Images of Expertise in Mathematics Teaching

Rosemary S. Russ, Bruce Sherin, and Miriam Gamoran Sherin

Abstract In this chapter we present a brief portrait of how researchers engaged in the study of mathematics teaching have understood teaching expertise, a portrait that is attentive to the diversity that has existed and continues to exist in the field. To do so we first adopt a historical perspective and attempt to capture some of the trends in how teaching expertise has been conceptualized, with an emphasis on how these trends were driven by broader changes in educational research. In particular, we trace the study of mathematics teaching through the traditions of process-product research, cognitive research, subject-specific cognitive research, situated cognition research, and design research. We then provide some sense for the diversity of perspectives and approaches to mathematics teaching that are currently prominent by presenting four images of mathematics teaching practice. We describe how researchers have tacitly conceived of mathematics teachers as either diagnosticians of students' thinking, conductors of classroom discourse, architects of curriculum, or river guides who are flexible in the moments of teaching. An awareness of these images of expertise will help the field both recognize and situate new images, allowing us to use them in productive ways to further understand the work of mathematics teaching.

Keywords Mathematics · Teaching · Expertise

Our charge in this chapter is to discuss how teaching expertise has been conceptualized by researchers engaged in the study of mathematics teaching. Although we accept that charge, we must note that there is no possibility of providing anything approaching a unitary account of mathematics teaching expertise, or of the research that seeks to understand that expertise. The problem is that teaching, as a profession, requires its practitioners to engage in a diverse constellation of

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R.S. Russ (✉)
School of Education and Social Policy, Northwestern University, Evanston, IL, USA
e-mail: r-russ@northwestern.edu

The authors Rosemary S. Russ, Bruce Sherin, Miriam Gamoran Sherin contributed equally to the writing of this chapter.

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tasks. Mathematics teachers must plan lessons, work with students individually and as a whole class, and they must present explanations, examples, and definitions. Similarly, mathematics teachers develop assessments, grade student work, and keep track of student progress. Complicating the situation still further is the problem that each of these tasks can, in practice, exhibit enormous variability.

This complexity requires that researchers studying mathematics teaching expertise, working as a field, adopt a divide-and-conquer approach. One way in which the field may divide up the undertaking is for individual researchers to work on different subsets of the "diverse constellation of tasks" faced by teachers. So, some researchers might choose to look at how teachers create lesson plans, while others might look at how they lead classroom discussions.

But the situation is a bit more complicated than this divide-and-conquer story suggests. The fact is that individual researchers may look at the problem of understanding teaching expertise from very different angles. Moreover, new perspectives percolate through the field, changing with time, and spreading from one researcher to another. As they do, the problem of understanding teaching expertise is divided and re-divided in such a way that the work of multiple researchers does not fit together cleanly.

The particular way that an individual researcher chooses to conceptualize and study mathematics teaching is likely influenced by a number of factors. First, there are the current trends in the broader landscape of education research – the perspectives that percolate through the field. A second and related influence is that researchers each have their own particular commitments to and assumptions about what aspects of the practice are important for successful teaching and learning. Third, researchers must choose components of teaching practices that are tractable and feasible to study.

In this chapter, we seek to present a brief portrait of how the field has understood mathematics teaching expertise, a portrait that is attentive to the variability and diversity that existed and continue to exist. We will do this in two ways. First, we adopt a historical perspective and attempt to capture some of the broad trends in how teaching expertise was conceptualized, with an emphasis on how these trends were driven by broader changes in the landscape of educational research. In describing these historical trends we treat the field as largely monolithic in its approach and emphases. Our focus then shifts to the present and, in doing so, we attempt to provide some sense for the diversity of perspectives and approaches to mathematics teaching expertise that are currently prominent. To paint a picture of this diversity, we present four images of mathematics teaching practice. In describing these images, we will also attempt to show how our current conceptions of teaching expertise continue to be influenced by perspectives that were prominent in the past. To do so, we first present a short teaching vignette from an eighth-grade mathematics classroom that we will use to ground the discussion throughout the chapter. To be clear, our goal is not to characterize expert mathematics teachers as a class of teachers distinct from novice teachers. Instead, we seek to describe several key aspects of the expertise involved in teaching mathematics today.

A Vignette of Mathematics Teaching: The Crowd Estimation Problem

It was December, and Mr. Louis' 4th period class was nearing the end of a unit on comparing and scaling (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1997b). At the beginning of class, students were given a picture of a 14 cm × 9 cm rectangle densely filled with dots (Fig. 1). Students were told to imagine that the picture was an aerial photograph of a crowd at a rally and that each dot represented a person. Their task was to estimate how many people attended the rally. Students began by working on the problem in small groups as the teacher circulated throughout the class. Mr. Louis then invited Tina's group to the front of the room to share their approach. Using the overhead projector to demonstrate, Tina explained that they divided the original rectangle into 126 small squares that were 1 cm × 1 cm. Then they counted 17 dots in one of the small squares. To estimate the total population, they multiplied 17 by 126. Tina concluded, "and we got 2,142. That was our approximate answer."

