

# Chapter 10

## The Location of Undesirable Facilities

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### 10.1 Introduction

Undesirable facilities are those facilities that have adverse effects on people or the environment. They generate some form of pollution, nuisance, potential health hazard, or danger to nearby residents; they also may harm nearby ecosystems. Examples are incinerators, landfills or sewage plants, airports, stadia, repositories of hazardous wastes, nuclear or chemical plants, prisons, and military installations. Although they provide some disservice to nearby residents, these facilities are necessary to society. In addition, there is often some travel involved to and from these facilities and an associated transportation cost that increases with distance from the population, which in turn suggests that they should be placed away but not very far away. The terms *semi-obnoxious* and *semi-desirable* have also been used for some of these facilities, but the undesirable features (perceived or real) of these facilities dominate the desirable ones. Since the analytical models used for locating these facilities do not change much with their degree of undesirability, as Erkut and Neuman (1989) suggested, we will use the term *undesirable* for all of them.

Since its inception, location theory has been dominated by models and methods for locating desirable facilities, such as warehouses, hospitals, and firehouses, which need to be placed close to the population centers receiving their services. This changed in the 1970s with the launching of undesirable facility location research. Several reasons are attributed to this late entry in the location literature, notably that most of the aforementioned undesirable facilities, such as airports, mega-stadia, and sewage, chemical, and nuclear plants, are the byproducts of the technological advances and industrialization of the second half of the twentieth century. In addition, both industrial waste and municipal waste increase with world population and economic development while the waste generated by some of these facilities is toxic and has to be disposed of safely.

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In the early 1970s, public environmental concerns triggered federal legislation in the United States, which, in turn, enhanced awareness of the potential hazards and generated a need for the systematic placement of these facilities to minimize their undesirable effects. Prior to the 1970s, the protection of basic air and water supplies was a matter mainly left to each state. During the 1970s, responsibility for clean air and water shifted to the federal government. The Environmental Protection Agency was created in 1970, and during the ensuing decade several regulatory laws were passed to protect human health and the environment from potential hazards of pollution and waste disposal. These included the *Clean Air Act* of 1970 and the *Safe Drinking Water Act* of 1974 for enforcing clean air and drinking water standards, the *Resource Conservation and Recovery Act* of 1976 for regulating the disposal of solid and hazardous wastes, the *Clean Water Act* of 1977 for eliminating releases of toxic substances into the water, and the *Comprehensive Environmental Response, Compensation, and Liability Act* of 1980 for protecting people from abandoned heavily contaminated toxic waste sites.

A suitable location objective for locating an undesirable facility is the maximization of some increasing function of the distance between the facility and the affected customers. Analogous to the *minisum* and *minimax* objectives, most popular for locating desirable facilities, the *maxisum* and *maximin* objectives are established for locating undesirable facilities; see, e.g., Eiselt and Laporte (1995). The *maxisum* objective maximizes the sum of distances (or average distance) between the facility and the customers, while the *maximin* objective maximizes the distance between the facility and the closest customer to it. Sometimes weights are assigned to customers to represent the relative incompatibility between a customer and the facility and *weighted distances* are used. The objectives for undesirable facilities are frequently referred to as *push objectives*, since they push the undesirable facility away from the customers, while the objectives for desirable (attractive) facilities are referred to as *pull objectives*, since they pull the facility closer to the customers. To avoid pushing the undesirable facility to an infinite distance from the customers, which does not make sense in a real life problem, the objectives for undesirable facilities have to be optimized within a bounded region, a distinct difference from the desirable facilities objectives. In addition, the optimization models for undesirable facility location are more difficult to solve. Unlike the desirable facility location models, undesirable facility location models are nonconvex, typically having many local optima.

Although Goldman and Dearing (1975) are credited with first discussing the concept of optimally locating “semi-desirable” or “partially noxious facilities” in a conference paper, Church and Garfinkel’s (1978) paper was the first published work on undesirable facility location. Their paper dealt with the *maxisum* location problem on a general network: they found a point of the network such that the sum of weighted shortest path distances from the nodes is maximized. They reduced the network solution space to a finite set of candidate points for optimality, consisting of the set of *bottleneck points* of the network and the leaf nodes of the network. Church and Garfinkel first showed that the *maxisum* objective renders

a nonconvex problem having many local optima, and that Hakimi's principle of optimality at a node does not hold. Exploiting the structure of the problem and utilizing bounds, their algorithm found the global optimum by partially enumerating local maxima. Their algorithm was adapted later for other undesirable facility location problems. This pioneering work stimulated a large body of research in undesirable facility location that complemented the desirable facility location literature.

The maximin location problem first appeared in the works of Shamos (1975) and Shamos and Hoey (1975) who studied the complexity of several fundamental problems in computational geometry. One of these problems is finding the *largest empty circle* of a given set of points in the plane, i.e., the circle that contains no points of the set, yet whose center is in the convex hull of the given points. The center of that circle is the maximin point as it maximizes the Euclidean distance to the closest point in the set. The solution is found by constructing the Voronoi diagram for the set of points. The first papers on the maximin location problem were published five years later by Dasarathy and White (1980) and Drezner and Wesolowsky (1980).

Building on their earlier work on pattern recognition, Dasarathy and White (1980) first formulated and solved the maximin problem with Euclidean distances for a feasible region, which is a convex polyhedron in  $k$ -dimensional space. They delineated their general algorithm for a 3-dimensional space. For the 2-dimensional space, they expanded Shamos and Hoey's Voronoi construction to account for optimality at the boundary of the feasible region. Their principal contributions are the characterization of the problem as nonlinear and nonconvex, the establishment of the properties of local optima using the Karush-Kuhn-Tucker optimality conditions, and the development of a general algorithm for solving the problem.

Drezner and Wesolowsky (1980) considered the same problem but with customer weights and a convex planar region defined by maximum distance constraints, one for each point (customer). Equivalently, the customers want the facility as far away as possible but within certain distance from them, which in turn signifies the semi-obnoxious character of the facility. Their optimization procedure was different from the one in Dasarathy and White (1980). They used a bisection search based on a graphical approach to approximate the optimal solution.

The above classical contributions, which cast the foundation of undesirable facility location theory during the late 1970s, are presented in this chapter. The detail and illustrative examples are helpful to introduce a beginner into the basic concepts of undesirable facility location research but also there is sufficient depth for the veteran researcher in the field to review and appreciate the classical contributions. An effort has been made to include major theoretical results, the thought process of the authors at the time, and the impact their work had on location literature. Although this is not a survey paper, major works that followed the classical contributions are surveyed.

This chapter is organized as follows: The classical contributions are presented in Sect. 10.2 and their impact is assessed in Sect. 10.3. The chapter ends with a summary and outlook of undesirable facility location research in Sect. 10.4.

## 10.2 The Classical Contributions

The classical contributions on the location of undesirable facilities can be classified according to the objective functions used in the respective optimization problems: *maxisum* and *maximin*. In an effort to minimize the adverse effects of the facility to be located, both of these objectives maximize some increasing function of the distance between the facility and the affected customers, namely the sum of distances and the minimum distance. The original papers appeared within a five-year period in the second half of the 1970s. We first present the classical contribution of Church and Garfinkel (1978) that utilizes the *maxisum* objective on a network. This work is followed by contributions that consider the *maximin* objective in continuous space: Shamos (1975) and Shamos and Hoey (1975), Dasarathy and White (1980), and Drezner and Wesolowsky (1980).

### 10.2.1 The *Maxisum* Problem on a Network: Church and Garfinkel (1978)

Let  $G = (N, A)$  be a connected and undirected graph with no loops or multiple arcs, where  $N$  is the set of  $n$  nodes and  $A$  is the set of  $m$  arcs. The nodes represent customers that exhibit some adverse interaction with the new facility. We want to find a point  $x$ ,  $x \in G$ , for locating the facility that maximizes

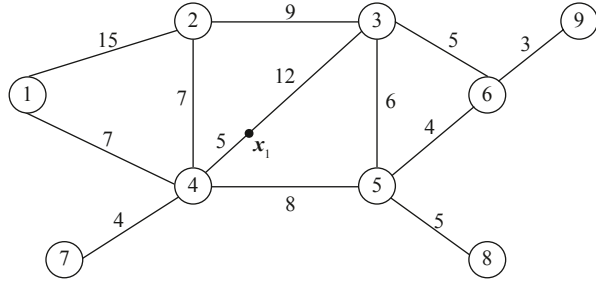
$$T(x) = \sum_{i \in N} w_i d(i, x), \quad (10.1)$$

where  $w_i \geq 0$  is the weight of node  $i$  and  $d(i, x)$  is the length of the shortest path between node  $i$  and  $x \in G$ .

Church and Garfinkel first formulated the above problem and named it *maxian* as it is identical to the *median* problem except that the objective is maximizing instead of minimizing. Thus, the solutions of these two problems find the two extreme values of  $T(x)$ . As they remark, this may help in evaluating how bad a given solution is with respect to any one of the two objectives.

Whereas the median problem attempts to find a location that is close to a given set of points, the *maxian* problem attempts to find a location that is as far as possible from these points. It should be noted that in the *maxian* problem, which is more often called *maxisum* problem, the type of facility to be located is not as important as is the adverse interaction between the given points and the new facility. In fact, Church and Garfinkel gave as an example of application the location of a house or a business—by no means undesirable—in a city among pockets of high crime incidence concentrated at the nodes of the network. The interaction between the nodes and the facility results in danger or potential harm to the facility that decreases with its distance from the nodes. In this example, the weight  $w_i$  represents the relative danger of node  $i$  to the facility.

**Fig. 10.1** A nine node network



For median problems, Hakimi (1964) proved that there exists a node which is optimal. This suggests a straightforward procedure for finding the optimal solution: evaluate objective function (10.1) at all nodes of the network and select the node(s) with the minimum value. Changing the optimization operator to “max” results in a surprisingly more complicated problem, in which the optimal point cannot be only at the nodes but also on the arcs of the network. Moreover, the maximum is a non-convex problem, thus possessing many local optima. One of the key results of this work is that the search for the optimal solution is reduced from the infinite number of points of network  $G$  to a finite set of candidate points, often referred to as a *Finite Dominating Set* of points (*FDS*). After a brief notation, the points of interest are introduced below and optimality properties are established.

An interior point  $x$  on an arc  $(i, j)$  divides it into two arc segments  $(i, x)$  and  $(j, x)$ . Denote the lengths of the two segments  $c(i, x)$  and  $c(j, x)$ , respectively, and denote point  $x$  by the arc  $(i, j)$  is on and its distance from node  $i$ :  $[(i, j); c(i, x)]$ . For example, in Fig. 10.1 (slightly modified from Church and Garfinkel 1978),  $x_1 = [(3, 4); 12]$ .

