\| F(x_n) - F\left(x^*\right) \right\|_{B_2} &= \left\| (F(x_n) - y) + \left(y - F\left(x^*\right)\right) \right\|_{B_2} \\ &= \left\| (F(x_n) - y) + \left(y - y^*\right) \right\|_{B_2} \\ &\leq \|F(x_n) - y\|_{B_2} + \|y - y^*\|_{B_2} \\ &\leq \sqrt{\delta^2 + \delta^{2\mu} \|x^* - x_0\|_1^2} + \delta. \end{aligned} \tag{1.56}$$

By Theorem 1.3, there exists the modulus of the continuity $\omega_F(z)$ of the operator

$$F^{-1} : F\left(\overline{K}(\delta, x_0)\right) \to \overline{K}(\delta, x_0).$$

By (1.56),

$$\|x_n - x^*\|_{B_1} \leq \omega_F\left(\sqrt{\delta^2 + \delta^{2\mu}\|x_0 - x^*\|_Q^2} + \delta\right). \tag{1.57}$$

Consider an arbitrary $\xi \in (0, 1)$. Then one can choose the number $\delta_0 = \delta_0(\xi)$ so small that

$$\omega_F\left(\sqrt{\delta^2 + \delta^{2\mu}\|x^* - x_0\|_Q^2} + \delta\right) \leq \xi\|x_0 - x^*\|_Q, \forall\delta \in (0, \delta_0). \tag{1.58}$$

The estimate (1.54) follows from (1.57) and (1.58). The proof for the case $\delta = 0$ is almost identical with the above. □

Thus, a simple conclusion from Theorem 1.8 is that if a uniqueness theorem holds for an ill-posed problem and the level of the error δ is sufficiently small, then the minimization of the Tikhonov functional leads to a refinement of the first guess x_0 even for a single value of the regularization parameter. This explains why the second stage of the two-stage numerical procedure of this book refines the solution obtained on the first stage.

In estimates (1.54) and (1.55) the number ξ is not specified. We now want to specify the dependence ξ from δ. To do this, we need to impose an additional assumption on the function $\omega(z)$. In fact, this assumption requires the proof of the Lipschitz stability of the problem (1.46) on the compact set $\overline{K}(\delta, x_0)$. However, in order to simplify the presentation, we do not prove the Lipschitz stability of CIPs in this book. We refer to, for example, works [14, 32, 33, 62, 79–81, 104, 161], where the Lipschitz stability was established for some CIPs via the method of Carleman estimates; see Sect. 1.10 for this method.

Corollary 1.8. *Assume that conditions of Theorem 1.8 are satisfied. Let $\omega_F(z)$ be the modulus of the continuity of the operator $F^{-1} : F\left(\overline{K}(\delta, x_0)\right) \to \overline{K}(\delta, x_0)$. Let the function $\omega_F(z)$ be such that $\omega_F(z) \leq Cz, \forall z \geq 0$ with a positive constant C* independent on z. *Then there exists a sufficiently small number $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ (1.54) becomes*

$$\|x_n - x^*\|_{B_1} \leq 3C\delta^{\mu}\|x_0 - x^*\|_Q,$$

and (1.55) becomes

$$\|x_n - x^*\|_{B_1} \leq 3C\delta^{\mu}\|x_0 - x^*\|_{B_1}.$$

In the case of the noiseless data with $\delta = 0$, one should replace δ^{μ} with α in these estimates.

Proof. It was assumed in Theorem 1.8 that $\|x^* - x_0\|_Q \neq 0$. Since, $\delta < \delta^\mu \|x^* - x_0\|_Q$ for sufficiently small δ, then for $\delta \in (0, \delta_0)$,

$$\sqrt{\delta^2 + \delta^{2\mu} \|x^* - x_0\|_Q^2} + \delta \leq \delta + \delta^\mu \|x^* - x_0\|_Q + \delta < 3\delta^\mu \|x^* - x_0\|.$$

Hence,

$$\omega_F \left(\sqrt{\delta^2 + \delta^{2\mu} \|x^* - x_0\|_Q^2} + \delta \right) \leq 3C\delta^\mu \|x^* - x_0\|, \delta \in \left(0, \overline{\delta}\right).$$

The rest of the proof follows from (1.57). □

1.9 Global Convergence in Terms of Definition 1.1.2.4

The goal of this section is to show that the two-stage numerical procedure of this book converges globally to the exact solution in the classical sense in terms of Definition 1.1.2.4. In other words, it converges globally within the frameworks of the pair of approximate mathematical models (M_1, M_2). Since we discuss in this section the two-stage numerical procedure (rather than the first stage only), we rely here on assumption 1.1.2. First, we need to prove that if the locally convergent numerical method of the second stage is based on the minimization of the Tikhonov functional, then it does not face the problem of local minima and ravines in a small neighborhood of the exact solution.

Consider a nonlinear ill-posed problem. Suppose that a numerical method for this problem is approximately globally convergent in terms of Definition 1.1.2.1. Then this method ends up with a good approximation $x_0 := x_{\text{glob}}$ for the element $x^* \in B$. The element x^* represents the unique exact solution of this problem within the framework of the approximate mathematical model M_1. To refine x_{glob}, we apply a locally convergent method satisfying conditions of Definition 1.1.2.3. Consider now the approximate mathematical model M_2 associated with the numerical method of this definition. Let the corresponding k-dimensional Banach space be $B_k \subseteq B, \dim B_k = k < \infty$. Let $x_{\text{glob}}, x^* \in B_k$. We want to refine the solution x_{glob}, which is obtained on the first stage, via the minimization of the Tikhonov functional in which the starting point of iterations would be $x_0 := x_{\text{glob}}$. We anticipate that this refinement would provide a better approximation for the exact solution x^*.

Almost any minimization procedure of a least squares functional is based on a version of the gradient method, which is a locally convergent one. The gradient method stops at a point where a certain norm of the gradient is close to zero. Hence, if this Tikhonov functional has local minima in any neighborhood of x^*, then any version of the gradient method can stop at any of those minima. However, it is unclear which of these minima should be selected as a regularized solution. On the other hand, a strongly convex functional does not have local minima. Furthermore, it is well known that if a functional is strongly convex on an open set and if it achieves its minimal value on this set, then the point of this minimum is unique, and the gradient method converges to it; see, for example, [128, 137].

Suppose that the Tikhonov functional is strongly convex in a certain small neighborhood of the point $x^* \in B_k$ (i.e., locally strongly convex). In addition, let both the regularized solution and x_{glob} belong to the interior of this neighborhood. Then local minima do not occur on the refinement stage, provided that x_{glob} is the starting point of iterations for this stage. Theorem 3.1 of [29] claims the local strong convexity of this functional in a small neighborhood of a regularized solution. In Theorem 1.9.1.2, we extend that result of [29] to the case of a small neighborhood of the exact solution x^*. In addition, we use here Theorem 1.8, which was not used in [29]. Based on Theorem 1.9.1.2, we conclude in Sect. 1.9.2 that the two-stage numerical procedure of this book converges globally in the classical sense in terms of Definition 1.1.2.4.

The local strong convexity of the Tikhonov functional was also proved in earlier publications [139, 140]. These works require the continuity of the second Fréchet derivative of the original operator F. Unlike this, we require the Lipschitz continuity of the first Fréchet derivative, which is easier to verify for CIPs.

1.9.1 The Local Strong Convexity

First, we remind the notion of the Fréchet derivative [113].

Definition 1.9.1 ([113]). Let B_1 and B_2 be two Banach spaces and $\mathcal{L}(B_1, B_2)$ be the space of bounded linear operators mapping B_1 into B_2. Let $G \subseteq B_1$ be a convex set containing interior points and $A : G \to B_2$ be an operator. Let $x \in G$ be an interior point of the set G. Let $x \in G$ be an interior point of G. Assume that

$$A(x+h) = A(x) + \left(A'(x), h\right) + \varepsilon(x,h), \forall h : x+h \in G,$$

where the operator $A'(x) \in \mathcal{L}(B_1, B_2)$ and $(A'(x), h)$ means that $A'(x)$ acts on h. Assume that

$$\lim_{\|h\|_{B_1} \to 0} \frac{\|\varepsilon(x,h)\|_{B_2}}{\|h\|_{B_1}} = 0.$$

Then the bounded linear operator $A'(x) : B_1 \to B_2$ is called the Fréchet derivative of the operator A at the point x.

Assume that the Fréchet derivative of the operator A exists for all interior points $x \in G$, and it is continuous in terms of the norm of the space $\mathcal{L}(B_1, B_2)$. Let interior points $x, z \in G$. Since G is convex, then the entire segment of the straight line connecting these two points also belongs to G. The following formula is valid:

$$A(x) - A(z) = \int_0^1 \left(A'(z + \theta(x-z)), x-z\right) d\theta. \tag{1.59}$$

Let B be a Banach space, $G \subseteq B$ be a convex set, and $J : G \to \mathbb{R}$ be a functional. The functional J is called *strongly convex* on the set G if there exists a constant $\kappa > 0$ such that for any two interior points $x, z \in G$ and for any number $\lambda \in [0, 1]$, the following inequality holds [128]:

$$\kappa \frac{\lambda (1-\lambda)}{2} \|x - z\|_B^2 + J(\lambda x + (1-\lambda) z) \le \lambda J(x) + (1-\lambda) J(z).$$

The following theorem is well known:

Theorem 1.9.1.1 ([128]). *Let H be a Hilbert space, $G \subseteq H$ be a convex set containing interior points, and $J : G \to \mathbb{R}$ be a functional. Suppose that this functional has the Fréchet derivative $J'(x) \in \mathcal{L}(H, \mathbb{R})$ for any interior point $x \in G$. Then the strong convexity of J on the set G is equivalent with the following condition:*

$$\left(J'(x) - J'(z), x - z\right) \ge 2\rho \|x - z\|^2, \forall x, z \in G, \tag{1.60}$$

where $\rho > 0$ is the strong convexity constant.

Consider now the case when $B_1 = H_1$ and $B_2 = H_2$ are two Hilbert spaces. In order not to work with a stronger norm of the regularization term in the Tikhonov functional, we assume that $\dim H_1 < \infty$ since all norms in a finite dimensional Banach space are equivalent. Denote norms in H_1 and H_2 as $\|\cdot\|$ and $\|\cdot\|_2$, respectively. The norm in the space of bounded linear operators $\mathcal{L}(H_1, H_2)$ we also denote in this section as $\|\cdot\|$ for brevity. It will always be clear from the context of this section whether the sign $\|\cdot\|$ is related to an element of H_1 or to an element of $\mathcal{L}(H_1, H_2)$. Let $G \subseteq H_1$ be a bounded closed convex set and $\widetilde{F} : G \to H_2$ be a continuous operator. Similarly with (1.46), consider the problem of the solution of the equation $\widetilde{F}(x) = y, x \in G$. We again assume that the element y (which we call "the data") is given with an error, $\|y - y^*\|_2 \le \delta$, where y^* is the exact right-hand side of this equation, which corresponds to its exact solution $x^* \in G$, $\widetilde{F}(x^*) = y^*$. It is convenient to replace in this section the operator $\widetilde{F}$ with $F : G \to H_2$ defined as $F(x) = \widetilde{F}(x) - y$. Hence, we consider the equation

$$F(x) = 0, x \in G, \tag{1.61}$$

where

$$\left\| F\left(x^*\right) \right\|_2 \le \delta. \tag{1.62}$$

Let the point $x_0 \in G$. Consider the Tikhonov functional corresponding to (1.61):

$$J_\alpha(x) = \frac{1}{2} \|F(x)\|_2^2 + \frac{\alpha}{2} \|x - x_0\|^2. \tag{1.63}$$

For any $\beta > 0$ and for any $x \in H_1$, denote $V_\beta(x) = \{z \in H_1 : \|x - z\| < \beta\}$.

Theorem 1.9.1.2. *Let H_1 and H_2 be two Hilbert spaces,* $\dim H_1 < \infty, G \subset H_1$ *be a closed bounded convex set containing interior points, and $F : G \to H_2$ be a continuous one-to-one operator. Let $x^* \in G$ be the exact solution of (1.61) with the exact data and $\delta \in (0,1)$ be the error in the data . Let (1.62) be satisfied and $V_1(x^*) \subset G$. Assume that for every $x \in V_1(x^*)$, the operator F has the Fréchet derivative $F'(x) \in \mathcal{L}(H_1, H_2)$. Suppose that this derivative is uniformly bounded and Lipschitz continuous in $V_1(x^*)$, i.e.,*

$$\|F'(x)\| \leq N_1, \ \forall x \in V_1(x^*), \tag{1.64}$$

$$\|F'(x) - F'(z)\| \leq N_2 \|x - z\|, \ \forall x, z \in V_1(x^*), \tag{1.65}$$

where $N_1, N_2 = const. > 0$. Let

$$\alpha = \alpha(\delta) = \delta^{2\mu}, \quad \forall \delta \in (0,1), \tag{1.66}$$

$$\mu = const. \in \left(0, \frac{1}{4}\right). \tag{1.67}$$

Then there exists a sufficiently small number $\delta_0 = \delta_0(N_1, N_2, \mu) \in (0,1)$ such that for all $\delta \in (0, \delta_0)$, the functional $J_{\alpha(\delta)}(x)$ is strongly convex in the neighborhood $V_{\alpha(\delta)}(x^*)$ of the exact solution x^* with the strong convexity constant $\alpha/4$. Next, let in (1.63) the first guess x_0 for the exact solution x^* be so accurate that

$$\|x_0 - x^*\| < \frac{\delta^{3\mu}}{3}. \tag{1.68}$$

Then there exists the unique regularized solution $x_{\alpha(\delta)}$ of (1.61) and $x_{\alpha(\delta)} \in V_{\delta^{3\mu}/3}(x^*)$. In addition, the gradient method of the minimization of the functional $J_{\alpha(\delta)}(x)$, which starts at x_0, converges to $x_{\alpha(\delta)}$. Furthermore, let $\xi \in (0,1)$ be an arbitrary number. Then there exists a number $\delta_1 = \delta_1(N_1, N_2, \mu, \xi) \in (0, \delta_0)$ such that

$$\left\|x_{\alpha(\delta)} - x^*\right\| \leq \xi \|x_0 - x^*\|, \ \forall \delta \in (0, \delta_1). \tag{1.69}$$

In other words, the regularized solution $x_{\alpha(\delta)}$ provides a better accuracy than the first guess x_0.

Remark 1.9.1. Consider now the noiseless case with $\delta = 0$. Then one should replace in this theorem $\delta_0 = \delta_0(N_1, N_2, \mu) \in (0,1)$ with $\alpha_0 = \alpha_0(N_1, N_2) \in (0,1)$ to be sufficiently small and require that $\alpha \in (0, \alpha_0)$.

Proof of Theorem 1.9.1.2. For any point $x \in V_1(x^*)$, let $F'^*(x)$ be the linear operator, which is adjoint to the operator $F'(x)$. By (1.63), the Fréchet derivative of the functional $J_\alpha(x)$ acts on the element $u \in H_1$ as

$$\left(J'_\alpha(x), u\right) = \left(F'^*(x) F(x) + \alpha(x - x_0), u\right), \ \forall x \in G, \forall u \in H_1.$$

Consider two arbitrary points $x, z \in V_{\delta^{3\mu}}(x^*)$. We have

$$\begin{aligned}\left(J_\alpha'(x) - J_\alpha'(z), x - z\right) &= \alpha \|x - z\|^2 + \left(F'^*(x) F(x) - F'^*(z) F(z), x - z\right) \\ &= \alpha \|x - z\|^2 + \left(F'^*(x) F(x) - F'^*(x) F(z), x - z\right) \\ &\quad + \left(F'^*(x) F(z) - F'^*(z) F(z), x - z\right).\end{aligned}$$

Denote

$$\begin{aligned}A_1 &= \left(F'^*(x) F(x) - F'^*(x) F(z), x - z\right), \\ A_2 &= \left(F'^*(x) F(z) - F'^*(z) F(z), x - z\right).\end{aligned}$$

Then

$$\left(J_\alpha'(x) - J_\alpha'(z), x - z\right) = \alpha \|x - z\|^2 + A_1 + A_2. \tag{1.70}$$

Estimate A_1, A_2 from the below. Since

$$A_1 = A_1 - \left(F'^*(x) F'(x)(x - z), x - z\right) + \left(F'^*(x) F'(x)(x - z), x - z\right),$$

then by (1.59),

$$\begin{aligned}A_1 &= \left(F'^*(x)\left(\int_0^1 \left(F'(z + \theta(x - z)) - F'(x), x - z\mathrm{d}\theta\right)\right), x - z\right) \\ &\quad + \left(F'^*(x) F'(x)(x - z), x - z\right).\end{aligned} \tag{1.71}$$

Since $\|A\| = \|A^*\|, \forall A \in \mathcal{L}(H_1, H_2)$, then using (1.64) and (1.65), we obtain

$$\begin{aligned}&\left|\left(F'^*(x)\left(\int_0^1 \left(F'(z + \theta(x - z)) - F'(x), x - z\mathrm{d}\theta\right)\right), x - z\right)\right| \\ &\leq \left\|F'(x)\right\| \int_0^1 \left\|F'(z + \theta(x - z)) - F'(x), x - z\right\|_2 \mathrm{d}\theta \cdot \|x - z\| \\ &\leq \frac{1}{2} N_1 N_2 \|x - z\|^3.\end{aligned}$$

Next,

$$\begin{aligned}\left(F'^*(x) F'(x)(x - z), x - z\right) &= \left(F'(x)(x - z), F'(x)(x - z)\right)_2 \\ &= \left\|F'(x)(x - y)\right\|_2^2 \geq 0,\end{aligned}$$

where $(,)_2$ is the scalar product in H_2. Hence, using (1.71), we obtain

$$A_1 \geq -\frac{1}{2} N_1 N_2 \|x - z\|^3 . \tag{1.72}$$

Now we estimate A_2 :

$$|A_2| \leq \|F(z)\|_2 \left\| F'(x) - F'(z) \right\| \|x - z\| \leq N_2 \|x - z\|^2 \|F(z)\|_2 .$$

By (1.59) and (1.64),

$$\|F(z)\|_2 \leq \left\| F(z) - F\left(x^*\right) \right\|_2 + \left\| F\left(x^*\right) \right\|_2 \leq N_1 \|z - x^*\| + \left\| F\left(x^*\right) \right\|_2 .$$

Hence, using (1.62), we obtain

$$|A_2| \leq N_2 \|x - z\|^2 \left(N_1 \|z - x^*\| + \left\| F\left(x^*\right) \right\|_2 \right) \leq N_2 \|x - z\|^2 \left(N_1 \delta^{3\mu} + \delta \right).$$

Thus,

$$A_2 \geq -N_2 \|x - z\|^2 \left(N_1 \delta^{3\mu} + \delta \right). \tag{1.73}$$

By (1.66) and (1.67), we can choose $\delta_0 = \delta_0 (N_1, N_2, \mu) \in (0, 1)$ and $\tau = \tau (N_1, N_2) \in (0, 1)$ so small that

$$\left(N_1 \delta^{3\mu} + \delta \right) \leq 2 N_1 \delta^{3\mu}. \tag{1.74}$$

Combining (1.73) and (1.74) with (1.66)–(1.70) and (1.72) and recalling that $x, z \in V_{\delta^{3\mu}}(x^*)$, we obtain

$$\begin{aligned} \left(J'_\alpha(x) - J'_\alpha(z), x - z \right) &\geq \|x - z\|^2 \left[\alpha - \frac{N_1 N_2}{2} \|x - z\| - 2 N_1 N_2 \delta^{3\mu} \right] \\ &\geq \|x - z\|^2 \left[\delta^{2\mu} - \frac{5}{2} N_1 N_2 \delta^{3\mu} \right] \\ &\geq \frac{\delta^{2\mu}}{2} \|x - z\|^2 = \frac{\alpha}{2} \|x - z\|^2 . \end{aligned}$$

Combing this with Theorem 1.9.1.1, we obtain the assertion about the strong convexity.

Since G is a closed bounded set in the finite dimensional space H_1, then there exists a minimizer $x_{\alpha(\delta)} \in G$ of the functional $J_{\alpha(\delta)}$ in (1.63). Combining (1.45) with (1.66), (1.67), and (1.68) and decreasing, if necessary, δ_0, we obtain for $\delta \in (0, \delta_0)$

$$\left\|x_{\alpha(\delta)} - x^*\right\| \leq \frac{\delta}{\sqrt{\alpha}} + 2\left\|x^* - x_0\right\| < \delta^{1-\mu} + \frac{2}{3}\delta^{3\mu}$$
$$= \frac{2}{3}\delta^{3\mu}\left(1 + \frac{3}{2}\delta^{1-4\mu}\right) < \frac{2}{3}\delta^{3\mu} \cdot \frac{3}{2} = \delta^{3\mu}.$$

Thus,

$$\left\|x_{\alpha(\delta)} - x^*\right\| < \delta^{3\mu}.$$

The latter implies that $x_{\alpha(\delta)} \in V_{\delta^{3\mu}}(x^*)$. Since the functional J_α is strongly convex on the set $V_{\delta^{2\mu}}(x^*)$, the set $V_{\delta^{3\mu}}(x^*) \subset V_{\delta^{2\mu}}(x^*)$ for sufficiently small δ and the minimizer $x_{\alpha(\delta)} \in V_{\delta^{3\mu}}(x^*)$, then this minimizer is unique. Furthermore, since by (1.68) the point $x_0 \in V_{\delta^{3\mu}}(x^*)$, then it is well known that the gradient method with its starting point at x_0 converges to the minimizer $x_{\alpha(\delta)}$; see, for example, [137].

