CHAPTER 7

THE TRANSPORT OF HEAT

In this chapter, we shall take a closer look at the transport of entropy. Simple aspects will be introduced that go beyond what we already studied in Chapter 4 (Section 4.6). This extends the treatment of thermal processes into the realm of phenomena which are missing from the theory of the thermodynamics of ideal fluids (Chapter 5). Many texts on thermodynamics and on heat transfer sharply distinguish between the two subjects, which only emphasizes that a unified presentation of all thermal phenomena is called for. While we will not achieve the stated goal in this chapter, the ground will be prepared for a theory of continuum thermodynamics of which we will get a first glimpse in Part III.

The first section of this chapter provides a qualitative description of the three types of entropy transport: *conduction*, *convection*, and *radiation*. It introduces the formulation of the law of balance of entropy for a uniform body. Then, simple applications of all three forms of heat transfer will be discussed, giving an overview of some practical problems. Flow systems, i.e., open systems where fluids transport dissolved substances, entropy and momentum will be introduced in Chapter 8. There we extend the notion of chemical potential to fluids in flow systems.

Entropy production in heat transfer will be considered as we go along, preparing the ground for the concept of minimization of irreversibility in thermal design which will be applied in Chapter 9.

7.1 TRANSPORT PROCESSES AND THE BALANCE OF ENTROPY

In this section, I will describe qualitatively the basic phenomena underlying the transport of heat. Simple observations tell us that entropy can flow in three different ways: *conduction*, *convection*, and *radiation*. Consideration of these types of transport will lead to the formulation of the law of balance of entropy in a more general form than previously encountered, and will yield a better understanding of the role of hotness in thermal processes. In the end, the equation of balance of entropy will contain terms describing the different modes of transport.

These types of transfer processes are found not only in thermal physics, but in other fields of the natural sciences as well. Momentum transports have been classified in the same manner in Chapter 3. For this reason alone, it is important to have a clear understanding of the nature of entropy transfer.

H.U. Fuchs, *The Dynamics of Heat: A Unified Approach to Thermodynamics and Heat Transfer*, Graduate Texts in Physics, DOI 10.1007/978-1-4419-7604-8_8,

7.1.1 Conductive Transport of Entropy

Heat one end of a metal rod over a flame; in a very short time the other end will feel hot as well. If you throw a hot stone in cold water, it will cool down while the water gets warmer. In a heat exchanger, a hot fluid flows through pipes, heating a cooler fluid which flows around the pipes. In all of these examples, entropy is removed from some bodies and added to others. Why else should some objects become colder while others heat up? The possibility of changing the temperature by compression, i.e., adiabatic processes, does not occur in these examples. Therefore we say that entropy has been transferred. Obviously, entropy flows from hotter to colder bodies.

How is entropy transported in these examples, and what are possible conditions for this process to occur? First, we observe that material transport cannot be involved. A piece of metal heated at one end retains its integrity. A hot stone does not dissolve in water, thereby spreading the entropy it contains. In the case of the heat exchanger, it is true that the fluids move; however, entropy must be transferred through the walls of the pipes. Heat therefore flows *through* bodies *without the help of a body transporting it*, and it flows from one body to another if the two are brought in *direct contact*. These are examples of heat conduction (*conduction* or *diffusion of entropy*).

An example that we studied in Chapter 4 tells us something about the role of temperature in the conductive transport of entropy. Two bodies having different temperatures are brought in thermal contact, and their hotnesses are monitored. It is found that the temperatures of the bodies change until they have become equal. As long as they are changing, entropy must be flowing: one of the bodies is cooled, the other is heated. In the end, however, the exchange stops. We conclude that entropy flows conductively as long as there is a difference of temperatures between the bodies exchanging heat, and that by itself, entropy flows only from hotter to colder objects.

Driving forces. This type of behavior is well known from a number of different physical phenomena. Connect two containers having different cross sections that are filled with water up to different levels; let the water flow between them. As a different example, connect two electrically charged spheres with a wire and monitor the electrical potential of each of the spheres. We know what will happen in both cases: the water levels in the containers will reach the same height, and the electric potentials of the two spheres will be the same after the process ends (Chapter 1). In each case, something flows as long as there is a difference of potentials, i.e., a *driving force*. In analogy to these well-known phenomena, we shall interpret the conductive transport of entropy as follows:

> *In conductive transport, entropy flows by itself through bodies from points of higher to points of lower temperature. In other words, entropy flows as long as there is a difference of temperatures, i.e., a thermal driving force.*

The balance of entropy. Conductive transport of heat is a prime example of an irreversible process (Section 4.6.3). A body conducting entropy produces more entropy at the same time. This must be so because in a steady-state process, the same amount of energy which enters the body at high temperature leaves it at a lower thermal level. Therefore, the current of entropy leaving the body must be larger than the one entering. Clearly then, the equation of balance of entropy must include the production term for entropy in addition to the term describing conductive transfer of heat into and out of the body:

$$\dot{S} = I_{S,cond} + \Pi_S \tag{7.1}$$

Here, $I_{S,cond}$ is the net current of entropy transported conductively with respect to the body in question. We call it the *conductive flux of entropy*.

Flow across surfaces. The conductive current is our way of describing a phenomenon in which we picture entropy to flow *across the surfaces of bodies*. If we are interested in the flow through a body we simply introduce imaginary surfaces inside. Again entropy flows across a surface where one part of a body touches another (Fig. 7.1). In this sense, conduction is a surface phenomenon, and it is rendered formal by a physical quantity, namely a flux I_S whose distribution over a surface is of prime interest (Chapters 11 and 13). We stress this point since a body can pick up or lose entropy in other ways (i.e., by radiation and sources of heat).

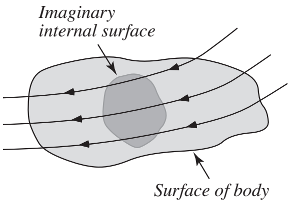

Figure 7.1: Conductive currents of entropy flow across surfaces. Such surfaces may be real surfaces of bodies or imaginary surfaces, such as those which we introduce to separate different parts of bodies. The flow lines in this figure do not reflect the fact that entropy is produced in conduction.

7.1.2 Transport of Entropy with Fluids: Convection

Northern Europe would be a pretty cold place to live were it not for the Gulf Stream, which transports huge amounts of heat from the Gulf of Mexico to the west coast of Europe. Our weather would be pretty dull were it not for the currents of hot or cold air in our atmosphere. These are just two important examples of a different mode of heat transfer. It is quite clear that in these cases entropy is transferred with the help of a material medium, like air or water. You can find examples all around you. Heated air rises from a radiator in a room; hot water which is pumped through the pipes of a central heating system delivers entropy to the radiators; water begins to boil at roughly 100°C, transporting entropy via a material current of steam. If entropy is carried by a material which is flowing we speak of *convective entropy transport*.

These examples demonstrate that convection is a very important phenomenon in our daily lives. We shall deal with some simple aspects of convection in Section 7.3. (More details will be provided in Chapters 8 and 14.) We are interested in a particular question at this point, the problem of the *driving force* of this type of heat transport.

The driving force. A difference of temperatures drives the conduction of entropy. You can easily see that this cannot be the driving force in the case of convection. The reason why hot water flows through pipes to your shower definitely cannot be found in a difference of temperatures: a pump drives the flow of water. The fact that the water is hot is immaterial to this transport phenomenon. We have to conclude that the cause of convective heat flow has to be sought in the driving force which lets the material substance (water, air, etc.) move: we know that this is a *pressure difference* set up by a pump or through some other device or process:

> *Entropy can be transported via a flowing substance. In this case, the flow of entropy is accidental. The driving force of the process is the difference of pressure which lets the material substance flow.*

There are some important examples of convection which might make us believe that a temperature difference must be the driving force of the process. Think of air rising above a hot radiator in a room. Also, the water circulating in a central heating system does so apparently because it is heated at one end (in the boiler). Indeed, the water does not flow if the heating is stopped.

Still, the immediate driving force for the flow of water (which is responsible for the transport of entropy in the system) is not a difference of temperatures but a *pressure difference caused by the heating*. The hot water in the boiler is slightly less dense than the surrounding liquid; therefore, it begins to rise as a consequence of buoyancy, which is a consequence of a pressure difference. The heating is responsible for the flow only in an indirect way. Another example is presented by our atmosphere: air can easily flow into a region where the temperature is higher. We call this phenomenon *free* or *natural convection* to distinguish it from convection induced by a pump, which is called *forced convection*.

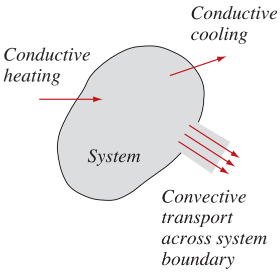

Figure 7.2: Entropy may flow across the boundary of a body either by conduction or by convection. In the former case, matter does not cross the surface and entropy flows through matter. In convection, a substance flows across the surface whereby entropy it contains is transported into or out of the system as well.

The balance of entropy. Since convection and conduction are obviously different types of entropy transport, we should distinguish between them. For this reason, we also introduce *convective currents* in the equation of continuity of entropy:

$$\dot{S} = I_{S,cond} + I_{S,conv} + \Pi_S \tag{7.2}$$

This equation tells us that the entropy content of a body can change as a consequence of two types of flow and the production of entropy (Fig. 7.2). Again, we are confronted with a surface phenomenon. Substance flows into and out of regions of space across surfaces, real or imagined. Just as in the case of conduction, we introduce fluxes to describe mathematically what is going on.

There is an important point to note. The transport of substance leads to changes of this quantity in regions of space influenced by the flow. Therefore, we have to be extremely careful to state what we are talking about, i.e., to identify the system (or element) for which we are performing a balance of entropy. So far, we have always used an identifiable material body as the physical system under consideration (Chapters 4 and 5). Such a body is assumed to retain its material integrity; i.e., it is not allowed to exchange matter with its surroundings. We shall continue to use the term *body* in this sense—an aggregate of matter which can always be identified and separated from the rest of the world. For a body such as a stone this identification is quite simple and clear. It is still simple in the case of air enclosed by rigid walls. In situations where matter flows, however, this becomes more difficult. Still, we may think of an identifiable amount of water moving with the flow of a river. This body of water is thought to be separated from the rest of the water by an imaginary surface which moves and deforms with the body (Fig. 7.3). If we have a body in mind, the balancing of quantities such as entropy always refers to this piece of matter. The time derivative of the entropy function (or of other functions) in Equ.(7.1) is taken for *the entropy of the body*. For this reason it is sometimes called a *material derivative*.

Control volumes. It is often more convenient to do the accounting with respect to a region of space rather than an identifiable body (Fig. 7.3). This is particularly true in cases where matter flows. Imagine a region of space surrounded by an imaginary surface. We often speak of a *control volume* and a *control surface* to distinguish it from bodies. A control surface may easily be penetrated by flows of matter, which leads to changes of the amount of substance in the control volume. This is the case if we consider convective currents, as we have done above. The time derivative of the entropy in Equ.(7.2) is not taken with respect to a body, but with respect to some control volume (which may be stationary or moving). The derivative, therefore, is of *the entropy of the control volume*. We will learn later how to distinguish mathematically between this derivative and a material one. By the way, systems which may exchange matter are called *open*, while those which do not are called *closed*. Bodies are closed systems by definition.

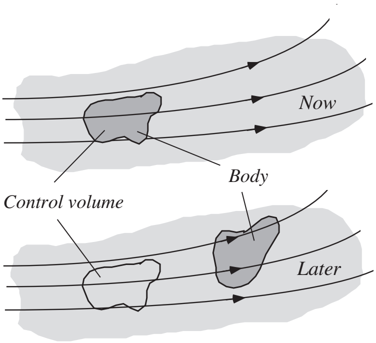

Figure 7.3: A body and a control volume in a general situation involving the flow of substance. A body moves and deforms with the flow. A control volume is any region of space, itself moving or stationary. In general, a control volume is penetrated by convective currents. In this example, the body and control volume occupy the same region of space initially.

7.1.3 Transport of Entropy with Radiation

It is obvious in some cases that entropy is transported neither by conduction nor by convection. Take the heat of the Sun, which travels to us through empty space, covering a distance of 150 million kilometers. It is clear that the Sun must radiate heat since it produces vast amounts of entropy all the time without changing noticeably. The transport cannot be via conduction. Also, there is no material substance which can act as a carrier of entropy in a convective process.

Heat which is emitted by warm bodies can even be photographed. You can see objects on infrared films. They look unfamiliar, but the process clearly is similar to photography with normal light. This suggests that there is a medium which transports heat in these cases after all. This medium would be similar to light. Indeed, this is the accepted picture: electromagnetic radiation (X-rays, ultraviolet, visible, infrared, or radio frequency) carries heat. Hot bodies emit electromagnetic radiation which then transports heat.

One group of phenomena is so pervasive that it makes us think that some bodies must radiate heat (entropy). (On closer inspection, however, you may realize that these phenomena are not the kind of proof we are looking for.) You can sit behind a glass window and feel the heat of the Sun's radiation. You can observe the same phenomenon when you sit by a fire; while all the heated air might go up the chimney, you still get hot sitting there. Meals can be kept hot by lamps, and again conduction or convection are not responsible for the flow of heat. The problem with these cases is that the heat felt by the bodies absorbing radiation may be produced inside them. Indeed, in the case of solar radiation, almost all the entropy which appears in a body absorbing the Sun's rays is created in the body (see Chapter 16 on solar radiation).

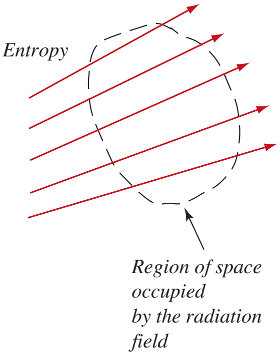

Figure 7.4: Imagine an empty region of space between the Sun and the Earth. Entropy is transported with radiation through this control volume. There is no difference between temperatures across the region.

Transport through the radiation field. We call this type of transport the *radiation of heat*. It is interpreted as the flow of entropy (and other quantities) through a physical system different from normal bodies, namely the *electromagnetic field*. The transport through the field takes place with radiation which can store and transport such quantities as entropy, momentum, and energy, just like ordinary materials. W can write down an equation of balance of entropy for the electromagnetic field in an otherwise empty control volume (Fig. 7.4):

$$\dot{S}_{field} = I_{S,rad} \tag{7.3}$$

The flow of entropy through the field is a surface phenomenon with currents flowing across imaginary surfaces drawn around regions of space. The amount of entropy in a region of space occupied by a radiation field changes as a consequence of the transport of entropy together with radiation into and out of the region (Fig. 7.4). The flow of heat through empty space is not dissipative. As a simple example, consider two imaginary spheres drawn concentrically around the Sun, the first near its surface, the second much further out. Later in this chapter and in Chapter 12 we will learn how to compute the flux of entropy through surfaces cutting through the radiation field. We will find that the same amount of entropy flows through both spheres in the same time span. Therefore the rate of production of entropy for a region of space which contains only the radiation field is zero.

It is interesting to ask whether we need a difference of temperatures for entropy to flow radiatively through the electromagnetic field. In fact, this is not the case. We associate the same temperature with the radiation which has just left the Sun and with the radiation that arrives at the Earth. In this sense, radiative transfer of entropy has much in common with convective transport. The driving force for the transport, if one is needed at all, is not the thermal driving force responsible for conduction. This is of profound importance for the determination of the relationship between fluxes of entropy and of energy (see Section 7.4). Put simply, there is a great difference between entropy flowing by itself in conductive transport, and entropy being carried by something else, be it water or radiation.

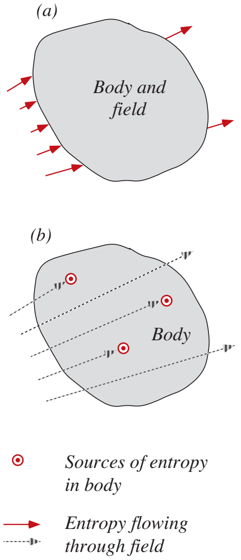

Figure 7.5: The same region of space as in Fig. 7.4 is now filled with matter such as air. Field and body can occupy the same space at the same time. For this reason their interaction takes place at every point inside the system. If we consider the material body only, we have to introduce sources of entropy where the body absorbs radiation from the field, and sinks where it emits entropy to the field.

7.1.4 Interaction of Bodies and Fields

Often, we are not interested in the transport of entropy through the radiation field but rather in the interaction of fields and bodies. The example of the Sun emitting radiation and of the radiation penetrating the Earth's atmosphere can tell us much about this interaction. The radiation which is not reflected back into space enters the atmosphere, where part of it is absorbed along the way to the surface of the Earth. We know from experience that only part of the radiation is absorbed; the rest reaches the surface. At the same time, the air must emit entropy since it cannot continually absorb radiation without getting hotter and hotter. Absorption and emission take place in every part of the atmosphere. This means that the radiation field pervades the air; it does not stop where the layer of air surrounding our planet begins. In other words, *the radiation field and the atmosphere occupy the same region of space* at the same time (Fig. 7.5).

The balance of entropy for body and field. To motivate the law of balance of entropy in the case of radiative transfer, we shall proceed in two steps. First, consider the combined system of matter and field occupying some region of space (Fig. 7.5a). In the case of solar radiation interacting with the Earth's atmosphere, this system absorbs some of the radiation flowing through the field. As far as the region of space is concerned, we have only radiative fluxes of entropy with respect to its surface. (Neglect for the moment that entropy may be conducted through air, and that air may flow through the system.) The entropy of the system may change only due to such radiative currents and the production of entropy in case of dissipation:

$$\dot{S} = I_{S,rad} + \Pi_S \tag{7.4}$$

Indeed, as we shall learn in Section 7.4, the absorption and emission of entropy are irreversible processes. For this reason, we may not neglect the production term.

However, we are often interested only in the balance of entropy with respect to the body alone (Fig. 7.5b). In this case, we have to consider the interaction of matter with the part of the field which occupies the same space. The interaction between the two, if it takes place at all, takes the form of absorption of radiation from the field by the body, or emission from the body to the field. Absorption and emission take place at every point in space occupied by the two systems. The properties of the body and of the field determine the amount of entropy which is absorbed or emitted.