Following the presentation, Mr. Louis turned to the class and asked, "What do people think about this group's method?" Among several comments from students, Robert responded that Tina's group would have gotten a more accurate estimate for the total population if they had used bigger squares. When prompted to elaborate, Robert explained that "with smaller squares there may be a bunch of dots packed into a small area. In just that particular area or something. Or there might have been not a lot of dots." Robert's point was that the number of dots in a larger square might be more representative of the density of the picture than the number of dots in a smaller square (see Fig. 2).

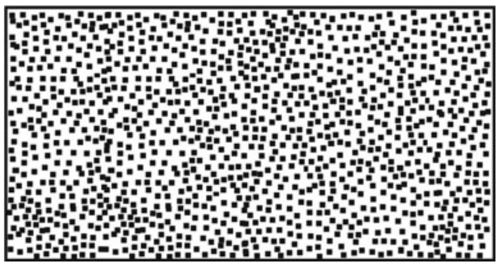

Fig. 1 Estimate the population of the crowd shown in the picture

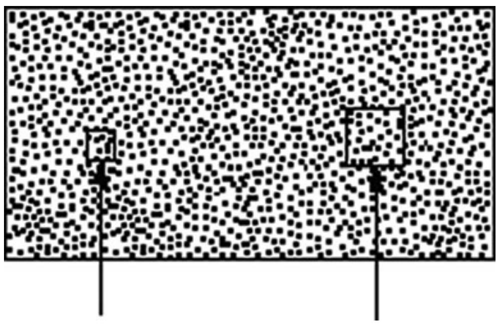

Fig. 2 Two proposed solutions to the crowd estimation problem

Mr. Louis again turned to the class for comments: “What do you think about what Robert just said?” Several students said that they agreed with Robert, including Amy, Jin, and Sal. In contrast, Jeff suggested an alternative method that involved finding the average number of dots in 10 small squares. “It would have been better if instead of …one small square… they took ten squares from all random spots that were small size and divided the total of all the groups by 10.” After a few minutes, Mr. Louis drew the class’ attention specifically to Robert’s and Jeff’s ideas. “We have two competing ideas here.” He drew a diagram to illustrate the different approaches and encouraged the students to compare and contrast the two methods. “Which way do you think would produce the most accurate estimate of the population?”

As the class discussed Robert’s and Jeff’s methods, students raised a number of issues including the role of averaging (“[For] a better estimate you have to have an average.”), the context in which the sample was drawn (“Robert’s methods would be better if…the big squares had the same number of dots each time.”) and the relationship between the samples (“Is Jeff’s method just… making the square ten times larger?”) While aware of the productive discussion taking place, Mr. Louis also realized that the bell would soon ring. He encouraged students to continue thinking about the lesson: “There’s still a lot of really rich math in here, so let’s try to think about what we think here.” He then assigned students their homework, which included describing a way to estimate the number of blades of grass on a football field and selecting an effective sample to use to determine the favorite rock band of students at their school.

Research Paradigms in the Study of Mathematics Teaching

The prevailing paradigms of research in a given field at any given time heavily influence and serve to organize the particulars of research carried out in that field. Research on teaching in general and mathematics teaching in particular is no exception. Here we review some of the major trends and traditions in the history of research on teaching to provide a background against which we can better understand current research on mathematics teaching expertise.

Process-Product Research

Early research on teaching was driven by a desire to identify relationships and find connections between classroom teaching and students’ learning. Described as “process-product research,” these studies sought to answer questions that took the general form of “What characteristics of teachers and teaching are linked, in some causally relevant way, to desired student outcomes?” (Floden, 2001, p. 7). To answer these questions this work focused largely on observable behaviors of teachers and students in classrooms. Researchers would choose particular behaviors or attributes of teachers to examine (e.g. "experience") and find ways to quantify those variables (e.g. number of years of teaching) while simultaneously observing and measuring outcome variables in students (e.g. scores on achievement tests). This work then aimed to discover effective teaching strategies by documenting large numbers of classrooms and identifying correlations and covariations between what the teachers did and what the students learned (Rex, Steadman, & Graciano, 2006).

One assumption of this work was that effective teaching strategies were domain general; researchers could look across teaching in different domains and make generalizations about what teaching expertise looked like overall. Thus data from mathematics classrooms was combined with data from science and history classrooms in order to perform these large-scale correlational studies. For example, one productive line of research grounded in the process-product tradition revolved around "wait time" (Rowe, 1974). In this work, the amount of time teachers wait after asking a question and before evaluating a student response was examined in relation to the frequency and complexity of students' responses. Other research in this tradition explored the influence of various classroom management techniques as well as the influence of different types of teacher questions on student responses (Dunkin & Biddle, 1974). In this work researchers observed teaching practice and compared it to measurable student outcomes using domain general variables such as "wait time," management techniques, or types of questioning.