Let $\xi \in (0, 1)$ be an arbitrary number. By Theorem 1.8 we can choose

$$\delta_1 = \delta_1(N_1, N_2, \mu, \xi) \in (0, \delta_0),$$

so small that

$$\left\|x_{\alpha(\delta)} - x^*\right\| \leq \xi\left\|x_0 - x^*\right\|, \forall\delta \in (0, \delta_1),$$

which proves (1.69). Hence, (1.68) implies that $x_{\alpha(\delta)} \in V_{\delta^{3\mu}/3}(x^*)$. □

1.9.2 The Global Convergence

One of main points of this book is the two-stage numerical procedure for certain CIPs, which addresses both central questions posed in the beginning of Sect. 1.1. This procedure was developed in [25–29, 115, 116, 160]. In this section, we briefly present some arguments showing that this procedure converges globally to the exact solution in terms of Definition 1.1.2.4. Corresponding theorems and numerical confirmations are presented in Chaps. 2–6. Consider one of CIPs of this book.

- On the first stage, a numerical method with the approximate global convergence property (Definition 1.1.2.1) ends up with a function $c_{\text{glob}}(x)$. Let $c^*(x) \in B$ be the exact solution of this CIP. Then corresponding approximate global convergence theorems of either Chap. 2 or Chap. 6 guarantee that the function c_{glob} provides a sufficiently good approximation for c^*.
- On the second stage, we use an approximate mathematical model M_2 to minimize the Tikhonov functional (1.63) associated with our CIP. In the case of the adaptive finite element method (FEM) this model basically means the assumption that the solution belongs to a finite dimensional space generated by all linear combinations of standard piecewise linear finite elements (see details in Chap. 4). This space is equipped with the norm $\|\cdot\|_{L_2(\Omega)}$. In the case when the Tikhonov functional is minimized via the finite difference method (FDM) (Chap. 6), this

model means a finite number of grid points in the finite difference scheme and a finite dimensional space associated with it; see, for example, [146] for this space. In any of these two cases, we have the finite dimensional Hilbert space H_1. We assume that $H_1 \subset B$ as a set, $\|\cdot\|_{H_1} \leq \|\cdot\|_B$ and $c_{\text{glob}} \in H_1$. Also, $c^* \in H_1$ (assumption 1.1.2). Following Definition 1.1.2.4 we assume that in (1.2) $x_n := c_n$ and $\varepsilon \in (0, \rho]$. Here, ρ is the number of Definition 1.1.2.3, and functions c_n are obtained in our iterative process of the numerical method of the first stage. Let $\delta \in (0, \delta_0)$ be the level of the error in the data. Let the number $\rho \in \left(0, \delta^{3\mu}/3\right)$, where the numbers δ_0, μ were defined in Theorem 1.9.1.2 Then (1.2) implies that

$$\left\|c_{\text{glob}} - c^*\right\|_{H_1} < \frac{\delta^{3mu}}{3},$$

which is exactly (1.68) with $x_0 := c_{\text{glob}}, x^* := c^*$. Theorem 1.9.1.2 implies that the regularized solution $c_{\alpha(\delta)}$ exists, and it is unique. Furthermore, (1.69) of Theorem 1.9.1.2 ensures that

$$\left\|c_{\alpha(\delta)} - c^*\right\|_{H_1} < \left\|c_{\text{glob}} - c^*\right\|_{H_1} < \frac{\delta^{3\mu}}{3}.$$

Next, again by Theorem 1.9.1.2, the gradient method of the minimization of the Tikhonov functional with its starting point c_{glob} converges to $c_{\alpha(\delta)}$. Thus, in the limiting case of $\delta \to 0$, we arrive at the exact solution c^*.

- Therefore the two-stage numerical procedure of this book converges globally in the classical sense within frameworks of the pair of approximate mathematical models (M_1, M_2), as described in Definition 1.1.2.4.
- In addition, extensive numerical and experimental studies of follow-up chapters demonstrate that conditions of the informal Definition 1.1.2.2 are also in place.

1.10 Uniqueness Theorems for Some Coefficient Inverse Problems

1.10.1 Introduction

This section is devoted to a short survey of currently known uniqueness theorems for CIPs with the data resulting from a single measurement. As it is clear from the construction of Sect. 1.7.1 as well as from Theorems 1.3, 1.8, and 1.9.1.2, the question of the uniqueness is a very important one for, for example, a justification of the validity of numerical methods for ill-posed problems. Before 1981, only the so-called "local" uniqueness theorems were known for multidimensional CIPs with single measurement data. The word "local" in this case means that it was assumed in these theorems that either the unknown coefficient is sufficiently small, or it is piecewise analytic with respect to at least one variable, or that this coefficient can be represented in a special form, or that the CIP is linearized near the constant background [124, 143]. The absence of "global" uniqueness results for these CIPs was one of main stumbling blocks of the entire theory of inverse problems at that time. The term "global" means here that the main assumption about the unknown coefficient should be that it belongs to one of main functional spaces, for example, C^k, H^k. In addition, one might probably impose some mild additional assumptions, for example, positivity. But one should not impose abovementioned "local" assumptions.

For the first time, the question about global uniqueness theorems was addressed positively and for a broad class of CIPs with single measurement data in the works of A.L. Bukhgeim and M.V. Klibanov in 1981. First, these results were announced in their joint paper [43]. The first complete proofs were published in two separate papers [44, 95] in the same issue of proceedings. This technique is now called the "Bukhgeim-Klibanov method." Currently, this method is the only one enabling for proofs of global uniqueness results for multidimensional CIPs with single measurement data. Note that the idea of the "elimination" of the unknown coefficient from the governing PDE via the differentiation, which is used in our approximately globally convergent numerical method (Chaps. 2 and 6), was originated by the Bukhgeim-Klibanov method.

The Bukhgeim-Klibanov method is based on the idea of applications of the so-called Carleman estimates to proofs of uniqueness results for CIPs. These estimates were first introduced in the famous paper of the Swedish mathematician Torsten Carleman in 1939 [50]. Roughly speaking, as soon as a Carleman estimate is valid for the operator of a PDE, then the Bukhgeim-Klibanov method leads to a certain uniqueness theorem for a corresponding CIP for this CIP. On the other hand, since Carleman estimates are known for three main types of partial differential operators of the second order (hyperbolic, parabolic, and elliptic), then this method is applicable to a wide class of CIPs. Since the publication of works [43, 44, 95] in 1981, many researchers have discussed this method in their publications. Because uniqueness is not the main topic of this book, we refer only to some samples of those publications in [14, 31–33, 45, 62, 79–81, 83, 84, 96–99, 102–104, 136]. We refer to [161] for a survey with a far more detailed list of references.

Although the Bukhgeim-Klibanov method is a very general one, there is a certain problem associated with it. This problem was viewed as a shortcoming at the time of the inception of this method. Specifically, it is required that at least one initial condition not to be zero in the entire domain of interest Ω. At the same time, the main interest in applications in, for example, the hyperbolic case, is when one of initial conditions is identically zero and another one is either the δ-function or that the wave field is initialized by the plane wave. The uniqueness question in the latter case remains a long-standing and well-known unsolved problem; see [58] for some progress in this direction.

On the other hand, the recent computational experience of the authors indicates that the above is only a mild restriction from the applied standpoint. Indeed, suppose that initial conditions for a hyperbolic equation are

$$u(x,0) = 0, u_t(x,0) = \delta(x - x_0) \tag{1.75}$$

where $x, x_0 \in \mathbb{R}^n$ and the source position $\{x_0\}$ is fixed. Then one can consider an approximation for the δ-function in the sense of distributions as

$$u^{\varepsilon}(x,0) = 0, u_t^{\varepsilon}(x,0) = \frac{1}{\left(\sqrt{\pi}\varepsilon\right)^n} \exp\left(-\frac{|x-x_0|^2}{\varepsilon^2}\right) \tag{1.76}$$

for a sufficiently small number $\varepsilon > 0$. Suppose that the domain Ω is located far from the source $\{x_0\}$, which is common in applications. Then the solution of the forward problem with initial conditions (1.76) differs negligibly from the case (1.75) for $x \in \Omega$. If a numerical method of solving this CIP is stable, as it is the case of algorithms of this book, then this negligible difference in the boundary data at $\partial\Omega$ will practically not affect computational results. On the other hand, in the case (1.76), uniqueness is restored. Therefore, the Bukhgeim-Klibanov method addresses properly the applied aspect of the uniqueness question for CIPs with single measurement data.

The single work where the problem of the zero initial condition was partially addressed is [112]. In this paper, the case of a single incident plane wave was considered. Derivatives with respect to variables, which are orthogonal to the direction of the propagation of this wave, are expressed via finite differences. Results of this work are presented in Sect. 1.11.

In Sect. 1.10, we prove uniqueness theorems for some CIPs for hyperbolic, parabolic, and elliptic PDEs using the Bukhgeim-Klibanov method. These theorems were published in somewhat different formulations in [43, 95–97, 99, 102]. For the sake of completeness, we also derive a Carleman estimate for the corresponding hyperbolic operator. Since this is an introductory chapter, we do not include here proofs of Carleman estimates for parabolic and elliptic operators and refer to Chap. 4 of [124] instead. In addition, the Carleman estimate for the Laplace operator is derived in Chap. 6 of this book. The only reason why we assume everywhere in Sect. 1.10 that the domain Ω is $\Omega = \{|x| < R\} \subset \mathbb{R}^n$, $R = \text{const.} > 0$ is our desire to simplify the presentation. Similar arguments can be considered for an arbitrary convex domain with a smooth boundary.

1.10.2 Carleman Estimate for a Hyperbolic Operator

Let $\Omega = \{|x| < R\} \subset \mathbb{R}^n$ and $T = \text{const.} > 0$. Denote

$$Q_T^{\pm} = \Omega \times (-T, T), S_T^{\pm} = \partial\Omega \times (-T, T), Q_T = \Omega \times (0, T), S_T = \partial\Omega \times (0, T).$$

Let $x_0 \in \Omega, \eta \in (0, 1)$. Consider the function $\psi(x, t)$:

$$\psi(x,t) = |x - x_0|^2 - \eta t^2. \tag{1.77}$$

We now introduce the Carleman weight function (CWF) by

$$W(x,t) = \exp[\lambda\psi(x,t)], \tag{1.78}$$

where $\lambda > 1$ is a large parameter which we will specify later. The level surfaces of the function $W(x,t)$ are hyperboloids $H_d = \left\{|x-x_0|^2 - \eta t^2 = d = \text{const}\right\}$. For $d \in (0, R^2)$, consider the domain G_d:

$$G_d = \left\{(x,t) : x \in \Omega, |x-x_0|^2 - \eta t^2 > d\right\} \subset Q_T^{\pm}. \tag{1.79}$$

Hence, $G_d \neq \varnothing$ and $\nabla_x \psi(x,t) \neq 0$ in $\overline{G}_d$. Define the hyperbolic operator L_0 as

$$L_0 u = c(x) u_{tt} - \Delta u. \tag{1.80}$$

The Carleman estimate for the operator L_0 is established in Theorem 1.10.2. As to the proof of this theorem, it should be kept in mind that proofs of Carleman estimates are always space consuming; see, for example, Chap. 4 of [124]. For brevity, we assume in Theorem 1.10.2 that the dimension of the space $\mathbb{R}^n$ is $n \geq 2$. An analog of this theorem for the case $n = 1$ can be proven similarly. This theorem was proven in [124] for the case $c \equiv 1$ and in [84, 102] for the case when the function c satisfies conditions (1.81) and (1.82). As it is clear from Theorem 1.10.2, the Carleman estimate for a partial differential operator depends only on its principal part.

Theorem 1.10.2. *Let $\Omega = \{|x| < R\} \subset \mathbb{R}^n, n \geq 2, x_0 \in \Omega$, and L_0 be the hyperbolic operator defined in (1.80). Suppose that in (1.80), the coefficient satisfies the following conditions:*

$$c \in C^1\left(\overline{\Omega}\right), c(x) \in [1, \overline{c}], \text{where } \overline{c} = const. \geq 1, \tag{1.81}$$

$$(x - x_0, \nabla c) \geq 0, \ \forall x \in \overline{\Omega}, \tag{1.82}$$

where $(\cdot,\cdot)$ denotes the dot product in $\mathbb{R}^n$. Let

$$P = P(x_0, \Omega) = \max_{x \in \overline{\Omega}} |x - x_0|. \tag{1.83}$$

Then there exist a sufficiently small number $\eta_0 = \eta_0(\overline{c}, P, \|\nabla c\|_{C(\overline{\Omega})}) \in (0,1]$ such that for any $\eta \in (0, \eta_0]$, one can choose a sufficiently large number $\lambda_0 = \lambda_0(\Omega, \eta, c, x_0) > 1$ and number $C = (\Omega, \eta, c, x_0) > 0$, such that for all $u \in C^2\left(\overline{G}_d\right)$ and for all $\lambda \geq \lambda_0$, the following pointwise Carleman estimate holds

$$(L_0 u)^2 W^2 \geq C\lambda\left(|\nabla u|^2 + u_t^2 + \lambda^2 u^2\right) W^2 + \nabla \cdot U + V_t, \text{ in } G_d, \tag{1.84}$$

where the CWF $W(x,t)$ is defined by (1.78) and components of the vector function (U,V) satisfy the following estimates:

$$|U| \leq C\lambda^3 \left(|\nabla u|^2 + u_t^2 + u^2\right) W^2, \tag{1.85}$$

$$|V| \leq C\lambda^3 \left[|t| \left(u_t^2 + |\nabla u|^2 + u^2\right) + (|\nabla u| + |u|)\, |u_t|\right] W^2. \tag{1.86}$$

In particular, (1.86) implies that if either $u(x,0) = 0$ or $u_t(x,0) = 0$, then

$$V(x,0) = 0. \tag{1.87}$$

Proof. In this proof, C denotes different positive constants depending on the same parameters as indicated in the conditions of this theorem. Also, in this proof, $O(1/\lambda)$ denotes different $C^1\left(\overline{Q_T^{\pm}}\right)$ functions such that

$$\left|O\left(\frac{1}{\lambda}\right)\right| \leq \frac{C}{\lambda}, \forall \lambda > 1, \tag{1.88}$$

and the same is true for the first derivatives of these functions. We use (1.88) in many parts of this proof below. Denote $v = u \cdot W$ and express the operator $L_0(u)$ in terms of the function v. Below $f_i := \partial_{x_i} f$. We have

$$u = v \cdot \exp\left[\lambda\left(\eta t^2 - |x - x_0|^2\right)\right],$$

$$u_t = (v_t + 2\lambda\eta t \cdot v) \exp\left[\lambda\left(\eta t^2 - |x - x_0|^2\right)\right],$$

$$u_{tt} = \left(v_{tt} + 4\lambda\eta t \cdot v_t + 4\lambda^2\left(\eta^2 t^2 + O\left(\frac{1}{\lambda}\right)\right) v\right) W^{-1},$$

$$u_i = [v_i - 2\lambda (x_i - x_{0i})\, v] \exp\left[\lambda\left(\eta t^2 - |x - x_0|^2\right)\right],$$

$$u_{ii} = \left[v_{ii} - 4\lambda (x_i - x_{0i})\, v_i + 4\lambda^2\left(|x - x_0|^2 + O\left(\frac{1}{\lambda}\right)\right) v\right] W^{-1}.$$

Hence,

$$\begin{aligned}(L_0 u)^2 W^2 &= (c(x)\, u_{tt} - \Delta u)^2 W^2 \\ &= \left\{\left[c(x)\, v_{tt} - \Delta v - 4\lambda^2\left(|x - x_0|^2 - c\eta^2 t^2 + O\left(\frac{1}{\lambda}\right)\right) v\right]\right. \\ &\quad \left. + 4\lambda c\eta t v_t + 4\lambda \sum_{i=1}^{n} (x_i - x_{0i})\, v_i\right\}^2.\end{aligned}$$

Denote

$$
\begin{aligned}
z_1 &= cv_{tt} - \Delta v - 4\lambda^2 \left(|x - x_0|^2 - c\eta^2 t^2 + O\left(\frac{1}{\lambda}\right)\right) v,\\
z_2 &= 4\lambda c\eta t \cdot v_t,\\
z_3 &= 4\lambda \sum_{i=1}^{n} (x_i - x_{0i})\, v_i .
\end{aligned}
$$

Then $(L_0 u)^2\, W^2 = (z_1 + z_2 + z_3)^2$. Hence,

$$
(L_0 u)^2\, W^2 \geq z_1^2 + 2z_1 z_2 + 2z_1 z_3. \tag{1.89}
$$

We estimate separately each term in the inequality (1.89) from the below in five steps.

Step 1. Estimate the term $2z_1z_2$. We have

$$
\begin{aligned}
2z_1 z_2 &= 8\lambda c\eta t \cdot v_t \left[cv_{tt} - \Delta v - 4\lambda^2 \left(|x - x_0|^2 - c\eta^2 t^2 + O\left(\frac{1}{\lambda}\right)\right) v\right]\\
&= \left[4\lambda c^2 \eta t \cdot v_t^2\right]_t - 4\lambda c^2 \eta v_t^2\\
&\quad + \sum_{i=1}^{n} (-8\lambda c\eta t \cdot v_t v_i)_i + \sum_{i=1}^{n} 8\lambda c\eta t \cdot v_{it} v_i\\
&\quad + 8\lambda\eta t \cdot v_t \sum_{i=1}^{n} c_i v_i + \left[-16\lambda^3 c\eta \left(t\, |x - x_0|^2 - c\eta^2 t^3 + tO\left(\frac{1}{\lambda}\right)\right) v^2\right]_t\\
&\quad + 16\lambda^3 c\eta \left(|x - x_0|^2 - 3c\eta^2 t^2 + O\left(\frac{1}{\lambda}\right)\right) v^2\\
&= -4\lambda c^2 \eta v_t^2 + \left(4\lambda c^2 \eta t \cdot v_t^2 + \sum_{i=1}^{n} 4\lambda c\eta t v_i^2 \right)_t - 4\lambda c\eta\, |\nabla v|^2\\
&\quad + 8\lambda\eta t \cdot v_t \sum_{i=1}^{n} c_i v_i + 16\lambda^3 c\eta \left[|x - x_0|^2 - 3c\eta^2 t^2 + O\left(\frac{1}{\lambda}\right)\right] v^2\\
&\quad + \nabla \cdot U_1 + \left[4\lambda c^2 \eta t v_t^2 - 16\lambda^3 c\eta \left(t\, |x - x_0|^2 - c\eta^2 t^2 + tO\left(\frac{1}{\lambda}\right)\right) v^2\right]_t
\end{aligned}
$$

Thus, we have obtained that

$$2z_1z_2 = -4\lambda c\eta\left(cv_t^2 + |\nabla v|^2\right) + 8\lambda\eta t \cdot v_t \sum_{i=1}^{n} c_i v_i$$
$$+16\lambda^3 c\eta\left[|x-x_0|^2 - 3c\eta^2 t^2 + O\left(\frac{1}{\lambda}\right)\right]v^2 + \nabla\cdot U_1 + (V_1)_t\,, \quad (1.90)$$

where the vector function (U_1, V_1) satisfies the following estimates:

$$|U_1| \le C\lambda^3\left(|\nabla u|^2 + u_t^2 + u^2\right)W^2, \quad (1.91)$$

$$|V_1| \le C\lambda^3\,|t|\left(u_t^2 + |\nabla u|^2 + u^2\right)W^2. \quad (1.92)$$

To include the function u in the estimate for $|U_1|\,,|V_1|\,,$ we have replaced v with $u = v\cdot W^{-1}.$

