1

Introduction

In this introductory chapter we give a short account of the contents of the book and discuss simple notions and examples of the fixed point theory to be developed and applied to more involved applications in later chapters. As an introduction to the fixed point theory and its applications let us recall two fixed point theorems on a nonempty closed and bounded subset P of $\mathbb{R}^m$, one purely topological (Brouwer's fixed point theorem) and one order-theoretically based. A point $x \in P$ is called a fixed point of a function $G : P \to P$ if $x = Gx$. We assume that $\mathbb{R}^m$ is equipped with Euclidean metric.

Theorem 1.1 (Brouwer's Fixed Point Theorem). *If P is a closed, bounded, and convex subset of $\mathbb{R}^m$, then every continuous function $G : P \to P$ has a fixed point.*

To formulate the purely order-theoretic fixed point theorem we equip $\mathbb{R}^m$ with the coordinatewise partial order '$\leq$', i.e., for $x, y \in \mathbb{R}^m$, we define $x \leq y$ if and only if $x_i \leq y_i$, $i = 1, \ldots, m$. A function $G : P \to P$ is called increasing if $x \leq y$ implies $Gx \leq Gy$. Further, we will need the notion of a *sup-center* of the set P, which is defined as follows: A point $c \in P$ is called a sup-center of P if $\sup\{c, x\} \in P$ for each $x \in P$. The next fixed point theorem is a special case of Corollary 2.41(a) of Chap. 2.

Theorem 1.2. *If P is a closed and bounded subset of $\mathbb{R}^m$ having a sup-center, then every increasing function $G : P \to P$ has a fixed point.*

Note that in Theorem 1.2 neither continuity of the fixed point operator nor convexity of the set P is needed. Let us give two examples of sets P that have the required properties of Theorem 1.2. The geometrical center $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$ of every set

$$P = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : \sum_{i=1}^{m} |x_i - c_i|^p \leq r^p\}, \; p, \, r \in (0, \infty), \tag{1.1}$$

S. Carl and S. Heikkilä, *Fixed Point Theory in Ordered Sets and Applications: From Differential and Integral Equations to Game Theory*,
DOI 10.1007/978-1-4419-7585-0_1, © Springer Science+Business Media, LLC 2011

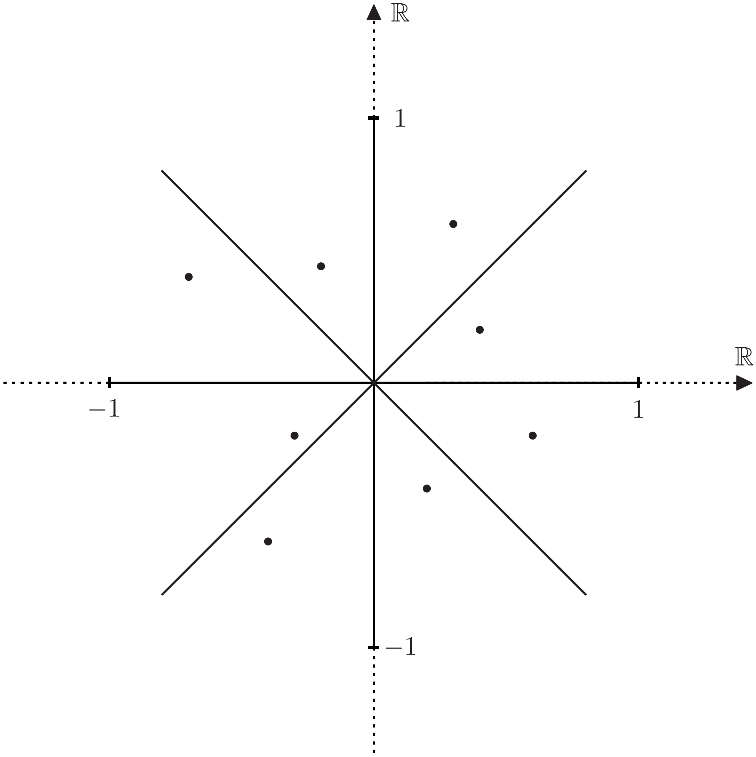

Fig. 1.1. Closed Bounded Set in $\mathbb{R}^2$ with $(0,0)$ as Sup-Center

is a sup-center of P. Because these sets are also closed and bounded, then every increasing mapping $G : P \to P$ has a fixed point. Notice that P is not convex if $0 < p < 1$, as assumed in Theorem 1.1. If P has the smallest element c, then c is a sup-center of P. If $m = 2$, a necessary and sufficient condition for a point $c = (c_1, c_2)$ of P to be a sup-center of P is that whenever a point $y = (y_1, y_2)$ of P and c are unordered, then $(y_1, c_2) \in P$ if $y_2 < c_2$ and $(c_1, y_2) \in P$ if $y_1 < c_1$. The second example of a set $P \subset \mathbb{R}^2$ is illustrated by Fig. 1.1, where P consists of all the solid lines and the isolated points. One easily verifies that $c = (0,0)$ is a sup-center.

Theorem 1.1 and Theorem 1.2 can be applied, e.g., in the study of the solvability of a finite system of equations. For simplicity consider the system

$$u = u(x,y), \quad v = v(x,y). \tag{1.2}$$

Assume that P is a closed and bounded subset of $\mathbb{R}^2$, and that $G = (u,v)$ maps P into itself. By Theorem 1.1 the system (1.2) has a solution if G is continuous and P is convex. But there is no constructive method to solve system (1.2) under these hypotheses. By Theorem 1.2 the system (1.2) has a solution if G is increasing and P is only assumed to possess a sup-center. As we shall see in Chap. 2 the proof of Theorem 1.2 is constructive. In the special case when strictly monotone sequences of the image $G[P]$ are finite, the following algorithm can be applied to obtain a solution of (1.2) when the sup-center of P is $c = (c_1, c_2)$. Maple commands 'fi;od' in the following program mean 'end if;end do'.

$u := u(x, y) : v := v(x, y) : x := c_1 : y := c_2 :$
for k **from** 0 **while** $\text{abs}(u - x) + \text{abs}(v - y) > 0$ **do**;
if $(u - x)(v - y) < 0$ **then** $x := max\{x, u\} : y := max\{y, v\}$
else $x := u : y := v$:**fi;od;**
sol $:= (x, y)$;

It is shown in Chap. 2 that the above algorithm can be applied to approximate a solution of (1.2) in the case when G is continuous and increasing, replacing G by its suitable upper and lower estimates.

Consider next generalizations of Theorem 1.1 and Theorem 1.2 to the case when P is a nonempty subset of an infinite-dimensional normed space E. The generalization of Brouwer's fixed point theorem to infinite-dimensional Banach spaces requires the compactness of the fixed point operator. As compact operators play a central role also in later chapters we recall their definition here for convenience, see, e.g., [62, 228].

Definition 1.3. *Let X and Y be normed spaces, and $T : D(T) \subseteq X \to Y$ an operator with domain $D(T)$. The operator T is called* **compact** *iff T is continuous, and T maps bounded sets into relatively compact sets. Compact operators are also called* **completely continuous**.

In Theorem 1.5 we assume that E is ordered by a closed and convex cone E_+ for which $-E_+ \cap E_+ = \{0\}$. A subset A of P is said to have a sup-center in P if there exists a $c \in P$ such that $\sup\{c, x\}$ exists in E and belongs to P for every $x \in A$.