Let us now consider our vignette from the perspective of a researcher working within the process-product paradigm. What slice of Mr. Louis' practice would be of interest to this researcher? Likely he would seek to isolate, observe, and quantify individual features of the instruction or of Mr. Louis himself that contributed to his students' success or failure in the classroom. For example, the data might be used to explore questions such as: Does the number of times a teacher asks his students to explain their ideas – as Mr. Louis does with Tina, Robert, and Jeff – impact the students' achievement on a test of the same material? When teachers have students work in small groups to solve problem – as Mr. Louis does with this problem – are students more likely to turn in a correct problem solution? Furthermore, in seeking to answer these questions, it would be assumed that the answers are not domain-specific. So data from Mr. Louis' mathematics class might be pooled with data from science and social studies classrooms.

A First Look into the Mind of the Teacher

By the 1980s a new paradigm grounded in the intellectual traditions of cognitive science and psychology began to drive research on teaching. Rather than observing and describing teacher behaviors, cognitivist researchers sought to generate accounts of teacher knowledge and thinking. This "approach to the study of teaching assume[d] that what teachers do is affected by what they think" (Clark & Yinger, 1987, p. 231). What might now seem relatively obvious was, against the backdrop of behaviorism, revelatory. Indeed, the promise seemed to be great. It was hoped that researchers might gain more traction in understanding teacher practice if instead of just directly describing behaviors, as research in the process-product tradition had done, research first tried to understand the thinking of the teacher that gave rise to that behavior. This work looked in particular at "three fundamental types of cognitive processes" of teachers including "studies of [teachers'] judgment and policy, of problem solving, and of decision making" (Shulman, 1986, pp. 23–24).

An example of cognitivist research on teaching is the study of teacher planning. For example, Peterson and Clark (1978) interviewed teachers following instruction as a way to explore the relationship between the teachers' goals for a lesson and their decisions about adapting the lesson during instruction. In related work, Clark and Yinger (1979) identified different goals that teachers had in mind as they planned for instruction (e.g., planning in order to structure a lesson versus planning in order to develop an appropriate assessment activity). For researchers within this paradigm, providing detailed accounts of teachers' cognition was essential to understanding and making sense of their classroom teaching. However, these accounts of teacher cognition were still domain general; differences in domains were not considered relevant for examining teacher thinking.

We return again to our vignette to demonstrate how early cognitivist researchers might have attempted to make sense of Mr. Louis' teaching. What slice of our vignette might they have chosen to focus on? To start, such researchers may have sought to uncover Mr. Louis' plans for instruction, his "lesson image" (Morine-Dershimer, 1978–79), and points in the lesson where Mr. Louis expected to shift from one activity to the next. They may have shown Mr. Louis portions of the lesson after instruction with the goal of having Mr. Louis reconstruct his thinking at particular points in time. How did he decide when to move from small group work to the group presentation? Did he have in mind a particular "wrap-up" for the lesson that he then abandoned given time constraints? As in the process-product tradition, these analyses of Mr. Louis' teaching would not be substantively affected by the fact that he teaches mathematics.

A Focus on Subject-Specific Teaching

As the cognitive revolution unfolded over the middle and latter twentieth century, one lesson was clear: Looking across multiple populations and diverse fields, it was repeatedly established that expertise is profoundly domain-specific (Glaser & Chi, 1988). To exhibit expertise in a domain, an expert must acquire a body of knowledge that is specific to that domain. As Shulman summarized it, "the thrust of the cognitive science research program in learning is subject matter specific rather than generic" (Shulman, 1986, p. 25). The implication of this work for teaching was doubly significant. First, it implied that we must expect teaching expertise to exhibit the same kind of domain-specificity as any other discipline. Second, and more subtly, teaching is a discipline that is concerned with helping others – students – to acquire expertise. If student reasoning depends on domains then what teachers must do to influence that reasoning will likely also depend on the domain. Thus, research into effectiveness in teaching, in addition to focusing on cognition, needed to focus in particular on domain-specific cognition.

Shulman (1986) led the field's advance into domain-specific cognitive research on teaching. In particular, he contrasted teachers' subject matter knowledge with what he called their "pedagogical content knowledge." Subject matter knowledge, according to Shulman, concerned one's understanding of the facts and concepts within a domain, while pedagogical content knowledge, on the other hand, had to do with an understanding of how to teach those facts and concepts. A wealth of researchers elaborated on Shulman's claims in the area of mathematics instruction, identifying pedagogical content knowledge in varied domains such as elementary fractions (Marks, 1989) and functions (Even, 1993). Others looked closely at the role of pedagogical content knowledge during instruction, making claims that the depth of one's pedagogical content knowledge is what characterizes the accomplished mathematics teacher (Borko et al., 1992; Putnam, 1992; Sherin, 2002). The assumption behind all of this work is that pedagogical content knowledge is inherently domain specific and crucial for successful teaching practice.