Step 2. We now estimate the term $2z_1z_3$. We have

$$2z_1z_3 = 8\lambda\sum_{i=1}^{n}(x_i - x_{0i})\,v_i\left[cv_{tt} - \Delta v - 4\lambda^2\left(|x-x_0|^2 - c\eta^2t^2 + O\left(\frac{1}{\lambda}\right)\right)v\right]$$
$$= \left(\sum_{i=1}^{n} 8c\lambda\,(x_i - x_{0i})\,v_iv_t\right)_t - \sum_{i=1}^{n} 8\lambda\,(x_i - x_{0i})\,cv_{it}v_t$$
$$-\sum_{j=1}^{n}\sum_{i=1}^{n} 8\lambda\,(x_i - x_{0i})\,v_iv_{jj}$$
$$+\sum_{i=1}^{n}\left[-16\lambda^3\,(x_i - x_{0i})\left(|x-x_0|^2 - c\eta^2t^2 + O\left(\frac{1}{\lambda}\right)\right)v^2\right]_i$$
$$+16\lambda^3\left[(n+2)\,|x-x_0|^2 - nc\eta^2t^2 + O\left(\frac{1}{\lambda}\right)\right]v^2$$
$$= \sum_{i=1}^{n}\left(-4\lambda\,(x_i - x_{0i})\,cv_t^2\right)_i + 4\lambda\,[nc + (x - x_0, \nabla c)]\,v_t^2$$
$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{n}\left(-8\lambda\,(x_i - x_{0i})\,v_iv_j\right)\right]_j + 8\lambda\,|\nabla v|^2$$
$$+\sum_{j=1}^{n}\sum_{i=1}^{n} 8\lambda\,(x_i - x_{0_i})\,v_{ij}v_j$$

$$+16\lambda^3\left[(n+2)|x-x_0|^2 - nc\eta^2t^2 + O\left(\frac{1}{\lambda}\right)\right]v^2$$

$$+\left(8c\lambda v_t\sum_{i=1}^{n}(x_i-x_{0i})\,v_i\right)_t$$

$$= 4\lambda\left[nc+(x-x_0,\nabla c)\right]v_t^2 + 8\lambda|\nabla v|^2$$

$$+\sum_{i=1}^{n}\left[\sum_{j=1}^{n}4\lambda\left(x_i-x_{0_i}\right)v_j^2\right]_i - 4\lambda|\nabla v|^2$$

$$+16\lambda^3\left[(n+2)|x-x_0|^2 - c\eta^2t^2 + O\left(\frac{1}{\lambda}\right)\right]v^2 + \nabla\cdot U_2 + (V_2)_t\,.$$

Since by (1.82), $(x-x_0,\nabla c)\geq 0$, then we obtain

$$2z_1z_3 \geq 4\lambda ncv_t^2 + 4\lambda|\nabla v|^2$$
$$+16\lambda^3\left[(n+2)|x-x_0|^2 - nc\eta^2t^2 + O\left(\frac{1}{\lambda}\right)\right]v^2$$
$$+\nabla\cdot U_2 + V_{2t}, \tag{1.93}$$

$$|U_2| \leq C\lambda^3\left(|\nabla u|^2 + u_t^2 + u^2\right)W^2, \tag{1.94}$$

$$|V_2| \leq C\lambda^3\left[|t|\left(|\nabla u|^2 + |u|^2\right) + (|\nabla u| + |u|)\,|u_t|\right]W^2. \tag{1.95}$$

Step 3. In this step, we estimate the term $2z_1z_2 + 2z_1z_3$. It follows from (1.82) that $|x-x_0|\leq P, \forall x\in\Omega$. On the other hand, since $|x-x_0|^2-\eta t^2 > d > 0$ in G_d, then $\eta|t|\leq P\sqrt{\eta}$ in G_d. This estimate as well as the Cauchy-Schwarz inequality imply that

$$8\lambda\eta t\cdot v_t\sum_{i=1}^{n}c_iv_i = -8\lambda\eta t\cdot v_t\,(\nabla c,\nabla v) \geq -8\lambda\eta\,|t|\cdot|v_t|\cdot|\nabla v|\cdot\|\nabla c\|_{C(\overline{\Omega})}$$
$$\geq -4\lambda\sqrt{\eta}P\,\|\nabla c\|_{C(\overline{\Omega})}\cdot\left(v_t^2+|\nabla v|^2\right). \tag{1.96}$$

Since by (1.81), $\overline{c}\geq 1$, then (1.90) and (1.96) imply that

$$2z_1z_2 \geq -4\lambda\left(\overline{c}^2\eta + \sqrt{\eta}P\,\|\nabla c\|_{C(\overline{\Omega})}\right)\left(v_t^2+|\nabla v|^2\right)$$
$$+16\lambda^3\eta\left[|x-x_0|^2 - 3c_1\eta^2t^2 + O\left(\frac{1}{\lambda}\right)\right]v^2 + \nabla\cdot U_1 + (V_1)_t\,. \tag{1.97}$$

Let $U_3 = U_1 + U_2, V_3 = V_1 + V_2$. Hence, (1.93)–(1.97) imply that

$$\begin{aligned}2z_1z_2 + 2z_1z_3 \geq\ & 4\lambda \left[nc - \left(\overline{c}^2\eta + \sqrt{\eta}P \left\|\nabla c\right\|_{C(\overline{\Omega})}\right)\right] v_t^2 \\ & + 4\lambda \left(1 - \left(\overline{c}^2\eta + \sqrt{\eta}P \left\|\nabla c\right\|_{C(\overline{\Omega})}\right)\right) |\nabla v|^2 \\ & + 16\lambda^3 \left[(n + 2 + \eta) |x - x_0|^2 - (n + 3\eta) \overline{c}\eta^2 t^2 + O\left(\frac{1}{\lambda}\right)\right] v^2 \\ & + \nabla \cdot U_3 + V_{3t}, \end{aligned} \tag{1.98}$$

$$|U_3| \leq C\lambda^3 \left(|\nabla u|^2 + u_t^2 + u^2\right) W^2, \tag{1.99}$$

$$|V_3| \leq C\lambda^3 \left[|t| \left(u_t^2 + |\nabla u|^2 + u^2\right) + (|\nabla u| + |u|) |u_t|\right] W^2. \tag{1.100}$$

Step 4. We now estimate the term z_1^2 from the below. We are doing this only in order to prove Corollary 1.10.2, since multipliers at $v_t^2, |\nabla v|^2, v^2$ in (1.98) are positive anyway for sufficiently small η. Let $b > 0$ be a number, which we will choose later. We have

$$\begin{aligned} z_1^2 &= \left[cv_{tt} - \Delta v - 4\lambda^2 \left(|x - x_0|^2 - c\eta^2 t^2 + O\left(\frac{1}{\lambda}\right)\right) v + \lambda b v\right]^2 \\ &= (2\lambda cbvv_t)_t - 2\lambda cbv_t^2 + \sum_{i=1}^{n} (-2\lambda bvv_i)_i \\ &\quad +2\lambda b |\nabla v|^2 - 8\lambda^3 b \left[|x - x_0|^2 - c\eta^2 t^2 + O\left(\frac{1}{\lambda}\right)\right] v^2. \end{aligned}$$

Since by (1.81), $c \geq 1$, then we obtain

$$\begin{aligned} z_1^2 \geq\ & 2\lambda b |\nabla v|^2 - 2\lambda cbv_t^2 \\ & - 8\lambda^3 b \left[|x - x_0|^2 - \eta^2 t^2 + O\left(\frac{1}{\lambda}\right)\right] v^2 + \nabla \cdot U_4 + V_{4t}, \end{aligned} \tag{1.101}$$

$$|U_4| \leq C\lambda^3 \left(|\nabla u|^2 + u_t^2 + u^2\right) W^2, \tag{1.102}$$

$$|V_4| \leq C\lambda^3 \left(|t| u^2 + |u_t| \cdot |u|\right) W^2. \tag{1.103}$$

Step 5. Finally, we estimate the term $z_1^2 + 2z_1z_2 + 2z_1z_3$, which is the right-hand side of (1.89). Summing up (1.98) and (1.101) and taking into account (1.99), (1.100), (1.102), and (1.103), we obtain

$$
\begin{aligned}
z_1^2 + 2z_1z_2 + 2z_1z_3 \geq 4\lambda &\left[\left(n - \frac{b}{2} \right) c - \left(\overline{c}^2 \eta + \sqrt{\eta} P \left\| \nabla c \right\|_{C(\overline{\Omega})} \right) \right] v_t^2 \\
&+ 4\lambda \left(1 + \frac{b}{2} - \left(\overline{c}^2 \eta + \sqrt{\eta} P \left\| \nabla c \right\|_{C(\overline{\Omega})} \right) \right) \left| \nabla v \right|^2 \\
&+ 16\lambda^3 \left[\left(n + 2 + \eta - \frac{b}{2} \right) \left| x - x_0 \right|^2 \right. \\
&\left. - \left(n\overline{c} + 3\eta\overline{c} - \frac{b}{2} \right) \eta^2 t^2 + O\left(\frac{1}{\lambda} \right) \right] v^2 \\
&+ \nabla \cdot U_5 + V_{5t},
\end{aligned} \tag{1.104}
$$

$$
\left| U_5 \right| \leq C\lambda^3 \left(\left| \nabla u \right|^2 + u_t^2 + u^2 \right) W^2, \tag{1.105}
$$

$$
\left| V_5 \right| \leq C\lambda^3 \left[\left| t \right| \left(u_t^2 + \left| \nabla u \right|^2 + u^2 \right) + \left(\left| \nabla u \right| + \left| u \right| \right) \left| u_t \right| \right] W^2. \tag{1.106}
$$

Choose now $b = 1$ and choose $\eta_0 = \eta_0 \left(\overline{c}, P, \left\| \nabla c \right\|_{C(\overline{\Omega})} \right) \in (0, 1)$ so small that

$$
\frac{3}{2} - \left(\overline{c}^2 \eta + \sqrt{\eta} P \left\| \nabla c \right\|_{C(\overline{\Omega})} \right) \geq 1, \forall \eta \in (0, \eta_0], \tag{1.107}
$$

$$
\left(n\overline{c} + 3\eta\overline{c} - \frac{1}{2} \right) \eta \leq n + \frac{3}{2} + \eta, \forall \eta \in (0, \eta_0]. \tag{1.108}
$$

Since $n \geq 2$ and $c \geq 1$, then (1.104) becomes

$$
\begin{aligned}
z_1^2 + 2z_1z_2 + 2z_1z_3 &\geq 2\lambda \left(v_t^2 + \left| \nabla v \right|^2 \right) \\
&+ 16\lambda^3 \left(\left| x - x_0 \right|^2 - \eta t^2 + O\left(\frac{1}{\lambda} \right) \right) v^2 + \nabla \cdot U_5 + V_{5t}.
\end{aligned} \tag{1.109}
$$

Since $\left| x - x_0 \right|^2 - \eta t^2 > d > 0$ in G_d, then replacing in (1.109) v with $u = vW^{-1}$ and using (1.88), (1.105), and (1.106) as well as the fact that λ is sufficiently large, we obtain (1.84)–(1.86). □

Corollary 1.10.2. *Assume now that in (1.80), the operator $L_0 u = u_{tt} - \Delta u$ and $n \geq 2$. Then, condition (1.82) holds automatically, and in Theorem 1.10.2,one can choose $\eta_0 = 1$.*

Proof. We now can set in (1.81) $\overline{c} := 1$. Since in the above proof $b = 1$, then we have in (1.104) for $n \geq 2$

$$\left(n-\frac{b}{2}\right)c-\left(\overline{c}^2\eta+\sqrt{\eta}P\,\|\nabla c\|_{C\left(\overline{\Omega}\right)}\right)=n-\left(\eta+\frac{1}{2}\right)\geq\frac{1}{2},$$

$$\left(1+\frac{b}{2}-\left(\overline{c}^2\eta+\sqrt{\eta}P\,\|\nabla c\|_{C\left(\overline{\Omega}\right)}\right)\right)=\frac{3}{2}-\eta\geq\frac{1}{2},$$

$$\begin{aligned}&\left(n+2+\eta-\frac{b}{2}\right)|x-x_0|^2-\left(n\overline{c}+3\eta\overline{c}-\frac{b}{2}\right)\eta^2t^2\\&=\left(n+\frac{3}{2}+\eta\right)|x-x_0|^2-\left(n-\frac{1}{2}+3\eta\right)\eta^2t^2\\&\geq|x-x_0|^2-\eta t^2>d.\end{aligned}$$

Therefore, (1.109) is satisfied for all $\eta\in(0,1)$. □

1.10.3 Estimating an Integral

Lemma 1.10.3 is very important for the Bukhgeim-Klibanov method.

Lemma 1.10.3. *Let the function $\varphi\in C^1[0,a]$ and $\varphi'(t)\leq -b$ in $[0,a]$, where $b=const>0$. For a function $g\in L_2(-a,a)$, consider the integral*

$$I(g,\lambda)=\int_{-a}^{a}\left(\int_0^t g(\tau)\,\mathrm{d}\tau\right)^2\exp\left[2\lambda\varphi\left(t^2\right)\right]\mathrm{d}t.$$

Then,

$$I(g,\lambda)\leq\frac{1}{4\lambda b}\int_{-a}^{a}g^2(t)\exp\left[2\lambda\varphi\left(t^2\right)\right]\mathrm{d}t.$$

Proof. We have for $t>0$

$$\begin{aligned}t\exp\left[2\lambda\varphi\left(t^2\right)\right]&=t\frac{4\lambda\varphi'\left(t^2\right)}{4\lambda\varphi'\left(t^2\right)}\exp\left[2\lambda\varphi\left(t^2\right)\right]\\&=\frac{1}{4\lambda\varphi'\left(t^2\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\exp\left[2\lambda\varphi\left(t^2\right)\right]\right\}\\&=-\frac{1}{4\lambda\varphi'\left(t^2\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left\{-\exp\left[2\lambda\varphi\left(t^2\right)\right]\right\}\\&\leq\frac{1}{4\lambda b}\frac{\mathrm{d}}{\mathrm{d}t}\left\{-\exp\left[2\lambda\varphi\left(t^2\right)\right]\right\}.\end{aligned}$$

Hence,

$$\int_0^a \left(\int_0^t g(\tau)\,\mathrm{d}\tau\right)^2 \exp\left(2\lambda\varphi\left(t^2\right)\right)\mathrm{d}t \leq \int_0^a \exp\left(2\lambda\varphi\left(t^2\right)\right) t \left(\int_0^t g^2(\tau)\,\mathrm{d}\tau\right)\mathrm{d}t$$

$$\leq \frac{1}{4\lambda b}\int_0^a \frac{\mathrm{d}}{\mathrm{d}t}\left[-\exp\left(2\lambda\varphi\left(t^2\right)\right)\right]\left(\int_0^t g^2(\tau)\,\mathrm{d}\tau\right)\mathrm{d}t$$

$$= -\frac{1}{4\lambda b}\exp\left(2\lambda\varphi\left(a^2\right)\right)\int_0^a g^2(\tau)\,\mathrm{d}\tau + \frac{1}{4\lambda b}\int_0^a g^2(\tau)\exp\left(2\lambda\varphi\left(t^2\right)\right)\mathrm{d}t$$

$$\leq \frac{1}{4\lambda b}\int_0^a g^2(\tau)\exp\left(2\lambda\varphi\left(t^2\right)\right)\mathrm{d}t.$$

Thus, we have obtained that

$$\int_0^a \exp\left(2\lambda\varphi\left(t^2\right)\right)\left(\int_0^t g(\tau)\,\mathrm{d}\tau\right)^2\mathrm{d}t \leq \frac{1}{4\lambda b}\int_0^a g^2(\tau)\exp\left(2\lambda\varphi\left(t^2\right)\right)\mathrm{d}t.$$

Similarly,

$$\int_{-a}^0 \exp\left(2\lambda\varphi\left(t^2\right)\right)\left(\int_0^t g(\tau)\,\mathrm{d}\tau\right)^2\mathrm{d}t \leq \frac{1}{4\lambda b}\int_{-a}^0 g^2(\tau)\exp\left(2\lambda\varphi\left(t^2\right)\right)\mathrm{d}t.$$

□

1.10.4 Cauchy Problem with the Lateral Data for a Hyperbolic Inequality with Volterra-Like Integrals

Recall that we assume in Sect. 1.10 that $\Omega = \{|x| < R\} \subset \mathbb{R}^n$. Let P be the number defined in (1.83) and $d = \text{const.} \in \left(0, P^2\right)$. Let G_d be the domain defined in (1.79). Define its subdomain G_d^+ as

$$G_d^+ = \left\{(x,t) : |x - x_0|^2 - \eta t^2 > d, t > 0, x \in \Omega\right\}. \tag{1.110}$$

Hence, $G_d^+ = G_d \cap \{t > 0\}$. Let

$$T > \sqrt{\frac{P^2 - d}{\eta}}. \tag{1.111}$$

Using (1.79) and (1.110), we obtain

$$G_d^+ \subset Q_T, \quad G_d^+ = G_d \cap \{t > 0\}, \tag{1.112}$$

$$\overline{G}_d^+ \cap \{t = T\} = \varnothing, \quad \overline{G}_d \cap \{t = \pm T\} = \varnothing. \tag{1.113}$$

Let $A_1 > 0$ and $A_2 \geq 0$ be two constants. Assume that the function $u \in C^2\left(\overline{Q}_T\right)$ satisfies the following hyperbolic inequality with Volterra-like integrals:

$$\begin{aligned} |c(x) u_{tt} - \Delta u| &\leq A_1 \Big(|\nabla u| + |u_t| + |u| \Big)(x,t) \\ &+ A_2 \int_0^t (|\nabla u| + |u_t| + |u|)(x,\tau)\,\mathrm{d}\tau, \text{ in } G_d^+. \end{aligned} \tag{1.114}$$

Also, let this function u has zero Cauchy data at the lateral side $S_T \cap \overline{G}_d^+$ of the domain G_d^+ :

$$u\,|_{S_T \cap \overline{G}_d^+} = \frac{\partial u}{\partial n}\,|_{S_T \cap \overline{G}_d^+} = 0. \tag{1.115}$$

In addition, we assume that

$$\text{either } u(x,0) = 0 \text{ or } u_t(x,0) = 0 \text{ for } x \in \overline{G}_d^+ \cap \{t = 0\}. \tag{1.116}$$

The goal of this section is to prove that conditions (1.114)–(1.116) imply that $u(x,t) \equiv 0$ in G_d^+. In particular, if $A_2 = 0$, then integrals are not present in (1.114). Hence, in this case, the corresponding hyperbolic equation

$$c(x) u_{tt} = \Delta u + \sum_{j=1}^{n+1} b_j(x,t) u_j + a(x,t) u \text{ in } G_d^+,$$

where $u_{n+1} := u_t$ with coefficients $b_j, c \in C\left(\overline{G}_d^{\,+}\right)$ can be reduced to the inequality (1.114). Hence, Theorem 1.10.4 implies uniqueness for this equation with the Cauchy data (1.115) and one of initial conditions (1.116). The reason why we introduce Volterra integrals in (1.114) is that they appear in the proof of Theorem 1.10.5.1 Furthermore, assume that (1.111) holds. This implies (1.112) and (1.113). Consider now inequality (1.114) with the Cauchy data (1.115) in the domain G_d (thus allowing $t < 0$). Then an obvious analog of Theorem 1.10.4 is also valid, and the proof is almost identical. In the case $A_2 = 0$, such an analog was published in [124].