Theorem 1.4 (Schauder's Fixed Point Theorem). *Let P be a nonempty, closed, bounded, and convex subset of the Banach space E, and assume that $G : P \to P$ is compact. Then G has a fixed point.*

Theorem 1.5 ([116]). *Let P be a subset of an ordered normed space E, and let $G : P \to P$ be increasing. If the weak closure of $G[P]$ has a sup-center in P, and if monotone sequences of $G[P]$ have weak limits in P, then G has a fixed point.*

If P is, e.g., the closed unit ball in l^2 defined by

$$l^2 = \{x = (x_1, x_2, \dots) : \sum_{i=1}^{\infty} |x_i|^2 < \infty\},$$

then the conclusion of Theorem 1.4 does not hold if $G : P \to P$ is only assumed to be continuous (see Kakutani's counterexample). Thus the result of Theorem 1.4 is not valid if the compactness hypothesis of G is missing. On the other hand, no compactness or continuity is assumed in Theorem 1.5, which is also a consequence of Proposition 2.40(a). The geometrical centers of bounded and closed balls of p-normed spaces l^p, ordered coordinatewise, and $L^p(\Omega)$, $1 \le p < \infty$, ordered a.e. pointwise, are their sup-centers. This is true also for closed and bounded balls of Sobolev spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, $1 < p < \infty$, ordered a.e. pointwise. Moreover, these balls are weakly sequentially closed and their monotone sequences have weak limits. Hence, if P is any of these balls, then every increasing function $G : P \to P$ has a fixed point by Theorem 1.5. To demonstrate the applicability of Theorem 1.4 and Theorem 1.5 let us consider two simple examples of elliptic Dirichlet boundary value problems with homogeneous boundary values.

Example 1.6.

$$-\Delta u(x) = f(x, u(x)) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$. Let us assume that f satisfies the following conditions:

(f1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e., $x \mapsto f(x, s)$ is measurable in Ω for all $s \in \mathbb{R}$, and $s \mapsto f(x, s)$ is continuous for almost all (a.a.) $x \in \Omega$.

(f2) The function f fulfills the following growth condition: there is a function $k \in L^2_+(\Omega)$ and a positive constant a such that for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$ we have
$$|f(x, s)| \le k(x) + a|s|.$$

By $L^2_+(\Omega)$ we denote the positive cone of all nonnegative functions of $L^2(\Omega)$. Setting $V_0 = W_0^{1,2}(\Omega)$, V_0^* its dual space, $\mathcal{A} = -\Delta$, and defining $\mathcal{A} : V_0 \to V_0^*$ by

$$\langle \mathcal{A}u, \varphi \rangle = \int_\Omega \nabla u \nabla \varphi \, dx, \quad \forall\, \varphi \in V_0,$$

then $\mathcal{A} : V_0 \to V_0^*$ is a strongly monotone, bounded, and continuous operator. Denoting by F the Nemytskij operator associated with f by

$$F(u)(x) = f(x, u(x)),$$

then, in view of (f1)–(f2), $F : L^2(\Omega) \to L^2(\Omega)$ is continuous and bounded. The compact embedding $i : V_0 \hookrightarrow L^2(\Omega)$ readily implies that the operator $\mathcal{F} = i^* \circ F \circ i : V_0 \to V_0^*$ (i^* is the adjoint operator of i) given by

$$\langle \mathcal{F}(u), \varphi \rangle = \int_\Omega F(u)\, \varphi \, dx, \quad \forall\, \varphi \in V_0$$

is completely continuous. With these notations the weak solution of (1.3) can be given the following form: Find $u \in V_0$ such that

$$\mathcal{A}u - \mathcal{F}(u) = 0 \quad \text{in} \quad V_0^*. \tag{1.4}$$

Since $\mathcal{F} : V_0 \to V_0^*$ is completely continuous and bounded, and $\mathcal{A} : V_0 \to V_0^*$ is strongly monotone, continuous, and bounded, it follows that $\mathcal{A} - \mathcal{F} : V_0 \to V_0^*$ is, in particular, continuous, bounded, and pseudomonotone. The classical theory on pseudomonotone operators due to Brezis and Browder (see, e.g., [229]) ensures that if $\mathcal{A} - \mathcal{F} : V_0 \to V_0^*$ is, in addition, coercive, then $\mathcal{A} - \mathcal{F} : V_0 \to V_0^*$ is surjective, which means that (1.4) has a solution, i.e., (1.3) has a weak solution. A sufficient condition to ensure coerciveness of $\mathcal{A} - \mathcal{F}$ is that the positive constant a in (f2) satisfies $a < \lambda_1$, where λ_1 is the first Dirichlet eigenvalue of $\mathcal{A} = -\Delta$, which is known to be positive and simple, see [6]. This can readily be verified by using (f2) and the following variational characterization of the first eigenvalue λ_1 by

$$\lambda_1 = \inf_{0 \neq v \in V_0} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Omega |v|^2 \, dx}.$$

Now we estimate as follows

$$\begin{aligned} \langle \mathcal{A}u - \mathcal{F}(u), u \rangle &\geq \int_\Omega |\nabla u|^2 \, dx - \|k\|_2 \|u\|_2 - a\|u\|_2^2 \\ &\geq \left(1 - \frac{a}{\lambda_1}\right) \|\nabla u\|_2^2 - \frac{\|k\|_2}{\sqrt{\lambda_1}} \|\nabla u\|_2, \end{aligned}$$

where $\|\cdot\|_2 = \|\cdot\|_{L^2(\Omega)}$. As $\|u\| = \|\nabla u\|_2$ is an equivalent norm in V_0, we see from the last estimate that

$$\frac{1}{\|\nabla u\|_2} \langle \mathcal{A}u - \mathcal{F}(u), u \rangle \to \infty \quad \text{as} \quad \|\nabla u\|_2 \to \infty,$$

which proves the coercivity, and thus the existence of solutions of (1.4).

An alternative approach to the existence proof for (1.4) that is closely related to the pseudomonotone operator theory is based on Schauder's fixed point theorem (see Theorem 1.4). To this end, problem (1.4) is transformed into a fixed point equation as follows: As $\mathcal{A} = -\Delta : V_0 \to V_0^*$ is a linear, strongly monotone, and bounded operator, it follows that the inverse $\mathcal{A}^{-1} : V_0^* \to V_0$ is linear and bounded, which allows us to rewrite (1.4) in the form: Find $u \in V_0$ such that

$$u = \mathcal{A}^{-1} \circ \mathcal{F}(u) \tag{1.5}$$

holds, i.e., that $u \in V_0$ is fixed point of the operator

$$G = \mathcal{A}^{-1} \circ \mathcal{F}.$$

Since under hypotheses (f1)–(f2), $\mathcal{F} : V_0 \to V_0^*$ is completely continuous, and $\mathcal{A}^{-1} : V_0^* \to V_0$ is linear and bounded, it readily follows that $G : V_0 \to V_0^*$ is continuous and compact. In order to apply Schauder's theorem we are going to verify that under the same assumption on a, i.e., $a < \lambda_1$, G maps a closed ball $B(0,R) \subset V_0$ into itself, which finally allows us to apply Schauder's theorem, and thus the existence of solutions of (1.4). Let $v \in B(0,R)$, and denote $u = Gv$. Then, by definition of the operator G, $u \in V_0$ satisfies

$$\int_\Omega \nabla u \nabla \varphi \, dx = \int_\Omega F(v)\, \varphi \, dx, \quad \forall\, \varphi \in V_0.$$

In particular, the last equation holds for $u = \varphi$, which yields

$$\begin{aligned} \|\nabla u\|_2^2 &= \int_\Omega F(v)\, u \, dx \le \|F(v)\|_2 \|u\|_2 \le \|k\|_2 \|u\|_2 + a\|v\|_2\|u\|_2 \\ &\le \frac{a}{\lambda_1}\|\nabla v\|_2 \|\nabla u\|_2 + \frac{\|k\|_2}{\sqrt{\lambda_1}}\|\nabla u\|_2, \end{aligned}$$

which yields (note $u = Gv$) the norm estimate in V_0

$$\|Gv\|_{V_0} \le \frac{a}{\lambda_1}\|\nabla v\|_2 + \frac{\|k\|_2}{\sqrt{\lambda_1}}, \quad \forall\, v \in V_0,$$

where $\|u\|_{V_0} := \|\nabla u\|_2$. Thus if $R > 0$ is chosen in such a way that

$$\frac{a}{\lambda_1} R + \frac{\|k\|_2}{\sqrt{\lambda_1}} \le R,$$

then G provides a mapping of $B(0,R)$ into itself. Such an R always exists, because $\frac{a}{\lambda_1} < 1$. This completes the existence proof via Schauder's fixed point theorem.

