11

Portfolio Theory

11.1 Trading Off Expected Return and Risk

How should we invest our wealth? Portfolio theory provides an answer to this question based upon two principles:

- we want to maximize the expected return; and
- we want to minimize the risk, which we define in this chapter to be the standard deviation of the return, though we may ultimately be concerned with the probabilities of large losses.

These goals are somewhat at odds because riskier assets generally have a higher expected return, since investors demand a reward for bearing risk. The difference between the expected return of a risky asset and the risk-free rate of return is called the *risk premium*. Without risk premiums, few investors would invest in risky assets.

Nonetheless, there are optimal compromises between expected return and risk. In this chapter we show how to maximize expected return subject to an upper bound on the risk, or to minimize the risk subject to a lower bound on the expected return. One key concept that we discuss is reduction of risk by diversifying the portfolio.

11.2 One Risky Asset and One Risk-Free Asset

We start with a simple example with one risky asset, which could be a portfolio, for example, a mutual fund. Assume that the expected return is 0.15 and the standard deviation of the return is 0.25. Assume that there is a *risk-free asset*, such as, a 90-day T-bill, and the risk-free rate is 6%, so the return on the risk-free asset is 6%, or 0.06. The standard deviation of the return on the risk-free asset is 0 by definition of "risk-free." The rates and returns here are annual, though all that is necessary is that they be in the same time units.

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We are faced with the problem of constructing an investment portfolio that we will hold for one time period, which is called the *holding period* and which could be a day, a month, a quarter, a year, 10 years, and so forth. At the end of the holding period we might want to readjust the portfolio, so for now we are only looking at returns over one time period. Suppose that a fraction w of our wealth is invested in the risky asset and the remaining fraction $1 - w$ is invested in the risk-free asset. Then the expected return is

$$E(R) = w(0.15) + (1 - w)(0.06) = 0.06 + 0.09w, \tag{11.1}$$

the variance of the return is

$$\sigma_R^2 = w^2\,(0.25)^2 + (1 - w)^2\,(0)^2 = w^2(0.25)^2,$$

and the standard deviation of the return is

$$\sigma_R = 0.25\,w. \tag{11.2}$$

To decide what proportion w of one's wealth to invest in the risky asset, one chooses either the expected return $E(R)$ one wants or the amount of risk σ_R with which one is willing to live. Once either $E(R)$ or σ_R is chosen, w can be determined.

Although σ is a measure of risk, a more direct measure of risk is actual monetary loss. In the next example, w is chosen to control the maximum size of the loss.

Example 11.1. Finding w to achieved a targeted value-at-risk

Suppose that a firm is planning to invest \$1,000,000 and has capital reserves that could cover a loss of \$150,000 but no more. Therefore, the firm would like to be certain that, if there is a loss, then it is no more than 15%, that is, that R is greater than -0.15. Suppose that R is normally distributed. Then the only way to guarantee that R is greater than -0.15 with probability equal to 1 is to invest entirely in the risk-free asset. The firm might instead be more modest and require only that $P(R < -0.15)$ be small, for example, 0.01. Therefore, the firm should find the value of w such that

$$P(R < -0.15) = \Phi\left(\frac{-0.15 - (0.06 + 0.09\,w)}{0.25\,w}\right) = 0.01.$$

The solution is

$$w = \frac{-0.21}{0.25\,\Phi^{-1}(0.01) + 0.9} = 0.4264.$$

In Chapter 19, \$150,000 is called the value-at-risk (= VaR) and $1 - 0.01 = 0.99$ is called the confidence coefficient. What was done in this example is to find the portfolio that has a VaR of \$150,000 with 0.99 confidence.

□

More generally, if the expected returns on the risky and risk-free assets are μ_1 and μ_f and if the standard deviation of the risky asset is σ_1, then the expected return on the portfolio is $w\mu_1 + (1-w)\mu_f$ while the standard deviation of the portfolio's return is $|w|\,\sigma_1$.

This model is simple but not as useless as it might seem at first. As discussed later, finding an optimal portfolio can be achieved in two steps:

1. finding the "optimal" portfolio of risky assets, called the "tangency portfolio," and
2. finding the appropriate mix of the risk-free asset and the tangency portfolio.

So we now know how to do the second step. What we still need to learn is how find the tangency portfolio.

11.2.1 Estimating $E(R)$ and σ_R

The value of the risk-free rate, μ_f, will be known since Treasury bill rates are published in sources providing financial information.

What should we use as the values of $E(R)$ and σ_R? If returns on the asset are assumed to be stationary, then we can take a time series of past returns and use the sample mean and standard deviation. Whether the stationarity assumption is realistic is always debatable. If we think that $E(R)$ and σ_R will be different from the past, we could subjectively adjust these estimates upward or downward according to our opinions, but we must live with the consequences if our opinions prove to be incorrect.

Another question is how long a time series to use, that is, how far back in time one should gather data. A long series, say 10 or 20 years, will give much less variable estimates. However, if the series is not stationary but rather has slowly drifting parameters, then a shorter series (maybe 1 or 2 years) will be more representative of the future. Almost every time series of returns is nearly stationary over short enough time periods.

11.3 Two Risky Assets

11.3.1 Risk Versus Expected Return

The mathematics of mixing risky assets is most easily understood when there are only two risky assets. This is where we start.

Suppose the two risky assets have returns R_1 and R_2 and that we mix them in proportions w and $1-w$, respectively. The return on the portfolio is $R_p = wR_1 + (1-w)R_2$. The expected return on the portfolio is $E(R_P) = w\mu_1 + (1-w)\mu_2$. Let ρ_{12} be the correlation between the returns on the two risky assets. The variance of the return on the portfolio is

$$\sigma_R^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\rho_{12}\,\sigma_1\sigma_2. \tag{11.3}$$

Note that $\sigma_{R_1,R_2} = \rho_{12}\sigma_1\sigma_2$.

Example 11.2. The expectation and variance of the return on a portfolio with two risky assets

If $\mu_1 = 0.14$, $\mu_2 = 0.08$, $\sigma_1 = 0.2$, $\sigma_2 = 0.15$, and $\rho_{12} = 0$, then

$$E(R_P) = 0.08 + 0.06w.$$

Also, because $\rho_{12} = 0$ in this example,

$$\sigma_{R_P}^2 = (0.2)^2\,w^2 + (0.15)^2\,(1-w)^2.$$

Using differential calculus, one can easily show that the portfolio with the minimum risk is $w = 0.045/0.125 = 0.36$. For this portfolio $E(R_P) = 0.08 + (0.06)(0.36) = 0.1016$ and $\sigma_{R_P} = \sqrt{(0.2)^2(0.36)^2 + (0.15)^2(0.64)^2} = 0.12$.