Sources due to absorption and emission. Absorption and emission of radiation are not surface phenomena, but rather *volumetric processes*. In the absorption of radiation by the Earth's atmosphere there is no flow of entropy through this material body. Entropy enters the material system via the field, which means that there are *no currents through matter* associated with this type of transport. It simply appears at every point depending upon the degree of interaction. If we write an equation of balance of entropy for the material body only, we have to represent the interaction using a *source term* instead of currents:

$$\dot{S}_{body} = \Sigma_{S,body} \tag{7.5}$$

Here, $\Sigma_{S,body}$ is the *entropy supply* or *source strength of entropy* which is the net time rate at which entropy enters or leaves the body as a result of the interaction. The equation of balance of entropy of the field, on the other hand, must take the form

$$\dot{S}_{field} = I_{S,rad} + \Sigma_{S,field} \tag{7.6}$$

If we now combine the last two equations, we obtain:

$$\dot{S}_{body} + \dot{S}_{field} = I_{S,rad} + \Sigma_{S,body} + \Sigma_{S,field} \tag{7.7}$$

Comparison of this expression with Equ.(7.4) tells us that entropy must have been produced as a result of emission (or absorption) of radiation by the field and absorption (or emission) by the body. In other words, more entropy is absorbed by the body than is emitted by the field. The relation between the two source rates and the rate of production of entropy must be given by

$$\Sigma_{S,body} + \Sigma_{S,field} = \Pi_S \tag{7.8}$$

with the entropy being produced as a result of the transfer between the field and the body (Fig. 7.6). Note that we have to distinguish between sources of entropy due to entropy production and transfer between fields and bodies. Both effects are volumetric, in contrast to flows (conductive and convective).

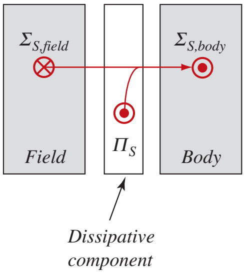

Figure 7.6: If we model the result of the transfer of radiation from the field to the body as uniform heating of the body, we have the problem of deciding where to include the source of irreversibility. The solution presented in the equations corresponds to introducing a dissipative component between the field and the body.

The general law of balance of entropy. If we now include the source term with the conductive and convective fluxes and the production of entropy in Equ.(7.2), we finally obtain the most general case of the equation of balance of entropy for a body:

$$\dot{S} = I_{S,cond} + I_{S,conv} + \Sigma_S + \Pi_S \tag{7.9}$$

This equation includes all the processes we are going to discuss. It expresses the fact that the entropy of a body may change as a result of three distinct types of transport: conductive, convective, and radiative, and the effects of irreversibility.

7.1.5 The Balance of Energy

One of the most important practical problems in the theory of heat transport is the determination of the fluxes and source terms of entropy in the equation of balance (see Equ.(7.9).) We have to find the constitutive laws which let us calculate these quantities in concrete situations. At this point, the energy principle will come to our aid. All three types of entropy transport are accompanied by the flow of energy. For this reason we should consider the law of balance of energy alongside that of entropy. Since energy is a conserved quantity, the amount stored in a body can change only by way of transfer to or from another system. The type of transfer of energy depends on the type of entropy flow. In the cases of conduction and convection, energy flows with entropy across system boundaries. This means that in these cases it is accounted for in terms of conductive or convective currents. If entropy is transferred radiatively, however, the interaction of bodies and fields leads to sinks or sources of energy in the body (or in the field). As a result of entropy transfer, energy either flows across system boundaries, or it pours into bodies via a radiation field. Therefore we distinguish between two types of currents and a source term of energy for material systems:

$$\dot{E}_{body} = I_{E,cond} + I_{E,conv} + \Sigma_{E,body} \tag{7.10}$$

For the radiation field alone, the equation of balance of energy must take the form

$$\dot{E}_{field} = I_{E,rad} + \Sigma_{E,field} \tag{7.11}$$

The last term on the right-hand side of Equ.(7.10) is the *source rate* or the *supply of energy*. Actually, in Equ.(7.10), we have neglected the transport of energy due to other processes such as mechanical ones. In the case of convective currents we cannot always do this. However, for the purpose of this chapter we shall regard such contributions as negligible compared to the other terms.

As we shall see, the relationship between entropy and energy in thermal transport phenomena will help us greatly in resolving the constitutive problem. To be specific, we are interested in a number of relationships, namely those between:

- Fluxes of entropy and energy in conduction.
- Fluxes of entropy and energy in convection.
- Fluxes of entropy and energy through the radiation field.
- Sources of entropy and energy.
- Energy and the production of entropy.

The following sections will in turn deal with different modes of transport. The one type of relationship between entropy and energy in thermal processes that we have considered so far (remember Equ.(4.21) in Chapter 4) is not of a general nature for transport. Rather, convection and radiation must lead to different expressions relating fluxes of entropy and energy.

In the following sections we shall introduce some simple aspects of all three modes of heat transfer to gain some idea of the breadth of applications. We will encounter a simple version of heat conduction, a discussion of the radiation of heat from surfaces, and an introduction to heat transfer from solid bodies to fluids (or vice versa). In later chapters, conduction, convection, and radiation will be presented in more depth.

QUESTIONS

1. What is the driving force for a conductive transport of entropy? What is it for a convective flow of entropy?
2. When (hot) water flows out of a tank, there is a convective entropy current leaving the system. Does this mean that the water in the tank gets colder?
3. In the case of flows and sources of entropy, entropy is transported from one system to another. So, what is the (geometric or spatial) difference between a current of entropy and an entropy source rate?
4. Both source rates of entropy and entropy production rates are volumetric quantities. What is their difference?
5. If a body of air in the atmosphere absorbs some solar radiation, are the energy source rates of the radiation field and of the air equal? Are the entropy source rates equal?

EXAMPLE 7.1. Conductive and convective fluxes of entropy.

Consider hot water flowing through a metal pipe as in the figure below. (a) Consider the interior of a part of the pipe as the system, and assume this control volume not to move or deform. Account for all fluxes of entropy penetrating the surface of this control volume. (b) Consider the water in the control volume at a particular instant to be the system. Follow this body of water in its motion and repeat the problem (a).

SOLUTION: a) We are dealing with a stationary control volume (*CV*) through which water and entropy are flowing. (See the upper part of Fig. Ex.1; water is flowing from left to right.) We have to find the currents flowing through the control surface and determine the fluxes associated with them.

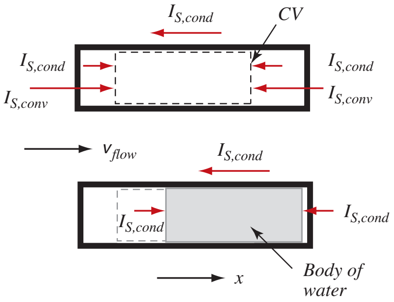

Figure Ex.1

First of all, entropy must be flowing radially outward through the pipe if the surroundings are cooler than the water. This means that we have a *conductive* current of entropy penetrating the cylindrical surface. Since the flow is outward, its flux will have a negative sign (symbolized by an arrow in the negative x-direction).

Second, because of the loss of heat through the walls of the pipe, the water entering the control volume will be warmer than the water leaving. We have a thermal driving force in the direction parallel to the axis of the pipe. At the control surface, there must be conductive currents of entropy through the water in its direction of flow. Therefore, we have a positive flux associated with the conductive current at the entrance to the control volume (left), and a negative flux due to the current leaving the system.

Finally, two convective fluxes are associated with the flow of water into and out of the control volume. Entropy stored in the water is carried across the surface of the system. Again the flux is positive at the inlet, and negative at the outlet. The flux at the entrance is larger in magnitude than the one at the outlet.

(b) If we follow a certain body of water in its motion, the system wall moves with it (lower part of Fig. Ex.1). Water does not flow across the surface of the body, which is represented by the shaded area. This means that there are no convective currents of entropy to be considered. The conductive currents still exist, and they are the same as the ones identified in (a).

7.2 CURRENTS OF ENTROPY IN CONDUCTION

We have considered the generic expressions for laws of balance of entropy and energy in the previous section. Now we shall turn our attention to the constitutive theories that will allow us to quantify entropy transfers. Let me begin with conduction.

7.2.1 Fourier's Law for Entropy

What factors does the current of entropy in conductive transport depend upon? If the temperature of a body changes from place to place, there must be *temperature gradients*. This is one factor upon that we expect the rate of flow of heat, the entropy current, to depend. The material through which the entropy flows must also play a role in the determination of the current. The influence of the material will be described by its *conductivity*. If the current of entropy depends upon the temperature gradient and the conductivity in the simplest possible way, we say that it obeys *Fourier's law*.

Fourier's law for a slab of matter. We can motivate the form of Fourier's law in a simple manner. The idea is borrowed from electricity, where we also have encountered phenomena having to do with conduction, namely the conduction of charge (Chapters 1). Consider the conduction of entropy through a slab of material as shown in Fig. 7.7. Assume that entropy flows only in one direction, and that the distribution of the current of entropy does not vary in a plane perpendicular to the flow. In other words we will consider only the simplest possible case of a flow field.

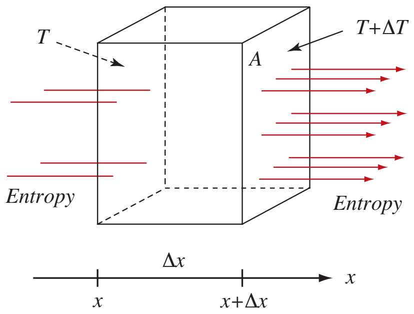

Figure 7.7: Entropy flows in one direction only through a slab of matter. We assume that the distribution of the current does not vary in planes perpendicular to the x-direction. There is a difference of temperatures between front and back faces which serves as the driving force of the flow of entropy.

The basic question is this: how does the current density of entropy depend upon the circumstances? From what we already know, the current of entropy through a body should depend upon the temperature difference across the body (in the direction of the flow of entropy) and a conductance that depends upon the geometry and the conductive properties of the body. The conductance is what we should be concerned with here. Clearly, for given temperature difference, the current doubles if the cross section A of the conducting body doubles: we simply have two equal conducting bodies in parallel. Secondly, we expect the current to halve if the thickness of the slab is doubled. Therefore,

$$I_S(x) = -k_S \frac{A}{\Delta x} \Delta T \tag{7.12}$$

The negative sign tells us that entropy flows into the body at x (Fig. 7.7); remember that the temperature difference is negative. The factor k_S quantifies the conducting property of the material; it is called the *entropy conductivity* of the substance.

This is *Fourier's law* of conduction. Naturally, the entropy conductivity is expected to depend upon the material the body is made up of, and on temperature (Fig. 7.8). Some values of entropy conductivities are given in Table 7.1.

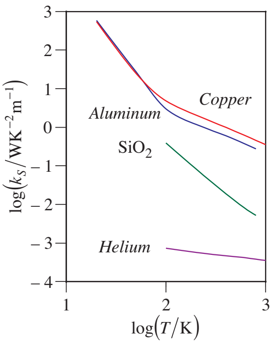

Figure 7.8: Some entropy conductivities k_S as functions of temperature. The values of the conductivities with respect to energy, i.e., those commonly listed in tables, are equal to k_S multiplied by the temperature of the material (Fig. 7.9).

Table 7.1: Conductivity of some materials

Substance	**Conditions**	**Conductivity (entropy)**	**Conductivity (energy)**
	300 K	k_S / $\mathrm{W \cdot K^{-2} m^{-1}}$	k_E / $\mathrm{W \cdot K^{-1} m^{-1}}$
Gases at atmospheric pressure			
Air	200 K	$9.05 \cdot 10^{-5}$	0.0181
	300 K	$8.73 \cdot 10^{-5}$	0.0262
	400 K	$8.41 \cdot 10^{-5}$	0.0337
Helium	200 K	$5.89 \cdot 10^{-4}$	0.1177
CO_2	250 K	$5.16 \cdot 10^{-5}$	0.0129
	300 K	$5.53 \cdot 10^{-5}$	0.0166
H_2O vapor	400 K	$6.52 \cdot 10^{-5}$	0.0261
	500 K	$6.76 \cdot 10^{-5}$	0.0339
	600 K	$7.03 \cdot 10^{-5}$	0.0422
Saturated liquids			
Mercury	293 K	$2.97 \cdot 10^{-2}$	8.69
Water	273 K	$2.02 \cdot 10^{-3}$	0.552
	293 K	$2.04 \cdot 10^{-3}$	0.597
	313 K	$2.01 \cdot 10^{-3}$	0.628
	333 K	$1.96 \cdot 10^{-3}$	0.651
	353 K	$1.89 \cdot 10^{-3}$	0.668
	373 K	$1.82 \cdot 10^{-3}$	0.680
Liquid metals			
Sodium	366 K	0.232	84.96

Table 7.1: Conductivity of some materials

Substance	Conditions	Conductivity (entropy)	Conductivity (energy)
	300 K	k_S / W · K^{-2}m^{-1}	k_E / W · K^{-1}m^{-1}
Solids at 20°C			
Aluminum		0.80	240
Brick (building brick)		0.0024	0.7
Bronze (75% Cu, 25% Sn)		0.089	26
Clay		0.0043	1.3
Concrete (cinder)		0.0026	0.76
Copper	300 K	1.32	400
	500 K	0.772	386
	800 K	0.458	366
Fiber (insulating board)		$1.6 \cdot 10^{-4}$	0.048
Glass (window)		0.0027	0.8
Glass fiber		$1.3 \cdot 10^{-4}$	0.038
Granite		0.0058–0.014	1.7–4.0
Iron	300 K	0.27	80
	500 K	0.12	61
Lead	300 K	0.12	35
Limestone	100–300°C		1.3
Paper		$3.6 \cdot 10^{-5}$	0.011
Pyrex		0.0046	1.4
Rubber (vulcanized, hard)		$4.3 \cdot 10^{-5}$	0.013
Sand		$8.9 \cdot 10^{-5}$	0.027
Sandstone		0.0061	1.83
Steel	1.0% C	0.15	43
	20% Ni	0.065	19
Tissue	Fat layer	0.00066	0.20
	Muscle	0.0014	0.41
Tungsten		0.556	163
Wood	Oak	0.00057	0.166
	White pine	0.00038	0.112

The differential form of Fourier's law. To prepare the ground for treating the continuous case (Part III), let me transform the expression for a conductive current of entropy. Experience tells us that the conductive transport of heat violates our assumption of spatial uniformity: temperatures must change from point to point inside the materials conducting entropy. Therefore, Equ.(7.12) is not the best possible expression for a conductive current of entropy. Also, if you remember the treatment of heat transfer in Chapter 4, you will notice that Equ.(7.12) cannot hold for every point inside the slab through which heat is conducted. Conduction is dissipative, meaning that entropy is produced as it flows through the body. In other words, the entropy current increases in magnitude in the direction of flow and we have to specify where we want to apply Equ.(7.12).

All these difficulties disappear in a continuum description of the phenomenon (Chapters 11 and 13). At this point, all we should do is take a first look at Fourier's law in the continuous form. In a first step, we introduce a measure of the distribution of the current over the surface through which it flows, namely the current density j_S. In our case, it is related to the magnitude of the flux I_S as follows:

$$|I_S| = A\,|j_S| \tag{7.13}$$

The meaning of the density of a current of entropy will be explained in more detail in Chapters 11 and 13. The unit of the current density of entropy is $\mathrm{W/(K \cdot m^2)}$.

Equ.(7.12) indicates how we can proceed. The term $\Delta T/\Delta x$ in Fourier's law is changed to the *temperature gradient dT/dx*. The conductivity keeps its meaning, so we arrive at

$$j_S = -k_S \frac{dT}{dx} \tag{7.14}$$

It is clear that the current of entropy must vanish if the temperature gradient is zero. In the simplest case, the current density will depend linearly upon the gradient. Also, entropy is not conducted if we deal with a perfect insulator whose conductivity is zero. For these reasons, Equ.(7.14) makes sense. It is perfectly analogous to what we have seen in the case of the conduction of momentum or electric charge (Chapter 3). Note that we have not yet solved the constitutive problem of the conductive transport of heat. While we now have a relation for the current of entropy appearing in the equation of balance, we still do not have an expression for the rate of generation of entropy.

7.2.2 The flow of entropy and energy in conduction

A current of energy is always associated with a current of entropy in conductive transport. According to Equ.(4.21), the former is equal to the latter multiplied by the temperature of the material at the point where the two flow together. This relation carries over to the continuous case:

$$j_{E,th} = T\, j_S \tag{7.15}$$

Just as there is a current density of entropy, there also must be a current density of energy. Equ.(7.15) holds for every point in a body through which entropy flows conductively. The validity of the generalization of $I_E = TI_S$ can be proved more rigorously (Chapter 12). For now, let us accept it as intuitively clear. If we introduce the definition of the *conductivity with respect to energy* (*thermal conductivity,* Fig. 7.9),

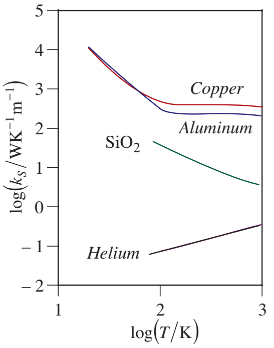

Figure 7.9: Some thermal conductivities k_E as functions of temperature. See Fig. 7.8 for the equivalent entropy conductances.

$$k_E = T\, k_S \tag{7.16}$$

Fourier's law can be expressed in terms of the thermal energy current:

$$j_{E,th} = -k_E \frac{dT}{dx} \tag{7.17}$$

Since we consider pure conduction of heat, only the thermal energy current appears in a process. Therefore, the expression for the balance of energy will take a particularly simple form.

7.2.3 Entropy Production in Conduction

The conductance of a fluidlike quantity is a prime example of an entropy producing process. Here are two ways to determine the entropy production rate in conduction of entropy.

Dissipation. If entropy enters the hot end of a slab of material and exits at the cold end, as shown in Fig. 7.7, entropy is produced inside. This is so because the entropy falls from a high to a low level thereby releasing energy which is dissipated. From what we have studied in Chapter 4 (Section 4.6.3) we know that

$$\Pi_S = -\frac{1}{T_\mathrm{L}} \Delta T\, I_S = \frac{1}{T_\mathrm{L}} \Delta T\, k_S \frac{A}{\Delta x} \Delta T$$

or

$$\Pi_S = \frac{1}{T_\mathrm{L}} k_S A \Delta x \left(\frac{\Delta T}{\Delta x}\right)^2 \tag{7.18}$$

The volume *density of the production rate* is introduced in the continuous case. It is obtained by dividing the expression in Equ.(7.18) by the volume of the slab:

$$\pi_S = \frac{1}{T} k_S \left(\frac{dT}{dx}\right)^2 \tag{7.19}$$

The term $T\pi_S$ is the *density of the dissipation rate* inside the material as a consequence of conduction.