Schauder's theorem fails if $\mathcal{F} : V_0 \to V_0^*$ lacks compactness, which may occur, e.g., when in (f2) a critical growth of the form

$$|f(x,s)| \le k(x) + a|s|^{2^*-1}$$

is allowed, where 2^* is the critical Sobolev exponent. Lack of compactness occurs also if (1.3) is studied in unbounded domains, or if $s \mapsto f(x,s)$ is no longer continuous. It is Theorem 1.5 that allows us to deal with these kinds of problems provided the fixed point operator G is increasing. In particular, if only continuity of G is violated, then neither monotone operator theory in the sense of Brezis–Browder–Lions–Minty nor fixed point theorems that assume as a least requirement the continuity of the fixed point operator can be applied. To give a simple example, where standard methods fail, consider the next example.

Example 1.7. Let Ω be as in the example before. We study the following discontinuous Dirichlet boundary value problem:

$$-\Delta u(x) = a[u(x)] + k(x) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

where $a > 0$ is some constant, $k \in L^2(\Omega)$, and $s \mapsto [s]$ stands for the integer function, i.e., $[s]$ denotes the greatest integer with $[s] \le s$. Apparently, in this case $f(x, s) := a[s] + k(x)$ is discontinuous in $s \in \mathbb{R}$. Set $\tilde{k}(x) = |k(x)| + 1$, then we have $\tilde{k} \in L^2_+(\Omega)$, and the following estimate holds

$$|f(x, s)| \le \tilde{k}(x) + a|s|.$$

Due to the structure of f the Nemytskij operator $F : L^2(\Omega) \to L^2(\Omega)$ is still well defined and bounded, however, F is no longer continuous. With the same notation as in Example 1.6 we can transform the elliptic problem (1.6) into the fixed point equation in V_0 of the form

$$u = \mathcal{A}^{-1} \circ \mathcal{F}(u). \tag{1.7}$$

The same estimate as in the previous example shows that the fixed point operator $G = \mathcal{A}^{-1} \circ \mathcal{F}$ maps a ball $B(0, \tilde{R}) \subset V_0$ into itself provided $a < \lambda_1$, and $\tilde{R} > 0$ is sufficiently large. Note, however, that the fixed point operator is no longer continuous. Now, we easily observe that $G : V_0 \to V_0$ is increasing with respect to the underlying natural partial order in V_0 defined via the order cone $L^2_+(\Omega)$. The latter is a simple consequence of the fact that $\mathcal{F} : V_0 \to V_0^*$ is increasing, and because of the inverse monotonicity of $\mathcal{A}^{-1}$, which is a consequence of the maximum principle for the Laplacian. Taking into account the comments after Theorem 1.5, we may apply Theorem 1.5 to ensure that G has a fixed point, which proves the existence of weak solutions for (1.6) provided $0 < a < \lambda_1$. It should be noted that the classical fixed point results for increasing self-mappings due to Amann, Tarski, and Bourbaki (see [228]) cannot be applied here.

Further applications of Theorem 1.5 to deal with elliptic problems that lack compactness are demonstrated in [48], where we prove existence results for elliptic problems with critical growth or discontinuity of the data.

The results of Theorem 1.4 and Theorem 1.5 can be extended to set-valued (also called multi-valued) mappings. Let us assume that P is a nonempty subset of a topological space X. In Theorem 1.9 we assume that X is equipped with such a partial ordering that the order intervals $[a, b] = \{x \in X : a \le x \le b\}$ are closed. Denote by 2^P the set of all subsets of P. An element x of P is called a *fixed point* of a set-valued mapping $\mathcal{F} : P \to 2^P$ if $x \in \mathcal{F}(x)$. We say that $\mathcal{F}$ is *increasing* if, whenever $x \le y$ in P, then for every $z \in \mathcal{F}(x)$ there exists a $w \in \mathcal{F}(y)$ such that $z \le w$, and for every $w \in \mathcal{F}(y)$ there exists a $z \in \mathcal{F}(x)$ such that $z \le w$.

Theorem 1.8 (Generalized Theorem of Kakutani). *A multi-valued function $\mathcal{F} : P \to 2^P$ has a fixed point if P is a nonempty, compact, and convex set in a locally convex Hausdorff space X, $\mathcal{F} : P \to 2^P$ is upper semi-continuous, and if the set $\mathcal{F}(x)$ is nonempty, closed, and convex for all $x \in P$.*

The following theorem is a consequence of Theorem 2.12, which is proved in Chap. 2.

Theorem 1.9. *A multi-valued function $\mathcal{F} : P \to 2^P$ has a fixed point if $\mathcal{F}$ is increasing, its values $\mathcal{F}(x)$ are nonempty and compact for all $x \in P$, chains of $\mathcal{F}[P]$ have supremums and infimums, and if $\overline{\mathcal{F}[P]}$ has a sup-center in P.*

In particular, if P is any set defined in (1.1), then every increasing mapping $\mathcal{F} : P \to 2^P$ whose values are nonempty closed subsets of $\mathbb{R}^m$ has a fixed point by Theorem 1.9. As a further consequence of Theorem 1.9 one gets the following order-theoretic fixed point result in infinite-dimensional ordered Banach spaces, which is useful in applications to discontinuous differential equations (see Theorem 4.37).

Theorem 1.10. *Let P be a closed and bounded ball in a reflexive lattice-ordered Banach space X, and assume that $\|x^+\| = \|\sup\{0, x\}\| \le \|x\|$ holds for all $x \in X$. Then every increasing mapping $\mathcal{F} : P \to 2^P$, whose values are nonempty and weakly sequentially closed, has a fixed point.*

To give an idea of how Theorem 1.10 can be applied to differential equations, let us consider a simple example.

Example 1.11. Consider the following slightly extended version of problem (1.6):

$$-\Delta u(x) = a[u(x)] + g(x, u(x)) + k(x) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.8}$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with the following growth condition

(g) There exist a positive constant b with $b < \lambda_1 - a$, and a $h \in L^2(\Omega)$, such that for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$

$$|g(x, s)| \le h(x) + b|s|$$

holds. Here a and λ_1 are as in Example 1.7

If we rewrite the right-hand side of equation (1.8) in the form

$$f(x, s, r) := a[r] + g(x, s) + k(x), \tag{1.9}$$

then we can distinguish between the continuous and discontinuous dependence of the right-hand side of (1.8). This allows an approach toward the existence of solutions of (1.8) by means of the multi-valued fixed point Theorem 1.9. Note, $s \mapsto f(x, s, r)$ is continuous, and $r \mapsto f(x, s, r)$ is discontinuous and monotone increasing. Let $v \in V_0$ be fixed, and consider the boundary value problem

$$-\Delta u(x) = f(x, u(x), v(x)) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{1.10}$$

As the function $(x, s) \mapsto f(x, s, v(x))$ with f defined in (1.9) is a Carathéodory function, one can apply the same approach as in Example 1.6 to get the existence of solutions for (1.10). For fixed $v \in V_0$, denote now by Gv the set of all solutions of (1.10). This provides a multi-valued mapping $G : V_0 \to 2^{V_0}$, and certainly any fixed point of G is a solution of the original boundary value problem (1.8), and vice versa. By similar estimates as in Examples 1.6 and 1.7 one can show that under the given assumptions, in particular due to $0 < a + b < \lambda_1$, there is a closed ball $B(0, R) \subset V_0$ such that the multi-valued mapping G maps $B(0, R)$ into itself. As V_0 is a reflexive lattice-ordered Banach space satisfying $\|u^+\| = \|\sup\{0, u\}\| \le \|u\|$ for all $u \in V_0$, for $G : B(0, R) \to 2^{B(0,R)}$ to possess a fixed point it is enough to show that $G : B(0, R) \to 2^{B(0,R)}$ is increasing, and that the images Gv are weakly sequentially closed, see Theorem 4.37. This will be demonstrated for more involved elliptic problems in Chap. 4.