Let us now return again to the case of Mr. Louis. What slice of his teaching would cognitivists committed to domain specificity examine? Researchers from this tradition would be particularly interested in the thinking that Mr. Louis does that is mathematical in nature. For example, they might ask: What knowledge did Mr. Louis use that allowed him to see Jeff and Robert's ideas as competing alternatives? What did Mr. Louis know about students' common misconceptions in mathematics that caused him to select this particular problem to help them understand sample size? For these researchers, answering such questions would likely involve examining the classroom activity in detail and interviewing Mr. Louis about his thinking both in the moment of instruction and during his planning.

A Situative Perspective on Teaching

Like all successful research paradigms, the cognitive perspective engendered a backlash of sorts. At the heart of this backlash was the sense that, in the cognitive tradition, too much explanatory emphasis was located on the in-the-moment cogitations of individual actors. Instead, it was argued, a perspective is needed in which the individual is understood as embedded in physical and social systems, spread over space and time. This perspective has been known by many names; in its more recent incarnations, the names *situated cognition* and *situative perspective* are common. Though the situated perspective surged to prominence in the 1980s and 1990s, it traced its lineage to older traditions, including the instrumental psychology of the Soviet psychologists, in which thinking was thought to arise first on an interpsychological plane (e.g., Vygotsky, 1978).

Adopting a situative perspective has led researchers to see the mathematics classroom as a place and community with a history, and to focus on interactions among teachers, students, and artifacts. Studying mathematics teaching expertise then involves studying, for example, the roles of participants in the classroom discourse (e.g., Moschkovich, 2007; Sfard, 2007), how artifacts and ideas are taken up among community members (e.g., O'Connor, 2001), and how the teacher establishes an environment in which responsibility for learning is shared among participants (e.g., Silver & Smith, 1996).

We return to our vignette once more to demonstrate the focus of attention of situative researchers. These researchers would be interested in questions such as: How does the interaction between Mr. Louis and his students give rise to the various approaches to the problem that are voiced? How do the artifacts and representations used in the classroom mediate or afford the learning that occurs? When and how is new knowledge and language appropriated by Mr. Louis' students? Close examination of students' work in the small groups and their discourse during the large group discussion would be crucial to this analysis, including studying issues of power and agency, the identities and roles the students and Mr. Louis develop or adopt during the course of the lesson, and the negotiation of norms of participation and representation in the classroom. In addition to analyses of this particular moment from Mr. Louis' teaching, those with a situative perspective would also be interested in the history of the class and the students themselves, and how that history impacts what occurs in that moment.

Design Research: Teaching as Curriculum Adaptation

Among the more recent trends to influence research on teaching expertise is what has been referred to as *design research.* Unlike the shifts described above, the design research perspective does not constitute a fundamental change in the way that human reasoning or social systems are understood. Rather, it represents a change in how we conceptualize the relationship between research and practice. In some respects, the relationship between research and practice is seen as *more* intimate. In design research, design and theory development are carried out in tandem, and the boundary between research and design is essentially eliminated (Edelson, 2002). In other respects, the relationship between theory and practice is understood to be loosened. It is explicitly recognized that designs are just that – designs – and that theories of learning do not come close to determining all aspects of an instructional design (Brown & Campione, 1996) Additionally, the design research perspective emphasizes that educational theories and designs must be portable in the sense that they can survive diffusion into the world.

The design research perspective can be seen as having a variety of impacts on the way we understand the nature of teacher expertise. Unlike the paradigms described in the preceding sections, this perspective does not draw our attention primarily to the reasoning and acting of the teacher. Instead, we are led to view the teacher through the lens of the larger instructional system in which the intentions of a curricular designer are brought to bear on students. More specifically, the teacher is understood as playing a particular role within this larger system, as the interpreter and applier of curriculum materials. Within this perspective, research on mathematics teaching focuses on patterns in teachers' use of curriculum materials. For example, Remillard (2005) examines the cognitive resources teachers bring to the work of enacting curricula in their classrooms. In other work, Sherin and Drake (2009) document the different ways that elementary mathematics teachers read, evaluate, and adapt a new mathematics curriculum.

We can return once more to our vignette with the design research perspective in mind. Design researchers entering Mr. Louis' class would likely not be content to observe and analyze only what occurred during his lesson enactment. Instead they would examine how the lesson was enacted as compared to how it was designed and seek to understand Mr. Louis' reasons for adapting the lesson as he did. The design researcher would be interested both in the adaptations Mr. Louis made while planning before the class and those he made in the moments of instruction. For example, Mr. Louis had students discuss their ideas with the entire group. A design researcher would examine the curriculum documents to identify whether this was a change from the original design. If so, why did Mr. Louis change this aspect of the design? When did he decide to change the lesson? Did this change maintain the original goals of the curriculum designers? Design researchers would seek to understand Mr. Louis' teaching as part of a system that includes not only the teacher and the students but also the curriculum designers and the curriculum itself.

Current Research on Mathematics Teaching: Four Images of Expertise

In the preceding sections, our perspective was historical; we attempted only to capture the trends in the field – changes to the broad landscape of education theory – and the new understandings of expertise in teaching that grew out of these changes. In doing so, we essentially treated the field as monolithic. Of course, at any point in time, there is variability among researchers. In this section, we turn to the present day, and we attempt to paint a picture of the variety that exists.

