16

The Capital Asset Pricing Model

16.1 Introduction to the CAPM

The *CAPM (capital asset pricing model)* has a variety of uses. It provides a theoretical justification for the widespread practice of passive investing by holding *index funds*.[1] The CAPM can provide estimates of expected rates of return on individual investments and can establish "fair" rates of return on invested capital in regulated firms or in firms working on a cost-plus basis.[2]

The CAPM starts with the question, what would be the risk premiums on securities if the following assumptions were true?

1. The market prices are "in equilibrium." In particular, for each asset, supply equals demand.
2. Everyone has the same forecasts of expected returns and risks.
3. All investors choose portfolios optimally according to the principles of efficient diversification discussed in Chapter 11. This implies that everyone holds a tangency portfolio of risky assets as well as the risk-free asset.
4. The market rewards people for assuming unavoidable risk, but there is no reward for needless risks due to inefficient portfolio selection. Therefore, the risk premium on a single security is not due to its "standalone" risk, but rather to its contribution to the risk of the tangency portfolio. The various components of risk are discussed in Section 16.4.

Assumption 3 implies that the market portfolio is equal to the tangency portfolio. Therefore, a broad index fund that mimics the market portfolio can be used as an approximation to the tangency portfolio.

The validity of the CAPM can only be guaranteed if all of these assumptions are true, and certainly no one believes that any of them are exactly true.

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[1] An index fund holds the same portfolio as some index. For example, an S&P 500 index fund holds all 500 stocks on the S&P 500 in the same proportions as in the index. Some funds do not replicate an index exactly, but are designed to track the index, for instance, by being cointegrated with the index.

[2] See Bodie and Merton (2000).

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Assumption 3 is at best an idealization. Moreover, some of the conclusions of the CAPM are contradicted by the behavior of financial markets; see Section 17.4.1 for an example. Despite its shortcomings, the CAPM is widely used in finance and it is essential for a student of finance to understand the CAPM. Many of its concepts such as the beta of an asset and systematic and diversifiable risks are of great importance, and the CAPM has been generalized to the widely used factor models in Chapter 17.

16.2 The Capital Market Line (CML)

The *capital market line* (CML) relates the excess expected return on an efficient portfolio to its risk. *Excess expected return* is the expected return minus the risk-free rate and is also called the risk premium. The CML is

$$\mu_R = \mu_f + \frac{\mu_M - \mu_f}{\sigma_M}\sigma_R, \tag{16.1}$$

where R is the return on a given efficient portfolio (mixture of the market portfolio [= tangency portfolio] and the risk-free asset), $\mu_R = E(R)$, μ_f is the risk-free rate, R_M is the return on the market portfolio, $\mu_M = E(R_M)$, σ_M is the standard deviation of R_M, and σ_R is the standard deviation of R. The risk premium of R is $\mu_R - \mu_f$ and the risk premium of the market portfolio is $\mu_M - \mu_f$.

In (16.1) μ_f, μ_M, and σ_M are constant. What varies are σ_R and μ_R. These vary as we change the efficient portfolio R. Think of the CML as showing how μ_R depends on σ_R.

The slope of the CML is, of course,

$$\frac{\mu_M - \mu_f}{\sigma_M},$$

which can be interpreted as the ratio of the risk premium to the standard deviation of the market portfolio. This is Sharpe's "reward-to-risk ratio." Equation (16.1) can be rewritten as

$$\frac{\mu_R - \mu_f}{\sigma_R} = \frac{\mu_M - \mu_f}{\sigma_M},$$

which says that the reward-to-risk ratio for any efficient portfolio equals that ratio for the market portfolio.

Example 16.1. The CML

Suppose that the risk-free rate of interest is $\mu_f = 0.06$, the expected return on the market portfolio is $\mu_M = 0.15$, and the risk of the market portfolio is $\sigma_M = 0.22$. Then the slope of the CML is $(0.15 - 0.06)/0.22 = 9/22$. The CML of this example is illustrated in Figure 16.1.

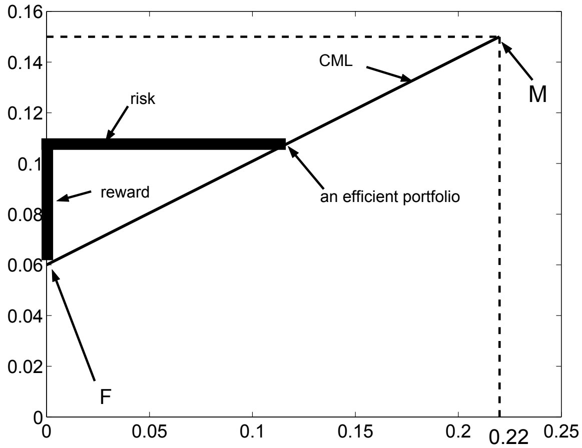

Fig. 16.1. *CML when $\mu_f = 0.06$, $\mu_M = 0.15$, and $\sigma_M = 0.22$. All efficient portfolios are on the line connecting the risk-free asset (F) and the market portfolio (M). Therefore, the reward-to-risk ratio is the same for all efficient portfolios, including the market portfolio. This fact is illustrated by the thick lines, whose lengths are the risk and reward for a typical efficient portfolio.*

□

The CML is easy to derive. Consider an efficient portfolio that allocates a proportion w of its assets to the market portfolio and $(1-w)$ to the risk-free asset. Then

$$R = wR_M + (1-w)\mu_f = \mu_f + w(R_M - \mu_f). \tag{16.2}$$

Therefore, taking expectations in (16.2),

$$\mu_R = \mu_f + w(\mu_M - \mu_f). \tag{16.3}$$

Also, from (16.2),

$$\sigma_R = w\sigma_M, \tag{16.4}$$

or

$$w = \frac{\sigma_R}{\sigma_M}. \tag{16.5}$$

Substituting (16.5) into (16.3) gives the CML.