Chapter 3 is devoted to comparison principles for, in general, multi-valued elliptic and parabolic variational inequalities, with an account of the main differences between them. Elliptic multi-valued variational inequalities of the following kind are considered: Let $K \subseteq W^{1,p}(\Omega)$ be a closed convex set. Find $u \in K$, $\eta \in L^q(\Omega)$, and $\xi \in L^q(\partial\Omega)$ satisfying:

$$\eta(x) \in \partial j_1(x, u(x)), \text{ a.e. } x \in \Omega, \quad \xi(x) \in \partial j_2(x, \gamma u(x)), \text{ a.e. } x \in \partial\Omega, \tag{1.11}$$

$$\langle Au - h, v - u\rangle + \int_\Omega \eta\,(v - u)\,dx + \int_{\partial\Omega} \xi\,(\gamma v - \gamma u)\,d\sigma \ge 0, \ \forall\, v \in K, \tag{1.12}$$

where $s \mapsto \partial j_k(x, s)$ are given by Clarke's generalized gradient of locally Lipschitz functions $s \mapsto j_k(x, s)$, $k = 1, 2$, γ is the trace operator, and A is some quasilinear elliptic operator of Leray–Lions type. As for parabolic multi-valued variational inequalities, the underlying solution space is

$$W = \{u \in X : \partial u/\partial t \in X^*\},$$

where $X = L^p(0, \tau; W^{1,p}(\Omega))$, and X^* is its dual space. Consider the time-derivative $L = \frac{\partial}{\partial t} : D(L) \to X^*$ as an operator from the domain $D(L)$ to X^* where $D(L)$ is given by

$$D(L) = \{u \in W : u(0) = 0\},$$

and let $K \subseteq X$ be closed and convex. The following general class of multi-valued parabolic variational inequalities is treated in Chap. 3: Find $u \in K \cap D(L)$, $\eta \in L^q(Q)$, and $\xi \in L^q(\Gamma)$ satisfying:

$$\eta(x, t) \in \partial j_1(x, t, u(x, t)), \text{ for a.e. } (x, t) \in Q, \tag{1.13}$$

$$\xi(x, t) \in \partial j_2(x, t, \gamma u(x, t)), \text{ for a.e. } x \in \Gamma, \text{ and} \tag{1.14}$$

$$\langle Lu + Au - h, v - u\rangle + \int_Q \eta\,(v - u)\,dxdt + \int_\Gamma \xi\,(\gamma v - \gamma u)\,d\Gamma \ge 0, \ \forall\, v \in K, \tag{1.15}$$

where $Q = \Omega \times (0,\tau)$ and $\Gamma = \partial\Omega \times (0,\tau)$. For both problems (1.11)–(1.12) and (1.13)–(1.15) we establish existence and comparison results in terms of appropriately defined sub- and supersolutions, and characterize their solution sets topologically and order-theoretically. We are demonstrating by a number of examples that the variational inequality problems (1.11)–(1.12) and (1.13)–(1.15) include a wide range of specific elliptic and parabolic boundary value problems and variational inequalities. In this sense, Chap. 3 is not only a prerequisite for Chaps. 4 and 5, but it is of interest on its own and can be read independently.

In Chaps. 4 and 5 we apply the fixed point results of Chap. 2 combined with the comparison results of Chap. 3 to deal with discontinuous single and multi-valued elliptic and parabolic problems of different kinds. In particular, we consider nonlocal, discontinuous elliptic and parabolic boundary value problems and multi-valued elliptic problems with discontinuously perturbed Clarke's generalized gradient. In the study of those problems, besides fixed point and comparison results, the existence of *extremal solutions* of certain associated auxiliary problems play an important role. Extremal solution results that are proved in Chap. 3 require rather involved techniques. These results are used to transform a given multi-valued elliptic or parabolic problem into a fixed point equation.

Differential and integral equations treated in Sects. 6.1–6.4 and 7.1–7.2 contain functions that are Henstock–Lebesgue (HL) integrable with respect to the independent variable. A function g from a compact real interval $[a,b]$ to a Banach space E is called HL *integrable* if there is a function $f : [a,b] \to E$, called a *primitive* of g, with the following property: To each $\epsilon > 0$ there corresponds such a function $\delta : [a,b] \to (0,\infty)$ that whenever $[a,b] = \cup_{i=1}^m [t_{i-1}, t_i]$ and $\xi_i \in [t_{i-1}, t_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all $i = 1,\dots,m$, then

$$\sum_{i=1}^{m} \|f(t_i) - f(t_{i-1}) - g(\xi_i)(t_i - t_{i-1})\| < \epsilon. \tag{1.16}$$

Criteria for HL integrability that are sufficient in most of our applications are given by the following lemma.

Lemma 1.12. *Given a function $g : [a,b] \to E$, assume there exists a continuous function $f : [a,b] \to E$ and a countable subset Z of $[a,b]$ such that $f'(t) = g(t)$ for all $t \in [a,b] \setminus Z$. Then g is HL integrable on $[a,b]$, and f is a primitive of g.*

Proof: Since Z is countable, it can be represented in the form $Z = \{x_j\}_{j\in\mathbb{N}}$. Let $\epsilon > 0$ be given. Since f is continuous, and the values of g have finite norms, then for every $x_j \in Z$ there exists a $\delta(x_j) > 0$ such that $\|f(\bar t) - f(t)\| < 2^{-j-1}\epsilon$, and $\|g(x_j)\|(\bar t - t) < 2^{-j-1}\epsilon$ whenever $x_j - \delta(x_j) < t \le x_j \le \bar t < x_j + \delta(x_j)$.

To each $\xi \in [a,b] \setminus Z$ there corresponds, since $f'(\xi)$ exists, such a $\delta(\xi) > 0$ that $\|f(\bar t) - f(t) - f'(\xi)(\bar t - t)\| < \epsilon(\bar t - t)/(b-a)$ whenever $\xi \in [t, \bar t] \subset$

$(\xi - \delta(\xi), \xi + \delta(\xi))$. Consequently, if $a = t_0 < t_1 < \cdots < t_m = b$, and if $\xi_i \in [t_{i-1}, t_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all $i = 1, \ldots, m$, then

$$\begin{aligned}
&\sum_{i=1}^{m} \|f(t_i) - f(t_{i-1}) - g(\xi_i)(t_i - t_{i-1})\| \\
&\leq \sum_{\xi_i = x_j \in Z} (\|f(t_i) - f(t_{i-1})\| + \|g(x_j)\|(t_i - t_{i-1})) \\
&\quad + \sum_{\xi_i \in [a,b] \setminus Z} \|f(t_i) - f(t_{i-1}) - f'(\xi_i)(t_i - t_{i-1})\| \leq 2\epsilon.
\end{aligned}$$

Thus g is HL integrable and f is its primitive. □

Remark 1.13. If the set Z in Lemma 1.12 is uncountable, an extra condition, called the Strong Lusin Condition (see Chap. 9), is needed to ensure HL integrability.