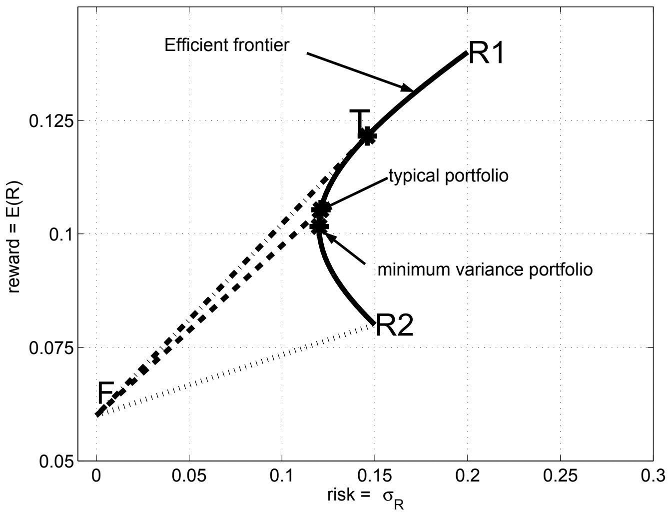

Fig. 11.1. *Expected return versus risk for Example 11.2. F = risk-free asset. T = tangency portfolio. R_1 is the first risky asset. R_2 is the second risky asset.*

The somewhat parabolic curve[1] in Figure 11.1 is the locus of values of $(\sigma_R, E(R))$ when $0 \le w \le 1$. The leftmost point on this locus achieves the minimum value of the risk and is called the *minimum variance portfolio*. The

[1] In fact, the curve would be parabolic if σ_R^2 were plotted on the x-axis instead of σ_R.

points on this locus that have an expected return at least as large as the minimum variance portfolio are called the *efficient frontier*. Portfolios on the efficient frontier are called *efficient portfolios* or, more precisely, *mean-variance efficient portfolios*.[2] The points labeled R_1 and R_2 correspond to $w = 1$ and $w = 0$, respectively. The other features of this figure are explained in Section 11.4. □

In practice, the mean and standard deviations of the returns can be estimated as discussed in Section 11.2.1 and the correlation coefficient can be estimated by the sample correlation coefficient. Alternatively, in Chapter 17 factor models are used to estimate expected returns and the covariance matrix of returns.

11.4 Combining Two Risky Assets with a Risk-Free Asset

Our ultimate goal is to find optimal portfolios combining many risky assets with a risk-free asset. However, many of the concepts needed for this task can be first understood most easily when there are only two risky assets.

11.4.1 Tangency Portfolio with Two Risky Assets

As mentioned in Section 11.3.1, each point on the efficient frontier in Figure 11.1 is $(\sigma_{R_P}, E(R_p))$ for some value of w between 0 and 1. If we fix w, then we have a fixed portfolio of the two risky assets. Now let us mix that portfolio of risky assets with the risk-free asset. The point F in Figure 11.1 gives $(\sigma_{R_P}, E(R))$ for the risk-free asset; of course, $\sigma_{R_P} = 0$ at F. The possible values of $(\sigma_{R_P}, E(R_p))$ for a portfolio consisting of the fixed portfolio of two risky assets and the risk-free asset is a line connecting the point F with a point on the efficient frontier, for example, the dashed line. The dotted line connecting F with R_2 mixes the risk-free asset with the second risky asset.

Notice that the dashed and dotted line connecting F with the point labeled T lies above the dashed line connecting F and the typical portfolio. This means that for any value of σ_{R_P}, the dashed and dotted line gives a higher expected return than the dashed line. The slope of each line is called its *Sharpe's ratio*, named after William Sharpe, whom we will meet again in Chapter 16. If $E(R_P)$ and σ_{R_P} are the expected return and standard deviation of the return on a portfolio and μ_f is the risk-free rate, then

$$\frac{E(R_P) - \mu_f}{\sigma_{R_P}} \tag{11.4}$$

[2] When a risk-free asset is available, then the efficient portfolios are no longer those on the efficient frontier but rather are characterized by Result 11.4.1 ahead.

is Sharpe's ratio of the portfolio. Sharpe's ratio can be thought of as a "reward-to-risk" ratio. It is the ratio of the reward quantified by the "excess expected return" to the risk as measured by the standard deviation.

A line with a larger slope gives a higher expected return for a given level of risk, so the larger Sharpe's ratio, the better regardless of what level of risk one is willing to accept. The point T on the parabola represents the portfolio with the highest Sharpe's ratio. It is the optimal portfolio for the purpose of mixing with the risk-free asset. This portfolio is called the *tangency portfolio* since its line is tangent to the efficient frontier.

Result 11.4.1 *The optimal or efficient portfolios mix the tangency portfolio with the risk-free asset. Each efficient portfolio has two properties:*

- *it has a higher expected return than any other portfolio with the same or smaller risk, and*
- *it has a smaller risk than any other portfolio with the same or higher expected return.*

Thus we can only improve (reduce) the risk of an efficient portfolio by accepting a worse (smaller) expected return, and we can only improve (increase) the expected return of an efficient portfolio by accepting worse (higher) risk.

Note that all efficient portfolios use the same mix of the two risky assets, namely, the tangency portfolio. Only the proportion allocated to the tangency portfolio and the proportion allocated to the risk-free asset vary.

Given the importance of the tangency portfolio, you may be wondering "how do we find it?" Again, let μ_1, μ_2, and μ_f be the expected returns on the two risky assets and the return on the risk-free asset. Let σ_1 and σ_2 be the standard deviations of the returns on the two risky assets and let ρ_{12} be the correlation between the returns on the risky assets.

Define $V_1 = \mu_1 - \mu_f$ and $V_2 = \mu_2 - \mu_f$, the excess expected returns. Then the tangency portfolio uses weight

$$w_T = \frac{V_1\sigma_2^2 - V_2\rho_{12}\,\sigma_1\sigma_2}{V_1\sigma_2^2 + V_2\sigma_1^2 - (V_1 + V_2)\rho_{12}\,\sigma_1\sigma_2} \tag{11.5}$$

for the first risky asset and weight $(1 - w_T)$ for the second.

Let R_T, $E(R_T)$, and σ_T be the return, expected return, and standard deviation of the return on the tangency portfolio. Then $E(R_T)$ and σ_T can be found by first finding w_T using (11.5) and then using the formulas

$$E(R_T) = w_T\mu_1 + (1 - w_T)\mu_2$$

and

$$\sigma_T = \sqrt{w_T^2\sigma_1^2 + (1-w_T)^2\sigma_2^2 + 2w_T(1-w_T)\rho_{12}\sigma_1\sigma_2}\,.$$

Example 11.3. The tangency portfolio with two risky assets

Suppose as before that $\mu_1 = 0.14$, $\mu_2 = 0.08$, $\sigma_1 = 0.2$, $\sigma_2 = 0.15$, and $\rho_{12} = 0$. Suppose as well that $\mu_f = 0.06$. Then $V_1 = 0.14 - 0.06 = 0.08$ and $V_2 = 0.08 - 0.06 = 0.02$. Plugging these values into formula (11.5), we get $w_T = 0.693$ and $1 - w_t = 0.307$. Therefore,

$$E(R_T) = (0.693)(0.14) + (0.307)(0.08) = 0.122,$$

and

$$\sigma_T = \sqrt{(0.693)^2(0.2)^2 + (0.307)^2(0.15)^2} = 0.146.$$

□

11.4.2 Combining the Tangency Portfolio with the Risk-Free Asset

Let R_p be the return on the portfolio that allocates a fraction ω of the investment to the tangency portfolio and $1-\omega$ to the risk-free asset. Then $R_p = \omega R_T + (1-\omega)\mu_f = \mu_f + \omega(R_T - R_f)$, so that

$$E(R_p) = \mu_f + \omega\{E(R_T) - \mu_f\} \quad \text{and} \quad \sigma_{R_p} = \omega\sigma_T.$$

Example 11.4. (Continuation of Example 11.2)

What is the optimal investment with $\sigma_{R_p} = 0.05$?