Capturing this diversity in a meaningful way is challenging. The perspectives adopted by researchers are changeable, and boundaries are never clear. To paint our picture we present four *images* of mathematics teaching expertise. Each of these images encapsulates an orientation toward mathematics teaching expertise, and each highlights some facets of expertise and ignores others. To help clarify the differences among these images, we will highlight the kinds of questions that each image might pursue relative to the crowd estimation lesson.

Mathematics Teacher as Diagnostician

One way to conceive of mathematics teaching today is that the central role of the teacher is as a diagnostician. The teacher, like a doctor or mechanic, must examine the mathematical thinking of students, look for symptoms (e.g., wrong or surprising answers), and diagnose their underlying cause (e.g., a faulty conceptualization). Thus, the emphasis here is on the need for teachers to be able to discern the meaning of the mathematical ideas and methods that students raise in class. This aspect of teaching practice has been referred to in a number of ways including “sizing up students’ ideas” (Ball, Lubienski, & Mewborn, 2001), “observing student reasoning” (Kazemi & Franke, 2004), and “drawing inferences about student talk” (Hammer & Schifter, 2001). The skill needed to interpret students’ mathematical ideas should not be underestimated (Even & Wallach, 2004). Students’ ideas can be quite complex, and students do not always articulate their thinking clearly. Furthermore, teachers are often expected to make sense of a student’s idea quite quickly and with little in the way of resources that might offer potential interpretations for the teacher to consider. Wallach and Even (2005), for example, warn of the potential for teachers to *under-hear* or *over-hear* as they work to make sense of the methods students share in class.

Looking at Data

Researchers who adopt the stance of mathematics teacher as diagnostician tend to look closely at interactions between teachers and students around specific mathematical content. They might focus, for example, on the questions teachers ask students about their ideas or on the explanations teachers provide about students’ methods. This approach is strongly connected to the cognitivist’s commitment to subject-specific cognition. The assumption is that the process of diagnosis involves looking at mathematical content in a very detailed and up-close manner.

Studying Teacher Expertise

A focus on the teacher as diagnostician leads researchers to several related lines of inquiry. One area of study examines what teachers understand about student thinking in particular mathematical domains. For example, Even and Tirosh (2002) discuss the extent to which seventh-grade teachers recognize students’ tendency to simplify algebraic expressions without regard to “like terms.” Similarly, Son and Crespo (2009) examine how elementary and secondary teachers reason about a novel student method for dividing fractions. Closely related to such research are investigations of what teachers’ themselves understand about various mathematics topics (see, for example, Borko et al., 1992; Stephens, 2008). The idea here is that the ways teachers diagnose students’ ideas rest heavily on the teachers’ own understanding of the mathematical content.

In other work, researchers aligned with the teacher-as-diagnostician perspective delineate categories of knowledge that support teachers’ ability to interpret students’ thinking. This line of inquiry builds directly on Shulman’s (1986) introduction of pedagogical content knowledge. For instance, Ball, Thames, and Phelps (2008) define “specialized content knowledge” – a “kind of unpacking of mathematics” (p. 402) that allows teachers to, for example, identify common student misconceptions and decide whether or not a novel student method is generalizeable. In other work, Ma (1999) explains that teachers who possess “knowledge packages” (p. 118) – collections of mathematical concepts that a teacher views as strongly connected – are able to provide in-depth, conceptually-based responses to scenarios describing student misconceptions. For these researchers, what is of interest is the kinds of mathematical knowledge that teachers draw on to successfully diagnose students' thinking.

A third line of inquiry revolves around researchers' efforts to help teachers become more effective diagnosticians. For example, the Cognitively Guided Instruction project organized professional development for elementary school teachers around students' understanding of addition and subtraction word problems (Carpenter, Fennema, Peterson, & Carey, 1988). Franke, Carpenter, Levi, and Fenemma (2001) report that, as a result, most participants learned to listen carefully to their students' ideas and that, in some cases, knowledge of students' thinking became generative for the teachers. In other words, teachers' ability to analyze students' strategies influenced the teachers' own learning of mathematics and informed their instructional decisions.

The Crowd Estimation Lesson

We now consider how a researcher focused on studying the ways mathematics teachers diagnose student thinking might examine the Crowd Estimation lesson. Of particular interest to the researcher would be ongoing evidence of Mr. Louis working to understand the ideas that students share in class. For example, during the initial presentation, Mr. Louis requested clarification of the group's approach, asking if Jen "would write some of this down for us" and explain, "What did you do after that?" Shortly after, when Robert suggested an alternative, Mr. Louis probed, "That's interesting. Why do you say that?" Similarly when Amy commented that Robert's method was good because of the bigger squares, Mr. Louis asked her to expand, "Why would that make a difference?" One way to understand Mr. Louis' frequent elaboration requests to students is that he is seeking more information from which to diagnose their thinking. Furthermore, the researcher might also be drawn to particular moments in the lesson where Mr. Louis appears to be drawing on his knowledge of mathematics to diagnose students' ideas. For example, Mr. Louis' understanding of ratio and proportion was likely an important resource in understanding the difference between Robert's and Jeff's methods. Similarly, his pedagogical content knowledge likely played a role in how he chose to represent Robert's and Jeff's methods visually for the class.

