CHAPTER 11

THERMODYNAMICS OF CONTINUOUS SYSTEMS

So far we have used laws of balance and expressions for processes, i.e., for flows, source rates and production rates, in their integral forms; in other words we have written and applied the appropriate equations for an entire body. Now we will justify and derive the proper equations of balance of mass, entropy, and momentum, for continuous bodies. This will prepare the ground for theories of thermodynamics of continuous processes which build upon these laws of balance and expressions for constitutive laws in analogy to how we have proceeded before in this book.

I would like to emphasize that the following development will not be rigorous in either the mathematical or the physical senses. Mathematical derivations will be performed for the simplest geometrical cases of purely one-dimensional transports only, and the physical scope of the laws will be limited to relatively simple phenomena.[1] Still, I will use ideas in their most fundamental form to motivate the laws of balance and some simple constitutive relations of continuum physics, and general results will be presented as extensions of simpler ones. The result will be an uncluttered presentation of a subject often deemed difficult for beginners.

11.1 LIFE AND MIGRATION OF LOCUSTS

Let me introduce the ideas and some simple mathematical forms of spatially continuous processes by discussing a particularly vivid case, namely, the migration of locusts. As mentioned above, I will make many simplifying assumptions, but this will not limit the importance of the results. The discussion will be exactly what we need as begin-

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[1] Neither restriction will fundamentally limit the applicability of the ideas. The general mathematical forms for more complicated three-dimensional flow processes have the same basic appearance as the simple one-dimensional ones. We will simply suggest that the derivations presented here will carry over to the more advanced cases. When terms representing particular physical phenomena are left out of an equation for simplicity, it is assumed that they could be included similarly to other terms for which the derivation is presented. Those of you who wish to see a more rigorous treatment of the subject should turn to books on fluid dynamics and continuum mechanics. See, for example, Malvern, 1969; Mase, 1970; Landau and Lifshitz, 1959; Lai, Rubin, and Krempl, 1978; and Whitaker, 1968.

H.U. Fuchs, *The Dynamics of Heat: A Unified Approach to Thermodynamics and Heat Transfer*, Graduate Texts in Physics, DOI 10.1007/978-1-4419-7604-8_12,

ners if we want to connect understanding to formalism.

First of all, assume that the space in which locusts live is a very thin strip on the ground effectively creating a single-dimensional domain (Fig. 11.1). The area (inside) of the strip will be called *volume* even though it is not three-dimensional, and the sides of the domain will be called *surface*. At any given moment, there are certain numbers of insects in different parts (elements) of their domain, and these numbers change from place to place and in the course of time. Let us assume that there are always so many insects that we cannot make out an individual one. Rather, we will see them as a continuous mass distributed in space.

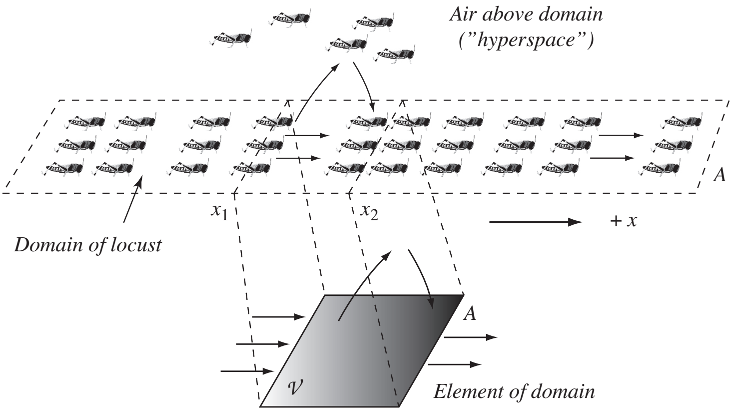

Figure 11.1: locusts in a single-dimensional domain (in the strip, there is never any variation perpendicular to the x-direction). The insects can move in the x-direction (positive or negative), and they can jump into or out of "hyperspace." A part of the domain (element) has been singled out for formulating the law of balance of locusts. The shades of gray indicate locust density, the arrows symbolize processes. A is the width of the domain perpendicular to x, and x_1 and x_2 denote the ends of the element chosen for analysis.

When the insects fly, they are no longer part of their space. We can imagine the air above the strip on which they live to be a "hyperspace" that allows the insects to disappear at a point and reappear at some other point without moving through the domain itself; this is a vivid description of radiative processes. In the domain itself, the insects can crawl in either direction, and they can be swept along with water if there is a stream in (parts of) their space. These processes are analogs of what we call conduction and convection in physics. Apart from these transport processes, the insects can be born and die. All the processes taken together are responsible for changing the numbers of insects in their space which leads to the law of balance for locusts.

There will be particular circumstances that determine the processes. Such circumstances are the numbers of insects at different locations, food resources, water on the ground, currents of air, etc. We can even imagine more complex phenomena where processes influence other processes or feed back on themselves, and just possibly the history of the system might have an impact upon what is going on later. Such circumstances are the basis of what we call constitutive laws.

11.1.1 Locust Density and Densities of Process Quantities

In a description of spatially continuous systems, the number of locusts and quantities denoting processes such as currents and production rates vary from point to point in the domain in which the insects live. At any given time, the quantities we use to describe the system are functions of position.

Density of locusts. Take the number of locusts at a point of the domain, i.e., at some location x. We want to assume that there is absolutely no variation perpendicular to the direction of x. Apparently, defining the number of insects at a point does not make much sense. Rather, we need to know the density of locusts as a measure of their distribution in space. From this density, the number of insects in the entire domain or a part thereof, i.e., in an element (Fig. 11.1), can be calculated. Assume the element to be fixed in space and call $\rho_L(x,t)$ the density of locusts which is a function of position and time. Then

$$n_L(t) = \int_{\mathcal{V}} \rho_L(x,t)\,dV = \int_{x_1}^{x_2} \rho_L(x,t)\,A\,dx \qquad \textbf{(11.1)}$$

is the number of insects in the element shown in Fig. 11.1. A is the surface (side) of this element which extends from x_1 to x_2; $\mathcal{V}$ is the two-dimensional volume of the element. Remember that the density of insects does not vary except in then x-direction.

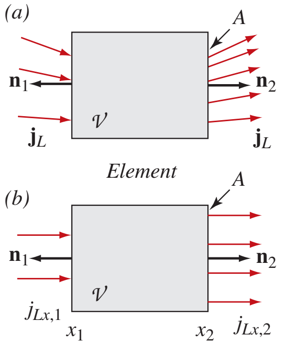

Figure 11.2: Distribution of currents on the surface of an element. (a) General case, (b) purely single dimensional transport. If the dot product of the current density vector and the surface normal vector of unit length are calculated, the component (including sign) of the current density is obtained.

Conductive transports. Now turn to insects crawling in their space, i.e., either in positive or in negative x-direction; we will deal with the other processes a little later. As far as our element is concerned, the insects crawl across its surfaces at x_1 and x_2 (see Fig. 11.2). The transports are described as conductive currents. Since the number of locusts crossing a boundary per unit of time can vary on that boundary—here, perpendicularly to the x-direction—the transport process is described by a *current density* $\mathbf{j}_L$ on the line of width A. Since the insects could crawl in any direction on the ground at the location of the boundary, a current density is a vector quantity. Naturally, we shall continue to assume that they move only in x-direction (positive or negative), and that the current density does not vary except in x-direction (Fig. 11.2, b). This will make the mathematics very simple: we only have an x-component j_{Lx} of the vector. The net current of locusts is determined as the integral of the current density over the total surface $\mathcal{A}$ of the element:

$$I_{L,i}(t) = -\int_{\mathcal{A}} \mathbf{j}_L \cdot \mathbf{n}_i(x_i,t)\,dA \qquad \textbf{(11.2)}$$

The component of $\mathbf{j}_L$ used to calculate the current is obtained by the dot product with the surface normal vector $\mathbf{n}_i$ at location x_i (Fig. 11.2).[2] In our strongly simplified case, the current turns out to be equal to $j_{Lx}A$ at x_1 and equal to $-j_{Lx}A$ at x_2. This still allows the insects to crawl in either positive or negative direction at the two surfaces of the element in Fig. 11.1; the direction is defined by the sign of j. The net conductive current is therefore

$$I_{L,net}(t) = -\left(j_{L,2}(t) - j_{L,1}(t)\right)A \qquad \textbf{(11.3)}$$

Births and deaths. For now, take a look at how to describe births and deaths, i.e., the processes of production and destruction of the species—very much like a chemical reaction (Chapter 6). What is the production rate of locusts $\Pi_L(t)$ in the element with

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[2] In mathematics, it is standard practice to choose the outward direction of a surface of a system as the positive direction. This means that the surface normal vector points outwards (see Fig. 11.2). Since we count a current going into a system as a positive quantity, this leads to the negative sign in Equ.(11.2).

volume $\mathcal{V}$? Since the process can vary spatially *inside* the system, it has to be described by a volume density—a quantity distributed in space. The resulting quantity for the element is the integral of the *production rate density* π_L over the volume:

$$\Pi_L(t) = \int_{\mathcal{V}} \pi_L(x,t)dV = \int_{x_1}^{x_2} \pi_L(x,t)A\,dx \tag{11.4}$$

As for everything else in this example, we assume that the production rate density varies only in x-direction.

11.1.2 A Law of Balance of Locusts

Now we are prepared to formulate a first version of a law of balance for the element under consideration. The number of locusts in the element changes because of migration on the ground (conduction) and births and deaths (production). The law of balance of locusts is therefore

$$\frac{d}{dt} n_L(t) = I_{L,net}(t) + \Pi_L(t) \tag{11.5}$$

n_L, I_L, and Π_L are given by Equations (11.1), (11.3), and (11.4). Two of these are volume integrals, the second—the expression for the net current—looks different. Interestingly, it can be transformed into a volume integral as well. For the simple case discussed here, the formalism can be described quite intuitively. The net current equals the difference of the current densities at the ends of the element. This difference can be obtained as the integral in the x-direction of the (spatial rate of) change of $j_x(x)$:

$$I_{L,net}(t) = -\left(j_L(x_2,t) - j_L(x_1,t)\right)A = -\int_{x_1}^{x_2} \frac{\partial j_{Lx}(x,t)}{\partial x} A\,dx$$

or

$$I_{L,net}(t) = -\int_{\mathcal{V}} \frac{\partial j_{Lx}(x,t)}{\partial x} dV \tag{11.6}$$

Now the pieces can be assembled and combined. Inserting Equations (11.1), (11.4), and (11.6) into (11.5) leads to

$$\frac{d}{dt}\int_{\mathcal{V}} \rho_L(x,t)dV = -\int_{\mathcal{V}} \frac{\partial j_{Lx}(x,t)}{\partial x} dV + \int_{\mathcal{V}} \pi_L(x,t)dV$$

Since the element is fixed in space, the time derivative of the integral on the left can be transformed into

$$\frac{d}{dt}\int_{\mathcal{V}} \rho_L(x,t)dV = \int_{\mathcal{V}} \frac{\partial \rho_L(x,t)}{\partial t} dV$$

Now we have

$$\int_{\mathcal{V}} \frac{\partial \rho_L(x,t)}{\partial t} dV + \int_{\mathcal{V}} \frac{\partial j_{Lx}(x,t)}{\partial x} dV - \int_{\mathcal{V}} \pi_L(x,t)dV = 0$$

or

$$\int_{\mathcal{V}}\left[\frac{\partial \rho_L(x,t)}{\partial t}+\frac{\partial j_{Lx}(x,t)}{\partial x}-\pi_L(x,t)\right]dV=0$$

There is one final step to be made which is based upon an interpretation of the last expression. The combined terms in brackets can basically take any values, meaning that their integral over the fixed volume of the element should not generally equal zero. However, since the integral does vanish, the term in brackets must be equal to zero for all values of x and t, so

$$\frac{\partial \rho_L(x,t)}{\partial t}+\frac{\partial j_{Lx}(x,t)}{\partial x}=\pi_L(x,t) \tag{11.7}$$

This is called the *local* or *differential form of the law of balance* (of locusts) in contrast to the integral form in Equ.(11.5). Since there are derivatives with respect to time and position, the equation is a partial differential equation for the density, production rate density, and current density. It still looks similar to the integral law of balance and, with a little bit of practice, can be read, written, and understood almost as easily as the simpler forms we have been using so far. We just have to get used to thinking in terms of densities and remember that the local change of the current density[3] matters—not the current density itself.