Answer: The maximum expected return with $\sigma_{R_p} = 0.05$ mixes the tangency portfolio and the risk-free asset such that $\sigma_{R_p} = 0.05$. Since $\sigma_T = 0.146$, we have that $0.05 = \sigma_{R_p} = \omega\,\sigma_T = 0.146\,\omega$, so that $\omega = 0.05/0.146 = 0.343$ and $1-\omega = 0.657$.

So 65.7% of the portfolio should be in the risk-free asset, and 34.3% should be in the tangency portfolio. Thus (0.343)(69.3%) = 23.7% should be in the first risky asset and (0.343)(30.7%) = 10.5% should be in the second risky asset. The total is not quite 100% because of rounding. The allocation is summarized in Table 11.1. □

Example 11.5. (Continuation of Example 11.2)

Now suppose that you want a 10% expected return. Compare

- the best portfolio of only risky assets, and

Table 11.1. *Optimal allocation to two risky assets and the risk-free asset to achieve* $\sigma_R = 0.05$.

Asset	Allocation (%)
risk-free	65.7
risky 1	23.7
risky 2	10.5
Total	99.9

- The best portfolio of the risky assets and the risk-free asset.

Answer: The best portfolio of only risky assets uses w solving $0.1 = w(0.14) + (1 - w)(0.08)$, which implies that $w = 1/3$. This is the *only* portfolio of risky assets with $E(R_p) = 0.1$, so by default it is best. Then

$$\sigma_{R_P} = \sqrt{w^2(0.2)^2 + (1 - w)^2(0.15)^2} = \sqrt{(1/9)(0.2)^2 + 4/9(0.15)^2} = 0.120.$$

The best portfolio of the two risky assets and the risk-free asset can be found as follows. First, $0.1 = E(R) = \mu_f + \omega\{E(R_T) - \mu_f\} = 0.06 + 0.062\,\omega = 0.06 + 0.425\,\sigma_R$, since $\sigma_{R_P} = \omega\,\sigma_T$ or $\omega = \sigma_{R_P}/\sigma_T = \sigma_{R_P}/0.146$. This implies that $\sigma_{R_P} = 0.04/0.425 = 0.094$ and $\omega = 0.04/0.062 = 0.645$. So combining the risk-free asset with the two risky assets reduces σ_{R_P} from 0.120 to 0.094 while maintaining $E(R_p)$ at 0.1. The reduction in risk is $(0.120 - 0.094)/0.094 = 28\%$, which is substantial. □

Table 11.2. *Minimum value of* σ_R *as a function of the available assets. In all cases, the expected return is* 0.1. *When only the risk-free asset and the second risky asset are available, then a return of* 0.1 *is achievable only if buying on margin is permitted.*

Available Assets	Minimum σ_R
first risky, risk-free	0.1
2nd risky, risk-free	0.3
Both riskies	0.12
All three	0.094

11.4.3 Effect of ρ_{12}

Positive correlation between the two risky assets increases risk. With positive correlation, the two assets tend to move together which increases the volatility of the portfolio. Conversely, negative correlation is beneficial since decreases risk. If the assets are negatively correlated, a negative return of one tends

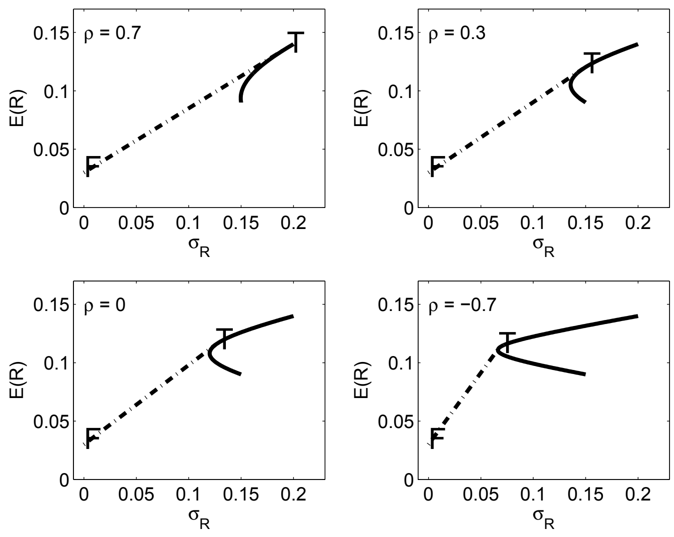

Fig. 11.2. *Efficient frontier and tangency portfolio when* $\mu_1 = 0.14$*,* $\mu_2 = 0.09$*,* $\sigma_1 = 0.2$*,* $\sigma_2 = 0.15$*, and* $\mu_f = 0.03$*. The value of* ρ_{12} *is varied from* 0.7 *to* -0.7*.*

to occur with a positive return of the other so the volatility of the portfolio decreases. Figure 11.2 shows the efficient frontier and tangency portfolio when $\mu_1 = 0.14$, $\mu_2 = 0.09$, $\sigma_1 = 0.2$, $\sigma_2 = 0.15$, and $\mu_f = 0.03$. The value of ρ_{12} is varied from 0.7 to -0.7. Notice that Sharpe's ratio of the tangency portfolio returns increases as ρ_{12} decreases. This means that when ρ_{12} is small, then efficient portfolios have less risk for a given expected return compared to when ρ_{12} is large.

11.5 Selling Short

Often some of the weights in an efficient portfolio are negative. A negative weight on an asset means that this asset is sold short. *Selling short* is a way to profit if a stock price goes *down*. To sell a stock short, one sells the stock without owning it. The stock must be borrowed from a broker or another customer of the broker. At a later point in time, one buys the stock and gives it back to the lender. This closes the short position.

Suppose a stock is selling at \$25/share and you sell 100 shares short. This gives you \$2500. If the stock goes down to \$17/share, you can buy the 100 shares for \$1700 and close out your short position. You made a profit of \$800 (ignoring transaction costs) because the stock went down 8 points. If the stock had gone up, then you would have had a loss.

Suppose now that you have \$100 and there are two risky assets. With your money you could buy \$150 worth of risky asset 1 and sell \$50 short of risky asset 2. The net cost would be exactly \$100. If R_1 and R_2 are the returns on risky assets 1 and 2, then the return on your portfolio would be

$$\frac{3}{2}R_1 + \left(-\frac{1}{2}\right) R_2.$$

Your portfolio weights are $w_1 = 3/2$ and $w_2 = -1/2$. Thus, you hope that risky asset 1 rises in price and risky asset 2 falls in price. Here, again, we have ignored transaction costs.