The CAPM says that the optimal way to invest is to

1. decide on the risk σ_R that you can tolerate, $0 \le \sigma_R \le \sigma_M$[3];
2. calculate $w = \sigma_R/\sigma_M$;
3. invest w proportion of your investment in an index fund, that is, a fund that tracks the market as a whole;

[3] In fact, $\sigma_R > \sigma_M$ is possible by borrowing money to buy risky assets on margin.

4. invest $1 - w$ proportion of your investment in risk-free Treasury bills, or a money-market fund.

Alternatively,

1. choose the reward $\mu_R - \mu_f$ that you want; the only constraint is that $\mu_f \le \mu_R \le \mu_M$ so that $0 \le w \le 1$[4];
2. calculate
$$w = \frac{\mu_R - \mu_f}{\mu_M - \mu_f};$$
3. do steps 3 and 4 as above.

Instead of specifying the expected return or standard deviation of return, as in Example 11.1 one can find the portfolio with the highest expected return subject to a guarantee that with confidence $1 - \alpha$ the maximum loss is below a prescribed bound M determined, say, by a firm's capital reserves. If the firm invests an amount C, then for the loss to be greater than M the return must be less than $-M/C$. If we assume that the return is normally distributed, then by (A.11), (16.3), and (16.4),

$$P\left(R < -\frac{M}{C}\right) = \Phi\left(\frac{-M/C - \{\mu_f + w(\mu_M - \mu_f)\}}{w\sigma_M}\right). \tag{16.6}$$

Thus, we solve the following equation for w:

$$\Phi^{-1}(\alpha) = \frac{-M/C - \{\mu_f + w(\mu_M - \mu_f)\}}{w\sigma_M}.$$

One can view $w = \sigma_R/\sigma_M$ as an index of the risk aversion of the investor. The smaller the value of w the more risk-averse the investor. If an investor has w equal to 0, then that investor is 100% in risk-free assets. Similarly, an investor with $w = 1$ is totally invested in the tangency portfolio of risky assets.[5]

16.3 Betas and the Security Market Line

The *security market line* (SML) relates the excess return on an asset to the slope of its regression on the market portfolio. The SML differs from the CML in that the SML applies to all assets while the CML applies only to efficient portfolios.

Suppose that there are many securities indexed by j. Define

$\sigma_{jM} =$ covariance between the returns on the jth security and the market portfolio.

[4] This constraint can be relaxed if one is permitted to buy assets on margin.

[5] An investor with $w > 1$ is buying the market portfolio on margin, that is, borrowing money to buy the market portfolio.

Also, define

$$\beta_j = \frac{\sigma_{jM}}{\sigma_M^2}. \tag{16.7}$$

It follows from the theory of best linear prediction in Section 14.10.1 that β_j is the slope of the best linear predictor of the jth security's returns using returns of the market portfolio as the predictor variable. This fact follows from equation (14.41) for the slope of a best linear prediction equation. In fact, the best linear predictor of R_j based on R_M is

$$\widehat{R}_j = \beta_{0,j} + \beta_j R_M, \tag{16.8}$$

where β_j in (16.8) is the same as in (16.7).

Another way to appreciate the significance of β_j uses linear regression. As discussed in Section 14.10, linear regression is a method for estimating the coefficients of the best linear predictor based upon data. To apply linear regression, suppose that we have a bivariate time series $(R_{j,t}, R_{M,t})_{t=1}^n$ of returns on the jth asset and the market portfolio. Then, the estimated slope of the linear regression regression of $R_{j,t}$ on $R_{M,t}$ is

$$\hat{\beta}_j = \frac{\sum_{t=1}^n (R_{j,t} - \overline{R}_j)(R_{M,t} - \overline{R}_M)}{\sum_{t=1}^n (R_{M,t} - \overline{R}_M)^2}, \tag{16.9}$$

which, after multiplying the numerator and denominator by the same factor n^{-1}, becomes an estimate of σ_{jM} divided by an estimate of σ_M^2 and therefore by (16.7) an estimate of β_j.

Let μ_j be the expected return on the jth security. Then $\mu_j - \mu_f$ is the *risk premium* (or *reward for risk* or *excess expected return*) for that security. Using the CAPM, it can be shown that

$$\mu_j - \mu_f = \beta_j(\mu_M - \mu_f). \tag{16.10}$$

This equation, which is called the security market line (SML), is derived in Section 16.5.2. In (16.10) β_j is a variable in the linear equation, not the slope; more precisely, μ_j is a linear function of β_j with slope $\mu_M - \mu_f$. This point is worth remembering. Otherwise, there could be some confusion since β_j was defined earlier as a slope of a regression model. In other words, β_j is a slope in one context but is the independent variable in the SML. One can estimate β_j using (16.9) and then plug this estimate into (16.10).