11.1.3 Some Simple Constitutive Relations

If the laws of balance are the trunk of a tree, constitutive relations are its branches. The image tells us something about the possible complexity of special material laws. I do not intend to create a biologically viable theory here, so I will present just a couple of the simplest possible relations that let us put branches onto Equ.(11.7).

Dying of locusts. Assume that in the time span of interest to us, locusts are not born; they only die. One of the simplest ideas for a model of a death rate is to set it proportional to the instantaneous number of insects living in a system. Therefore, the production rate density must be proportional to the density of the insects:

$$\pi_L(x,t)=-f_D\rho_L(x,t) \tag{11.8}$$

f_D is a factor describing the speed at which a population would die out according to this relation—it is similar to a decay constant in radioactive decay.

Crawling of insects. Another simple idea used in biology that looks similar to examples found in physics and chemistry, deals with the migration of animals. Assume that locusts tend to migrate in their space in the direction of lower insect density. Describing this idea with formal tools is quite simple (Fig. 11.3). Imagine the density as a function of position to form a kind of landscape with the density as the level. Where the landscape is steeper, the flow will be stronger. The quantity measuring how steep

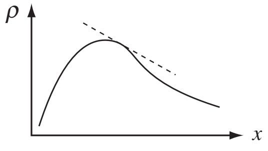

Figure 11.3: The density of locusts as a function of position creates a kind of landscape. The slope of this landscape determines the strength of a diffusive flow.

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[3] The partial derivative of the single component of the current density is the single-dimensional form of what is called the divergence of the current density vector (see Section 11.2.5).

the density landscape is, is called the *gradient* of the density, $\partial\rho_L / \partial x$. So, the conductive current density of locusts may be set proportional to this gradient:

$$j_{Lx}(x,t) = -k\frac{\partial\rho_L(x,t)}{\partial x} \tag{11.9}$$

The factor k is Equ.(11.7) a kind of conductivity or diffusion coefficient. The minus sign tells us that the flow is in the direction of decreasing density. Even though the gradient looks just like the divergence (Equ.(11.7)) in a single spatial dimension, gradient and divergence are two very different mathematical quantities.

The field equation for density. If the particular relations for the production rate density and the current density are introduced into the equation of balance, Equ.(11.7), we obtain a partial differential equation for the density:

$$\frac{\partial\rho_L(x,t)}{\partial t} + \frac{\partial}{\partial x}\left(-k\frac{\partial\rho_L(x,t)}{\partial x}\right) = -f_D\rho_L(x,t) \tag{11.10}$$

In continuum physics, an equation of this type is called a *field equation*. There are two types of side conditions that have to be specified for a concrete solution: we need an *initial condition* and *boundary conditions*. The initial condition might specify the density in the entire domain at an initial point of time. In the simple case discussed here, there will be boundary conditions at either end of the domain, i.e., at $x = 0$ and $x = L$. The conditions can take different forms. For example, the density may be prescribed at either end of the space, or the flows may be given, or we may have a mixture of these forms. Depending upon all the details, the differential equation and its side conditions need to be solved numerically.

11.1.4 Radiation and Convection

To finish this introductory example, let me briefly describe how we can deal with the other two types of processes: radiation and convection of locusts. The former will be identified with flying locusts, the latter may be thought of as occurring if the insects are swept along with a stream of water.

Radiative process. When the insects fly off, they leave their domain at the spot where they were and lift off into what I called "hyperspace." When they land again, they appear in the domain at a point. The process looks clearly like what we know from radiation (see Chapters 7 and 12). Radiation is described by source rates. All we have to do to apply the idea to the continuous case is to introduce a *source rate density* σ. The source rate of locusts in an element (as in Fig. 11.1) is determined as the volume integral of the source rate density:

$$\Sigma_L(t) = \int_{\mathcal{V}} \sigma_L(x,t)dV \tag{11.11}$$

This is equivalent to the relation between production rate and production rate density defined in Equ.(11.4). Indeed, from a mathematical point of view, there is no difference between production and radiation for the system under consideration. The difference lies in the fact that, in the case of radiation, the insects come from a different space—they are not born, i.e., not created out of nothing.

Convective transport. As we have seen in Chapter 8, convective transports are described by currents. This means that the mathematical forms used for conductive currents apply as well: there is a current density (a function distributed over the surface of an element) and the divergence of this current density appears in the law of balance.

The only difference is in the particular form of the current. Imagine the locusts to be caught by a stream, and their density in the water is ρ_L. We can directly apply the ideas developed in Section 8.2.1 (see Fig. 8.3 and Equ.(8.3)). In the purely one-dimensional case, the convective current of insects turns out to be equal to $I_{L,conv} = A\rho_L v$, where v is the flow speed of the water. The *convective current density* $j_{L,conv}$ of locusts must then equal

$$j_{L,conv} = \rho_L v \tag{11.12}$$

This form introduces a new quantity, namely, the flow speed that must be obtained from some other information. In continuum physics, flow speed would be the solution of a continuum model of the fluid. This means that a model involving convective processes is generally a coupled problem dealing with the species in question and the fluids that lead to convection.

General form of the law of balance of locusts. We can now combine the expressions for the additional processes with the law of balance of locusts. The integral form of the law of balance is

$$\frac{d}{dt} n_L(t) = I_{L,cond,net}(t) + I_{L,conv,net}(t) + \Sigma_L(t) + \Pi_L(t) \tag{11.13}$$

If we apply the same reasoning that led to Equ.(11.7), we obtain

$$\frac{\partial \rho_L}{\partial t} + \frac{\partial}{\partial x}\left(j_{Lx,cond} + \rho_L v\right) = \sigma_L + \pi_L \tag{11.14}$$

for the most general local form of the law of balance of locusts. Again, with a little practice, one can understand such equations quite easily.

In the following sections, we are going to study a generalized version of the case of locusts, but applied to the phenomena of physics we are accustomed to. First, the terms needed for a continuum description will be introduced, then the laws of balance of mass, momentum, and entropy will be formulated before we finally turn to an example of a continuum theory of thermodynamics. Such a theory involves constitutive laws in addition to the laws of balance.

11.2 DENSITIES AND CURRENT DENSITIES

Equations of balance for a body relate the rate of change of system content, currents across the surface of the system, and possibly, source rates and rates of production. They tell us how quantities such as mass, entropy, and momentum of a body change as a result of flow processes, and where appropriate, as a consequence of absorption and production. While in integral form, these quantities refer to an entire body, in continuous processes they have to be transformed to reflect their distribution in space. Entering these distributions into the well-known forms of the equations of balance will yield the desired result, namely equations of balance for continuous processes.

11.2.1 The Density of Fluidlike Quantities

Figure 11.4: The density of a quantity tells us about its distribution over a volume. Where the density is higher, more "stuff" is contained in a part of space. If the density changes in space, we have to integrate the distribution over volume to obtain the amount of the quantity contained in a system.

We have come across an example of a density in our discussion of locusts. The density of a fluidlike quantity is our means of telling how the quantity is distributed over a given volume. The concept is easily grasped: if it is constant, the density multiplied by the volume tells us how much of the quantity is contained in the volume; if it is variable, we simply have to integrate the density over the volume to obtain the desired result (Fig. 11.4). A density is commonly introduced for the mass of a body. Integrating the mass density ρ_m over volume yields the mass of the body contained in the volume:

$$m = \int_{\mathcal{V}} \rho_m \, dV \tag{11.15}$$

This carries over to all quantities of a similar type, including entropy, momentum, and energy. If we write ρ_Q for the density of a particular quantity, the amount Q of this quantity contained in a given volume is calculated according to

$$Q = \int_{\mathcal{V}} \rho_Q \, dV \tag{11.16}$$

For our current purpose, this definition is applied to entropy, mass, and momentum of a system. It has been presented for mass in Equ.(11.15); as is customary in this special case, we will not use the subscript m to denote mass: the symbol ρ is used for the mass density of a body. The density of a particular quantity is often written in terms of the *specific* value $q = Q/m$ of the quantity and the mass density of the material containing the quantity:

$$\rho_Q = \rho q \tag{11.17}$$

Using this form, Equ.(11.16) becomes

$$Q = \int_{\mathcal{V}} \rho q dV \tag{11.18}$$

Entropy. Entropy is the fluidlike quantity of thermal processes. If we apply the general expressions formulated above, we obtain the following equations for the entropy of a system:

$$S = \int_{\mathcal{V}} \rho_S \, dV = \int_{\mathcal{V}} \rho s dV \tag{11.19}$$

Momentum. For momentum, we obtain a perfectly analogous equation, at least in the one-dimensional case:

$$p = \int_{\mathcal{V}} \rho_p \, dV = \int_{\mathcal{V}} \rho v dV \tag{11.20}$$

The velocity v of matter represents its specific momentum, i.e., its momentum per mass. Since momentum is a vector, we either have to write Equ.(11.20) as one vector equation or as three component equations (in a Cartesian coordinate system):

$$\mathbf{p} = \int_{\mathcal{V}} \rho \mathbf{v} \, dV \tag{11.21}$$

or

$$\begin{aligned} p_x &= \int_{\mathcal{V}} \rho v_x \, dV \\ p_y &= \int_{\mathcal{V}} \rho v_y \, dV \\ p_z &= \int_{\mathcal{V}} \rho v_z \, dV \end{aligned} \qquad (11.22)$$

If we work with the components of momentum in a Cartesian coordinate system,[4] we can deal with definitions and laws as if momentum consisted of three independent scalar quantities with properties similar to those of mass or entropy.

11.2.2 Source Rate Densities and Production Rate Densities

Some of the fluidlike quantities change as a consequence of source or production processes. Sources are used to describe the interaction of bodies and fields where quantities are transported into and out of bodies without crossing their surfaces. Quantities transferred in this way either originate or end up in another system; they are not produced in this manner. Since a source rate tells us the rate at which a fluidlike quantity appears or disappears inside a body, and since this rate may be different in different parts of a body, we again introduce the concept of a distribution function, this time of the source rate over a volume. In other words, we use the spatial density σ_Q of the source rate Σ_Q to quantify its distribution. The relation between the source rate and its density is known from the density of a quantity:

$$\Sigma_Q = \int_{\mathcal{V}} \sigma_Q \, dV = \int_{\mathcal{V}} \rho f_Q \, dV \qquad (11.23)$$

where f_Q is the specific source rate (source rate per mass). If you wish to use this expression for a particular physical quantity such as entropy or momentum,[5] you have to substitute that quantity for Q. Remember, in the case of momentum you have to write three independent equations for each of the Cartesian components of the momentum vector.

