Chapter 7
Equilibria in Competitive Location Models

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7.1 Introduction

Whereas the usual location models locate facilities based on the wishes and objectives of a single decision maker, competitive location models consider the location of facilities that are under the jurisdiction of more than one decision maker. The economist Hotelling (1929) was the first to introduce competition into location models. His results stood unchallenged for fifty years, until d'Aspremont et al. (1979) corrected an inconsistency that invalidated Hotelling's main result. Nonetheless, this has not diminished the originality and importance of the original contribution, and it is also the reason why the present paper reviews Hotelling's contribution and its impact on location models with multiple decision makers.

Arguably, the best way to deal with competitive location models is to assess their components. Most prominent among them are the number of decision makers involved, the pricing policy, the rules of the game, and the behavior of the customers. Eiselt et al. (1993) provide a taxonomy and annotated bibliography that includes these features. Rather than restating their description, I will only very briefly summarize the main features. The most prominent *pricing policies* include *mill pricing*, where the price at each branch is fixed by the decision maker and customers provide for their own transportation, *spatial price discrimination*, where the firm sets the price a customer will be charged for the goods that are delivered to his place, and *uniform delivered pricing*, in which case all customers will receive the good for the same price (which typically means that customers located closer to a branch of the firm will subsidize those farther away). Other policies such as *zone pricing* may also be investigated.

The rules of the game are more complex. They essentially include rules that govern the process of decision making. In particular, they specify whether the firms' decisions are made sequentially or simultaneously. In case of pure location competi-

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tion, i.e., the case in which all firms compete only in terms of locations, a sequential process would indicate that, say, firm *A* locates first, followed by firm *B*, then firm *C*, and so forth. This location with *foresight* is discussed in Chap. 8 of this volume. The distinguishing feature of the sequential location process is its asymmetry. The first firm, being aware of the fact that other firms will locate after it has chosen locations for its own branches, will take this knowledge into account and use what Teitz (1968) called "conservative maximization." Subsequent firms will also attempt to guard themselves against firms that follow but, at the same time, will take the locations of already existing firms into account. This chapter deals exclusively with simultaneous location.

The situation becomes more complex when variables other than location exist. Many authors, including Hotelling (1929) in his seminal work, allow the firms to not only choose locations for their branches, but also to determine their prices. One possibility is to require that all firms make their choices simultaneously. Most authors, however (including Hotelling) use a two-stage process: in the first stage, all firms simultaneously choose their respective locations. Once these choices have been made, they are revealed to all firms. In the second stage, all firms then simultaneously determine the prices they want to charge. This sequence has been chosen as the much more permanent location decision comes first, followed by the price decision, which can easily be adjusted or modified later on. Furthermore, when making a decision in Stage 1, firms will anticipate the price competition in Stage 2. Such a game will be solved by backward recursion: for each pair of locations, the two firms will independently determine their optimal prices. Given those prices, firms will then—again independently—determine their optimal locations.

One question that arises rather naturally in all of these models is whether or not the set of locations that arises from such a process is stable. The concept applied here is the Nash (sometimes also referred to as Cournot-Nash) equilibrium. Loosely speaking, a Nash equilibrium is a situation in which none of the firms has an incentive (meaning can improve its objective) by unilaterally changing any of its parameters, be it location, price, quantity, or any of the other variables in the model. Most papers, especially in the economic literature, investigate whether such an equilibrium exists in the model under considerations, and, if so, if it is unique. While simple Nash equilibria can be determined in pure location competition, the two-stage "first location, then price" game requires a refinement of the equilibrium concept. The optimality concept that applies in such a procedure is Selten's (1975) subgame perfect Nash equilibrium.

In addition to locating facilities such as warehouses, retail stores, fast food outlets, gas stations, or other facilities of this nature, it has also been suggested to use location models for seemingly unrelated problems such as the design of brands, the determination of positions for political candidates, or the allocation of tasks to employees. The main features of these nonphysical location models are described below.

First consider the design of products, which is typically referred to as the *brand positioning problem*. In this application, we first define a continuous "feature space," in which each dimension represents a specific feature of the class of products under consideration. For example, in the case of automobiles these features could include horsepower, maximal speed (or, alternatively, acceleration), and gas mileage. Clearly, it is required that each feature under consideration be quantitative. Also note the correlation between some of the factors, e.g., horsepower and gas mileage. The products are then also mapped into space according to their features. This is followed by the mapping of (potential) customers, who are also mapped into the feature space by their respective ideal points, i.e., the product features they would like best. It then stands to reason that a customer will evaluate a product based on the distance between his own ideal point and the location of the product. The reason is that, just like physical distances, the distance between potential customer and product in a feature space expresses the disutility of a customer for that product. And, continuing that line of argument, a potential customer will choose the product that is closest to his own ideal point. One problem associated with this model is the existence of features such as price and gas consumption, which have an ideal point that is zero (or, if you will, negative infinity). Anderson et al. (1982) suggest an "outside game," a construct that allows the meaningful inclusion of such features in the model.

Another somewhat similar application is found in the area of political science. While the spatial analysis of political scenarios is not at all new—consider the classical contributions by Downs (1957) and Black (1958)—advances in location analysis helped tremendously to improve modeling and the solution of political models. Models of this nature first construct an "issue space," an n-dimensional space in which each dimension represents a political issue that is deemed relevant in an election. One of the key problems of the analysis is the quantification and measurability of issues, such as domestic policies, economic policies, etc. Candidates and likely voters are then mapped into this space by way of their ideal point (for the voters) and their stand on the issues (for the candidates) respectively, and assuming that—following some metric—voters will vote for the candidate closest to their own ideal point. That way, it is possible to determine the number of voters that will vote for each of the candidates and, more importantly, how each of the candidates should redefine his stand on the issues so as to maximize the number of votes he will obtain. In addition to the aforementioned difficulty of measurability there is also the determination of the ideal points of millions of voters. In their seminal contribution on the subject, Rusk and Weisberg (1976) used more or less well-defined groups such as "policemen," "urban rioters," "Republicans," "Democrats," and others to determine their average ideal point and, with the help of the variance determined by a sample, define a "cloud" around this ideal point that will then represent the voters in this group. The authors get around the problem of measurability of the axes by applying a multidimensional scaling technique (see, for example, Kruskal 1964). Additional contributions can be found in the other papers in the edited volume by Niemi and Weisberg (1976). It is also worth pointing out that one of the few features of this model that makes the political positioning simpler than the Hotelling's original scenario is the absence of prices in the model.