The SML says that the risk premium of the jth asset is the product of its beta (β_j) and the risk premium of the market portfolio ($\mu_M - \mu_f$). Therefore, β_j measures both the riskiness of the jth asset and the reward for assuming that riskiness. Consequently, β_j is a measure of how "aggressive" the jth asset is. By definition, the beta for the market portfolio is 1; i.e., $\beta_M = 1$. This suggest the rules-of-thumb

$$\begin{aligned} \beta_j > 1 &\Rightarrow \text{"aggressive,"} \\ \beta_j = 1 &\Rightarrow \text{"average risk,"} \\ \beta_j < 1 &\Rightarrow \text{"not aggressive."} \end{aligned}$$

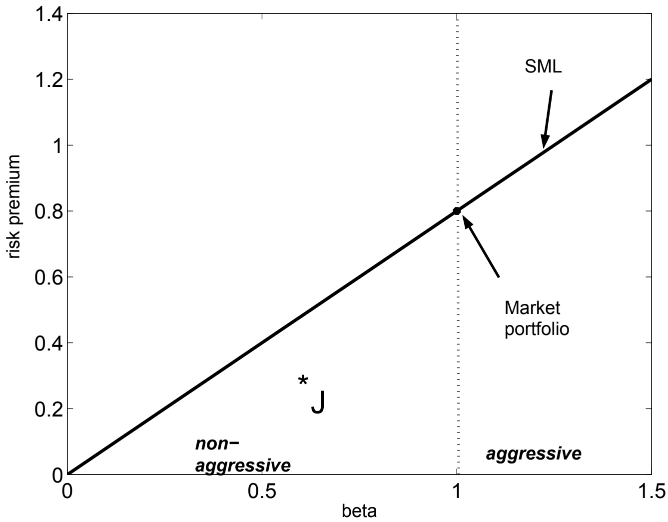

Fig. 16.2. *Security market line (SML) showing that the risk premium of an asset is a linear function of the asset's beta. J is a security not on the line and a contradiction to the CAPM. Theory predicts that the price of J decreases until J is on the SML. The vertical dotted line separates the nonaggressive and aggressive regions.*

Figure 16.2 illustrates the SML and an asset J that is not on the SML. This asset contradicts the CAPM, because according to the CAPM all assets are on the SML so no such asset exists.

Consider what would happen if an asset like J did exist. Investors would not want to buy it because, since it is below the SML, its risk premium is too low for the risk given by its beta. They would invest less in J and more in other securities. Therefore, the price of J would decline and *after* this decline its expected return would increase. After that increase, the asset J would be on the SML, or so the theory predicts.

16.3.1 Examples of Betas

Table 16.1 has some "five-year betas" taken from the Salomon, Smith, Barney website between February 27 and March 5, 2001. The beta for the S&P 500 is given as 1.00; why?

16.3.2 Comparison of the CML with the SML

The CML applies only to the return R of an efficient portfolio. It can be arranged so as to relate the excess expected return of that portfolio to the excess expected return of the market portfolio:

Table 16.1. Selected stocks and in which industries they are. Betas are given for each stock (Stock's β) and its industry (Ind's β). Betas taken from the Salomon, Smith, Barney website between February 27 and March 5, 2001.

Stock (symbol)	Industry	Stock's $\boldsymbol{\beta}$	Ind's $\boldsymbol{\beta}$
Celanese (CZ)	Synthetics	0.13	0.86
General Mills (GIS)	Food—major diversif	0.29	0.39
Kellogg (K)	Food—major, diversif	0.30	0.39
Proctor & Gamble (PG)	Cleaning prod	0.35	0.40
Exxon-Mobil (XOM)	Oil/gas	0.39	0.56
7-Eleven (SE)	Grocery stores	0.55	0.38
Merck (Mrk)	Major drug manuf	0.56	0.62
McDonalds (MCD)	Restaurants	0.71	0.63
McGraw-Hill (MHP)	Pub—books	0.87	0.77
Ford (F)	Auto	0.89	1.00
Aetna (AET)	Health care plans	1.11	0.98
General Motors (GM)	Major auto manuf	1.11	1.09
AT&T (T)	Long dist carrier	1.19	1.34
General Electric (GE)	Conglomerates	1.22	0.99
Genentech (DNA)	Biotech	1.43	0.69
Microsoft (MSFT)	Software applic.	1.77	1.72
Cree (Cree)	Semicond equip	2.16	2.30
Amazon (AMZN)	Net soft & serv	2.99	2.46
Doubleclick (Dclk)	Net soft & serv	4.06	2.46

$$\mu_R - \mu_f = \left(\frac{\sigma_R}{\sigma_M}\right)(\mu_M - \mu_f). \tag{16.11}$$

The SML applies to *any* asset and like the CML relates its excess expected return to the excess expected return of the market portfolio:

$$\mu_j - \mu_f = \beta_j(\mu_M - \mu_f). \tag{16.12}$$

If we take an efficient portfolio and consider it as an asset, then μ_R and μ_j both denote the expected return on that portfolio/asset. Both (16.11) and (16.12) hold so that

$$\frac{\sigma_R}{\sigma_M} = \beta_R.$$

16.4 The Security Characteristic Line

Let R_{jt} be the return at time t on the jth asset. Similarly, let $R_{M,t}$ and $\mu_{f,t}$ be the return on the market portfolio and the risk-free return at time t. The *security characteristic line* (sometimes shortened to the characteristic line) is a regression model:

$$R_{j,t} = \mu_{f,t} + \beta_j(R_{M,t} - \mu_{f,t}) + \epsilon_{j,t}, \tag{16.13}$$

where $\epsilon_{j,t}$ is $N(0, \sigma^2_{\epsilon,j})$. It is often assumed that the $\epsilon_{j,t}$s are uncorrelated across assets, that is, that $\epsilon_{j,t}$ is uncorrelated with $\epsilon_{j',t}$ for $j \neq j'$. This assumption has important ramifications for risk reduction by diversification; see Section 16.4.1.