If one sells a stock short, one is said to have a *short position* in that stock, and owning the stock is called a *long position*.

11.6 Risk-Efficient Portfolios with *N* Risky Assets

In this section, we use quadratic programming to find efficient portfolios with an arbitrary number of assets. An advantage of quadratic programming is that it allows one to impose constraints such as limiting short sales.

Assume that we have N risky assets and that the return on the ith risky asset is R_i and has expected value μ_i. Define

$$\boldsymbol{R} = \begin{pmatrix} R_1 \\ \vdots \\ R_N \end{pmatrix}$$

to be the random vector of returns,

$$E(\boldsymbol{R}) = \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_N \end{pmatrix},$$

and $\boldsymbol{\Sigma}$ to be the covariance matrix of $\boldsymbol{R}$.

Let

$$\boldsymbol{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix}$$

be a vector of portfolio weights so that $w_1 + \cdots + w_N = \mathbf{1}^{\mathsf{T}}\boldsymbol{\omega} = 1$, where

$$\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

is a column of N ones. The expected return on the portfolio is

$$\sum_{i=1}^{N} \omega_i \mu_i = \boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{\mu}. \tag{11.6}$$

Suppose there is a target value, μ_P, of the expected return on the portfolio. When $N = 2$, the target expected returns is achieved by only one portfolio and its w_1-value solves $\mu_P = w_1\mu_1 + w_2\mu_2 = \mu_2 + w_1(\mu_1 - \mu_2)$. For $N \geq 3$, there will be an infinite number of portfolios achieving the target μ_P. The one with the smallest variance is called the "efficient" portfolio. Our goal is to find the efficient portfolio.

The variance of the return on the portfolio with weights $\boldsymbol{w}$ is

$$\boldsymbol{w}^{\mathsf{T}}\boldsymbol{\Sigma}\boldsymbol{w}. \tag{11.7}$$

Thus, given a target μ_P, the efficient portfolio minimizes (11.7) subject to

$$\boldsymbol{w}^{\mathsf{T}}\boldsymbol{\mu} = \mu_P \tag{11.8}$$

and

$$\boldsymbol{w}^{\mathsf{T}}\boldsymbol{1} = 1. \tag{11.9}$$

Quadratic programming is used to minimize a quadratic objective function subject to linear constraints. In applications to portfolio optimization, the objective function is the variance of the portfolio return. The objective function is a function of N variables, such as, the weights of N assets, that are denoted by an $N \times 1$ vector $\boldsymbol{x}$. Suppose that the quadratic objective function to be minimized is

$$\frac{1}{2}\boldsymbol{x}^{\mathsf{T}}\boldsymbol{D}\boldsymbol{x} - \boldsymbol{d}^{\mathsf{T}}\boldsymbol{x}, \tag{11.10}$$

where $\boldsymbol{D}$ is an $N \times N$ matrix and $\boldsymbol{d}$ is an $N \times 1$ vector. The factor of 1/2 is not essential but is used here to keep our notation consistent with `R`. There are two types of linear constraints on $\boldsymbol{x}$, inequality and equality constraints. The linear inequality constraints are

$$\boldsymbol{A}_{\text{neq}}^{\mathsf{T}}\boldsymbol{x} \geq \boldsymbol{b}_{\text{neq}}, \tag{11.11}$$

where $\boldsymbol{A}_{\text{neq}}$ is an $m \times N$ matrix, $\boldsymbol{b}_{\text{neq}}$ is an $m \times 1$ vector, and m is the number of inequality constraints. The equality constraints are

$$\boldsymbol{A}_{\text{eq}}^{\mathsf{T}}\boldsymbol{x} = \boldsymbol{b}_{\text{eq}}, \tag{11.12}$$

where $\boldsymbol{A}_{\text{eq}}$ is an $n \times N$ matrix, $\boldsymbol{b}_{\text{eq}}$ is an $n \times 1$ vector, and n is the number of equality constraints. Quadratic programming minimizes the quadratic objective function (11.10) subject to linear inequality constraints (11.11) and linear equality constraints (11.12).

To apply quadratic programming to find an efficient portfolio, we use $\boldsymbol{x} = \boldsymbol{w}$, $\boldsymbol{D} = 2\boldsymbol{\Sigma}$, and $\boldsymbol{d}$ equal to an $N \times 1$ vector of zeros so that (11.10) is $\boldsymbol{w}^{\mathsf{T}}\boldsymbol{\Sigma}\boldsymbol{w}$, the return variance of the portfolio. There are two equality constraints, one that the weights sum to 1 and the other that the portfolio return is a specified target μ_P. Therefore, we define

$$\boldsymbol{A}_{\text{eq}}^{\mathsf{T}} = \begin{pmatrix} \boldsymbol{1}^{\mathsf{T}} \\ \boldsymbol{\mu}^{\mathsf{T}} \end{pmatrix}$$

and

$$\boldsymbol{b}_{\text{eq}} = \begin{pmatrix} 1 \\ \mu_P \end{pmatrix},$$

so that (11.12) becomes

$$\begin{pmatrix} \mathbf{1}^{\mathsf{T}}\boldsymbol{w} \\ \boldsymbol{\mu}^{\mathsf{T}}\boldsymbol{w} \end{pmatrix} = \begin{pmatrix} 1 \\ \mu_P \end{pmatrix},$$

which is the same as constraints (11.8) and (11.9).

Investors often wish to impose additional inequality constraints. If an investor cannot or does not wish to sell short, then the constraints

$$\boldsymbol{w} \geq \mathbf{0}$$

can be used. Here $\mathbf{0}$ is a vector of zeros. In this case $\boldsymbol{A}_{\text{neq}}$ is the $N \times N$ identical matrix and $\boldsymbol{b}_{\text{neq}} = \mathbf{0}$.

To avoid concentrating the portfolio in just one or a few stocks, an investor may wish to constrain the portfolio so that no w_i exceeds a bound λ, for example, $\lambda = 1/4$ means that no more than 1/4 of the portfolio can be in any single stock. In this case, $\boldsymbol{w} \leq \lambda\mathbf{1}$ or equivalently $-\boldsymbol{w} \geq -\lambda\mathbf{1}$, so that $\boldsymbol{A}_{\text{neq}}$ is minus the $N \times N$ identity matrix and $\boldsymbol{b}_{\text{neq}} = -\lambda\mathbf{1}$. One can combine these constraints with those that prohibit short selling.

To find the efficient frontier, one uses a grid of values of μ_P and finds the corresponding efficient portfolios. For each portfolio, σ_P^2, which is the minimized value of the objective function, can be calculated. Then one can find the minimum variance portfolio by finding the portfolio with the smallest value of the σ_P^2. The efficient frontier is the set of efficient portfolios with expected return above the expected return of the minimum variance portfolio. One can also compute Sharpe's ratio for each portfolio on the efficient frontier and the tangency portfolio is the one maximizing Sharpe's ratio.