Theorem 1.10.4. *Let $\Omega = \{|x| < R\} \subset \mathbb{R}^n, n \geq 2$, and $x_0 \in \Omega$. Assume that $d \in \left(0, P^2\right)$ and the inequality (1.111) holds with the constant $\eta := \eta_0 = \eta_0\left(\overline{c}, P, \|\nabla c\|_{C(\overline{\Omega})}\right) \in (0,1]$ of Theorem 1.10.2. Suppose that the function $u \in$*

$C^2\left(\overline{G}\,_d^+\right)$ *satisfies conditions (1.114)–(1.116) and that the coefficient* $c(x)$ *satisfies conditions (1.81), (1.82). Then*

$$u(x,t) = 0 \text{ in } G_d^+. \tag{1.117}$$

In particular, if in (1.110) $x_0 = 0$ *and* $d = 0$, *then*

$$u(x,t) = 0 \text{ in } Q_T. \tag{1.118}$$

In addition, if $c(x) \equiv 1$ *and in (1.110)* $x_0 = 0$ *and* $d = 0$, *then it is sufficient for (1.118) to replace (1.111) with*

$$T > R, \eta = 1. \tag{1.119}$$

Proof. We note first that the boundary of the domain G_d^+ consists of three parts:

$$\begin{aligned}
\partial G_d^+ &= \cup_{i=1}^3 \partial_i G_d^+,\\
\partial_1 G_d^+ &= \left\{|x-x_0|^2 - \eta_0 t^2 = d, t > 0, |x| < R\right\} \subset Q_T,\\
\partial_2 G_d^+ &= \left\{|x-x_0|^2 - \eta_0 t^2 > d, t > 0, |x| = R\right\} \subset S_T,\\
\partial_3 G_d^+ &= \left\{|x-x_0|^2 > d, t = 0, |x| < R\right\}.
\end{aligned} \tag{1.120}$$

Hence, the hypersurface $\partial_1 G_d^+$ is a level surface of the CWF W. Let the function $g \in L_2\left(G_d^+\right)$. Then (1.77), (1.78), and (1.120) imply that

$$\begin{aligned}
\int\limits_{G_d^+} \left(\int\limits_0^t g(x,\tau)\,\mathrm{d}\tau\right)^2 W^2 \mathrm{d}x\mathrm{d}t &= \int\limits_{\partial_3 G_d^+} \exp\left(2\lambda\,|x-x_0|^2\right)\\
&\quad \times \left[\int\limits_0^{t(x)} \left(\int\limits_0^t g(x,\tau)\,\mathrm{d}\tau\right)^2 \mathrm{e}^{-2\lambda\eta t^2}\mathrm{d}t\right]\mathrm{d}x,
\end{aligned}$$

$$t(x) = \frac{\sqrt{|x-x_0|^2 - d}}{\sqrt{\eta}}.$$

Hence, applying Lemma 1.10.3 to the inner integral

$$\int\limits_0^{t(x)} \left(\int\limits_0^t g(x,\tau)\,\mathrm{d}\tau\right)^2 \mathrm{e}^{-2\lambda\eta t^2}\mathrm{d}t,$$

we obtain

$$\int\limits_{G_d^+} \left(\int\limits_0^t g\left(x,\tau\right) \mathrm{d}\tau \right)^2 W^2 \mathrm{d}x\mathrm{d}t \le \frac{1}{4\eta\lambda} \int\limits_{G_d^+} g^2 W^2 \mathrm{d}x\mathrm{d}t, \quad \forall g \in L_2\left(G_d^+\right). \tag{1.121}$$

Multiply both sides of the inequality (1.114) by the function $W(x,t)$ with sufficiently large parameter $\lambda > 1$. Then, square both sides, integrate over the domain G_d^+, and use (1.121). We obtain with a constant $A = A(A_1, A_2, \eta) > 0$

$$\int\limits_{G_d^+} (cu_{tt} - \Delta u)^2 W^2 \mathrm{d}x\mathrm{d}t \le A \int\limits_{G_d^+} \left(|\nabla u|^2 + u_t^2 + u^2 \right) W^2 \mathrm{d}x\mathrm{d}t. \tag{1.122}$$

We now can apply Theorem 1.10.2 to estimate the left-hand side of (1.122) from the below. Integrating the inequality (1.84) over the domain G_d^+ using (1.85)–(1.87), (1.115), (1.116), and (1.120) and applying the Gauss' formula, we obtain for sufficiently large $\lambda \ge \lambda_0 > 1$

$$\int\limits_{G_d^+} (cu_{tt} - \Delta u)^2 W^2 \mathrm{d}x\mathrm{d}t \ge C\lambda \int\limits_{G_d^+} \left(|\nabla u|^2 + u_t^2 + \lambda^2 u^2 \right) W^2 \mathrm{d}x\mathrm{d}t$$

$$-C\lambda^3 \mathrm{e}^{2\lambda d} \int\limits_{\partial_1 G_d^+} \left(|\nabla u|^2 + u_t^2 + \lambda^2 u^2 \right) W^2 \mathrm{d}S.$$

Comparing this with (1.122), we obtain

$$C\lambda \int\limits_{G_d^+} \left(|\nabla u|^2 + u_t^2 + \lambda^2 u^2 \right) W^2 \mathrm{d}x\mathrm{d}t - C\lambda^3 \mathrm{e}^{2\lambda d} \int\limits_{\partial_1 G_d^+} \left(|\nabla u|^2 + u_t^2 + \lambda^2 u^2 \right) W^2 \mathrm{d}S$$

$$\le A \int\limits_{G_d^+} \left(|\nabla u|^2 + u_t^2 + u^2 \right) W^2 \mathrm{d}x\mathrm{d}t.$$

Hence, choosing a sufficient large $\lambda_1 > \lambda_0$, we obtain for $\lambda \ge \lambda_1$ with a new constant $C > 0$

$$\lambda \int\limits_{G_d^+} \left(|\nabla u|^2 + u_t^2 + \lambda^2 u^2 \right) W^2 \mathrm{d}x\mathrm{d}t \le C\lambda^3 \mathrm{e}^{2\lambda d} \int\limits_{\partial_1 G_d^+} \left(|\nabla u|^2 + u_t^2 + \lambda^2 u^2 \right) \mathrm{d}S. \tag{1.123}$$

Consider a sufficiently small number $\varepsilon > 0$ such that $d + \varepsilon < P^2$. Then by (1.112), $G_{d+\varepsilon}^+ \subset Q_T$. Obviously, $G_d^+ \subset G_{d+\varepsilon}^+$. Hence, replacing in the left-hand side of the inequality (1.123) G_d^+ with $G_{d+\varepsilon}^+$, we strengthen this inequality. Also, $W^2(x,t) \geq e^{2\lambda(d+\varepsilon)}$ in $G_{d+\varepsilon}^+$. Hence, we obtain from (1.123)

$$e^{2\lambda(d+\varepsilon)} \int\limits_{G_{d+\varepsilon}^+} u^2 dxdt \leq Ce^{2\lambda d} \int\limits_{\partial_1 G_d^+} \left(|\nabla u|^2 + u_t^2 + \lambda^2 u^2\right) dS.$$

Dividing this inequality by $e^{2\lambda(d+\varepsilon)}$, we obtain

$$\int\limits_{G_{d+\varepsilon}^+} u^2 dxdt \leq Ce^{-2\lambda\varepsilon} \int\limits_{\partial_1 G_d^+} \left(|\nabla u|^2 + u_t^2 + \lambda^2 u^2\right) dS. \tag{1.124}$$

Setting in (1.124) $\lambda \to \infty$, we obtain $u = 0$ in $G_{d+\varepsilon}^+$. Since $\varepsilon > 0$ is an arbitrary sufficiently small number, then (1.117) is true.

Consider now the case when in (1.110) $x_0 = 0$ and $d = 0$. Then $P = R$ and by (1.111),

$$T > \frac{R}{\sqrt{\eta_0}}. \tag{1.125}$$

Consider a sufficiently small number $\varepsilon \in \left(0, R^2\right)$. Then by (1.112), $G_\varepsilon^+ \subset Q_T$ and by (1.125),

$$T > \frac{\sqrt{R^2 - \varepsilon}}{\sqrt{\eta_0}}.$$

Hence, (1.117) implies that $u = 0$ in G_ε^+. Hence, $u = 0$ in G_0^+. Next, since $x_0 = 0$, then it follows from (1.122) that $\overline{G_0^+} \cap \{t = 0\} = \{|x| < R\} = \Omega$. Hence,

$$u(x,0) = u_t(x,0) = 0, x \in \Omega. \tag{1.126}$$

Next, denote $cu_{tt} - \Delta u := f$. Hence, by (1.114),

$$\begin{aligned} 2u_t(cu_{tt} - \Delta u) &= 2u_t f \leq u_t^2 + f^2 \\ &\leq u_t^2 + \overline{A}\left[|\nabla u|^2 + u_t^2 + u^2 + \int_0^t \left(|\nabla u|^2 + u_t^2 + u^2\right)(x,\tau)\, d\tau\right], \end{aligned}$$

with a certain positive constant $\overline{A}$. Hence, we now can work with $2u_t(cu_{tt} - \Delta u)$ as it is done in the standard energy estimate for a hyperbolic PDE [119]. In doing so, we can use one of zero boundary conditions (1.115) at S_T and zero initial conditions (1.126). This way, we obtain $u = 0$ in Q_T, which proves (1.118). The case $c \equiv 1$, including (1.119), follows from Corollary 1.10.2 and (1.118). □

The proof of the following corollary can be obtained via a slight modification of the proof of Theorem 1.10.4.

Corollary 1.10.4. *Assume that in Theorem 1.10.4 the domain G_d^+ is replaced with the domain G_d, the integral in (1.114) is replaced with*

$$\left| \int_0^t \left(|\nabla u| + |u_t| + |u| \right)(x,\tau)\, d\tau \right|,$$

and that the rest of conditions of Theorem 1.10.4, except of (1.116), is in place. Then conclusions (1.117)–(1.119) of Theorem 1.10.4 still hold with the replacement of the pair $\left(G_d^+, Q_T\right)$ with the pair $\left(G_d, Q_T^{\pm}\right)$.

1.10.5 Coefficient Inverse Problem for a Hyperbolic Equation

The Hyperbolic Coefficient Inverse Problem. Let the function $u \in C^2\left(\overline{Q}_T\right)$ satisfies the following conditions:

$$c(x)\, u_{tt} = \Delta u + \sum_{|\alpha| \le 1} a_\alpha(x)\, D_x^\alpha u, \text{ in } Q_T, \tag{1.127}$$

$$u(x,0) = f_0(x), \quad u_t(x,0) = f_1(x), \tag{1.128}$$

$$u|_{S_T} = p(x,t), \quad \frac{\partial u}{\partial n}|_{S_T} = q(x,t), \tag{1.129}$$

where functions $a_\alpha, c \in C\left(\overline{Q}_T\right)$, and $c \ge 1$. Determine one of coefficients of (1.127).

The CIP (1.127)–(1.129) is the problem with the single measurement data because only a single pair (f_0, f_1) of initial conditions is used.

Theorem 1.10.5.1. *Let the coefficient $c(x)$ in (1.127) satisfies conditions (1.81) and (1.82). In addition, let coefficients $a_\alpha \in C\left(\overline{\Omega}\right)$. Let the domain $\Omega = \{|x| < R\} \subset \mathbb{R}^n, n \ge 2$. Consider two cases:*

Case 1. The coefficient $c(x)$ is unknown, and all other coefficients $a_\alpha(x)$ are known. In this case, we assume that

$$\Delta f_0(x) + \sum_{|\alpha| \le 1} a_\alpha(x)\, D_x^\alpha f_0(x) \ne 0 \textit{ for } x \in \overline{\Omega}. \tag{1.130}$$

Then for a sufficiently large $T > 0$, there exists at most one pair of functions (u, c) satisfying (1.127)–(1.129) and such that $u \in C^4\left(\overline{Q}_T\right)$.

Case 2. The coefficient $a_{\alpha_0}(x)$ is unknown, and the rest of coefficients is known. In this case, we assume that

$$D_x^{\alpha_0} f_0(x) \neq 0 \, for \; x \in \overline{\Omega}.$$

Then for a sufficiently large $T > 0$, there exists at most one pair of functions (u, a_{α_0}) satisfying (1.127)–(1.129) and such that $u \in C^{3+|\alpha_0|}\left(\overline{Q}_T\right)$.

If in (1.128) $f_0(x) \equiv 0$, then conditions of these two cases should be imposed on the function $f_1(x)$, the required smoothness of the function u should be increased by one and the above statements about uniqueness would still hold.

Proof. First, we note that if $f_0(x) \equiv 0$, then one should consider in this proof u_t instead of u, and the rest of the proof is the same as the one below. We prove this theorem only for Case 1, since Case 2 is similar. Assume that there exist two solutions (u_1, c_1) and (u_2, c_2). Denote $\widetilde{u} = u_1 - u_2, \widetilde{c} = c_1 - c_2$. Since

$$c_1 u_{1tt} - c_2 u_{2tt} = c_1 u_{1tt} - c_1 u_{2tt} + (c_1 - c_2)\, u_{2tt} = c_1 \widetilde{u}_{tt} + \widetilde{c} u_{2tt},$$

then (1.127)–(1.129) lead to

$$L\widetilde{u} = c_1(x)\widetilde{u}_{tt} - \Delta\widetilde{u} - \sum_{j=1}^{n} a_\alpha(x)\, D_x^\alpha \widetilde{u} = -\widetilde{c}(x)\, H(x,t)\,, \text{ in } Q_T, \tag{1.131}$$

$$\widetilde{u}(x,0) = 0, \widetilde{u}_t(x,0) = 0, \tag{1.132}$$

$$\widetilde{u}|_{S_T} = \frac{\partial \widetilde{u}}{\partial n}|_{S_T} = 0, \tag{1.133}$$

$$H(x,t) := u_{2tt}(x,t)\,. \tag{1.134}$$

Setting in (1.127) $c := c_2, u := u_2, t := 0$ and using (1.128), (1.130), and (1.134), we obtain

$$H(x,0) = c_2^{-1}(x)\left(\Delta f_0(x) + \sum_{|\alpha|\leq 1} a_\alpha(x)\, D_x^\alpha f_0(x)\right) \neq 0 \text{ for } x \in \overline{\Omega}.$$

Hence, there exists a sufficiently small positive number ε, such that

$$H(x,t) \neq 0 \text{ in } \overline{Q}_\varepsilon = \overline{\Omega} \times \; [0,\varepsilon]\,. \tag{1.135}$$

Now, we eliminate the unknown coefficient $\widetilde{c}(x)$ from (1.131). We have

$$-\widetilde{c}(x) = \frac{L\widetilde{u}}{H(x,t)} \text{ in } \overline{Q}_\varepsilon.$$

Hence,

$$\frac{\partial}{\partial t}\left[-\widetilde{c}\,(x)\right] = \frac{\partial}{\partial t}\left[\frac{L\widetilde{u}}{H\,(x,t)}\right] = 0 \text{ in } \overline{Q}_{\varepsilon}.$$

Or

$$L\widetilde{u}_t = \frac{H_t}{H}\,(L\widetilde{u}) \text{ in } \overline{Q}_{\varepsilon}. \tag{1.136}$$

Denote

$$h\,(x,t) = \frac{H_t}{H}\,(x,t)\,. \tag{1.137}$$

Since $u \in C^4\left(\overline{Q}_T\right)$, then (1.134), (1.135), and (1.137) imply that

$$h \in C^2\left(\overline{Q}_{\varepsilon}\right). \tag{1.138}$$

Introduce a new function $v\,(x,t)$:

$$v\,(x,t) = \widetilde{u}_t\,(x,t) - h\widetilde{u}\,(x,t) \tag{1.139}$$

Considering (1.139) as an ordinary differential equation with respect to $\widetilde{u}\,(x,t)$ and using (1.132) as well as (1.137), we obtain

$$\widetilde{u}\,(x,t) = \int_0^t K\,(x,t,\tau)\,v\,(x,\tau)\,\mathrm{d}\tau, \tag{1.140}$$

$$K\,(x,t,\tau) = \frac{H\,(x,t)}{H\,(x,\tau)} \in C^2\left(\overline{\Omega} \times [0,\varepsilon] \times [0,\varepsilon]\right), \tag{1.141}$$

$$v\,(x,0) = 0. \tag{1.142}$$

Using (1.139)–(1.141), we obtain the following formulas in $\overline{Q}_{\varepsilon}$:

$$\begin{aligned} c_1\,(\widetilde{u}_t)_{tt} - hc_1\widetilde{u}_{tt} &= c_1\,(\widetilde{u}_t - h\widetilde{u})_{tt} + 2c_1h_t\widetilde{u}_t + c_1h_{tt}\widetilde{u} \\ &= c_1v_{tt} + 2c_1h_tv + 2c_1h_t\int_0^t K_t\,(x,t,\tau)\,v\,(x,\tau)\,\mathrm{d}\tau \\ &\quad + c_1h_{tt}\int_0^t K\,(x,t,\tau)\,v\,(x,\tau)\,\mathrm{d}\tau, \end{aligned}$$

$$\begin{aligned}\Delta \widetilde{u}_t - h\Delta \widetilde{u} &= \Delta\left(\widetilde{u}_t - h\widetilde{u}\right) + 2\nabla h \nabla \widetilde{u} + \Delta h \widetilde{u} \\ &= \Delta v + 2\nabla h \nabla \left(\int_0^t K(x,t,\tau)\, v(x,\tau)\, d\tau \right) \\ &\quad + \Delta h \int_0^t K(x,t,\tau)\, v(x,\tau)\, d\tau.\end{aligned}$$

By (1.136),

$$L\widetilde{u}_t - hL\widetilde{u} = 0 \text{ in } Q_\varepsilon. \tag{1.143}$$

Hence, substituting the recent formulas in (1.143) and using boundary conditions (1.133) and the initial condition (1.142), we obtain the following inequality:

$$|c_1(x)\, v_{tt} - \Delta v| \le M \left[|\nabla v|(x,t) + |v|(x,t) + \int_0^t \left(|\nabla v| + |v|\right)(x,\tau)\, d\tau \right] \text{ in } \overline{Q}_\varepsilon,$$

$$v\,|_{S_\varepsilon} = \frac{\partial v}{\partial n}\,|_{S_\varepsilon} = 0, \tag{1.144}$$

$$v(x,0) = 0,$$

where $M > 0$ is a constant independent on v, x, t.

Let $\eta_0 = \eta_0\left(\overline{c}, R, \|\nabla c\|_{C(\overline{\Omega})}\right) \in (0,1]$ be the number considered in Theorems 1.10.2 and 1.10.4. Consider now the domain $G^+_{\eta_0\varepsilon^2}$ defined as

$$G^+_{\eta_0\varepsilon^2} = \left\{(x,t) : |x|^2 - \eta_0 t^2 > R^2 - \eta_0\varepsilon^2, t > 0, |x| < R\right\}.$$

Then, $G^+_{\eta_0\varepsilon^2} \subset \overline{Q}_\varepsilon$. Hence, we can apply now Theorem 1.10.4 to conditions (1.144). Thus, we obtain $v(x,t) = 0$ in $G^+_{\eta_0\varepsilon^2}$. Hence, by (1.140) $\widetilde{u}(x,t) = 0$ in $G^+_{\eta_0\varepsilon^2}$. Therefore, setting $t = 0$ in (1.131) and using (1.135), we obtain

$$\widetilde{c}(x) = 0 \text{ for } x \in \left\{|x| \in \left(\sqrt{R^2 - \eta_0\varepsilon^2}, R\right)\right\}. \tag{1.145}$$

Substitute this in (1.131) and use (1.132) and (1.133). We obtain

$$L\widetilde{u} = c_1(x)\widetilde{u}_{tt} - \Delta\widetilde{u} - \sum_{j=1}^{n} a_\alpha(x)\, D_x^\alpha \widetilde{u} = 0, \tag{1.146}$$

$$\text{for } (x,t) \in \left\{|x| \in \left(\sqrt{R^2 - \eta_0 \varepsilon^2}, R\right)\right\} \times (0,T), \tag{1.147}$$

$$\widetilde{u}(x,0) = 0, \widetilde{u}_t(x,0) = 0, \tag{1.148}$$

$$\widetilde{u}|_{S_T} = \frac{\partial \widetilde{u}}{\partial n}|_{S_T} = 0. \tag{1.149}$$

Consider an arbitrary number $t_0 \in (0, T - \varepsilon)$. And consider the domain $G_{\eta_0 \varepsilon^2}(t_0)$:

$$G_{\eta_0 \varepsilon^2}(t_0) = \left\{(x,t) : |x|^2 - \eta_0 (t - t_0)^2 > R^2 - \eta_0 \varepsilon^2, t > 0, |x| < R\right\}.$$

Hence, in this domain $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \cap \{t > 0\}$. Since $t_0 \in (0, T - \varepsilon)$, then $t \in (0, T)$ in this domain. Hence,

$$G_{\eta_0 \varepsilon^2}(t_0) \subset \left\{|x| \in \left(\sqrt{R^2 - \eta_0 \varepsilon^2}, R\right)\right\} \times (0,T).$$

Hence, we can apply Corollary 1.10.4 to the domain $G_{\eta_0 \varepsilon^2}(t_0)$ and conditions (1.146)–(1.149). Therefore, $\widetilde{u}(x,t) = 0$ for $(x,t) \in G_{\eta_0 \varepsilon^2}(t_0)$. Since t_0 is an arbitrary number of the interval $(0, T - \varepsilon)$, then, varying this number, we obtain that

$$\widetilde{u}(x,t) = 0 \text{ for } (x,t) \in \left\{|x| \in \left(\sqrt{R^2 - \eta_0 \varepsilon^2}, R\right)\right\} \times (0, T - \varepsilon).$$

Therefore, we now can replace in (1.131)–(1.134) sets Q_T, S_T with sets

$$Q_T^{\varepsilon} = \left\{|x| < \sqrt{R^2 - \eta_0 \varepsilon^2}\right\} \times (0, T - \varepsilon),$$

$$S_T^{\varepsilon} = \left\{|x| = \sqrt{R^2 - \eta_0 \varepsilon^2}\right\} \times (0, T - \varepsilon),$$

and repeat the above proof. Hence, we obtain instead of (1.145) that

$$\widetilde{c}(x) = 0 \text{ for } x \in \left\{|x| \in \left(\sqrt{R^2 - 2\eta_0 \varepsilon^2}, R\right)\right\}.$$

Since $\varepsilon > 0$ is sufficiently small, we can always choose ε such that $R^2 = k\eta_0 \varepsilon^2$ where $k = k(R, \varepsilon) \geq 1$ is an integer. Suppose that

$$T > k\varepsilon = \frac{R}{\sqrt{\eta_0}\varepsilon}.$$

Hence, we can repeat this process k times until the entire domain $\Omega = \{|x| < R\}$ would be exhausted. Thus, we obtain after k steps that $\widetilde{c}(x) = 0$ in Ω. Thus, the right-hand side of (1.131) is identical zero. This, (1.131)–(1.133) and the standard energy estimate imply that $\widetilde{u}(x,t) = 0$ in Q_T. □

A slightly inconvenient point of Theorem 1.10.5.1 is that the observation time T is assumed to be sufficiently large. Our experience of working with experimental data (Chaps. 5 and 6) indicates that this is not a severe restriction in applications. Indeed, usually the outcoming signal can be measured for quite a long time. Still, it is possible to restrict the value of T to the same one as in Theorem 1.10.4 via imposing the condition $f_1(x) \equiv 0$. This was observed in [80, 81]. The proof of Theorem 1.10.5.2 partially repeats arguments of [80, 81].