The same idea applies to the phenomenon of the production of a quantity. Again, the rate of production may vary over the volume of a system, in which case we should introduce the spatial density π_Q of the rate of production Π_Q:

$$\Pi_Q = \int_{\mathcal{V}} \pi_Q \, dV \qquad (11.24)$$

Of the quantities we are dealing with in this chapter—mass, entropy, momentum, and energy—only entropy admits a (strictly non-negative) production term.

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[4] For reasons of simplicity, whenever multidimensional forms of equations of balance will be written in component form, a rectangular Cartesian coordinate system will be chosen. More general coordinate systems are treated in books on continuum mechanics (for example, Malvern, 1969).

[5] As you know, there is no source term of mass if we use mass as a measure of amount of substance (as we currently do). Substances cannot be transferred radiatively. If we actually had to deal with mass as in gravitational processes, the equivalence of mass and energy tells us that there could be radiative sources of mass in a system.

11.2.3 Current Densities

The current density (or flux density) of a fluidlike quantity is used as the measure of the distribution of a current over a surface, just as the density is used to describe the distribution of a fluidlike quantity in space (see Fig. 11.5).

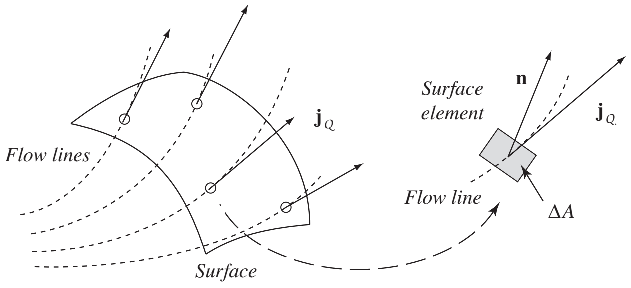

Figure 11.5: Flow lines cutting through a surface present an image of the distribution of a current. The distribution is measured in terms of the current density over the surface. Note that the flux density of a scalar quantity is a vector with a direction tangential to the flow line. The orientation of this vector with respect to the surface normal vector **n** is used when calculating the flux of the quantity transported through space.

Let us start with the simplest possible case of a current density and its associated current or flux, namely that of a flow perpendicular to a flat surface having a constant value of the current density on the surface. Naturally, in this case the flux is calculated simply as the product of the absolute value of the current density and the surface area. With a given orientation of the surface, the value of the flux is taken to be *negative* if the current flows in the direction of the normal vector representing the orientation. This means that, in this simple case, we can write the flux in the following form:

$$I_Q = -A\,\mathbf{j}_Q \cdot \mathbf{n} \tag{11.25}$$

The dot denotes the scalar product of the vectors $\mathbf{j}_Q$ and **n**. As before, for a surface of a body, we will take the orientation positive for outward direction (Fig. 11.6). We can now relax the condition that the flow must be perpendicular to the flat surface. The same form as in Equ.(11.25) still applies; and it does so as well for a small part of a curved surface cut by arbitrary flow lines as depicted in Fig. 11.5. In general, then, the flux of an arbitrary flow field cutting through a curved surface must be given by the surface integral

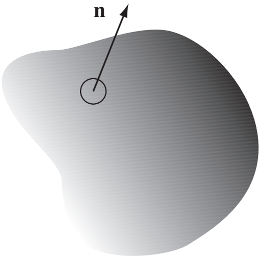

Figure 11.6: The orientation of the surface of a body is taken to be positive for the outward direction. The normal vector **n** points away from the body.

$$I_Q = -\int_{\mathcal{A}} \mathbf{j}_Q \cdot \mathbf{n}\, dA \tag{11.26}$$

where $\mathcal{A}$ is the surface under consideration. This surface may be the closed surface of a body or a part thereof.

Remember that we have to distinguish between two different types of currents if we are dealing with material systems,[6] namely those associated with convective and conductive transports. If a fluidlike quantity flows through matter, the flux is said to be conductive; if it flows with matter, we call the transport convective. Conductive transports are quantified in terms of a conductive current density $\mathbf{j}_Q^{(c)}$. The convective cur-

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[6] There is a third type of current or flux associated with the transport of a fluidlike quantity through a field.

rent, on the other hand, can be expressed in terms of the specific value of the quantity transported by matter and the flux of mass; see Section 8.2 of Chapter 8:

$$I_{Q,conv} = qI_m \tag{11.27}$$

All we must do to obtain the continuum form of this relation is to calculate the flux of mass for the spatially variable case. You know that the flux of mass is equal to the product of the density of the material and its flux of volume. The latter quantity must be given by the integral of the volume flux density over the surface:

$$I_V = -\int_{\mathcal{A}} \mathbf{j}_V \cdot \mathbf{n}dA = -\int_{\mathcal{A}} \mathbf{v} \cdot \mathbf{n}dA \tag{11.28}$$

As you can see, the volume flux density is equal to the speed of flow:

$$\mathbf{j}_V = \mathbf{v} \tag{11.29}$$

Therefore, the flux density of mass is the product of the density and the speed of flow:

$$\mathbf{j}_m = \rho\mathbf{v} \tag{11.30}$$

This tells us that the convective current density of a quantity Q is given by the product of the specific quantity and the mass flux density:

$$\mathbf{j}_{Q,conv} = q\rho\mathbf{v} \tag{11.31}$$

Now, we have the means of expressing the total flux density of a particular quantity, namely by adding up the conductive and the convective parts:

$$\mathbf{j}_Q = q\rho\mathbf{v} + \mathbf{j}_Q^{(c)} \tag{11.32}$$

and, for stationary control volumes, the flux turns out to be

$$I_Q = -\int_{\mathcal{A}} \left(q\rho\mathbf{v} + \mathbf{j}_Q^{(c)}\right) \cdot \mathbf{n}dA \tag{11.33}$$

11.2.4 Current Densities of Mass, Entropy, and Momentum

To obtain the expressions pertaining to the particular quantities with which we are dealing, we simply have to replace the general quantity Q by either mass, entropy, or momentum. Since mass (amount of substance) does not have a conductive flux, the total flux density is given by the expression in Equ.(11.30):

$$\mathbf{j}_m = \rho\mathbf{v} \tag{11.34}$$

Entropy is a fairly simple case since this quantity is a scalar just like mass or electric charge. The specific entropy is abbreviated by s, while the conductive current density of entropy is written as $\mathbf{j}_S^{(c)}$:

$$\mathbf{j}_S = s\rho\mathbf{v} + \mathbf{j}_S^{(c)} \tag{11.35}$$

Momentum, on the other hand, presents us with a more complicated case since we have to deal with a vectorial quantity. The problem is simplified if we treat each of the components of the vector independently. As you recall from the brief presentation in Chapter 3, a component of momentum can be thought of as flowing through matter much like entropy or charge do. The flow of each of the components of momentum results in a flow field like those shown in Chapter 3 and below in Fig. 11.7.

Here are the equations for a single component of momentum. You can build the complete result for all three components by combining the parts. If we choose the *x*-component, the specific value of *x*-momentum p_x is the *x*-component of the velocity. Therefore, we have

$$\mathbf{j}_{px} = \rho v_x \mathbf{v} + \mathbf{j}_{px}^{(c)} \tag{11.36}$$

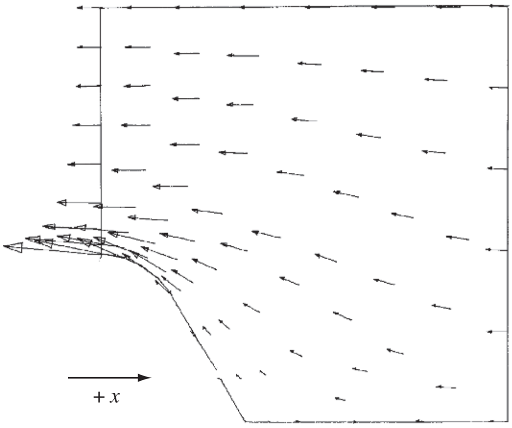

Figure 11.7: Flow pattern of one component of momentum resulting from tension in a flat strip having a notch. The component of momentum whose flow is depicted here is the one identified with the direction of tension. See also Fig. 3.8.

In this case it might be instructive to present all three components of the current density:

$$\begin{aligned} j_{pxx} &= \rho v_x v_x + j_{pxx}^{(c)} \\ j_{pxy} &= \rho v_x v_y + j_{pxy}^{(c)} \\ j_{pxz} &= \rho v_x v_z + j_{pxz}^{(c)} \end{aligned} \tag{11.37}$$

These quantities have a simple graphical representation; j_{pxx}, for example, represents the current density of *x*-momentum flowing in *x*-direction, while j_{pxy} is the current density of *x*-momentum flowing in *y*-direction (see Fig. 11.7) Since there are three components of current density vectors belonging to the three components of momentum, a total of nine components[7] form the momentum current density tensor.

11.2.5 Transformation of a Surface Integral (Divergence Theorem)

In Section 11.1.2, we transformed a surface integral into an integral over the volume bounded by the surface. We treated the simple example of purely one-dimensional migration of locusts. Since the number of locusts is a scalar quantity, its current density is a vector describing the three possible directions of flow of this fluidlike quantity. If the current density vector has only one component, then locusts move in only one direction. In this case, the locusts flux is

$$I_{Lx} = -\int_{\mathcal{A}} \mathbf{j}_L \cdot \mathbf{n}_x \, dA \overset{def}{=} -\int_{\mathcal{A}} j_{Lx} \, dA$$

The second form on the right has been introduced to shorten the notation. This integral can be transformed into a volume integral according to

$$I_L = -\int_{\mathcal{A}} j_{Lx} \, dA = -\int_{\mathcal{V}} \frac{\partial}{\partial x} j_{Lx} \, dV$$

We used this relation to derive the local form of the equation of balance of locusts above in Section 11.1.2 (see Equ.(11.6)). In this form, the transformation is the simplest example of what is called the *divergence theorem* or *Gauss's theorem*. Let me briefly write down this relation without giving a proof.[8] If we define a current density vector $\mathbf{j}_Q$ on the closed surface of a body, the surface integral can be transformed into an integral over the volume enclosed by the surface:

$$\int_{\mathcal{A}} \mathbf{j}_Q \cdot \mathbf{n} dA = \int_{\mathcal{V}} \nabla \cdot \mathbf{j}_Q dV \tag{11.38}$$

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[7] This quantity cannot be represented as a vector anymore; rather, it is a tensor which may be written in matrix form

$$\mathcal{J}_p = \begin{pmatrix} \rho v_x v_x + j_{pxx}^{(c)} & \rho v_x v_y + j_{pxy}^{(c)} & \rho v_x v_z + j_{pxz}^{(c)} \\ \rho v_y v_x + j_{pyx}^{(c)} & \rho v_y v_y + j_{pyy}^{(c)} & \rho v_y v_z + j_{pyz}^{(c)} \\ \rho v_z v_x + j_{pzx}^{(c)} & \rho v_z v_y + j_{pzy}^{(c)} & \rho v_z v_z + j_{pzz}^{(c)} \end{pmatrix}$$