It is assumed that the shortest path distances between nodes are known. Ahuja et al. (1993) demonstrated that they can be effectively computed by several algorithms. Table 10.1 contains the shortest path distances between the nodes of the above network,  $d(i, k)$ , and the sum of distances from a node  $k$  to all nodes,

$$T(k) = \sum_{i \in N} d(i, k).$$

Let  $x$  be an interior point of arc  $(i, j)$ . If there exists a node  $k$  such that

$$d(k, i) + c(i, x) = d(k, j) + c(j, x), \tag{10.2}$$

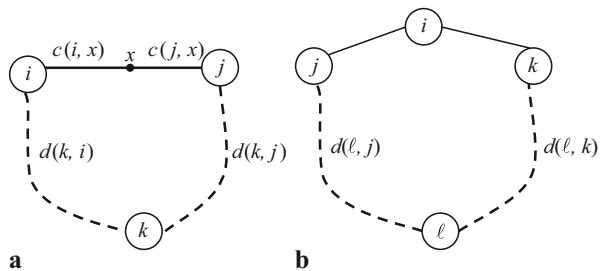
$x$  is called *arc bottleneck point* with respect to node  $k$  and  $B_A(k)$  denotes the set of bottleneck points generated by node  $k$  on  $A$ . This is illustrated in Fig. 10.2a, where the shortest paths from node  $k$  to  $i$  and  $j$  are shown as broken lines. Since  $c(i, x), c(j, x) > 0$  and  $c(i, x) + c(j, x) = c(i, j)$ , it follows from (10.2) that arc  $(i, j)$  contains an arc bottleneck point with respect to node  $k$  if and only if  $|d(k, i) - d(k, j)| < c(i, j)$ . By letting

$$p(k) = d(i, k) - d(j, k), \tag{10.3}$$

**Table 10.1** Node-to-node shortest path distance matrix  $d(i, k)$  and  $T(k)$

Node $i$	Node $k$									$T(k)$
	1	2	3	4	5	6	7	8	9	
1	0	14	21	7	15	19	11	20	22	129
2	14	0	9	7	15	14	11	20	17	107
3	21	9	0	14	6	5	18	11	8	92
4	7	7	14	0	8	12	4	13	15	80
5	15	15	6	8	0	4	12	5	7	72
6	19	14	5	12	4	0	16	9	3	82
7	11	11	18	4	12	16	0	17	19	108
8	20	20	11	13	5	9	17	0	12	107
9	22	17	8	15	7	3	19	12	0	103

**Fig. 10.2** Bottleneck points



the condition for arc  $(i, j)$  to contain an arc bottleneck point generated by node  $k$  can be written as

$$|p(k)| < c(i, j). \tag{10.4}$$

Note that  $p(k)$  measures how much farther away node  $i$  is from  $k$  than from  $j$ . The range of values of  $p(k)$  can be found by the triangle inequality for shortest path distances. Since  $d(i, k) \leq d(i, j) + d(j, k)$  and  $d(j, k) \leq d(j, i) + d(i, k)$ , substituting in (10.3) we obtain  $-d(i, j) \leq p(k) \leq d(i, j)$ , or

$$|p(k)| \leq d(i, j). \tag{10.5}$$

If  $d(i, j) < c(i, j)$ , (10.4) and (10.5) imply that every node  $k \in N$  has an arc bottleneck point on arc  $(i, j)$ . This is later illustrated for arc  $(1, 2)$  of Fig. 10.1.

Inequality (10.4) implies that no shortest path from  $k$  to  $i$  or from  $k$  to  $j$  contains arc  $(i, j)$ . Clearly, a bottleneck point on arc  $(i, j)$ , with respect to node  $k$ , is associated with a cycle formed by the shortest path from node  $k$  to node  $i$ , arc  $(i, j)$ , and the shortest path from node  $j$  back to node  $k$ , as shown in Fig. 10.2a. The bottleneck point is the point in a cycle that is the farthest away from node  $k$ . For example,  $x_1 \in B_A(6)$  in Fig. 10.1 as node 6 and point  $x_1$  are the endpoints of two equidistant paths,  $\{(6, 3), (3, x_1)\}$  and  $\{(x_1, 4), (4, 5), (5, 6)\}$ , and inequality (10.4) is satisfied as  $|5 - 12| < 17$ . By considering all cycles that contain node 6, in Fig. 10.1, and identifying arcs on those cycles containing bottleneck points using inequality

(10.4), the complete set  $B_A(6)$  can be derived:  $B_A(6) = \{(3, 5); 2.5\}, \{(3, 4); 12\}, \{(2, 4); 2.5\}, \{(1, 2); 5\}$ . Since there is a unique path between leaf node 9 and node 6,  $B_A(9) = B_A(6)$ . Note that it is possible for arc  $(i, j)$  to contain a bottleneck point with respect to each one of its vertices  $i$  and  $j$ . This happens if and only if  $d(i, j) < c(i, j)$ , i.e., the shortest path between the end nodes of an arc is not that same arc. In Fig. 10.1, arc  $(1, 2)$  satisfies that condition, as  $d(1, 2) = 14 < c(1, 2) = 15$  and the associated cycle is  $\{(1, 2), (2, 4), (4, 1)\}$ . Therefore, point  $[(1, 2); 14.5]$  is in  $B_A(1)$  and point  $[(1, 2); 0.5]$  is in  $B_A(2)$ .

Bottleneck points can also appear on nodes. If there exist distinct arcs  $(i, j)$  and  $(i, k)$  incident to node  $i$  and a node  $\ell \neq i$  such that  $d(\ell, j) + c(j, i) = d(\ell, k) + c(k, i)$ , then node  $i$  is a *node bottleneck point* with respect to node  $\ell$ , denoted by  $i \in B_N(\ell)$ , and illustrated in Fig. 10.2b. For example, node 2 is in  $B_N(5)$  in Fig. 10.1. There are two equidistant paths from node 5 to node 2, one containing arc  $(2, 3)$  and the other  $(2, 4)$ , whose union forms a cycle. Each arc or node bottleneck point is associated with a cycle in  $G$  that contains the point. Conversely, Church and Garfinkel show that every cycle in  $G$  contains a bottleneck point with respect to every node in the cycle. This result suggests a method for finding all bottleneck points of a network: find all cycles in  $G$  and for every node in a cycle identify the corresponding bottleneck point.

Let  $B_A = \bigcup_{k \in N} B_A(k)$  denote the set of all arc bottleneck points,  $B_N = \bigcup_{k \in N} B_N(k)$  the set of all node bottleneck points, and  $B = B_A \cup B_N$  the set of all bottleneck points of  $G$ . Let  $D$  denote the set of dangling (leaf) nodes of  $G$ . In the network of Fig. 10.1, for example,  $D = \{7, 8, 9\}$ . Since bottleneck points are defined by cycles,  $D$  and  $B$  have no elements in common. The following theorem reduces the solution space from an infinite set (network  $G$ ) to a finite set of candidate points for optimality consisting of the set of leaf nodes and the set of bottleneck points of  $G$ .

**Theorem 1:** *There exists a point  $\hat{x}$  which maximizes (10.1) such that  $\hat{x} \in X^* = D \cup B$ .*

*Proof:* Church and Garfinkel show that for every point  $x \in G$ ,  $x \notin D \cup B$ , there exists an  $x \in D \cup B$  with a better objective value. Consider first an interior point  $x$  of arc  $(i, j)$ ,  $x \notin B_A$ . Then, within the  $\varepsilon$ -neighborhood of  $x$ , the sum in (10.1) can be decomposed into two, one over nodes  $k \in N_i(x)$  and one over  $k \in N_j(x)$ , where  $N_i(x)$  and  $N_j(x)$  are nodes  $k \in N$  whose shortest path to  $x$  includes segment  $(i, x)$  and  $(j, x)$ , respectively:

$$T(x) = \sum_{k \in N_i(x)} w_k d(k, x) + \sum_{k \in N_j(x)} w_k d(k, x).$$

For a point  $x_\varepsilon$  in the interior of arc  $(i, j)$ , such that  $c(j, x_\varepsilon) = c(j, x) + \varepsilon$ , for  $\varepsilon > 0$  and infinitesimal,  $N_i(x)$  and  $N_j(x)$  remain unchanged and therefore,

$$T(x_\varepsilon) - T(x) = \left[ \sum_{k \in N_j(x)} w_k - \sum_{k \in N_i(x)} w_k \right] \varepsilon = q(x)\varepsilon.$$

Assuming without loss of generality that  $q(x) \geq 0$ , it follows that  $T(x_\epsilon) - T(x)$  increases with  $\epsilon$  until  $x_\epsilon$  reaches an arc bottleneck point or node  $i$ .

Consider now a point  $x$  that is on a node and  $x \notin D \cup B_N$ . Similarly, in this case a path from  $x$  can be found on  $G$  of increasing objective value until a point of  $D \cup B$  is encountered. □

On a given arc  $(i, j) \in G$ , there exist at most  $n$  bottleneck points identified by cycles containing arc  $(i, j)$  and each node  $k \in G$ . Therefore, there exist at most  $mn$  bottleneck points. Since the number of leaves in a network is at most  $n$ , the size of the set containing the optimal solution is  $O(mn)$ .

Since bottleneck points occur only on cycles of  $G$  and a tree network has no cycles, the following corollary follows from Theorem 1.

**Corollary 1:** *If the network is a tree, there exists an optimal point which is a leaf node.*

A straightforward approach for solving (10.1) is to find the best point on each edge, and then compare these points and select the optimal point in  $G$ .

The shortest path distance between a node  $k \in N$  and a point  $y \in (i, j)$ , is  $d(k, y) = \min\{d(i, k) + c(i, y), d(j, k) + c(j, y)\}$ . It is maximized when  $d(i, k) + c(i, y) = d(j, k) + c(j, y)$ . Substituting  $c(i, y) = c(i, j) - c(j, y)$ , the point on arc  $(i, j)$  with the maximum distance from node  $k$ , denoted by  $y(k)$ , is at a distance from node  $j$ ,  $c(j, y) = \frac{1}{2}[d(i, k) - d(j, k) + c(i, j)]$ . After it is simplified using (10.3), it becomes:

$$c(j, y) = \frac{1}{2} [p(k) + c(i, j)]. \tag{10.6}$$

In other words, for a given arc  $(i, j)$  the length of  $(j, y)$  is increasing with  $p(k)$ . The greater the value of  $p(k)$ , the further  $y(k)$  is from node  $j$ . If  $p(k) = c(i, j)$ ,  $y(k) = i$ , while if  $p(k) = -c(i, j)$ ,  $y(k) = j$ . Therefore, if we reorder nodes  $k \in N$  in increasing magnitude of  $p(k)$  they will map to  $y(k)$  points in the same order on arc  $(j, i)$ , according to (10.6). To reorder the nodes  $k \in N$  for arc  $(i, j)$  we re-index them by  $r(k)$  in terms of increasing  $p(k)$ , i.e.,  $r(k_2) > r(k_1) \rightarrow p(k_2) \geq p(k_1)$ . Clearly,  $p(i) = -d(i, j)$  and  $p(j) = d(i, j)$ , and we can let  $r(i) = 1$  and  $r(j) = n$ .

Table 10.2 contains  $p(k)$  and  $r(k)$ ,  $k \in N$ , for arc  $(i, j) = (1, 2)$  of Fig. 10.1. The distance of point  $y(k)$  from node 2,  $c(2, y)$ , is also computed according to (10.6). Note that  $d(1, 2) < c(1, 2)$  and therefore  $|p(k)| < c(1, 2) = 15$  for every  $k \in N$ . Based on an earlier observation, every node has a bottleneck point on arc  $(1, 2)$  although not all of them are distinct.

We want to express the objective function (10.1) at some point  $y \in (i, j)$ . Consider two consecutive  $y(k_1)$  and  $y(k_2)$  points, i.e.,  $r(k_1) = t$  and  $r(k_2) = t + 1$ ,  $t = 0, \dots, n$ ,

**Table 10.2** Bottleneck points on arc (1, 2)

$k$	1	2	3	4	5	6	7	8	9
$p(k)$	-14	14	12	0	0	5	0	0	5
$r(k)$	1	9	8	2	3	6	4	5	7
$c(2, y)$	.5	14.5	13.5	7.5	7.5	10	7.5	7.5	10
$T(y(k))$	111.5	133.5	140.5	160.5	160.5	158	160.5	160.5	158



where  $r(k) = 0$  and  $r(k) = n + 1$  are associated with node  $j$  and  $i$ , respectively. For notational simplicity, use  $y = c(j, y)$ . In other words,  $y$  represents both a point  $y \in (i, j)$  and its distance from  $j$  on arc  $(i, j)$ . For  $y \in (y(k_1), y(k_2))$ ,

$$d(k, y) = \begin{cases} d(k, i) + c(i, j) - y, & \text{for } \{k|r(k) \leq t\} \\ d(k, j) + y, & \text{for } \{k|r(k) \geq t + 1\} \end{cases} \quad (10.7)$$

and the objective function  $T(y)$  can now be expressed as

$$T(y) = \sum_{k|r(k) \geq t+1} w_k d(k, j) + \sum_{k|r(k) \leq t} w_k [d(k, i) + c(i, j)] + W(t)y, \quad (10.8)$$

where

$$W(t) = \sum_{k|r(k) \geq t+1} w_k - \sum_{k|r(k) \leq t} w_k, \quad t = 0, \dots, n \quad (10.9)$$

is the gradient of  $T(y)$ . When  $y(k_1)$  and  $y(k_2)$  are distinct points,  $T(y)$  is a line segment with slope  $W(t)$ . When  $y(k_1) = y(k_2)$ , the line segment becomes a degenerate point. The number of different line segments of  $T(y)$  for  $y \in (i, j)$  depends on the number of distinct numbers  $y(k)$ . It can be as low as 1 (whole arc), when  $|p(k)| = c(i, j)$  for all  $k \in N$ , and as high as  $n + 1$ , when all values  $y(k)$  are distinct bottleneck points. The former happens for arc (4, 5) of Fig. 10.1. Clearly, the slope  $W(t)$  is nonincreasing with increasing  $t$ , as one scans consecutive arc segments  $(y(k_1), y(k_2))$  from node  $j$  to node  $i$  on arc  $(j, i)$ . Since it is also continuous,  $T(y)$  is piecewise linear and concave, and the theorem below follows.