Theorem 1.10.5.2. *Assume that all conditions of Theorem 1.10.5.1 are satisfied. In addition, assume that in (1.128) the function* $f_1(x) \equiv 0$. *Then Theorem 1.10.5.1 remains valid if*

$$T > \frac{R}{\sqrt{\eta_0}}. \tag{1.150}$$

In particular, if $c(x) \equiv 1$, *then it is sufficient to have* $T > R$.

Proof. Similarly with the proof of Theorem 1.10.5.1, we consider now only for Case 1. We keep notations of Theorem 1.10.5.1Consider the function $w(x,t) = \widetilde{u}_{tt}(x,t)$. Then (1.131)–(1.134) imply that

$$c_1(x)\, w_{tt} - \Delta w - \sum_{j=1}^{n} a_\alpha(x)\, D_x^\alpha w = -\widetilde{c}(x)\, \partial_t^4 u_2, \text{ in } Q_T, \tag{1.151}$$

$$w_t(x,0) = 0, \tag{1.152}$$

$$w|_{S_T} = \frac{\partial w}{\partial n}|_{S_T} = 0, \tag{1.153}$$

$$w(x,0) = -\widetilde{c}(x)\, p(x), \tag{1.154}$$

$$p(x) = c_1^{-1}(x) \left(\Delta f_0(x) + \sum_{|\alpha| \leq 1} a_\alpha(x)\, D_x^\alpha f_0(x) \right) \neq 0 \text{ for } x \in \overline{\Omega}. \tag{1.155}$$

Hence, it follows from (1.154) and (1.155) that

$$-\widetilde{c}(x) = \frac{w(x,0)}{p(x)} = \frac{1}{p(x)} \left[w(x,t) - \int_0^t w_t(x,\tau)\, d\tau \right].$$

Substituting this formula in (1.151), using Theorem 1.10.4, (1.150), (1.152), and (1.153) and proceeding similarly with the proof of Theorem 1.10.4, we obtain that $\widetilde{c}(x) = 0$ in Ω and $w(x,t) = \widetilde{u}(x,t) = 0$ in Q_T. □

1.10.6 The First Coefficient Inverse Problem for a Parabolic Equation

Consider the Cauchy problem for the following parabolic equation:

$$c(x)\, u_t = \Delta u + \sum_{|\alpha| \le 1} a_\alpha(x)\, D_x^\alpha u, \text{ in } D_T^{n+1} = \mathbb{R}^n \times (0,T), \tag{1.156}$$

$$u(x,0) = f_0(x), \tag{1.157}$$

$$c, a_\alpha \in C^\beta(\mathbb{R}^n),\ c(x) \ge 1,\ f_0 \in C^{2+\beta}(\mathbb{R}^n),\ \beta \in (0,1). \tag{1.158}$$

So (1.156) and (1.157) is the forward problem. Given conditions (1.158), this problem has unique solution $u \in C^{2+\beta,1+\beta/2}\left(\overline{D}_T^{n+1}\right)$ [120]. Just as in Sect. 1.10.5, assume that $\Omega = \{|x| < R\} \subset \mathbb{R}^n, n \ge 2$. Let $\Gamma \subseteq \partial\Omega$ be a part of the boundary of the domain Ω, $T = \text{const.} > 0$ and $\Gamma_T = \Gamma \times (0,T)$.

The First Parabolic Coefficient Inverse Problem. Suppose that one of coefficients in (1.156) is unknown inside the domain Ω and is known outside of it. Also, assume that all other coefficients in (1.156) are known, and conditions (1.157), (1.158) are satisfied. Determine that unknown coefficient inside Ω, assuming that the following functions $p(x,t)$ and $q(x,t)$ are known,

$$u\mid_{\Gamma_T} = p(x,t), \quad \frac{\partial u}{\partial n}\mid_{\Gamma_T} = q(x,t). \tag{1.159}$$

It is yet unclear how to prove a uniqueness theorem for this CIP "straightforwardly." The reason is that one cannot extend properly the solution of the problem (1.156) and (1.157) in $\{t < 0\}$. Thus, the idea here is to consider an associated CIP for a hyperbolic PDE using a connection between these two CIPs via an analog of the Laplace transform. Next, Theorem 1.10.5.1will provide the desired uniqueness result.

That associated hyperbolic Cauchy problem is

$$v_{tt} = \frac{1}{c(x)}\left(\Delta v + \sum_{|\alpha| \le 1} a_\alpha(x)\, D_x^\alpha v\right) \text{ in } D_\infty^{n+1} = \mathbb{R}^n \times (0,\infty), \tag{1.160}$$

$$v|_{t=0} = 0,\ v_t|_{t=0} = f_0(x). \tag{1.161}$$

In addition to (1.158), we assume that the coefficients $c(x), a_\alpha(x)$ and the initial condition $f_0(x)$ are so smooth that the solution v of the problem (1.160) and (1.161) is such that (a) $v \in C^4(\overline{D}_\infty^{n+1})$ if the function $c(x)$ is unknown and (b) $v \in C^{3+|\alpha|}(\overline{D}_\infty^{n+1})$ if the function $c(x)$ is known and any of functions $a_\alpha(x)$ is unknown.

Consider an interesting Laplace-like transform which was proposed, for the first time, by K.G. Reznickaya in 1973 [142] and was widely used since then [102, 123, 124]. Namely, one can easily verify the following connection between solutions u and v of parabolic and hyperbolic Cauchy problems (1.156), (1.157) and (1.160), (1.161)

$$u(x,t) = \frac{1}{2t\sqrt{\pi t}} \int_0^\infty \exp\left[-\frac{\tau^2}{4t}\right] \tau v(x,\tau)\,d\tau. \tag{1.162}$$

Since the transformation (1.162) is one-to-one (as an analog of the Laplace transform), the following two functions $\overline{p}(x,t)$ and $\overline{q}(x,t)$ can be uniquely determined from functions (1.159):

$$v|_{\Gamma_\infty} = \overline{p}(x,t), \quad \frac{\partial v}{\partial n}|_{\Gamma_\infty} = \overline{q}(x,t). \tag{1.163}$$

Therefore, the first parabolic CIP is reduced to the hyperbolic CIP (1.160), (1.161) and (1.163). At the same time, the inversion of the transformation (1.162) is a severely ill-posed procedure. Hence, this inversion cannot be used for computations.

We are almost ready now to apply Theorem 1.10.5.1 The only thing left to do is to replace Γ_∞ in (1.163) with S_∞. To do this, we observe that, using (1.159) and the fact that the unknown coefficient is given outside of the domain Ω, one can uniquely determine the function $u(x,t)$ for $(x,t) \in (\mathbb{R}^n \diagdown \Omega) \times (0,T)$. This is because of the well known uniqueness result for the Cauchy problem for the parabolic equation with the Cauchy data given at a part of the lateral boundary [124]. Therefore, we can uniquely determine functions u, $\partial_n u$ at S_T. This means in turn that we can replace in (1.163) Γ_∞ with S_∞. Hence, Theorem 1.10.5.1 leads to Theorem 1.10.6.

Theorem 1.10.6. *Assume that conditions (1.158) hold. Also, assume that coefficients $c(x), a_\alpha(x)$ and the initial condition $f_0(x)$ are so smooth that the solution v of the problem (1.160) and (1.161) is such that:*

(a) $v \in C^4(\overline{D}_\infty^{n+1})$ *if the function $c(x)$ is unknown*
and

(b) $v \in C^{3+|\alpha|}(\overline{D}_\infty^{n+1})$ *if any of functions $a_\alpha(x)$ is unknown. Let $\Omega = \{|x| < R\} \subset \mathbb{R}^n, n \geq 2$. Suppose that conditions of either of Cases 1 or 2 of Theorem 1.10.5.1 hold. Also, assume that coefficients of (1.160) and the initial condition (1.157) are so smooth that the smoothness of the solution $v(x,t)$ of the hyperbolic Cauchy problem (1.160) and (1.161) required in Theorem 1.10.5.1 is guaranteed. Then, conclusions of 1.10.5.1 are true with respect to the CIP (1.156)–(1.159).*

1.10.7 *The Second Coefficient Inverse Problem for a Parabolic Equation*

Theorem 1.10.6 has two inconvenient points. First, one needs to reduce the parabolic CIP to the hyperbolic CIP via inverting the transform (1.162). Second, one needs to use a special form of the elliptic operator in (1.156). The coefficient $c(x)$ in the principal part of this operator must satisfy conditions (1.81) and (1.82). Although these conditions are satisfied for the case $c(x) \equiv 1$, still the question remains whether it is possible to prove uniqueness of a CIP for the case of a general parabolic operator of the second order. It is shown in this section that the latter is possible, provided that one can guarantee the existence of the solution of the parabolic PDE for both positive and negative values of t. This condition is always used in studies of CIPs for parabolic PDEs via the Bukhgeim-Klibanov method; see for example, [33, 62, 79, 161].

Let $\Omega \subset \mathbb{R}^n$ be either finite or infinite domain with the piecewise smooth boundary $\partial\Omega$, $\Gamma \subset \partial\Phi$ be a part of this boundary, and $T = \text{const} > 0$. Denote

$$Q_T^{\pm} = G \times (-T, T), \Gamma_T^{\pm} = \Gamma \times (-T, T).$$

Let L be the following elliptic operator in $P_T^{\pm}$:

$$Lu = \sum_{|\alpha| \leq 2} a_{\alpha}(x,t) D_x^{\alpha} u, (x,t) \in P_T^{\pm}, \tag{1.164}$$

$$a_{\alpha} \in C^1\left(\overline{Q_T^{\pm}}\right), \tag{1.165}$$

$$\mu_1 |\xi|^2 \leq \sum_{|\alpha|=2} a_{\alpha}(x,t) \xi^{\alpha} \leq \mu_2 |\xi|^2; \quad \mu_1, \mu_2 = \text{const.} > 0, \tag{1.166}$$

$$\forall \xi \in \mathbb{R}^n, \forall (x,t) \in \overline{Q}_T^{\pm}. \tag{1.167}$$

The Second Parabolic Coefficient Inverse Problem. Assume that one of coefficients a_{α_0} of the operator L is independent of t, $a_{\alpha_0} := a_{\alpha_0}(x)$ and is unknown in Ω, whereas all other coefficients of L are known in $Q_T^{\pm}$. Let the function $u \in C^{4,2}\left(\overline{Q}\,_T^{\pm}\right)$ satisfy the parabolic equation

$$u_t = Lu + F(x,t), \text{ in } Q_T^{\pm}. \tag{1.168}$$

Determine the coefficient $a_{\alpha_0}(x)$ for $x \in \Omega$ assuming that the function $F(x,t)$ is known in $Q_T^{\pm}$ and that the following functions $f_0(x)$, $p(x,t)$, and $q(x,t)$ are known as well:

$$u(x,0) = f_0(x), x \in \Omega, \tag{1.169}$$

$$u|_{\Gamma_T^{\pm}} = p(x,t), \frac{\partial u}{\partial n}|_{\Gamma_T^{\pm}} = q(x,t). \tag{1.170}$$

Prior to the formulation of the uniqueness theorem for this problem, we present the Carleman estimate for the principal part L_{p} of the parabolic operator L in (1.164):

$$L_{\mathrm{p}}u = u_t - \sum_{|\alpha|=2} a_\alpha(x,t) D_x^\alpha u.$$

We assume for brevity only that

$$\Omega \subset \{x_1 > 0\} \text{ and } \Gamma = \{x \in \mathbb{R}^n : x_1 = 0, |\overline{x}| \leq A\}, A = \text{const.} > 0, \tag{1.171}$$

where $\overline{x} = (x_2, ..., x_n)$. Let $\xi \in (0,1)$. Consider the function

$$\psi(x,t) = x_1 + \frac{|\overline{x}|^2}{A^2} + \frac{t^2}{T^2} + \xi. \tag{1.172}$$

Let $\gamma \in (\xi, 1)$. Consider the domain H_γ :

$$H_\gamma = \{(x,t) : x_1 > 0, \psi(x,t) < \gamma\}. \tag{1.173}$$

Let $\lambda, \nu > 1$ be two large parameters which we will choose later. In the domain H_γ we consider the following function φ, which is the CWF for the operator L_{p} :

$$\varphi(x,t) = \exp\left[\lambda \psi^{-\nu}(x,t)\right].$$

Lemma 1.10.7 was proven in [124] for the case when terms with $1/\lambda$ were not involved in (1.174). However, these terms can still be incorporated if using ideas of the proof of the second fundamental inequality for elliptic operators of Ladyzhenskaya [119].

Lemma 1.10.7. *Let functions $a_\alpha(x,t), |\alpha| = 2$ satisfy conditions (1.165)–(1.167) and:*

$$\max_{|\alpha|=2} \||\nabla_{x,t} a_\alpha|\|_{C\left(\overline{Q}_T^{\pm}\right)} \leq B = \text{const.}$$

Then, there exist sufficiently large numbers $\lambda_0 = \lambda_0(\xi, \gamma, \mu_1, \mu_2, B) > 1$, $\nu_0 = \nu_0(\xi, \gamma, \mu_1, \mu_2, B) > 1$ such that for $\nu := \nu_0$, for all $\lambda \geq \lambda_0$, and for all functions $u \in C^{2,1}(\overline{Q}_T^{\pm})$, the following pointwise Carleman estimate holds for the operator L_{p}:

$$\left(L_{\mathrm{p}}u\right)^2 \varphi^2 \geq \frac{C}{\lambda}\left(u_t^2 + \sum_{|\alpha|=2}\left(D_x^\alpha u\right)^2\right)\varphi^2 + C\lambda |\nabla u|^2 \varphi^2 + C\lambda^3 |\nabla u|^2 \varphi^2$$
$$+\nabla \cdot U + V_t, \quad (x,t) \in H_\gamma, \tag{1.174}$$

where the vector function (U, V) *satisfies the following estimate:*

$$|(U,V)| \leq C\lambda^3 \left[\sum_{|\alpha|\leq 2} \left(D_x^\alpha u\right)^2 + u_t^2 \right] \varphi^2.$$

Here, the constant $C = C(\beta, \gamma, \mu_1, \mu_2, B) > 0$ *is independent on* λ, u.

Theorem 1.10.7. *Assume that (1.171) holds, the unknown coefficient* $a_{\alpha_0}(x)$ *is independent on* t, *and that* $D_x^{\alpha_0} f_0(x) \neq 0$ *in* $\overline{\Omega}$. *Then there exists at most one solution* $(a_{\alpha_0}, u) \in C^1\left(\overline{\Omega}\right) \times C^{4,2}\left(\overline{P}_T^{\pm}\right)$ *of the inverse problem (1.164)–(1.170).*

Proof. Let $B_1 = \|u\|_{C^{4,2}\left(\overline{Q}_T^{\pm}\right)}$. Let $\theta \in (0,1)$ be a parameter which we will choose later. We change variables now only because coefficients in the principal part $L_{\rm p}$ of the operator $\partial_t - L$ depend on t. If they would be independent on t, we would not need this change of variables. Change variables in (1.168) as $(t', x') = \left(\theta t, \sqrt{\theta} x\right)$ and keep the same notations for new functions, new domains, and new variables for brevity. In new variables we have

$$\max_{\overline{P}_T^{\pm}} |\partial_t a_\alpha(x,t)| \leq \theta B, \ \max_{\overline{P}_T^{\pm}} |\nabla_x a_\alpha(x,t)| \leq \sqrt{\theta} B, |\alpha| = 2, \tag{1.175}$$

$$\max_{\overline{P}_T^{\pm}} |\partial_t a_\alpha(x,t)| \leq \theta B', |\alpha| \leq 1, \tag{1.176}$$

where the number B is defined in Lemma 1.10.7 and B' is another positive constant independent on θ. In particular, (1.175) means that the constant $C > 0$ in Lemma 1.10.7 remains the same after this change of variables. Conditions (1.168)–(1.170) become

$$u_t = \sum_{|\alpha|=2} a_\alpha(x,t) D_x^\alpha u + \sum_{|\alpha|\leq 1} \left(\sqrt{\theta}\right)^{|\alpha|-2} a_\alpha(x,t) D_x^\alpha u + \theta F(x,t), \tag{1.177}$$

$$u(x,0) = f_0(x), \tag{1.178}$$

$$u|_{\Gamma_T^{\pm}} = p(x,t), \quad \partial_{x_1} u|_{\Gamma_T^{\pm}} = -\sqrt{\theta} q(x,t). \tag{1.179}$$

Assume that there exist two pairs of functions satisfying conditions of this theorem:

$$\left(a_{\alpha_0}^{(1)}, u_1\right), \left(a_{\alpha_0}^{(2)}, u_2\right), b(x) = a_{\alpha_0}^{(1)}(x) - a_{\alpha_0}^{(2)}(x), \widetilde{u} = u_1 - u_2.$$

Then, similarly with the proof of Theorem 1.10.5.1, we obtain from (1.177)–(1.179)

$$\widetilde{u}_t - \sum_{|\alpha|=2,\alpha\neq\alpha_0} a_\alpha(x,t) D_x^\alpha \widetilde{u} - \sum_{|\alpha|\leq 1} \left(\sqrt{\theta}\right)^{|\alpha|-2} a_\alpha(x,t) D_x^\alpha u$$

$$-\left(\sqrt{\theta}\right)^{|\alpha_0|-2} a_{\alpha_0}^{(1)}(x) D_x^{\alpha_0}\widetilde{u} = -\left(\sqrt{\theta}\right)^{|\alpha_0|-2} b(x) D_x^{\alpha_0} u_2, \tag{1.180}$$

$$\widetilde{u}(x,0) = 0, \tag{1.181}$$

$$\widetilde{u}|_{\Gamma_T^\pm} = 0, \quad \partial_{x_1}\widetilde{u}|_{\Gamma_T^\pm} = 0. \tag{1.182}$$

Since (1.171) holds, we can assume without loss of generality that $H_\gamma \subset P_T^\pm$. Next, since $D_x^{\alpha_0} f_0(x) \neq 0$ in $\overline{\Omega}$, we can assume without loss of generality that there exists a constant $d > 0$ such that in old variables $D_x^{\alpha_0} f_0(x) \geq 2d > 0$. Hence, in new variables $|D_x^{\alpha_0} f_0(x)| \geq 2\theta^{\alpha_0/2} d$. Therefore, we can choose in (1.172) and (1.173) $\gamma - \xi > 0$ so small that in new variables

$$D_x^{\alpha_0} u_2(x,t) \geq \theta^{\alpha_0/2} d \text{ in } \overline{H}_\gamma. \tag{1.183}$$

In addition,

$$\left|D^{\alpha+\alpha_0} u_2(x,t)\right| \leq d_1 \text{ in } \overline{H}_\gamma, \ \forall \alpha \in \{|\alpha| \leq 2\}, \tag{1.184}$$

where the constant $d_1 > 0$ is independent on θ, as long as $\theta \in (0,1)$.