Compared with Lebesgue and Bochner integrability, the definition of HL integrability is easier to understand because no measure theory is needed. Moreover, all Bochner integrable (i.e., in real-valued case Lebesgue integrable) functions are HL integrable, but not conversely. For instance, HL integrability encloses improper integrals. Consider the real-valued function f defined on $[0, 1]$ by $f(0) = 0$ and $f(t) = t^2 \cos(1/t^2)$ for $t \in (0, 1]$. This function is differentiable on $[0, 1]$, whence f' is HL integrable by Lemma 1.12. However, f' is not Lebesgue integrable on $[0, 1]$. More generally, let t be called a singular point of the domain interval of a real-valued function that is not Lebesgue integrable on any subinterval that contains t. Then (cf. [167]) there exist "HL integrable functions on an interval that admit a set of singular points with its measure as close as possible but not equal to that of the whole interval."

If g is HL integrable on $[a, b]$, it is HL integrable on every closed subinterval $[c, d]$ of $[a, b]$. The *Henstock–Kurzweil* integral of g over $[c, d]$ is defined by

$$^K\!\!\int_c^d g(s)\, ds := f(d) - f(c), \quad \text{where } f \text{ is a primitive of } g.$$

The main advantage of the Henstock–Kurzweil integral is its applicability for integration of highly oscillatory functions that occur in quantum theory and nonlinear analysis. This integral provides a tool to construct a rigorous mathematical formulation for Feynman's path integral, which plays an important role in quantum physics (see, e.g., [143, 182]).

On the other hand, as stated in [98, p.13], the most important factor preventing a widespread use of the Henstock–Kurzweil integral in engineering, mathematics, and physics has been the lack of a natural Banach space structure for the class of HL integrable functions, even in the case when $E = \mathbb{R}$. However, if E is ordered, the validity of the dominated and monotone convergence theorems, which we prove for order-bounded sequences of HL integrable functions (see Chap. 9), considerably improve the applicability of the

Henstock–Kurzweil integral in Nonlinear Analysis. Combined with fixed point theorems in ordered normed spaces presented in Chap. 2, these convergence theorems provide effective tools to solve differential and integral equations that contain HL integrable valued functions and discontinuous nonlinearities. All this will be discussed in detail in Chaps. 6 and 7 and shows once more the importance of the order structure of the underlying function spaces. In particular, the above stated lack of a Banach space structure causes no problems in our studies. Moreover, as the following simple example shows, the ordering allows us to determine the smallest and greatest solutions of such equations.

Example 1.14. Determine the smallest and the greatest continuous solutions of the following Cauchy problem:

$$y'(t) = q(t, y(t), y) \text{ for a.e. } t \in J := [0, 4], \quad y(0) = 0, \tag{1.17}$$

where

$$\begin{cases} q(t, x, y) = p(t)\,\mathrm{sgn}(x) + h(y)(1 + \cos(t)), \\ p(t) = \left|\cos\left(\frac{1}{t}\right)\right| + \frac{1}{t}\mathrm{sgn}\Big(\cos\left(\frac{1}{t}\right)\Big)\sin\left(\frac{1}{t}\right), \\ h(y) = \left[2\arctan\Big(\int_1^4 y(s)\,ds\Big)\right], \quad \mathrm{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases} \\ [x] = \max\{n \in \mathbb{Z} : n \le x\}. \end{cases} \tag{1.18}$$

Note that the bracket function, called the greatest integer function, occurs in the function h.

Solution: If $y \in C(J, \mathbb{R})$ and $y(t) > 0$ when $t > 0$, then $\mathrm{sgn}(y(t)) = 1$ when $t > 0$. Thus

$$q(t, y(t), y) = q_y(t) := p(t) + h(y)(1 + \cos(t)), \quad t \in J.$$

The function $f_y : J \to \mathbb{R}$, defined by

$$f_y(0) = 0, \;\; f_y(t) = t\left|\cos\left(\frac{1}{t}\right)\right| + h(y)(t + \sin(t)), \quad t \in (0, 4],$$

is continuous, and $f_y'(t) = q_y(t)$ if $t \in (0, 4]$ and $t \neq \frac{1}{(2n+1)\pi}$, $n \in \mathbb{N}_0$. Thus q_y is HL integrable on J and f_y is its primitive by Lemma 1.12. This result and the definitions of f_y, q_y and the Henstock–Kurzweil integral imply that

$$Gy(t) := {}^K\!\!\int_0^t q(s, y(s), y)\,ds = f_y(t), \quad t \in J.$$

Moreover, $h(y) = \left[2\arctan\Big(\int_1^4 y(s)\,ds\Big)\right] \le 3$ for every $y \in C(J, \mathbb{R})$. Thus, defining

$$y^*(t) = t\left|\cos\left(\frac{1}{t}\right)\right| + 3(t + \sin(t)),\ t \in (0,4],\ y^*(0) = 0,$$

then $f_y(t) \le y^*(t)$ for all $t \in J$ and $y \in C(J,\mathbb{R})$. On the other hand, it is easy to show that $h(y^*) = \left[2\ \arctan\left(\int_1^4 y^*(s)\,ds\right)\right] = 3$. Consequently,

$$y^*(t) = f_{y^*}(t) = Gy^*(t), \quad t \in J.$$

It follows from the above equation by differentiation that

$$(y^*)'(t) = f'_{y^*}(t) = q_{y^*}(t) = q(t, y^*(t), y^*), \quad t \in (0,4],\ t \neq \frac{1}{(2n+1)\pi},\ n \in \mathbb{N}_0.$$

Moreover $y^*(0) = 0$, so that y^* is a solution of problem (1.17). The above reasoning shows also that if $y \in C(J,\mathbb{R})$ is a solution of problem (1.17), then $y(t) \le y^*(t)$ for every $t \in J$. Thus y^* is the greatest continuous solution of problem (1.17).

By similar reasoning one can show that the smallest solution of the Cauchy problem (1.17) is

$$y_*(t) = -t\left|\cos\left(\frac{1}{t}\right)\right| - 4(t + \sin(t)),\ t \in (0,4],\ y_*(0) = 0.$$

The function $(t,x,y) \mapsto q(t,x,y)$, defined in (1.18), has the following properties.

- It is HL integrable, but it is neither Lebesgue integrable nor continuous with respect to the independent variable t if $x \neq 0$, because p is not Lebesgue integrable.
- Its dependence on all the variables t, x, and y is discontinuous, since the signum function sgn, the greatest integer function $[\cdot]$, and the function p are discontinuous.
- Its dependence on the unknown function y is nonlocal, since the integral of function y appears in the argument of the arctan-function.
- Its dependence on x is not monotone, since p attains positive and negative values in a infinite number of disjoint sets of positive measure. For instance, $y^*(t) > y_*(t)$ for all $t \in (0,4]$, but the difference function $t \mapsto q(t, y^*(t), y^*) - q(t, y_*(t), y_*) = 2p(t) + 7(t + \sin(t))$ is neither nonnegative-valued nor Lebesgue integrable on J.