Mathematics Teacher as Conductor

A second way to conceive of mathematics teaching expertise is to imagine the teacher as a conductor, directing and shaping the classroom discourse. By many accounts, discourse is an essential component of mathematics instruction today (e.g., National Council of Teachers of Mathematics, 2000). Classroom discourse communities support student participation in important mathematical practices including explanation, argumentation, and justification. Furthermore, research has demonstrated that classrooms in which students regularly talk about mathematics provide valuable access to multiple ways of thinking about and solving problems.

At the same time, managing classroom discourse effectively is not a simple task. As Stein, Engle, Smith, and Hughes (2008) explain, "A key challenge that mathematics teacher's face. . .is to orchestrate whole-class discussions. . .in ways that advance the mathematical learning of the whole class" (p. 314).

Looking at Data

Researchers who draw on the perspective of mathematics teacher as conductor typically focus their investigations on the conversations that take place during class. They often look closely, for example, at who speaks and when, how teachers elicit comments from students, the kinds of questions teachers (and students) ask, and what counts as a valid explanation in a given discussion. This perspective draws heavily on both the cognitive and situative paradigms for teaching. Discourse is thought to involve thinking and meaning making on the part of the teacher; at the same time discourse arises from communities and marks membership in that community (Moschkovich, 2007).

Studying Teacher Expertise

Despite a common focus on the teachers' role in classroom discourse, researchers adopting this stance explore several different lines of inquiry. First, a number of studies investigate stages through which teachers move as they develop their abilities to effectively facilitate mathematical discourse. For instance, Smith (2000) describes key phases in the development of a middle-school teachers' questioning techniques. In other work, Hufferd-Ackles, Fuson, and Sherin (2004) introduce a four-step process of developing a "math-talk-learning community" (p. 4) in which discourse shifted from teacher-directed to student-directed, and from a focus on answers to a focus on mathematical thinking. The emphasis in all this work is on the development of the teacher's expertise as conductor of classroom discourse.

A second, related, approach concerns the teacher's use of classroom norms for communicating about mathematical ideas. Emphasis is on what Yackel and Cobb (1996) define as "sociomathematical norms," shared understandings of what "counts as mathematically different, mathematically sophisticated, mathematically efficient, and mathematically elegant" (p. 461). Of interest then, is uncovering how teachers lay the groundwork for establishing such norms. For example, early in the year Lampert (2001) explicitly encouraged students to add to one another's ideas in order to establish among the class appropriate ways to respond to, and even challenge, a person's ideas.

Third, some researchers who align with the teacher-as-conductor perspective primarily investigate patterns in the ways that teachers engage in classroom discourse. For example, Forman, Larreamendy-Joerns, Stein, and Brown (1998) describe the use of *revoicing* and *filtering* in order to highlight and clarify students' contributions. In other work, Williams and Baxter (1996) and Nathan and Knuth (2003) illustrate teachers' use of *analytic* and *social* scaffolding design to support worthwhile discussions of student mathematical thinking during class discussions. This approach to research emphasizes the different strategies used repeatedly by teachers in their role as conductors.

The Crowd Estimation Lesson

So how would researchers aligned with the teacher-as-conductor view of teaching expertise analyze the Crowd Estimation lesson? To start, a single lesson would not provide sufficient evidence to allow for an investigation of the development of whole-class discourse in Mr. Louis' classroom. While the researcher might be able to draw a few related conclusions from the data (e.g., that this type of discourse was familiar to students), without access to a series of discussions facilitated by Mr. Louis such analysis would be difficult.

At the same time, the lesson does provide a rich context for examining other aspects of the teacher-as-conductor perspective. First, there is evidence of several sociomathematical norms in place, norms that are mediated by the teacher. For example, Mr. Louis elicited multiple solutions to the estimation task from students, and each strategy was allotted time for discussion. Researchers might also explore patterns in Mr. Louis' discourse with the class. For example, Mr. Louis consistently encouraged students to comment on each other's ideas. He did this by following up a student's comment with a general question to the class: "What you guys think about Robert's idea?" "What do other people think?" He also regularly asked students to explain each other's ideas and strategies: "Can someone summarize what John said?" "What is Jared trying to say?" In exploring these patterns, researchers would try to characterize the nature of the expertise needed to effectively take on the role of teacher as conductor.

Mathematics Teacher as Architect

A third way to conceive of mathematics teaching expertise is that of the teacher as architect. Of central concern in this perspective is the teacher's role in selecting and implementing curriculum materials. Curricula are viewed as the primary vehicle through which policy and reform recommendations reach students (Sykes, 1990). Yet at the same time, a wealth of research emphasizes that curricula are not teacher-proof, and that instead, as teachers use curricula they necessarily interpret and adapt the materials for their own use (Lappan, 1997a). The perspective of teacher-as-architect emphasizes that effectively supporting student learning of mathematics requires expertise on the part of the teacher both in choosing tasks to use with students as well as deciding how those tasks should be carried out.