The workload allocation problem follows a similar logic. Here, tasks and employees are mapped into an ability space that expresses their requirements and abilities, respectively. The idea is to allocate tasks to employees so as to minimize the distance between employee and task, matching the requirements of the tasks and the employees' abilities as closely as possible. A close match may be desired to increase job satisfaction and hence avoid high job turnovers, absenteeism, and other work-related problems. Again, some of the main problems related to these applications are the quantifications of the abilities and the determination of an appropriate distance function. Readers are referred to Schmalensee and Thisse (1988) for their survey on applications in feature spaces and ability spaces. For a recent reference, see Eiselt and Marianov (2008a).

The contributions surveyed in this paper all have one feature in common: they all emphasize the analysis of equilibria in competitive location models. Other aspects of competitive location models are dealt with in Chaps. 8 and 9 of this volume.

7.2 Hotelling (1929): Competitive Location on a Linear Market

Hotelling starts his paper with a critical evaluation of past contributions. Of interest are particularly the embedding of his own work into the framework provided by Bertrand and Cournot. The discussion of a duopoly dates back to Cournot (1838). In his model, Cournot considers a duopoly with both firms competing on the same market with the same product. The variable costs have been normalized to zero (we may assume that they have been deducted from the price that the firms charge), and the two firms face a common demand function. The duopolists compete in quantities and the resulting solution is a Cournot-Nash equilibrium. Bertrand (1883), on the other hand, has duopolists competing in prices. Such competition is very intense, as even a slight undercutting will revert the entire market to the cheaper firm. While Hotelling's contribution is in the footsteps of these two (and other) predecessors, its novelty is that he includes competition in space, while his predecessors' models were set in a spaceless economy.

Hotelling's basic model includes a space in the form of a closed line segment of length ℓ. It is worth noting that Hotelling justified the choice of a line segment by referring to it as "main street" or a stretch of a transcontinental railroad. Later authors have claimed that Hotelling's "justification" of the "linear market" was based on "two ice cream vendors on a beach," an example never envisaged by Hotelling but put forth by later contributors.

Customer demand is distributed uniformly along the line at a unit density, so that the total demand equals ℓ. The demand is assumed to be completely inelastic. Two competing firms face the task of simultaneously locating one facility each and setting the price for a homogeneous product. Both firms use mill pricing, so that customers have to drive to the facility of their choice, pay for the product at the facility, and then ship it home: their full price includes the mill price charged at the facility and the transportation costs for shipping the good from the facility to

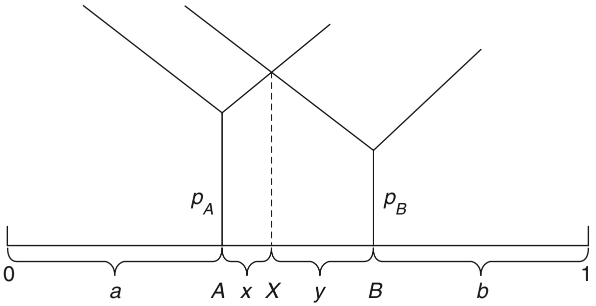


Fig. 7.1 Price functions of duopolists on a line segment

their home. Given a homogeneous (standardized) good, customers are indifferent between purchasing the good from either facility, so that they will choose the facility from which they can obtain the good for the lower full price, regardless of how distant the closest facility is. The transportation costs are assumed to be linear in the distance. The two firms are assumed to have equal cost functions, which have been normalized to zero.

Formally, define the market as a line segment of length ℓ and assume that firm A is located a units from the left end of the market, while firm B is located at a distance of b from the right end of the market. The only condition is that firm A is located to the left of firm B (which does not restrict generality, as this situation, if violated, can always been achieved by exchanging the names of the facilities). The facilities charge mill prices of p_A and p_B, respectively, and the unit transportation costs are c. Figure 7.1 shows the present situation. Each of the Y-shaped functions shows the full price (the mill price plus transportation costs) customers have to pay if they purchase from the facility in question: the stem of the "Y" is the mill price, and the slope of the two branches of the "Y" is the unit transportation cost c. Given that the good is homogeneous, customers will purchase from the source with the lower full price, i.e., the lower envelope of the branches of the two "Ys." This results in a marginal customer X (Hotelling did not use the expression), who is defined as a customer indifferent between purchasing from firm A or from firm B. Clearly, all customers to the left of the marginal customer can buy the good more cheaply from firm A, while those to the right of X can purchase the good more cheaply from firm B. This will define firm A's market area from the left end of the market to the marginal customer, while firm B's market area extends from the marginal customer to the right end of the market.

Authors who followed Hotelling usually refer to the region to the left of A as "A's hinterland," the region to the right of B as "B's hinterland," and the area between firms A and B as the "competitive region." (It appears that Smithies (1941) was the first author to use these terms.) The two hinterlands are of length a and b, and the competitive region is divided by the marginal customer X into pieces of lengths x and y, respectively. Formally, we have

$$a + x + y + b = \ell, \tag{7.1}$$

and the marginal customer X is defined as a place at which prices are equal, i.e., $p_A + cx = p_B + cy$ with unit transportation costs c. Solving this system of two equations for x and y, we obtain

$$x = \tfrac{1}{2}[\ell - a - b + \tfrac{1}{c}(p_B - p_A)] \tag{7.2}$$

and

$$y = \tfrac{1}{2}[\ell - a - b + \tfrac{1}{c}(p_A - p_B)], \tag{7.3}$$

so that the profits are

$$\pi_A = p_A q_A = p_A(a + x) = \tfrac{1}{2}(\ell + a - b)p_A + \tfrac{p_B}{2c} p_A - \tfrac{1}{2c} p_A^2 \tag{7.4}$$

and

$$\pi_B = p_B q_B = p_B(b + y) = \tfrac{1}{2}(\ell - a + b)p_B + \tfrac{p_A}{2c} p_B - \tfrac{1}{2c} p_B^2. \tag{7.5}$$

For any given values of ℓ, a, b, and π_i, $i = A, B$, iso-profit lines can be plotted in p_A, p_B space as hyperbolas. Since each duopolist will adjust his own price so as to maximize his profit, we can take partial derivatives

$$\frac{\partial \pi_A}{\partial p_A} = \tfrac{1}{2}(\ell + a - b) + \frac{p_B}{2c} - \tfrac{1}{c} p_A = 0 \tag{7.6a}$$

or

$$p_A^* = \tfrac{1}{2}c(\ell + a - b) + \tfrac{1}{2}p_B \tag{7.6b}$$

and

$$\frac{\partial \pi_B}{\partial p_B} = \tfrac{1}{2}(\ell - a + b) + \frac{p_A}{2c} - \tfrac{1}{c} p_B = 0 \tag{7.7a}$$

or

$$p_B^* = \tfrac{1}{2}c(\ell - a + b) + \tfrac{1}{2}p_A. \tag{7.7b}$$

(Note that $\frac{\partial^2 \pi_A}{\partial p_A^2} < 0$ and $\frac{\partial^2 \pi_B}{\partial p_B^2} < 0$, so that these conditions determine a local maximum).