The negative conductive part of this quantity is commonly called the *stress tensor*

$$\mathcal{T} = \begin{pmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{pmatrix} = -\begin{pmatrix} j_{pxx}^{(c)} & j_{pxy}^{(c)} & j_{pxz}^{(c)} \\ j_{pyx}^{(c)} & j_{pyy}^{(c)} & j_{pyz}^{(c)} \\ j_{pzx}^{(c)} & j_{pzy}^{(c)} & j_{pzz}^{(c)} \end{pmatrix}$$

while the complete quantity would be called the *momentum current tensor*. The surface integral of a row of the tensor (for one of the components of the coordinate systems) is called the component of the surface force

$$F_x = \int_{\mathcal{A}} \mathbf{T}_x \cdot \mathbf{n} dA$$

($\mathbf{T}_x$ is the first row of the stress tensor), while the surface integral for the stress tensor is the surface force vector

$$\mathbf{F} = \int_{\mathcal{A}} \mathcal{T} \cdot \mathbf{n} dA$$

where $\nabla \cdot \mathbf{j}_Q$ is called the *divergence* of $\mathbf{j}_Q$. In rectangular Cartesian coordinates

$$\nabla \cdot \mathbf{j}_Q = \frac{\partial}{\partial x} j_{Qx} + \frac{\partial}{\partial y} j_{Qy} + \frac{\partial}{\partial z} j_{Qz} \tag{11.39}$$

The divergence of a vector written in component form is often abbreviated as follows:

$$\frac{\partial}{\partial x} j_{Qx} + \frac{\partial}{\partial y} j_{Qy} + \frac{\partial}{\partial z} j_{Qz} \equiv \frac{\partial}{\partial x_i} j_{Qi} \tag{11.40}$$

In this notation it is assumed that a summation is carried out over all indices which appear twice in the same term; x_i, $i = 1,2,3$ stands for the three components (x,y,z) of the coordinate system. In this form, the divergence looks like the expression used in single-dimensional cases. In fact, the simplest examples usually suggest the proper form of more complicated cases.

11.3 THE BALANCE OF MASS

Let us start with the first of the three fluidlike quantities for which we have to obtain laws of balance, namely the amount of substance. The balance of amount of substance is a necessary prerequisite for formulating theories applicable to fluid or otherwise deformable media. If we wish to quantify convective currents associated with processes in open systems, we have to be able to write down the currents of amount of substance. For practical reasons, however, engineers commonly use mass as a substitute for amount of substance, and as long as there are no chemical reactions taking place inside the material, there is no problem in doing so. Therefore, we will use a formulation based on mass.

In the previous section we introduced the concepts and tools needed to formulate the continuum forms of the laws of balance of fluidlike quantities. Starting with the integrated form of the balance of mass

$$\dot{m} = I_m \tag{11.41}$$

we can easily show how to obtain the appropriate local equation applicable to the continuous case. Let us apply this law to a stationary control volume of simple shape (Fig. 11.8) and assume the flow field to be one-dimensional. In this equation, m is the mass inside the control volume, while I_m is the net current of mass across the surface of the control volume. We shall replace the mass by the volume integral of the mass density, and the flux by the surface integral of the flux density, as in Equ.(11.26). With Equ.(11.34) this leads to

$$\frac{d}{dt}\int_{\mathcal{V}} \rho dV + \int_{\mathcal{A}} \rho v dA = 0 \tag{11.42}$$

If we use the divergence theorem for the surface integral and apply the time derivative

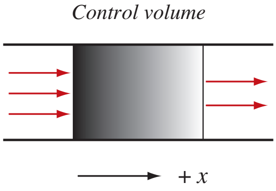

Figure 11.8: Simple one-dimensional flow with respect to an open and stationary control volume. Imagine a fluid flowing in the x-direction only.

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8. For a derivation of the divergence theorem see Marsden and Weinstein (1985), Vol. III, p. 927.

to the integrand of the first integral, we obtain

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} dV + \int_{\mathcal{V}} \frac{\partial}{\partial x} (\rho v) dV = 0$$

or

$$\int_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) \right] dV = 0$$

Since the integral must be zero for arbitrary volumes *V*, the last expression can only be satisfied if the terms in brackets are equal to zero:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) = 0 \tag{11.43}$$

You can easily apply the transformations to the more general three-dimensional case

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho v_i) = 0 \tag{11.44}$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{11.45}$$

This looks very similar to the simpler expression. In contrast to Equ.(11.42) which is the integral form of the law of balance of mass, Equ.(11.43) and its counterpart in Equations (11.44) or (11.45) represent the local or differential form of this law. The balance of mass often is called the equation of continuity.

11.4 THE BALANCE OF ENTROPY

Entropy is a scalar quantity just like mass, so the derivation of the local form of the law of balance should lead to a result similar to what we have just seen. Consider as we did in Fig. 11.8, the flow of a fluid in the *x*-direction only. As far as entropy is concerned, we will include conductive and convective transports in the derivation, and production of entropy in irreversible processes. Sources of entropy from radiation, however, will be excluded here. The integral form of the equation of balance of entropy for the control volume in Fig. 11.8 then looks like

$$\dot{S} = I_{S,conv} + I_{S,cond} + \Pi_S \tag{11.46}$$

If we introduce densities and current densities as in Section 11.2, the law becomes

$$\frac{d}{dt} \int_{\mathcal{V}} (\rho s) dV + \int_{\mathcal{A}} \left(s \rho v + j_S^{(c)} \right) dA = \int_{\mathcal{V}} \pi_S dV \tag{11.47}$$

s is the *specific entropy* of the fluid, $j_S^{(c)}$ and π_S represent the conductive *entropy current density* and the *density of the entropy production rate*, respectively. Remember that we are dealing with a purely one-dimensional case. If we now apply the transformation of the surface integral, we obtain

$$\int_V \left[\frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x}\left(s\rho v + j_S^{(c)} \right) - \pi_S \right] dV = 0$$

The expression in brackets must be zero, which yields the local form of the law of balance:

$$\frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x}\left(s\rho v + j_S^{(c)} \right) = \pi_S \tag{11.48}$$

The general three-dimensional case can be written in a form that looks just like the one derived for purely one-dimensional transports. Applying the divergence theorem to the generalized form of Equ.(11.47) yields

$$\frac{\partial}{\partial t}(\rho s) + \nabla \cdot \left(s\rho \mathbf{v} + \mathbf{j}_S^{(c)} \right) = \pi_S \tag{11.49}$$

or

$$\frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x_i}\left(s\rho v_i + j_{Si}^{(c)} \right) = \pi_S \tag{11.50}$$

Extending this result to include the effects of sources from radiation is pretty simple. How this is done will be demonstrated below for the case of momentum (remember that gravity leads to sources of momentum).

11.5 THE BALANCE OF MOMENTUM

Basically, the law of balance of momentum is derived analogously to what you have seen so far. While the fundamental ideas do not change, the current case can be rather complex if we try to deal with it in the most general form. It is therefore all the more important to discuss the simplest possible nontrivial case. Fortunately, purely one-dimensional flow of momentum is meaningful in physical terms, so let us deal with this case in some detail.

One-dimensional convective transport of momentum is a simple concept: if a fluid flows in one direction only, it carries only one single component of momentum. The case of one-dimensional conductive transport is just as well known. Let the direction of fluid flow define the spatial component we are talking about. Having the same component of momentum flowing through the fluid simply means that the material is under compression or tension in the same direction. A frictionless fluid flowing through a straight pipe demonstrates what we mean: the conductive momentum current density of the component parallel to the pipe's axis is the pressure of the fluid.

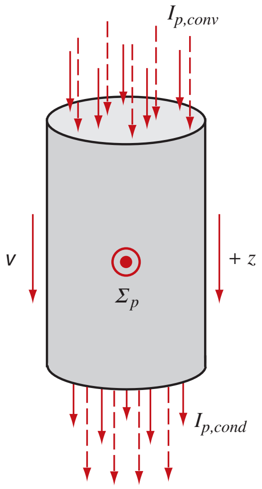

Figure 11.9: Flow lines depicting the convective and conductive transports of momentum are shown together with a source of momentum due to the interaction of the fluid with the gravitational field. The fluid is flowing downward leading to the convective downward flow of momentum together with the fluid (dashed lines). Since the material is under compression, momentum flows conductively in the positive direction (downward; solid lines). With the positive direction as chosen, the gravitational field supplies momentum to the fluid.

In addition to conductive and convective modes of transport, we will consider sources of momentum due to the interaction of the fluid with a field. If you imagine the fluid flowing through a vertical pipe (Fig. 11.9), the action of the gravitational field leads to the flow of momentum of the same (vertical) component directly into or out of the body.

If we collect the different terms, the integral equation of balance of momentum for the z-direction looks like

$$\dot{p} = I_{p,conv} + I_{p,cond} + \Sigma_p \tag{11.51}$$

where p stands for the z-component of momentum. As before, we introduce the proper densities and flux densities and obtain

$$\frac{d}{dt}\int_{\mathcal{V}}(\rho v)dV + \int_{\mathcal{A}}\left(\rho v v + j_p^{(c)}\right)dA = \int_{\mathcal{V}}\sigma_p dV \tag{11.52}$$

Now, we can apply the divergence theorem to this equation:

$$\int_{\mathcal{V}}\left[\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}\left(\rho v v + j_p^{(c)}\right) - \sigma_p\right]dV = 0$$

leading to the desired differential form of balance of momentum for the purely one-dimensional example discussed so far:

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}\left(\rho v v + j_p^{(c)}\right) = \sigma_p \tag{11.53}$$

The general case requires two additional steps. First, for a component of momentum, we have to treat the momentum flux as a vector; this yields an equation similar to Equ.(11.49) for the component. Second, we have to write analogous equations for the other two components of momentum:

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho v_x) + \nabla\cdot\left(\rho v_x \mathbf{v} + \mathbf{j}_{px}^{(c)}\right) &= \sigma_{px}\\
\frac{\partial}{\partial t}(\rho v_y) + \nabla\cdot\left(\rho v_y \mathbf{v} + \mathbf{j}_{py}^{(c)}\right) &= \sigma_{py}\\
\frac{\partial}{\partial t}(\rho v_z) + \nabla\cdot\left(\rho v_z \mathbf{v} + \mathbf{j}_{pz}^{(c)}\right) &= \sigma_{pz}
\end{aligned} \tag{11.54}$$

These three equations can be written as a single one using the momentum current tensor introduced in Footnote 7. While the result can be presented in a compact form, the actual equations are rather lengthy and difficult to read in their component form.

Let us briefly return to the meaning of the conductive momentum flux and the source term in Equ.(11.53). For a simple fluid, the purely one-dimensional case of conductive momentum transport describes the state of compression of that fluid. In other words, the current density is the pressure of the fluid:

$$j_p^{(c)} = P \tag{11.55}$$

The source term, on the other hand, arises as the result of the gravitational interaction. As described in Section 3.3, the gravitational field supplies momentum to the body, leading to a source rate, which in integrated form, must be equal to the weight of the body. The source density must therefore equal

$$\sigma_p = \rho g \tag{11.56}$$

which shows that the specific source rate f_p introduced in Equ.(11.23) is the gravitational field strength g. The influence of fields other than the gravitational field can be included in the same manner. Forces of this nature are called body forces.

11.6 THE MATERIAL DERIVATIVE AND LAWS OF BALANCE

The local form of the equations of balance derived above has a couple of features which need explaining. The derivatives occurring in relations such as Equ.(11.48) or Equ.(11.53) are partial derivatives, which means that, for example, the time derivative must be taken at a fixed position. This form is often called the *Eulerian form* of the partial differential equations describing the balance of fluidlike quantities. The second feature is related to the first: the equations explicitly contain convective currents.