Looking at Data

Researchers who adopt the perspective of mathematics teachers as architects tend to look closely at one or more of several different activities in which teachers engage around the use of curriculum materials. For instance, researchers may focus on the process through which teachers plan for instruction, or reflect on lessons post-instruction. Alternatively, they may investigate particular components of curriculum implementation. What is of interest is the reasoning that teachers engage in as they design instruction. The teacher-as-architect stance draws on the perspectives of both situated cognition and design research. In line with situated cognition, this approach recognizes that curriculum materials are mediating tools used by teachers to accomplish their goals (Brown, 2009). In addition, in line with design research is the idea that teachers are consumers and adapters of designs, as well as designers of classroom activity themselves. Even when using published curriculum materials, the process through which teachers take the *page as written* and move to the *lesson as enacted* can be thought of as a process of design (Silver, Ghousseini, Charalambous, & Mills, 2009) As Brown (2009) explains, "Teaching by design is not so much a conscious choice as an inevitable reality" (p. 19).

Studying Teacher Expertise

The focus on mathematics teacher as architect has increased in popularity over the last 15 years and has resulted in several related lines of inquiry. One approach examines the extent to which the mathematics activities selected by teachers represent *cognitively demanding tasks* – "problems that promote conceptual understanding and the development of thinking, reasoning, and problem-solving skills" (Stein et al., 2008, p. 315). Along the same lines are studies that examine whether teachers maintain a high level of cognitive demand as a task is carried out (e.g. Smith, 2000). Such research seeks to understand the expertise needed to carry out a mathematics lesson in ways that maintain the integrity of the planned lesson.

In other work, researchers characterize teachers' typical approaches to using mathematics curriculum materials. For example, Remillard and Bryans (2004) define one group of teachers as "thorough piloters" who allowed the published materials to generally guide the structure of lessons in contrast to another group's "intermittent and narrow" use of the same curriculum (p. 375). Similarly, Nicol and Crespo (2006) identified different ways that teachers adapted a traditional mathematics curriculum: by extending activities suggested by the text or by creating new problems and questions to insert in lessons. By looking at the impact of these different approaches on instruction, researchers attempt to uncover some of the expertise involved in designing and implementing effective mathematics lessons.

The Crowd Estimation Lesson

A researcher drawing on the teacher-as-architect perspective would likely find several aspects of the Crowd Estimation lesson of interest. One issue might be how Mr. Louis organized the lesson – with students initially working in groups, then a student presentation followed by a whole class discussion, and finally with related homework problems assigned. How did this structure serve to meet Mr. Louis' goals for the lesson? Did Mr. Louis consider brainstorming strategies as a whole class first, and then having students work in groups to pursue some of the strategies in more detail? How did the mathematical content of the lesson as well as students' experience with similar tasks influence his decisions? The researcher might also want to explore Mr. Louis' choice to have Jen's group present their method to the class. Earlier, Mr. Louis circulated throughout the room as students worked on the task in groups. Were there certain features of Jen's group's method that Mr. Louis wanted the class to see, and wanted the class to see first? How did his choice of Jen's group enable or constrain Mr. Louis to move forward with his planned goals for the lesson? While the lesson itself might provide some evidence related to these issues, the researcher would likely want to interview Mr. Louis to examine both of these issues in depth. In doing so, the researcher would endeavor to uncover ways in which Mr. Louis' expertise enabled him to serve in the role of lesson architect – designing and carrying out the lesson in ways intended to support student learning.

Mathematics Teachers as River Guide

A fourth way to conceive of mathematics teaching expertise is that of the teacher as a river guide, as one whose job it is to be flexible in the moment. Like a river guide, a teacher has a carefully crafted plan; the "river" in this case is a lesson that has been carefully reviewed and whose contingencies have been considered. Yet the river guide's true expertise comes to light during the ride, when the rapids change, or a paddler makes an unexpected move. It is the river guide who must respond quickly and effectively. In the same way, teaching expertise can be viewed as being responsive to the context, to students, and to what occurs in the moment (Berliner, 1994). Our use of the river guide metaphor is intended to emphasize that teachers are on the river with the students. We think of them not just as leading students down the river but also as actively engaged with students in the journey.

Looking at Data

Researchers who adopt the perspective of mathematics teacher as river guide typically focus their investigations on the interactions in the classroom. In particular, they try to identify moments of instruction in which teachers make on-the-fly decisions about how to proceed. Through videotapes of instruction and/or interviews with teachers, the researcher will explore, for example, moments in which teachers deviate from their planned lessons, respond to unexpected student ideas, or adapt an activity in the midst of instruction. This perspective draws on both the cognitive and situated paradigms for teaching. From the cognitive perspective, the teacher's expertise as river guide is reflected in the teachers' understanding of subject matter, students, and so on. From the situated perspective, it is reflected in the way expert teachers react to and fluidly operate within changes in the setting and context.