The expressions (7.6b) and (7.7b) are usually (although not by Hotelling) referred to as reaction functions of the two firms. In particular, if firm B were to set any price p_B, then firm A would react by setting its price to a level specified by relation (7.6b). Similarly, firm B will react by using relation (7.7b) to any price p_A set by its competitor A.

Solving for the prices p_A and p_B results in the equilibrium prices

$$\bar{p}_A = c\left(\ell + \frac{a-b}{3}\right) \tag{7.8a}$$

and

$$\bar{p}_B = c\left(\ell - \frac{a-b}{3}\right), \tag{7.8b}$$

and the quantities at equilibrium are

$$\bar{q}_A = a + x = {}^1\!/\!_2\left(\ell + \frac{a-b}{3}\right) \tag{7.9a}$$

and

$$\bar{q}_B = b + y = {}^1\!/\!_2\left(\ell - \frac{a-b}{3}\right). \tag{7.9b}$$

This can best be explained graphically. Hotelling's original example involves values of $\ell = 35$, $a = 4$, $b = 1$, $c = 1$, and it is shown in Fig. 7.2. Given his numerical example, the optimality conditions result in the reaction functions $p_A^* = 19 + {}^1\!/\!_2 p_B$ and $p_B^* = 16 + {}^1\!/\!_2 p_A$, respectively. Solving the two linear equations results in the equilibrium prices $\bar{p}_A = 36$ and $\bar{p}_B = 34$.

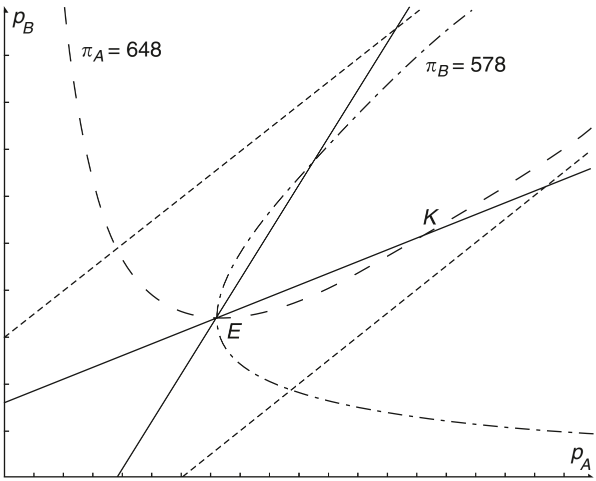
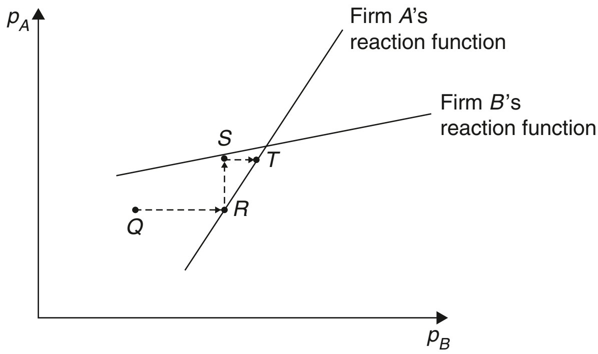


Fig. 7.2 Reaction functions of the duopolists and equilibrium

Fig. 7.3 Price adjustments over time



The lines with short dashes define a "corridor" between the lines $p_B \leq p_A + 30$ and $p_B \geq p_A - 30$. This corridor is the set of price combinations in which the price difference is no larger than the cost of shipping one unit from one facility to the other. In other words, it is the area within which neither competitor cuts out its opponent. The solid lines represent the reaction functions that result from the optimality conditions. The steeper line is firm A's reaction function, while the flatter line is firm B's reaction function. The broken line with long dashes denotes the set of price combinations that result in $\pi_A = 648$ (the profit that results from the equilibrium prices at point E, *viz*., $p_A = 36$ and $p_B = 34$). Finally, the broken and dotted line is the set of price combinations that result in $\pi_B = 578$.

Hotelling then describes a procedure in which the two firms start with non-equilibrium prices that they subsequently adjust in sequential fashion. For simplicity, the two reaction functions are shown again in Fig. 7.3, where E again denotes the equilibrium point. Suppose now that the two firms charge prices so as to realize point Q. Given this combination of (below equilibrium) prices, either of the firms has an incentive to change (here: raise) its price. Suppose that firm A will react first. Firm A will assume that, at least for some time, its competitor will not react. This assumption was later referred to as "zero conjectural variation" by Eaton and Lipsey (1975). Furthermore, firm A will act without any foresight and consequently move from point Q to the point on firm A's reaction function, which is point R. Once this has been accomplished, firm B will react and move to the point on its reaction function, *viz*., point S. Then firm A reacts again by moving to point T, and so on. The price adjustment from points in any of the three other cones is similar. Note also the similarity of the adjustment process here to that in the famed cobweb theorem in economic theory.

At this point, Hotelling remarks in a footnote that the above conclusions are true only as long as the difference in price does not exceed the cost of shipping one unit from A to B or vice versa. Formally, the condition is

$$|p_A - p_B| \leq c(\ell - a - b). \tag{7.10}$$

If this condition is not satisfied, the equilibrium is not point E but some other point. It is important to note that Hotelling does indeed realize that his computations are valid only for a certain range of prices (price differences, to be exact). However, he does not elaborate. Hotelling's result, the clustering of the duopolists at the center of the market has also been referred to as the *principle of minimal differentiation* (in reference to product design and the political model introduced in the beginning of this paper) or *Hotelling's law*.

An interesting case of cooperation results. Starting again at the equilibrium point E in Fig. 7.2, assume that firm A is willing to forego profits in the near future and moves out of point E by raising its price, and moving to the right. Behaving optimally, firm B will again move towards its point on its reaction function by increasing its price as well. As long as firm A's price increase was modest, the point that will be realized will be located on firm B's reaction function to the left of point K. This point does provide both firm A *and* firm B with higher profits than at equilibrium. However, the solution is inherently unstable (similar to the well-known *Prisoner's dilemma*), as firm A has an incentive to increase its profit even more by moving onto its own reaction function. Such a move will, however, result in sequential price adjustments that ultimately lead back to the equilibrium solution E.

Part II of Hotelling's paper deals with a variety of extensions of his basic model, as well as alternative explanations. He first notes that the profits at equilibrium are

$$\bar{\pi}_A = {}^1\!/\!_2 c\left(\ell + \frac{a-b}{3}\right)^2 \tag{7.11a}$$

and

$$\bar{\pi}_B = {}^1\!/\!_2 c\left(\ell - \frac{a-b}{3}\right)^2. \tag{7.11b}$$

Given that, it is apparent that the profit of both firms increases with increasing unit transportation costs c. In other words, rather than promoting better means of transportation, the two firms would fare better if transportation were to be made more difficult. The reason is that if transportation were very difficult, each firm could behave as a local monopolist and charge monopolist's prices. It is important to point out that while higher transportation costs as applied to shipments from the firms to their customers do, in fact, increase profits, they will have a detrimental effect on the variable costs as they also apply to shipments from the firms' suppliers to the firms. These costs were neglected in the model. This means that the argument regarding the parameter c is better explained by the existence of tariffs.