It is possible to transform the laws of balance using the balance of mass and to obtain alternative forms which no longer contain the convective currents. As such, the resulting equations look as if they were written for closed systems for which convection does not play a role. Instead, there appears a combination of derivatives of the density of the quantity investigated, a combination which can be interpreted as a *material derivative*. Let us see how this happens.

Take as an example the simple law of balance of entropy in Equ.(11.48). If you take the derivative of some of the products occurring in the equation, you get

$$\rho \frac{\partial s}{\partial t} + s \frac{\partial \rho}{\partial t} + s \frac{\partial(\rho v)}{\partial x} + \rho v \frac{\partial s}{\partial x} + \frac{\partial j_s^{(c)}}{\partial x} = \pi_S$$

or

$$\rho \left[\frac{\partial s}{\partial t} + v \frac{\partial s}{\partial x} \right] + s \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} \right] + \frac{\partial j_s^{(c)}}{\partial x} = \pi_S$$

According to the law of balance of mass presented in Equ.(11.43), the term in the second pair of brackets must equal zero. Therefore, the transformed law of balance looks like

$$\rho \left[\frac{\partial s}{\partial t} + v \frac{\partial s}{\partial x} \right] + \frac{\partial j_s^{(c)}}{\partial x} = \pi_S \tag{11.57}$$

The convective current density has disappeared. Introducing the operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \tag{11.58}$$

which is commonly called the *material time derivative* or the *substantial derivative*, Equ.(11.57) can be written as follows:

$$\rho \frac{Ds}{Dt} + \frac{\partial j_S^{(c)}}{\partial x} = \pi_S \tag{11.59}$$

The time rate of change of the quantity s expressed by the material derivative is the one measured by someone carried along with the fluid rather than by someone at a fixed location.

11.7 THE ENERGY PRINCIPLE

The law of balance of energy should be considered separately. The reason for this is not that the energy principle is more important than any of the other fundamental laws but that we should again stress that energy is special in that it appears equally in all physical phenomena and not just in one special subject area. The formal expression of this idea was applied in Chapter 10, where the energy principle was understood to put restrictions upon the processes in question. It will be used again further below in Section 11.8, where the theory of thermodynamics of viscous heat-conducting fluids will be presented.

Since energy is exchanged in all types of physical processes, it is almost impossible, and certainly not realistic, to try to present a totally general form of the law of balance. The following discussion is in line with the area of applications considered here, namely the thermomechanics of nonreacting fluid systems.

11.7.1 Energy Density and Energy Current Densities

Consider again a fluid flowing through a control volume such as in Fig. 11.8, and let all transports be purely one-dimensional. In addition to mass, entropy, and momentum, energy is transported across the faces of the control volume, and just like the other quantities, it can also be stored in the system. Since energy cannot be produced, we will have to consider the density of energy in the fluid, energy density currents across the surface, and sources of energy due to radiative transports.

The energy density of a fluid. A fluid contains an amount of energy that depends upon its state, which in this case, can be described by temperature, pressure, and velocity. It is common to associate a part of the energy content with the state of motion of the material (i.e., the kinetic energy) while the rest is lumped together as the internal energy of the fluid. This distinction was made in Chapter 8 where we discussed simple applications of flow processes. The density of energy of the fluid is therefore written as the sum of two parts, the density of internal energy, and the density of kinetic energy. If we use instead the specific values of internal energy and of kinetic energy, the expression takes the following form:

$$\rho e = \rho\left(u + \frac{1}{2}v^2\right) \tag{11.60}$$

Here, e is the specific energy, while $u = e^{(i)}$ stands for the specific internal energy (intrinsic part of e). Naturally, the second term on the right-hand side represents the kinetic energy per mass. The energy of a fluid system can then be written as follows:

$$E = \int_{\mathcal{V}} \rho\left(u + \frac{1}{2}v^2\right) dV \tag{11.61}$$

Energy current densities. Energy can be carried into or out of a control volume across the surface of the system. Such processes are described in terms of currents. Since the exchange of energy may be the result of numerous different interactions, we should be specific and discuss only those phenomena which are important to our current theme. Other cases can be treated in analogy to what we do here. Since energy can be transferred convectively and conductively, we can say that

$$j_E = j_{E,conv} + j_{E,cond} \tag{11.62}$$

In general, these current densities are vectors. The convective transport is the result of the fluid carrying stored energy across the system boundary. We need the convective energy flux density to quantify this process; it can be expressed in terms of the specific energy and the flux of mass:

$$j_{E,conv} = \rho v e = \rho v\left(u + \frac{1}{2}v^2\right) \tag{11.63}$$

The conductive transport of energy presents a slightly more complicated problem, since we might have to consider all sorts of transfer processes. Limiting our attention to thermal and mechanical phenomena, however, leaves us with only two possibilities: energy may be either added or withdrawn together with entropy in heating and cooling, or it may flow across the system boundary as the result of the conductive exchange of momentum:

$$j_{E,cond} = j_{E,th} + j_{E,mech} \tag{11.64}$$

The former possibility gets short treatment; since we shall try to establish the relationship between transports of entropy and of energy, we cannot say more about this case so we simply introduce a thermal energy current density. The second term in this equation, on the other hand, is well known from mechanics. The exchange of energy is directly tied to the conductive flow of momentum, and to the speed of the fluid at the surface of the system:

$$j_{E,mech} = v j_p^{(c)} \tag{11.65}$$

In three dimensions, a component of the energy flux density vector is obtained for each component of momentum by calculating the scalar product of the momentum current vector and the velocity. Adding up all the terms introduced so far yields the desired expression for the energy current density:

$$j_E = \rho v\left(u + \frac{1}{2}v^2\right) + v j_p^{(c)} + j_{E,th} \tag{11.66}$$

As mentioned before, this equation has to be changed or extended if other processes are to be taken into consideration.

The total energy flux with respect to the control volume in Fig. 11.8 is obtained by integrating the expression in Equ.(11.66) over the surface elements perpendicular to the direction of flow:

$$I_E = -\int_{\mathcal{A}}\left[\rho v\left(u + \frac{1}{2}v^2\right) + v j_p^{(c)} + j_{E,th}\right] dA \tag{11.67}$$

Sources of energy. Energy can be directly supplied to the interior of a body or a control volume as a consequence of its interaction with fields. Three cases of interest to us are the supply of energy together with momentum if we include the action of a gravitational field; sources of energy due to the absorption of electromagnetic radiation;

and sources of energy such as those related to nuclear decay or chemical reactions inside the system. Describing these effects calls for introducing an energy supply density such that

$$\Sigma_E = \int_{\mathcal{V}} \sigma_E \, dV \tag{11.68}$$

Gravitational interaction and absorption of radiation lead to a specific expression for the source density, namely

$$\sigma_E = \rho\left(f_p v + r\right) \tag{11.69}$$

where $f_p = g$ is the specific source rate of momentum (the specific body force) and r denotes the specific rate of absorption of energy together with radiation.

11.7.2 The Balance of Energy

Writing down the law of balance of energy as it applies to a fairly general case of the thermomechanics of fluid systems, is as simple as the examples we have seen before. The integral statement

$$\dot{E} = I_E + \Sigma_E \tag{11.70}$$

can be written using the densities and the current densities:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\left(u + \frac{1}{2}v^2\right) dV + \int_{\mathcal{A}} \left[\rho v\left(u + \frac{1}{2}v^2\right) + v j_p^{(c)} + j_{E,th}\right] dA = \int_{\mathcal{V}} \rho\left(f_p v + r\right) dV \tag{11.71}$$

Applying the integral transformation to the surface integral, collecting all the terms on one side, and noting that the resulting volume integral must vanish identically, yields the local form of the law of balance of energy:

$$\frac{\partial}{\partial t}\rho\left(u + \frac{1}{2}v^2\right) + \frac{\partial}{\partial x}\left[\rho v\left(u + \frac{1}{2}v^2\right) + v j_p^{(c)} + j_{E,th}\right] = \rho\left(f_p v + r\right) \tag{11.72}$$

Together with the laws of balance derived above for mass, entropy, and momentum, this equation will furnish the starting point for the development of our example of continuum thermodynamics in Section 11.8.

11.7.3 The Balance of Energy in the Material Form

As mentioned in Section 11.6, the local equations of balance are often written using the material derivative, Equ.(11.58), in which case they look like laws of balance written for a particular body, i.e., for a closed system. Let us first motivate this particular form for supply-free processes before attempting a derivation.

Imagine following a fluid in motion. For observers flowing along with a particular body, the momentum of the material is always zero and, as a corollary, the kinetic energy vanishes; they see only the internal energy of the fluid. The exchange of energy with the environment takes two forms. The momentum flowing through the body does so with or against a gradient of the potential of motion, i.e., with or against the gradient of the velocity at the location of the body. The rate per volume at which energy is released is equal to the product of the current density of momentum and the gradient of the velocity. A similar result holds for the thermal interaction; only here we do not know the relationship between the flux of energy and the flux of entropy. The rate of release of energy per volume in the body is equal to the gradient of the energy flux density. Adding up the terms should yield

$$\rho \frac{Du}{Dt} + j_p{}^{(c)} \frac{\partial v}{\partial x} + \frac{\partial j_{E,th}}{\partial x} = 0 \qquad \textbf{(11.73)}$$

The second term in this equation is called the *stress power.*

This result is a consequence of Equ.(11.72), and the laws of balance of mass and of momentum Equations (11.43) and (11.53); remember that we do not include any of the source terms. In a first step, the law of balance of energy becomes

$$\begin{aligned} &\rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \rho v \frac{\partial u}{\partial x} + u \frac{\partial(\rho v)}{\partial x} \\ &\qquad + \frac{\partial}{\partial x}\left[\rho v\left(\frac{1}{2}v^2\right)\right] + \frac{\partial}{\partial x}\left(v j_p{}^{(c)}\right) + \frac{\partial j_{E,th}}{\partial x} = 0 \end{aligned}$$

Together, the second and the fifth terms vanish because of the law of balance of mass, while the first and the fourth terms yield the product of the density and the material derivative of the internal energy. Therefore we have

$$\rho \frac{Du}{Dt} + j_p{}^{(c)} \frac{\partial v}{\partial x} + \frac{\partial j_{E,th}}{\partial x} + \left\{ \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x}\left[\rho v\left(\frac{1}{2}v^2\right)\right] + v \frac{\partial j_p{}^{(c)}}{\partial x} \right\} = 0$$

If we can show that the expression inside the braces in this expression is zero, then the proof of Equ.(11.73) is complete. Now, the spatial derivative of the momentum current density can be replaced using the law of balance of momentum:

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x}\left[\rho v\left(\frac{1}{2}v^2\right)\right] + v \frac{\partial j_p{}^{(c)}}{\partial x} \\ &\qquad = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x}\left[\rho v\left(\frac{1}{2}v^2\right)\right] + v\left[-\frac{\partial}{\partial t}(\rho v) - \frac{\partial}{\partial x}(\rho v v)\right] \end{aligned}$$

After some lengthy algebra, we find that this expression is equal to

$$-\frac{1}{2}v^2\left(\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x}\right)$$

This quantity vanishes because of the law of balance of mass. Naturally, for the three-dimensional case, the derivation is more involved, but the result still holds in a form similar to the one stated above.