Let

$$L_1\widetilde{u} = \sum_{|\alpha|=2,\alpha\neq\alpha_0} a_\alpha(x,t) D_x^\alpha \widetilde{u} + \sum_{|\alpha|\leq 1,\alpha\neq\alpha_0} \left(\sqrt{\theta}\right)^{|\alpha|-2} a_\alpha(x,t) D_x^\alpha \widetilde{u}$$

$$+\left(\sqrt{\theta}\right)^{|\alpha_0|-2} a_{\alpha_0}^{(1)}(x) D_x^{\alpha_0}\widetilde{u}.$$

Using (1.180), we obtain

$$-\left(\sqrt{\theta}\right)^{|\alpha_0|-2} b(x) = \frac{\widetilde{u}_t - L_1\widetilde{u}}{D_x^{\alpha_0} u_2} \text{ in } H_\gamma.$$

Differentiating this equality with respect to t, we obtain

$$(\widetilde{u}_t - L_1\widetilde{u})_t - g(x,t)(\widetilde{u}_t - L_1\widetilde{u}) = 0 \text{ in } H_\gamma, \tag{1.185}$$

$$g(x,t) = \partial_t \ln\left(D_x^{\alpha_0} u_2\right). \tag{1.186}$$

We have

$$(L_1\widetilde{u})_t = \sum_{|\alpha|=2,\alpha\neq\alpha_0} a_\alpha(x,t)\, D_x^\alpha \widetilde{u}_t + \sum_{|\alpha|\leq 1,\alpha\neq\alpha_0} \left(\sqrt{\theta}\right)^{|\alpha|-2} a_\alpha(x,t)\, D_x^\alpha \widetilde{u}_t$$
$$+ \left(\sqrt{\theta}\right)^{|\alpha_0|-2} a_{\alpha_0}^{(1)}(x)\, D_x^{\alpha_0}\widetilde{u}_t + \sum_{|\alpha|=2,\alpha\neq\alpha_0} \partial_t\left(a_\alpha(x,t)\right) D_x^\alpha \widetilde{u}$$
$$+ \sum_{|\alpha|\leq 1,\alpha\neq\alpha_0} \left(\sqrt{\theta}\right)^{|\alpha|-2} \partial_t\left(a_\alpha(x,t)\right) D_x^\alpha \widetilde{u}.$$

Hence, (1.185) implies that

$$\left(\widetilde{u}_{tt} - g\widetilde{u}_t\right) - \sum_{|\alpha|=2,\alpha\neq\alpha_0} a_\alpha(x,t)\left(D_x^\alpha\widetilde{u}_t - gD_x^\alpha\widetilde{u}\right)$$
$$- \sum_{|\alpha|\leq 1,\alpha\neq\alpha_0} \left(\sqrt{\theta}\right)^{|\alpha|-2} a_\alpha(x,t)\left(D_x^\alpha\widetilde{u}_t - gD_x^\alpha\widetilde{u}\right)$$
$$-\left(\sqrt{\theta}\right)^{|\alpha_0|-2} a_{\alpha_0}^{(1)}(x)\left(D_x^{\alpha_0}\widetilde{u}_t - gD_x^{\alpha_0}\widetilde{u}\right)$$
$$- \sum_{|\alpha|=2,\alpha\neq\alpha_0} \partial_t\left(a_\alpha(x,t)\right) D_x^\alpha\widetilde{u}$$
$$- \sum_{|\alpha|\leq 1,\alpha\neq\alpha_0} \left(\sqrt{\theta}\right)^{|\alpha|-2} \partial_t\left(a_\alpha(x,t)\right) D_x^\alpha\widetilde{u} = 0, \text{ in } H_\gamma. \tag{1.187}$$

Now, use the formula $gD_x^\alpha\widetilde{u} = D_x^\alpha(g\widetilde{u}) + lot$, where lot is a linear combination of derivatives of the function $\widetilde{u}$ whose order is less than $|\alpha|$. Then

$$D_x^\alpha\widetilde{u}_t - gD_x^\alpha\widetilde{u} = D_x^\alpha\left(\widetilde{u}_t - g\widetilde{u}\right) + lot,\ \widetilde{u}_{tt} - g\widetilde{u}_t = D_t\left(\widetilde{u}_t - g\widetilde{u}\right) + g_t\widetilde{u}. \tag{1.188}$$

Denote $v = \widetilde{u}_t - p\widetilde{u}$. Then, (1.181) and (1.186) imply that

$$\widetilde{u}(x,t) = \int_0^t K(x,t,\tau)\, v(x,\tau)\, \mathrm{d}\tau, \text{ in } H_\gamma, \tag{1.189}$$

$$K(x,t,\tau) = \frac{D_x^{\alpha_0} u_2(x,t)}{D_x^{\alpha_0} u_2(x,\tau)}. \tag{1.190}$$

It follows from (1.183), (1.184), and (1.190) that

$$\left|D_x^\alpha K(x,t,\tau)\right| \leq M, |\alpha| \leq 2 \text{ for } (x,t), (x,\tau) \in H_\gamma. \tag{1.191}$$

Here and below in this proof, M denotes different positive constants independent on the function v and the parameter $\theta \in (0,1)$.

Hence, using (1.175), (1.176), and (1.187)–(1.191), we obtain

$$\left|L_{\mathrm{p}} v\right| = \left| v_t - \sum_{|\alpha|=2} a_\alpha (x,t) D_x^\alpha v \right| \leq M\theta \sum_{|\alpha|=2} \left| \int_0^t |D_x^\alpha v| (x,\tau) \,\mathrm{d}\tau \right|$$

$$+ \frac{M}{\theta^2} \sum_{|\alpha|\leq 1} |D_x^\alpha v| + \frac{M}{\theta^2} \sum_{|\alpha|\leq 1} \left| \int_0^t |D_x^\alpha v| (x,\tau) \,\mathrm{d}\tau \right|, \text{ in } H_\gamma, \tag{1.192}$$

$$v|_{\Gamma_T^\pm \cap \overline{H}_\gamma} = 0, \quad \partial_{x_1} v|_{\Gamma_T^\pm \cap \overline{H}_\gamma} = 0. \tag{1.193}$$

Now, we are ready to apply the Carleman estimate of Lemma 1.10.7, assuming that parameters $\nu := \nu_0, \lambda_0$ are the same as ones in this lemma and that $\lambda \geq \lambda_0$. Multiply both sides of the inequality (1.192) by the function $\varphi(x,t)$, then square it and integrate over the domain H_γ. We obtain

$$\int_{H_\gamma} \left(L_{\mathrm{p}} v\right)^2 \varphi^2 \mathrm{d}x\mathrm{d}t \leq M\theta^2 \sum_{|\alpha|=2} \int_{H_\gamma} \left(\int_0^t |D_x^\alpha v| (x,\tau) \,\mathrm{d}\tau \right)^2 \varphi^2 \mathrm{d}x\mathrm{d}t$$

$$+ \frac{M}{\theta^4} \int_{H_\gamma} \left(\int_0^t (|\nabla v| + |v|)(x,\tau) \,\mathrm{d}\tau \right)^2 \varphi^2 \mathrm{d}x\mathrm{d}t$$

$$+ \frac{M}{\theta^4} \int_{H_\gamma} \left(|\nabla v|^2 + v^2 \right) \varphi^2 \mathrm{d}x\mathrm{d}t. \tag{1.194}$$

Using Lemma 1.10.3, we obtain from (1.194)

$$\int_{H_\gamma} (L_0 v)^2 \varphi^2 \mathrm{d}x\mathrm{d}t \leq \frac{M\theta^2}{\lambda} \sum_{|\alpha|=2} \int_{H_\gamma} \left(D_x^\alpha v\right)^2 \varphi^2 \mathrm{d}x\mathrm{d}t + \frac{M}{\theta^4} \int_{H_\gamma} \left(|\nabla v|^2 + v^2 \right) \varphi^2 \mathrm{d}x\mathrm{d}t. \tag{1.195}$$

On the other hand, using (1.174) and (1.193), we obtain

$$\int_{H_\gamma} \left(L_{\mathrm{p}} v\right)^2 \varphi^2 \mathrm{d}x\mathrm{d}t \geq \frac{C}{\lambda} \sum_{|\alpha|=2} \int_{H_\gamma} \left(D_x^\alpha v\right)^2 \varphi^2 \mathrm{d}x\mathrm{d}t + C \int_{H_\gamma} \left(\lambda |\nabla v|^2 + \lambda^3 v^2 \right) \varphi^2 \mathrm{d}x\mathrm{d}t$$

$$- C\lambda^3 \int_{\partial_1 H_\gamma} \sum_{|\alpha|\leq 2} \left(D_x^\alpha v\right)^2 \varphi^2 \mathrm{d}x\mathrm{d}t, \tag{1.196}$$

where $\partial_1 H_\gamma = \{(x,t) : \psi(x,t) = \gamma, x_1 > 0\}$. Hence,

$$\varphi^2 = \exp\left(2\lambda\gamma^{-\nu}\right) \text{ on } \partial_1 H_\gamma. \tag{1.197}$$

Choose $\theta \in (0,1)$ so small that $M\theta^2 \leq C/2$. Then comparing (1.195) with (1.196) and taking into account (1.197), we obtain for sufficiently large $\lambda \geq \lambda_1(\theta) > 1$

$$\frac{1}{\lambda}\sum_{|\alpha|=2}\int_{H_\gamma}\left(D_x^\alpha v\right)^2\varphi^2\mathrm{d}x\mathrm{d}t+\int_{H_\gamma}\left(\lambda\,|\nabla v|^2+\lambda^3 v^2\right)\varphi^2\mathrm{d}x\mathrm{d}t$$
$$\leq M\lambda^3\exp\left(2\lambda\gamma^{-\nu}\right)\int_{\partial_1 H_\gamma}\sum_{|\alpha|\leq 2}\left(D_x^\alpha v\right)^2\varphi^2\mathrm{d}S. \tag{1.198}$$

Let $\varepsilon \in (0,\gamma-\xi)$ be an arbitrary number. Then $H_{\gamma-\varepsilon} \subset H_\gamma$ and $\varphi^2(x,t) \geq \exp\left[2\lambda(\gamma-\varepsilon)^{-\nu}\right]$ in $H_{\gamma-\varepsilon}$. Hence, (1.198) implies that

$$\int_{H_{\gamma-\varepsilon}} v^2\mathrm{d}x\mathrm{d}t \leq M\exp\left\{-2\lambda\left[(\gamma-\varepsilon)^{-\nu}-\gamma^{-\nu}\right]\right\}\int_{\partial_1 H_\gamma}\sum_{|\alpha|\leq 2}\left(D_x^\alpha v\right)^2\varphi^2\mathrm{d}S.$$

Setting here $\lambda \to \infty$, we obtain that the right-hand side of this inequality tends to zero, which implies that $v(x,t) = 0$ in $H_{\gamma-\varepsilon}$. Since $\varepsilon \in (0,\gamma-\xi)$ is an arbitrary number, then $v(x,t) = 0$ in H_γ. Hence, (1.189) implies that $\widetilde{u}(x,t) = 0$ in H_γ. Next, (1.180) and (1.183) imply that $b(x) = 0$ in $H_\gamma \cap \{t=0\}$. Hence, $a_{\alpha_0}^{(1)}(x) = a_{\alpha_0}^{(2)}(x)$ in $H_\gamma \cap \{t=0\}$. Therefore, applying the same method to the homogeneous equation (1.180) with boundary conditions (1.182) and changing, if necessary, variables as $(t'',x'') = (t-t_0,x)$ with appropriate numbers $t_0 \in (-T,T)$, we obtain that

$$\widetilde{u}(x,t) = 0 \text{ for } (x,t) \in \left\{x_1+\frac{|\overline{x}|^2}{A^2}<\gamma-\xi\right\}\times(-T,T). \tag{1.199}$$

It is clear that changing x variables by rotations of the coordinate system as well as by shifting the location of the origin and proceeding similarly with the above, we can cover the entire time cylinder $Q_T^{\pm}$ by domains, which are similar with the one in (1.199). Thus, $a_{\alpha_0}^{(1)}(x) = a_{\alpha_0}^{(2)}(x)$ in G and $u_1(x,t) = u_2(x,t)$ in $Q_T^{\pm}$. □

1.10.8 *The Third Coefficient Inverse Problem for a Parabolic Equation*

Let L be the elliptic operator in $\mathbb{R}^n$, whose coefficients depend only on x:

$$Lu = \sum_{|\alpha|\leq 2} a_\alpha(x)\,D_x^\alpha u, \tag{1.200}$$

$$a_\alpha \in C^{2+\beta}\left(\mathbb{R}^n\right),\ \beta\in(0,1) \tag{1.201}$$

We assume that

$$\mu_1 |\xi|^2 \leq \sum_{|\alpha|=2} a_\alpha (x) \xi^\alpha \leq \mu_2 |\xi|^2, \forall x, \xi \in \mathbb{R}^n. \tag{1.202}$$

Consider the following Cauchy problem

$$u_t = Lu \text{ in } D_T^{n+1}, \quad u \in C^{4+\beta,2+\beta/2}\left(\overline{D}_T^{n+1}\right), \tag{1.203}$$

$$u|_{t=0} = f(x) \in C^{4+\beta}(\mathbb{R}^n) \tag{1.204}$$

It is well known that the problem (1.203) and (1.204) has unique solution [120].

The Third Parabolic Coefficient Inverse Problem. Let $T_0 \in (0, T)$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that the coefficient $a_{\alpha_0}(x)$ of the operator L is known inside Ω and is unknown outside of Ω. Assume that the initial condition $f(x)$ is also unknown. Determine both the coefficient $a_{\alpha_0}(x)$ for $x \in \mathbb{R}^n \diagdown \Omega$ and the initial condition $f(x)$ for $x \in \mathbb{R}^n$, assuming that the following function $F(x)$ is known:

$$F(x) = u(x, T_0), x \in \mathbb{R}^n. \tag{1.205}$$

Theorem 1.10.8. *Assume that conditions (1.200)–(1.202) hold, all coefficients of the operator L belong to $C^\infty(\Omega)$, and*

$$D^{\alpha_0} F(x) \neq 0, \text{ in } \mathbb{R}^n \diagdown \Omega.$$

Then, there exists at most one pair of functions $(a_{\alpha_0}(x), u(x,t))$ satisfying conditions (1.203)–(1.205).

Proof. Consider the solution of the following hyperbolic Cauchy problem:

$$v_{tt} = Lv \text{ in } D_\infty^{n+1},$$

$$v(x,0) = 0, v_t(x,0) = f(x).$$

Then the Laplace-like transform (1.162) connects functions u and v. Hence, for any $x \in \mathbb{R}^n$ the function $u(x,t)$ is analytic with respect to the variable $t > 0$ as a function of a real variable. We now show that the function $u(x,t)$ can be uniquely determined for $(x,t) \in \Omega \times (0,T)$. Since all coefficients $a_\alpha \in C^\infty(\Omega)$, then the solution u of the Cauchy problem (1.203) and (1.204) is $u \in C^\infty(\Omega \times (0,T))$ [69]. Hence, using (1.203) and (1.205), we obtain

$$D_t^{k+1} u(x, T_0) = L^k [F(x)], x \in \Omega, k = 0, 1, \ldots$$

Thus, one can uniquely determine all t derivatives of the function $u(x,t)$ at $t = T_0$ for all $x \in \Omega$. Hence, the analyticity of the function $u(x,t)$ with respect to

t implies that this function can be uniquely determined for $(x,t) \in \Omega \times (0,T)$. Hence, Theorem 1.10.7 implies that the coefficient $a_{\alpha_0}(x)$ is uniquely determined in the domain $\mathbb{R}^n \diagdown \Omega$. To establish that the initial condition $f(x)$ is also uniquely determined, we refer to the well-known theorem about the uniqueness of the solution of the parabolic equation with reversed time [69, 124]. □

1.10.9 A Coefficient Inverse Problem for an Elliptic Equation

We now consider an elliptic analog of the second parabolic CIP. Let $\Omega \subset \mathbb{R}^n$ be either finite or infinite convex domain with the piecewise smooth boundary $\partial\Omega$ and let $\Gamma \subset \partial\Omega$ be a part of this boundary. Let $T = \text{const} > 0$. Denote again

$$Q_T^{\pm} = \Omega \times (-T,T), \Gamma_T^{\pm} = \Gamma \times (-T,T).$$

Let L be an elliptic operator in $Q_T^{\pm}$:

$$Lu = \sum_{|\alpha| \leq 2} a_\alpha(x,t) D_x^\alpha u, (x,t) \in Q_T^{\pm}, \tag{1.206}$$

$$a_\alpha \in C^1\left(\overline{Q_T^{\pm}}\right), \tag{1.207}$$

$$\mu_1 |\xi|^2 \leq \sum_{|\alpha|=2} a_\alpha(x,t) \xi^\alpha \leq \mu_2 |\xi|^2; \quad \mu_1, \mu_2 = \text{const.} > 0 \tag{1.208}$$

$$\forall \xi \in \mathbb{R}^n, \forall (x,t) \in \overline{Q_T^{\pm}}. \tag{1.209}$$

Coefficient Inverse Problem for an Elliptic Equation. Let the function $u \in C^2\left(\overline{Q_T^{\pm}}\right)$ satisfies the following conditions:

$$u_{tt} + Lu = F(x,t) \text{ in } Q_T^{\pm}, \tag{1.210}$$

$$u(x,0) = f_0(x) \text{ in } \Omega, \tag{1.211}$$

$$u|_{\Gamma_T^{\pm}} = p(x,t), \quad \frac{\partial u}{\partial n}|_{\Gamma_T^{\pm}} = q(x,t). \tag{1.212}$$

Assume that the coefficient $a_{\alpha_0}(x)$ of the operator L is independent of t and is unknown in Ω and all other coefficients are known in $Q_T^{\pm}$. Determine the coefficient $a_{\alpha_0}(x)$ from conditions (1.206)–(1.212).

Theorem 1.10.9. *Assume that $D_x^{\alpha_0} f_0(x) \neq 0$ in $\overline{\Omega}$. Then, there exists at most one pair of functions $(a_{\alpha_0}(x), u(x,t))$ such that conditions (1.206)–(1.212) hold and, in addition, the function $u \in C^3\left(\overline{Q_T^{\pm}}\right)$.*

Proof. Lemma 1.10.7 remains valid if the parabolic operator L_p in (1.174) is replaced with the elliptic operator [124]:

$$u_{tt} + \sum_{|\alpha|=2} a_\alpha (x,t) D_x^\alpha u.$$

Therefore, the proof is completely similar with the proof of Theorem 1.10.7. □

1.11 Uniqueness for the Case of an Incident Plane Wave in Partial Finite Differences

We present in this section the result of the paper [112]. Unlike all uniqueness theorems of Sect. 1.10, we assume now that initial conditions equal zero in the entire domain of interest. At the same time, we assume that the underlying hyperbolic PDE is written in the form of finite differences with respect to those variables which are orthogonal to the direction of propagation of the incident plane wave. Derivatives with respect to other variables are understood in the conventional form. In addition, we assume that grid step sizes in finite differences are bounded from the below. In fact, this assumption is quite often used in computations of CIPs.

Both classical forward problems for PDEs and ill-posed problems are routinely solved numerically by the FDM, see; for example, [114–116, 146] as well as Sects. 6.8 and 6.9. Therefore, it is important to prove uniqueness theorems for CIPs for the case when they are written in finite differences. However, there is a fundamental difference between classical forward problems and nonclassical ill-posed problems. Indeed, since classical forward problems are well-posed, then it makes sense to investigate convergence of the FDM when the spatial step grid step size h_{sp} tends to zero; see, for example, [146] for such results.

However, in the case of ill-posed problems, there is no point sometimes to investigate the convergence of FDM-based numerical methods when the spatial step size h_{sp} tends to zero. This is because in the ill-posed case, h_{sp} should usually be limited from the below by an a priori chosen constant, $h_{sp} \geq \overline{h} = \text{const.} > 0$. The constant $\overline{h}$ is usually chosen in numerical experiments. The reason of this limitation is that h_{sp} serves as an implicit regularization parameter in the *discrete* case of the FDM. Because of this, h_{sp} cannot be significantly decreased. The same observation takes place in numerical studies of Chap. 6; see Sect. 6.8.1 as well as [114, 116].

For the sake of brevity, we consider here only the 3D case. Theorems 1.11.1.1 and 1.11.1.2 below have almost identical formulations and proofs for the n-D case with $n \geq 2$. Below, $\mathbf{x} = (x,y,z) \in \mathbb{R}^3$. Let the function $a \in C^2(\mathbb{R}^3)$ and is bounded in $\mathbb{R}^3$ together with its derivatives. Consider the Cauchy problem

$$u_{tt} = \Delta u + a(\mathbf{x})\, u, (\mathbf{x},t) \in \mathbb{R}^3 \times (0,T), \tag{1.213}$$

$$u(\mathbf{x},0) = 0, u_t(\mathbf{x},0) = \delta(z). \tag{1.214}$$

Conditions (1.213) and (1.214) mean that the wave field u is initialized by the plane wave at the plane $\{z=0\}$. This plane wave propagates along the z-axis. Let $A_x, A_y, T = \text{const.} > 0$. Define the strip G as

$$G = \left\{\mathbf{x} : x \in (0, A_x), y \in \left(0, A_y\right)\right\}, \ G_T = G \times (0, T),$$
$$S_T = \left\{z = 0, x \in (0, A_x), y \in \left(0, A_y\right)\right\} \times (0, T).$$

Coefficient Inverse Problem 1.11. *Assume that the function $a(\mathbf{x})$ is unknown in G. Determine the coefficient $a(\mathbf{x})$ for $x \in G$, assuming that the following two functions $r(x,t), s(x,t)$ are given*:

$$u\,|_{S_T} = r(\mathbf{x}, t), \ u_z\,|_{S_T} = s(\mathbf{x}, t). \tag{1.215}$$

The question of the uniqueness of this CIP is a well-known long-standing open problem. Note that (1.215) is the backscattering data. The main challenge is the single measurement, not the backscattering. For the first time, this question was addressed in [34]. However, infinitely many measurements were considered in these references. The second class of uniqueness results for the case of single measurement data with zero initial conditions are ones when the unknown coefficient is represented as a finite sum of a Fourier-like series:

$$a(x, y, z) = \sum_{k=1}^{N} b_k(x, y)\, a_k(z), N < \infty, \tag{1.216}$$

where functions a_k, b_k are unknown. The main restriction here is $N < \infty$. This kind of results follows from a special method of the integral geometry, which was developed in [124]; see Sect. 6.3 of [124].

In this section, we prove uniqueness theorem for a closely related inverse problem. Specifically, we assume that derivatives with respect to (x, y) are written via finite differences with the grid step sizes (h_1, h_2). Numbers h_1, h_2 do not tend to zero. However, derivatives with respect to z, t are written in the usual form. The uniqueness Theorem 1.11.1.2 uses these assumptions. Since finite differences are often used in computations, then Theorem 1.11.1.2 seems to be more attractive for computations than the assumption (1.216).