Let $\mu_{j,t} = E(R_{j,t})$ and $\mu_{M,t} = E(R_{M,t})$. Taking expectations in (16.13) we get,

$$\mu_{j,t} = \mu_{f,t} + \beta_j(\mu_{M,t} - \mu_{f,t}),$$

which is equation (16.10), the SML, though in (16.10) it is not shown explicitly that the expected returns can depend on t. The SML gives us information about expected returns, but not about the variance of the returns. For the latter we need the characteristic line. The characteristic line is said to be a *return-generating process* since it gives us a probability model of the returns, not just a model of their expected values.

An analogy to the distinction between the SML and characteristic line is this. The regression line $E(Y|X) = \beta_0 + \beta_1 X$ gives the expected value of Y given X but not the conditional probability distribution of Y given X. The regression model

$$Y_t = \beta_0 + \beta_1 X_t + \epsilon_t \quad \text{and} \quad \epsilon_t \sim N(0, \sigma^2)$$

does give us this conditional probability distribution.

The characteristic line implies that

$$\sigma_j^2 = \beta_j^2 \sigma_M^2 + \sigma^2_{\epsilon,j},$$

that

$$\sigma_{jj'} = \beta_j \beta_{j'} \sigma_M^2$$

for $j \neq j'$, and that

$$\sigma_{Mj} = \beta_j \sigma_M^2.$$

The total risk of the jth asset is

$$\sigma_j = \sqrt{\beta_j^2 \sigma_M^2 + \sigma^2_{\epsilon,j}}.$$

The squared risk has two components: $\beta_j^2 \sigma_M^2$ is called the *market* or *systematic component of risk* and $\sigma^2_{\epsilon,j}$ is called the *unique, nonmarket,* or *unsystematic component of risk.*

16.4.1 Reducing Unique Risk by Diversification

The market component cannot be reduced by diversification, but the unique component can be reduced or even eliminated by sufficient diversification.

Suppose that there are N assets with returns $R_{1,t}, \ldots, R_{N,t}$ for holding period t. If we form a portfolio with weights $w_1, \ldots, w_N$, then the return of the portfolio is

$$R_{P,t} = w_1 R_{1,t} + \cdots + w_N R_{N,t}.$$

Let $R_{M,t}$ be the return on the market portfolio. According to the characteristic line model $R_{j,t} = \mu_{f,t} + \beta_j(R_{M,t} - \mu_{f,t}) + \epsilon_{j,t}$, so that

$$R_{P,t} = \mu_{f,t} + \left(\sum_{j=1}^{N} \beta_j w_j\right)(R_{M,t} - \mu_{f,t}) + \sum_{j=1}^{N} w_j \epsilon_{j,t}.$$

Therefore, the portfolio beta is

$$\beta_P = \sum_{j=1}^{N} w_j \beta_j,$$

and the "epsilon" for the portfolio is

$$\epsilon_{P,t} = \sum_{j=1}^{N} w_j \epsilon_{j,t}.$$

We now assume that $\epsilon_{1,t}, \ldots, \epsilon_{N,t}$ are uncorrelated. Therefore, by equation (7.11),

$$\sigma^2_{\epsilon,P} = \sum_{j=1}^{N} w_j^2 \sigma^2_{\epsilon,j}.$$

Example 16.2. Reduction in risk by diversification

Suppose the assets in the portfolio are equally weighted; that is, $w_j = 1/N$ for all j. Then

$$\beta_P = \frac{\sum_{j=1}^{N} \beta_j}{N},$$

and

$$\sigma^2_{\epsilon,P} = \frac{N^{-1}\sum_{j=1}^{N} \sigma^2_{\epsilon,j}}{N} = \frac{\overline{\sigma}^2_{\epsilon}}{N},$$

where $\overline{\sigma}^2_{\epsilon}$ is the average of the $\sigma^2_{\epsilon,j}$.

If $\sigma^2_{\epsilon,j}$ is a constant, say σ^2_{ϵ}, for all j, then

$$\sigma_{\epsilon,P} = \frac{\sigma_{\epsilon}}{\sqrt{N}}. \tag{16.14}$$

For example, suppose that σ_{ϵ} is 5%. If $N = 20$, then by (16.14) $\sigma_{\epsilon,P}$ is 1.12%. If $N = 100$, then $\sigma_{\epsilon,P}$ is 0.5%. There are approximately 1600 stocks on the NYSE; if $N = 1600$, then $\sigma_{\epsilon,P} = 0.125\%$. □

16.4.2 Are the Assumptions Sensible?

A key assumption that allows nonmarket risk to be removed by diversification is that $\epsilon_{1,t}, \ldots, \epsilon_{N,t}$ are uncorrelated. This assumption implies that *all* correlation among the cross-section[6] of asset returns is due to a single cause and that cause is measured by the market index. For this reason, the characteristic line is a "single-factor" or "single-index" model with $R_{M,t}$ being the "factor."

This assumption of uncorrelated ϵ_{jt} would not be valid if, for example, two energy stocks are correlated over and beyond their correlation due to the market index. In this case, unique risk could not be eliminated by holding a large portfolio of all energy stocks. However, if there are many market sectors and the sectors are uncorrelated, then one could eliminate nonmarket risk by diversifying across all sectors. All that is needed is to treat the sectors themselves as the underlying assets and then apply the CAPM theory.

Correlation among the stocks in a market sector can be modeled using a factor model; see Chapter 17.

16.5 Some More Portfolio Theory

In this section we use portfolio theory to show that $\sigma_{j,M}$ quantifies the contribution of the jth asset to the risk of the market portfolio. Also, we derive the SML.