11.8 THERMODYNAMICS OF A VISCOUS HEAT CONDUCTING FLUID

The bulk of engineering work in the fields of fluid dynamics and heat transfer is built upon the basis of the theory of heat-conducting viscous fluids, the subject we will discuss in this section. Except for constructing the constitutive laws applicable to the field, we have already done most of the preparations necessary for the following development. We must only assemble the parts and derive the consequences of the assumptions, much like we did for the uniform viscous fluid in Chapter 10. If the following lines look forbidding at first, keep in mind that it is the amount of algebra needed which creates this impression. The results can be understood intuitively in terms of what we have learned in the previous chapters.

The assumptions which the theory is based upon can be divided into three groups. The first, the laws of balance of mass, entropy, momentum, and energy, has been discussed above; we only must assemble them in the form which applies to our case. The second is made up of the preliminary expressions for the constitutive laws which will be introduced in Section 11.8.2 and the third group deals with the assumption of ideal walls which we will not go into again (see Section 10.1 for more details).

11.8.1 The Laws of Balance

The laws of balance used in the field of nonreactive heat-conducting viscous fluids are those of mass, entropy, momentum, and energy. For the following derivation, we will stay with the simple one-dimensional case discussed in the previous sections. For source-free processes, the appropriate forms of the first three laws are

$$\begin{aligned}
&\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=0\\
&\frac{\partial}{\partial t}(\rho s)+\frac{\partial}{\partial x}\left(s\rho v+j_S{}^{(c)}\right)=\pi_S\\
&\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}\left(\rho v v+j_p{}^{(c)}\right)=0
\end{aligned}\tag{11.74}$$

while the balance of energy looks as follows

$$\frac{\partial}{\partial t}\left[\rho\left(u+\frac{1}{2}v^2\right)\right]+\frac{\partial}{\partial x}\left[\rho v\left(u+\frac{1}{2}v^2\right)+v j_p{}^{(c)}+j_{E,th}\right]=0\tag{11.75}$$

In component notation for three dimensions, the equations of balance for the three main fluidlike quantities are as follows:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_j}\left(\rho v_j\right)=0\\
&\frac{\partial}{\partial t}(\rho s)+\frac{\partial}{\partial x_j}\left(s\rho v_j+j_{Sj}{}^{(c)}\right)=\pi_S\\
&\frac{\partial}{\partial t}(\rho v_i)+\frac{\partial}{\partial x_j}\left(\rho v_i v_j+j_{pij}{}^{(c)}\right)=0
\end{aligned}\tag{11.76}$$

The balance of energy takes the form

$$\frac{\partial}{\partial t}\left[\rho\left(u+\frac{1}{2}v^2\right)\right]+\frac{\partial}{\partial x_j}\left[\rho v_j\left(u+\frac{1}{2}v^2\right)+v_i j_{pij}{}^{(c)}+j_{E,th,j}\right]=0 \qquad \textbf{(11.77)}$$

Remember that we have to sum over the indices which appear twice in an expression ($j = 1,2,3$).

As you know from all the examples in the previous chapters, equations of balance do not solve a problem by themselves. Moreover, they contain as their most important quantities those which are often not accessible to measurement. If we wish to specify the condition of a fluid in terms of density, velocity, and temperature at different points in space for different times, i.e., if we wish to obtain the functions $\rho(\mathbf{x},t)$, $\mathbf{v}(\mathbf{x},t)$, and $T(\mathbf{x},t)$, we will obviously have to eliminate quantities such as entropy and energy from the equations. This goal is achieved if we know the proper constitutive laws with whose help the equations of balance can be transformed into field equations for the measurable quantities.

11.8.2 Constitutive Relations for a Navier-Stokes-Fourier Fluid

In this section we will present the constitutive laws which govern the behavior of viscous heat-conducting fluids in a strongly simplified form. It is possible to include all the interesting physical effects in the purely one-dimensional case. This reduces the complexity of the calculations considerably. We will need preliminary forms of the constitutive laws for the specific entropy and energy and for the flux densities of entropy, momentum, and energy (for the case of heating). A suitable thermal equation of state must be added to these.[9]

In many instances, gradients of temperature and velocity in fluids can be assumed to be small, which leads to particularly simple forms of the constitutive laws with only linear dependence on the gradients. We shall assume that

$$s = s(\rho,T) \qquad \textbf{(11.78)}$$

and

$$u = u(\rho,T) \qquad \textbf{(11.79)}$$

for the specific quantities, and

$$\begin{aligned} j_p{}^{(c)} &= P(\rho,T) - \mu'(\rho,T)\frac{\partial v}{\partial x} \\ j_S{}^{(c)} &= -k_S(\rho,T)\frac{\partial T}{\partial x} \\ j_{E,th} &= -\beta(\rho,T)\frac{\partial T}{\partial x} \end{aligned} \qquad \textbf{(11.80)}$$

for the fluxes of momentum, entropy, and energy (for heating).[10] Here, μ' is a viscosity

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[9] For more details on the proper constitutive laws, you should turn to books such as those by Müller (1985) and Malvern (1969).

which may include the bulk viscosity with the normal viscosity μ of the fluid, while k_S is the thermal conductivity with respect to entropy (i.e., the entropy conductivity);[11] β is a function that will be related to k_S later. We have used constitutive expressions of these or similar forms before for the simple materials treated in previous chapters. Remember that the terms involving gradients of temperature or velocity are the local forms of our simpler expressions for conductive transports: a conductive flow is one that depends upon the difference of its own potential. The first equation, Equ.$(11.80)_1$, reminds us of the special pressure law used in Section 10.1.1 where we dealt with a uniform viscous fluid. Velocity gradients in one dimension mean that the fluid is either being compressed or expanded, leading to viscous effects. The other two expressions, Equations $(11.80)_{2,3}$, are Fourier's law of heat conduction.

11.8.3 Evaluation of the Energy Principle

We wish to be able to derive more detailed results about the constitutive laws and quantities of a Navier-Stokes-Fourier fluid. As you remember, the laws of balance serve as restrictions upon these relations. In particular, all processes must satisfy the energy principle in addition to the laws of balance of amount of substance (mass), entropy, and momentum. In Chapter 10, the approach using Lagrange multipliers was introduced; we will now use it for the present case. The laws of balance are satisfied simultaneously if and only if the following relation holds:

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[10] Actually, the preliminary forms are slightly more complicated. First, one exploits the principle of material frame indifference and representation theorems for isotropic functions to obtain rather general expressions for the constitutive laws (Müller, p. 4–7). These expressions are linearized, leading to

$$\begin{aligned}
s &= s(\rho,T) + a(\rho,T)d_{ii} \\
u &= u(\rho,T) + b(\rho,T)d_{ii} \\
j_{pij}^{(c)} &= P(\rho,T)\delta_{ij} - \lambda(\rho,T)d_{kk}\delta_{ij} - 2\mu(\rho,T)d_{ij} \\
j_S^{(c)} &= -k_s(\rho,T)\frac{\partial T}{\partial x} \\
j_{E,th} &= -\beta(\rho,T)\frac{\partial T}{\partial x}
\end{aligned}$$

for the three-dimensional case of the Navier-Stokes-Fourier fluid. When the equations of balance are exploited, the functions a and b turn out to be equal to zero, meaning that the entropy and the density take the same form in equilibrium as in nonsteady-state processes. To simplify the derivation, we have omitted the terms relating to the gradient of velocity from Equ.(11.78). Equ.$(11.80)_1$ is the purely one-dimensional version of the expression for the stress tensor for nonvanishing bulk viscosity, where

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

is the symmetric part of the velocity gradient tensor, and λ and μ are two independent parameters characterizing the influence of viscosity of the fluid.

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\rho\left(u+\frac{1}{2}v^2\right)\right]+\frac{\partial}{\partial x}\left[\rho v\left(u+\frac{1}{2}v^2\right)+v\,j_p{}^{(c)}+j_{E,th}\right]\\&\quad-\lambda^{\rho}\left[\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)\right]-\lambda^{S}\left[\frac{\partial}{\partial t}(\rho s)+\frac{\partial}{\partial x}\left(s\rho v+j_S{}^{(c)}\right)-\pi_S\right]\\&\quad-\lambda^{v}\left[\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}\left(\rho v v+j_p{}^{(c)}\right)\right]=0\end{aligned} \qquad \textbf{(11.81)}$$

This expression is obtained by multiplying each of the equations of balance of mass, entropy, and momentum by its own Lagrange multiplier, which here are called λ^{ρ}, λ^{S}, and λ^{v}, respectively, and then subtracting them from the energy principle Equ.(11.77). For the following calculations, note that the constitutive quantities in the laws of balance are functions of ρ, T, dT/dx, and dv/dx. You can verify this by looking at Equations (11.78) and (11.80). We should now calculate the derivatives with respect to time and to position in Equ.(11.81), keeping in mind the dependencies just mentioned. This yields a long expression[12] with terms containing the derivatives

$$\frac{\partial v}{\partial t}\;,\;\frac{\partial^2 v}{\partial x^2}\;,\;\frac{\partial\rho}{\partial t}\;,\;\frac{\partial\rho}{\partial x}\;,\;\frac{\partial T}{\partial t}\;,\;\frac{\partial^2 T}{\partial x^2}\;,\;\text{ and }\;\frac{\partial T}{\partial x}\;,\;\frac{\partial v}{\partial x}$$

(If we had included the terms that depend upon the gradient of velocity in the constitutive expressions for specific entropy and energy (Footnote 10), the mixed derivative of the velocity with respect to time and position would also need to be taken into account.) While terms containing the first six of these are expressly linear in these derivatives (they do not occur again in the terms multiplying them), this is not true for the last two derivatives, since these are among the list of independent variables of the constitutive quantities. We separately assemble all the terms that are dependent upon the derivatives listed above.[13] Noting that the expressions multiplying the first six of the derivatives must vanish identically (otherwise the expression in Equ.(11.81) may be violated), we obtain six conditions which must be satisfied by the fluid, plus one re-

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[11] The bulk viscosity is defined as

$$\kappa=\lambda+\frac{2}{3}\mu$$

where λ and μ have been introduced in the constitutive relations of the Navier-Stokes fluid in Footnote 10. In terms of the deviators

$$j_{pij}{}'=j_{pij}-P^{*}\delta_{ij}\quad,\quad d_{ij}{}'=d_{ij}-\frac{1}{3}d_{kk}\delta_{ij}$$

the (conductive) momentum current can be written as follows:

$$j_{pij}{}'=-2\mu d_{ij}{}'\quad,\quad P^{*}=P-\kappa d_{kk}$$

(see Malvern (1967), p. 299). The pressure P^{*} includes a bulk viscosity term in addition to the pressure P in equilibrium. For fluids with vanishing bulk viscosity, the pressure term to be used is the normal pressure, and the viscosity of a “one-dimensional” gas depends only upon the normal viscosity μ.

sidual equation containing the rate of production of entropy; the second of these is identical to the first, so we have only five conditions:

$$\begin{aligned}
&\nu = \lambda^{v}\\
&\rho\frac{\partial u}{\partial\rho} + u - \frac{1}{2}\nu^2 - \lambda^{\rho} - \lambda^{S}s - \lambda^{S}\rho\frac{\partial s}{\partial\rho} = 0\\
&\frac{\partial j_E}{\partial\rho} - \lambda^{S}\frac{\partial j_S{}^{(c)}}{\partial\rho} = 0\\
&\frac{\partial u}{\partial T} - \lambda^{S}\frac{\partial s}{\partial T} = 0\\
&\frac{\partial j_E}{\partial T_{,x}} - \lambda^{S}\frac{\partial j_S{}^{(c)}}{\partial T_{,x}} = 0
\end{aligned} \tag{11.82}$$

and the following residual equation involving the gradients of temperature and velocity:

$$\begin{aligned}
&\frac{\partial T}{\partial x}\left\{\frac{\partial j_E}{\partial T} - \lambda^{S}\frac{\partial j_S{}^{(c)}}{\partial T}\right\}\\
&\qquad + \frac{\partial\nu}{\partial x}\left\{\rho\left(u - \frac{1}{2}\nu^2 - \lambda^{\rho} - \lambda^{S}s\right) + j_p{}^{(c)}\right\} + \lambda^{S}\pi_S = 0
\end{aligned} \tag{11.83}$$

This relation will be used to determine the rate of production of entropy as a consequence of heat conduction and of viscosity.