The paper then examines the case in which one firm's location (without loss of generality assume this is firm A) has fixed its location and firm B now chooses its own location. Given its profit at equilibrium as shown in relation (7.11b), it is apparent that firm B's profit increases with increasing value of b. In other words, it will pay firm B to locate as close to its competitor as possible. This is again the "agglomeration result" (or "principle of minimum differentiation" as it became known later). However, Hotelling again notes the problem that occurs when the two facilities are sufficiently close so that one firm can cut out its opponent.

Another interesting result relates to the total profit of the two firms, which are, say, governed by a central planner. Formally, we have

$$\pi_A + \pi_B = c\left[\ell^2\left(\frac{a-b}{3}\right)^2\right], \tag{7.12}$$

indicating that it would benefit the planner to have the two facilities locate at sites that are as different from each other as possible, i.e., maximizing $a - b$.

The next few paragraphs of the paper examine the relationship between the solution arrived at by profit maximization as opposed to the solution that optimizes some social objective. The social objective chosen is the minimization of total transportation costs. For simplicity, consider the left end of the market between 0 and A an interval of length a. The transportation costs in this interval for all shipments to the facility at point A are $\int_{t=0}^{a} ctdt = \frac{1}{2}ca^2$. Applying this result to all intervals, *viz.*, those from *0* to A (an interval of length a), from A to the marginal customer X (an interval of length x), from the marginal customer X to facility B (an interval of length y), and finally the interval from facility B to the end of the market (an interval of length b), results in total transportation costs

$$TTC = \tfrac{1}{2}c\left(a^2 + b^2 + x^2 + y^2\right). \tag{7.13}$$

Given fixed locations of the facilities A and B, the values of a and b are fixed as well, and so is $x + y$. Then $x^2 + y^2$ is minimized, if $x = y$. This, in turn, is only satisfied, if $p_A = p_B$, which, while entirely possible under the direction of a central planner or commissar, is an outcome that is highly unlikely under competition. It does, however, indicate that social planners will prefer equal prices charged at the facilities. Assume now that $a \neq b$. Without loss of generality, let $a > b$, which, given individual profit maximization, implies that at equilibrium, $\bar{p}_A > \bar{p}_B$, see relation (7.8a). This means that some customers in the competitive region, although they are located closer to facility A, will make their purchases and resulting shipments from facility B. This results in higher transportation costs as if they were to make their purchases at facility A, which renders this solution not "socially optimal." In fact, Hotelling states, "Consequently some buyers will ship their purchases from B's store, though they are closer to A's and socially it would be more economical for them to buy from A." This clearly indicates Hotelling's allocation rule assumes that customers purchase their goods from the source that offers the lowest full price (even though he may not advocate this practice). This is worth pointing out since some authors use the term "Hotelling's allocation" to mean the allocation of a customer to his closest facility, which is not correct.

If the facilities can be moved at will, the social optimum again minimizes the function shown in (7.13) with a, b, x, and y all variable and the single constraint that $a + b + x + y = \ell$ plus the nonnegativity constraints. At optimum, all variables assume equal values ($a = b = x = y = \frac{1}{4}\ell$), so that the two facilities are located at the quartiles of the market. The highest transportation cost paid by any customer in this arrangement is then ¼ℓ. In contrast, competition will have the two facilities cluster at the center of the market (Hotelling again notes the "unimportant qualification" that deals with the possibility of one competitor cutting out its opponent), so the highest possible full price is ½ℓ. The author uses this as an example of "wastefulness of private profit-seeking management."

Another extension deals with additional firms. In the case of individual profit maximization, Hotelling notes that the third firm will locate "close to *A* and *B*, but not between them." In some sense, this anticipates the analyses performed later by Lerner and Singer (1937) and subsequently by Eaton and Lipsey (1975). For more facilities, Hotelling asserts that clustering will occur, but no specifics are given. The case of social optimization for three facilities is again easy: the facilities locate symmetrically at $\frac{1}{6}\ell$, $\frac{3}{6}\ell$, and $\frac{5}{6}\ell$, respectively.

Hotelling then extends the range of applicability of his model from scenarios that involve the physical transportation of items to multidimensional spaces (today referred to as *feature spaces*), in which each dimension symbolizes a (quantifiable) feature of (a class of) products. He uses one dimension to distinguish between different brands of cider, and the attribute of the cider that identifies the particular brand is its sweetness. What used to be facilities in the competitive location model discussed above now represents brands of cider. Customers are again distributed along the line segment, such that each customer is represented by its "most preferred point" (or "ideal point") on the line, such as the point that represents the sweetness of cider that this customer desires most. The distance between a customer's ideal point and a brand is then a measure that expresses the customer's disutility associated with buying and consuming that particular brand of cider.

The results of the preceding analysis, *viz*., the clustering in case of individual profit maximization, then imply "excessive sameness." Hotelling credits this in part to standardization and economies-of-scale in the production process, but also to the results derived in this study. The main lesson for a firm that intends to enter the market with a new product is *not* to make the product identical to existing products (in which case Bertrand price competition would ensue, driving down prices), but design a product that differs slightly from existing products by locating the brand in feature space close, but not too close, to existing brands. Hotelling's assertion that the similarities of political platforms of Republican and Democratic parties (which are again represented by their main issues in *issue space*) can also be attributed to the effects studied here are not valid *per se*, as political models do not involve prices, thus reducing the model to a much simpler version. Some remarks regarding political models are provided in the next section in this chapter as well as Chap. 19 in this volume.

Some further generalization and extensions are discussed. First, Hotelling affirms that demand densities other than the uniform demand distribution used in his analysis provide "no essential change in conclusion." In the case of buyers being located in a two-dimensional plane, the market areas of the two firms are divided by a hyperbola. In case of more than two facilities, the market areas will be bounded by arcs of hyperbolas. In multidimensional spaces (such as feature spaces), the demand density is typically not uniform and it occurs within a finite bounded region. Here, not all facilities need to belong to the same firm. There is a general tendency among outsiders to move inward and approach the cluster, which is again the agglomeration result of this paper. This result is asserted, but not proven. For more on market areas and their use in location planning, see the Chaps. 18 and 19 in this volume.