16.5.1 Contributions to the Market Portfolio's Risk

Suppose that the market consists of N risky assets and that $w_{1,M}, \ldots, w_{N,M}$ are the weights of these assets in the market portfolio. Then

$$R_{M,t} = \sum_{i=1}^{N} w_{i,M} R_{i,t},$$

which implies that the covariance between the return on the jth asset and the return on the market portfolio is

$$\sigma_{j,M} = \mathrm{Cov}\left(R_{j,t}, \sum_{i=1}^{N} w_{i,M} R_{i,t} \right) = \sum_{i=1}^{N} w_{i,M} \sigma_{i,j}. \tag{16.15}$$

Therefore,

$$\sigma_M^2 = \sum_{j=1}^{N} \sum_{i=1}^{N} w_{j,M} w_{i,M} \sigma_{i,j} = \sum_{j=1}^{N} w_{j,M} \left(\sum_{i=1}^{N} w_{i,M} \sigma_{i,j} \right) = \sum_{j=1}^{N} w_{j,M} \sigma_{j,M}. \tag{16.16}$$

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[6] "Cross-section" of returns means returns across assets within a *single* holding period.

Equation (16.16) shows that the contribution of the jth asset to the risk of the market portfolio is $w_{j,M}\sigma_{j,M}$, where $w_{j,M}$ is the weight of the jth asset in the market portfolio and $\sigma_{j,M}$ is the covariance between the return on the jth asset and the return on the market portfolio.

16.5.2 Derivation of the SML

The derivation of the SML is a nice application of portfolio theory, calculus, and geometric reasoning. It is based on a clever idea of putting together a portfolio with two assets, the market portfolio and the ith risky asset, and then looking at the locus in reward-risk space as the portfolio weight assigned to the ith risky asset varies.

Consider a portfolio P with weight w_i given to the ith risky asset and weight $(1 - w_i)$ given to the market portfolio. The return on this portfolio is

$$R_{P,t} = w_i R_{i,t} + (1 - w_i) R_{M,t}.$$

The expected return is

$$\mu_P = w_i \mu_i + (1 - w_i)\mu_M, \tag{16.17}$$

and the risk is

$$\sigma_P = \sqrt{w_i^2 \sigma_i^2 + (1 - w_i)^2 \sigma_M^2 + 2 w_i (1 - w_i) \sigma_{i,M}}. \tag{16.18}$$

As we vary w_i, we get the locus of points on (σ, μ) space that is shown as a dashed curve in Figure 16.3.

It is easy to see geometrically that the derivative of this locus of points evaluated at the tangency portfolio (which is the point where $w_i = 0$) is equal to the slope of the CML. We can calculate this derivative and equate it to the slope of the CML to see what we get. The result is the SML.

We have from (16.17)

$$\frac{d\,\mu_P}{d\,w_i} = \mu_i - \mu_M,$$

and from (16.18) that

$$\frac{d\,\sigma_P}{d\,w_i} = \frac{1}{2}\sigma_P^{-1}\left\{2 w_i \sigma_i^2 - 2(1 - w_i)\sigma_M^2 + 2(1 - 2w_i)\sigma_{i,M}\right\}.$$

Therefore,

$$\frac{d\,\mu_P}{d\,\sigma_P} = \frac{d\,\mu_P / d\,w_i}{d\,\sigma_P / d\,w_i} = \frac{(\mu_i - \mu_M)\sigma_P}{w_i \sigma_i^2 - \sigma_M^2 + w_i \sigma_M^2 + \sigma_{i,M} - 2 w_i \sigma_{i,M}}.$$

Next,

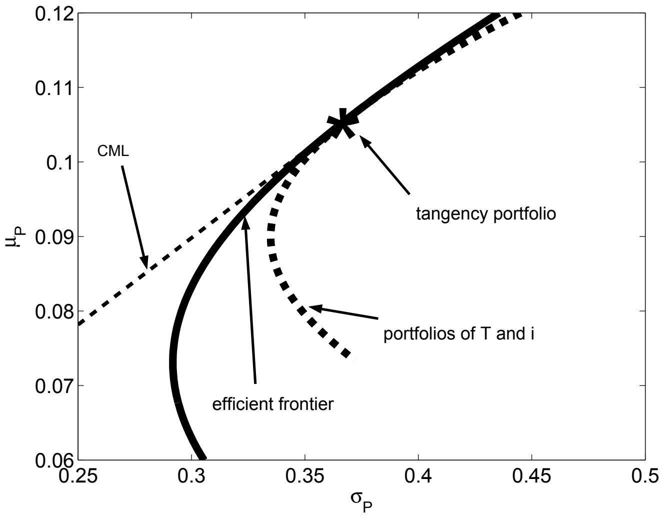

Fig. 16.3. *Derivation of the SML. The market portfolio and the tangency portfolio are equal according to the CAPM. The dashed curve is the locus of portfolios combining asset i and the market portfolio. The dashed curve is to the right of the efficient frontier and intersects the efficient frontier at the tangency portfolio. Therefore, the derivative of the dashed curve at the tangency portfolio is equal to the slope of the CML, since this curve is tangent to the CML at the tangency portfolio.*

$$\left.\frac{d\,\mu_P}{d\,\sigma_P}\right|_{w_i=0} = \frac{(\mu_i - \mu_M)\sigma_M}{\sigma_{i,M} - \sigma_M^2}.$$

Recall that $w_i = 0$ is the tangency portfolio, the point in Figure 16.3 where the dashed locus is tangent to the CML. Therefore,

$$\left.\frac{d\,\mu_P}{d\,\sigma_P}\right|_{w_i=0}$$

must equal the slope of the CML, which is $(\mu_M - \mu_f)/\sigma_M$. Therefore,

$$\frac{(\mu_i - \mu_M)\sigma_M}{\sigma_{i,M} - \sigma_M^2} = \frac{\mu_M - \mu_f}{\sigma_M},$$

which, after some algebra, gives us

$$\mu_i - \mu_f = \frac{\sigma_{i,M}}{\sigma_M^2}(\mu_M - \mu_f) = \beta_i(\mu_M - \mu_f),$$

which is the SML given in equation (16.10).