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[12] There are essentially four parts associated with the four laws of balance included in the combined expression of Equ.(11.81). Together they read:

$$\begin{aligned}
&\rho\frac{\partial u}{\partial\rho}\frac{\partial\rho}{\partial t} + \rho\frac{\partial u}{\partial T}\frac{\partial T}{\partial t} + u\frac{\partial\rho}{\partial t} + \rho\nu\frac{\partial\nu}{\partial t} + \frac{1}{2}\nu^2\frac{\partial\rho}{\partial t} + \rho\nu\frac{\partial u}{\partial\rho}\frac{\partial\rho}{\partial x} + \rho\nu\frac{\partial u}{\partial T}\frac{\partial T}{\partial x} + u\nu\frac{\partial\rho}{\partial x}\\
&+\rho\nu^2\frac{\partial\nu}{\partial x} + \nu\frac{1}{2}\nu^2\frac{\partial\rho}{\partial x} + \rho\left(u + \frac{1}{2}\nu^2\right)\frac{\partial\nu}{\partial x} + j_p{}^{(c)}\frac{\partial\nu}{\partial x} + \nu\frac{\partial j_p{}^{(c)}}{\partial\rho}\frac{\partial\rho}{\partial x} + \nu\frac{\partial j_p{}^{(c)}}{\partial T}\frac{\partial T}{\partial x}\\
&+\nu\frac{\partial j_p{}^{(c)}}{\partial\nu_{,x}}\frac{\partial^2\nu}{\partial x^2} + \frac{\partial j_E}{\partial\rho}\frac{\partial\rho}{\partial x} + \frac{\partial j_E}{\partial T}\frac{\partial T}{\partial x} + \frac{\partial j_E}{\partial T_{,x}}\frac{\partial^2 T}{\partial x^2}\\
&-\lambda^{\rho}\left[\frac{\partial\rho}{\partial t} + \rho\frac{\partial\nu}{\partial x} + \nu\frac{\partial\rho}{\partial x}\right]\\
&-\lambda^{S}\left[s\frac{\partial\rho}{\partial t} + \rho\frac{\partial s}{\partial\rho}\frac{\partial\rho}{\partial t} + \rho\frac{\partial s}{\partial T}\frac{\partial T}{\partial t} + s\nu\frac{\partial\rho}{\partial x} + s\rho\frac{\partial\nu}{\partial x} + \rho\nu\frac{\partial s}{\partial\rho}\frac{\partial\rho}{\partial x} + \rho\nu\frac{\partial s}{\partial T}\frac{\partial T}{\partial x}\right.\\
&\qquad\qquad\left. + \frac{\partial j_S{}^{(c)}}{\partial\rho}\frac{\partial\rho}{\partial x} + \frac{\partial j_S{}^{(c)}}{\partial T}\frac{\partial T}{\partial x} + \frac{\partial j_S{}^{(c)}}{\partial T_{,x}}\frac{\partial^2 T}{\partial x^2} - \pi_S\right]\\
&-\lambda^{v}\left[\rho\frac{\partial\nu}{\partial t} + \nu\frac{\partial\rho}{\partial t} + 2\rho\nu\frac{\partial\nu}{\partial x} + \nu^2\frac{\partial\rho}{\partial x} + \frac{\partial j_p{}^{(c)}}{\partial\rho}\frac{\partial\rho}{\partial x} + \frac{\partial j_p{}^{(c)}}{\partial T}\frac{\partial T}{\partial x} + \frac{\partial j_p{}^{(c)}}{\partial\nu_{,x}}\frac{\partial^2\nu}{\partial x^2}\right]
\end{aligned}$$

Here, $T_{,x}$ and $\nu_{,x}$ are the gradients of temperature and velocity, respectively.

11.8.4 Determination of the Lagrange Multipliers

A first look at Equ.$(11.82)_1$ tells us that the Lagrange multiplier associated with the equation of balance of momentum is equal to the velocity, a result which should not surprise us at all: the velocity is the potential of motion.

Next, we are going to introduce the constitutive expressions for the fluxes of entropy and of energy in heating, Equations $(11.80)_{2,3}$, into Equ.$(11.82)_5$. This yields a relation between the conductivity and the factor β:

$$\lambda^S k_S = \beta$$

which tells us that the Lagrange multiplier associated with entropy may depend only upon density and temperature. It also yields an equivalent relation between the fluxes themselves. You can introduce this latter relation in Equ.$(11.82)_3$ which leads to the following result:

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13. After collecting the different terms, the expression in Footnote 12 becomes

$$\begin{aligned}
&\frac{\partial v}{\partial t}\left\{v\rho-\lambda^v\rho\right\}+\frac{\partial^2 v}{\partial x^2}\left\{v\frac{\partial j_p{}^{(c)}}{\partial v_{,x}}-\lambda^v\frac{\partial j_p{}^{(c)}}{\partial v_{,x}}\right\}+\\
&\frac{\partial\rho}{\partial t}\left\{\rho\frac{\partial u}{\partial\rho}+u+\frac{1}{2}v^2-\lambda^\rho-\lambda^S s-\lambda^S\rho\frac{\partial s}{\partial\rho}-\lambda^v v\right\}+\\
&\frac{\partial\rho}{\partial x}\left\{\rho v\frac{\partial u}{\partial\rho}+uv+v\frac{1}{2}v^2+v\frac{\partial j_p{}^{(c)}}{\partial\rho}+\frac{\partial j_E}{\partial\rho}-\lambda^\rho v\right.\\
&\qquad\left.-\lambda^S\left[sv+\rho v\frac{\partial s}{\partial\rho}+\frac{\partial j_s{}^{(c)}}{\partial\rho}\right]-\lambda^v\left[v^2+\frac{\partial j_p{}^{(c)}}{\partial\rho}\right]\right\}+\\
&\frac{\partial T}{\partial t}\left\{\rho\frac{\partial u}{\partial T}-\lambda^S\rho\frac{\partial s}{\partial T}\right\}+\frac{\partial^2 T}{\partial x^2}\left\{\frac{\partial j_E}{\partial T_{,x}}-\lambda^S\frac{\partial j_s{}^{(c)}}{\partial T_{,x}}\right\}+\\
&\frac{\partial T}{\partial x}\left\{\rho v\frac{\partial u}{\partial T}+v\frac{\partial j_p{}^{(c)}}{\partial T}+\frac{\partial j_E}{\partial T}-\lambda^S\left[\rho v\frac{\partial s}{\partial T}+\frac{\partial j_s{}^{(c)}}{\partial T}\right]-\lambda^v\frac{\partial j_p{}^{(c)}}{\partial T}\right\}+\\
&\frac{\partial v}{\partial x}\left\{\rho v^2+\rho\left(u+\frac{1}{2}v^2\right)+j_p{}^{(c)}-\lambda^\rho\rho-\lambda^S s\rho-\lambda^v 2\rho v\right\}+\lambda^S\pi_s=0
\end{aligned}$$

Each of the first six expressions in braces must vanish; otherwise Equ.(11.81) may be violated. This leaves the residual equation

$$\begin{aligned}
&\frac{\partial T}{\partial x}\left\{\rho v\frac{\partial u}{\partial T}+v\frac{\partial j_p{}^{(c)}}{\partial T}+\frac{\partial j_E}{\partial T}-\lambda^S\left[\rho v\frac{\partial s}{\partial T}+\frac{\partial j_S{}^{(c)}}{\partial T}\right]-\lambda^v\frac{\partial j_p{}^{(c)}}{\partial T}\right\}+\\
&\frac{\partial v}{\partial x}\left\{\rho v^2+\rho\left(u+\frac{1}{2}v^2\right)+j_p{}^{(c)}-\lambda^\rho\rho-\lambda^S s\rho-\lambda^v 2\rho v\right\}+\lambda^S\pi_s=0
\end{aligned}$$

which will let us calculate the rate of production of entropy as a consequence of conduction of heat and viscosity.

$$\frac{\partial j_E}{\partial \rho} - \lambda^S \frac{\partial}{\partial \rho}\left(\frac{1}{\lambda^S} j_E\right) = 0 \quad \Rightarrow \quad \frac{\partial \lambda^S}{\partial \rho} = 0$$

This proves that the Lagrange multiplier may be a function only of temperature, and we have

$$j_E = \lambda^S(T) j_S{}^{(c)} \tag{11.84}$$

The residual equation now takes the following form:

$$\lambda^S \pi_S = -j_S{}^{(c)} \frac{d\lambda^S}{dT} \frac{\partial T}{\partial x} - \frac{\partial v}{\partial x}\left\{\rho\left(u - \frac{1}{2}v^2 - \lambda^\rho - \lambda^S s\right) + j_p{}^{(c)}\right\} \tag{11.85}$$

This result has important consequences. Introducing the constitutive law for the flux density of entropy and for the current density of momentum yields

$$\lambda^S \pi_S = k_S \frac{d\lambda^S}{dT} \frac{\partial T}{\partial x} \frac{\partial T}{\partial x} - \rho\left(u - \frac{1}{2}v^2 + \frac{P}{\rho} - \lambda^\rho - \lambda^S s\right)\frac{\partial v}{\partial x} + \mu' \frac{\partial v}{\partial x}\frac{\partial v}{\partial x}$$

In equilibrium, where the gradients of temperature and of velocity are zero, the rate of production of entropy will vanish. Moreover, for the rate of production of entropy to be minimal in equilibrium, its derivatives with respect to the gradients must vanish:

$$\left.\frac{\partial \pi_S}{\partial T_{,x}}\right|_E = 0 \quad \text{and} \quad \left.\frac{\partial \pi_S}{\partial v_{,x}}\right|_E = 0$$

(E stands for equilibrium). The first condition is satisfied identically, while the latter delivers an interesting relation for the Lagrange multiplier associated with mass:

$$\lambda^\rho = u - \frac{1}{2}v^2 + \frac{P}{\rho} - \lambda^S s \tag{11.86}$$

As you may have noted, the Lagrange multiplier belonging to mass (amount of substance) already looks very similar to the chemical potential of the simple fluids discussed in Chapters 6 and 8.

11.8.5 Determination of the Lagrange Multiplier for Entropy

If we manage to determine the multiplier associated with entropy, the essential results of this section will have been derived. We shall know the relation (11.84) between flux densities of entropy and of energy in heating; the expression (11.85) for the rate of production of entropy; and the Lagrange multiplier for mass, Equ.(11.86), which is indeed going to tell us that the latter is the chemical potential.