An important extension concerns the elasticity of demand. So far, it has been assumed that firms offer a product for which the demand is fixed, i.e., completely inelastic. While this may occur in the case of essential goods, it is highly unlikely for most products. One of the central questions is whether the price or the quantity should be a variable. So far in the analysis, the quantities have been restricted to the constant ℓ. Given elastic demand, this limitation no longer applies and prices or quantities can be used as independent variables. Hotelling asserts that even with elastic demand, the results derived above will remain "qualitatively true," even though there will be less of a tendency to cluster.

7.3 The Impact of Hotelling's Contribution

Hotelling's original paper has sparked controversy, as well as a flurry of papers written about his model and similar scenarios. In their survey and taxonomy, Eiselt et al. (1993) already list about a hundred papers on the subject. Since then, at least another hundred contributions have been published. It is possible to broadly distinguish between two types of contributions: those that deal with the existence of Nash equilibria, and those that examine von Stackelberg solutions. There is no doubt that the impact of Hotelling's paper has been felt by both streams. However, this chapter will only survey those papers that deal with Nash equilibria; von Stackelberg solutions are examined in detail in Chaps. 8 and 9 of this volume. This chapter will follow the developments of those works that can be considered continuations and refinements of Hotelling's work. Most contributions in this area are made by economists, and their tool of choice is game theory.

Those who followed in Hotelling's footsteps generalized his model in various directions. These directions include (but are by no means limited to)

- different spaces
- $n > 2$ facilities,
- different assumptions about competitors' behavior
- different transport cost functions and different pricing policies,
- different assumptions concerning customer behavior,

and other generalizations. A few of the many milestones are highlighted below.

Probably the earliest contribution to deal with Hotelling models is put forward by Lerner and Singer (1937). The authors point to Hotelling's assumption of fixed demand and the customers' willingness to pay any amount to satisfy their demand as one of the main deficiencies of his model, particularly when applying his argument to favor a social/socialist solution as more efficient than a capitalist solution. The authors thus introduce a "demand price," defined as the highest amount customers are prepared to pay to satisfy their demand. The authors also criticize Hotelling's assumption that stipulates that a facility planner uses more information when choosing a location than when setting a price. The reason is that the location decision in stage 1 of the two-stage "first location, then price" game is made with the assumption that the opponent's price will be what results from a long line of price adaptations. However, in stage 2 this knowledge is no longer assumed to exist. In contrast, Lerner and Singer assumed that a firm's planner will not react when his opponent moves closer and takes a part of his customers, but he will react when undercut so that all of his customers are not supplied by his opponent. This is a concept, a variant of which Eaton and Lipsey (1975) referred to as "zero conjectural variation." This assumption leads to locations at about 3/8 away from the respective ends of the market. A further analysis in the paper assumed again that a firm, whose competitor is in the process of relocating, does not react except if undercut. The last part of the paper dealt with a Hotelling model with fixed and equal prices, resulting in pure location competition. The authors identified a large number of equilibrium locations for $n \geq 2$ facilities. Two competing firms will have a unique equilibrium solution by clustering at the center of the market; this is the "minimum differentiation" result Hotelling envisaged for his own model. The case of three firms is interesting: the two peripheral firms crowd in on the firm between them in order to gain a higher market share until the central firm has no market share left. It then "leapfrogs" to the outside, becomes a peripheral facility itself, and starts moving inwards as well. Teitz (1968) referred to this later as "dancing equilibria," which really means that this case has no equilibrium. For four or more firms locating on the linear market, their locations are at $\frac{1}{2\lceil n\rceil}, \frac{3}{2\lceil n\rceil}, \ldots, \frac{2\lceil n\rceil-1}{2\lceil n\rceil}$. Finally, in their analysis of the model with price discrimination, the equilibrium locations are at $\frac{1}{2n}, \frac{3}{2n}, \ldots, \frac{2n-1}{2n}$, which happens to be socially optimal in that it minimizes the total transportation costs. An interesting feature of this result is that a customer closer to a facility will have to pay more than one that is more remote from a firm. The reason is that the level of competition close to a firm is fairly low, which increases the price.

Smithies (1941) continued where Lerner and Singer (1937) left off. His particular interest were the assumptions concerning the behavior of the competitors. In particular, Smithies did not believe that competitive price cutting was a reasonable policy, as it would lead to an all-out price war. Given a price-quantity relation, his model included three cases that exhibited different levels of cooperation. In the first case, facilities would charge the same price and would locate symmetrically. This "full quasi-cooperation," as the author called it. This case includes little, if any competition, and it is not surprising that the results would be the same as if a monopolist were to locate two plants. The second behavioral assumption was for both firms to charge identical prices but compete in locations. Finally, case 3 exhibited "full competition" in the sense that both firms independently optimized their prices and locations. The results were examined according to their dependence on freight rates and changes in marginal costs. Kohlberg and Novshek's (1982) contribution followed Smithies in many respects in that each relocating facility would assume that its competitors would keep their locations and prices at the present level, except if undercutting occurred, in which case the firm that was undercut would reduce its price to its marginal cost. The main result was that there exists a certain length of market below which there is no equilibrium, while in case of longer markets, there exists a unique location-price Nash equilibrium for which the authors provide a necessary and sufficient condition. Along similar lines is the analysis by Stevens (1961), who was probably the first author to use matrix games for a discretized version of Hotelling's game. Given elastic demand similar to Smithies, the result was still central agglomeration.

Another generalization concerns locations on a circle. While the space may appear somewhat contrived, the results indicated not only the fragility of Hotelling equilibria, but also some of the special features of the linear market that are lost on a circle: hinterlands, for instance, are specific to linear markets (and tree networks, for that matter), but they do not exist on circles or on general networks. On a circle, multiple equilibria exist for all cases with two or more facilities, given rectangular demand density functions. Finally, some locational patterns on a disk were investigated regarding their equilibrium status. Based on simulation attempts, the authors conjectured that there is no equilibrium for $n > 2$ facilities.

The aforementioned contribution by Eaton and Lipsey is one of the papers most frequently referred to in the context of Hotelling's result, even though their model is quite different from Hotelling's contribution. Their work first restated the results obtained by Lerner and Singer (1937) before performing a variety of sensitivity analyses on the problem. Their first model was Hotelling's linear market with uniform demand density and the zero conjectural variation, i.e., no foresight. Model 2 was the same as Model 1, but with no zero conjectural variation. It results in minimax strategies, and as such anticipates the results by Prescott and Visscher (1977) that are presented in Chap. 8 of this volume. Finally, their third model is again similar to Model 1, but with the assumption of uniform demand density relaxed. The result for two firms was similar (the facilities will cluster at the median of the density function), and there was no equilibrium for three firms, and there may not be equilibria for more than three firms either, given a condition on the demand function. In particular, the authors proved that for an equilibrium to exist, it is necessary that the number of firms on the market is no more than twice the number of modes in the demand distribution.