The basic theory of Banach-valued HL integrable functions needed in Chaps. 6 and 7 is presented in Chap. 9. However, readers who are interested in Real Analysis may well consider the functions to be real-valued. For those readers who are familiar with Bochner integrability theory, notice that *all the theoretical results of Chaps. 6 and 7 where HL integrability is assumed remain valid if HL integrability is replaced by Bochner integrability. As far as the authors know, even the so obtained special cases are not presented in any other book.*

In Sect. 6.5 we study functional differential equations equipped with a functional initial condition in an ordered Banach space E. There we need fixed point results for an increasing mapping $G : P \to P$, where P is a subset of the Cartesian product of the space $L^1(J, E)$ of Bochner integrable functions from $J := [t_0, t_1]$ to E and the space $C(J_0, E)$ of continuous functions from $J_0 := [t_0 - r, t_0]$ to E. The difficulties one is faced with in the treatment of the considered problems are, first, that only a.e. pointwise weak convergence in $L^1(J, E)$ is available, and second, monotone and bounded sequences of the pointwise ordered space $C(J_0, E)$ need not necessarily have supremums and infimums in $C(J_0, E)$. The following purely order-theoretic fixed point theorem, which is proved in Chap. 2, is the main tool that will allow us to overcome the above described difficulties.

Theorem 1.15. *Let G be an increasing self-mapping of a partially ordered set P such that chains of the range $G[P]$ of G have supremums and infimums in P, and that the set of these supremums and infimums has a sup-center. Then G has minimal and maximal fixed points, as well as fixed points that are increasing with respect to G.*

This fixed point theorem will be applied, in particular, in Sects. 6.5 and 7.3 to prove existence and comparison results for solutions of operator equations in partially ordered sets, integral equations, as well as implicit functional differential problems in ordered Banach spaces. It is noteworthy that the data of the considered problems, i.e., operators and functions involved, are allowed to depend discontinuously on all their arguments. Moreover, we do not suppose the existence of subsolutions and/or supersolutions in the treatment.

The abstract order-theoretic fixed poind theory developed in Chap. 2 has been proved to be an extremely powerful tool in dealing with Nash equilibria for normal form games, which is the subject of Chap. 8.

John Nash invented in [185] an equilibrium concept that now bears his name. Because of its importance in economics, Nash earned the Nobel Prize for Economics in 1994. In [185] Nash described his equilibrium concept in terms of game theory as follows:

> *"Any N-tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the N strategy spaces of the players. One such N-tuple counters another if the strategy of each player in countering N-tuple yields the highest possible expectation for its player against the $N-1$ strategies of the other player in the countered N-tuple. A self-countering N-tuple is called an equilibrium point."*

To convert this description into a mathematical concept, we utilize the following notations. Let $\Gamma = \{S_1, \dots, S_N, u_1, \dots, u_N\}$ be a *finite normal-form game*, where the *strategy set* S_i for player i is finite, and the *utility function* u_i of player i is real-valued and defined on $S = S_1 \times \cdots \times S_N$. Using the notations

$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$ and $u_i(s_1, \ldots, s_N) = u_i(s_i, s_{-i})$, strategies $s_1^*, \ldots, s_N^*$ form a *pure Nash equilibrium* for Γ if and only if

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \text{ for all } s_i \in S_i \text{ and } i = 1, \ldots, N.$$

This definition implies that *the strategies of players form a pure Nash equilibrium if and only if no player can improve his/her utility by changing the strategy when all the other players keep their strategies fixed.*

Besides economics, this equilibrium concept has been applied in other social and behavioral sciences, biology, law, politics, etc., cf. [83, 109, 177, 193, 210, 218, 220, 224]. The Nash equilibrium has found so many applications partly because it can be usefully interpreted in a number of ways (cf. [144]). For instance, in human interaction (social, economic, or political) the utilities of players (individuals, firms, or politicians/parties) are mutually dependent on actions of all players. The Nash equilibrium provides an answer to the question of what is an optimal action for every player. The following simple thought experiment describes the usefulness of the Nash equilibrium concept and its relation to democratic decision procedure.

> The traffic board of Sohmu hires a consultant to make a traffic plan for the only crossroad of town having traffic lights. Traffic should be as safe as possible. The consultant seeks Nash safety equilibria and finds two: either every passenger goes toward green light, or, all passengers go toward red light. He suggests to the board that one of these alternatives should be chosen. The state council votes on the choice. Every council member votes according to his/her preferences: Green, Red, or Empty. The result is in Nash equilibrium with respect to the opinions of the council members.

The above thought experiment implies that the concept of Nash equilibrium harmonizes with democratic decision making. It also shows that if actions are in Nash equilibrium, they may give the best result for every participant. It is not a matter of a zero-sum game where someone loses when someone else wins.

Nash used in [186] a version of Theorem 1.8 (see [148]) to prove the existence of Nash equilibrium for a finite normal-form game. Because of finiteness of strategy sets, the application of Kakutani's fixed point theorem was not possible without extensions of S_i to be homeomorphic with convex sets. Thus he extended the strategy sets to contain also strategies that are called *mixed strategies*. This means that players i are allowed to choose independently randomizations of strategies of S_i, that is, each mixed strategy σ_i is a probability measure over S_i. The values of utilities $\mathcal{U}_i$, $i = 1, \ldots, N$, are then the expected values:

$$\mathcal{U}_i(\sigma_1, \ldots, \sigma_N) = \sum_{(s_1,\ldots,s_N)\in S} \sigma_1(\{s_1\}) \cdots \sigma_m(\{s_N\}) u_i(s_1, \ldots, s_N).$$

According to Nash's own interpretation stated above, a *mixed Nash equilibrium* for Γ is a profile of mixed strategies, one for each N players, that has the property that each player's strategy maximizes his/her expected utility against the given strategies of the other players. To put this into a mathematical form, denote $\sigma_{-i} = (\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_N)$ and $\mathcal{U}_i(\sigma_1, \dots, \sigma_N) = \mathcal{U}_i(\sigma_i, \sigma_{-i})$, and let Σ_i denote the set of all mixed strategies of player i. We say that mixed strategies $\sigma_1^*, \dots, \sigma_N^*$ form a *mixed Nash equilibrium* for Γ if

$$\mathcal{U}_i(\sigma_i^*, \sigma_{-i}^*) = \max_{\sigma_i \in \Sigma_i} \mathcal{U}_i(\sigma_i, \sigma_{-i}^*) \quad \text{for all} \quad i = 1, \dots, N.$$

As for a variety of areas where the concept of Nash equilibrium is applied, see [144, 183] and the references therein.

In Chap. 8 we present some recent results dealing with Nash equilibria for normal-form games. Our study is focused on games with strategic complementarities, which means roughly speaking that the best response of any player is increasing in actions of the other players. Sections 8.1 and 8.2 are devoted especially to those readers who are interested only in finite games. In section 8.1 we prove the existence for the smallest and greatest pure Nash equilibria of a normal-form game whose strategy spaces S_i are finite sets of real numbers, and the real-valued utility functions u_i possess a finite difference property. If the utilities $u_i(s_1, \dots, s_N)$ are also increasing (respectively decreasing) in s_j, $j \neq i$, the utilities of the greatest (respectively the smallest) pure Nash equilibrium are shown to majorize the utilities of all pure Nash equilibria. An application to a pricing problem is given.

Our presentation of Sects. 8.2–8.5 has three main purposes.

1. In order to avoid "throwing die" in the search for Nash equilibria, it would be desirable that Γ has a pure Nash equilibrium whose utilities majorize the utilities of all other Nash equilibria for Γ, including mixed Nash equilibria. In such a case it would be of no benefit to seek possible mixed Nash equilibria. If Γ is a finite normal-form game, every player who has at least two pure strategies has uncountably many mixed strategies. On the other hand, the set of its pure strategies, as well as the ranges of its utilities, are finite for each player. Thus one can find pure Nash equilibria in concrete situations by finite methods. In Sect. 8.2 we prove that finite normal-form games, which are supermodular in the sense defined in [218, p. 178], possess the above described desirable properties. The proof is constructive and provides a finite algorithm to determine the most profitable pure Nash equilibrium for Γ. This algorithm and Maple programming is applied to calculate the most profitable pure Nash equilibrium and the corresponding utilities for some concrete pricing games. Proposition 8.61 deals also with finite normal-form games.