Simultaneous balances of entropy and energy. Here is a type of derivation which will prove very useful in many applications. The first approach just outlined requires us to have direct knowledge of the amount of energy dissipated, and the temperature at which dissipation takes place. If we do not have this (or are unsure about the values), there is an indirect method that uses the balance of energy to determine the missing constitutive relation, i.e., the production rate of entropy.

Consider a resistive element similar to the slab of matter in Fig. 7.7. The laws of balance of entropy and of energy for this body take the forms

$$\begin{aligned} 0 &= I_{S1} + I_{S2} + \Pi_S \\ 0 &= I_{E,th1} + I_{E,th2} \end{aligned}$$

Since we assume that the resistive element does not store entropy or energy, rates of change of entropy and energy must equal zero. Together with the relations between current of entropy and energy,

$$\begin{aligned} I_{E,th1} &= T_1 I_{S1} \\ I_{E,th2} &= T_2 I_{S2} \end{aligned}$$

we arrive at

$$\begin{aligned}\Pi_S &= -\left(I_{S1} + I_{S2}\right) = -\left(\frac{1}{T_1} I_{E,th1} + \frac{1}{T_2} I_{E,th2}\right) \\ &= -\left(\frac{1}{T_1} I_{E,th1} - \frac{1}{T_2} I_{E,th1}\right) = -\frac{T_2 - T_1}{T_2}\frac{1}{T_1} I_{E,th1} = -\frac{T_2 - T_1}{T_2} I_{S1}\end{aligned}$$

which is equivalent to what we have derived before. Here we did not assume knowledge of the dissipation rate; the rate of dissipation is part of the combined laws regarding entropy and energy in a specific situation.

7.2.4 Calculation of Conductances or Resistances

In Section 4.6.1, the conductance (or its inverse, the resistance) for overall heat transfer through a series of transfer layers has been introduced. The definition carries over to a single (conductive) layer. Hence,

$$G_{S,cond} = k_S \frac{A}{\Delta x} \tag{7.20}$$

The resistance of such a layer is therefore equal to

$$R_{S,cond} = \frac{\Delta x}{k_S A} \tag{7.21}$$

The entropy transfer coefficient h_S (Equ.(4.45)) can be calculated from this, and all of this can be converted to energy related quantities (Equ.(4.47)). According to Section 4.6.4, conductances or resistances can be calculated for composite layers. It is customary to do this for the energy flow resistances or conductances (by assuming them to be independent of temperature). For a *series* of layers we get

$$R_{E,cond} = \sum R_{E,i} \tag{7.22}$$

and for *parallel* layers

$$G_{E,cond} = \sum G_{E,i} \tag{7.23}$$

Remember that these rules have to be taken with a grain of salt. Thermal properties such as conductances typically vary with temperature (Table 7.1 and Fig. 7.8). If we wish to do the calculations directly for the entropy conductances we have to take into account entropy production which, for series connections, leads to Equ.(4.55). If temperature differences are relatively small, and if property values are relatively uncertain, we can use Equ.(7.22) directly for the total entropy resistance as well.

7.2.5 A Dynamical Model of Conduction in a Copper Bar

To demonstrate the utility of the assumptions behind Fourier's law, let us work on a dynamical model of conduction in a long copper bar. An example of experimental data was presented in Chapter 4 (Section 4.1, Fig. 4.5) where a copper bar was stuck in ice water at one end and heated electrically at the other. First, before the electric heater was turned on, temperatures went down, those close to the end in ice water reacted first, then those farther away. With the heating on, temperatures went up, first those close to the heater, then those further down the line.

Even though we do not have the tools yet to treat the continuous case found in nature, we can still produce a useful model based on uniform bodies by dividing the copper bar into several elements. We view each element as a store for entropy having its temperature (which we associate with the center of the element; Fig. 7.10, top). The material between two center points is taken as the conductive piece (or resistive element) between two storage units.

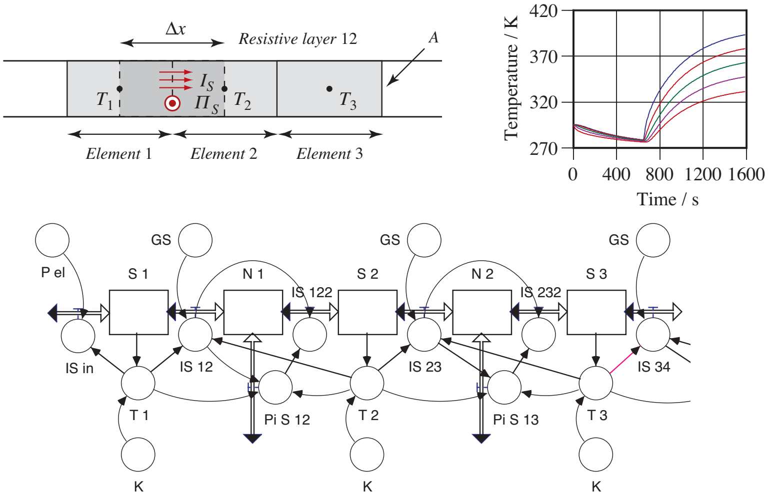

Figure 7.10: A long conducting bar is divided into elements (top left). An element is a storage unit for entropy. The material between (the centers of) two elements is modeled as a resistive element. Bottom: Section of a diagram of a dynamical model for this system. Note the entropy stores and the nodes (used to express the balance of entropy for the resistors). Top right: Simulation results for a particular case (compare to Fig. 4.5).

Length and cross section of an element define the geometry needed to calculate entropy capacitances (Chapter 4) of elements and conductances of resistors (Equ.(7.20)). The model can be simulated with proper initial and boundary conditions (see the graph in Fig. 7.10).

QUESTIONS

6. How large is the entropy current through a 10 m by 10 m sandstone wall which is 0.5 m thick if the temperature difference is 20°C?
7. Assume constant (temperature independent) entropy resistances of a couple of layers placed in series. Why can't we simply add them to get the total resistance?
8. Consider two conducting bars like the one in Fig. 7.10 made of two different materials. They are geometrically equivalent and have equal conductivities and specific entropy capacitances. The second material has a higher density. How would the behavior of the second material differ from that shown in Fig. 7.10?
9. Temperatures have been measured at 8 points along the copper bar in Fig. 7.10. Does this mean the dynamical model should be made of 8 elements?

EXAMPLE 7.2. Melting a block of ice insulated in glass fiber.

A cube of ice having a volume of 1.0 m^3 is in a box insulated by 10 cm thick glass fiber. It is left in an environment at 30°C. Estimate how long will it take for the cube to melt.

SOLUTION: Treat the layer of glass fiber as a flat blanket having a surface area of 6 m^2 (this is the surface area of the cube of ice). According to Table 7.1, the entropy conductance of glass fiber is 1.3 · 10–4 W/(K^2 · m). Therefore, the entropy current from the environment to the ice is

$$I_S = -k_S \frac{A}{\Delta x}\Delta T = -1.3 \cdot 10^{-4}\,\frac{6.0}{0.10}(0-30)\frac{\text{W}}{\text{K}} = 0.23\frac{\text{W}}{\text{K}}$$

The 1.0 m^3 of ice corresponds to 910 kg. The specific latent entropy of ice is l_f = 22/0.018 J/(K · kg) (Chapter 5, Table 5.1). Therefore, the latent entropy of the block is 1.1 · 10^6 J/K. The time taken to supply this much entropy to the volume of ice is

$$\Delta t = \frac{S_e}{I_S} = \frac{1.1 \cdot 10^6}{0.23}\text{s} \approx 5 \cdot 10^6 \text{s}$$

This corresponds to close to 60 days. Naturally, the model assumes that the melt water stays at a temperature of 0°C and transmits entropy easily (i.e., it is assumed the melt water does not blanket the ice in addition to the sheets of glass fiber).

EXAMPLE 7.3. Heating of an integrated circuit.[1]

A silicon chip is attached to an isothermal surface called a *header*. The top of the chip is covered uniformly by a power device which dissipates energy at a rate of 50 W. Thirty thin gold wires connect the top with the header. Assume the entropy to be conducted down through the chip in one direction only. What will the steady-state temperature be at the top of the chip, if the header is kept at a temperature of 25°C?

The chip has a surface area of 0.51 cm by 0.51 cm. It is composed of three layers. The first is made out of silicon with a thickness of 0.051 cm. The chip carries a thin layer of gold at the bottom; its thickness is 0.010 cm. Between these two a thin layer of silicon dioxide forms, which has a thickness of 0.00013 cm. The thermal conductivities with respect to energy are 88 W/(K · m), 312 W/(K · m), and 0.157 W/(K · m), respectively. The gold wires are 0.130 cm long and have a diameter of 0.0254 cm.

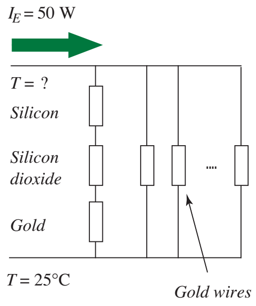

Figure Ex.3

SOLUTION: The device represents a thermal circuit with elements in parallel and in series (see figure). A constant thermal driving force is maintained over the circuit. We must figure out the total thermal resistance offered by the circuit to the flow of entropy and energy. The chip has a resistance of

$$\begin{aligned} R_{E,chip} &= \sum_{i=1}^{3} \frac{1}{k_{E,i}}\frac{\Delta x_i}{A} \\ &= \frac{1}{0.0051^2}\left(\frac{0.051 \cdot 10^{-2}}{88} + \frac{1.3 \cdot 10^{-6}}{0.157} + \frac{0.010 \cdot 10^{-2}}{312}\right)\frac{\text{K}}{\text{W}} = 0.554 \text{ K/W} \end{aligned}$$

This device is connected in parallel with thirty identical wires. Therefore, the total resistance is calculated to be

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[1] P. Ridgely (1987).

$$\frac{1}{R_{E,tot}} = \sum_{i=1}^{31} \frac{1}{R_{E,i}} = 30\frac{k_E A}{\Delta x} + \frac{1}{R_{E,chip}}$$

$$= 30\frac{312 \cdot \pi(1.27 \cdot 10^{-4})^2}{0.0013}\frac{\mathrm{W}}{\mathrm{K}} + \frac{1}{0.554}\frac{\mathrm{W}}{\mathrm{K}} = 2.17\frac{\mathrm{W}}{\mathrm{K}}$$

This corresponds to a resistance of 0.461 K/W. With the energy flux given, we can calculate the thermal driving force, i.e., the difference of temperatures between the top and the bottom of the chip:

$$\Delta T = R_E I_{E,th}$$
$$= 0.461\mathrm{K/W} \cdot 50\mathrm{W} = 23.0\mathrm{K}$$

The temperature at the top of the chip is 48°C. Without the gold wires it would be 53°C.

EXAMPLE 7.4. A model of two uniform bodies in thermal contact

The faces of two identical cylinders, one heated to a high temperature T_{1i} and the other to a low temperature T_{2i}, are brought in direct thermal contact. (a) Calculate their temperatures as functions of time by treating them as spatially uniform bodies. (b) Compare the result to a finite element computation of the temperatures of their centers. (c) Calculate the entropy produced.

SOLUTION: (a) The model can be constructed along the lines of the model presented in Fig. 7.10. We have two bodies touching directly (see Fig. Ex.4.1). They are treated as spatially uniform stores of entropy. We can imagine the entropy to travel on average from the center of the warmer to the center of the colder body, so we use the matter between the two centers as the conductive transfer layer.

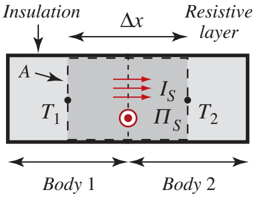

Figure Ex.4.1

The model can be constructed using the entropy or the energy balances for the bodies; remember that the conductor is treated as an element that does not store entropy. Let us use the balance of energy:

$$C\dot{T}_1 = -I_E \quad , \quad T_1(0) = T_{1i}$$
$$C\dot{T}_2 = I_E \quad , \quad T_2(0) = T_{2i}$$

C is the temperature coefficient of energy of a body, and *CdT/dt* is equal to *dE/dt* of a body. The bodies are taken to be identical, with different initial temperatures. The energy current flowing from the hotter to the cooler body is expressed by

$$I_E = \frac{k_E A}{\Delta x}(T_1 - T_2)$$

A is the cross section of the bodies (the surface area where they are touching), k_E is the thermal conductivity, and Δx is a distance which we take to be the distance of the centers of the two bodies. We now divide the differential equations by the temperature coefficients of energy, and then subtract the second from the first. This leads to a differential equation for the difference of temperatures $\Delta T = T_1 - T_2$:

$$\frac{d}{dt}\Delta T = -2\frac{k_E A}{\Delta x C}\Delta T$$

whose solution turns out to be

$$\Delta T = \Delta T_i \exp(-t/\tau) \quad , \quad \tau = \frac{\Delta x C/2}{k_E A}$$

where ΔT_i is the initial temperature difference of the bodies (see Fig. Ex.4.1).

(b) A long bar has an initial temperature distribution with its left half at 400 K and its right half at 200 K. The first of the diagrams (Fig. Ex.4.2, left) depicts the temperatures at evenly spaced points throughout the bar as a function of time (result of an FE computation). The second graph shows the average temperature of the two halves of the bar (circles) and the solution computed according to the result in (a) with Δx equal to half of the length of the bar. (Values are: $k_E = 1$, $C = 5$, $\Delta x = 5$, $A = 1$.) A judicious choice of Δx can make the solutions quite similar, and the rough model can serve as an estimate of what is happening in the bar.

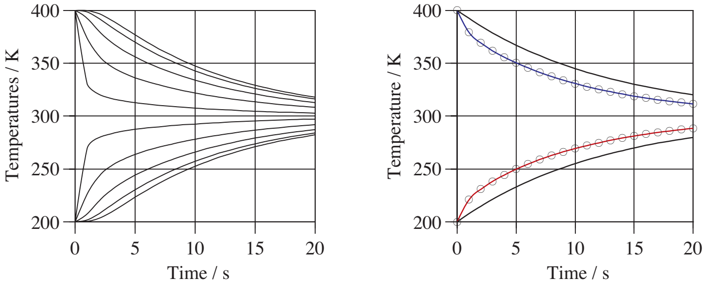

Figure Ex.4.2

(c) The irreversibility is measured in terms of the production of entropy. The rate of generation of entropy is expressed by

$$\Pi_S = I_E\left(\frac{1}{T_2} - \frac{1}{T_1}\right) = \frac{k_E A}{\Delta x}\left(T_1 - T_2\right)\frac{T_1 - T_2}{T_1 T_2}$$
$$= \frac{k_E A}{\Delta x}\left(T_1 - T_2\right)^2 \frac{1}{\left(T_f + \Delta T/2\right)\left(T_f - \Delta T/2\right)}$$

If we introduce the solution into this formula, we obtain

$$\Pi_S = \frac{k_E A}{\Delta x}\frac{4\exp(-2t/\tau)}{4\left(T_f/\Delta T_i\right)^2 - \exp(-2t/\tau)}$$

Integration of this expression over time (from zero to infinity) leads to[2]

$$S_{prod} = C\ln\left(\frac{4\left(T_f/\Delta T_i\right)^2}{4\left(T_f/\Delta T_i\right)^2 - 1}\right) = C\ln\left(\frac{4T_f^2}{4T_f^2 - \Delta T_i^{\,2}}\right) = C\ln\left(\frac{T_f^2}{T_{1i}T_{2i}}\right)$$

The last form is equivalent to the one obtained from thermostatic considerations alone, i.e., by applying balances of entropy and energy to the total process of equilibration of two identical bodies (Chapter 4).

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[2] The solution is obtained from

$$\int \frac{\exp(ax)}{b + c\exp(ax)}dx = \frac{1}{ac}\ln\left(b + c\exp(ax)\right)$$

EXAMPLE 7.5. The flow of heat through the mantle of the Earth.

The total flux of energy from the interior of the Earth through its surface can be estimated from the values of the temperature gradient in the crust and the conductivity with respect to energy. Their values are 0.06 K/m and 1 W/ (K · m), respectively. Assume that the entire flux is conducted from the core at a depth of 3400 km through the solid mantle (the radius of the Earth is 6400 km). Take as an average thermal conductivity the one found for the upper crust. According to these assumptions, how large would the temperature of the core of the Earth be?

SOLUTION: First we have to calculate the energy flux at the surface of the Earth. According to Equ.(7.17) it must be

$$I_{E,th} = 4\pi R^2 k_E \frac{dT}{dr} = 3.1 \cdot 10^{13}\ \mathrm{W}$$

R is the radius of the Earth. The energy flux out of the Earth is about 5000 times smaller than the one we receive from the Sun. We have applied Fourier's law, which was motivated for flat geometry. This is certainly allowed in the case of purely radial flow. We need only replace the normal temperature gradient by its radial counterpart. The following development, however, changes because of the differences in geometry.

If we knew the thermal resistance R_E of the Earth's mantle we could easily calculate the temperature difference necessary to conduct this current from the bottom of the mantle up to the surface. Since the conducting body is not flat, the surface area through which conduction is taking place varies constantly. Therefore, let us write Fourier's law in the form

$$\left|I_{E,th}\right| = \frac{dT}{dR_E}$$

with

$$\frac{dR_E}{dr} = \frac{1}{k_E A}$$

This means that we have to calculate the thermal resistance by integration. For a spherical shell with inner and outer radii r_i and r_0, respectively, and with constant k_E we get:

$$R_E = \int_0^{R_E} dR_E = \int_{r_i}^{r_0} \frac{dr}{k_E A} = \frac{1}{4\pi k_E}\int_{r_i}^{r_0}\frac{dr}{r^2} = \frac{1}{4\pi k_E}\left(\frac{1}{r_i} - \frac{1}{r_0}\right)$$

Note that the thermal resistance is of the form given in Equ.(7.20), with $\Delta x = r_0 - r_i$ and $A = 4\pi r_i r_0$. According to Equ.(4.46), the difference of temperatures between the core-mantle boundary and the surface must be

$$\Delta T = R_E I_{E,th} = 4.36 \cdot 10^5\,\mathrm{K}$$

This value is rather far off from the estimated temperature difference of some 3000 K. From seismic measurements we know that the mantle is solid, which limits the temperature below the melting point of rocks. A good number of reasons can be given to explain this huge discrepancy. The value of the conductivity might be wrong. (However, it will not be all that far off.) The entropy flowing out through the surface of the Earth might not come from the core; it might be produced in the mantel and the crust by radioactive decay; this is indeed the case (see Chapter 13). The flow of entropy varies with time; in our case, however, this does not change the result much because of the long time scale. Finally, the entropy might be transported not by conduction but by radiation and convection. This is true as well; it is mostly convection which transports entropy through the mantle, even though the mantle is solid! Over very long time scales, the material of the mantle is deformable and it moves; this process apparently is responsible for continental drift.