The authors then tackled the much more complex problem of equilibria in two-dimensional space. Again, they avoided boundary problems by considering a disk. Due to the difficulty of the problem even with fixed and equal prices, they investigated a number of patterns that are potential candidates for equilibria and determine whether or not they are indeed equilibria. The first pattern has facilities located on a circle around the center of the disk. This pattern self-destructs immediately as soon as individual firms (re-) optimize their location. Pattern 2 is similar, except with one facility at the center of the disk. This pattern also turn out to be unstable. Finally, pattern 3 is the Löschian honeycomb pattern that consists of hexagons. (Details concerning Lösch's work are found in Chap. 20 of this volume.) This pattern also self-destructs immediately as individual firms optimize their locations, thus the authors conjectured that there exists no equilibrium pattern on a disk with $n > 2$

facilities. (The case of $n = 2$ facilities is easily dispensed of: the two facilities cluster at the center of the market, a simple pairing observed on the linear market for $n \geq 4$ facilities.) For a discussion of the case of competition in bounded two-dimensional space, readers are referred to Chap. 19 of this volume.

Probably the most important contribution following Hotelling's work is the short paper by d'Aspremont et al. (1979), published fifty years after the original work appeared. It first pointed out an error in Hotelling's original work that resulted in the wrong conclusion: not only does the duopoly model described by Hotelling not have an equilibrium at the center of the market (central agglomeration), but the model does not have an equilibrium anywhere. Hotelling was aware that his results would need some refinements (see his footnote referred to above), but he was not aware of the severity of the consequences. Actually, the equilibrium he computed for facilities that are located closely together is wrong. However, d'Aspremont et al. (1979) were not the first to recognize that there were problems with Hotelling's analysis. To quote the earlier work by Prescott and Visscher (1977):

> The difficulty with this solution concept, as others have noted (Smithies 1941, Eaton 1976, and Salop 1979) is that when locations in Nash are sufficiently close, Nash equilibrium prices will not exist.

Without resorting to formalities, the lack of an equilibrium can readily be seen by the following arguments. Consider any locational arrangement that has the two facilities not clustered together. First of all, there is an incentive for firm A to move closer to its opponent until the right branch of its Y-shaped full price function coincides with that of firm B. Similarly, firm B has an incentive to move to the left until the left arm of its Y-shaped full price function coincides with that of firm A. Once that has been achieved (note that there is no clustering of the facilities yet), the firm with the lower mill price could lower its price by an arbitrarily small amount and, in doing so, be cheaper on the entire market. In doing so, its profit would jump up, meaning that the cheaper facility certainly has an incentive to undercut its opponent. The more expensive facility could now react by lowering its price so as to undercut its opponent (which is Bertrand's price competition). Once prices have reached a very low level, it would benefit either of the two facilities to move significantly far away from its opponent so as to enjoy a local monopoly and the associated positive profits.

D'Aspremont et al. (1979) used a more formal argument. The authors first proved that any equilibrium if it exists at all, it either has $a + b = \ell$ (both facilities locate at the center at the market), in which case both prices are equal to zero (the Bertrand solution), or $a + b < \ell$, in which case the price difference must satisfy

$$|\bar{p}_A - \bar{p}_B| < c(\ell - a - b). \tag{7.14}$$

Condition (7.14) expresses the requirement that the difference in prices is less than the cost required to ship one unit from one facility to another. If this condition were violated, it would imply that the lower-price facility is able to cut out its opponent and capture the entire market. Clearly, this cannot be an equilibrium solution as the higher-priced facility would be left without a zero profit that it could increase by undercutting its opponent in turn.

We are now able to present a formal expression for the existence of an equilibrium. Recall that the equilibrium profits were determined in relations (7.11a) and (7.11b) as

$$\bar{\pi}_A = {}^1\!/\!_2 c\left(\ell + \frac{a-b}{3}\right)^2 \quad \text{and} \quad \bar{\pi}_B = {}^1\!/\!_2 c\left(\ell - \frac{a-b}{3}\right)^2.$$

An equilibrium can then only exist if and only if a firm's equilibrium profit is larger than the profit it would obtain if it were to slightly undercut its opponent by some small value ε. If for instance, firm A were to undercut firm B, then its profit would be $p_A\ell$, as it captures the entire market. Assuming that firm A undercuts firm B by setting its price to $p_A = p_B - c(\ell - a - b) - \varepsilon$ with some $\varepsilon > 0$, while firm B charges its equilibrium price $\bar{p}_B$ specified in relation (7.8b), firm A's profit would be $\pi_A = [\bar{p}_B - c(\ell - a - b) - \varepsilon]\ell$. Clearly, an equilibrium can only exist if undercutting does not result in a higher profit than the equilibrium profit. Formally, an equilibrium will exist, if $\bar{\pi}_A \geq \pi_A$, or, equivalently,

$${}^1\!/\!_2 c\left(\ell + \frac{a-b}{3}\right)^2 \geq [\bar{p}_B - c(\ell - a - b) - \varepsilon]\ell.$$

Applying some standard algebraic transformations and repeating the process for firm B, undercutting firm A, we obtain the necessary and sufficient existence conditions for equilibria as

$$\left(\ell + \frac{a-b}{3}\right)^2 \geq \tfrac{4}{3}\ell(a + 2b) \tag{7.15a}$$

and

$$\left(\ell + \frac{a-b}{3}\right)^2 \geq \tfrac{4}{3}\ell(2a + b). \tag{7.15b}$$

Note that for symmetric equilibria $a = b$, so that the conditions (7.15a) and (7.15b) reduce to $a = b \leq {}^1\!/\!_4\ell$. This means that the condition requires the two facilities being located outside the first and third quartiles, which is, of course, not satisfied by Hotelling's "central agglomeration" result.

The authors continued to examine a model that is identical to that investigated by Hotelling, except that it uses quadratic transportation costs of the type $c(\text{distance})^2$. While physical transportation is unlikely to exhibit such cost function, models with nonphysical spaces very well may. The result is not only that this model does have a unique equilibrium, but that at equilibrium, we have maximum (rather than minimum) differentiation with both firms locating at the respective ends of the market. This is but one indication of the instability of Hotelling models in general. This point was driven home even further by Anderson (1988), who considered a

Hotelling model with a linear quadratic transportation cost function of the type c_1(distance) + c_2(distance)^2. This type of cost function was first introduced by Gabszewicz and Thisse (1986). With this cost function, there exists an equilibrium only if $c_1 = 0$, i.e., the function has no linear part at all, regardless how small. However, for certain pairs of locations with the duopolists located close together, there is a price equilibrium. In case only pure strategies are allowed in stage 1 but mixed strategies are permitted in stage 2, an equilibrium exists only if the transport costs are "sufficiently" convex as expressed by the relation of parameters a and b. The Hotelling model with linear-quadratic transportation costs was picked up again by Hamoudi and Moral (2005).