**Theorem 2:** *A best point  $y^*(k)$  on arc  $(i, j)$  satisfies  $r^*(k) = \min\{r(k)|W(r(k)) \leq 0, r(k) = 1, \dots, n\}$ , and its distance from node  $j$  is given by (10.6).*

It is possible that  $W(r(k)) = 0$  at the best point  $y^*(k)$ . Then every point on the arc segment between  $y(k^*)$  and  $\{y(\ell)|r(\ell) = r^*(k) + 1\}$  maximizes  $T(y)$  on arc  $(i, j)$ .

Although Church and Garfinkel allude to the concavity property of the objective function on an arc, they do not explicitly state it. To illustrate the “piecewise linear and concave” property of the objective function  $T(y)$  over an arc, objective values at all  $y(k) \in (2, 1)$  are calculated using (10.8) and (10.9). Equal weights,  $w_k = 1, k \in N$ , are assumed. The objective values  $T(y(k))$  are displayed in the last row of Table 10.2. In Fig. 10.3,  $T(y)$  is plotted for arc (1, 2) using the data of Table 10.2. Point  $y = 0$  and point  $y = 15$  correspond to nodes 2 and 1, respectively. The objective value at node 2 and 1 is 107 and 129, respectively, taken from the last column of Table 10.1.

The *unweighted maxian problem* is the maxian problem in which all weights  $w_i$  are equal to 1. As is shown below, the procedure for finding the best point on an arc can be greatly simplified when weights are equal.

Note in (10.9) that as point  $y$  on arc  $(i, j)$  moves from one interval  $[y(k_1), y(k_2)]$  to the next  $[y(k_2), y(k_3)]$ , such that  $r(k_1) = t, r(k_2) = t + 1$  and  $r(k_3) = t + 2$ , the gradient

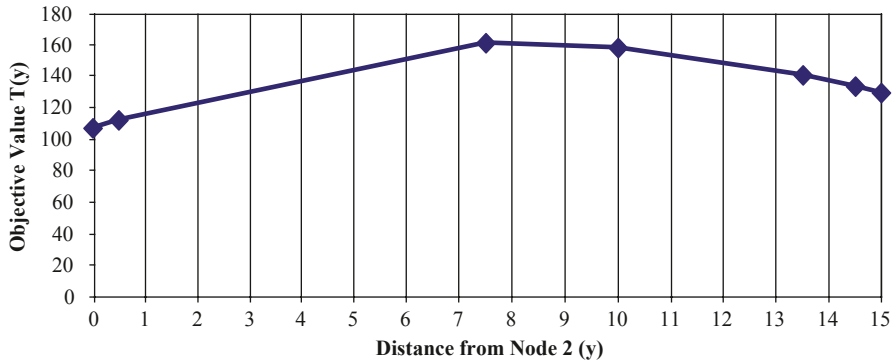


Fig. 10.3 Objective function value over arc (2, 1)

$W(t)$  decreases by  $2w_{k_2}$ . If all weights are equal to 1 the gradient decreases by 2. For our 9-node example,  $W(t) = 9 - 2t$ ,  $t = 0, \dots, 9$ . The following corollary simplifies the procedure for finding the best point on an arc.

**Corollary 2:** A best point  $y^*(k)$  on arc  $(i, j)$  satisfies

$$r^*(k) = \begin{cases} 1/2n, & \text{if } n \text{ is even} \\ 1/2(n + 1), & \text{if } n \text{ is odd} \end{cases}$$

and its distance from node  $j$  is given by (10.6).

The best point on an arc can be found in  $O(n \log n)$  time by sorting the  $n$  points with respect to increasing values of  $p(k)$ . Therefore, the total time required to find the optimal maximum point in an unweighted network is  $O(mn \log n)$ .

Consider again arc (1, 2) of the network of Fig. 10.1. Since  $n$  is odd, the best point  $y^*(k)$  on arc (1, 2) is associated with  $r^*(k) = 1/2(n + 1) = 5$ . From Table 10.2 we find that the arc bottleneck point  $y^*(k)$  is generated by node  $k = 8$  and is at distance  $y = 7.5$  units from node 2 (see also Fig. 10.3). Note that, in addition to node 8, the best point on arc (1, 2) is the bottleneck point of nodes 4, 5 and 7.

Bounds of the objective function over an arc can be found as follows. Since  $T(y)$  is concave over each arc  $(i, j)$ , its minimum occurs at an endpoint, i.e., at one of the two nodes,  $i$  or  $j$ , or both. Therefore, a lower bound of  $T(y)$  over arc  $(i, j)$ ,  $\underline{T}(i, j)$ , is specified in the following relation

$$\underline{T}(i, j) = \min \left\{ \sum_{k \in N} w_k d(k, i), \sum_{k \in N} w_k d(k, j) \right\}. \tag{10.10}$$

A good lower bound of the optimal objective value over  $G$  is  $\underline{T}$ . It is obtained by comparing the objective values of the nodes of  $G$ :

$$\underline{T} = \max_{i \in N} \sum_{k \in N} w_k d(k, i) \tag{10.11}$$

To find an upper bound of  $T(y)$  over an arc  $(i, j)$ ,  $\bar{T}(i, j)$ , we consider the upper bound of  $d(k, y)$ ,  $y \in (i, j)$ . The maximum distance point on arc  $(i, j)$  from node  $k$  is point  $y(k)$ , found earlier. Its distance from node  $k$  is  $d(k, j) + c(j, y)$ , where  $c(j, y)$  is given by (10.6), such that

$$d(k, y) \leq 1/2[c(i, j) + d(k, i) + d(k, j)].$$

Multiplying both sides of this expression by  $w_k$  and taking the summation for all  $k \in N$ , we obtain

$$\begin{aligned} T(y) &\leq \bar{T}(i, j) = 1/2 \left[ c(i, j) \sum_{k \in N} w_k + \sum_{k \in N} w_k d(k, i) + \sum_{k \in N} w_k d(k, j) \right] \\ &= 1/2 \left[ c(i, j) \sum_{k \in N} w_k + T(i) + T(j) \right]. \end{aligned} \quad (10.12)$$

For the unweighted maxian problem (with  $w_k = 1$ ,  $k \in N$ ), the upper bound reduces to

$$\bar{T}(i, j) = 1/2 [nc(i, j) + T(i) + T(j)]. \quad (10.13)$$

The search for the optimal solution starts with the node  $\hat{x}$  associated with  $\underline{T}$ . This is the incumbent solution. Upper bounds on all arcs,  $\bar{T}(i, j)$ , are used to eliminate at the outset as many arcs as possible. After finding the best point on an arc, the lower bound is updated and is used to eliminate additional arcs. When all remaining arcs have been considered, the incumbent is the optimal solution.

### Algorithm 1: Maxisum Problem on a Network

- Step 1:* Compute a lower bound on  $T(x^*)$ ,  $\underline{T}$ , using (10.11) and identify the point  $\hat{x} \in G$  at which it occurs.
- Step 2:* Compute the upper bounds on all arcs  $(i, j) \in A$ ,  $\bar{T}(i, j)$ , using (10.12).
- Step 3:* Eliminate every arc  $(i, j) \in A$  for which  $\bar{T}(i, j) < \underline{T}$  from further consideration. Let the set of remaining arcs be  $A'$ .
- Step 4:* Until  $A' = \emptyset$ , repeat.
- Step 4.1:* Let the arc  $(i, j) \in A'$  with the largest  $\bar{T}(i, j)$  be  $(u, v)$ . Find the best point on arc  $(u, v)$  and denote it by  $y^*$  and its objective value by  $T(y^*)$ . Set  $A' \leftarrow A' - (u, v)$ .
- Step 4.2:* If  $T(y^*) > \underline{T}$ , update  $\underline{T} \leftarrow T(y^*)$  and  $\hat{x} \leftarrow y^*$ , and eliminate remaining arcs  $(i, j) \in A'$  for which  $\bar{T}(i, j) < \underline{T}$ .
- Step 5:* The optimal solution is  $x^* = \hat{x}$  and  $T(x^*) = \underline{T}$ .

**Table 10.3** Objective value upper bounds on arcs

Arc $(i, j)$	(1, 2)	(1, 4)	(2, 3)	(2, 4)	(3, 4)	(3, 5)
$\bar{T}(i, j)$	185.5	136	140	125	162.5	109
Arc $(i, j)$	(3, 6)	(4, 5)	(4, 7)	(5, 6)	(5, 8)	(6, 9)
$\bar{T}(i, j)$	109.5	112	112	95	112	106

**Table 10.4** Bottleneck points on arc (3, 4)

$K$	1	2	3	4	5	6	7	8	9
$p(k)$	14	2	-14	14	-2	-7	14	-2	-7
$r(k)$	7	6	1	8	4	2	9	5	3
$c(4, y)$	15.5	9.5	1.5	15.5	7.5	5	15.5	7.5	5
$T(y(k))$								125.5	

Algorithm 1 is illustrated for the unweighted maximin of the network of Fig. 10.1. Node 1 provides a lower bound  $\underline{T} = T(1) = 129$  from the last column of Table 10.1. Upper bounds on the arcs  $\bar{T}(i, j)$  have been calculated according to (10.13) and are shown in Table 10.3. According to Step 3 of the algorithm, all arcs except  $A' = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$  are eliminated. Of the remaining, arc (1, 2) is selected for its largest upper bound (185.5). The best point on arc (1, 2) was found earlier at its midpoint with objective value 160.5. The new  $\underline{T} = 160.5$  allows us to eliminate all remaining arcs except (3, 4).

To find the best point on arc (3, 4) we construct Table 10.4, from which  $r^*(k) = 5$  occurs for  $k = 8$ . The best point is  $[(4, 3); 7.5]$  with objective value 125.5, calculated directly from (10.8) and (10.9). This is inferior to the incumbent point, which is optimal because there are no other remaining arcs to consider. Therefore, the optimal point is  $[(1, 2); 7.5]$  with objective value 160.5.

### 10.2.2 The Maximin Location Problem in Continuous Space with Euclidean Distances

Analogous to *minimax* problem, the *maximin* objective attempts to find a location for an undesirable facility that minimizes the adverse impact on the most affected customer, which is the one closest to the facility. The maximin objective was first used in continuous space with Euclidean distances. Three original contributions for undesirable facility location are analyzed in this subsection. The earliest work by Shamos (1975) and Shamos and Hoey (1975) characterized the unweighted maximin problem in the plane as an interesting problem of computational geometry, whose solution is the byproduct of the construction of a Voronoi diagram. Although they suggested the use of the maximin objective for undesirable facility location, Dasarathy and White (1980) and Drezner and Wesolowsky (1980) first formulated the maximin problem with practical feasible regions making it a suitable location model for undesirable facilities.