First, we prove in Lemma 1.11.3 a new Carleman estimate, which is significantly different from conventional Carleman estimates of Sect. 1.10. The main new element here is that a certain integral over the characteristic curve contains only nonnegative terms with large parameters involved. Usually, the positivity of surface integrals is not the case of Carleman estimates. This new Carleman estimate enables us to apply a new idea, compared with the method of Sect. 1.10. That new idea is generated by the second line of (1.244) in combination with (1.254). Indeed, in all previous publications about the Bukhgeim-Klibanov method, t-integrals of the Volterra type were used; see Sect. 1.10. Unlike this, we do not use those integrals in the proof of Theorem 1.11.1.2.

Discrete Carleman estimates are attracting an interest of researchers [37,38,105]. However, they were not yet used for proofs of uniqueness of discrete CIPs. A discrete Carleman estimate is not used here.

1.11.1 Results

Consider partitions of intervals $x \in (0, A_x)$, $y \in \left(0, A_y\right)$ in small subintervals with step sizes h_1 and h_2, respectively:

$$0 = x_0 < x_1 < ... < x_{N_1} = A_x, \; 0 = y_0 < y_1 < ... < y_{N_2} = A_y, \quad (1.217)$$

$$x_i - x_{i-1} = h_1, \; y_j - y_{j-1} = h_2, \; h := (h_1, h_2), \; h_0 = \min(h_1, h_2); N_1, N_2 > 2. \quad (1.218)$$

Hence, we have obtained the grid

$$G_h = \{(x, y) : x = ih_1, y = jh_2\}_{(i,j)=(0,0)}^{(N_1,N_2)}.$$

Consider a vector function $f^h(z,t)$ defined on this grid:

$$f^h(z,t) = \left\{f_{i,j}(z,t)\right\}_{(i,j)=(0,0)}^{(N_1,N_2)}.$$

For two vector functions $f^h(z,t)$, $g^h(z,t)$, define

$$g^h(z,t)\, f^h(z,t) := \left\{k_{i,j}(z,t)\right\}_{(i,j)=(0,0)}^{(N_1,N_2)}, \; k_{i,j}(z,t) = g_{i,j}(z,t) \cdot f_{i,j}(z,t). \quad (1.219)$$

Denote

$$\left(f^h(z,t)\right)^2 := \sum_{(i,j)=(0,0)}^{(N_1,N_2)} f_{i,j}^2(z,t).$$

We define finite difference second derivatives $\partial_{x,h}^2 f^h(z,t)$ and $\partial_{y,h}^2 f^h(z,t)$ with respect to x and y, respectively, in the usual way as

$$\partial_{x,h}^2 f^h(z,t) = \left\{\partial_{x,h}^2 f_{i,j}(z,t)\right\}_{(i,j)=(0,0)}^{(N_1,N_2)},$$

$$\partial_{y,h}^2 f^h(z,t) = \left\{\partial_{y,h}^2 f_{i,j}(z,t)\right\}_{(i,j)=(0,0)}^{(N_1,N_2)},$$

$$\partial^2_{x,h} f_{i,j}(z,t) := \frac{1}{h_1^2}\begin{cases} f_{i-1,j}(z,t) - 2f_{i,j}(z,t) + f_{i+1,j}(z,t), \text{ if } i \neq 0, i \neq N_1, \\ f_{i,j}(z,t) - 2f_{i+1,j}(z,t) + f_{i+2,j}(z,t), \text{ if } i = 0, \\ f_{i,j}(z,t) - 2f_{i-1,j}(z,t) + f_{i-2,j}(z,t), \text{ if } i = N_1, \end{cases}$$

and similarly for $\partial^2_{y,h} f^h(z,t)$. Hence, if a function $g(x,y,z,t)$ has continuous derivatives up to the fourth order with respect to x, then $\partial^2_{x,h} g_{i,j}(z,t)$ approximates $g_{xx}(x,y,z,t)$ at the point $(x,y) = (ih_1, jh_2)$ with the accuracy $O\left(h_1^2\right), h_1 \to 0$ in the case when $ih_1 \neq 0, A_x$. And it approximates with the accuracy $O(h_1), h_1 \to 0$ in the case when $ih_1 = 0, A_x$. This is similar for the y derivative. Next, we define the finite difference Laplace operator as

$$\begin{aligned}
\Delta_h f_{i,j}(z,t) &:= \partial_z^2 f_{i,j}(z,t) + \Delta_{h,x,y} f_{i,j}(z,t), \\
\Delta_{h,x,y} f_{i,j}(z,t) &:= \partial^2_{x,h} f_{i,j}(z,t) + \partial^2_{y,h} f_{i,j}(z,t), \\
\Delta_h f^h(z,t) &:= \left\{\Delta_h f_{i,j}(z,t)\right\}_{(i,j)=(0,0)}^{(N_1,N_2)} \\
&:= \left(\partial_z^2 f_{i,j}(z,t)\right)_{(i,j)=(0,0)}^{(N_1,N_2)} + \left\{\Delta_{h,x,y} f_{i,j}(z,t)\right\}_{(i,j)=(0,0)}^{(N_1,N_2)} \\
&:= \partial_z^2 f^h(z,t) + \Delta_{h,x,y} f^h(z,t).
\end{aligned}$$

Define

$$a^h(z) := \left\{a_{i,j}(z)\right\}_{(i,j)=(0,0)}^{(N_1,N_2)}.$$

Rewrite the problem (1.213), (1.214) in the finite difference form as

$$u_{tt}^h = \Delta^h u^h + a^h(z) u^h, \quad (z,t) \in \mathbb{R} \times (0,T), \tag{1.220}$$

$$u^h(z,0) = 0, u_t^h(z,0) = \delta(z), \tag{1.221}$$

where the product $a^h(z) u^h$ is understood as in (1.219).

Coefficient Inverse Problem 1.11.1.1. Let the vector function $u^h(z,t)$ be the solution of the problem (1.220) and (1.221). Determine the vector function $a^h(z)$ assuming that the following two vector functions $r^h(t), s^h(t)$,

$$r^h(t) = \left\{r_{i,j}(t)\right\}_{(i,j)=(0,0)}^{(N_1,N_2)}, \; s^h(t) = \left\{s_{i,j}(t)\right\}_{(i,j)=(0,0)}^{(N_1,N_2)}, \tag{1.222}$$

are given:

$$u^h(0,t) = r^h(t), \; u_z^h(0,t) = s^h(t), \; t \in (0,T). \tag{1.223}$$

Theorem 1.11.1.1. *Let the vector function $a^h(z) \in C^1(\mathbb{R})$ and is bounded in $\mathbb{R}$. Then, there exists unique solution of the forward problem (1.220) and (1.221) of the form*

$$u^h(z,t) = \left\{u_{i,j}(z,t)\right\}_{(i,j)=(0,0)}^{(N_1,N_2)},$$

where

$$u_{i,j}(z,t) = \frac{1}{2}H(t-|z|) + \overline{u}_{i,j}(z,t), (i,j) \in [0,N_1] \times [0,N_2]. \tag{1.224}$$

In (1.223), $H(z)$ is the Heaviside function,

$$H(z) = \begin{cases} 1 \text{ if} & z \geq 0, \\ 0 \text{ if} & z < 0 \end{cases}$$

and the function $\overline{u}_{i,j}$ is such that

$$\overline{u}_{i,j} \in C^3(t \geq |z|), \overline{u}_{i,j}(z,t) = 0 \text{ for } t \in (0,|z|]. \tag{1.225}$$

Theorem 1.11.1.2. *Let $R > 0$ be an arbitrary number and $T > 2R$. Assume that there exist two pairs of vector functions $\left(u_1^h(z,t), a_1^h(z)\right)$, $\left(u_2^h(z,t), a_2^h(z)\right)$ such that $a_1^h, a_2^h \in C^1(\mathbb{R})$ and vector functions u_1^h and $u_2^h$are solutions of the problem (1.220) and (1.221) of the form (1.224) and (1.225) with $a^h := a_1^h$ and $a^h := a_2^h$, respectively. In addition, assume that both vector functions u_1^h, u_2^h satisfy the same conditions (1.223). Then $a_1^h(z) = a_2^h(z)$ for $|z| < R$ and*

$$u_1^h(z,t) = u_2^h(z,t) \text{ for } (z,t) \in \{|z| < R, t \in (0, T-|z|)\}. \tag{1.226}$$

1.11.2 Proof of Theorem 1.11.1.1

Denoting temporarily $f_{i,j}(z,t) = \Delta_{h,x,y}u_{i,j} + a_{i,j}u_{i,j}$, rewrite (1.220) and (1.221) as

$$\partial_t^2 u_{i,j} = \partial_z^2 u_{i,j} + f_{i,j}(z,t), \ (i,j) \in [0,N_1] \times [0,N_2], \tag{1.227}$$

$$u_{i,j}(z,0) = 0, \ \partial_t u_{i,j}(z,0) = \delta(z). \tag{1.228}$$

Using D'Alembert formula, we derive from (1.227) and (1.228) that

$$u_{i,j}(z,t) = \frac{1}{2}H(t-|z|) + \frac{1}{2}\int_0^t \mathrm{d}\tau \int_{\tau-t+z}^{t-\tau+z} \left(\Delta_{h,x,y}u_{i,j} + a_{i,j}u_{i,j}\right)(\xi,\tau)\,\mathrm{d}\xi, \tag{1.229}$$

for $(i,j) \in [0,N_1] \times [0,N_2]$. In (1.229), the integration is carried out over the triangle $\Delta(z,t)$ in the (ξ,τ)-plane, where the triangle $\Delta(z,t)$ has vertices at $(\xi_1,\tau_1) = (z-t,0), (\xi_2,\tau_2) = (z,t)$, and $(\xi_3,\tau_3) = (z+t,0)$. If we consider the set of equations (1.229) considered for $(i,j) \in [0,N_1] \times [0,N_2]$, then we obtain a linear system of coupled Volterra-like integral equations. Hence, this system can be solved iteratively:

$$u_{i,j}^{(n)}(z,t) = \frac{1}{2}H(t-|z|) + \frac{1}{2}\int_0^t d\tau \int_{\tau-t+z}^{t-\tau+z} \left(\Delta_{h,x,y} u_{i,j}^{(n-1)} + a_{i,j} u_{i,j}^{(n-1)}\right)(\xi,\tau)\, d\xi,$$

for $(i,j) \in [0,N_1] \times [0,N_2]$. Let

$$\max_{i,j} \sup_R \left|a_{i,j}(z)\right| \leq M, M = \text{const.} > 0.$$

The standard technique for Volterra equations leads to the following estimate:

$$\left|u_{i,j}^{(n)}(z,t)\right| \leq \sum_{n=0}^{\infty} \frac{(Ct)^n}{n!}, z \in \mathbb{R}, t > 0, \ \forall (i,j) \in [0,N_1] \times [0,N_2], \tag{1.230}$$

where the constant $C = C(h,M)$. Hence, there exists a solution of the integral equation (1.229) such that this solution is continuous for $t \in [0,|z|]$ and for $t \geq |z|$.

Next, let in (1.229) $t < |z|$. Then the triangle $\Delta(z,t)$ is located below $\{\tau = |\xi|\}$ and above $\{\tau = 0\}$. Hence, we obtain from (1.229)

$$u_{i,j}(z,t) = \frac{1}{2}\int_0^t d\tau \int_{\tau-t+z}^{t-\tau+z} \left(\Delta_{h,x,y} u_{i,j} + a_{i,j} u_{i,j}\right)(\xi,\tau)\, d\xi,$$
$$\text{for } t < |z|, \ (i,j) \in [0,N_1] \times [0,N_2]. \tag{1.231}$$

Iterating (1.231), we obtain similarly with (1.230) that

$$\left|u_{i,j}(z,t)\right| \leq \frac{(Ct)^n}{n!}, n = 1,2,...; \ (i,j) \in [0,N_1] \times [0,N_2].$$

Hence, $u_{i,j}(z,t) = 0$ for $t < |z|$. The same way uniqueness of the problem (1.229) can be proven.

Let

$$\overline{u}_{i,j}(z,t) = u_{i,j}(z,t) - \frac{1}{2}H(t-|z|).$$

Since $\Delta_{h,x,y}[H(t-|z|)] = 0$ and $\overline{u}_{i,j}(z,t) = 0$ for $t < |z|$, then the integration in (1.229) is actually carried out over the following domain:

$$\{(\xi,\tau) : |\xi| < \tau < t - |z-\xi|\} = \left\{(\xi,\tau) : \xi \in \left(\frac{z-t}{2}, \frac{z+t}{2}\right), \right.$$
$$\left. \tau \in (|\xi|, t-|z-\xi|)\right\}.$$

Hence, (1.229) leads to the following equation for $\overline{u}_{i,j}(z,t)$:

$$\overline{u}_{i,j}(z,t) = \frac{1}{2}\int\limits_{\frac{z-t}{2}}^{\frac{z+t}{2}} d\xi \int\limits_{|\xi|}^{t-|z-\xi|} \left(\Delta_{h,x,y}\overline{u}_{i,j} + a_{i,j}\overline{u}_{i,j}\right)(\xi,\tau)\, d\tau$$
$$+\frac{1}{2}\int\limits_{\frac{z-t}{2}}^{\frac{z+t}{2}} a_{i,j}(\xi)(t-|z-\xi|-|\xi|)\, d\xi,\ t>|z|,\ \ (i,j)\in[0,N_1]$$
$$\times[0,N_2].$$

Differentiating these equations, we obtain that

$$\overline{u}_{i,j}(z,t)\in C^3(t\geq|z|),\ \forall\,(i,j)\in[0,N_1]\times[0,N_2]$$

Thus, the solution $\left\{u_{i,j}(z,t)\right\}_{(i,j)=(0,0)}^{(N_1,N_2)}$ of the system of equations (1.229) satisfies conditions (1.227), (1.228), (1.224), and (1.225).

It follows from Theorem 1.11.1.1 that we can consider functions $u_{i,j}(z,t)$ only above the characteristic line $\{t=|z|\}$ in the (z,t) plane. Hence, consider new functions $w_{i,j}(z,t) = u_{i,j}(z,t+z), z>0$. The domain $\{t>z, z>0\}$ becomes now $\{t>0, z>0\}$. Using (1.220), (1.222)–(1.224), and (1.225), we obtain

$$\partial_z^2 w_{i,j} - 2\partial_z\partial_t w_{i,j} = -\Delta_{h,x,y}w_{i,j} + a_{i,j}(z)\,w_{i,j},\,(z,t)\in\{z,t>0\}, \quad (1.232)$$

$$w_{i,j}(z,0)=\tfrac{1}{2}, \quad (1.233)$$

$$w_{i,j}(0,t)=r_{i,j}(t),\ \partial_z w_{i,j}(0,t)=s_{i,j}(t),\ t\in(0,T), \quad (1.234)$$

$$w_{i,j}\in C^3(z,t\geq 0), \quad (1.235)$$

$$(i,j)\in[0,N_1]\times[0,N_2]. \quad (1.236)$$

1.11.3 The Carleman Estimate

Consider parameters α,β,ν where

$$\alpha\in\left(0,\frac{1}{2}\right),\beta,\nu>0.$$

Also, let $\lambda>1$ be a sufficiently large parameter. We will choose λ later. Consider functions $\psi(z,t)$ and $\varphi(z,t)$ defined as

$$\psi(z,t)=z+\alpha t+1,\ \varphi(z,t)=\exp\left(\lambda\psi^{-\nu}\right). \quad (1.237)$$

Define the domain $D_{\beta,\alpha}$ as

$$D_{\beta,\alpha} = \{(z,t) : z,t > 0, \psi(z,t) < 1+\beta\}. \tag{1.238}$$

The boundary of this domain is

$$\partial D_{\beta,\alpha} = \cup_{i=1}^{3} \partial_i D_{\beta,\alpha}, \tag{1.239}$$

$$\partial_1 D_{\beta,\alpha} = \{t = 0, z \in (0,\beta)\}, \tag{1.240}$$

$$\partial_2 D_{\beta,\alpha} = \left\{z = 0, 0 < t < \frac{\beta}{\alpha}\right\}, \tag{1.241}$$

$$\partial_3 D_{\beta,\alpha} = \{z,t > 0, \psi(z,t) = 1+\beta\}, \tag{1.242}$$

$$\varphi(z,t) \mid_{\partial_3 D_{\beta,\alpha}} = \exp\left[\lambda(1+\beta)^{-\nu}\right] = \min_{\overline{D}_{\beta,\alpha}} \varphi(z,t). \tag{1.243}$$

It follows from (1.234) that when applying the Carleman estimate of Lemma 1.11.3 in the proof of Theorem 1.11.1.2, we will have Dirichlet and Neumann data at $\partial_2 D_{\beta,\alpha}$. At $\partial_3 D_{\beta,\alpha}$ the function $\varphi(z,t)$ attains its minimal value, which is one of the key points of any Carleman estimate. However, we will not have any data at $\partial_1 D_{\beta,\alpha}$ when applying Lemma 1.11.3. Note that $\partial_1 D_{\beta,\alpha}$ is not a level curve of the function $\varphi(z,t)$. Still, we prove that the integral over $\partial_1 D_{\beta,\alpha}$, which occurs in the Carleman estimate due to the Gauss' formula, contains only nonnegative terms with the large parameters λ, λ^3; see the second line of (1.244). The latter is the main new feature of Lemma 1.11.3.

Lemma 1.11.3 (Carleman estimate). *Let $\alpha \in (0, 1/2)$ and $\beta, \nu > 0$. Then, there exist constants $\lambda_0 = \lambda_0(\alpha,\beta,\nu) > 1, C = C(\alpha,\beta,\nu) > 0$ such that the following Carleman estimate is valid:*

$$\begin{aligned}\int\limits_{D_{\beta,\alpha}} (u_{zz} - 2u_{zt})^2 \varphi^2 \mathrm{d}z\mathrm{d}t &\geq C\lambda \int\limits_{D_{\beta,\alpha}} \left(u_z^2 + u_t^2 + \lambda^2 u^2\right) \varphi^2 \mathrm{d}z\mathrm{d}t \\ &+ C\lambda \int\limits_{\partial_1 D_{\beta,\alpha}} \left(u_z^2 + \lambda^2 u^2\right)(z,0)\, \varphi^2(z,0)\, \mathrm{d}z \\ &- C\lambda^3 \exp\left[2\lambda(\beta+1)^{-\nu}\right] \int\limits_{\partial_3 D_{\beta,\alpha}} \left(u_z^2 + u_t^2 + u^2\right) \mathrm{d}S,\end{aligned} \tag{1.244}$$

$$\forall u \in \left\{u : u \in C^2\left(\overline{D}_{\beta,\alpha}\right), u\mid_{\partial_2 D_{\beta,\alpha}} = \partial_z u \mid_{\partial_2 D_{\beta,\alpha}} = 0\right\},\ \forall \lambda \geq \lambda_0. \tag{1.245}$$

Proof. In this proof, $C = C(\alpha,\beta,\nu) > 0$ denotes different positive constants. Consider a new function $v = u\varphi$ and express $u_{zz} - 2u_{zt}$ via v. By (1.237),

$$
\begin{aligned}
u &= v\exp\left(-\lambda\psi^{-\nu}\right),\\
u_z &= \left(v_z + \lambda\nu\psi^{-\nu-1}v\right)\exp\left(-\lambda\psi^{-\nu}\right),\\
u_{zz} &= \left[v_{zz} + 2\lambda\nu\psi^{-\nu-1}v_z + \lambda^2\nu^2\psi^{-2\nu-2}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v\right]\exp\left(-\lambda\psi^{-\nu}\right),\\
u_{zt} &= \left[v_{zt} + \alpha\lambda\nu\psi^{-\nu-1}v_z + \lambda\nu\psi^{-\nu-1}v_t + \alpha\lambda^2\nu^2\psi^{-2\nu-2}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v\right]\\
&\quad\times\exp\left(-\lambda\psi^{-\nu}\right),\\
u_{zz}-2u_{zt} &= \left[v_{zz}-2v_{zt} + (1-2\alpha)\lambda^2\nu^2\psi^{-2\nu-2}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v\right]\\
&\quad\times\exp\left(-\lambda\psi^{-\nu}\right) + \left[2(1-\alpha)\lambda\nu\psi^{-\nu-1}v_z - 2\lambda\nu\psi^{-\nu-1}v_t\right]\exp\left(-\lambda\psi^{-\nu}\right).
\end{aligned}
$$