2. Theorem 1.9 along with other fixed point theorems presented in Chap. 2 are applied in Sects. 8.3 and 8.4 to derive existence and comparison results for exremal Nash equilibria of normal-form games in more general settings. For instance, the result for finite supermodular games proved in Sect. 8.2 is extended in Sect. 8.4 to the case when the strategy spaces S_i are compact sublattices of complete and separable ordered metric spaces. The easiest case is when strategy spaces are subsets of $\mathbb{R}$. In fact, it has been shown recently (cf.

[86]) that when the strategies of a supermodular normal-form game are real numbers, the mixed extension of that game is supermodular, its equilibria form a non-empty complete lattice, and its extremal equilibria are in pure strategies when mixed strategies are ordered by first order stochastic dominance. A problem that arises when the strategies are not in $\mathbb{R}$ is described in [86, Chap. 3] as follows:

> "When the strategy spaces are multidimensional, the set of mixed strategies is not a lattice. This implies that we lack the mathematical structure needed for the theory of complementarities. We need lattice property to make sense of increasing best responses when they are not real-valued. Multiple best responses are always present when dealing with mixed equilibria and there does not seem a simple solution to the requirement that strategy spaces be lattices."

In particular, classical fixed point theorems in complete lattices are not applicable, even for finite normal-form games having multidimensional strategy spaces. Moreover, in such cases the desirable comparison results between pure and mixed strategies cannot be obtained by the methods used, e.g., in [86, 180, 217, 218, 222, 223]. The results of Theorems 2.20 and 2.21 and their duals provide tools to overcome the problem caused by the above stated non-lattice property. Our results imply that the smallest and greatest pure Nash equilibria of supermodular games form lower and upper bounds for all possible mixed Nash equilibria when the set of mixed strategies is ordered by first-order stochastic dominance. These lower and upper bounds have the important property that they are the smallest and greatest rationalizable strategy profile, as shown in [179]. In particular, if these bounds are equal, then there is only one pure Nash equilibrium, which is also a unique rationalizable strategy profile, and no properly mixed Nash equilibria exist. In [218, Sect. 4] the following eight examples of supermodular games are presented: Pricing game with substitute products, production game with complementary products, multimarket oligopoly, arms race game, trading partner search game, optimal consumption game with multiple products, facility location game, and minimum cut game. The first example is studied here more closely. In this example the greatest Nash equilibrium has also the desirable property that the utilities of the greatest Nash equilibrium majorize the utilities of all other Nash equilibria, including mixed Nash equilibria. Concrete examples are solved by using Maple programming.

3. Another property, which restricts the application of the original result of Nash, is that the utility functions are assumed to be real-valued. This requires that differences of values of u_i can be estimated by numbers. The results of Sects. 8.2 and 8.4 are proved in the case when the values of u_i are in ordered vector spaces E_i. Thus we can consider the cases when the values of utilities are random variables by choosing $E_i = L^2(\Omega_i, \mathbb{R})$, ordered a.e. pointwise. If the strategy spaces are finite, the only extra hypothesis is that the ranges of $u_i(\cdot, s_{-i})$ are directed upward. As an application the pricing game is extended

to the form where the values of the utility functions are in $\mathbb{R}^{m_i}$. Concrete examples are solved also in this case.

But in the definition of pure Nash equilibrium a partial ordering of the values of u_i is sufficient. The fixed point theorems presented in Chap. 2 apply to derive existence results for extremal pure Nash equilibria of normal-form games when both the strategy spaces S_i and ranges of utility functions u_i are partially ordered sets. Such results are presented in Sect. 8.3. We also present results dealing with monotone comparative statics, i.e., conditions that ensure the monotone dependence of extremal pure Nash equilibria on a parameter that belongs to a poset. As for applications, see, e.g., [10, 11, 180, 218]. These results can be applied to cases where the utilities of different players are evaluated in different ordinal scales, where all the values of the utility functions need not even be order-related. Thus the way is open for new applications of the theory of Nash equilibrium, for instance, in social and behavioral sciences. In such applications the term ‘utility’ would approach to one of its meanings: “the greatest happiness of the greatest number.”

A necessary condition for the existence of a Nash equilibrium for Γ is that the functions $u_i(\cdot, s_{-i})$ have maximum points. When the ranges of these functions are in partially ordered sets that are not topologized, the classical hypotheses, like upper semicontinuity, are not available. Therefore we define a new concept, called upper closeness, which ensures the existence of required maximum points. Upper semicontinuity implies upper closeness for real-valued functions.

In Sect. 8.5 we study the existence of undominated and weakly dominating strategies of normal-form games when the ranges of the utility functions are in partially ordered sets. A justification for Sects. 8.2–8.5 is a philosophy of economic modeling stated in [13]: ”The weakest sufficient conditions for robust conclusions is particularly important to economists.” Concrete examples are presented.

The existence of winning strategies for a pursuit and evasion game is proved in Sect. 8.6. As an introduction to the subject consider a finite pursuit and evasion game. Game board P is a nonempty subset of $\mathbb{R}^2$, equipped with coordinatewise partial ordering $\leq$. Assume that to every position $x \in P$ of player p (pursuer) there corresponds a nonempty subset $\mathcal{F}(x) \subseteq P$ of possible positions y of player q (quarry). The only rule of the game is:

(R) If (x_n, y_n) and (x_{n+1}, y_{n+1}) are consecutive positions of a play, then $y_n \leq y_{n+1}$ whenever $x_n < x_{n+1}$, and $y_{n+1} \leq y_n$ whenever $x_{n+1} < x_n$.

We say that a strategy of p is a winning strategy if the use of it yields capturing, i.e., after a finite number of move pairs p and q are in the same position. Player p has a winning strategy if the following conditions hold:

(i) The set $F[P] = \bigcup\{\mathcal{F}(x) : \ x \in P\}$ has a sup-center $c \in P$.

(ii) If $x \le y$ in P, then for every $z \in \mathcal{F}(x)$ for which $z \le y$ there exists such a $w \in \mathcal{F}(y)$ that $z \le w$, and for every $w \in \mathcal{F}(y)$ satisfying $x \le w$ there exists such a $z \in \mathcal{F}(x)$ that $z \le w$.
(iii) Strictly monotone sequences of $\mathcal{F}[P]$ are finite.

The existence of a winning strategy for p can be justified as follows. Player p starts from $x_0 = c$, and q starts from $y_0 \in \mathcal{F}(x_0)$. If $x_0 = y_0$, then p wins. Otherwise, let x_n and $y_n \in \mathcal{F}(x_n)$ denote positions of p and q after nth move pair of a play. If $x_n \neq y_n$, then p moves to $x_{n+1} = \sup\{c, y_n\}$ if x_n and y_n are unordered, and to $x_{n+1} = y_n$ if x_n and y_n are ordered. If $x_n < x_{n+1}$, then q must obey rule (R) and choose a position y_{n+1} of $\mathcal{F}(x_{n+1})$ such that $y_n \le y_{n+1}$, which is possible due to condition (ii). If $x_{n+1} < x_n$, then similarly q obeying the rule (R) can choose by condition (ii) $y_{n+1} \in \mathcal{F}(x_{n+1})$ so that $y_{n+1} \le y_n$. Condition (iii) ensures that every play which follows these rules stops after a finite number of moves to the situation where $x_m = y_m$.