### 10.2.2.1 Shamos (1975) and Shamos and Hoey (1975): The Origins of the Maximin Problem

The maximin location problem first appeared in the works of Shamos and Hoey, who studied the complexity of several fundamental problems in computational geometry. The maximin problem is stated as follows: Given a set  $N$  of  $n$  points  $a_i$  in  $\mathbb{R}^2$ , find the largest empty circle that contains no points of the set yet whose center  $x$  is interior to the convex hull of the points,  $CH(N)$ . Equivalently,

$$\max_{x \in CH(N)} \min_{i \in N} d(i, x), \quad (10.14)$$

where  $d(i, x) = \|a_i - x\|$  is the Euclidean distance between point  $a_i$  and  $x$ . The center of such a circle is the unweighted maximin point. Since such a point is farthest away from the closest customer point, it is suitable for the location of an undesirable facility, such as a source of pollution. For the same reason, the maximin point is suitable for locating a new business—albeit a desirable facility—that does not wish to compete for territory with established outlets represented by existing points. The solution point is restricted to a bounded feasible region,  $CH(N)$ , because otherwise it is going to be at an infinite distance from the customers. Moreover, Shamos and Hoey characterized the new problem as the dual of the (unweighted) minimax problem, posed much earlier by Sylvester (1857), which found the smallest circle enclosing all points of set  $N$ . The minimax objective was thoroughly investigated during the 1970s as an alternative to the minisum objective for the location of “desirable” facilities. Shamos and Hoey solved the maximin problem by constructing the Voronoi diagram of the  $a_i$  points.

Associated with each point  $a_i$ ,  $1 \leq i \leq n$ , there exists a polygon  $V_i$ , called a Voronoi polygon, with the following property: if  $x \in V_i$ , then  $\|a_i - x\| \leq \|a_j - x\|$ ,  $1 \leq j \leq n$ . The polygon  $V_i$  is the intersection of halfplanes containing  $a_i$ , where the halfplanes are determined by the perpendicular bisectors of the line segments joining  $a_i$  and  $a_j$ ,  $j \neq i$ . The edges of the Voronoi polygons, some of which are unbounded halflines, are called Voronoi edges and their vertices Voronoi vertices. A Voronoi vertex is the common point of at least three Voronoi polygons, i.e., is equidistant from at least three  $a_i$  points. The circle drawn with its center at a Voronoi vertex and its radius the distance to its equidistant points contains no  $a_i$  points in its interior; it is an empty circle. The Voronoi vertex associated with the largest empty circle is the optimal solution to (10.14). The interior points of a Voronoi edge are equidistant from exactly two (neighboring) points  $a_i$ . The union of the boundaries of the Voronoi polygons is called a Voronoi diagram. The union of the Voronoi diagram and the interior sets of all Voronoi polygons constitute  $\mathbb{R}^2$ . The Voronoi diagram uses all relevant proximity information and is constructed very efficiently in  $O(n \log n)$  time. The maximin problem in one dimension reduces to finding a pair of two consecutive points on a line that are farthest apart. Shamos and Hoey observe that this problem is also solved in  $O(n \log n)$  time.

### 10.2.2.2 Dasarathy and White (1980): The Unweighted Maximin Problem in a Bounded Convex Region

Dasarathy and White (1980) first formulated the unweighted maximin problem for a general feasible region  $S$  that is a bounded and convex polyhedron in  $\mathbb{R}^k$ ,

$$\max_{x \in S} \min_{i \in N} d(i, x), \quad (10.15)$$

where  $d(i, x) = \|a_i - x\|$  is the Euclidean distance between point  $a_i$  and  $x$  in  $\mathbb{R}^k$  and  $S$  is described by a set of  $m$  linear constraints, so that  $S = \{x | c_j x \leq b_j, 1 \leq j \leq m\}$ .

The authors described a number of applications of this problem, not necessarily all in location. Viewing (10.15) as the problem of finding the largest hypersphere centered in  $S$ , whose interior is free of points  $a_i$ , they put forward some applications in information theory and in pattern recognition. It appears that these applications influenced the authors to cast the maximin problem in a higher dimensional space and not in the 2-dimensional space where most location applications are found. Needless to say, the location application of the maximin problem (10.15) had the greatest impact in future undesirable location literature. If the  $a_i$  points represent  $n$  cities in a region  $S$  and a highly polluting industry is to be located within  $S$ , the maximin problem will find its location such that the amount of pollutants reaching any city is minimized. It is assumed that the pollutant dispersion is uniform in all directions and the amount of pollutants reaching each city is a monotonically decreasing function of the distance between the city and that industry. Modeling the spread of pollutants in conjunction with the facility that generates them was studied later by Karkazis and Papadimitriou (1992) and Karkazis and Boffey (1994). Note that unlike the maxisum objective, which attempts to minimize the unpleasant collective impact to all customers, the maximin objective attempts to minimize the impact to the most adversely affected customer, making it an equity measure relative to that customer.

Dasarathy and White view the maximin problem also as a covering problem. Consider, for example, the  $a_i$  points being the locations of  $n$  radar stations and the convex set  $S$  the region monitored by these stations. Then (10.15) finds the minimum (of the maximum) power required by the stations such that each point in  $S$  is monitored by one or more of the stations. It is assumed that the required power of a station is a monotonically increasing function of the distance over which it can receive or send signals.

Letting  $z$  represent the square of the objective function in (10.15), the maximin problem can be converted to a standard nonlinear programming formulation:

$$\text{Max } z \quad (10.16)$$

$$\text{s.t. } z - \|a_i - x\|^2 \leq 0, \quad 1 \leq i \leq n \quad (10.17)$$

$$c_j x \leq b_j, \quad 1 \leq j \leq m \quad (10.18)$$

The above problem described by (10.16)–(10.18) is clearly not a convex programming problem due to constraints (10.17). Therefore, it may have several local optima one has to enumerate explicitly or implicitly to find the global optimum. The properties of a local optimum can be explored by constructing the necessary Karush-Kuhn-Tucker conditions for a local optimum at  $(x^*, z^*)$ . Let the Lagrangian multipliers for constraints (10.17) and (10.18) be  $v_i^* \geq 0$ ,  $1 \leq i \leq n$ , and  $u_j^* \geq 0$ ,  $1 \leq j \leq m$ , respectively. Then, in addition to the feasibility conditions (10.17) and (10.18) the following conditions should be satisfied at  $(x^*, z^*)$ :

$$\sum_{i=1}^n v_i^* = 1, \quad (10.19)$$

$$\text{s.t. } \sum_{i=1}^n 2v_i^*(a_i - x^*) - \sum_{j=1}^m u_j^* c_j = 0, \quad (10.20)$$

$$v_i^* (\|a_i - x^*\|^2 - z^*) = 0, \quad 1 \leq i \leq n, \quad (10.21)$$

$$u_j^*(c_j x^* - b_j) = 0, \quad 1 \leq j \leq m. \quad (10.22)$$

In addition, a local optimum either lies on the boundary of the feasible region (Case b) or not (Case a). These two cases are analyzed below to reveal the properties of local optima of (10.15).

*Case a:* If  $x^*$  does not lie on the boundary of  $S$ , none of constraints (10.18) are binding at  $x^*$ , which in turn forces all  $u_j^* = 0$  by (10.22). In that case, (10.19) and (10.20) indicate that  $x^*$  lies in the convex hull of the  $a_i$  points,  $\text{CH}(N)$ . Furthermore, in expressing the convex combination, only the multipliers  $v_i^*$  that are associated with points that are equidistant from  $x^*$  need to be positive, due to (10.21). Equivalently,  $x^*$  can be expressed as a convex combination of the points  $a_i$  that lie on the surface of the optimal hypersphere. Since  $k + 1$  or fewer points suffice to express the convex combination of more than  $k$  points in  $\mathbb{R}^k$  (Caratheodory's theorem restated in Hadley 1964), a local optimum in  $\text{CH}(N)$  is equidistant from at least  $k + 1$  points  $a_i$ .

*Case b:* If  $x^*$  lies on the boundary of  $S$ , one or more (ignoring degenerate cases, up to  $k$ ) of constraints (10.18) are binding at  $x^*$ . Let  $d$ ,  $0 \leq d \leq k - 1$ , be the dimension of the smallest facet  $F$ ,  $F \subset S$ , on which  $x^*$  lies. Assume now that at most  $d$  of constraints (10.17) are binding at  $(x^*, z^*)$ , or equivalently, at most  $d$  of the points  $a_i$  are equidistant from  $x^*$ . Draw the projections of these equidistant points  $a_i$  on the affine space  $A$  of  $F$  (a hyperplane of the same dimension that includes  $F$ ). Since the number of such projections on  $F$  results in at most  $d$  points in  $A$ , there exists a hyperplane  $H$  of  $A$  of a dimension lower than  $d$  that passes through them. If point  $x^*$  is moved away from  $H$  by an infinitesimal distance, still lying on  $F$ , the distance from the equidistant points increases and therefore the objective function  $z^*$  increases,

thus contradicting the optimality of  $(x^*, z^*)$ . Therefore  $x^*$  should be equidistant from at least  $d + 1$  nearest  $a_i$  points.

The results of the above two cases are summarized in the following theorem:

**Theorem 3:** *The optimal solution  $x^*$  of the maximin problem (10.15) either lies on the boundary of the convex polyhedron  $S$  or in the convex hull of the  $a_i$  points  $CH(N)$ . If it lies in  $CH(N)$ ,  $x^*$  is equidistant from at least  $(k + 1)$  nearest  $a_i$ . If it lies on the boundary of  $S$ ,  $x^*$  is equidistant from at least  $(d + 1)$  nearest  $a_i$ , where  $d$  is the dimension of the smallest facet on which  $x^*$  lies.*

Similar to the case of the maxisum problem on a network as formulated in relation (10.1), the above theorem reduces the feasible region to a finite candidate set of solutions containing the optimal point of the maximin problem. These candidate points are either within  $CH(N)$ , analogous to Church and Garfinkel’s bottleneck points, or remote points of the boundary of the feasible region, analogous to the leaves of a network.

The above theorem suggests a method for identifying candidate points on  $CH(N)$  and on the boundary of  $S$  as follows:

1. The point that is equidistant from every combination of  $k + 1$  points  $a_i$  is found and checked for feasibility using (10.17) and (10.18). Similarly, the center and radius of the hypersphere that passes through these  $k + 1$  points is found and checked if the center is in  $S$  and there are no points in the interior of the hypersphere.  $\binom{n}{k + 1}$  combinations of points  $a_i$  are considered and for each one of them a system of  $k$  simultaneous linear equations is solved for the  $k$  components of  $x$ .
2. The point of each facet  $F$  of the boundary of  $S$  that is equidistant from every combination of  $d + 1$  points  $a_i$  is found, where  $d$  is the dimensionality of  $F$ . For each facet  $F$  of dimensionality  $d$ ,  $\binom{n}{d + 1}$  combinations of points  $a_i$  are considered, and for each one of them a system of  $k$  simultaneous linear equations are solved, of which  $d$  linear equations stipulate that  $x$  is equidistant from  $d + 1$  points  $a_i$  and  $k - d$  equations define the facet  $F$ .

As in the algorithm for the maxisum problem, bounds are used so that not all candidate points are explicitly generated. Dasarathy and White used a lower bound  $L$  and an upper bound  $U$  on the global  $z^*$  to eliminate facets from further consideration and to forgo the feasibility test if a generated point in  $CH(N)$  has an objective value  $z$  that falls outside these bounds. As lower bound on the global value of the objective  $z^*$ , the objective value of the current best solution is used. A good initial lower bound  $L_0$  can be obtained by evaluating all extreme points  $e_j$  of  $S$ , and at the same time taking care of the examination of 0-dimensionality facets:

$$L_0 = \max_j \min_i \|a_i - e_j\|^2. \tag{10.23}$$



Dasarathy and White computed an upper bound  $U$  on the global  $z^*$  by maximizing the Lagrangian of the problem for any nonnegative multipliers  $v_i$ ,

$$U = \max_{(x \in S, z)} \left\{ z + \sum_i v_i (\|a_i - x\|^2 - z) \right\}. \tag{10.24}$$

By letting  $\sum v_i = 1$ , they developed an efficient algorithm for computing  $U$ . They also used upper bounds on the objective value  $z$  on facets in an effort to eliminate them. A facet  $F$  is eliminated from further consideration if the square distances between some  $a_i$  and all the extreme points of  $F$  are smaller than the current best objective value  $L$ . Similarly, an upper bound on the objective value  $z$  on facet  $F$  is

$$\min_{i \in N} \left\{ U_i(F) = \max_{e_j \in F} \|a_i - e_j\|^2 \right\}. \tag{10.25}$$

Although Dasarathy and White provided an algorithm for a convex polyhedron  $S$  in three dimensions ( $k = 3$ ), a general algorithm is presented below for any  $k \geq 2$ . This maximin algorithm below can be easily modified for other distance metrics and for weighted distances.