Example 11.6. Finding the efficient frontier, tangency portfolio, and minimum variance portfolio using quadratic programming

The following `R` program uses the returns on three stocks, GE, IBM, and Mobil, in the `CRSPday` data set in the `Ecdat` package. The function `solve.QP` in the `quadprog` package is used for quadratic programming. `solve.QP` combines $\boldsymbol{A}_{\text{eq}}^{\mathsf{T}}$ and $\boldsymbol{A}_{\text{neq}}^{\mathsf{T}}$ into a single matrix `Amat` by stacking $\boldsymbol{A}_{\text{eq}}^{\mathsf{T}}$ on top of $\boldsymbol{A}_{\text{neq}}^{\mathsf{T}}$. The parameter `meq` is the number of rows of $\boldsymbol{A}_{\text{eq}}^{\mathsf{T}}$. $\boldsymbol{b}_{\text{eq}}$ and $\boldsymbol{b}_{\text{neq}}$ are handled analogously. In this example, there are no inequality constraints, so $\boldsymbol{A}_{\text{neq}}^{\mathsf{T}}$ and $\boldsymbol{b}_{\text{neq}}$ are not needed, but they are used in the next example.

The efficient portfolio is found for each of 300 target values of μ_P between 0.05 and 0.14. For each portfolio, Sharpe's ratio is found and the logical vector `ind` indicates which portfolio is the tangency portfolio maximizing Sharpe's ratio. Similarly, `ind2` indicates the minimum variance portfolio. It is assumed that the risk-free rate is 1.3%/year.

```
library(Ecdat)
library(quadprog)
data(CRSPday)
R = 100*CRSPday[,4:6]
mean_vect = apply(R,2,mean)
cov_mat = cov(R)
sd_vect = sqrt(diag(cov_mat))

Amat = cbind(rep(1,3),mean_vect)  # set the constraints matrix
muP = seq(.05,.14,length=300)  # set of 300 possible target values
                              # for the expect portfolio return
sdP = muP # set up storage for std dev's of portfolio returns
weights = matrix(0,nrow=300,ncol=3) # storage for portfolio weights

for (i in 1:length(muP))  # find the optimal portfolios for
                           # each target expected return
{
bvec = c(1,muP[i])  # constraint vector
result =
   solve.QP(Dmat=2*cov_mat,dvec=rep(0,3),Amat=Amat,bvec=bvec,meq=2)
sdP[i] = sqrt(result$value)
weights[i,] = result$solution
}
postscript("quad_prog_plot.ps",width=6,height=5)
plot(sdP,muP,type="l",xlim=c(0,2.5),ylim=c(0,.15),lty=3)  #  plot
          # the efficient frontier (and inefficient portfolios
          # below the min var portfolio)
mufree = 1.3/253 # input value of risk-free interest rate
points(0,mufree,cex=4,pch="*")  # show risk-free asset
sharpe =( muP-mufree)/sdP # compute Sharpe's ratios
ind = (sharpe == max(sharpe)) # Find maximum Sharpe's ratio
options(digits=3)
weights[ind,] #  print the weights of the tangency portfolio
lines(c(0,2),mufree+c(0,2)*(muP[ind]-mufree)/sdP[ind],lwd=4,lty=2)
           # show line of optimal portfolios
points(sdP[ind],muP[ind],cex=4,pch="*") # show tangency portfolio
ind2 = (sdP == min(sdP)) # find the minimum variance portfolio
points(sdP[ind2],muP[ind2],cex=2,pch="+") # show min var portfolio
ind3 = (muP > muP[ind2])
lines(sdP[ind3],muP[ind3],type="l",xlim=c(0,.25),
   ylim=c(0,.3),lwd=2)  #  plot the efficient frontier
text(sd_vect[1],mean_vect[1],"GE",cex=1.5)
text(sd_vect[2],mean_vect[2],"IBM",cex=1.5)
text(sd_vect[3],mean_vect[3],"Mobil",cex=1.5)
graphics.off()
```

The plot produced by this program is Figure 11.3. The program prints the weights of the tangency portfolio, which are

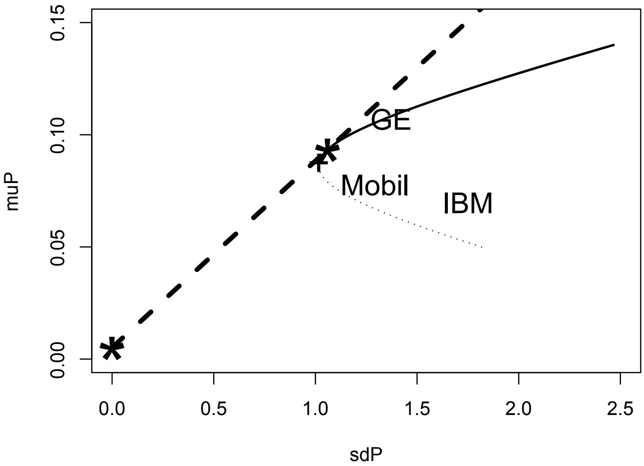

Fig. 11.3. *Efficient frontier (solid), line of efficient portfolios (dashed) connecting the risk-free asset and tangency portfolio (asterisks), and the minimum variance portfolio (plus) with three stocks (GE, IBM, and Mobil). The three stocks are also shown on reward-risk space.*

```
> weights[ind,] # Find tangency portfolio
[1] 0.5512 0.0844 0.3645
```

□

Example 11.7. Finding the efficient frontier, tangency portfolio, and minimum variance portfolio with no short selling using quadratic programming

In this example, Example 11.6 is modified so that short sales are not allowed. Only three lines of code need to be changed. When short sales are prohibited, the target expected return on the portfolio must lie between the smallest and largest expected returns on the stocks. This is enforced by the following change:

```
muP = seq(min(mean_vect)+.0001,max(mean_vect)-.0001,length=300)
```

To enforce no short sales, an $\boldsymbol{A}_{\text{neq}}$ matrix is needed and is set equal to a 3×3 identity matrix:

```
Amat = cbind(rep(1,3),mean_vect,diag(1,nrow=3))
                            # set the constraints matrix
```

Also, $\boldsymbol{b}_{\text{neq}}$ is set equal to a three-dimensional vector of zeros:

```
bvec = c(1,muP[i],rep(0,3))
```

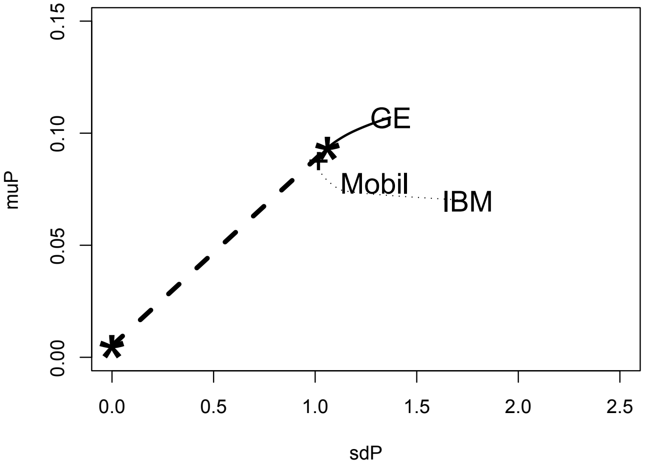

Fig. 11.4. *Efficient frontier (solid), line of efficient portfolios (dashed) connecting the risk-free asset and tangency portfolio (asterisks), and the minimum variance portfolio (plus) with three stocks (GE, IBM, and Mobil) with short sales prohibited.*

The new plot is shown in Figure 11.4. Since the tangency portfolio in Example 11.6 had all weights positive, the tangency portfolio is unchanged by the prohibition of short sales. The efficient frontier is changed since without short sales, it is impossible to have expected returns greater than the expected return of GE, the stock with the highest expected return. In contrast, when short sales are allowed, there is no upper bound on the expected return (or on the risk).