The derivation roughly goes as follows. We now use the condition of existence of ideal walls. Analogous to what we did in Section 10.1.4, we derive the result that for two fluids separated by an ideal wall the Lagrange multiplier must be equal:

$$\lambda_{\mathrm{I}}^S(T) = \lambda_{\mathrm{II}}^S(T)$$

This means that the Lagrange multiplier is a universal function; i.e., it must be the same for different Navier-Stokes-Fourier fluids.

We can derive an expression for the rate of change of the specific entropy by combining Equ.(11.82)$_2$ and Equ.(11.82)$_4$. Taking into consideration the result derived in Equ.(11.86) we arrive at

$$\dot{s} = \frac{1}{\lambda^S}\left(\frac{\partial u}{\partial \rho} - \frac{P}{\rho^2}\right)\dot{\rho} + \frac{1}{\lambda^S}\frac{\partial u}{\partial T}\dot{T} \tag{11.87}$$

Since we assume the functions to be sufficiently smooth, this implies an integrability condition of the form

$$\frac{\partial}{\partial T}\left\{\frac{1}{\lambda^S}\left(\frac{\partial u}{\partial \rho} - \frac{P}{\rho^2}\right)\right\} = \frac{\partial}{\partial \rho}\left\{\frac{1}{\lambda^S}\frac{\partial u}{\partial T}\right\}$$

Keep in mind that the Lagrange multiplier depends only upon temperature; you can now derive the following differential equation for the multiplier:

$$\frac{1}{\lambda^S}\frac{d\lambda^S}{dT} = \frac{\partial}{\partial T}\left(\frac{P}{\rho^2}\right)\left(\frac{P}{\rho^2} - \frac{\partial u}{\partial \rho}\right)^{-1} \tag{11.88}$$

Since this relation must hold for all types of fluids, it should also hold for the ideal gas, in which case the right-hand side turns out to be equal to the inverse of the ideal gas temperature. Therefore, we finally have the important result

$$\lambda^S = T \tag{11.89}$$

As in previous examples of materials, the thermal potential is the ideal gas temperature for the Navier-Stokes-Fourier fluid as well.

11.8.6 The Results for Navier-Stokes-Fourier Fluids

We can now assemble the results of the theory. Expressions for the relation between fluxes of entropy and energy, for the rate of production of entropy, and for the Lagrange multiplier for mass (i.e., the chemical potential) can be written in their final form:

$$j_{E,th} = T\, j_S^{(c)} \tag{11.90}$$

$$\pi_S = \frac{1}{T}k_S\frac{\partial T}{\partial x}\frac{\partial T}{\partial x} + \frac{1}{T}\mu'\frac{\partial v}{\partial x}\frac{\partial v}{\partial x} \tag{11.91}$$

$$\mu_m = u - \frac{1}{2}v^2 + \frac{P}{\rho} - T\,s \tag{11.92}$$

Equ.(11.91) also shows that the viscosity may not be a negative quantity (the rate of production of entropy should be positive even if no conduction of heat is present), and

Equ.(11.90) lets us calculate the factor β and the energy current due to conduction:

$$\beta = k_E = T\,k_S \quad \Rightarrow \quad j_{E,th} = -k_E \frac{\partial T}{\partial x} \tag{11.93}$$

The index m in Equ.(11.92) reminds us that the chemical potential of the fluid has been written for mass instead of for amount of substance. Multiplying the expression by the molar mass delivers the result we know from the treatment of uniform fluids in Chapter 8. From Equ.(11.87), we can also derive the Gibbs Fundamental Form for the heat-conducting viscous fluids investigated here. Interestingly, it is the same as the one known from the theory of uniform ideal fluids:

$$\dot{u} = T\,\dot{s} + \frac{P}{\rho^2}\,\dot{\rho} \tag{11.94}$$

Here, it is written using the density of the fluid instead of the volume of a particular body. The fact that some results look the same for reversible and for irreversible fluids is often interpreted as meaning that Navier-Stokes-Fourier fluids do not deviate much from equilibrium (a condition called *local thermodynamic equilibrium*). We normally get such results for materials having linear constitutive relations like Fourier's law or Newton's law of viscosity. A simple case of where all of this does not hold is the example of heat conduction with inductive behavior.

11.8.7 The Thermal Energy Equation for the One-Dimensional NS Fluid

It is common to present the balance of energy in the material form with the constitutive quantities introduced. The resulting relation is called the thermal energy equation. If you start with Equ.(11.73) and use the constitutive relations in the form

$$\begin{aligned} u &= u(\rho,T) \\ j_p{}^{(c)} &= P(\rho,T) - \mu'(\rho,T)\frac{\partial v}{\partial x} \\ j_{E,th} &= -k_E(\rho,T)\frac{\partial T}{\partial x} \end{aligned}$$

then the balance of energy becomes

$$\rho\frac{Du(\rho,T)}{Dt} + \left(P(\rho,T) - \mu'(\rho,T)\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x} + \frac{\partial}{\partial x}\left(-k_E(\rho,T)\frac{\partial T}{\partial x}\right) = 0$$

or

$$\rho\left[\frac{\partial u}{\partial \rho}\frac{D\rho}{Dt} + \frac{\partial u}{\partial T}\frac{DT}{Dt}\right] + P\frac{\partial v}{\partial x} - \mu'\left(\frac{\partial v}{\partial x}\right)^2 - \frac{\partial}{\partial x}\left(k_E\frac{\partial T}{\partial x}\right) = 0 \tag{11.95}$$

It is interesting to specialize this result to an incompressible fluid. Because of this new condition, the first term with the rate of change of the density disappears and so do the terms involving pressure and viscosity. (In one dimension, the speed of the incompressible fluid may not change with position.) If we assume constant fluid properties, we finally get

$$\rho c\left(\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial x}\right)-k_E\,\frac{\partial^2 T}{\partial x^2}=0 \qquad \textbf{(11.96)}$$

With the exception of the second term, which is the result of fluid flow, i.e., of convection, this equation is identical to the one we will derive for time-dependent conduction of heat through a stationary body (see Chapter 13). An incompressible fluid with all quantities (mass, entropy, and momentum) flowing in only a single direction, displays the effects of the transport of entropy both due to conduction and to convection. Since there are no velocity gradients and no shear forces (requiring sideways flow of momentum), there are no viscous effects, and the momentum flux density (i.e., the pressure) must remain constant in the direction of flow. As a result, there is no reference to the mechanical processes to be found anymore in the resulting energy equation. This will not be the case for two-dimensional incompressible flow which we will study in the following section.

11.9 LESSONS OF CONTINUUM THERMODYNAMICS

Even though we have only had a very brief encounter with continuum thermodynamics up to this point, we can draw some important conclusions.

In view of our normal tendency to divide our description of the world into small, neat compartments, maybe the most important of the insights gained from continuum thermodynamics is that the classical field of thermodynamics and the subject of heat transfer form a natural unity. Despite all the claims of text books that heat transfer and thermodynamics do not mix, we do not have to artificially separate the two. Thermodynamics of uniform bodies can be presented as a theory of the dynamics of heat using the proper laws of balance of entropy, momentum, energy, etc., and associated constitutive relations; these same equations of balance are then used as the starting point for investigations of the flow of heat.

Problems in engineering and in the sciences are not solved if we put up high walls around each subject. Rather, if we accept the images inherent in continuum physics for the description of simpler examples as well, these walls are brought down. Introductory physics and thermodynamics would profit considerably from a careful look at what continuum processes have to tell us. Students who have grown up with this kind of mental picture will more easily venture into the exciting combined field of thermodynamics and heat transfer.

Continuum processes are clearly irreversible. While the quest for a description of reversible processes is understandable, and while theories of the dynamics of such processes can be built and applied successfully, the belief that reversibility requires equilibrium in the sense of static conditions has led classical thermodynamics into a tight corner out of which it can escape only if the lessons of continuum physics are learned. If you have followed the development of ideas in this book, you will have seen that everyday phenomena suggest an image of heat as the fundamental extensive thermal quantity leading directly to the equation of balance of entropy. Applying this law to continuum processes is therefore natural and straightforward, and its success tells us that self-imposed limitations in classical thermodynamics are unjustified and unnecessary.

The development of continuum physics in the 20th century should give us the courage to strengthen and renew our teaching of classical physics, to expose students of the sciences and engineering to the foundations of the classical fields, so that these foundations may be applied to the solution of current and future problems.

EXERCISES AND PROBLEMS

1. Consider the model of the life and migration of locusts (Section 11.1). Show that, on purely mathematical grounds, we cannot distinguish between production rate densities and source rate densities. What is their difference, then?
2. Create a uniform systems representation of the spatially continuous case of the law of balance of the locust model. Divide the one-dimensional space into a number of elements. Include migration on the ground and births and deaths. (a) Sketch the diagram of a system dynamics model. (b) Write the equation(s) for a single element of this pseudo-finite-element model. (c) Compare the equation(s) to the continuous case. Can the uniform model be recovered from the continuous case, or vice-versa? (d) In the uniform case, it appears to be impossible to distinguish between (conductive) flows and birth or death rates on purely mathematical grounds. Why? What is their difference, then?
3. Consider the locust model as in Equ.(11.10). (a) Write the equation for constant conductivity and no births and deaths. (b) What will the solution of this equation be for the time-independent (steady-state) case? What (and how many) boundary conditions are needed? How does the solution depend upon the particular boundary conditions you chose?
4. Consider the locust model as in Problem 3 but take a non-zero death rate as in Equ.(11.10); the death rate is assumed to be independent of time and space. What will the steady-state solution of this equation for constant conductivity be? What (and how many) boundary conditions are needed?
5. Add food as a new species to the one-dimensional locust model. The food resource cannot move, it can only be produced and destroyed. (a) Formulate the law of balance of food for the spatially and temporally continuous case. (b) Write constitutive relations for the production of food (example: logistic growth) and for destruction (consumption), and add these to the law of balance.
6. Show that the component form of the general equation of balance of momentum should be written as follows:

$$\frac{\partial}{\partial t}\left(\rho v_i\right)+\frac{\partial}{\partial x_j}\left(\rho v_i v_j + j_{pij}{}^{(c)}\right)=\sigma_{pi}$$

   Write all of the equations in fully expanded form.
7. Show that the law of balance of entropy in the form of Equ.(11.59) looks as if it represented the case of pure conductive transport of entropy. Does the law expressed by Equ.(11.59) exclude the phenomenon of convection? Can you explain the name *material derivative* given to the operator in Equ.(11.58)?
8. Write the law of balance of mass using the material derivative. Do the same for the simple case of pure one-dimensional transport of momentum.
9. In the purely one-dimensional case, gradients and divergences look the same. Explain their differences and write gradient (of a scalar function) and divergence (of a vector field) for a two-dimensional case.
10. Derive the general three-dimensional form of the substantial (material) derivative. Write the result both in coordinate-independent form and by using Cartesian coordinates.
11. Consider a flat car filled with water and travelling horizontally. Water flows out of the bottom through a hole. Determine the flux densities and the fluxes of momentum with respect

to a stationary control volume.

12. Perform a direct derivation of the balance of energy for purely one-dimensional convective and conductive flow of entropy as in Equ.(11.96). (*Hint:* Use a stationary control volume and compute convective and conductive currents of energy associated with entropy transfer only.)
13. Derive the complete expression for the rate of production of entropy for general three-dimensional flow of a Navier-Stokes-Fourier fluid.