Shaked (1982) considered a mixed strategy version the Hotelling model with fixed and equal prices and three competitors. Customers were uniformly distributed on the line. Following the result by Dasgupta and Maskin (1986), the solution would be doubly symmetric: both firms use the same mixed strategies, and the strategy is symmetric about ½ℓ. In particular, firms avoid locations in the extreme quartiles and choose locations instead in the central half of the market with equal probability. Osborne and Pitchik (1986) followed this line of investigation. Their model has fixed and equal prices, allows nonuniform demand distributions, and let the firms use mixed strategies. The authors first noted the well-known sensitivity of the model. For instance, for $n \geq 5$ facilities, the model does not have an equilibrium if the customer distribution is either strictly convex or strictly concave, regardless how close the distribution is to uniformity. The main results are: for $n \geq 3$, the game has a symmetric mixed strategy equilibrium and if the customer distribution is symmetric about the center of the market, so is the mixed strategy equilibrium; for $n = 3$, a unique equilibrium exists with one firm at the center of the market and the other two firms using mixed strategies for their locations.

The contribution by Kohlberg (1983) is different, as this appears to be the first paper that includes factors other than price and location. In particular, Kohlberg's model included not only the transportation cost (here interpreted as travel time), but also the time spent waiting at a facility. The waiting time is assumed to be increasing with the facility's market share. The author then proved that, while there is a unique equilibrium in the case of duopolists with both of them locating at the center of the market, there exist no equilibria for $n > 2$ facilities. Silva and Serra (2007) picked up the model but solved an optimization problem in discrete space; however, they do not investigate equilibria.

De Palma et al. (1985) took a different route. In their analysis, they employed Hotelling's original model with locations and prices variable, a linear market of length ℓ, and a uniform demand, but their model included n facilities and a random utility function that expresses the customers' evaluation of customer preferences. The authors put their model in the context of product placement with n products to be located on a line segment that determines the products' feature. A customer's (dis-) like of a product is expressed as a function of the distance between the customer's ideal point on the line and the product's location. The main assumption of their paper was that products and customers are heterogeneous. In particular, customers value purchasing a product according to the function

$$\text{(random utility)} = \text{(valuation of product)} - \text{(price)} - \text{(unit disutility cost } c\text{)} \times \text{(distance)} + \mu\varepsilon_i,$$

where $\mu > 0$ denotes the degree of heterogeneity of customer tastes (so that $\mu = 0$ equals homogeneous tastes), and the random variable ε that has a zero mean and unit variance. It turns out that heterogeneity in the logit function removes discontinuities in the products' profit functions.

After first considering only the location and then only the price model, the authors proved that in the location-price model, for $n > 2$ products and a degree of heterogeneity of $\mu < ½c\ell(1–2/n)$, there is no agglomerated Nash equilibrium, meaning an equilibrium with all facilities locating at the same point. However, if $\mu \geq c\ell$, central agglomeration with equal prices is a Nash equilibrium. In other words, large values of $\mu/c\ell$ lead to clustering, whereas small values of $\mu/c\ell$ result in dispersion. There are no results regarding the existence and the nature of other equilibria. Some tests revealed that equilibria may exist for $n = 3$. In summary, if all customers have very similar tastes, then there exists no equilibrium with similar products, while in case of very diversified customer tastes, products will tend to be the same. One may look at the result from the following angle: if tastes are very similar, then the firms have to diversify the products to appeal to different segments of the customer base, while in case of significantly diverse tastes, all products can occupy a similar position in feature space.

A follow-up of their 1985 paper was provided by De Palma et al. (1987a). The assumptions were again a linear market, fixed and equal prices, a linear transportation cost function, and the same random utility function shown above with μ again denoting the degree of heterogeneity in customers' tastes. Numerical computations reveal the following results: for $\mu/c < 0.157$, no symmetric equilibria exist; for $\mu/c \in [0.157; 1/6]$, only symmetric dispersed equilibria exist; for $\mu/c \in [1/6; 0.27]$, agglomerated and symmetric dispersed equilibria exist; for $\mu/c \geq 0.27$, only agglomerated equilibria exist. As far as an interpretation goes, consider a competitive location model in product (or feature) space. Here, less wealthy customers tend to be nondiscriminating, meaning that they tend not to care that much if a product is not exactly as they would like it to be. This implies that the value of c is small for this group, implying more heterogeneity and a larger value of μ. We can therefore associate a large value of μ/c for less affluent groups, while wealthier groups may be characterized by a small value of μ/c. The results of this study then indicate that less affluent customers with a large μ/c value will end up with products that are very similar to each other, while wealthy customers will face a market segment whose products are significantly different. This can, for instance, be observed in the automobile market, though to a much lesser extent today than ten or twenty years ago.

De Palma et al. (1987b) considered a competitive location model on a linear market that uses uniform delivered pricing. Apart from this feature, the usual Hotelling assumptions apply. Given the reasonable assumption that consumers purchase the product from the firm that offers the lowest delivered price and assuming that the products are perfectly homogeneous, the analysis indicates that there is no location—price equilibrium. The authors then changed the assumption concerning customer behavior. First of all, they assumed that customer tastes are homogeneous with a degree μ, which is taken as the standard deviation of the distribution of consumer tastes. This is the same assumption made in their earlier papers. The authors then proved that the model has indeed an equilibrium, as long as the degree of heterogeneity is sufficiently large, *viz*., $\mu \geq c\ell/8$. At that equilibrium, central agglomeration occurs. It was also shown that the result generalizes to n firms, in which case the existence condition is $\mu \geq [(n-1)/n](c\ell/4)$. At equilibrium, all firms are clustered at the center of the market and the equilibrium prices are independent of the number of facilities. Comparing the results with those obtained by De Palma et al. (1985) for mill pricing, it turns out that the mill price charged at the facility plus the transportation cost equals the uniform delivered price in this model, and that customers close to the facilities (in particular those inside the first and third quartiles) prefer mill pricing over uniform delivered pricing. The firms' profits are identical in both cases.

The paper by Labbé and Hakimi (1991) considered a network with customers located at the nodes. The delivered prices charged by the firms and paid at the nodes depend on the total quantity of the homogeneous good supplied by the duopolists at the node. The demand-price function is linear and has a negative slope. The authors use a two-stage procedure: in the first stage, firms choose their locations; in the second stage, they determine their production quantities. This feature was quite distinct from other contributions that use locations and prices as variables, whereas this work considers competition in locations and quantities (which is thus much closer to Cournot's original work, rather than Bertrand's unstable price competition). Employing the usual recursion, the authors prove that for any fixed pair of locations, the quantity game has an equilibrium. Under a condition that requires that it is always profitable to supply any market on the graph with a positive quantity of goods, a locational Nash equilibrium exists at the nodes of the graph. If this condition is not satisfied, the authors provided examples demonstrating that a locational Nash equilibrium either does not exist at all, or may exist on the edges of the graph.