7.3 THE NATURE OF HEAT TRANSFER AT A SOLID–FLUID BOUNDARY

Now we will introduce some aspects of heat transfer across interfaces separating different types of bodies. Consider a hot solid body submersed in some fluid. Entropy is conducted through the body to its surface from where it enters the fluid and is carried away convectively. The transport of entropy from the solid to the fluid, or vice-versa, is of interest in the design of heat exchangers, in the loss of heat from a building, in thermal solar collectors, in household appliances, and in many other applications. Another important case is heat transfer between liquids and gases. Just think of the interaction between the Earth’s atmosphere and the oceans, which has received much attention recently. Questions concerning the balance of energy and entropy, and of carbon dioxide and other trace gases, are of vital interest in environmental, atmospheric, and oceanographic sciences.

7.3.1 Boundary Layers

For now, let us limit our attention to the flow of entropy from solids to fluids. The transport mechanism is usually a mixture of conduction, convection, and radiation. (The last will be treated in Section 7.4.) Entropy flows through a hot body to its surface from where it somehow enters the fluid. For example, consider a viscous fluid flowing along a flat plate as in Fig. 7.11. The hydrodynamic phenomenon is described by the velocity of the fluid in the vicinity of the plate. The conditions in the undisturbed fluid are given by the free stream values of velocity and temperature far from the plate. Due to viscosity, the speed of flow is reduced to zero at the surface of the body. It is found that the velocity changes in a direction perpendicular to the surface from the value of zero to the free stream velocity further away in the undisturbed flow. The velocity gradient is confined to a thin *hydrodynamic* or *velocity boundary layer* in which all the interesting action takes place (Fig. 7.12a). The thickness of the boundary layer is zero at the leading edge of the plate and it increases with increasing distance along the surface. The boundary layer is defined to extend to points where the velocity has reached 99% of the free stream value. Typically, in the situation described, it has a thickness of the order of only a few millimeters.

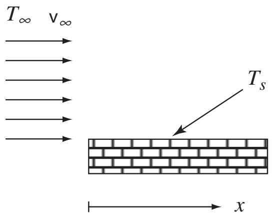


Figure 7.11: A fluid flows past a long flat plate. The fluid has free stream values of velocity and temperature far from the plate. The surface temperature of the solid body is assumed to be constant and different from the free stream value of the fluid. As a result, entropy and energy will be carried across the interface by the combined action of conduction and convection.

Now consider the temperature of the fluid. At the surface of the solid, the fluid is at rest and its temperature is that of the solid surface, which in general, is different from the free stream value. Therefore temperature gradients must develop perpendicularly to the surface; i.e., the temperature changes from the surface value to the free stream value, this time in a thin *thermal boundary layer* (Fig. 7.12b).

Again the thickness of this boundary layer increases along the plate from a value of zero at the leading edge. We can understand the importance of the conditions in the boundary layer for the transport of entropy. At the surface of the solid, entropy is transferred into the fluid in the conductive mode only. This allows us to write the entropy flux density in terms of the conductivity of the fluid and the temperature gradient in the fluid at the surface:

$$j_S = -k_{Sf} \left. \frac{dT}{dy} \right|_{y=0} \tag{7.24}$$

(see Equ.(7.14)). The index *f* refers to the fluid. Naturally, conditions in the fluid layer change right away, so Equ.(7.24) holds only for the surface ($y = 0$).

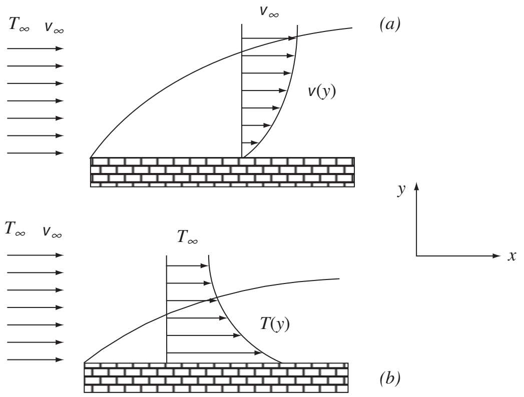


Figure 7.12: Velocity (a) and temperature (b) boundary layers develop at the surface of the solid body. The velocity is zero right at the surface, from where its value increases to the free stream velocity. The distance over which the quantity changes marks the extent of the boundary layer which increases along the plate. The temperature is equal to the surface temperature for $y = 0$. It decreases (or increases) from the surface to obtain the free stream value. The thicknesses of the velocity and temperature boundary layers are not the same.

7.3.2 Convective Heat Transfer Coefficient

Entropy and energy which enter the fluid conductively will be carried away with the flow of matter. The entropy and energy currents crossing the interface must depend in some way upon the physical state of the fluid and the temperatures at the surface of the body as well as far away from it. The process is a rather complex phenomenon. That is why it is commonly described in a strongly simplified manner. The entropy flux density is expressed in terms of the difference of temperatures between the surface of the solid body and the fluid far from the surface, and a coefficient which summarizes the complexity of the physical state of the fluid:

$$j_S\Big|_{y=0} = h_S\left(T_s - T_\infty\right) \tag{7.25}$$

h_S is called the (local) *convective entropy transfer coefficient*, while T_s and T_∞ represent the temperature of the surface and of the undisturbed fluid, respectively. The coefficient depends upon the details of the fluid flow. It has to be either calculated on the basis of a complete hydrodynamic theory, or measured in experiments. The expression in Equ.(7.25) is equal to the entropy flux at the interface, which allows us to equate the flux densities in Equations (7.24) and (7.25). This leads to an expression for the entropy transfer coefficient:

$$h_S = -\frac{k_{Sf}}{T_s - T_\infty}\frac{dT}{dy}\bigg|_{y=0} \tag{7.26}$$

Both the conductivity of the fluid and the temperature difference can be taken to be constant. Therefore, the convective transfer coefficient depends upon the temperature gradient of the fluid at the surface of the solid, which is determined by the conditions in the boundary layer. Experience tells us that a hot body submersed in a flowing medium cools much faster than in a still fluid. Therefore, the rate of transfer of entropy from a solid into a liquid or gas crucially depends upon the state of motion of the fluid. The type of flow plays an important role as well. We have to distinguish between *laminar* and *turbulent* flows on the one hand, and forced and free convection on the other. The rate of entropy transfer is very different in these cases. It is clear that we have not really solved the problem of convective heat transfer; we have simply shifted it to the task of determining the transfer coefficient from a theory combining motion and heat transfer.

The temperature gradient at the surface of the solid obviously diminishes with increasing thickness of the layer, which leads us to conclude that the local value of the transfer coefficient decreases along the plate. The coefficient is often replaced by an average value. In this case we can relate the entire entropy flux to the change of temperature and the *average transfer coefficient* h_{Sa}:

$$I_S\Big|_{y=0} = h_{Sa}\,A\left(T_s - T_\infty\right) \tag{7.27}$$

A is the total surface of the body. In this simplified form the constitutive law of convective entropy transfer commonly serves as a boundary condition for the conductive transport of entropy through the solid body.

Naturally, the entropy flux varies across the boundary layer. Entropy will be produced in the fluid due to both conduction and viscous friction. These are two of the possible dissipative processes taking place in the general type of fluid considered here.

The exchange of energy with entropy at the interface. The entropy current which is expressed by Equ.(7.27) is carried across the surface of the solid body by conduction alone, so it is possible to give a simple form of the energy flux entering or leaving the solid. In conduction, the entropy and energy currents are related by the local temperature. Therefore the energy flux at the surface is equal to

$$I_{E,th}\Big|_{y=0} = h_a\,A\left(T_s - T_\infty\right) \tag{7.28}$$

where

$$h_a = T_s h_{Sa} \tag{7.29}$$

is called *heat transfer coefficient*. We need to know the average transfer coefficient for concrete applications. Such values are listed in Table 7.2 for a few situations.

Table 7.2: Heat transfer coefficients with respect to energy [a]

Substance	**Transport mode**	h_a / W · K^{-1}m^{-2}
Air	Free convection	6–30
Air in rooms	Inside wall	8
	Window	8
	Floors and ceilings	6–8
Superheated steam or air	Forced convection	30–300
Oil	Forced convection	60–1800
Water	Forced convection	300–6000
	Boiling	3000–60000
Steam	Condensing	6000–120000

a. Order of magnitude, including the effect of radiation at the boundary.

From the foregoing we can define conductances or resistances. The expression for an energy flow *conductance* of a convective layer is

$$G_{E,conv} = h_a A \tag{7.30}$$

The inverse of this quantity is the (energy) *resistance*. Corresponding entropy conductances are calculated by dividing G_E by the appropriate temperatures. Once we know how to calculate conductances or resistances of single layers, we can find the overall conductance or resistance for compound layers just as in the case of conduction alone (Equ.(7.22) and Equ.(7.23)).

7.3.3 Overall (Total) Heat Transfer Coefficient

How large is the flux of entropy or energy through the wall of a building or through the insulation of a pipe? Obviously we are dealing with multilayer situations in which both conduction through solids and convection at solid–fluid boundaries occur. This situation was discussed in Chapter 4 (Section 4.6.4 and Fig. 4.44).

Consider heat transfer through a wall of a building. The transfer of energy is described in terms of the total difference of temperatures between the inside and the outside of the building, the surface area A, and an *overall heat transfer coefficient* h_{tot}:

$$I_{E,th} = h_{tot} A\left(T_1 - T_2\right) \tag{7.31}$$

$h_{tot} A$ is the overall conductance which can be calculated exactly as for compound conductive layers (Equ.(7.22) and Equ.(7.23)). Hence, for a *series of layers* (as in Fig. 4.44), the total heat transfer coefficient is

$$\frac{1}{h_{tot}} = \sum_{i=1}^{N} \frac{1}{h_{ai}} + \sum_{j=1}^{M} \frac{\Delta x_j}{k_{Ej}} \tag{7.32}$$

where the Δx_i are the thicknesses of each of the conducting layers which have conductivities k_{Ei}. There are N transition layers and M conductive ones.

QUESTIONS

10. If air is heated inside a room, would you expect heat to reach the walls by conduction through the air or rather by convective mixing of the air in the room?
11. Consider air flowing along a (heated) plate. Why should we expect the convective transfer coefficient to change along the plate (in flow direction?)
12. Consider hot air flowing along a cool plate. What will the temperature layer in the direction of flow look like?
13. If a box is insulated by 10 cm or more of glass fiber, do we still have to take into consideration convective transfer on the outside?
14. Consider hot water inside a thin-walled aluminum can cooling in the environment. Why is the convective transfer coefficient from the outside surface of the can to the air nearly equal to the total transfer coefficient from water to air?
15. What is the (entropy or energy) transfer coefficient for a conductive layer? How does it differ from the transfer coefficient for convection?

EXAMPLE 7.6. Surface temperature of a central heating radiator.

To heat a 100 m^2 apartment in an older not very well insulated building we need 1600 liters of heating oil per year. One liter of oil yields about 36 MJ when burned. Assume that the entire floor of the apartment is used for floor heating, and that the heating is on for one third of the year. (a) What is the average entropy current from the floor to the air? (b) How much warmer than the air will the surface of the floor have to be? Use a floor temperature of 300 K.

SOLUTION: (a) The entropy current is calculated from the energy current which is obtained from the energy delivered by the oil during about 10^7 s. For one square meter, we have

$$j_S = \frac{1}{T_{floor}}\frac{I_E}{A} = \frac{1}{T_{floor}}\frac{E_{e,th}}{A\Delta t} \approx \frac{1600\cdot 36\cdot 10^6}{300\cdot 100\cdot 10^7}\frac{\mathrm{W}}{\mathrm{K\cdot m^2}} = 0.20\frac{\mathrm{W}}{\mathrm{K\cdot m^2}}$$

(b) The entropy current calculated in (a) must be transferred from the floor to the air. According to Table 7.2, the entropy transfer coefficient from a floor to air is about 8/300 W/($K^2\cdot m^2$). With Equ.(7.25), we have:

$$\Delta T = \frac{j_S}{h_{Sa}} \approx \frac{0.2}{8/300}\mathrm{K} = 7.5\mathrm{K}$$

EXAMPLE 7.7. Surface temperatures of a single pane window in winter.

Consider a window having a metal frame. The window measures 1.20 m by 2.00 m. The glass has a thickness of 3.0 mm, and a conductivity with respect to energy of 1.0 W/(K·m). Take the convective transfer coefficients inside and outside to be 8.0 W/(K·m^2) and 12.0 W/(K·m^2), respectively. The metal frame is 3.0 cm wide around the window, and 5.0 mm thick. The conductivity is 220 W/(K·m) (aluminum), and the transfer coefficients inside and outside are taken to be 30 W/(K·m^2) and 50 W/(K·m^2), respectively. The temperature on the inside is 20°C; on the outside it is – 10°C. (a) What are the temperatures of a single pane window inside and outside in winter? (b) Calculate the flux of energy through the window if it has a metal frame. (c) How large is the total transfer coefficient of the window?

SOLUTION: (a) The energy flux through the glass is given by Equations (7.31) and (7.32):

$$\begin{aligned} I_{E,th} &= h_{tot}A\Delta T \\ &= \left[\frac{1}{8.0} + \frac{0.0030}{1.0} + \frac{1}{12.0}\right]^{-1} 2.4\cdot 30\ \mathrm{W} = 340\ \mathrm{W} \end{aligned}$$

We use this value to calculate the temperature drop from the inside to the surface of the window:

$$\Delta T = \frac{I_{E,th}}{h_{a1}A} = \frac{340}{8.0\cdot 2.4}\mathrm{K} = 17.7\mathrm{K}$$

which makes the temperature on the inside of the window 2°C. The same consideration for the thermal boundary layer outside delivers a temperature drop of 11.8 K. This means that the change of temperature through the glass is very small, and the outside surface has approximately the same temperature as the surface on the inside.

(b) The metal frame adds to the energy current. (It is in parallel with the window pane.) The surface area of the frame is roughly 0.19 m^2. Just as above, we calculate the energy current:

$$I_{E,th} = \left[\frac{1}{30} + \frac{0.0050}{220} + \frac{1}{50}\right]^{-1} 0.19\cdot 30\mathrm{W} = 110\mathrm{W}$$

The total energy current turns out to be 450 W which is very large. Note that the metal frame has a strong influence despite its small surface area.

(c) The total transfer coefficient is given by Equ.(7.32):

$$h_{tot} = \frac{I_{E,th}}{A\Delta T} = \frac{450}{2.6 \cdot 30}\frac{\text{W}}{\text{K} \cdot \text{m}^2} = 5.8\frac{\text{W}}{\text{K} \cdot \text{m}^2}$$

Good windows achieve a much smaller value of this coefficient (by as much as a factor of 10).

EXAMPLE 7.8. A surprising effect of insulation.

A metal pipe is to be insulated. It is found that, at least in principle, the insulation can have the opposite effect of what we would expect: the current of heat through the walls and the insulation of the pipe increases! How is this possible? Determine the conditions for the maximum heat flow.

SOLUTION: The thermal resistance of the insulation is made up of the resistance of the layer of insulation itself, and of the effect of convection at its surface. While the resistance of the insulating cylindrical shell grows with increasing thickness, the resistance due to the thermal boundary layer decreases because of the increase of surface area. There will be a minimum value of the total resistance at a certain outer radius of the insulation, depending on the material properties.

First we need an expression for the thermal resistance of a cylindrical shell (Fig. Ex.8). We proceed as in the case of a spherical shell (Example 7.5):

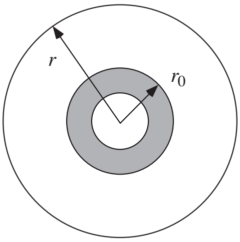


Figure Ex.8

$$R_E = \int_{r_0}^{r} \frac{1}{2\pi k_E L}\frac{dr}{r} = \frac{1}{2\pi k_E L}\ln\left(\frac{r}{r_0}\right)$$

The total resistance of insulation and boundary layer is

$$R_{tot} = R_{insulation} + R_{conv} = \frac{1}{2\pi k_E L}\ln\left(\frac{r}{r_0}\right) + \frac{1}{2\pi h_a L r}$$

Its minimum is found by setting its derivative with respect to the radial variable equal to zero. We obtain

$$r_{min} = k_E / h_a$$

The value of r for which the thermal energy current becomes largest does not depend upon the radius of the pipe. However, r_0 certainly must be smaller than the quantity just calculated. For normal values of the constitutive quantities, the pipe (and the insulation) must be rather thin. One might imagine the effect to play a role, for example, when ice starts to build up around thin branches or fibres in plants. An interesting suggestion has been made concerning the improvement of heat transfer through the air–water or air–air heat exchanger of a heat pump. At the cold end of the device, ice tends to build up at the surface, normally reducing the effectiveness of the pump. The geometry of the device could possibly be such that frost building up at its surface would lead to an increase of the rate of heat transfer.

EXAMPLE 7.9. Formation of ice on the surface of a lake.

Ice forms on the surface of a lake while the temperature of the air is – 10°C. How long does it take from the time ice begins to form for the sheet to reach a thickness of 20 cm? Take the temperature of the air to be constant. The convective transfer coefficient from ice to the air is 10 W/

(K·m^2). The conductivity with respect to energy of ice is 2.2 W/(K·m). Neglect the transfer from the water to the ice.

SOLUTION: For ice to form at the surface of the lake, the water must have reached a temperature of 0°C. Heat flows from the water into the air, first directly, and later through the ice; therefore water will freeze. If we can calculate the rate of formation of ice, we can determine the rate at which the thickness of the sheet grows.