□

11.7 Resampling and Efficient Portfolios

When N is small, the theory of portfolio optimization can be applied using sample means and the sample covariance matrix as in the previous examples. However, the effects of estimation error, especially with larger values of N, can result in portfolios that only appear efficient. This problem will be investigated in this section.

Example 11.8. The global asset allocation problem

One application of optimal portfolio selection is allocation of capital to different market segments. For example, Michaud (1998) discusses a global asset allocation problem where capital must be allocated to "U.S. stocks and government/corporate bonds, euros, and the Canadian, French, German, Japanese, and U.K. equity markets." Here we look at a similar example where we allocate capital to the equity markets of 10 different countries. Monthly returns for these markets were calculated from MSCI Hong Kong, MSCI Singapore, MSCI Brazil, MSCI Argentina, MSCI UK, MSCI Germany, MSCI Canada, MSCI France, MSCI Japan, and the S&P 500. "MSCI" means "Morgan Stanley Capital Index." The data are from January 1988 to January 2002, inclusive, so there are 169 months of data.

Assume that we want to find the tangency portfolio that maximizes Sharpe's ratio. The tangency portfolio was estimated using sample means and the sample covariance as in Example 11.6, and its Sharpe's ratio is estimated to be 0.3681. However, we should suspect that 0.3681 must be an overestimate since this portfolio only maximizes Sharpe's ratio using estimated parameters, not the true means and covariance matrix. To evaluate the possible amount of overestimation, one can use the bootstrap. As discussed in Chapter 6, in the bootstrap simulation experiment, the sample is the "true population" so that the sample mean and covariance matrix are the "true parameters," and the resamples mimic the sampling process. Actual Sharpe's ratios are calculated with the sample means and covariance matrix, while estimated Sharpe's ratio use the means and covariance matrix of the resamples.

First, 250 resamples were taken and for each the tangency portfolio was estimated. Resampling was done by sampling rows of the data matrix as discussed in Section 7.11. For each of the 250 tangency portfolios estimated from the resamples, the actual and estimated Sharpe's ratios were calculated. Boxplots of the 250 actual and 250 estimated Sharpe's ratios are in Figure 11.5(a). In this figure, there is a dashed horizontal line at height 0.3681, the actual Sharpe's ratio of the true tangency portfolio. One can see that all 250 estimated tangency portfolios have actual Sharpe's ratios below this value, as they must since the actual Sharpe's ratio is maximized by the true tangency portfolio, not the estimated tangency portfolios.

From the boxplot on the right-hand side of (a), one can see that the estimated Sharpe's ratios overestimate not only the actual Sharpe's ratios of the estimated tangency portfolios but also the somewhat larger (and unattainable) actual Sharpe's ratio of the true (but unknowable) tangency portfolio.

□

There are several ways to alleviate the problems caused by estimation error when attempting to find a tangency portfolio. One can try to find more accurate estimators; the factor models of Chapter 17 and Bayes estimators of

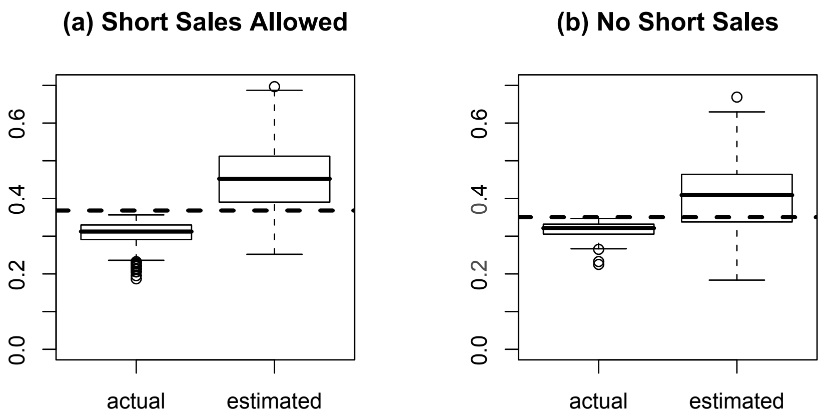

Fig. 11.5. *Bootstrapping estimation of the tangency portfolio and its Sharpe's ratio. (a) Short sales allowed. The left-hand boxplot is of the actual Sharpe's ratios of the estimated tangency portfolios for* 250 *resamples. The right-hand boxplot contains the estimated Sharpe's ratios for these portfolios. The horizontal dashed line indicates Sharpe's ratio of the true tangency portfolio. (b) Same as (a) but with short sales not allowed.*

Chapter 20 (see especially Example 20.12) do this. Another possibility is to restrict short sales.

Portfolios with short sales aggressively attempt to maximize Sharpe's ratio by selling short those stocks with the smallest estimated mean returns and having large long positions in those stocks with the highest estimated mean returns. The weakness with this approach is that it is particularly sensitive to estimation error. Unfortunately, expected returns are estimated with relatively large uncertainty. This problem can be seen in Figure 11.6, which contains KDEs of the bootstrap distributions of the mean returns, and Table 11.3, which has 95% confidence intervals for the mean returns. The percentile method is used for the confidence intervals, so the endpoints are the 2.5 and 97.5 bootstrap percentiles. Notice for Singapore and Japan, the confidence intervals include both positive and negative values. In the figure and the table, the returns are expressed as percentage returns.

Example 11.9. The global asset allocation problem: short sales prohibited

This example repeats the bootstrap experimentation of Example 11.8 with short sales prohibited by using inequality constraints such as in Example 11.7.