**Algorithm 2: Maximin Problem in a Convex Polyhedron**

- Step 1:* Find the lower bound  $L_0$  with the corresponding extreme point  $e_i$  of  $S$  and the upper bound  $U$  on the global  $z^*$  using (10.23) and (10.24), respectively.  $L \leftarrow L_0, x^* \leftarrow e_i$  ( $L$  keeps track of the current best local optimum,  $L \leq z^* \leq U$ ).
- Step 2:* (Search for the best local optimum interior to  $CH(N)$ .) Consider all combinations of the points  $a_i$  taken  $k + 1$  at a time. For each one of these combinations find the point  $x$  that is equidistant from the points  $a_i$  and its square distance  $z$ . If  $L < z \leq U$  and  $x$  is feasible  $L \leftarrow z, x^* \leftarrow x$ . Repeat Step 2 for all  $\binom{n}{k + 1}$  combinations of the  $a_i$  points.
- Step 3:* (Search for local optima on the boundary of  $S$ .) Set  $d \leftarrow k - 1$ .
- Step 3.1:* For every facet  $F$  of dimensionality  $d$ , repeat: If a point  $a_i$  exists,  $1 \leq i \leq n$ , such that  $U_i(F) \leq L$ , eliminate  $F$  from further consideration. If  $F$  is not eliminated, consider all combinations of the  $a_i$  points taken  $d + 1$  at a time. For each one of these combinations, find the point  $x \in A$  ( $A$  is the affine space of  $F, F \subset A$ ) that is equidistant from the points  $a_i$  and its square distance  $z$ . If  $L < z \leq U$  and  $x \in F$ , set  $L \leftarrow z, x^* \leftarrow x$ .
- Step 3.2:* Set  $d \leftarrow d - 1$ . If  $d > 1$ , go to Step 3.1.
- Step 4:* The optimal solution is  $(x^*, L)$ .

*Algorithm 2* assumes that the facet structure of  $S$  is known. For a 3-dimensional space, Dasarathy and White described an  $O(m^2 \log m)$  algorithm to obtain the face structure. The worst-case complexity of *Algorithm 2* is  $O(n^{k+2})$ . For a 2-dimensional feasible region, there is a lower worst-case complexity algorithm that utilizes the Voronoi diagram of the  $a_i$  points.

For  $S \subset \mathbb{R}^2$  (i.e.,  $k = 2$ ), *Algorithm 2* can be simplified as follows. Step 1 considers all the 0-dimensional facets of  $S$  and selects the best extreme point of  $S$  as a starting solution, and its  $z$ -value as the starting lower bound on the global  $z^*$ . Since the nearest  $a_i$  point to a vertex of a convex polygon having  $m$  edges can be determined in  $O(\log^2 n)$  time after  $O(n \log n)$  preprocessing (Shamos and Hoey 1975), Step 1 can be executed in  $O(m \log^2 n + n \log n)$  time. The local optima sought in Step 2 are among the Voronoi vertices of the Voronoi diagram of the  $a_i$  points. If a Voronoi vertex is in  $S$  it is a local optimum. Shamos and Hoey (1975) provided an  $O(\log m)$  algorithm for determining if a given point is within an  $m$ -edge polygon. The  $O(n)$  vertices of the Voronoi diagram can be generated in  $O(n \log n)$  time and tested for feasibility in  $O(n \log m)$  time. In Step 3, only the edges of the polygon  $S$  ( $d = 1$ ) have to be searched for local optima. The points of the edges that are equidistant from  $a_i$  points taken two at a time are the intersections of the Voronoi edges with the edges of  $S$ . A Voronoi edge can intersect the boundary of  $S$  at most twice. Using a binary search, Dasarathy and White found the intersections of the Voronoi edges with the edges of  $S$  in  $O(n \log m)$  time.

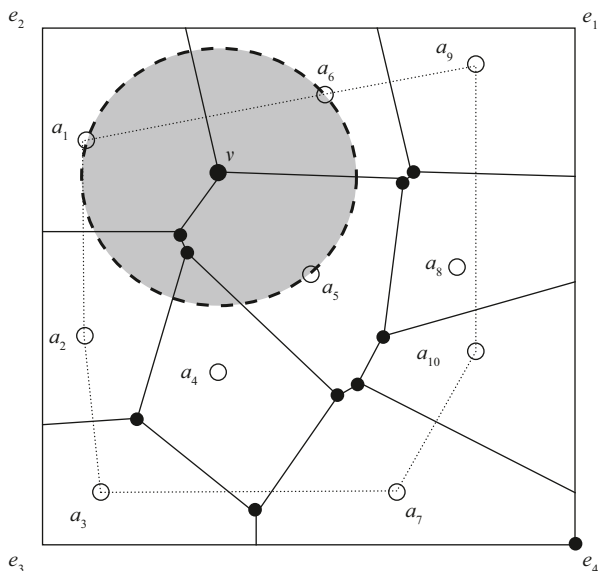
In summary, the optimum point of the maximin problem in a convex polygon  $S$  can be a vertex of  $S$ , a Voronoi vertex, or an intersection of a Voronoi edge with an edge of  $S$ . The required effort to solve it is  $O(m \log^2 n + n \log n + n \log m)$ .

Figure 10.4 shows a set  $N$  of 10 points,  $a_1, \dots, a_{10}$ , within a square region  $S$ . The boundary of  $CH(N)$  is displayed by dotted line segments. The Voronoi diagram of the points consists of 10 vertices and 19 edges. There are 8 intersections of Voronoi edges with the edges of  $S$ . The solution to (10.14), i.e., the maximin point in  $CH(N)$ , is vertex  $v$  of the Voronoi diagram, which is the center of the largest empty circle, shown in Fig. 10.4. The extreme point  $e_4$  is the maximin point in  $S$ , solution to (10.15), with objective value  $\|a_7 - e_4\|$ .

### 10.2.2.3 Drezner and Wesolowsky (1980): Weighted Maximin Problem with Maximum Distance Constraints

The difference in this contribution compared to the previous work is that positive weights  $w_i$  are assigned to customers and the solution method does not search the feasible region for local optima to find the best one(s), but instead progressively reduces the feasible space to trap the global optimum in an infinitesimal area. The feasible region is a convex bounded planar area defined by the intersection of circles, each having at their center a customer point  $a_p$  and radius  $r_i$  representing the maximum distance the facility can be from customer  $i$  with  $i \in N$ . Clearly, the authors had in mind the location of a semi-obnoxious facility, which is pushed away by each customer to a different degree depending on its weight, but at the same time is wanted within certain distance from each customer. The problem can be formulated as

**Fig. 10.4** Example problem using the Voronoi diagram



$$\max_{x \in S} \min_{i \in N} w_i d(i, x), \tag{10.26}$$

where  $d(i, x) = \| a_i - x \|$  is the Euclidean distance between point  $a_i$  and  $x \in \mathbb{R}^2$  and  $S$  is a set of  $n$  maximum distance constraints,  $S = \{x | d(i, x) \leq r_i, 1 \leq i \leq n\}$ .

The solution methodology is graphical in nature and is speeded up by a bisection search. Consider some objective value  $z$ ,  $z = \min_{i \in N} w_i d(i, x)$ . The points of the plane with better objective value than  $z$  are outside of the union of circles having centers  $a_i$  and radii  $z/w_i$ , or  $C(z) = \{x | d(i, x) \geq \frac{z}{w_i}\}$ . Starting with a relatively small value of  $z$ , one can solve the problem interactively by increasing  $z$  until the last point in  $S$  is covered, or  $S \cap C(z)$  is an infinitesimally small area. In fact, Brady and Rosenthal (1980) used this interactive graphical approach on the computer to solve constrained minimax problems. Instead of an interactive approach, Drezner and Wesolowsky used an efficient bisection search as follows. At some iteration, let  $\underline{z}$  be the objective value of the best solution found so far (lower bound on  $z^*$ ) and  $\bar{z}$  be an upper bound on  $z^*$ . A new objective value is generated and a procedure is used to find out if a point  $x$  exists in  $S \cap C(z)$  with that  $z$ -value. If it does,  $\bar{z} \leftarrow z$ , otherwise,  $\underline{z} \leftarrow z$ . The iterations continue until  $(\bar{z} - \underline{z})$  becomes smaller than a small preset constant. The solution  $x$  associated with  $\bar{z}$  is close to the best point  $x^*$  within an approximation.

### 10.3 Impact of the Original Papers

The above classical contributions stimulated a large body of research in undesirable facility location that complemented the existing (desirable) location literature. Up to that time, pull objective location models, such as minisum (median) and minimax

(center), dominated the location literature. The introduction of the push objective location models leveled the field of location science and opened it to new methods, applications and location problems. An outstanding example has been the launch of a new class of location problems that utilize a combination of push and pull objectives to find locations that best trade off the conflicting objectives. This section describes the immediate impact of the original papers on the location literature in the period of 10–15 years that followed as well as the major works that were afterwards influenced by the classical works and contributed to the undesirable location literature.

### 10.3.1 *The Impact of Church and Garfinkel's Contribution*

The pioneering work of Church and Garfinkel initiated the field of undesirable facility location by introducing the *maxisum* location problem and distinguishing it from the existing (desirable) location problems of the time. As Goldman (2006) notes, such a “three letter change” (substitution of max for min) might seem innocuous, but in fact substantially increases the difficulty to carry out the optimization. Church and Garfinkel showed that the new problem is nonconvex and thus may have many local optima, so that it is necessary to generate all or at least a subset of them by improving bounds on the optimal objective value to find the global optimum. In fact, many algorithms that were developed later for variations of the *maxisum* and the *maximin* objectives resemble Church and Garfinkel's algorithm. Similarly, the “existence of a finite candidate set of points containing the global optimum” that resonates throughout the undesirable location literature originated in this paper. Among the points in that set are local optima that arise due to the nonconvex property of the undesirable facility location problem which, in turn, render the “Hakimi property” of a network invalid. Finally, new terms were coined in their paper to enrich the location lexicon: “obnoxious” and “semi-obnoxious” facilities and “bottleneck points” of a network. In the remainder of this subsection we will include early contributions that built on the work of Church and Garfinkel.

Ting (1984) dealt with the *maxisum* problem on trees and developed an  $O(n)$  algorithm by using a special data structure. This is an improvement over Church and Garfinkel's  $O(n^2)$  algorithm for trees. Minieka (1983) addressed the unweighted *maxisum* problem and essentially developed the same algorithm as Church and Garfinkel to find the *antimedial* of the network, as he named the solution of the *maxisum* problem. In the same paper, Minieka studied another max-type problem,  $\max_{x \in G} \max_{i \in N} d(i, x)$ , whose solution named the *anticenter* of the network.

Hansen et al. (1981) considered a more general *maxisum* problem on a continuous and bounded feasible region  $S$  with  $S \in \mathbb{R}^2$ . By modeling the nuisance from the obnoxious facility located at  $x$  to a population center  $i$  as decreasing and continuous function of their distance,  $D_i[d(i, x)]$ , they actually formulated a minisum model, named the *anti-Weber problem*, as the counterpart of the Weber problem with the objective

$$\min_{x \in S} T(x) = \sum_{i \in N} D_i[d(i, x)], \quad (10.27)$$

where  $d(i, x)$  can be any distance metric, of which the most commonly used are the Euclidean, rectilinear and Tchebycheff metrics.