The equation for the current of entropy leaving the water as it turns into ice is Equ.(5.2) written for the mass of the water:

$$I_S = l_{f,ice} \Pi_{m,water}$$

$l_{f,ice}$ is the specific latent entropy of fusion of ice (Table 5.1). The rate of production (destruction) of mass of water equals the negative of the rate of production of mass of ice which is equal to the rate of change of mass of ice:

$$\frac{dm_{ice}}{dt} = -\Pi_{m,water}$$

Combining these rules leads to

$$\frac{dm_{ice}}{dt} = -\frac{I_S}{l_{f,ice}}$$

The current is determined by the rules discussed in this section:

$$I_S = \frac{1}{T_{water}} \frac{\Delta T}{R_E} = \frac{1}{T_{water}} \left(T_{air} - T_{water}\right) A \left[\frac{1}{h_a} + \frac{x}{k_E}\right]^{-1}$$

x is the instantaneous thickness of the ice sheet. If ice has not formed yet, the total transfer coefficient is the convective transfer coefficient alone. Furthermore, we can express the rate of change of the mass of the ice in terms of the rate of change of its thickness:

$$\frac{dm}{dt} = \rho_{ice} A \frac{dx}{dt}$$

In summary we get the following differential equation for the thickness of the sheet of ice:

$$\rho_{ice} A \frac{dx}{dt} = -\frac{1}{l_{f,ice}} \frac{1}{T_{water}} \left(T_{air} - T_{water}\right) A \left[\frac{1}{h_a} + \frac{x}{k_E}\right]^{-1}$$

Separating the variables and integrating, we obtain

$$\int_0^t dt = -\frac{l_{f,ice} T_{water} \rho_{ice}}{h_a k_E \left(T_{air} - T_{water}\right)} \int_0^x \left(k_E + h_a x\right) dx$$

For the sheet of ice to grow to a thickness of 0.20 m, we have to wait for a time

$$\begin{aligned} t &= \frac{l_{f,ice} T_{water} \rho_{ice}}{h_a k_E \left(T_{water} - T_{air}\right)} \frac{1}{2h_a} \left[\left(k_E + h_a x\right)^2 - k_E^2\right] \\ &= \frac{1220 \cdot 273 \cdot 920 \cdot \left(4.2^2 - 2.2^2\right)}{10 \cdot 2.2 \cdot (10 - 0) \cdot 2.0 \cdot 10} \mathrm{s} = 8.8 \cdot 10^5 \mathrm{s} \end{aligned}$$

This corresponds to about 10 days, which seems to be a pretty reasonable time span. It decreases noticeably if h_a increases because of stronger winds.

EXAMPLE 7.10. Cooling hot water in cold thick-walled container.

Hot water is poured into a thick walled cylindrical PVC container (Fig. Ex.10, left) that is well sealed at the top and at the bottom. The water is stirred continuously by a magnetic stirrer. As the water cools, temperatures of the water, the outside surface of the container, and the air are measured as functions of time (Fig. Ex.10.1, center; right: enlargement of the initial phase). The temperature of the air is almost constant.

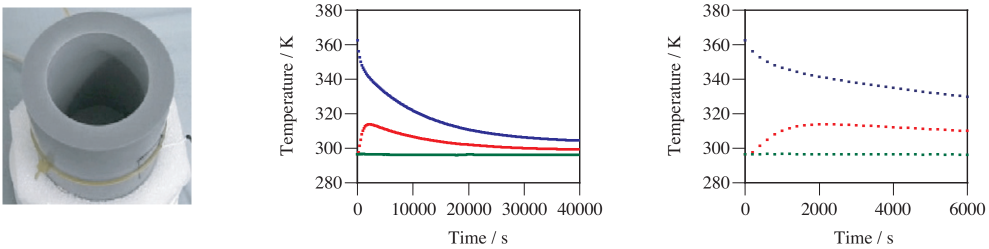


Figure Ex.10.1

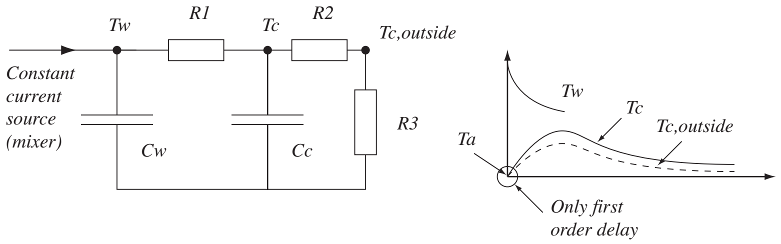
Experimental data: Power of the mixer: 1.0 W; mass of water: 0.30 kg; height of PVC cylinder: 0.105 m; inner radius: 3.00 cm; outer radius: 4.65 cm; density of PVC: 1400 kg/m^3; convective transfer coefficient PVC to air: 12 W/(K·m^2); the convective transfer coefficient from water to PVC is much higher.

(a) Explain the temperature-time diagram in words (note: the temperature of the water initially drops faster than would be expected from a simple exponential decay; the temperature of the outside surface of the container rises with a delay, etc.). (b) Think of the simplest possible model that might explain these observations. Sketch an electric circuit that represents your model. Sketch the corresponding diagram of a dynamical model. (Use the energy representation for your model.) (c) Formulate all equations for your model (without calculating conductances in detail). (d) Use experimental data to estimate the thermal properties of PVC (conductivity and specific heat).

SOLUTION: (a) T_w decreases since water loses entropy to the container. T_c increases first because of high gain of entropy. Then the container loses more and more entropy to the environment: T_c goes down. T_w decreases faster at the beginning than later because the container gets warmer. T_c(outside surface) has a second order delay since entropy takes time to flow through the wall. T_w and T_c stay above T_a because of entropy production by mixer.

(b) The simplest possible model has a capacitor each for water and for the container (Fig. Ex.10.2, left; with only a capacitor for water, there is no independent temperature for the container wall to compute). Voltages represent temperatures of uniform bodies; so T_w is the temperature of the water, T_c is the temperature at the center of the container wall.

Figure Ex.10.2



The resistor between the capacitors represents the transitional layers from the water to the center of the PVC wall. R_2 symbolizes the conductive resistance from the center of the wall to its outer surface, whereas R_3 is used for the convective transition from the surface to the air. From what we know of simple electric or hydraulic *RC* models (Chapter 1), the expected behavior of the temperatures is as in Fig. Ex.10.2, right. Gross features of the real behavior are there, but the delay in the temperature of the outside surface of the wall is absent. We need a better model with the wall divided into two or morel (radial) elements (Fig. Ex.10.3, top). The bottom part of Fig. Ex.10.3 shows a diagram of a dynamical model representing the circuit. In this model, T_{C2} and $T_{C,outside}$ show the expected delay.

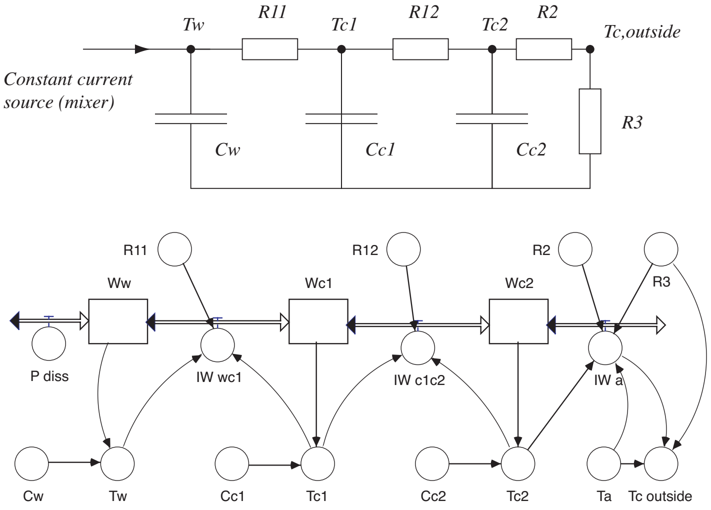


Figure Ex.10.3

(c) The simpler two-node model suffices for demonstrating the physical and mathematical structure. Its equations of balance (of energy) are:

$$\frac{dE_w}{dt} = \mathcal{P}_{mix} - I_{E,wc} \quad , \quad \frac{dE_c}{dt} = I_{E,wc} - I_{E,ca}$$

With capacitive relations

$$T_w = \frac{1}{C_w} E_w \quad , \quad T_c = \frac{1}{C_c} E_c$$

and flow laws

$$I_{E,wc} = \frac{1}{R_1}\left(T_w - T_c\right) \quad , \quad I_{E,ca} = \frac{1}{R_2 + R_3}\left(T_c - T_a\right)$$

the model is complete (except for initial conditions). The surface temperature of the container can be calculated from the results of this model:

$$T_{c,outside} = T_a + R_3 I_{E,ca}$$

(d) It is possible to formulate conditions for the missing material properties of PVC, i.e., its conductivity and its specific heat. The conditions are obtained from considerations of special circumstances. Let us start with the steady state which obtains after a long time (data in Fig. Ex.10.1, center):

$$I_E(to\ air) = I_E(through\ wall) = \mathcal{P}_{mix}$$

The energy current through the wall can be expressed as follows:

$$I_E(\mathit{through\ wall}) = k_{Wall}\frac{A_{Wall}}{\Delta x_{Wall}}\Delta T$$

Let us assume that the wall is flat. Since we know this current from steady-state conditions (it is equal to the power of the mixer), we can calculate the conductivity of PVC:

$$k_{Wall} = \frac{I_E \Delta x}{A\,\Delta T} = \frac{1.0\cdot 0.0165}{2\pi\cdot 0.040\cdot 0.105\cdot 4.3}\frac{\mathrm{W}}{\mathrm{K\cdot m}} \approx 0.15\frac{\mathrm{W}}{\mathrm{K\cdot m}}$$

The capacitance (specific heat) of the container is found as follows. Consider the balance of energy for container for a period of time from 0 s to 200 s:

$$\begin{aligned}\Delta E_c &= \left|\Delta E_{Water}\right| - \left|E_{Loss\ from\ wall}\right|\\ &= mc\left|\Delta T_{water}\right| - A_c h_{loss}\left|\Delta\overline{T}_{surface\ to\ air}\right|\Delta t\\ &= 0.30\cdot 4200\cdot 21\,\mathrm{J} - 2\pi\cdot 0.0465\cdot 0.105\,\mathrm{m}^2\cdot 12\frac{\mathrm{W}}{\mathrm{K\cdot m^2}}\cdot 13\,\mathrm{K}\cdot 2000\,\mathrm{s}\\ &= 17000\,\mathrm{J}\end{aligned}$$

In the first 2000 s, the temperature of the water (0.30 kg) drops by about 21 K. The energy loss of the wall is calculated from its average surface temperature relative to the ambient, its surface area and the convective transfer coefficient. The specific heat is related to the change of energy of the body:

$$\Delta E_c = c_c m_c \Delta T_c \qquad \Delta T_c \approx 30\,\mathrm{K}$$

$$c_c = \frac{\Delta E}{m_c \Delta T_c} = \frac{17000\,\mathrm{J}}{\rho_{\mathrm{PVC}}\pi\left(0.0465^2 - 0.030^2\right)\cdot 0.105\,\mathrm{m}^3\cdot 30\,\mathrm{K}} \approx 1000\frac{\mathrm{J}}{\mathrm{K\cdot kg}}$$

ΔT_c is estimated as follows. Initially, T_c is equal to ambient temperature (25°C). At $t = 2000$ s, the outside of the container is 41°C, on the inside it is nearly as warm as the water (69°C). This gives the container an average temperature of 55°C at $t = 2000$ s. These results agree rather well with values from tables.

7.4 BLACKBODY RADIATION FROM OPAQUE SURFACES

Next let us consider how a body radiates heat into its surroundings. Even though radiation is a rather complex phenomenon, one case can be treated fairly simply—the emission (and absorption) of radiation by an opaque body. Emission and absorption are volumetric processes as discussed in Section 7.1; still the emission from an opaque body looks like the flow of radiation from a surface. Let me describe briefly how this happens.

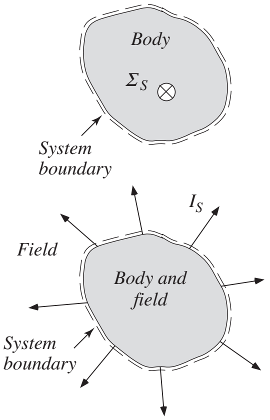


Figure 7.13: Radiation from the surface of an opaque body. There is a constant exchange of radiation between matter and field in the space occupied by both (top). Outside the body we simply see the radiation which is traveling through the field. The radiation seen outside effectively comes from a very thin layer at the surface of the material body (bottom).

A warm body emits radiation to the field occupying the same region, leading to a sink of radiation with respect to the body (and a source with respect to the field). Radiative transport through the field inside the system boundary in Fig. 7.13 is rather complicated. Radiation is emitted and reabsorbed constantly at such rates that the net effect is a flow of heat from hotter to cooler points. Since the body is assumed to be (just about) opaque to the radiation, what we see outside the system must originate from a relatively thin layer at the surface of the body. Outside the space occupied by matter, however, we have a simpler situation. There, radiation is traveling away from the region where it was emitted. If we surround the system by a surface, radiation effectively flows through the field through this boundary.

7.4.1 Hemispherical emission by blackbody surfaces

We are interested in an expression for the fluxes of entropy and energy from the surface of a body which is emitting radiation to its surroundings. In this section, we will discuss only the simplest cases, starting with radiation from the surfaces of *black bodies*. A black body is defined as one which absorbs all the radiation falling upon it. In Section 5.4, we studied blackbody radiation inside a cavity. On the basis of what is known about such radiation, we can motivate the form of the law for radiative transfer from the surface of a body such as the one in Fig. 7.14. If the surface layers of an opaque body have properties which lead to blackbody radiation, the radiation will be the same as if it had originated from a cavity deep inside the system. Since entropy and energy are carried away by radiation, their flux densities must be related to their (volume) densities inside the radiation field. From Section 5.4, Equations (5.83) and (5.86), we know that the density of entropy of blackbody radiation is proportional to the third power of its temperature, while the energy density depends upon the fourth power of the temperature. For this reason the rates at which these quantities flow with radiation from the surface of a body have the same dependence on temperature. It is customary to introduce the *hemispherical rate of emission of energy of a black body* or the *hemispherical emissive power of a black body* $\mathcal{E}_b$, which is defined as the amount of energy emitted by the surface of a black body per unit time and *per unit surface area*. We expect a law of the form:

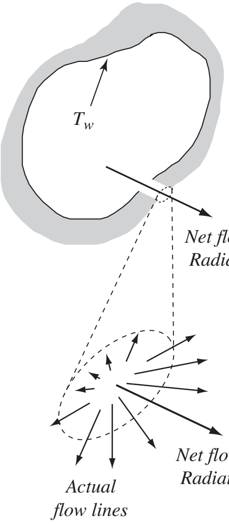


Figure 7.14: Blackbody radiation from the surface layers of an opaque blackbody is the same as that originating from the inside of a cavity in a body (Section 5.4).

$$\mathcal{E}_b = \sigma T^4 \tag{7.33}$$

The index *b* denotes black body radiation. Its counterpart, the *rate of emission of entropy*, $\mathcal{E}_{Sb}$, is expressed by

$$\mathcal{E}_{Sb} = \frac{4}{3}\sigma T^3 \tag{7.34}$$

This is, again, valid for blackbody radiation only. These expressions will be derived in Chapter 12. The constant σ introduced in these relations is called the *Stefan-Boltzmann constant*, and has the value $5.67 \cdot 10^{-8}$ W/(m$^2 \cdot$K^4). $\mathcal{E}_b$ has units W/m^2 and the units of $\mathcal{E}_{Sb}$ are W/K/m^2.

The rates of emission introduced here look very much like current densities that are used to describe the distribution of flows over a surface (see Equ.(7.13)). However, they are not real current densities since the type of transport is rather different from what we know from conduction (i.e., diffusion). In diffusion of charge, substances, or entropy through a material, the fluidlike quantity is transported at one point only in a single direction. Radiation, however, travels in all direction from a point (Fig. 7.14, bottom). Radiation can penetrate itself, meaning it can and will cross through a point in a field in and from all directions. Still, we can integrate the contributions of the transport of, say, entropy or energy over all directions and so obtain the total transfer of the quantity (per unit time and per unit area). This is what the rates of emission in Equ.(7.33) and Equ.(7.34) represent. When this does not cause a confusion, I will call these quantities current densities (fluxes per unit area).

If the rates of emission (the current densities) are constant over a surface, the currents of entropy and energy from the entire surface of a body are obtained simply by multiplying the emission rates by the surface area A:

$$I_{S,rad} = -\frac{4}{3}\sigma A T^3 \tag{7.35}$$

$$I_{E,rad} = -\sigma A T^4 \tag{7.36}$$

You should notice an important point: the relation between currents (fluxes) of entropy and of energy, which applies to the heating or cooling of a body (Equ.(4.21)), does not hold in the case of radiative fluxes through the radiation field. Heating and cooling of the material body, however, take the form of sources and sinks of entropy and energy for which the simple and direct relation between source rates of entropy and energy holds; see Equ.(7.41).

Net radiative energy flux for a black body radiator. In general, the expressions in Equations (7.35) and (7.36) are not the net fluxes, since the body might absorb heat from another piece of matter radiating towards it. It is instructive to derive the net energy flux for a black body totally surrounded by another black body at a different temperature T_w (such as the small piece of matter in the cavity in Fig. 7.15). Remember that the radiation field set up by the walls of the cavity is isotropic and the same at every point inside. This means that a point at the surface of the small body surrounded by the walls sees blackbody radiation coming at it at the same rate from all directions. Therefore the amount of energy per second and per unit surface area radiated toward, i.e., incident upon the body is

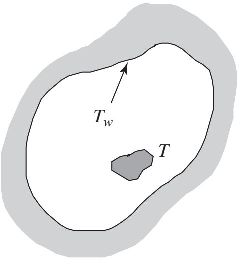


Figure 7.15: A cavity serves as a container of blackbody radiation. The walls are at a temperature T_w, while the small body inside the cavity has a temperature T.

$$\mathcal{G} = \sigma T_w^4 \tag{7.37}$$

$\mathcal{G}$ is called the *irradiance* of the surface. Since a black body absorbs all the radiation falling upon its surface, the net flux of energy with respect to the chunk of matter inside the cavity is given by

$$I_{E,rad,net} = -\sigma A\left[T^4 - T_w^4\right] \tag{7.38}$$

which is the difference between the rates of emission and absorption, i.e., the difference between (7.33) and (7.37). This result will be extended to bodies other than black bodies in Section 7.4.4.