16.6 Estimation of Beta and Testing the CAPM

16.6.1 Estimation Using Regression

Recall the security characteristic line

$$R_{j,t} = \mu_{f,t} + \beta_j(R_{M,t} - \mu_{f,t}) + \epsilon_{j,t}. \tag{16.19}$$

Let $R^*_{j,t} = R_{j,t} - \mu_{f,t}$ be the excess return on the jth security and let $R^*_{M,t} = R_{M,t} - \mu_{f,t},$ be the excess return on the market portfolio. Then (16.19) can be written as

$$R^*_{j,t} = \beta_j R^*_{M,t} + \epsilon_{j,t}. \tag{16.20}$$

Equation (16.20) is a regression model without an intercept and with β_j as the slope. A more elaborate model is

$$R^*_{j,t} = \alpha_j + \beta_j R^*_{M,t} + \epsilon_{j,t}, \tag{16.21}$$

which includes an intercept. The CAPM says that $\alpha_j = 0$ but by allowing $\alpha_j \neq 0$, we recognize the possibility of mispricing.

Given time series $R_{j,t}$, $R_{M,t}$, and $\mu_{f,t}$ for $t = 1, \ldots, n$, we can calculate $R^*_{j,t}$ and $R^*_{M,t}$ and regress $R^*_{j,t}$ on $R^*_{M,t}$ to estimate α_j, β_j, and $\sigma^2_{\epsilon,j}$. By testing the null hypothesis that $\alpha_j = 0$, we are testing whether the jth asset is mispriced according to the CAPM.

As discussed in Section 12.2.2, when fitting model (16.20) or (16.21) one should use daily data if available, rather than weekly or monthly data. A more difficult question to answer is how long a time series to use. Longer time series give more data, of course, but models (16.20) and (16.21) assume that β_j is constant and this might not be true over a long time period.

Example 16.3. Estimation of α and β for Microsoft

As an example, daily closing prices on Microsoft and the S&P 500 index from November 1, 1993, to April 3, 2003, were used. The S&P 500 was taken as the market price. Three-month T-bill rates were used as the risk-free returns.[7] The excess returns are the returns minus the T-bill rates.

```
Call:
lm(formula = EX_R_msft ~ EX_R_sp500)

Residuals:
      Min         1Q    Median        3Q       Max
-0.152863 -0.011146 -0.000764  0.010887  0.151599

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.000914   0.000409    2.23    0.026 *
EX_R_sp500  1.247978   0.035425   35.23   <2e-16 ***
---
```

[7] Interest rates are return rates. Thus, we use the T-bill rates themselves as the risk-free returns. One does *not* take logs and difference the T-bill rates as if they were prices. However, the T-bill rates were divided by 100 to convert from a percentage and then by 253 to convert to a daily rate.

```
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.0199 on 2360 degrees of freedom
Multiple R-squared: 0.345,      Adjusted R-squared: 0.344
F-statistic: 1.24e+03 on 1 and 2360 DF,  p-value: <2e-16
```

For Microsoft, we find that $\hat{\beta} = 1.25$ and $\hat{\alpha} = 0.0009$. The estimate of α is very small and, although the p-value for α is 0.026, we can conclude that for practical purposes, α is essentially 0. The estimate of σ_ϵ is the root MSE which equals 0.0199.

Notice that the R^2 (R-sq) value for the regression is 34.5%. The interpretation of R^2 is the percent of the variance in the excess returns on Microsoft that is due to excess returns on the market. In other words, 34.5% of the risk is due to systematic or market risk ($\beta_j^2 \sigma_M^2$). The remaining 65.5% is due to unique or nonmarket risk (σ_ϵ^2).

If we assume that $\alpha = 0$, then we can refit the model using a no-intercept model.

```
Call:
lm(formula = EX_R_msft ~ EX_R_sp500 - 1)

Residuals:
      Min        1Q    Median        3Q       Max
-0.151945 -0.010231  0.000148  0.011803  0.152476

Coefficients:
           Estimate Std. Error t value Pr(>|t|)
EX_R_sp500   1.2491     0.0355    35.2   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.0199 on 2361 degrees of freedom
Multiple R-squared: 0.345,      Adjusted R-squared: 0.344
F-statistic: 1.24e+03 on 1 and 2361 DF,  p-value: <2e-16
```

With no intercept $\hat{\beta}$, $\hat{\sigma}_\epsilon$ and R^2 are nearly the same as before—forcing a nearly zero intercept to be exactly zero has little effect. □

16.6.2 Testing the CAPM

Testing that α equals 0 tests only one of the conclusions of the CAPM. Accepting this null hypothesis only means that the CAPM has passed one test, not that we should now accept it as true.[8] To fully test the CAPM, its other conclusions should also be tested. The factor models in Section 17.3 have been used to test the CAPM and fairly strong evidence against the CAPM

[8] In fact, acceptance of a null hypothesis should never be interpreted as proof that the null hypothesis is true.

has been found. Fortunately, these factor models do provide a generalization of the CAPM that is likely to be useful for financial decision making.