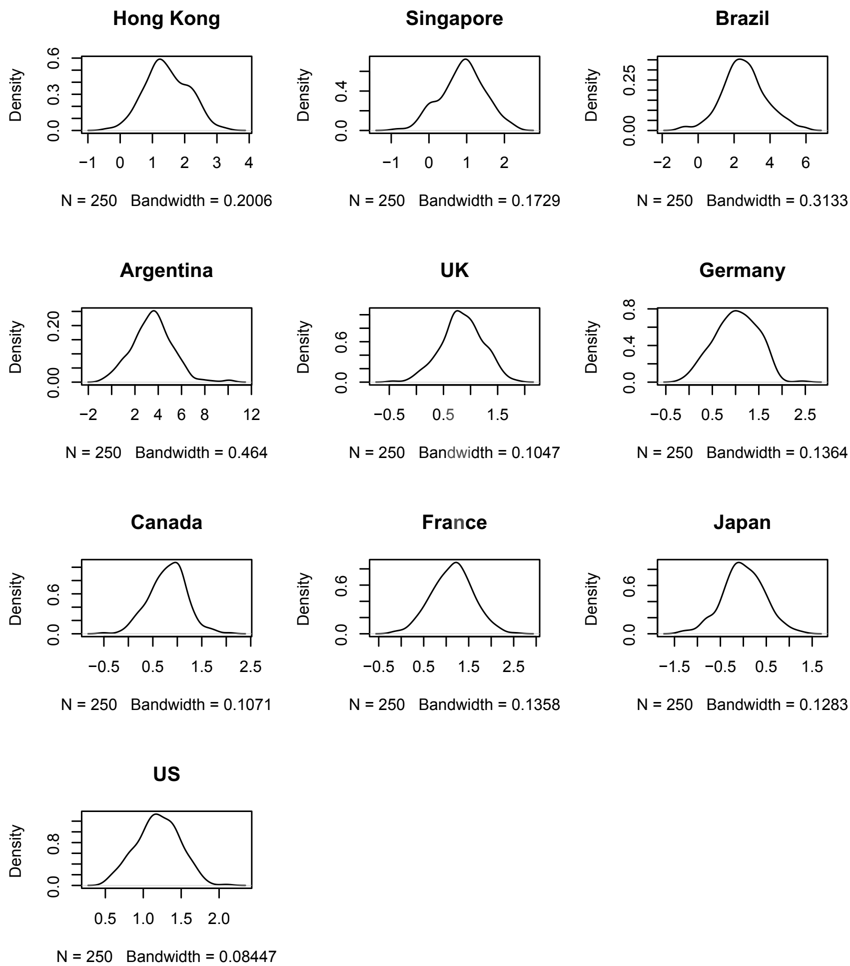

Fig. 11.6. *Kernel density estimates of the bootstrap distribution of the sample mean return for global asset allocation problem. Returns are expressed as percentages.*

With short sales not allowed, the actual Sharpe's ratio of the true tangency portfolio is 0.3503, which is only slightly less than when short sales are allowed.

Boxplots of actual and apparent Sharpe's ratios are in Figure 11.5(b). Comparing Figures 11.5(a) and (b), one sees that prohibiting short sales has two beneficial effects—Sharpe's ratios actually achieved are slightly higher with no short sales allowed compared to having no constraints on short sales. In fact, the mean of the 250 actual Sharpe's ratios is 0.3060 with short sales allowed and 0.3169 with short sales prohibited. Moreover, the overestimation of Sharpe's ratio is reduced by prohibiting short sales—the mean apparent

Table 11.3. 95% *percentile-method bootstrap confidence intervals for the mean returns of the* 10 *countries.*

Country	2.5%	97.5%
Hong Kong	0.186	2.709
Singapore	−0.229	2.003
Brazil	0.232	5.136
Argentina	0.196	6.548
UK	0.071	1.530
Germany	0.120	1.769
Canada	0.062	1.580
France	0.243	2.028
Japan	−0.884	0.874
U.S.	0.636	1.690

Sharpe's ratio is 0.4524 [with estimation error (0.4524−0.3681) = 0.0843] with short sales allowed by only 0.4038 [with estimation error (0.4038 − 0.3503) = 0.0535] with short sales prohibited. However, these effects, though positive, are only modest and do not entirely solve the problem of overestimation of Sharpe's ratio.

□

Example 11.10. The global asset allocation problem: Shrinkage estimation and short sales prohibited

In Example 11.9, we saw that shrinkage estimation can increase Sharpe's ratio of the estimated tangency portfolio, but the improvement is only modest. Further improvement requires more accurate estimation of the mean vector or the covariance matrix of the returns.

This example investigates possible improvements from shrinking the 10 estimated means toward each other. Specifically, if $\overline{Y}_i$ is the sample mean of the ith country, $\overline{Y} = (\sum_{i=1}^{10} \overline{Y}_i)/10$ is the grand mean (mean of the means), and α is a tuning parameter between 0 and 1, then the estimated mean return for the ith country is

$$\widehat{\mu}_i = \alpha \overline{Y}_i + (1-\alpha)\overline{Y}. \tag{11.13}$$

The purpose of shrinkage is to reduce the variance of the estimator, though the reduced variance comes at the expense of some bias. Since it is the mean of 10 means, $\overline{Y}$ is much less variable than any of $\overline{Y}_1, \ldots, \overline{Y}_{10}$. Therefore, $\mathrm{Var}(\widehat{\mu}_i)$ decreases as α is decreased toward 0. However,

$$E(\widehat{\mu}_i) = \alpha\mu_i + \frac{1-\alpha}{10}\sum_{i=1}^{10}\mu_i \tag{11.14}$$

so that, for any $\alpha \neq 1$, $\widehat{\mu}_i$ is biased, except under the very likely circumstance that $\mu_1 = \cdots = \mu_{10}$. The parameter α controls the bias–variance tradeoff. In this example, $\alpha = 1/2$ will be used for illustration and short sales will not be allowed.

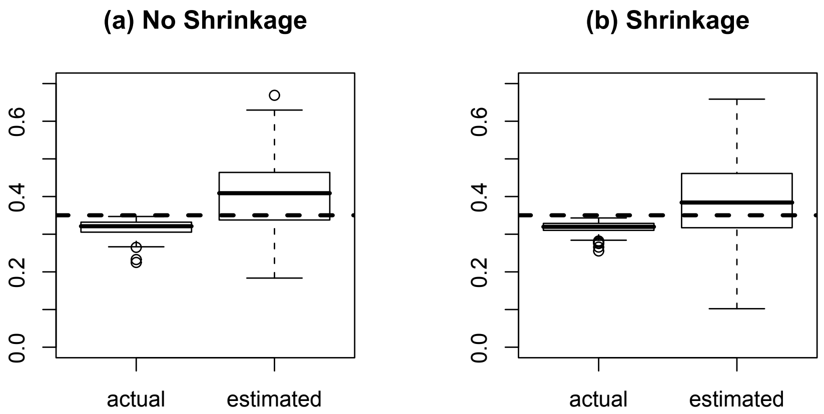

Fig. 11.7. *Bootstrapping estimation of the tangency portfolio and its Sharpe's ratio. Short sales not allowed. (a) No shrinkage. The left-hand boxplot is of the actual Sharpe's ratios of the estimated tangency portfolios for* 250 *resamples. The right-hand boxplot contains the estimated Sharpe's ratios for these portfolios. The horizontal dashed line indicates Sharpe's ratio of the true tangency portfolio. (b) Same as (a) but with shrinkage.*

Figure 11.7 compares the performance of shrinkage versus no shrinkage. Panel (a) contains the boxplots that we saw in panel (b) of Figure 11.5 where $\alpha = 1$. Panel (b) has the boxplots when the tangency portfolio is estimated using $\alpha = 1/2$. Compared to panel (a), in panel (b) the actual Sharpe's ratios are somewhat closer to the dashed line indicating Sharpe's ratio of the true tangency portfolio. Moreover, the estimated Sharpe's ratios in (b) are smaller and closer to the true Sharpe's ratios, so there is less overoptimization—shrinkage has helped in two ways.