7.4.2 Cooling and Heating of Bodies by Emission and Absorption

As discussed in Section 7.1.4, bodies can emit and absorb radiation. These processes are the result of the interaction of bodies and fields which occupy the same region of space. Therefore, emission and absorption are volumetric phenomena calling for source rates for their formal description. (This discussion could be extended to surfaces if we consider a thin layer around a body absorbing and emitting radiation. We can also extend the formalism by introducing absorption and emission rates per unit surface area.) Here, I will motivate the relationship between the source rates and the rates of production of entropy on the one hand, and the source rates of energy accompanying the processes on the other.

Take the model of a uniform body at temperature *T*. For the sake of argument, let the body emit entropy and energy to the field occupying the same region of space. (The reasoning also applies to the case of absorption of radiation.) The rates of emission of these two quantities are equal to the rates of change of the entropy and energy of the body if there are no other modes of transfer present:

$$\dot{S} = \Sigma_S \tag{7.39}$$

$$\dot{E} = \Sigma_E \tag{7.40}$$

Remember that processes are reversible in the model of uniform processes. Since the rates of change are related by the Gibbs Fundamental Form for a simple body (see, for example, Equ.(4.35)), the rates of emission satisfy the equation

$$\Sigma_E = T\,\Sigma_S \tag{7.41}$$

which means that the rate of emission of entropy to the field inside the system is equal to the rate of energy emitted divided by the temperature of the body. This result holds for the chunk of matter occupying the region of space in Fig. 7.16. It neglects the fact that entropy is produced as a result of the emission (and absorption) of radiation. However, as far as the body is concerned, this point is immaterial; it does not affect the balance of entropy for the chunk of matter, since our model associates the irreversibility with an additional element between the body and the field (see Fig. 7.6). For the body, emission of heat to a field has the same effect as cooling by conduction as a result of direct contact of a uniform body with its surroundings.

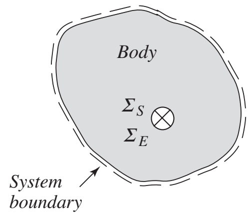


Figure 7.16: A material body and a radiation field occupy the same region of space (only the body is shown). Emission of radiation means that the body loses entropy and energy (and other quantities) at every point inside the field. In this view, the body and field are two separate physical systems.

We should be interested in the rate of production of entropy as a result of the emission or the absorption of radiation. If the processes of emission and absorption are irreversible, the source rates of entropy for the body and the field are not the same. We have expressed this point in Equ.(7.8):

$$\Sigma_S = -\Sigma_{S,field} + \Pi_S \tag{7.42}$$

Together with Equ.(7.41), we can express this in the form valid for the source rate of energy:

$$\Sigma_E = T\left(-\Sigma_{S,field} + \Pi_S\right) \tag{7.43}$$

Consider the case of no entropy being supplied by radiation which is absorbed by a body. Under these circumstances, all the entropy leading to the heating of the body must have been produced, the process being completely irreversible. Put differently, all the energy supplied to the body via the field has been dissipated, and Equ.(7.43) is formally equivalent to the expression for the relationship between the rate at which energy is dissipated and entropy is generated (see Equ.(4.23) of Chapter 4).

In summary, we may interpret the results of emission and absorption of entropy in terms of the cooling or heating of bodies. The only difference from the case treated so far, i.e., heating by conductive surface currents, is that we have to deal with source rates of entropy and energy, Equ.(7.41), instead of fluxes (Equ.(4.21) of Chapter 4). In this model, irreversibility is associated with an extra element placed between the field and the body (Fig. 7.6).

7.4.3 Emission, Absorption, and the Production of Entropy

Irreversibility of emission. Next, we should discuss the rate of production of entropy in the emission and absorption of radiation. Consider the case of emission: the rate at which entropy is emitted by the body to the field is smaller than the rate at which it flows away from the surface of the body (Fig. 7.13). This can be shown quite easily. Consider the model of a uniform body at temperature T for which

$$\Sigma_E = T\Sigma_s \tag{7.44}$$

On the other hand, the flux of entropy through the field at the surface of the body is four-thirds this quantity; see Equations (7.35) and (7.36):

$$I_{S,rad} = \frac{4}{3}\frac{I_{E,rad}}{T} = \frac{4}{3}\frac{\Sigma_E}{T}$$

or

$$I_{S,rad} = \frac{4}{3}\Sigma_S \tag{7.45}$$

We have to conclude that more entropy leaves the space occupied by the body than has been emitted by the body to the field. Therefore, entropy must have been produced in the volume occupied by radiating matter. In the same manner, we can prove that it is impossible for a body to just absorb entropy from a source at the same temperature. Entropy would have to be destroyed, which we know to be impossible. Therefore, it is impossible for a body to absorb entropy at the same temperature without emitting entropy at the same time.

Irreversibility of emission and absorption of two interacting bodies. Consider now the example of combined emission and absorption in a quantitative manner. Take two bodies having geometries such as in Fig. 7.15: the walls of a cavity completely surround a smaller body. Assume both surfaces to be black bodies. Let the smaller body be the hotter one, with temperature T_1 and surface area A. Then the net flux of energy from the smaller to the larger surface is given by Equ.(7.38) where T_2 is the temperature of the enclosure. Now we can compute the rate of production of entropy in the two radiative interfaces between bodies and field (as in Fig. 7.6) combined. With the help of Equ.(7.43) we find that

$$\Pi_S = \Pi_{S1} + \Pi_{S2} = \frac{1}{T_1}\Sigma_{E1} + \Sigma_{S,field1} + \frac{1}{T_2}\Sigma_{E2} + \Sigma_{S,field2}$$

The entropy emitted by one body to the field travels through the field and disappears from the field at the location of the second body, so the source rates for the field cancel, so we have

$$\Pi_S = \Sigma_{E1}\left(\frac{1}{T_1} - \frac{1}{T_2}\right) \tag{7.46}$$

Σ_{E1} corresponds to the flux of energy with respect to body 1. This expression is larger than zero since, if $T_1 > T_2$ (Σ_{E1} is negative). The flux of energy will be calculated according to Equ.(7.38) which transforms the production rate of entropy into

$$\begin{aligned}\Pi_S &= \frac{\sigma A}{T_1 T_2}\left(T_1 - T_2\right)\left(T_1^4 - T_2^4\right)\\ &= \frac{\sigma A}{T_1 T_2}\left(T_1 - T_2\right)\left(T_1^2 - T_2^2\right)\left(T_1^2 + T_2^2\right)\\ &= \frac{\sigma A}{T_1 T_2}\left(T_1 - T_2\right)^2\left(T_1 + T_2\right)\left(T_1^2 + T_2^2\right)\end{aligned} \tag{7.47}$$

This expression is larger than zero as long as one of the bodies is hotter than the other and it vanishes if they have the same hotness. Therefore, it does not matter which of the bodies we assume to possess the higher temperature. Emission and absorption of entropy by bodies at different temperatures is necessarily dissipative. We could have performed the computation of the balance of entropy for each of the bodies separately. Using the result for one of them, you can convince yourself that the particular statements made above regarding the irreversibility of emission and the impossibility of absorption without simultaneous emission are correct.

The irreversibility of radiative processes is not limited to the absorption and emission of (blackbody) radiation. Conversion of monochromatic radiation into blackbody radiation and the scattering of radiation have to be added to the list of irreversible processes. The fact that irreversibility necessarily accompanies radiative transfer is of importance for power engineering (see Chapters 9, 15, and 16).

A surface in the light of the Sun. Now consider a particularly relevant case, that of (the surface of) a black body at temperature T in the light of the Sun (Fig. 7.17). Entropy flows with solar radiation toward this surface and, since the body is a black radiator, is completely absorbed. Because of this process of absorption, there is an entropy production rate of Π_{S1}. Let $\mathcal{G}$ be the irradiance of the surface. Solar radiation is nearly black body radiation, so the entropy current incident upon the body of surface area A is

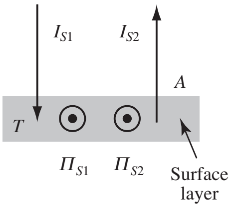


Figure 7.17: Entropy fluxes and entropy production rates for a surface layer of a body in the light of the Sun.

$$I_{S1} = \frac{4}{3} A \frac{\mathcal{G}}{T_{sun}} \tag{7.48}$$

This results from Equations (7.35) and (7.36). T_{sun} is the temperature of the surface of the Sun that emits the radiation which we receive; at the same time this is the temperature of the radiation (see Chapters 12 and 16). If the body only absorbed light, the balance of entropy would be

$$\dot{S} = I_{S1} + \Pi_{S1}$$

so the entropy production rate due to absorption equals

$$\Pi_{S1} = \dot{S} - I_{S1} = \frac{1}{T} I_{E,rad1} - I_{S1} = \frac{1}{T} A\mathcal{G} - I_{S1} = \frac{1}{T} A\mathcal{G} - \frac{4}{3} A \frac{\mathcal{G}}{T_{sun}}$$

The last step is a consequence of Equ.(7.48). We arrive at

$$\Pi_{S1} = \frac{1}{T T_{sun}}\left(T_{sun} - \frac{4}{3} T\right) A\mathcal{G} \tag{7.49}$$

for the production rate of entropy due to absorption of sunlight. Note that this expression becomes negative if T approaches T_{sun}. This tells us that we are not allowed to just consider absorption of radiation by a body; we always have to add entropy production due to emission. (For the case of Sun and Earth, there is no problem since the surface temperature of our planet is so much lower than that of the Sun.)

Since the body is warm at temperature T, it emits an entropy current equal to

$$I_{S2} = -A\sigma\frac{4}{3}T^3 \tag{7.50}$$

Equ.(7.45) tells us that a quarter of this current comes from entropy produced (and not from entropy emitted):

$$\Pi_{S2} = \frac{1}{3}A\sigma T^3 \tag{7.51}$$

The total entropy production rate allows us to make an interesting observation. Summing up the contributions from absorption and emission and setting $T = T_{sun}$, yields

$$\Pi_S = \frac{1}{3}A\sigma T_{sun}^3 - \frac{1}{3}\frac{1}{T_{sun}}A\mathcal{G}$$

which is equal to zero since there is no net exchange of radiation between the Sun and the Earth (with both at T_{sun}). If we accept this, we see that $\mathcal{G}$ must be equal to $\sigma T_{sun}{}^4$, meaning we can bring a body to T_{sun} if the incoming radiation has the intensity of sunlight at the surface of the Sun (which we can get by concentrating the Sun's light).

7.4.4 Radiative Properties of Gray Surfaces

So far we have limited ourselves to the case of blackbody radiation. We should now take a closer look at the emission and absorption of heat from surfaces not having blackbody properties. Since a black surface was defined as one which absorbs all incident radiation, we will now be concerned with *gray* surfaces, which absorb only a fraction thereof.

The absorptivity of gray surfaces. To describe the absorptive properties of a surface we need two quantities. The first was introduced above—the *irradiance* $\mathcal{G}$ which measures the total flow of energy incident on a surface per unit area. We need the *rate of absorption of energy per unit area* abbreviated by $\mathcal{A}$. If $\mathcal{A}$ equals $\mathcal{G}$, the surface is a perfect absorber; if it is smaller than $\mathcal{G}$, the surface is said to be gray. We introduce the *absorptivity* a, the ratio of rate of absorption and irradiance, so

$$\mathcal{A} = a\mathcal{G} \tag{7.52}$$

For a black body $a = 1$. Sometimes, the *reflectivity* ρ of an opaque surface is introduced in place of the absorptivity. The energy not absorbed is reflected (Fig. 7.18), which means that the reflectivity and the absorptivity are related by

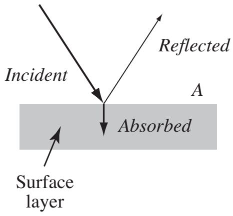


Figure 7.18: Incident ray, absorption and reflection.

$$\rho = 1 - a \tag{7.53}$$

Kirchhoff's Law. The interesting question now concerns the emissivity of nonblack surfaces. How does it compare to blackbody radiation? Let us once more consider the radiation inside a cavity. Since the field in the cavity is that of blackbody radiation, the irradiance in this enclosed space is the emissive power of a black body having the temperature of the walls:

$$\mathcal{G} = \mathcal{E}_b \tag{7.54}$$

In other words, the irradiance is the hemispherical power in a blackbody field, Equ.(7.33). Whether the walls are black surfaces does not matter. If they are not, the combination of emittance and reflectance still leads to radiation with blackbody properties. This is the reason why the radiation in a cavity is called *blackbody radiation*. Since the material of which the walls of the cavity are made does not play a role, the only factor determining the radiation in the cavity is the temperature of the walls.

An arbitrary body with absorptivity a is now introduced into the cavity (Fig. 7.19). After some time, stationary conditions will have been reached, the temperature of the small body will be the temperature of radiation in the cavity, and the emissive power of the body will be the fraction of the irradiance absorbed by the body. Since the radiation in the cavity is that of a black body, the emissive power of the body inside the cavity must equal the product of absorptivity and blackbody emissive power:

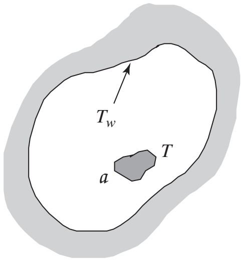


Figure 7.19: A body in a black body radiation field of the cavity. Its surface has an absorptivity equal to a.

$$\mathcal{E} = a\mathcal{E}_b \tag{7.55}$$

This is a form of *Kirchhoff's law*. It states that *the emissive power of a body is a fraction of the emissive power of a black body at the same temperature, where the fraction is the absorptivity*.

If we introduce the *emissivity* e of the body as the fraction of the emissive power of a black body at the same temperature,

$$\mathcal{E} = e\mathcal{E}_b \tag{7.56}$$

we can state Kirchhoff's law by saying that *the emissivity is equal to the absorptivity*. (This statement has to be qualified: they are equal at the same frequency of light; see Section 7.4.6.) Values of the absorptivity (or emissivity) of some materials are listed in Table 7.3. In general, they depend upon the temperature of the surface (Fig. 7.20).

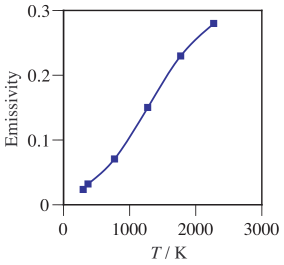


Figure 7.20: Emissivity of tungsten, as a function of temperature.

The net radiant flux for a gray body surrounded by blackbody radiation. If the small body in the cavity has a different temperature, then the rate at which energy is absorbed by the body is not equal to the rate at which it is emitted. The net source rate of radiant energy is calculated to be

$$I_{E,rad} = -A\left(\mathcal{E} - a\mathcal{G}\right) \tag{7.57}$$

A is the surface area of the body in the cavity. Since the emissive power is given by Kirchhoff's law, and since the irradiance in the cavity is equal to the emittance of the walls, we ultimately obtain the following expression for the flux of radiant energy from the surface of the body at temperature T irradiated from a distant surface at temperature T_w which completely surrounds it:

$$I_{E,rad} = -aA\left[\mathcal{E}_b(T) - \mathcal{E}_b(T_w)\right] \tag{7.58}$$

The rate is the difference between the emission and absorption rates as calculated for a blackbody surface, multiplied by the absorptivity of the surface. The blackbody emissive power is the same as that given by Equ.(7.55). In summary, the laws stated here allow us to calculate the flux from a surface area *A* which has a temperature *T*, and which is subject to radiation from surroundings at a temperature T_w:

$$I_{E,rad} = -eA\sigma\left(T^4 - T_w^4\right) \tag{7.59}$$

(the absorptivity has been replaced by the emissivity). Remember that this equation holds only for the particular geometry used in the example: the body is completely surrounded by the walls of the cavity (actually, a large space serves the same purpose; the condition is simply that the radiation of the body is not reflected back). As a result, all the radiation emitted by the walls will be incident upon the body and vice versa. For different geometries, where only part of the radiation emitted by either of the bodies strikes the other surface, the result is much more complicated. In such cases, it is customary to write the result in just about the same form with an additional factor (called the *shape factor*) taking care of the difference (Chapter 12).

Table 7.3: Emissivities and solar absorptivities

Substance		**Emissivity**[a] e	**Absorptivity for solar radiation** a
Aluminum	Polished	0.03	0.09
	Anodized	0.84	0.14
	Foil	0.05	0.15
Brick	Red (Purdue)	0.93	0.63
Concrete		0.88	0.60
Earth	Plowed field		0.75
Galvanized sheet metal	Clean, new	0.13	0.65
	Oxidized, weathered	0.28	0.80
Glass, window		0.92	
Ice	Smooth	0.97	
Paints	Black (Parsons)	0.98	0.98
	White, acrylic	0.90	0.26
	White, zinc oxide	0.93	0.16
Paper	White	0.95	0.28
Sandstone		0.85	
Snow	Fine particles, fresh	0.82	0.13
	Ice granules	0.89	0.33
Soot, coal		0.95	
Water	Deep	0.96	
Wood	Sawdust	0.75	
	Oak, planed	0.90	

a. For a temperature of roughly 300 K.

Radiant exchange between extended parallel plates. Here, we will derive the relation for the case of two extended gray surfaces facing each other in such a way that all the radiation originating from one of the bodies is intercepted by the other (Fig. 7.21). This geometry is found, for example, in flat-plate solar collectors. The two plates will be distinguished by indices 1 and 2. Their radiative properties will be expressed using the emissivities (absorptivities), and the reflectivities. The derivation is of use not only for its result but for the concepts and procedures as well.

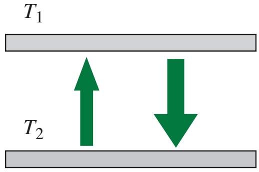


Figure 7.21: Radiant heat exchange between two extended parallel plates. If their temperatures are different, there will be a net flux of energy and entropy from the hotter to the cooler of the plates. (Energy flows are shown here.)