Similar to Church and Garfinkel's (1978) Theorem 1 above, Hansen et al. established a theorem that reduces the feasible region that contains the optimal location to the union of two sets. The first set, analogous to the set of bottleneck points of a network, consists of the points of  $S$  that are in the convex hull  $CH(N)$  of the points, i.e.,  $S \cap CH(N)$ . The second set, analogous to the leaf nodes of a network, consists of the points of  $S - CH(N)$  that are remote from  $CH(N)$ . A point  $y \in Y$  is said to be remote from set  $X$  if there exists  $x \in X$  such that the straight halfline starting from  $x$  and passing through  $y$  contains no point of  $Y$  beyond  $y$ . The results of this theorem are used by Hansen et al. to rationalize the locational pattern of nuclear power plants in France. Some power plants are at interior locations while many others are located at the border of France with Germany and Belgium or on the Atlantic coast. They solved the anti-Weber problem by a branch-and-bound method, similar to the *Big Square-Small Square* algorithm developed earlier by the same authors for the generalized Weber problem.

For the special case where  $D_i$  is a linear function of distance  $d(i, x)$ , relation (10.27) reduces to the (ordinary) maxisum objective on the plane:

$$\max_{x \in S} T(x) = \sum_{i \in N} w_i d(i, x). \quad (10.28)$$

For the maxisum problem (10.28), Hansen et al. reduced the set containing the optimal solution even further by excluding all interior points of  $S$ :

**Theorem 4:** *For the maxisum problem in the plane there exists an extreme point of the convex hull of the feasible region  $S$  that is optimal.*

The proof follows directly from the convexity property of the objective function. When the feasible region is approximated by a nonconvex polygon  $S$  with  $m$  vertices, Melachrinoudis and Cullinane (1986a) described a simple  $O(mn)$  algorithm for finding the weighted maxisum point by evaluating the vertices of  $CH(S)$ .

Theorem 4 states that the maxisum point is at remote points of the boundary of the feasible region. This result is analogous to Church and Garfinkel's result for trees where the optimal point is one of the leaves of the tree. The feasible region  $S$  therefore has to be bounded, otherwise the optimal solution of the maxisum problem is "out at infinity." It is even possible that the optimal location is at an existing facility point as Eiselt and Laporte (1995) illustrated in the following example. Consider the case of a square feasible region with four equally weighted customer points at the corners of the square. According to Theorem 4, the optimal maxisum locations are at the extreme points of the feasible region which coincide with the customer locations. Prescribing always a boundary solution and sometimes even a customer's location for the undesirable facility does not make the maxisum model very attractive for use in the plane. However, the maxisum objective is very useful

in a multiobjective setting when it is combined with a pull objective as described later in this section.

### 10.3.2 *The Impact of the Original Maximin Location Papers*

The original papers on the maximin location problem, directly or indirectly, had an impact on the undesirable facility location works that followed during the 1980s. A number of variations of the maximin problem with Euclidean distances have been solved using a solution approach similar to Dasarathy and White's for generating local optima by using the Karush-Kuhn-Tucker optimality conditions. Melachrinoudis (1985) and Melachrinoudis and Cullinane (1985) extend the weighted maximin problem to nonconvex regions and to regions that enclose forbidden areas, respectively. They provided an example for locating a toxic dump in the state of Massachusetts, which was represented by a nonconvex bounded planar region with forbidden areas for the facility around cities, wetlands, rivers, lakes, and ecosystems. The forbidden areas were approximated by the union of circles. Weights assigned to the customers, such as cities and towns, reflected the population size. The most important customer point, the city of Boston ( $i = 6$ ), was assigned a weight of 1, while the weight of the population center  $i$  was calculated relative to Boston by the formula  $w_i = (N_i/N_6)^{-1}$ , where  $N_i$  denotes the population of city  $i$ .

Since it has not been elaborated in the location literature, it is important to note here that unlike the weight of a desirable facility, the maximin weight is a decreasing function of the degree of incompatibility between a customer and the facility. For example, consider the simple case of a one-dimensional feasible region in the interval  $[0, 1]$ , with customer  $A$  at point 0 having weight 1 and customer  $B$  at point 1 having weight 3. The maximin point is at 0.75, meaning that the customer with the lower weight ( $A$ ) pushes the facility further away than the customer with the higher weight. By the way, the minimax point happens to be the same in this very small example, thus the customer with the higher weight pulls the desirable facility closer to it.

To explain this counterintuitive property of the maximin weights, consider the generalization of the weighted maximin problem of (10.26), where  $S \subset \mathbb{R}^k$ ,  $k \geq 1$ . Let the optimal point be  $x^*$  and the optimal objective value be  $z^*$ . Theorem 3, generalized for the weighted maximin problem, states that  $x^*$  is equidistant (in a weighted sense) from a subset of customers,  $N'$ , and  $|N'|$  depends on the dimensionality of  $S$  and on whether  $x^*$  lies in  $CH(N)$  or on the boundary of  $S$ . Theorem 3 and (10.26) imply that  $d(i, x) = z^*/w_i$ ,  $i \in N'$ , and  $d(i, x) > z^*/w_i$ ,  $i \in N - N'$ . Therefore, the distance of the maximin point from every point  $i \in N'$  is inversely proportional to its weight  $w_i$ , while the distance from each of the remaining points  $i \in N - N'$  is greater than a lower bound that is inversely proportional to its weight  $w_i$ .

The above property of the maximin weights is probably the reason some authors do not consider weights with the maximin problem. Karkazis (1988) studied an un-

weighted Euclidean maximin problem in which the facility was to be located within a polygonal region  $S$  but as far away as possible from any point of the boundary of protected areas. These were more generally defined forbidden regions than in Melachrinoudis and Cullinane (1985). Although there were no apparent customers, the optimization approach—similar to the geometrical approach of Shamos and Hoey (1975)—suggests that the customers constitute an infinite set represented by the boundaries of the protected areas. The solution amounts to finding the largest (empty) circle that contains no points of the protected areas yet whose center is in  $S$ .

Melachrinoudis and Smith (1995) extended the Voronoi method of Dasarathy and White (1980) and developed an  $O(mn^2)$  algorithm for the weighted maximin problem. For two points  $a_k, a_\ell$  having weights  $w_k, w_\ell$  such that  $w_k > w_\ell$ , the loci of weighted equidistant points is the Apollonius circle. This circle has center on the line connecting  $a_k$  and  $a_\ell$  at point  $o_{k\ell}$ , and radius  $\gamma_{k\ell}$ , both expressed in terms of the weights ratio,  $r_{k\ell} = w_\ell/w_k$ , in (10.29). The edges of the weighted Voronoi diagram are therefore circular segments or whole circles.

$$o_{k\ell} = \frac{a_k - r_{k\ell}^2 a_\ell}{1 - r_{k\ell}^2}, \quad \text{and} \quad \gamma_{k\ell} = \frac{r_{k\ell} \|a_k - a_\ell\|}{1 - r_{k\ell}^2}. \quad (10.29)$$

Melachrinoudis and Cullinane (1986b) developed a minimax model for undesirable facility location. The model seeks a facility location that minimizes the maximum weighted inverse square distance over all customers, or

$$\min_{x \in S} \max_{i \in N} \{w_i/d^2(i, x)\}. \quad (10.30)$$

The objective is justified in many situations since the concentration of pollutants such as noise or radiation follows the inverse square law, see Poynting's Theorem in Lipscomb and Taylor (1978). A customer weight represents the degree of incompatibility between the customer and the facility and unlike with the maximin objective, the higher the weight of the customer, the further away the facility is pushed. Similar to the one for the maximin problem, an  $O(n^4)$  algorithm was developed for a convex polygonal region  $S$ , while for a nonconvex feasible region, composed of many disjointed nonconvex sets representing irregular land and islands, a graphical computer procedure was suggested as in Drezner and Wesolowsky (1980). The minimax problem in (10.30) was shown by Erkut and Öncü (1991) to be equivalent to the maximin problem with weights  $w_i^{-1/2}$ , implying the above mentioned inverse relationship between the magnitude of a maximin weight and the degree of incompatibility it represents. Their proof used a more general formulation with an arbitrary exponent, i.e.,  $d^q(i, x)$ , in which case the weights in the equivalent maximin problem were  $w_i^{-1/q}$ . The negative exponent explains the opposite behavior of weights in the two problems.

A minimax objective was also developed by Hansen et al. (1981) for the location of an undesirable facility in which, however, a general continuous and decreasing function of distances was used. The authors named the problem the *anti-Rawls*

*problem*, since the objective can be characterized as an equity measure to the worst-off customer. When the function of distances is linear, the minimax reduces to the maximin problem. The authors used a simple method called *Black and White*, which is similar to the  $\epsilon$ -approximation method of Drezner and Wesolowsky (1980).

For the rectilinear maximin location problem in a convex polygon  $S$ , Melachrinoudis and Cullinane (1986a) and Melachrinoudis (1988) developed optimality properties similar to those described by Dasarathy and White, except that the convex hull  $CH(N)$  is replaced by the smallest rectangle  $H$  whose sides are parallel to the two coordinate directions and encases all customers. Thus, local optima exist in the union of two sets, the boundary of  $S$  and  $S \cap H$ .

### 10.3.3 Major Contributions on Undesirable Facility Location that Followed the Classical Works

Following the classical contributions, numerous papers on undesirable facility location problems have been published in the last thirty years. A few of them, which built on the classical contributions and those on which the classical contributions had a direct or indirect impact, were reviewed in the previous two subsections. In this subsection, a short survey of major works that followed the classical contributions is presented. This short and by no means all-inclusive survey includes representative works with similar distance metrics and solution spaces as well as multiobjective approaches. A comprehensive survey of undesirable facility location models, though less contemporary, can be found in Erkut and Neuman (1989) Eiselt and Laporte (1995) and Plastria (1996).

The classical contribution of Church and Garfinkel (1978) itself was followed by only an algorithmic refinement. Their algorithm requires  $O(mn^2)$  time to find the weighted maxisum point on a general network. By making use of the observation that  $T(x)$  in (10.1) is a piecewise linear and concave function of  $x$  on a given arc, Tamir (1991) briefly suggested an improvement leading to an  $O(mn)$  algorithm. Colebrook et al. (2005) described a complete algorithm of this improved complexity by making use of the above concavity property and by computing efficiently in  $O(n)$  time a new upper bound of  $T(x)$  over an arc. Their experimental results showed that the improved algorithm, compared to Church and Garfinkel's, ran in about half the time and processed about 25% fewer arcs due to tighter upper bounds on the arcs.

The unweighted maximin problem on a network admits a trivial solution, in that the optimal is the midpoint of the longest arc of the network. The weighted maximin problem on a network,  $\max_{x \in G} \min_{i \in N} w_i d(i, x)$ , has similar properties to the maxisum problem. It is nonconvex and it has a unique local optimum on each arc (Melachrinoudis and Zhang 1999). In addition to the set of arc bottleneck points, the finite set of candidates includes the set of center bottleneck points. For a complete coverage of finite dominating sets to the maximin and other location problems on



networks with general monotone or non-monotone distance functions, see Hooker et al. (1991). The algorithm for solving the maximin problem on networks is similar to Algorithm 1: searching arcs for local maxima, updating the lower bound and eliminating arcs using upper bounds on arcs. For each unfathomed arc, a linear program with two variables can be constructed which can be solved very efficiently by an  $O(n)$  algorithm. Melachrinoudis and Zhang (1999) and Berman and Drezner (2000) independently provided  $O(mn)$  algorithms by using  $O(n)$  algorithms for linear programming problems of Dyer (1984) and Megiddo (1982), respectively.