Denote

$$
y_1 = \left[v_{zz}-2v_{zt} + (1-2\alpha)\lambda^2\nu^2\psi^{-2\nu-2}\left(1-\tfrac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v\right],
$$

$$
y_2 = 2(1-\alpha)\lambda\nu\psi^{-\nu-1}v_z,
$$

$$
y_3 = 2\lambda\nu\psi^{-\nu-1}v_t.
$$

Hence,

$$
\left(u_{zz}-2u_{zt}\right)^2\varphi^2 \geq 2y_1y_2 - 2y_1y_3. \tag{1.246}
$$

We have

$$
\begin{aligned}
2y_1y_2 &= 4(1-\alpha)\lambda\nu\psi^{-\nu-1}v_z\left[v_{zz}-2v_{zt} + (1-2\alpha)\lambda^2\nu^2\psi^{-2\nu-2}\right.\\
&\qquad\left.\times\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v\right]\\
&= \partial_z\left(2(1-\alpha)\lambda\nu\psi^{-\nu-1}v_z^2\right) + 2(1-\alpha)\lambda\nu(\nu+1)\psi^{-\nu-2}v_z^2\\
&\quad+\partial_t\left(-4(1-\alpha)\lambda\nu\psi^{-\nu-1}v_z^2\right) - 4\alpha(1-\alpha)\lambda\nu(\nu+1)\psi^{-\nu-2}v_z^2\\
&\quad+\partial_z\left[2(1-\alpha)(1-2\alpha)\lambda^3\nu^3\psi^{-3\nu-3}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v^2\right]\\
&\quad+6(1-\alpha)(1-2\alpha)\lambda^3\nu^3(\nu+1)\psi^{-3\nu-4}\left(1-\frac{(2\nu+3)}{3\lambda\nu}\psi^{\nu}\right)v^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
2y_1y_2 &= 2(1-\alpha)(1-2\alpha)\lambda\nu(\nu+1)\psi^{-\nu-2}v_z^2\\
&\quad+6(1-\alpha)(1-2\alpha)\lambda^3\nu^3(\nu+1)\psi^{-3\nu-4}\left(1-\frac{(2\nu+3)}{3\lambda\nu}\psi^{\nu}\right)v^2
\end{aligned}
$$

$$
\begin{aligned}
&+\partial_t\left(-4(1-\alpha)\lambda\nu\psi^{-\nu-1}v_z^2\right) \\
&+\partial_z\Bigg[2(1-\alpha)\lambda\nu\psi^{-\nu-1}v_z^2+2(1-\alpha)(1-2\alpha)\lambda^3\nu^3\psi^{-3\nu-3} \\
&\qquad\times\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v^2\Bigg].
\end{aligned}
\tag{1.247}
$$

Next, we estimate $-2y_1y_3$:

$$
\begin{aligned}
-2y_1y_3 &= -4\lambda\nu\psi^{-\nu-1}v_t\left[v_{zz}-2v_{zt}+(1-2\alpha)\lambda^2\nu^2\psi^{-2\nu-2}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v\right] \\
&= \partial_z\left(-4\lambda\nu\psi^{-\nu-1}v_tv_z\right)+4\lambda\nu\psi^{-\nu-1}v_{zt}v_z-4\lambda\nu(\nu+1)\psi^{-\nu-2}v_tv_z \\
&\quad+\partial_z\left(4\lambda\nu\psi^{-\nu-1}v_t^2\right)+4\lambda\nu(\nu+1)\psi^{-\nu-2}v_t^2 \\
&\quad+\partial_t\left[-2(1-2\alpha)\lambda^3\nu^3\psi^{-3\nu-3}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v^2\right] \\
&\quad-6\alpha(1-2\alpha)\lambda^3\nu^3(\nu+1)\psi^{-3\nu-4}\left(1-\frac{(2\nu+3)}{3\lambda\nu}\psi^{\nu}\right)v^2.
\end{aligned}
$$

Next,

$$
4\lambda\nu\psi^{-\nu-1}v_{zt}v_z=\partial_t\left(2\lambda\nu\psi^{-\nu-1}v_z^2\right)+2\alpha\lambda\nu(\nu+1)\psi^{-\nu-2}v_z^2.
$$

Hence,

$$
\begin{aligned}
-2y_3y_1 &= 2\lambda\nu(\nu+1)\psi^{-\nu-2}\left(\alpha v_z^2-2v_tv_z+2v_t^2\right) \\
&\quad-6\alpha(1-2\alpha)\lambda^3\nu^3(\nu+1)\psi^{-3\nu-4}\left(1-\frac{(2\nu+3)}{3\lambda\nu}\psi^{\nu}\right)v^2 \\
&\quad+\partial_t\left[2\lambda\nu\psi^{-\nu-1}v_z^2-2(1-2\alpha)\lambda^3\nu^3\psi^{-3\nu-3}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v^2\right] \\
&\quad+\partial_z\left[-4\lambda\nu\psi^{-\nu-1}v_tv_z+4\lambda\nu\psi^{-\nu-1}v_t^2\right].
\end{aligned}
\tag{1.248}
$$

Summing up (1.247) and (1.248) and taking into account (1.246), we obtain

$$
\begin{aligned}
&(u_{zz}-2u_{zt})^2\varphi^2 \geq 2y_2y_1-2y_3y_1 \\
&\quad= 2\lambda\nu(\nu+1)\psi^{-\nu-2}\left[\left(1-2\alpha+3\alpha^2\right)v_z^2-2v_tv_z+2v_t^2\right] \\
&\quad\quad+6(1-2\alpha)^2\lambda^3\nu^3(\nu+1)\psi^{-3\nu-4}\left(1-\frac{(2\nu+3)}{3\lambda\nu}\psi^{\nu}\right)v^2 \\
&\quad\quad+\partial_t\Bigg[-2(1-2\alpha)\lambda\nu\psi^{-\nu-1}v_z^2-2(1-2\alpha)\lambda^3\nu^3\psi^{-3\nu-3}
\end{aligned}
$$

$$\times \left(1 - \frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right) v^2 \Bigg]$$
$$+\partial_z \left[2(1-\alpha)\lambda\nu\psi^{-\nu-1}v_z^2 + 2(1-\alpha)(1-2\alpha)\lambda^3\nu^3\psi^{-3\nu-3} \right.$$
$$\left. \times \left(1 - \frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right) v^2 \right]$$
$$+\partial_z \left[-4\lambda\nu\psi^{-\nu-1}v_t v_z + 4\lambda\nu\psi^{-\nu-1}v_t^2\right]. \tag{1.249}$$

For any $\alpha \in (0, 1/2)$, there exists a constant $C_1 = C_1(\alpha) > 0$ such that

$$\left(1 - 2\alpha + 3\alpha^2\right) a^2 - 2ab + 2b^2 \geq C_1 \left(a^2 + b^2\right), \ \forall a, b \in \mathbb{R}.$$

Hence, integrating (1.249) over D_β and using (1.239)–(1.243), and (1.245) as well as the Gauss's formula, we obtain

$$\int_{D_{\beta,\alpha}} (u_{zz} - 2u_{zt})^2 \varphi^2 \mathrm{d}z\mathrm{d}t \geq 2\lambda\nu(\nu+1) C_1 \int_{D_{\beta,\alpha}} \left(v_z^2 + v_t^2\right)\psi^{-\nu-2}\mathrm{d}z\mathrm{d}t$$
$$+6(1-2\alpha)^2\lambda^3\nu^3(\nu+1)\int_{D_{\beta,\alpha}} \psi^{-3\nu-4}\left(1 - \frac{(2\nu+3)}{3\lambda\nu}\psi^{\nu}\right) v^2 \mathrm{d}z\mathrm{d}t$$
$$+\int_{\partial_1 D_{\beta,\alpha}} \left[2(1-2\alpha)\lambda\nu\psi^{-\nu-1}v_z^2 + 2(1-2\alpha)\lambda^3\nu^3\psi^{-3\nu-3}\right.$$
$$\left.\times\left(1 - \frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right) v^2\right]\mathrm{d}z$$
$$+\int_{\partial_3 D_{\beta,\alpha}} \left[-2(1-2\alpha)\lambda\nu\psi^{-\nu-1}v_z^2 - 2(1-2\alpha)\lambda^3\nu^3\psi^{-3\nu-3}\right.$$
$$\left.\times\left(1 - \frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right) v^2\right]\cos(n,t)\,\mathrm{d}S$$
$$+\int_{\partial_3 D_{\beta,\alpha}} \left[2(1-\alpha)\lambda\nu\psi^{-\nu-1}v_z^2 + 2(1-\alpha)(1-2\alpha)\lambda^3\nu^3\psi^{-3\nu-3}\right.$$
$$\left.\times\left(1 - \frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right) v^2\right]\cos(n,z)\,\mathrm{d}S$$
$$+\int_{\partial_3 D_{\beta,\alpha}} \left[-4\lambda\nu\psi^{-\nu-1}v_t v_z + 4\lambda\nu\psi^{-\nu-1}v_t^2\right]\cos(n,z)\,\mathrm{d}S. \tag{1.250}$$

Here, $\cos(n,t)$ is the cosine of the angle between the unit outward normal vector n at $\partial_3 D_\beta$ and the positive direction of t axis and similarly $\cos(n,z)$. Since the number $\nu > 0$ is fixed, we can incorporate it in the constant C. Change variables back in (1.250) replacing v with $u = v\varphi$. Then we obtain (1.244) for sufficiently large $\lambda \geq \lambda_0(\nu,\beta)$. □

1.11.4 Proof of Theorem 1.11.1.2

We consider in this proof only the case $z \in \{z > 0\}$ since the case $z \in \{z < 0\}$ is similar. Assume that there exist two pairs of vector functions:

$$\left(u^{1,h}(z,t), a^{1,h}(z)\right) \text{ and } \left(u^{2,h}(z,t), a^{2,h}(z)\right)$$

satisfying conditions of this theorem. Then for $z,t > 0$ there exist two pairs of functions:

$$\left(w^{1,h}(z,t), a^{1,h}(z)\right) \text{ and } \left(w^{2,h}(z,t), a^{2,h}(z)\right),$$

where

$$w^{1,h}(z,t) = u^{1,h}(z,t+z) \text{ and } w^{2,h}(z,t) = u^{2,h}(z,t+z).$$

Denote

$$\widetilde{w}^h(z,t) = w^{1,h}(z,t) - w^{2,h}(z,t) = \left\{\widetilde{w}_{i,j}(z,t)\right\}_{(i,j)=(0,0)}^{(N_1,N_2)},$$

$$\widetilde{a}^h(z) = a^{1,h}(z) - a^{2,h}(z) = \left\{\widetilde{a}_{i,j}(z)\right\}_{(i,j)=(0,0)}^{(N_1,N_2)}.$$

Then, (1.232)–(1.236) imply that

$$\widetilde{w}^h_{zz} - 2\widetilde{w}^h_{zt} = -\Delta_{h,x,y}\widetilde{w}^h + a^{1,h}(z)\widetilde{w}^h + \widetilde{a}^h(z)\, w^{2,h}(z,t), \quad (z,t) \in \{t > 0, z > 0\}, \tag{1.251}$$

$$\widetilde{w}^h(z,0) = 0, \tag{1.252}$$

$$\widetilde{w}^h(0,t) = 0, \ \partial_z\widetilde{w}^h(0,t) = 0, \ t \in (0,T),$$
$$+\widetilde{w}^h \in C^3(\mathbb{R} \times [0,T]). \tag{1.253}$$

Hence, setting $t = 0$ in (1.251) and using (1.252), we obtain

$$\widetilde{a}^h(z) = -4\partial_z\partial_t\widetilde{w}^h(z,0). \tag{1.254}$$

Let

$$\widetilde{v}^h(z,t) = \partial_t\widetilde{w}^h(z,t), \ v^{2,h}(z,t) = \partial_t w^{2,h}(z,t). \tag{1.255}$$

Differentiating (1.251) with respect to t and using (1.253) and (1.254), we obtain for $(z,t) \in \{t > 0, z > 0\}$

$$\widetilde{v}^h_{zz} - 2\widetilde{v}^h_{zt} = -\Delta_{h,x,y}\widetilde{v}^h + a^{1,h}(z)\widetilde{v}^h - 4\partial_z\widetilde{v}^h(z,0)\,v^{2,h}(z,t)\,, \tag{1.256}$$

$$\widetilde{v}^h(0,t) = 0,\ \partial_z\widetilde{v}^h(0,t) = 0,\ t \in (0,T)\,. \tag{1.257}$$

Since $T > 2R$, then $(0, R/T) \subset (0, 1/2)$. In (1.237), choose an arbitrary $\alpha \in (R/T, 1/2)$ and an arbitrary $\nu > 0$. Next, set in (1.238) $\beta := R$. Consider (1.256) for the function $\widetilde{v}_{i,j}$ for an arbitrary pair $(i,j) \in [0, N_1] \times [0, N_2]$. Square both sides of the latter equation, multiply by the function $\varphi^2(z,t)$, and integrate over $D_{R,\alpha}$. We obtain with a constant

$$M = M\left(h_0, \left\|a^{1,h}\right\|_{C[0,R]}, \left\|v^{2,h}\right\|_{C\left(\overline{D}_R\right)}\right) > 0,$$

depending on listed parameters

$$\int\limits_{D_{R,\alpha}} \left(\partial_z^2\widetilde{v}_{i,j} - 2\partial_z\partial_t\widetilde{v}_{i,j}\right)^2 \varphi^2 \mathrm{d}z\mathrm{d}t$$
$$\leq M \int\limits_{D_{R,\alpha}} \left[\widetilde{v}^h(z,t)\right]^2 \varphi^2 dzdt + M \int\limits_{D_{R,\alpha}} \left[\widetilde{v}_{i,j}(z,0)\right]^2 \varphi^2 \mathrm{d}z\mathrm{d}t. \tag{1.258}$$

Since the function $\varphi^2(z,t)$ is decreasing with respect to t, we obtain from (1.241) and (1.258)

$$\int\limits_{D_{R,\alpha}} \left(\partial_z^2\widetilde{v}_{i,j} - 2\partial_z\partial_t\widetilde{v}_{i,j}\right)^2 \varphi^2 \mathrm{d}z\mathrm{d}t$$
$$\leq M \int\limits_{D_{R,\alpha}} \left[\widetilde{v}^h(z,t)\right]^2 \varphi^2 \mathrm{d}z\mathrm{d}t + M_1 \int\limits_{\partial_1 D_{R,\alpha}} \left[\widetilde{v}_{i,j}(z,0)\right]^2 \varphi^2(z,0)\,\mathrm{d}z, \tag{1.259}$$

where the constant $M_1 = M_1(M, R, \alpha) > 0$.

Applying Lemma 1.11.3 to the left-hand side of (1.259) and using (1.257), we obtain

$$C\lambda \int\limits_{D_{R,\alpha}} \left[\left(\partial_z\widetilde{v}_{i,j}\right)^2 + \left(\partial_t\widetilde{v}_{i,j}\right)^2 + \lambda^2\left(\widetilde{v}_{i,j}\right)^2\right]\varphi^2 \mathrm{d}z\mathrm{d}t$$
$$+C\lambda \int\limits_{\partial_1 D_{R,\alpha}} \left[\left(\partial_z\widetilde{v}_{i,j}\right)^2 + \lambda^2\left(\widetilde{v}_{i,j}\right)^2\right](z,0)\,\varphi^2(z,0)\,\mathrm{d}z$$

$$-C\lambda^3 \exp\left(2\lambda\left(R+1\right)^{-\nu}\right) \int\limits_{\partial_3 D_{R,\alpha}} \left[\left(\partial_z \widetilde{v}_{i,j}\right)^2 + \left(\partial_t \widetilde{v}_{i,j}\right)^2 + \left(\widetilde{v}_{i,j}\right)^2\right] \mathrm{d}S$$

$$\leq M \int\limits_{D_{R,\alpha}} \left[\widetilde{v}^h\left(z,t\right)\right]^2 \varphi^2 \mathrm{d}z\mathrm{d}t + M_1 \int\limits_{\partial_1 D_{R,\alpha}} \left[\widetilde{v}_{i,j}\left(z,0\right)\right]^2 \varphi^2\left(z,0\right) \mathrm{d}z. \quad (1.260)$$

Choose a sufficiently large number $\lambda_0 > 1$ such that

$$\max\left(M, M_1\right) < \frac{C\lambda_0^3}{2}. \quad (1.261)$$

Then with a new constant $C > 0$, we obtain from the estimate (1.260)

$$C\lambda \int\limits_{D_{R,\alpha}} \left[\left(\partial_z \widetilde{v}_{i,j}\right)^2 + \left(\partial_t \widetilde{v}_{i,j}\right)^2 + \lambda^2\left(\widetilde{v}_{i,j}\right)^2\right] \varphi^2 \mathrm{d}z\mathrm{d}t$$

$$+C\lambda^3 \int\limits_{\partial_1 D_{R,\alpha}} \left(\widetilde{v}_{i,j}\right)^2\left(z,0\right) \varphi^2\left(z,0\right) \mathrm{d}z$$

$$-C\lambda^3 \exp\left[2\lambda\left(R+1\right)^{-\nu}\right] \int\limits_{\partial_3 D_{R,\alpha}} \left[\left(\partial_z \widetilde{v}_{i,j}\right)^2 + \left(\partial_t \widetilde{v}_{i,j}\right)^2 + \left(\widetilde{v}_{i,j}\right)^2\right] \mathrm{d}S$$

$$\leq M \int\limits_{D_{R,\alpha}} \left[\widetilde{v}^h\left(z,t\right)\right]^2 \varphi^2 \mathrm{d}z\mathrm{d}t. \quad (1.262)$$

Summing up estimates (1.262) with respect to $(i, j) \in [0, N_1] \times [0, N_2]$ and using (1.261), we obtain a stronger estimate:

$$\int\limits_{D_{R,\alpha}} \left(\widetilde{v}^h\right)^2 \varphi^2 \mathrm{d}z\mathrm{d}t \leq C \exp\left[2\lambda\left(R+1\right)^{-\nu}\right] \int\limits_{\partial_3 D_{R,\alpha}} \left[\left(\widetilde{v}_z^h\right)^2 + \left(\widetilde{v}_t^h\right)^2 + \left(\widetilde{v}^h\right)^2\right] \mathrm{d}S. \quad (1.263)$$

Let $\varepsilon \in (0, R)$ be an arbitrary number. By (1.237) and (1.238),

$$\varphi^2\left(z,t\right) > \exp\left[2\lambda\left(R+1-\varepsilon\right)^{-\nu}\right] \text{ in } D_{R-\varepsilon},\ D_{R-\varepsilon} \subset D_R.$$

Hence, making the estimate (1.263) stronger, we obtain

$$\exp\left[2\lambda\left(R+1-\varepsilon\right)^{-\nu}\right] \int\limits_{D_{R-\varepsilon,\alpha}} \left(\widetilde{v}^h\right)^2 \mathrm{d}z\mathrm{d}t \leq C \exp\left[2\lambda\left(R+1\right)^{-\nu}\right]$$

$$\times \int\limits_{\partial_3 D_{R,\alpha}} \left[\left(\widetilde{v}_z^h\right)^2 + \left(\widetilde{v}_t^h\right)^2 + \left(\widetilde{v}^h\right)^2\right] \mathrm{d}S$$

or

$$\int\limits_{D_{R-\varepsilon,\alpha}} \left(\widetilde{v}^h\right)^2 \mathrm{d}z\mathrm{d}t \leq C \exp\left\{-2\lambda\left[(R+1-\varepsilon)^{-\nu} - (R+1)^{-\nu}\right]\right\}$$
$$\times \int\limits_{\partial_3 D_{R,\alpha}} \left[\left(\widetilde{v}_z^h\right)^2 + \left(\widetilde{v}_t^h\right)^2 + \left(\widetilde{v}^h\right)^2\right] \mathrm{d}S.$$

Setting here $\lambda \to \infty$, we obtain

$$\int\limits_{D_{R-\varepsilon,\alpha}} \left(\widetilde{v}^h\right)^2 \mathrm{d}z\mathrm{d}t = 0. \tag{1.264}$$

Since $\varepsilon \in (0, R)$ is an arbitrary number, then (1.264) implies that

$$\widetilde{v}^h(z,t) = 0 \text{ in } D_{R,\alpha}.$$

Since by (1.254),

$$\widetilde{a}^h(z) = -4\partial_z\partial_t\widetilde{w}^h(z,0) = -4\partial_z\widetilde{v}^h(z,0),$$

then $\widetilde{a}^h(z) = 0$ for $z \in (0, R)$. Thus, the function $a^h(z)$ is uniquely determined for $z \in \{|z| < R\}$.

Equations (1.220) represent a coupled system of 1D wave-like equations. Conditions (1.221) and (1.223) are Cauchy data for this system at $\{t = 0\}$ and $\{z = 0, t \in (0, T)\}$, respectively. Because of the 1D case, the time variable can be treated as the spatial variable and vice versa. Hence, treating for a moment z as the time variable and t as the spatial variable and recalling that the vector function $a^h(z)$ is known for $z \in \{|z| < R\}$, one can apply the standard energy estimate to (1.220), (1.221), and (1.223) for $\{z \in (0, R)\}$. It follows from this estimate that the vector function $u^h(z,t)$ is uniquely determined in the domain

$$(z,t) \in \{z \in (0, R), t \in (0, T - z)\}.$$

Similarly, the function $u^h(z,t)$ is uniquely determined in the domain

$$(z,t) \in \{z \in (-R, 0), t \in (0, T + z)\}.$$

Thus, (1.226) is established. □