The competitive location model investigated by Eiselt and Laporte (1993) included three firms, each attempting to maximize its own market share. The demand is located at the vertices of a tree. Contrary to the linear market, in which three market-share maximizing facilities end up without ever finding an equilibrium, the paper outlined under what conditions equilibria exist. In particular, there may be an agglomerated equilibrium with all facilities locating at the median of the tree, a semi-agglomerated equilibrium with two facilities locating at the median, while the third facility chooses an adjacent site, a dispersed equilibrium, in which the three facilities locate at three mutually adjacent vertices (one of which is the median), or no equilibrium. Loosely speaking, the more evenly the weights are distributed on the tree, the more likely it is that an equilibrium exists.

The focus of the contribution by Bhadury and Eiselt (1995) was the usual equilibrium—no equilibrium dichotomy. The paper proposed a measure that indicates not only whether or not an equilibrium exists, but how stable or unstable the solution is. While the paper demonstrated the computation of the measure in a tree network, it applies to all competitive location models. There are two cases to be considered. In the first case, at least one Nash equilibrium exists. The measure then determines the effort that is required to convince at least one of the firms to move out of its equilibrium location. Clearly, if it takes a large subsidy to make a firm move out of its present location, the situation can be considered very stable. On the other hand, in case no equilibria exist, a tax for any moves (or, alternatively, moving costs) will indicate how much it takes to stop a firm from relocating. If this amount is substantial, it indicates that much effort is needed to stop the firms from relocating, so that the situation is far from an equilibrium and as such is highly unstable. A continuous measure of this nature contains much more information than the usual existence/non-existence analysis.

Eiselt and Bhadury (1998) considered the problem of reachability of equilibria, given that they actually exist. Their space is a tree network with demand occurring at the nodes. Two competing firms locate one branch each at the nodes of a tree. They charge fixed, but not necessarily equal, mill prices. The authors developed necessary and sufficient criteria for the existence of equilibrium locations on a tree. Given that equilibrium locations exist, the paper then examined whether or not a sequential and repeated relocation procedure that starts at an arbitrary location will eventually lead to the equilibrium. The authors first demonstrated that, in general bi-matrix games with an arbitrary starting point, a Nash equilibrium, even if its exists, may not be reached. They then described a “reasonable” optimization procedure. In this process, one of the duopolists optimizes his own location, given his opponents present location. The assumption is that his opponent does not react, at least not for some time, so that the planner can reap the benefit of his own relocation. In the next step, the firm that relocated is now fixed at the site it chose and its opponent optimizes his location. This sequential process terminates when repeated reoptimization does not change the locations. The main result of the paper was that an equilibrium will be reached in this process, provided a proper tie-breaking rule is used. Table 7.1 summarizes some of the highlights in the analysis of Hotelling models.

Table 7.1 Some of the major contributions to Hotelling’s model

Authors	Year	Major aspect of the model
Hotelling	1929	The basic model
Lerner and Singer	1937	Hotelling results for $n > 2$
Smithies	1941	Different behavioral assumptions
Eaton and Lipsey	1975	Equilibria with $n > 2$, 2-D results
d’Aspremont et al.	1979	Hotelling was wrong, quadratic cost function
Shaked	1982	Firms use mixed strategies
Kohlberg	1983	A model with waiting time
De Palma et al.	1985, 1987a	Customers use probabilistic choice rule
De Palma et al.	1987b	The model with uniform delivered pricing
Andersson	1988	Linear-quadratic transportation costs
Labbé and Hakimi	1991	Equilibria on networks
Eiselt and Laporte	1993	Three facilities on a tree
Bhadury and Eiselt	1995	Stability of equilibria
Eiselt and Bhadury	1998	Reachability of equilibria

In summary then, what has Hotelling's contribution done for location science? First and foremost, it has alerted the location science community (by which I include all interested parties from regional scientists to mathematicians, engineers, and computer scientists) to the interdependencies of different factors of location planning, and it has provided insight into location models. While, for instance, it will be virtually impossible to compute Nash equilibria for any real location scenarios, the decision makers now know which factors are required to stabilize a solution and which will lead to instability. Similarly, decision makers know that competition means having to look over their shoulders and anticipate a reaction, and the hundreds of contributions that have followed Hotelling's original analysis have enabled decision makers to know what to look for: adaptations of prices, quantities, attractiveness of their facilities, and many others. Another area in which Hotelling's work has impacted the field is in the—still somewhat underdeveloped—area of nonphysical location. Much more work is needed to develop brand positioning models, the assignment of tasks to employees in ability space, and the positioning of political candidates in issue space to a point where they become viable tools for practical location problems.

7.4 Future Work

As highlighted in the above sections, much work has been done in the field of competitive location models. Below, I will list a few of the areas that appear to offer promising research leads.

1. *Models with additional parameters*. While in the original contributions firms were competing in location and price, additional factors exist that may be taken into consideration. One such possibility is weights that symbolize the attractiveness of firms or brands. In the retail context, the attractiveness of a store may be expressed in terms of floor space, opening hours, (perceived) friendliness of staff, and similar factors. Attraction functions have been used for a long time, such as in the original work by Huff (1964). In the locational context, models with attraction functions are also not new, as witnessed by the contributions by Eiselt and Laporte (1988, 1989), Drezner (1994), and Eiselt and Marianov (2008b). Another recent contribution that uses repeated optimization with a Huff-style attraction function is put forward by Fernández et al. (2007). However, none of these models discusses equilibrium issues. Another feature that may be included is the choice of technology.
2. An interesting aspect is *asymmetric models*, i.e., models in which competing firms have either different objective functions, use different pricing policies, or have different perceptions of existing demand structures. The paper by Thisse and Wildasin (1995) is a step in this direction, as it includes not only competing duopolists, but also a public facility. A model with different pricing policies on a linear market has been put forward by Eiselt (1991).

3. An obvious extension concerns the discussion of competitive location in 2- or *higher-dimensional spaces*. It is questionable, though, if this is a promising route: experience with two-dimensional models, even if price competition is ignored altogether, has shown it to be very difficult. Some results with have been obtained by Irmen and Thisse (1998). More details concerning pure location competition can be found in Chap. 19 of this volume.
4. A different angle concerns the product with market segmentation. It refers to firms competing in different markets. Again, these markets could either be separated in physical space or in abstract feature or issue spaces in nonphysical applications. Especially in the context of product design, it would be very interesting to see whether or not there are instances in which a firm will decide not to compete in some of market.
5. The issue of data *aggregation* in the context of competitive location models has recently been put forward by Plastria and Vanhaverbeke (2007). The discussion is still in its infancy and it remains to be seen if conclusive results can be obtained.

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