The first paper on the maximin problem using the rectilinear metric was published by Drezner and Wesolowsky (1983). Since the rectilinear distance is piecewise linear, the problem can be linearized. The feasible region is divided into rectangular segments by drawing horizontal and vertical lines through each customer point and a linear optimization problem is solved for each one of the  $O(n^2)$  linear programming problems. Upper bounds for each region are used to reduce the number of linear programs that need to be solved. Mehrez et al. (1986) proposed a new upper bound for that purpose which was further improved by Appa and Giannikos (1994). Sayin (2000) formulated the rectilinear maximin location problem as a mixed integer program that can be solved by any standard MIP solver. Nadirler and Karasakal (2008) simplified the mixed integer programming formulation and improved further the bounds to increase the computational performance of a branch and bound algorithm very similar to the *Big Square-Small Square* algorithm of Hansen et al. (1981) and the generalized Big Square-Small Square method of Plastria (1992).

As was mentioned earlier, undesirable facility location problems provide some service to the community and some travel may be required to and from it. Therefore, in addition to minimizing the undesirable effects on populations, the minimization of transportation cost is of interest. This gives rise to a bi-objective problem for locating undesirable facilities. Depending on the application, either the minimax or the minisum (desirable facility) objective can be combined with the maximin or maxisum (undesirable facility) objective. An advantage of solving a bi-objective undesirable facility location problem is that one can obtain the whole efficient frontier, i.e., the set of points that exhibit the complete tradeoff between the two objectives, including the two points that optimize the individual objectives. For a formal definition of efficient points and other concepts in multicriteria optimization, see Steuer (1989).

The first bi-objective problem for undesirable facility location was formulated by Mehrez et al. (1985). The authors combined the minimax and maximin unweighted objectives using rectilinear distances. They generated the whole efficient set by examining the intersections of any two lines forming the equirectilinear distances between every pair of customer points or boundary edges of the feasible region. Also using rectilinear distances, Melachrinoudis (1999) combined the minisum and maximin objectives and solved the problem by generating a series of  $O(n^2)$  linear programs as in Drezner and Wesolowsky (1983), but instead of solving the linear programs by the simplex method, he constructed the whole efficient frontier by reducing each linear program to simple variable ranges using the Fourier-Motzkin

elimination process. Brimberg and Juel (1998) used Euclidean distances in a bi-objective problem that combined the minisum objective and a second minisum objective (undesirable objective), which had the Euclidean distances raised to a negative power. They outlined an algorithm for generating the efficient set by solving the weighted-sum of the two objectives with varying weights. Skriver and Andersen (2001) solved the same problem using the Big Square-Small Square method and generated an approximation of the efficient set. A similar bi-objective model was developed by Yapicioglu et al. (2007) where the second minisum objective was modified further to model undesirable effects with distance. They approximated the effects at a distance  $d(i, x)$  from the facility as a piecewise linear and decreasing function of  $d(i, x)$ ; up to a certain distance, they argued, the obnoxious effects are constant, then decreasing with distance in a piecewise fashion, while beyond a certain distance the effects are nonexistent. Particle Swarm Optimization is used to approximate the efficient set.

Melachrinoudis and Xanthopoulos (2003) solved the Euclidean distance location problem with the minisum and maximin objectives. They developed the whole trajectory of the efficient frontier by a combination of a problem that optimizes the weighted sum of the objectives and the Voronoi diagram of the customer points. Using Karush-Kuhn-Tucker optimality conditions showed that this trajectory is not necessarily continuous and may consist of (a) a parametric curve of the weighted-sum of the objectives starting at the minisum point and ending at the boundary of its Voronoi polygon, (b) segments of the Voronoi edges as the weight of the maximin objective is increasing while the weight of the minisum objective is decreasing, and (c) segments of the boundary until the maximin point is reached.

A different type of undesirable facility location problem is the *minimal covering problem* in which the undesirable effects of a facility are evident only within certain distance from it, referred to as the circle of influence. Given  $n$  populations of size  $w_i$ ,  $i = 1, \dots, n$ , that are concentrated in  $n$  points on the plane, the location of an undesirable facility is to be found within a feasible planar region  $S$  to minimize the population covered within a certain distance  $r$  from the facility. This problem was introduced by Drezner and Wesolowsky (1994) who, in addition to the circle, determined the rectangle that contains the minimum total population. Berman et al. (1996) extended the problem to the network space. By considering the radius of the circle as a continuous variable and second objective, Plastria and Carrizosa (1999) solved the problem with two objectives. First, to maximize the radius  $r$  of a circle with center the point  $x$  at which the facility is to be located, and second, to minimize the popula-

tion covered in that circle,  $\sum_{d(i,x) < r} w_i$ . They developed polynomial algorithms for generating all efficient discs  $(x, r)$  whose number they show is finite. The trade-off information of efficient solutions can provide answers to interesting coverage questions, such as finding the facility location that minimizes the population covered within a given radius (previously defined as minimal covering problem) or finding the largest circle not covering more than a given total population. They considered a feasible region of any shape in the plane and the results can be extended to a planar network.

A more recent approach for locating an undesirable facility is with expropriation. The rationale is that in certain cases there is no point in the feasible region that is far enough from all customers to locate the undesirable facility. One possibility to resolve this issue is by buying (or compensating) some of the customers. Berman et al. (2003) introduced two models for the *location problem with expropriation*. In the first model, a location on a network was sought that maximizes the minimum distance (maximin) from the facility to the non-expropriated customer points, subject to a given expropriation budget. In the second model, the expropriation cost was minimized while ensuring that the facility is located at least certain distance away from all non-expropriated customer points. Berman and Wang (2007) added a second objective to the last model: the minimization of transportation cost. The two cost objectives were added into one, so the resulting problem is treated as a single objective problem. For a planar feasible region and rectilinear metric, they identified a finite dominating set that contains the optimal solution.

There are not many papers on undesirable facility location on networks using two objectives. Zhang and Melachrinoudis (2001) formulated the first bi-objective problem on a network by combining the maxisum objective with the maximin objective. Using the piecewise linear and concave property of both objectives on an arc, they developed fathoming rules for eliminating inefficient arcs and arc segments. Unfathomed arc segments were mapped onto the 2-dimensional objective space and a direct search was undertaken to construct the nondominated set, followed by the efficient set, which was shown to consist of discontinuous arc segments. Hamacher et al. (2002) developed several multicriteria models for undesirable facility location problems on a network with minisum and center objectives, and proposed methods for solving them.

A general model for the undesirable facility location problem with Euclidean distances in a polygonal feasible region  $S$  was presented by Saameno et al. (2006). By setting a parameter to certain values the model reduces to problems already defined: maximin, maxisum and bi-objective maximin/maxisum problems. In addition, the model reduces to the  $r$ -anticentrum problem, which maximizes the weighted sum of distances between the undesirable facility and its  $r$  closest customers. The maximin and maxisum problems are special cases of the  $r$ -anticentrum problem for  $r = 1$  and  $r = n$ , respectively. The authors generalized the properties of local optima developed by Dasarathy and White (1980) and Melachrinoudis and Smith (1995) for the whole class of objectives. They identified a finite dominating set consisting of the set of vertices of  $S$ ,  $V$ , the set of intersections of weighted bisectors (10.29) of customer points with the edges of  $S$ ,  $W$ , and the set of intersections of the weighted bisectors taken two at a time,  $I$ . The finite dominating set was obtained in  $O(nm^2 + m^4)$ . In their algorithm they generated all candidate points in the set, eliminated some of them using the Lipschitzian property of the objective function, and evaluated the remaining points to obtain the optimal solution.

In addition to the above classes of models for undesirable facility location there are other models, such as multifacility, discrete, and location and routing models, which followed the classical contributions; we cannot elaborate upon these papers due to the limited space in this chapter. For the interested reader, some excellent

papers and surveys are available. For the *p*-dispersion problem, Chandrasekaran and Daughety (1981), Kuby (1987), Erkut (1990), and Pisinger (2006); for the *p*-defense problem, Moon and Chaudhry (1984), Kincaid (1992), and Klein and Kincaid (1994); for generic *discrete multifacility* undesirable location problems, Chhajed and Lowe (1994). For locating multiple undesirable facilities on graphs using maxisum and maximin objectives, Tamir (1988, 1991); using coverage objectives, Berman and Huang (2008); using expropriation, Berman and Wang (2007). A recent survey for *location and routing* problems that includes undesirable facility location and routing of hazardous wastes can be found in Nagy and Salhi (2007).

## 10.4 Summary and Outlook

The advent of more stringent environmental standards, the resurgence of environmental groups and a greater awareness of the public of the potential dangers of pollution in the early 1970s generated a research need for systematically locating polluting and environmentally hazardous facilities. Undesirable facility location research began with the pioneering work of Church and Garfinkel (1978). Their work on the maxian (maxisum) problem was analyzed in detail followed by the first works on the Euclidean maximin problem of Dasarathy and White (1980) and Drezner and Wesolowsky (1980).

Church and Garfinkel (1978) first formulated a model for locating an undesirable facility on a network by replacing the *min* with the *max* operator in the median model that had dominated the location literature since the seminal work of Hakimi (1964). Unlike the median model, they demonstrated that the maxian model is hard to solve because it is nonconvex and typically has many local optima, a characteristic of undesirable facility location problems. They showed that local optima occur on the cycles (bottleneck points) and on the leaves of the network and developed a simple solution procedure that decomposes the network into its arcs in the search for the global optimum. Arcs were considered for fathoming using bounds, and the local maxisum point was found on unfathomed arcs by utilizing the concavity property of the objective function. This algorithm became a standard for future algorithms in undesirable facility location. For example, instead of arcs, parts of the feasible region are considered for fathoming in the Big Square-Small Square method or individual facets of the feasible region in Dasarathy and White's (1980) algorithm that partially enumerates local maxima. The work of Church and Garfinkel had an enormous impact by stimulating research and establishing the field of undesirable facility location in the 1980s.

Undesirable facility location in the continuous space has its origins in the *largest empty circle* problem, one of several problems Shamos (1975) and Shamos and Hoey (1975) studied in computational geometry. Dasarathy and White (1980) were the first to define the maximin problem using Euclidean distances as a nonconvex and nonlinear problem, derive the properties of the local optima using the Karush-Kuhn-Tucker optimality conditions, and identify a finite dominating set. They

proved that the global optimum is either in the convex hull of the customer points or on the boundary of the feasible region and developed an algorithm for searching those parts of the feasible region. As in Church and Garfinkel (1978), they used upper bounds to fathom facets of the feasible region and updated the lower bound on the optimal objective value using the current best feasible solution. For a 2-dimensional feasible region, they extended Shamos and Hoey's Voronoi diagram approach to search for local optima at the boundary of the feasible region.

Drezner and Wesolowsky (1980) first formulated the weighted maximin location problem. The 2-dimensional feasible region is the intersection of circles each having its center at a customer point and radius equal to the maximum distance the undesirable facility can be located away from that customer, implying that the facility performs some service to the customer and has to be within reach. Their solution approach is different from previous ones. It is graphical in nature and is implemented on the computer with a bisection search of the feasible region. Although the bisection search seems very efficient for this feasible region, it has not been used generally. The contributions by Dasarathy and White and by Drezner and Wesolowsky incited a large body of research in undesirable facility location using the maximin objective with various distance metrics and solution spaces. Unlike the maxisum objective, the maximin objective does not limit the optimum to the boundary and excludes customer points for locating the undesirable facility. Its use represents obvious advantages over the maxisum objective in continuous spaces.

Important works that followed the original papers were analyzed with special attention given to location models or methods that extended the classical contributions, such as considering single facility location models on network and planar space and under multiple objectives. It was not the purpose of this chapter to survey the literature on undesirable facility location, and therefore many important papers have not been included. A complete survey of this area is important and its time is due, so therefore it is suggested that such an effort be undertaken in the near future.

Regarding future research directions, consider what has been accomplished so far, what has not and what can be accomplished given the technological advances and the changing needs of society. The location literature is full of elegant mathematical models which admit neat solution algorithms. As ReVelle and Eiselt (2005) point out, the "location field is active from a research perspective but when it comes to applications it appears to be a significant deficit, at least as compared to other, similar fields." It is known that real life problems are complex with nasty feasible regions and multiple objectives that may not be necessarily functions of straightforward distance metrics such as Euclidean or rectilinear. When it comes to undesirable facilities, pollution density or its effects often are neither symmetric nor linear functions of distance. Very often, a real feasible region is not a simple polygonal area but the union of many disjoint regions.