Studying Teacher Expertise

A focus on the teacher as river guide leads researchers to engage in several related lines of inquiry. One approach involves exploring the nature of improvisation as it is exhibited in the act of teaching. For instance, Sawyer (2004) defines teaching as "improvisational performance" and examines the knowledge teachers draw on as they "think quickly and creatively" during instruction (p. 15). In other work, Heaton (2000) studied the process through which her own mathematics teaching was transformed as she came to "appreciate teaching as an improvisational activity" (p. 60).

> [Today] I moved away from the scripted lesson and made a move that went beyond asking children to explain their thinking. I was connected to the work of teaching in ways that I had not experienced before in mathematics. . .For a moment I was no longer in role of silent bystander. I took control. I knew what I was doing. For a moment, I was teaching. (p. 59)

The emphasis here is the idea that teaching expertise necessarily involves improvisation, deciding in the moment how to respond to the unfolding lesson.

Another approach that draws on the notion of teacher as river guide involves trying to model the on-the-fly decision-making process in which teachers engage. For the example, Schoenfeld (1998) illustrates that a mathematics teacher's actions can be modeled as a reaction to existing beliefs, knowledge, and goals. In particular, he demonstrates how these resources come into play when something unexpected happens in the classroom. Relatedly, Artzt and Armour-Thomas (2002) suggest that teachers engage in cycles of active monitoring and regulating during instruction that are mediated by their beliefs, knowledge, and goals. This line of work emphasizes the role of cognitive resources in enabling teachers to quickly and effectively respond to classroom activity.

Third, some researchers who align with the teacher-as-river-guide perspective focus specifically on the *noticing* that mathematics teachers engage in during instruction (Jacobs, Lamb, & Philipp, 2010; Mason, 1998; van Es & Sherin, 2008). The idea is that because the classroom is a complex environment with multiple events happening at the same time, the teacher cannot pay equal attention to all that is taking place. Instead, a key component of teacher expertise involves deciding where to focus one's attention and, according to Mason (2002) preparing oneself to attend to particular kinds of events. Building on Goodwin (1994), Sherin (2007) refers to this as "teacher's professional vision" – the ability of teachers to identify significant events in the classroom. In this strand of work researchers examine teachers' abilities to parse and make sense of classroom activity, which in turn allow teachers to be responsive to issues as they arise.

The Crowd Estimation Lesson

Returning to the Crowd Estimation Lesson, how might researchers aligned with the teacher-as-river-guide perspective examine the lesson? One event that would likely capture their attention is Mr. Louis' decision to put Robert's and Jeff's ideas before the class for comparison and further elaboration. "We have two competing ideas here." This is certainly a decision made by the teacher in the moment of instruction; Mr. Louis could not have known beforehand precisely what ideas would be raised in class, and in what ways. Instead, in the midst of instruction, with all that is taking place, Mr. Louis likely recognized some features of Robert's and Jeff's strategies that he believed would be worthwhile for the class to investigate. "We have Robert who says this. . .take a larger sample.. . . Jeff said something a little different. 'Take 10 squares like this and average them together.' What is Jeff saying. . .that we do?" In exploring this episode from the lesson the researcher would try to uncover what about Robert and Jeff's ideas peaked the teacher's attention in that moment and how the teacher quickly made the decision to juxtapose those ideas against one another.

Discussion

"It was December, and Mr. Louis' 4th period class was nearing the end of a unit . . ..." Thus began our summary of a single episode from a mathematics classroom. Throughout this chapter, we only used this one vignette as a reference point. But even this short vignette was enough to support numerous perspectives on the mathematics teaching expertise possessed by Mr. Louis.

In some respects, this chapter may be understood as a "review of the literature." But the expansive nature of our subject matter (how the field has conceptualized mathematics teaching expertise) and the limits of space (the usual chapter in an edited volume) required that our "review" take a somewhat non-traditional form. This was particularly true of our portrait of research on mathematics teaching expertise as it exists in the present day. There, our review centered around four "images" of the mathematics teacher: *diagnostician, conductor, architect,* and *river guide.* Looking back at these images, we realize that the need to be concise has led us to undertake a productive exercise. We have come to believe that it is productive to see researchers as adopting one or more of a moderately small number of images of mathematics teaching. This recognition helps us to understand some of the diversity in the field, as well as why research has clustered in some areas. To be clear, while we have described these images as independent, they are certainly related. Moving forward, we can imagine it would also be productive for the field to explore the ways that these four images are related.

Another way to understand the ideas presented in this chapter is as a "meta" analysis of mathematics education research on teaching expertise. Just as it is useful for our students of mathematics to be aware of their own thinking, we believe that it is useful for us as researchers to be aware of the perspectives that we adopt in our work, whether explicit or implicit. We expect that this will be particularly important as our field continues to move forward. As new paradigms for understanding the complex environment of the classroom emerge, so also will new images of expertise. An awareness of those images of expertise that currently exist will help us both recognize and situate new images, allowing us to use them in productive ways to further understand the work of mathematics teaching.

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