Often, as an alternative to regression using excess returns, the returns on the asset are regressed on the returns on the market. When this is done, an intercept model should be used. In the Microsoft data when using returns instead of excess returns, the estimate of beta changed hardly at all.

16.6.3 Interpretation of Alpha

If α is nonzero, then the security is mispriced, at least according to the CAPM. If $\alpha > 0$ then the security is underpriced; the returns are too large on average. This is an indication of an asset worth purchasing. Of course, one must be careful. If we reject the null hypothesis that $\alpha = 0$, all we have done is to show that the security was mispriced *in the past.* Since for the Microsoft data we accepted the null hypothesis that α is zero, there is no evidence that Microsoft was mispriced.

Warning: If we use returns rather than excess returns, then the intercept of the regression equation does *not* estimate α, so one cannot test whether α is zero by testing the intercept.

16.7 Using the CAPM in Portfolio Analysis

Suppose we have estimated beta and σ_ϵ^2 for each asset in a portfolio and also estimated σ_M^2 and μ_M for the market. Then, since μ_f is also known, we can compute the expectations, variances, and covariances of all asset returns by the formulas

$$\begin{aligned}
\mu_j &= \mu_f + \beta_j(\mu_M - \mu_f), \\
\sigma_j^2 &= \beta_j^2 \sigma_M^2 + \sigma_{\epsilon j}^2, \\
\sigma_{jj'} &= \beta_j \beta_{j'} \sigma_M^2 \text{ for } j \neq j'.
\end{aligned}$$

There is a serious danger here: These estimates depend heavily on the validity of the CAPM assumptions. Any or all of the quantities beta, σ_ϵ^2, σ_M^2, μ_M, and μ_f could depend on time t. However, it is generally assumed that the betas and σ_ϵ^2s of the assets as well as σ_M^2 and μ_M of the market are independent of t so that these parameters can be estimated assuming stationarity of the time series of returns.

16.8 Bibliographic Notes

The CAPM was developed by Sharpe (1964), Lintner (1965a,b), and Mossin (1966). Introductions to the CAPM can be found in Bodie, Kane, and Marcus

(1999), Bodie and Merton (2000), and Sharpe, Alexander, and Bailey (1999). I first learned about the CAPM from these three textbooks. Campbell, Lo, and MacKinlay (1997) discuss empirical testing of the CAPM. The derivation of the SML in Section 16.5.2 was adapted from Sharpe, Alexander, and Bailey (1999). Discussion of factor models can be found in Sharpe, Alexander, and Bailey (1999), Bodie, Kane, and Marcus (1999), and Campbell, Lo, and MacKinlay (1997).

16.9 References

Bodie, Z., and Merton, R. C. (2000) *Finance*, Prentice-Hall, Upper Saddle River, NJ.

Bodie, Z., Kane, A., and Marcus, A. (1999) *Investments*, 4th ed., Irwin/McGraw-Hill, Boston.

Campbell, J. Y., Lo, A. W., and MacKinlay, A. C. (1997) *The Econometrics of Financial Markets*, Princeton University Press, Princeton, NJ.

Lintner, J. (1965a) The valuation of risky assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics*, **47**, 13–37.

Lintner, J. (1965b) Security prices, risk, and maximal gains from diversification. *Journal of Finance*, **20**, 587–615.

Mossin, J. (1966) Equilibrium in capital markets. *Econometrica*, **34**, 768–783.

Sharpe, W. F. (1964) Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance*, **19**, 425–442.

Sharpe, W. F., Alexander, G. J., and Bailey, J. V. (1999) *Investments*, 6th ed., Prentice-Hall, Upper Saddle River, NJ.

16.10 `R` Lab

In this lab, you will fit model (16.19). The S&P 500 index will be a proxy for the market portfolio and the 90-day Treasury rate will serve as the risk-free rate.

This lab uses the data set `Stock_FX_Bond_2004_to_2006.csv`, which is available on the book's website. This data set contains a subset of the data in the data set `Stock_FX_Bond.csv` used elsewhere.

The `R` commands needed to fit model (16.19) will be given in small groups so that they can be explained better. First run the following commands to read the data, extract the prices, and find the number of observations:

```
dat = read.csv("Stock_FX_Bond_2004_to_2006.csv",header=T)
prices = dat[,c(5,7,9,11,13,15,17,24)]
n = dim(prices)[1]
```

Next, run these commands to convert the risk-free rate to a daily rate, compute net returns, extract the Treasury rate, and compute excess returns for the market and for seven stocks. The risk-free rate is given as a percentage so the returns are also computed as percentages.

```
dat2 =  as.matrix(cbind(dat[(2:n),3]/365,
   100*(prices[2:n,]/prices[1:(n-1),] - 1)))
names(dat2)[1] = "treasury"
risk_free = dat2[,1]
ExRet = dat2[,2:9] - risk_free
market = ExRet[,8]
stockExRet = ExRet[,1:7]
```

Now fit model (16.19) to each stock, compute the residuals, look at a scatterplot matrix of the residuals, and extract the estimated betas.

```
fit_reg = lm(stockExRet~market)
summary(fit_reg)
res = residuals(fit_reg)
pairs(res)
options(digits=3)
betas=fit_reg$coeff[2,]
```

Problem 1 *Would you reject the null hypothesis that alpha is zero for any of the seven stocks? Why or why not?*