In the course of the derivation, we will need an expression for the total flux of energy per unit area emanating from each of the plates. Since the plates have gray surfaces, they will not absorb all the radiation falling upon them; rather, part of the radiation will be reflected. It is common to call the total flux per unit area, i.e., the *sum of what is emitted and what is reflected*, the *radiosity* $\mathcal{B}$ of the surface. For the plates, the radiosities are

$$\begin{aligned} \mathcal{B}_1 &= e_1 \sigma T_1^4 + \rho_1 \mathcal{G}_1 \\ \mathcal{B}_2 &= e_2 \sigma T_2^4 + \rho_2 \mathcal{G}_2 \end{aligned} \tag{7.60}$$

The reflectivities are related to the absorptivities (emissivities) of a surface, and the irradiance of one of the plates is the radiosity of the other. Therefore,

$$\begin{aligned} \mathcal{B}_1 &= e_1 \sigma T_1^4 + (1 - e_1)\mathcal{B}_2 \\ \mathcal{B}_2 &= e_2 \sigma T_2^4 + (1 - e_2)\mathcal{B}_1 \end{aligned}$$

If we insert the radiosities expressed by Equ.(7.60) into this result, we obtain the following relations for the radiosities of the parallel plates:

$$\begin{aligned} \mathcal{B}_1 &= e_1 \sigma T_1^4 + (1 - e_1)\left(e_2 \sigma T_2^4 + \rho_2 \mathcal{G}_2\right) \\ \mathcal{B}_2 &= e_2 \sigma T_2^4 + (1 - e_2)\left(e_1 \sigma T_1^4 + \rho_1 \mathcal{G}_1\right) \end{aligned}$$

Now, the net flux density of energy radiated from plate 1 to plate 2 is the difference of the radiosities:

$$\mathcal{B}_1 - \mathcal{B}_2 = e_1 e_2 \sigma T_1^4 - e_1 e_2 \sigma T_2^4 + (1 - e_1)(1 - e_2)(\mathcal{G}_2 - \mathcal{G}_1)$$

Since $\mathcal{B}_1 - \mathcal{B}_2 = \mathcal{G}_2 - \mathcal{G}_1$, this is equivalent to

$$(\mathcal{B}_1 - \mathcal{B}_2)\left[1 - (1 - e_1)(1 - e_2)\right] = e_1 e_2 \sigma \left(T_1^4 - T_2^4\right)$$

A little algebra finally yields the expression for the net energy flux flowing from the hotter to the cooler of the two parallel plates:

$$I_{E,rad,net} = \frac{\sigma A\left(T_1^4 - T_2^4\right)}{\dfrac{1}{e_1} + \dfrac{1}{e_2} - 1} \tag{7.61}$$

For blackbody surfaces, the emissivities are equal to 1 and we regain the simpler expression already derived in Equ.(7.38).

7.4.5 The Heat Loss Coefficient for Flat-Plate Solar Collectors

Solar collectors provide a very nice application of the kind of heat transfer we have discussed so far. Basically, all three modes of transport take place in the process, which leads to loss of heat from a collector to the surroundings, with radiation and convection at interfaces being the most important.

Solar collectors receive radiation from the Sun; they lose heat to the surroundings when heated above the level of ambient temperature; and they remove heat via fluid flow through the device (Fig. 7.22). Here, we will consider the problem of calculating heat loss. Removal of heat will be dealt with in Chapter 8, while details of the absorption of radiation will be the subject of Chapter 16.

To define the problem of the exchange of entropy and energy with the surroundings, take a closer look at Fig. 7.22. A typical collector consists of an absorber for solar radiation, a duct for the fluid below the absorber which is insulated at the back, and possibly one or two glass covers to reduce top heat loss. The latter process will be the subject of interest in this section. Naturally, heat may also be lost to the back and to the sides, but these effects will not be considered here.

Assume a collector having a single cover made of a sheet of glass. Heat loss from the absorber plate to the environment is the result of the combined effects of radiation and convection from the plate to the cover and from the cover to the air surrounding the collector. Radiation and convection act as parallel modes of transfer from one body to the next, while the transports from the absorber to the cover, and from the cover to the surroundings are in series. Therefore, the combined effect of all modes of transfer can be described by the simple equivalent circuit also shown in Fig. 7.22.

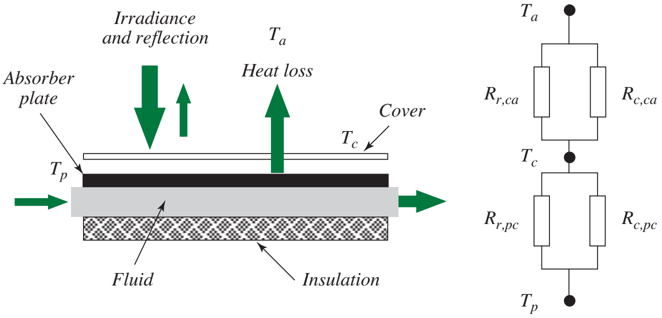


Figure 7.22: Heat loss of a solar collector occurs mostly through the top. The entropy and energy not carried away by the fluid will be transferred to the surroundings. The network on the right symbolizes the combined effect of radiation and convection from the absorber plate to the cover (here it is a single sheet of glass), and from the cover to the surroundings. Subscripts *p*, *c*, and *a* denote the absorber plate, the cover, and the ambient air, respectively, while *r* and *c* stand for radiation and convection.

If we use heat transfer coefficients *h* instead of resistances *R* to calculate the total energy flux due to loss, we can write

$$I_{E,loss} = A_c U_t \left(T_p - T_a\right) \tag{7.62}$$

where

$$\frac{1}{U_t} = \frac{1}{h_{pc}} + \frac{1}{h_{ca}} = \frac{1}{h_{r,pc} + h_{c,pc}} + \frac{1}{h_{r,ca} + h_{c,ca}} \tag{7.63}$$

A_c is the net surface area of the collector (essentially the surface area of the absorber), and U_t symbolizes the total heat transfer coefficient for top loss. Note that conductive transport through the thin glass cover has been neglected in this analysis. If we wish to compute the heat loss coefficient, we have to be able to quantify the heat transfer coefficients for convection and for radiation. While the former pose a problem which we cannot solve at this point,[3] the latter can be expressed in terms of what we have learned so far.

The radiative heat transfer coefficient from the absorber to the cover can be written in the form which will be derived in Example 7.13. If we apply the result derived for radiation between two parallel plates, Equ.(7.61), we obtain

$$h_{r,pc} = \frac{\sigma\left(T_p^2 + T_c^2\right)\left(T_p + T_c\right)}{1/e_1 + 1/e_2 - 1} \tag{7.64}$$

Obviously, in order to calculate this heat transfer coefficient, we need to know the temperatures of both the absorber and the cover. While the former has to be specified, the latter must be obtained as part of the solution of the problem. Now we still need the radiative transfer coefficient for transport from the cover to the surroundings. Radiation occurs between the cover, which has a particular emittance, and the sky for which we use an equivalent blackbody temperature[4] T_{sky}. Therefore, the coefficient turns out to be

$$h_{r,ca} = \sigma e_c\left(T_c^2 + T_{sky}^2\right)\left(T_c + T_{sky}\right) \tag{7.65}$$

See Equ.(7.59). The temperatures strongly depend upon operating conditions, while the convective heat transfer coefficient from the cover to the ambient air is a function of wind speed. Typical values for heat loss coefficients of collectors of the type described are around 5 W/(K·m^2).

7.4.6 Selective Absorbers and Emitters

According to Kirchhoff's law, the emissivity of a surface equals its absorptivity (Section 7.4.4). So why are there two numbers in Table 7.3 for a material, one for its emissivity and one for its absorptivity for solar radiation?

At first sight, Kirchhoff's law seems to pose a riddle. If emissivities equal absorptivities, an application of the laws of radiative heat transfer seem to indicate that bodies should have the same steady-state temperature irrespective of their emissivity/absorptivity. If they are better absorbers of solar radiation, they are better emitters of their own radiation which leads to the same result in balance. However, we know that this is not the case. Different bodies attain different temperatures in Sunlight, the most famous case being our planet whose radiative properties seem to be changing as a result of human activity which is assumed to change the Earth's temperature. Simpler and clearer cases are known from everyday life. Black surfaces get hot faster than shiny white ones when exposed to the Sun.

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[3] We simply have to assume reasonable convective heat transfer coefficients for both transfer from the cover to the surrounding air, and from the absorber to the cover. See Duffie and Beckman (1991; p. 160–176) for a detailed discussion of the problem.

[4] See Duffie and Beckman (1991), p. 158.

The reason is simple: Kirchhoff's law applies, but it applies separately for different types of radiation (see Chapter 14; a and e are equal, but their values change with the wavelength or frequency of the light).

At this point, we shall consider a simple case and apply a steady-state energy balance to find the temperature of a body exposed to sunlight. Take a body completely surrounded by air. The air acts as a black body environment which means that the radiative interaction of body and air lead to a flux of energy of the form of Equ.(7.59). There is an additional interaction because of sunlight. If A_s is the surface area effectively exposed to the Sun, and if $\mathcal{G}_s$ and a_s are the irradiance of solar light and the absorptivity of the surface with respect to sunlight, the rate of absorption will be

$$\Sigma_{E,abs} = a_s A_s \mathcal{G}_s \tag{7.66}$$

If we also allow for convection between the body and the air, the balance of energy in steady-state takes the form

$$a_s A_s \mathcal{G}_s = eA\sigma\left(T^4 - T_{air}^4\right) + Ah\left(T - T_{air}\right) \tag{7.67}$$

Disregard convection for the moment. It is clear that since a_s and e are not equal in general, the steady-state temperature T of the body will depend upon the ratio of emissivity to solar absorptivity. Take a perfect black body ($a = e = 1$ for all wavelengths) or bodies where $a_s = e$ as reference. A body with $a_s/e > 1$ should be warmer than the reference whereas it should be cooler if $a_s/e < 1$.

7.4.7 The Dynamics of an Incandescent Bulb

Here is a nice little application of a dynamical system involving radiation. A small light bulb is connected to a power supply whose voltage is quickly increases from zero and then reduced back to zero equally fast. Voltage across and electric current through the bulb are measured as functions of time (Fig. 7.23, left). If the data is plotted in a current-voltage diagram, we obtain an interesting dynamical characteristic of the bulb (mostly of its tungsten filament; Fig. 7.23, right) which is rather different from the typical steady-state characteristic curves of incandescent bulbs.

Actually, the last leg of the odd shaped characteristic line (part (2) in Fig. 7.23, right) is close to the steady-state characteristic curve; it is the first part that deviates from expectations. The reason for this is simply the drastic changes of temperature of the filament (and other parts of the bulb) as a result of increasing and decreasing dissipation of the energy released in the electric process and the radiating away of the entropy of the hot wire. For the first part of the process, the wire is still cold which leaves the resistivity of the material at a low value. Therefore, the electric current increases faster than in standard experiments where the voltage is raised very slowly step by step. After the peak of dissipation, the wire is hot and the characteristic follows more closely what we know from simple steady-state experiments.

A strongly simplified model of the phenomenon is shown in Fig. 7.24 (left). At its center, it uses the balance of entropy of the tungsten wire in the bulb. There are two main processes taking place: entropy is produced in the wire and is radiated away. The entropy of the body defines its temperature which, in turn, is used to express the constitutive laws for the production rate and the source rate of entropy. Details of the model concern the resistivity and the emissivity of tungsten which are functions of temperature (the temperature dependence of the entropy capacitance has not been included).

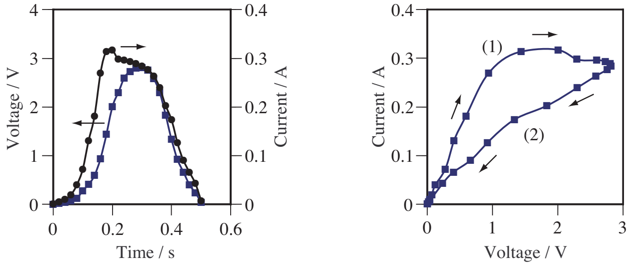

Figure 7.23: Data of voltage and electric current for a small incandescent bulb that is quickly turned on and off, as functions of time (left) and in the form of the current-voltage characteristic (right).

The law of balance of entropy for the tungsten wire includes the production rate due to dissipation of energy and a source rate as a result of radiation:

$$\dot{S} = \Pi_S + \Sigma_{S,rad} \tag{7.68}$$

We need three constitutive laws to complete the model: one for the temperature-entropy relation of a body made of tungsten (Section 4.5), one for the production rate of entropy (Section 4.4.2), and the last for the radiation of entropy by the hot wire. Fundamentally, each of the relations includes a temperature dependent material property, namely, specific entropy capacitance in the first case, electric resistivity in the second, and emissivity in the third.

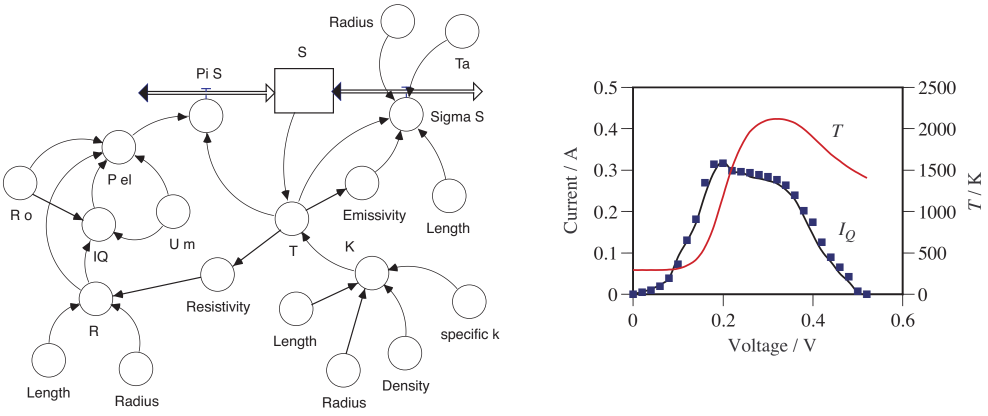

Figure 7.24: Diagram of a simple dynamical model of the tungsten wire (left) and data (squares) and simulation results for the electric current through the wire and the temperature of the wire (right).

In the present model, I have chosen not to include the temperature dependence of the entropy capacitance. Rather, an average value of κ has been estimated as follows. The specific heat of tungsten varies between 130 J/(K·kg) and 180 J/(K·kg) in the expected range of temperatures (300 K to 3000 K). We can divide an average value of the specific heat by an average value of the temperature to obtain a first approximation to the entropy capacitance:

$$K_{av} = m\left(c/T\right)_{av} \tag{7.69}$$

The mass m of the wire is obtained from the density and the volume where the latter is calculated from the length and radius of a cylindrical body (the radius will turn out to be a critical unknown parameter of the model). With a constant entropy capacitance, the entropy temperature relation is simply

$$S = K_{av}T \tag{7.70}$$

Now we need an expression for the entropy production rate in the tungsten wire. The first step is simple: the production rate equals the electric dissipation rate divided by the temperature of the wire. The latter is obtained from the entropy, Equ.(7.70), and the former equals the voltage times the electric current. The voltage across the wire is a part of the measured voltage U_m that will be used as an input to the model (see the model diagram in Fig. 7.24). It represents only a part of U_m since there are connecting elements (copper connectors) to the tungsten filament inside the bulb. They will be given a constant (unknown) resistance R_0, another simplification in our model:

$$U_T = \frac{R}{R+R_0}U_m$$

where R and U_T are the resistance of and the voltage across the filament, respectively. The electric current through the bulb will be

$$I_Q = \frac{1}{R+R_0}U_m$$

so the entropy production rate turns out to be equal to

$$\Pi_S = \frac{1}{T}\frac{R}{\left(R+R_0\right)^2}U_m^2 \tag{7.71}$$

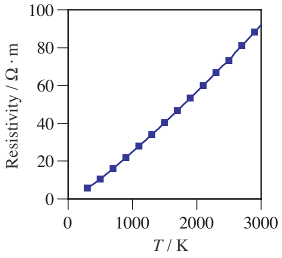

Figure 7.25: Electric resistivity of tungsten, as a function of temperature.

The second important material property of the filament concerns its resistance or rather, its *resistivity*. It can be introduced as an interpolation function of data in Fig. 7.25. The resistance of the wire is

$$R = \rho_{el}\frac{l}{\pi r^2} \tag{7.72}$$

Finally, the rate of emission of entropy from the tungsten wire must be found. The source rate of entropy equals the source rate of energy divided by the temperature (Equ.(7.44)). The energy source rate equals the energy flow which can be obtained from Equ.(7.59):

$$\Sigma_S = \frac{1}{T}aA\sigma\left(T^4 - T_a^4\right) \tag{7.73}$$

The absorptivity a equals the emissivity which is obtained from data as in Fig. 7.20.

This pretty much concludes the dynamical model. It can be used to predict the electric current through the bulb which can be compared to measurements (see the diagram on the right in Fig. 7.24). In addition, the model yields results on the temperature of the filament. This confirms the initial idea about why the bulb behaves in the manner observed. The temperature of the wire rises slowly at first and then stays high beyond the point in time where the lamp is turned off again.

The good agreement between simulation and measurements should not be taken as a sign of the model's perfection. It suffers from several shortcomings, but clearly it gives an understanding of the most basic phenomena. Structurally, we have left out at least a couple of important elements. For one, the thermal and electric dynamics of the connecting (copper) elements should be included in the model similarly to how this was done for the filament. Secondly, the entropy radiated from the filament is intercepted at least partly by the glass of the bulb; we might want to include the glass as an additional element in our model. Then, as mentioned before, the entropy capacitance should not be taken as constant—the range of temperatures is too great. Moreover, to get a halfway reliable model we should measure the geometric properties of the filament and connecting wires as carefully as possible.