The correspondence $x \mapsto \mathcal{F}(x)$ can be considered also as a set-valued mapping from P to the set $2^P \setminus \emptyset$ of nonempty subsets of P. Since the final positions $x = x_m$ of p and $y = y_m$ of q after a play satisfy $x = y \in \mathcal{F}(x)$, the above reasoning shows that $\mathcal{F}$ has a fixed point under conditions (i)–(iii).

To see that the pursuit–evasion game and the fixed point problem formulated above are different if one of the conditions (i)–(iii) is violated, choose $P = \{a, b\}$, where $a = (0, 1)$ and $b = (1, 0)$. If $\mathcal{F}(a) = \mathcal{F}(b) = P$, then both a and b are fixed points of $\mathcal{F}$. If x_0 is any of the points of P from which p starts, then q can start from the other point of P, and p cannot capture q. In this example conditions (ii) and (iii) hold, but (i) is not valid. This lack can yield also a nonexistence of a fixed point of $\mathcal{F}$ even in the single-valued case, as we see by choosing P as above and defining $\mathcal{F}(a) = \{b\}$ and $\mathcal{F}(b) = \{a\}$. Also in this case conditions (ii) and (iii) are valid. The above results hold true also when P is a partially ordered set (poset), positive and negative directions being determined by a partial ordering of P.

The finite game introduced above is generalized in Sect. 8.6, where we study the existence of winning strategies for pursuit and evasion games that are of ordinal length. The obtained results are then used to study the solvability of equations and inclusions in ordered spaces. Monotonicity hypotheses, like (ii) above, are weaker than those assumed in Chap. 2.

As for the roots of the methods used in the proofs of Theorems 1.2, 1.5, 1.9, 1.15, and related theorems in Chap. 2, and in Sect. 8.6, we refer to Ernst Zermelo's letter to David Hilbert, dated September 24, 1904. This letter contains the first proof that *every set can be well-ordered*, i.e., every set P has such a partial ordering that each nonempty subset A of P has the minimum. The proof in question was published in the same year in *Mathematische Annalen* (see [231]). The influence of that proof is described in [94, p.84] as follows: "The powder keg had been exploded through the match lighted by Zermelo in his first proof of well-ordering theorem." To find out what in that proof was so shocking we give an outline of it. The notations are changed to reveal a recursion principle that is implicitly included in Zermelo's proof. That principle forms a cornerstone to proofs of the main existence and comparison results of this book, including the fixed point results stated above. As for the concepts related to ordering, see Chap. 2.

Let P be a nonempty set, and let f be a *choice function* that selects from the complement $P \setminus U$ of every proper subset U of P an element $f(U)$ ($= \gamma(P \setminus U)$, where γ is "eine Belegung" in Zermelo's proof). We say that a nonempty subset A of P is an *f-set* if A has such an order relation $<$ that the following conditions holds:

(i) $(A, <)$ is *well-ordered*, and if $x \in A$, then $x = f(A^{<x})$, where $A^{<x} = \{y \in A : y < x\}$.

Applying a comparison principle for well-ordered sets proved by Georg Cantor in 1897 (see [36]) one can show that *if $A = (A, <)$ and $B = (B, \prec)$ are f-sets and $A \not\subseteq B$, then B is an initial segment of A, and if $x, y \in B$, then $x \prec y$ if and only if $x < y$.* Using these properties it is then elementary to verify that *the union C of f-sets is an f-set, ordered by the union of the orderings of f-sets.*

We have $C = P$, for otherwise $A = C \cup \{f(C)\}$ would be an f-set, ordered by the union of the ordering of C and $\{(y, f(C)) : y \in C\}$, contradicting the fact that C is the union of all f-sets. Thus P is an f-set, and hence well ordered.

The proof is based on three principles. One of them ensures the existence of a choice function f. After his proof Zermelo mentions that "Die Idee, unter Berufung auf dieses Prinzip eine *beliebige* Belegung der Wohlordnung zu grunde zu legen, verdanke ich Herrn Erhard Schmidt." This principle, which is a form of the Axiom of Choice, caused the strongest reactions against Zermelo's proof, because there exists no constructive method to determine f for an arbitrary infinite set P. Another principle used in the proof is Cantor's comparison principle for well-ordered sets. A third principle is hidden in the construction of the union C of f-sets: Because C is an f-set, then $x \in C$ implies $x = f(C^{<x})$. Conversely, if $x = f(C^{<x})$, then $x \in P = C$. Consequently,

$$\text{(A)} \qquad \boxed{x \in C \iff x = f(C^{<x}).}$$

In Zermelo's proof f was a choice function. Recently, this special instance is generalized to the following mathematical method, called the **Chain Generating Recursion Principle** (see [112, 133]).

Given any nonempty partially ordered set $P = (P, <)$, a family $\mathcal{D}$ of subsets of P with $\emptyset \in \mathcal{D}$ and a mapping $f : \mathcal{D} \to P$, there is exactly one well-ordered chain C of P such that (A) holds. Moreover, if $C \in \mathcal{D}$, then $f(C)$ is not a strict upper bound of C.

In the proof of this result only elementary properties of set theory are used in [112, 133]. In particular, neither the Axiom of Choice nor Cantor's comparison principle are needed. To get this book more self-contained we give another proof in the Preliminaries, Chap. 2.

To give a simple example, let $\mathcal{D}$ be the family of all finite subsets of the set $P = \mathbb{R}$ of real numbers, and $f(U)$, $U \in \mathcal{D}$, the number of elements of U. By the Chain Generating Recursion Principle there is exactly one subset C of $\mathbb{R}$ that is well-ordered by the natural ordering of $\mathbb{R}$ and satisfies (A). The elements of C are values of f, so that $C \subseteq \mathbb{N}_0 = \{0, 1, \dots\}$. On the other hand, $\mathbb{N}_0$ is a well-ordered subset of $\mathbb{R}$, and $n = f(\mathbb{N}_0^{<n})$, $n \in \mathbb{N}_0$. Thus $\mathbb{N}_0$ is an f-set, whence $\mathbb{N}_0 \subseteq C$. Consequently, $C = \mathbb{N}_0$, so that (A) generates the set of natural numbers.

More generally, given $(P, <, \mathcal{D}, f)$, condition (A) can be considered formally as a 'recursion automate' that generates exactly one well-ordered set C. The amount of admissible quadruples $(P, <, \mathcal{D}, f)$ is so big that no set can accommodate them.

The first elements of C satisfying (A) are

$$x_0 := f(\emptyset), \dots, x_{n+1} := f(\{x_0, \dots, x_n\}), \quad \text{as long as} \quad x_n < f(\{x_0, \dots, x_n\}). \tag{1.19}$$

If $x_{n+1} = x_n$ for some n, then $x_n = \max C$. This property can be used to derive algorithmic methods that apply to determine exact or approximative solutions for many kinds of concrete discontinuous nonlocal problems, as well as to calculate pure Nash equilibria and corresponding utilities for finite normal-form games. The Chain Generating Recursion Principle is applied in this book to introduce generalized iteration methods, which provide the basis for the proofs of our main fixed point theorems including Theorems 1.2, 1.5, 1.9, and 1.15. They are applied to prove existence and comparison results for a number of diverse problems such as, e.g., operator equations and inclusions, partial differential equations and inclusions, ordinary functional differential and integral equations in ordered Banach spaces involving singularities, discontinuities, and also non-absolutely integrable functions. Moreover, these abstract fixed point results are shown to be useful and effective tools to prove existence results for extremal Nash equilibria for normal-form games, and to study the existence of winning strategies for pursuit and evasion games.