Problem 2 *Use model (16.19) to estimate the expected excess return for all seven stocks. Compare these results to using the sample means of the excess returns to estimate these parameters. Assume for the remainder of this lab that all alphas are zero. (*Note: *Because of this assumption, one might consider reestimating the betas and the residuals with a no-intercept model. However, since the estimated alphas were close to zero, forcing the alphas to be exactly zero will not change the estimates of the betas or the residuals by much. Therefore, for simplicity, do not reestimate.)*

Problem 3 *Compute the correlation matrix of the residuals. Do any of the residual correlations seem large? Could you suggest a reason why the large correlations might be large? (Information about the companies in this data set is available at Yahoo Finance and other Internet sites.)*

Problem 4 *Use model (16.19) to estimate the covariance matrix of the excess returns for the seven companies.*

Problem 5 *What percentage of the excess return variance for UTX is due to the market?*

Problem 6 *An analyst predicts that the expected excess return on the market next year will be 4%. Assume that the betas estimated here using data from 2004–2006 are suitable as estimates of next year's betas. Estimate the expected excess returns for the seven stocks for next year.*

16.11 Exercises

1. What is the beta of a portfolio if $E(R_P) = 16\%$, $\mu_f = 5.5\%$, and $E(R_M) = 11\%$?
2. Suppose that the risk-free rate of interest is 0.03 and the expected rate of return on the market portfolio is 0.14. The standard deviation of the market portfolio is 0.12.
   (a) According to the CAPM, what is the efficient way to invest with an expected rate of return of 0.11?
   (b) What is the risk (standard deviation) of the portfolio in part (a)?
3. Suppose that the risk-free interest rate is 0.023, that the expected return on the market portfolio is $\mu_M = 0.10$, and that the volatility of the market portfolio is $\sigma_M = 0.12$.
   (a) What is the expected return on an efficient portfolio with $\sigma_R = 0.05$?
   (b) Stock A returns have a covariance of 0.004 with market returns. What is the beta of Stock A?
   (c) Stock B has beta equal to 1.5 and $\sigma_\epsilon = 0.08$. Stock C has beta equal to 1.8 and $\sigma_\epsilon = 0.10$.
      i. What is the expected return of a portfolio that is one-half Stock B and one-half Stock C?
      ii. What is the volatility of a portfolio that is one-half Stock B and one-half Stock C? Assume that the ϵs of Stocks B and C are independent.
4. Show that equation (16.15) follows from equation (7.8).
5. True or false: The CAPM implies that investors demand a higher return to hold more volatile securities. Explain your answer.
6. Suppose that the riskless rate of return is 4% and the expected market return is 12%. The standard deviation of the market return is 11%. Suppose as well that the covariance of the return on Stock A with the market return is $165\%^2$.[9]
   (a) What is the beta of Stock A?
   (b) What is the expected return on Stock A?

[9] If returns are expressed in units of percent, then the units of variances and covariances are percent-squared. A variance of $165\%^2$ equals 165/10,000.

(c) If the variance of the return on Stock A is $220\%^2$, what percentage of this variance is due to market risk?

7. Suppose there are three risky assets with the following betas and $\sigma^2_{\epsilon_j}$.

j	β_j	$\sigma^2_{\epsilon_j}$
1	0.9	0.010
2	1.1	0.015
3	0.6	0.011

Suppose also that the variance of $R_{Mt} - \mu_{ft}$ is 0.014.

(a) What is the beta of an equally weighted portfolio of these three assets?
(b) What is the variance of the excess return on the equally weighted portfolio?
(c) What proportion of the total risk of asset 1 is due to market risk?

8. Suppose there are two risky assets, call them C and D. The tangency portfolio is 60% C and 40% D. The expected yearly returns are 4% and 6% for assets C and D. The standard deviations of the yearly returns are 10% and 18% for C and D and the correlation between the returns on C and D is 0.5. The risk-free yearly rate is 1.2%.

(a) What is the expected yearly return on the tangency portfolio?
(b) What is the standard deviation of the yearly return on the tangency portfolio?
(c) If you want an efficient portfolio with a standard deviation of the yearly return equal to 3%, what proportion of your equity should be in the risk-free asset? If there is more than one solution, use the portfolio with the higher expected yearly return.
(d) If you want an efficient portfolio with an expected yearly return equal to 7%, what proportions of your equity should be in asset C, asset D, and the risk-free asset?

9. What is the beta of a portfolio if the expected return on the portfolio is $E(R_P) = 15\%$, the risk-free rate is $\mu_f = 6\%$, and the expected return on the market is $E(R_M) = 12\%$? Make the usual CAPM assumptions including that the portfolio alpha is zero.

10. Suppose that the risk-free rate of interest is 0.07 and the expected rate of return on the market portfolio is 0.14. The standard deviation of the market portfolio is 0.12.

(a) According to the CAPM, what is the efficient way to invest with an expected rate of return of 0.11?
(b) What is the risk (standard deviation) of the portfolio in part (a)?

11. Suppose there are three risky assets with the following betas and $\sigma^2_{\epsilon_j}$ when regressed on the market portfolio.

j	β_j	$\sigma^2_{\epsilon_j}$
1	0.7	0.010
2	0.8	0.025
3	0.6	0.012

Assume ϵ_1, ϵ_2, and ϵ_3 are uncorrelated. Suppose also that the variance of $R_M - \mu_f$ is 0.02.

(a) What is the beta of an equally weighted portfolio of these three assets?

(b) What is the variance of the excess return on the equally weighted portfolio?

(c) What proportion of the total risk of asset 1 is due to market risk?