QUESTIONS

16. Why is the measure of the flow of entropy through the radiation field not a standard current? Why is the measure of the rate of emission from a body to a field (or vice-versa) a source rate and not a current?
17. Why does the basic relation $I_E = TI_S$ not apply to radiation flowing through the radiation field? Does the analogous relation hold for sources and sinks of energy and entropy in radiative interaction between bodies and fields?
18. What simple argument shows that emission of radiation is dissipative?
19. Sunlight, i.e., the light flowing through the radiation field from Sun to Earth, has a temperature of nearly 6000 K. The light radiated by our planet has a temperature close to 300 K. What is the ratio of the entropy per unit energy for these two types of radiation?
20. If, on balance, the Earth radiates away as much energy as it receives from the Sun, how much entropy is produced by the planet?
21. Consider an imaginary sphere having a large radius (such as the distance from Sun to Earth) drawn around the Sun and consider one square meter of this surface through which the Sun's light is flowing. Are the values of $\mathcal{E}$ and $\mathcal{E}_S$ the same as at the surface of the Sun?
22. Consider the opaque surface of a warm body surrounded by air. There is radiation alongside convective entropy transfer. Are the processes in parallel or in series?
23. Imagine a spherical rock in space (a planet without atmosphere) exposed to the light of the Sun. In steady-state it will attain a certain temperature. Does this temperature depend upon the emissivity of its surface?
24. Does galvanized sheet metal get hotter in sunlight than a surface painted black?

EXAMPLE 7.11. Surface temperature of the Sun.

The radiation originating in the thin surface layer of the Sun has properties almost like those of a black body. (a) Using the solar constant (1370 W/m^2), the distance from the Earth to the Sun ($1.5 \cdot 10^8$ km), and the Sun's radius (700,000 km), derive the temperature of its surface. (b) Calculate the rate at which entropy is emitted by the total surface.

SOLUTION: (a) The solar constant $\mathcal{G}_{sc}$ is the value of the energy flux per unit normal area at the distance of the Earth. Using this value we calculate the emissive power of the Sun (called the luminosity L):

$$L = 4\pi d^2 \mathcal{G}_{sc} = 4\pi \cdot \left(1.5 \cdot 10^{11}\,\text{m}\right)^2 \cdot 1370\,\text{W}/\text{m}^2 = 3.9 \cdot 10^{26}\,\text{W}$$

The Sun approximates a black body which does not receive radiation from the surrounding space. Therefore, Equ.(7.36) applies to the relation between emissive power and temperature, leading to a value of

$$T = \left(\frac{L}{4\pi R_{sun}^2 \sigma}\right)^{1/4} = \left(\frac{3.9 \cdot 10^{26}\,\text{W}}{4\pi \cdot \left(7.0 \cdot 10^8\,\text{m}\right)^2 \cdot \sigma}\right)^{1/4} = 5770\,\text{K}$$

for the surface temperature of the Sun.

(b) The entropy flux flowing away from the Sun through the field is given by

$$I_S = \frac{4}{3} A \sigma T_{sun}^3 = 8.95 \cdot 10^{22}\,\text{W}/\text{K}$$

EXAMPLE 7.12. Surface temperature of the Earth.

Model the Earth as (a) a black body of uniform temperature, absorbing radiation from the Sun and emitting radiation to outer space. How large is the value of the temperature attained by the surface of this body in steady state? (b) Repeat this for a gray surface and again (c) for a black radiator that absorbs 70% of sunlight (because of reflection by snow and clouds).

SOLUTION: (a) We can use Equ.(7.67) without convection and $T = 0$ K for the environment. This yields

$$A_s \mathcal{G}_{sc} = A \sigma T^4$$

A_s is the projected surface of a the sphere (a circle) and A is the surface of the planet. Inserting and solving for T leads to

$$T = \left(\frac{\mathcal{G}_{sc}}{4\sigma}\right)^{1/4} = \left(\frac{1370\,\text{W}/\text{m}^2}{4\sigma}\right)^{1/4} = 279\,\text{K}$$

(b) For non-black bodies we have

$$a_s A_s \mathcal{G}_{sc} = e A \sigma T^4$$

Since $a_s = e$ for gray surfaces, the new condition leads to the same result for the temperature of the planet, $T = 279$ K.

(c) If the Earth radiates like a black body, $e = 1$. However, if it reflects 30% of incoming radiation, we may set $a_s = 0.7$ (it is as if the planet were a selective absorber/radiator). Now we have

$$T = \left(\frac{a_s \mathcal{G}_{sc}}{4 e \sigma}\right)^{1/4} = \left(\frac{0.7 \cdot 1370\,\text{W}/\text{m}^2}{4\sigma}\right)^{1/4} = 255\,\text{K}$$

All three results are too low. The mean surface temperature of our planet is more like 288 K, leaving us with the problem of how to explain this difference (see Chapter 9, Section 9.6).

EXAMPLE 7.13. The radiative heat transfer coefficient.

Write the equation for the exchange of energy between a black body and its surroundings, Equ.(7.38), in a form which resembles the equation of convective heat transfer at a solid–fluid boundary. How would you write the overall heat transfer coefficient, including convection?

SOLUTION: It is possible to transform the term involving the difference of the fourth powers of the temperatures in such a way that the difference of temperatures occurs in the equation:

$$\begin{aligned} I_{E,rad,net} &= \sigma A\left[T_1^4 - T_2^4\right] \\ &= \sigma A\left(T_1^2 + T_2^2\right)\left(T_1^2 - T_2^2\right) \\ &= \sigma A\left(T_1^2 + T_2^2\right)\left(T_1 + T_2\right)\left(T_1 - T_2\right) \end{aligned}$$

Comparison with the desired form, Equ.(7.27), shows that

$$h_{rad} = \sigma\left(T_1^2 + T_2^2\right)\left(T_1 + T_2\right)$$

Obviously, the radiative heat transfer coefficient strongly depends upon the temperatures involved.

If convection is present as well, we are dealing with a case of parallel flow of heat. The flux of energy is equal to the sum of the radiative and the convective fluxes. Therefore, the overall heat transfer coefficient must be equal to the sum of the radiative and convective transfer coefficients.

EXAMPLE 7.14. Absorption of solar radiation: the balance of entropy.

A body absorbs a fraction f of the energy current associated with solar radiation I_E intercepted by it. Represent the losses to the environment in terms of a total heat transfer coefficient h (which includes radiation). Assume that solar radiation does not carry any entropy. (Because of the high temperature associated with solar radiation, this assumption is quite applicable here.) (a) Calculate the sum of the rates of entropy generation due to absorption of radiation and losses. (b) Show that you obtain the same result using the balance of entropy for the body if you take the system boundary to coincide with the environment at temperature T_a. (c) Compare the magnitude of the effects for a body with a surface area of 1.0 m^2 at a temperature of 50°C absorbing 80% of an energy flux of 1000 W/m^2 in an environment of 20°C. The heat transfer coefficient has a value of 10 W/(K·m^2).

SOLUTION: For the solution of the problem we will need the equation of balance of energy for the body:

$$\dot{E} = \Sigma_E - I_{E,loss} = f\,I_E - hA\left(T - T_a\right)$$

(a) Entropy production is due to two distinct irreversible processes, the absorption of radiation and heat transfer to a colder body (the environment). Since the energy of solar radiation absorbed is dissipated, the rate of production due to absorption of radiation is

$$\Pi_{S,1} = \Sigma_E / T$$

The rate of production of entropy as a result of heat transfer, on the other hand, is given by

$$\Pi_{S,2} = hA\left(T - T_a\right)\left(\frac{1}{T_a} - \frac{1}{T}\right)$$

(b) If we consider the body as our system and draw the system boundary at the location of the environment at temperature T_a, we include the part responsible for heat transfer in the system. In this case, the equation of balance of entropy takes the form

$$\dot{S} = I_S + \Pi_S$$

Remember that the radiation is assumed not to deliver any entropy, so there is no source term. Now we have

$$\Pi_S = \frac{\dot{E}}{T} - \frac{I_{E,loss}}{T_a} = \frac{f I_E - hA(T - T_a)}{T} + \frac{hA(T - T_a)}{T_a}$$

This result is equivalent to what we obtained by calculating the rates of production independently.

(c) Inserting the numbers into the expression obtained in (a) gives values of 2.5 W/K and 0.095 W/K, respectively. This tells us something about the relative irreversibilities of the processes (absorption and heat loss): the former is much larger. If we wanted to optimize a system by minimizing entropy production, we have to be able to quantify different contributions to irreversibility (see Chapter 9 for a discussion of this approach).

EXERCISES AND PROBLEMS

1. Sunlight passes in one direction through a gas inside a long cylinder. The flux of entropy at the surface where the light is entering has a magnitude of 5.0 W/K. At the opposite end, the flux of the current of entropy leaving the body is 4.0 W/K. (a) Determine the net flux of entropy with respect to the region of space occupied by the body. (b) At what (minimal) rate is the entropy of the body changing? (c) What is the value of the source rate of entropy for the field? How large is the flux of entropy with respect to the material body?
2. A copper bar of length 0.50 m and cross section 10.0 cm^2 has a temperature of 500 K at one end and 300 K at the other. As heat flows through the bar in steady state, measurements indicate that the temperature varies linearly along the bar. (a) Determine the temperature gradient. Take the direction of entropy flow to be positive. (b) Estimate the current densities of entropy and of energy for the center of the bar using the values read from Fig. 7.8. How large is the conductivity with respect to energy? (c) Divide the bar into two equal parts. With this current of entropy flowing, what is the flux of entropy at the surface where the parts touch with respect to the part from where the entropy is flowing?
3. An immersion heater in a water kettle is hooked up to 220 V. Its electrical resistance is 160 Ω at a temperature of 20°C; the temperature coefficient of the resistance is $4 \cdot 100^{-3}$ K^{-1}. If the heat transfer coefficient between heater and water is 100 W/(K·m^2) and the surface area of the heater is 0.020 m^2, how large will the energy current from the heater to the water be? How does the situation change if a layer of mineral deposit builds up around the heater?
4. Show that the energy current transmitted through a cylindrical shell of length L having inner and outer radii r_1 and r_2 is

$$I_E = \pi L \left[\frac{1}{2 r_1 h_1} + \frac{1}{2 r_2 h_2} + \frac{1}{2 k_E} \ln\left(\frac{r_2}{r_1}\right) \right]^{-1} (T_1 - T_2)$$

   where h_1 and h_2 are the inner and the outer convective heat transfer coefficients. The temperatures of the fluids on the inside and the outside are T_1 and T_2.
5. A cylindrical volume of rock below ground has been heated uniformly to 50°C while the rest of the rock has a temperature of 10°C. (This might be done in solar seasonal heat storage applications.) Use the average values for granite for the properties of the rock. (a) Make

the following model for heat loss from the cylindrical area to the surroundings. While the temperatures of the storage area and the surroundings remain uniform, heat flows through a cylindrical mantle with inner and outer radii equal to half and to twice the radius of the storage cylinder, respectively. Estimate the energy current due to heat loss for a radius of 5.0 m and a length of the cylindrical space of 40 m. (b) How large should the radius be made for heat loss over a period of half a year not to exceed one quarter of the energy stored in the cylinder?

6. A sheet of metal with a selective surface of 2.0 m^2 lies horizontally on the ground. The bottom of the sheet is well insulated. In the visible part of the spectrum the emission coefficient of the metal is 0.90, while in the infrared it is 0.30. Take the ambient temperature to be 20°C. The Sun stands 50° above the horizon, and 70% of the radiation outside the atmosphere penetrates the air. (Assume all the radiation from the sky to be direct and not diffuse.) (a) Neglecting convection, how large should the temperature of the metal sheet be in the light of the Sun? (b) Now take into consideration convective heat transfer at the upper surface of the sheet. The convective heat transfer coefficient is assumed to be 14 W/($K \cdot m^2$). Calculate the temperature attained by the sheet under these conditions.

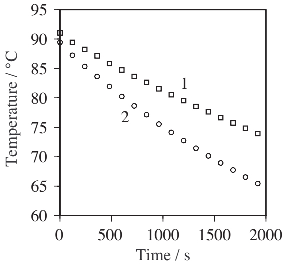

Figure P.7

7. Hot water is left to cool in a thin-walled aluminum can. In a first experiment, the aluminum is highly polished (curve number 1 in Fig. P.7). In the second experiment (curve 2), the can is painted black. Assume the convective transfer from the water to the can to be highly efficient. Data: Mass of water: 0.476 kg; surface area of the can: 0.0325 m^2; Ambient temperature: 21.6°C. (a) Determine the rate of change of entropy of the water in Experiment 1 at $t = 500$ s? (b) Assume radiation to be negligible in Experiment 1. What are the convective entropy and energy transfer coefficients for the surface of the can (can to air). (c) Use the data of Experiment 2 to determine the emissivity of the black surface.

8. Normally, the surface temperatures of stars are derived from their colors or their spectra. However, it is also possible to calculate this quantity from the intensity of their light (i.e., from the irradiance at the surface of the Earth), and from their angular diameter as seen from the Earth. Angular diameters of some nearby stars can be determined with the aid of interferometric methods. In the case of the star Sirius in the constellation of Canis Majoris, these values are $8.6 \cdot 10^{-8}$ W/m^2, and $6.12 \cdot 10^{-3}$ arc seconds, respectively.

9. Consider the Earth as a uniform body. (a) How large is the rate at which entropy appears in the atmosphere, biosphere, and the oceans of the Earth if we take their temperature to be 300 K? The solar constant outside the atmosphere is 1.36 kW/m^2. 30% of the radiation is directly reflected back into space. (b) How large is the flux of entropy through the radiation field just before radiation is absorbed? (c) How large is the rate of production of entropy on the planet as a result of absorption? (d) How large is the rate of entropy generation overall?

10. A photovoltaic panel with an area of 1.0 m^2 is exposed to constant solar radiation having an energy current of 800 W/m^2. Initially, panel and cells are at ambient temperature (300 K). The panel has a heat capacity (energy capacity) of 1000 J/K. The absorption coefficient of the panel for sunlight is 0.85. The emission coefficient of the panel for thermal radiation is 1. Energy goes directly to the air as well (the heat transfer coefficient is 12 W/($K \cdot m^2$)). The electric efficiency of the panel decreases with temperature according to

$$\eta = 0.15 - b\left(T - T_a\right) \quad , \quad b = 1.667 \cdot 10^{-3}\,\mathrm{K}^{-1}$$

The efficiency is defined as the ratio of electric power and energy current of sunlight (not the absorption rate!). (a) What is the electric power right at the beginning? (b) Formulate the law of balance of energy of the panel in general (instantaneous) form. (c) What is the rate of change of temperature of the panel right at the beginning? (d) Determine the steady-state temperature of the panel resulting after a period of time. (e) Sketch as precisely as possible, the temperature and the electric power of the panel as functions of time.

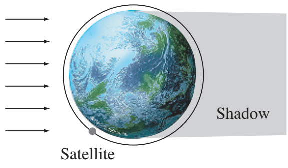

Figure P.11.1

11. A spherical satellite with a radius of 0.50 m moves in a low orbit around the Earth (Fig. P.11.1). Approximately half the time it is exposed to the Sun's light (the solar constant is

1370 W/m^2). In the Earth's shadow it is irradiated by the earth itself. The satellite is a thin aluminum shell. (a) The Earth absorbs approximately 70% of the energy of the incident light of the Sun. The energy is then uniformly reradiated over the entire surface. What is the energy flow of the earth's radiation per square meter? (b) Calculate the highest and the lowest steady-state temperatures reached by the satellite. This temperature is uniform over the entire surface. Assume that the satellite is a black body radiator. When it is in sunlight, ignore the Earth's radiation. (c) Determine the mass of the satellite with the help of the temperature as a function of time (see Fig. P.11.2).

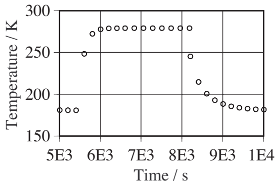

Figure P.11.2

12. In solar energy applications, parabolic troughs are used to focus light upon absorbers of cylindrical shape. Calculate the heat loss coefficient of such an absorber. Consider it to be made of a metal pipe having a diameter of 5.5 cm, surrounded by a thin glass cover with an outer diameter of 8.5 cm. The annulus between the pipe and the cover is evacuated. Take the convective heat transfer coefficient at the surface of the cover to be 35 W/(K·m^2). The emissivities of glass and the metal pipe are 0.88 and 0.92, respectively. Present the result as a function of absorber temperature for an ambient temperature of 20°C.

13. A bottle of white wine is placed in a refrigerator whose inner temperature we take to be constant at 0°C. How long will it take for the temperature of the wine to decrease from an initial value of 20°C to the desired 8°C? Treat the wine as a uniform system of mass 0.75 kg and use the constitutive quantities of water. The bottle is made out of glass with a thickness of 5.0 mm. The height and the diameter of the main body of the bottle are 25 cm and 8 cm, respectively; neglect its bottom and its neck and treat the mantle as a flat layer. The convective transfer coefficients inside and outside are 200 W/(K·m^2) and 10 W/(K·m^2), respectively.

14. A spherical thin-walled water tank has a volume of 1.0 m^3. The water inside is kept at a constant temperature of 60°C by heating it with an energy current equal to 1.0 kW. The ambient temperature is 15°C. How long will it take for the water to reach a temperature of 40°C after the heater has been turned off?

15. A body of water having a volume of 1.0 m^3 loses heat to its surroundings. The temperatures are 80°C and 20°C for the water (initially) and the environment, respectively. The product of total heat transfer coefficient and surface area is 60 W/K. (a) How long does it take for the temperature difference between the water and the surroundings to decrease to half its initial value? (b) How large is the rate of production of entropy right at the beginning? (c) How much entropy is produced in total from the beginning until the water has cooled down completely? (d) How much energy could have been released by an ideal Carnot engine operating between the water and the environment as the water cools to ambient temperature?

16. To maintain an inner temperature of 20°C in a building situated in a 0°C environment, the required heating load is 5 kW. Without heating, the house is found to cool down as follows: every day, its temperature decreases by 1/5 of the temperature difference to the environment. (a) Determine the product of surface area and total heat transfer coefficient. (b) Model the building as a single node system. Calculate its temperature coefficient of energy. (c) Assume the temperature inside the building to be 12°C. Calculate the heating power necessary if you wish the temperature to rise by 1°C per hour.

17. A tall, well insulated cylinder of radius 0.75 m contains 10,000 kg of water. The lower 3500 kg has a temperature of 20°C, while the temperature of the rest of the water is 80°C. Such stratification may be approximately attained while charging a hot water storage tank in solar applications. (a) Estimate how long it will take for the difference of the temperatures of the two segments of water to decrease to 30°C. (*Hint:* Model the segments as uniform bodies; for the thermal resistance take a distance from the center of the hotter to the center of the cooler part.) (b) Calculate the initial rate of production of entropy. (c) How large is the initial rate of loss of available power?