The parameters of the problem, such as customer weights, may change over time depending on the population size, technological developments, and legislation for hazardous wastes and facility standards. Multiple stakeholders and decision makers are usually involved in undesirable facility location decisions; therefore, more realistic, integrated and robust location models need to be developed that relate to the

practitioners' concerns. New technological tools such as geographical information systems are readily available, together with versatile optimization tools and vast computer power to make the task easier. Without discouraging the development of elegant mathematical models that admit creative solution procedures, researchers should be encouraged to tackle real-life problems with creative formulations, even if they have to solve them for a near-optimal solution by a standard optimization software package or a heuristic procedure.

Although in this chapter we reviewed single facility location on continuous or network spaces, discrete location models appear to be more realistic from a practitioner's decision making point of view, maybe because it is more natural to compare the merits of given sites rather than find one among an infinite number of possible sites. As a strategic decision, the facility location process usually involves two stages: one approach is to evaluate many candidate sites in the first stage and come up with a few using constraints and minimum requirements, and in the second stage to select a site using multiple criteria optimization, as in Min et al. (1997); another approach is to generate a small number of candidate sites in the first stage with analytical models and in the second stage to use discrete multiobjective tools to select the candidate site, as suggested by Erkut and Neuman (1989) and Plastria (1992).

An interesting non-geographical area in which undesirable facility location models could be applied is product design; see, e.g., Goldman (2006). The attributes of a product, such as physical dimensions, expected lifetime, and cost can be regarded as coordinates in the attribute or design space. Given the existing products in the market (points in space), a company may want to design a new product to differentiate from the existing ones as an alternative to purchase, yet not make it very different. The new product has the properties of a semi-desirable facility that needs to be located in the design space away from existing points but within reach.

Finally, location researchers should adapt their models to fill new needs of the society and use tools made available by new technologies. Some examples are applications in telecommunications and especially wireless networks, homeland security, environment change and global warming (Francis 2008), and use of geographical information systems (Murray and Church 2008).

## References

- Appa GM, Giannikos I (1994) Is linear programming necessary for single facility location with maximin of rectilinear distance? *J Oper Res Soc* 45:97–107
- Ahuja RK, Magnanti TL, Orlin JB (1993) *Network flows: theory, algorithms and applications*. Prentice Hall, Englewood Cliffs
- Berman O, Drezner Z (2000) A note on the location of an obnoxious facility on a network. *Eur J Oper Res* 120:215–217
- Berman O, Huang R (2008) The minimum weighted covering location problem with distance constraints. *Comput Oper Res* 35:356–372
- Berman O, Wang Q (2007) Locating semi-obnoxious facilities with expropriation: minisum criterion. *J Oper Res Soc* 58:378–390

- Berman O, Drezner Z, Wesolowsky GO (1996) Minimum covering criterion for obnoxious facility location on a network. *Networks* 28:1–5
- Berman O, Drezner Z, Wesolowsky GO (2003) The expropriation location model. *J Oper Res Soc* 54:769–776
- Brady SD, Rosenthal RE (1980) Interactive computer graphical solutions of constrained minimax problems. *AIIE Trans* 12:241–248
- Brimberg J, Juel H (1998) A bicriteria model for locating a semi-desirable facility in the plane. *Eur J Oper Res* 106:144–151
- Chandrasekaran R, Daughety A (1981) Location on tree networks:  $p$ -centre and  $n$ -dispersion problems. *Math Oper Res* 6:50–57
- Chhajer D, Lowe TJ (1994) Solving structured multifacility location problems efficiently. *Transp Sci* 28:104–115
- Church RL, Garfinkel RS (1978) Locating an obnoxious facility on a network. *Transp Sci* 2:107–118
- Colebrook M, Gutierrez J, Sicilia J (2005) A new bound and an  $O(mn)$  algorithm for the undesirable 1-median problem (maxian) on networks. *Comput Oper Res* 32:309–325
- Dasarathy B, White LJ (1980) A maxmin location problem. *Oper Res* 28:1385–1401
- Drezner Z, Wesolowsky GO (1980) A maximin location problem with maximum distance constraints. *AIIE Transactions* 12:249–252
- Drezner Z, Wesolowsky GO (1983) Location of an obnoxious facility with rectangular distances. *J Reg Sci* 23:241–248
- Drezner Z, Wesolowsky GO (1994) Finding the circle or rectangle containing the minimum weight of points. *Locat Sci* 2:83–90
- Dyer ME (1984) Linear time algorithms for two and three-variable linear programs. *SIAM J Comput* 13:31–45
- Eiselt HA, Laporte G (1995) Objectives in location problems. In: Drezner Z (ed) *Facility location, a survey of applications and methods*. Springer, Berlin, pp 151–180
- Erkut E (1990) The discrete  $p$ -dispersion problem. *Eur J Oper Res* 46:48–60
- Erkut E, Neuman S (1989) Analytical models for locating undesirable facilities. *Eur J Oper Res* 40:275–291
- Erkut E, Öncü TS (1991) A parametric 1-maximin location problem. *J Oper Res Soc* 42:49–55
- Francis RL (2008) A discussion of some location problems global warming can cause. *Abstract, ISOLDE* 11:139
- Goldman AJ (2006) Optimal facility-location. *J Res Nat Inst Stand Technol* 111:97–101
- Goldman AJ, Dearing PM (1975) Concepts of optimal location for partially noxious facilities. *Bull Oper Res Soc Am* 23(Suppl 1):B-31
- Hadley G (1964) *Nonlinear and dynamic programming*. Addison-Wesley, Reading
- Hakimi SL (1964) Optimal location of switching centers and the absolute centers and medians of a graph. *Oper Res* 12:450–459
- Hamacher HW, Labbé M, Nickel S, Skriver AJV (2002) Multicriteria semi-obnoxious network location problems (MSNLP) with sum and center objectives. *Ann Oper Res* 110:33–53
- Hansen P, Peeters D, Thisse JF (1981) On the location of an obnoxious facility. *Sistemi Urbani* 3:299–317
- Hooker JN, Garfinkel RS, Chen CK (1991) Finite dominating sets for network location problems. *Oper Res* 3:100–118
- Karkazis J (1988) The general unweighted problem of locating obnoxious facilities on the plane. *Belgian J Oper Res Stat Comput Sci* 28:43–49
- Karkazis J, Boffey C (1994) Modeling pollution spread. *Stud Locat Anal* 7:91–104
- Karkazis J, Papadimitriou B (1992) A branch and bound algorithm for location of facilities causing atmospheric pollution. *Eur J Oper Res* 58:363–373
- Kincaid RK (1992) Good solutions to discrete noxious location problems via metaheuristics. *Ann Oper Res* 40:265–281
- Klein CM, Kincaid RK (1994) The discrete anti- $p$ -center problem. *Transp Sci* 28:77–79

- Kuby MJ (1987) Programming models for facility dispersion: the  $p$ -dispersion and maximum dispersion models. *Geogr Anal* 19:315–329
- Lipscomb DM, Taylor AC Jr (1978) Noise control, handbook of principles and practices. Van Nostrand Reinhold, New York
- Megiddo N (1982) Linear-time algorithms for linear programming in  $\mathbb{R}^3$  and related problems. *SIAM J Comput* 4:759–776
- Mehrez A, Sinuany-Stern Z, Stulman A (1985) A single facility location problem with a weighted maximin-minimax rectilinear distance. *Comput Oper Res* 12:51–60
- Mehrez A, Sinuany-Stern Z, Stulman A (1986) An enhancement of the Drezner-Wesolowsky algorithm for single facility location with maximin of rectilinear distance. *J Oper Res Soc* 37:971–977
- Melachrinoudis E (1985) Determining an optimal location for an undesirable facility in a work-room environment. *Appl Math Model* 9:365–369
- Melachrinoudis E (1988) An efficient computational procedure for the rectilinear maximin location problem. *Transp Sci* 22:217–223
- Melachrinoudis E (1999) Bicriteria location of a semi-obnoxious facility. *Comput Ind Eng* 37:581–593
- Melachrinoudis E, Cullinane TP (1985) Locating an undesirable facility within a geographical region using the maximin criterion. *J Reg Sci* 25:115–127
- Melachrinoudis E, Cullinane TP (1986a) Locating an undesirable facility within a polygonal region. *Ann Oper Res* 6:137–145
- Melachrinoudis E, Cullinane TP (1986b) Locating an undesirable facility with a minimax criterion. *Eur J Oper Res* 24:239–246
- Melachrinoudis E, Smith JM (1995) An  $O(mn^2)$  algorithm for the maximin problem in  $E^2$ . *Oper Res Lett* 18:25–30
- Melachrinoudis E, Xanthopoulos Z (2003) Semi-obnoxious single facility location in Euclidean space. *Comput Oper Res* 30:2191–2209
- Melachrinoudis E, Zhang FG (1999) An  $O(mn)$  algorithm for the 1-maximin problem on a network. *Comput Oper Res* 26:849–869
- Min H, Melachrinoudis E, Wu X (1997) Dynamic expansion and location of an airport: a multiple objective approach. *Transp Res—Part A* 31:403–417
- Minieka E (1983) Anticenters and antimedianes of a network. *Networks* 13:359–364
- Moon ID, Chaudhry SS (1984) An analysis of network location problems with distance constraints. *Manag Sci* 30:290–307
- Murray AT, Church RL (2008) Location analysis and GIS. Abstract, ISOLDE 11:105
- Nadirler D, Karasakal E (2008) Mixed integer programming-based solution procedure for single-facility location with maximin of rectilinear distance. *J Oper Res Soc* 59:563–570
- Nagy G, Salhi S (2007) Location-routing: issues, models and methods. *Eur J Oper Res* 177:649–672
- Pisinger D (2006) Upper bounds and exact algorithms for the  $p$ -dispersion problems. *Comput Oper Res* 33:1380–1398
- Plastria F (1992) GBSS: The generalized big square small square method for planar facility location. *Eur J Oper Res* 62:163–174
- Plastria F (1996) Optimal location of undesirable facilities: a selective overview. *JORBEL* 36:109–127
- Plastria F, Carrizosa E (1999) Undesirable facility location with minimal covering objectives. *Eur J Oper Res* 121:302–319
- Revelle CS, Eiselt HA (2005) Location analysis: a synthesis and survey. *Eur J Oper Res* 165:1–19
- Saameno JJ, Guerrero C, Munoz J, Merida E (2006) A general model for the undesirable single facility location problem. *Oper Res Lett* 34:427–436
- Sayin S (2000) A mixed integer programming formulation for the 1-maximin problem. *J Oper Res Soc* 51:371–375
- Shamos MI (1975) Geometric complexity. Proceedings of the seventh ACM symposium on Theory of Computing, pp 224–233



- Shamos MI, Hoey D (1975) Closest-point problems. 16th annual symposium on foundations of computer science, pp 151–162
- Skriver AJV, Andersen KA (2001) The bicriterion semi-obnoxious location (BSL) problem solved by an  $\varepsilon$ -approximation. *Eur J Oper Res* 146:517–528
- Steuer ER (1989) Multiple criteria optimization: theory, computation, and application. Krieger, Malabar
- Sylvester JJ (1857) A question in the geometry of situation. *Q J Pure Appl Math* 1:79
- Tamir A (1988) Improved complexity bounds for center location problems on networks by using dynamic data structures. *SIAM J Discrete Math* 1:377–396
- Tamir A (1991) Obnoxious facility location on graphs. *SIAM J Discrete Math* 4:550–567
- Ting SS (1984) A linear time algorithm for maxisum facility location on tree networks. *Transp Sci* 18:76–84
- Yapicioglu H, Smith AE, Dozier G (2007) Solving the semi-desirable facility location problem using the bi-objective particle swarm. *Eur J Oper Res* 177:733–749
- Zhang FG, Melachrinoudis E (2001) The maximin-maxisum network location problem. *Comput Optim Appl* 19:209–234