The next step might be selection of α to optimize performance of shrinkage estimation. Doing this need not be difficult, since different values of α can be compared by bootstrapping.

□

There are other methods for improving the estimation of the mean vector and estimation of the covariance matrix can be improved as well, for example, by using the factor models in Chapter 17 or Bayesian estimation as in Chapter 20. Moreover, one need not focus on the tangency portfolio but could, for example, estimate the minimum variance portfolio. Whatever the focus of estimation, the bootstrap can be used to compare various strategies for improving the estimation of the optimal portfolio.

11.8 Bibliographic Notes

Markowitz (1952) was the original paper on portfolio theory and was expanded into the book Markowitz (1959). Bodie and Merton (2000) provide an elementary introduction to portfolio selection theory. Bodie, Kane, and Marcus (1999) and Sharpe, Alexander, and Bailey (1999) give a more comprehensive treatment. See also Merton (1972). Formula (11.5) is derived in Example 5.10 of Ruppert (2004).

Jobson and Korkie (1980) and Britten-Jones (1999) discuss the statistical issue of estimating the efficient frontier; see the latter for additional recent references. Britten-Jones (1999) shows that the tangency portfolio can be estimated by regression analysis and hypotheses about the tangency portfolio can be tested by regression F-tests. Jagannathan and Ma (2003) discuss how imposing constraints such as no short sales can reduce risk.

11.9 References

Bodie, Z., and Merton, R. C. (2000) *Finance*, Prentice-Hall, Upper Saddle River, NJ.

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11.10 R Lab

11.10.1 Efficient Equity Portfolios

This section uses daily stock prices in the data set `Stock_FX_Bond.csv` that is posted on the book's website and in which any variable whose name ends with "AC" is an adjusted closing price. As the name suggests, these prices have been adjusted for dividends and stock splits, so that returns can be calculated without further adjustments. Run the following code which will read the data, compute the returns for six stocks, create a scatterplot matrix of these returns, and compute the mean vector, covariance matrix, and vector of standard deviations of the returns. Note that returns will be percentages.

```
dat = read.csv("Stock_FX_Bond.csv",header=T)
prices = cbind(dat$GM_AC,dat$F_AC,dat$CAT_AC,dat$UTX_AC,
   dat$MRK_AC,dat$IBM_AC)
n = dim(prices)[1]
returns =  100*(prices[2:n,]/prices[1:(n-1),] - 1)
pairs(returns)
mean_vect = apply(returns,2,mean)
cov_mat = cov(returns)
sd_vect = sqrt(diag(cov_mat))
```

Problem 1 *Write an* R *program to find the efficient frontier, the tangency portfolio, and the minimum variance portfolio, and plot on "reward-risk space" the location of each of the six stocks, the efficient frontier, the tangency portfolio, and the line of efficient portfolios. Use the constraints that* $-0.1 \leq w_j \leq 0.5$ *for each stock. The first constraint limits short sales but does not rule them out completely. The second constraint prohibits more than 50% of the investment in any single stock. Assume that the annual risk-free rate is 3% and convert this to a daily rate by dividing by 365, since interest is earned on trading as well as nontrading days.*

Problem 2 *If an investor wants an efficient portfolio with an expected daily return of 0.07%, how should the investor allocate his or her capital to the six stocks and to the risk-free asset? Assume that the investor wishes to use the tangency portfolio computed with the constraints* $-0.1 \leq w_j \leq 0.5$*, not the unconstrained tangency portfolio.*

Problem 3 *Does this data set include Black Monday?*

11.11 Exercises

1. Suppose that there are two risky assets, A and B, with expected returns equal to 2.3% and 4.5%, respectively. Suppose that the standard deviations of the returns are $\sqrt{6}\%$ and $\sqrt{11}\%$ and that the returns on the assets have a correlation of 0.17.
   (a) What portfolio of A and B achieves a 3% rate of expected return?
   (b) What portfolios of A and B achieve a $\sqrt{5.5}\%$ standard deviation of return? Among these, which has the largest expected return?
2. Suppose there are two risky assets, C and D, the tangency portfolio is 65% C and 35% D, and the expected return and standard deviation of the return on the tangency portfolio are 5% and 7%, respectively. Suppose also that the risk-free rate of return is 1.5%. If you want the standard deviation of your return to be 5%, what proportions of your capital should be in the risk-free asset, asset C, and asset D?
3. (a) Suppose that stock A shares sell at $75 and stock B shares at $115. A portfolio has 300 shares of stock A and 100 of stock B. What are the weights w and $1 - w$ of stocks A and B in this portfolio?
   (b) More generally, if a portfolio has N stocks, if the price per share of the jth stock is P_j, and if the portfolio has n_j shares of stock j, then find a formula for w_j as a function of $n_1, \ldots, n_N$ and $P_1, \ldots, P_N$.
4. Let $\mathcal{R}_P$ be a return of some type on a portfolio and let $\mathcal{R}_1, \ldots, \mathcal{R}_N$ be the same type of returns on the assets in this portfolio. Is

$$\mathcal{R}_P = w_1\mathcal{R}_1 + \cdots + w_N\mathcal{R}_N$$

   true if $\mathcal{R}_P$ is a net return? Is this equation true if $\mathcal{R}_P$ is a gross return? Is it true if $\mathcal{R}_P$ is a log return? Justify your answers.
5. Suppose one has a sample of monthly log returns on two stocks with sample means of 0.0032 and 0.0074, sample variances of 0.017 and 0.025, and a sample covariance of 0.0059. For purposes of resampling, consider these to be the "true population values." A bootstrap resample has sample means of 0.0047 and 0.0065, sample variances of 0.0125 and 0.023, and a sample covariance of 0.0058.
   (a) Using the resample, estimate the efficient portfolio of these two stocks that has an expected return of 0.005; that is, give the two portfolio weights.
   (b) What is the estimated variance of the return of the portfolio in part (a) using the resample variances and covariances?
   (c) What are the actual expected return and variance of return for the portfolio in (a) when calculated with the true population values (e.g., with using the original sample means, variances, and covariance)?

6. Stocks 1 and 2 are selling for \$100 and \$125, respectively. You own 200 shares of stock 1 and 100 shares of stock 2. The weekly returns on these stocks have means of 0.001 and 0.0015, respectively, and standard deviations of 0.03 and 0.04, respectively. Their weekly returns have a correlation of 0.35. Find the covariance matrix of the weekly returns on the two stocks, the mean and standard deviation of the weekly returns on the portfolio, and the one-week VaR(0.05) for your portfolio.