Chapter 8
Simply Connected Domains

8.1 The General Cauchy Closed Curve Theorem

As we have seen, it can happen that a function f is analytic on a closed curve C and yet $\int_C f \neq 0$. Perhaps the simplest such example was given by

$$\int_{|z|=1} \frac{1}{z} dz = 2\pi i.$$

On the other hand, the Closed Curve Theorem—6.3—showed that if f is analytic throughout a disc, the integral around any closed curve is 0. We now seek to determine the most general type of domain in which the Closed Curve Theorem is valid. Note that the domain of analyticity of $f(z) = 1/z$ is the punctured plane. We will see that it is precisely the existence of a "hole" at $z = 0$ which allowed the above counterexample. The property of a domain which assures that it has no "holes" is called simple connectedness. The formal definition is as follows.

8.1 Definition

A region D is *simply connected* if its complement is "connected within ϵ to ∞." That is, if for any $z_0 \in \tilde{D}$ and $\epsilon > 0$, there is a continuous curve $\gamma(t)$, $0 \le t < \infty$ such that

(a) $d(\gamma(t), \tilde{D}) < \epsilon$ for all $t \ge 0$,
(b) $\gamma(0) = z_0$,
(c) $\lim_{t \to \infty} \gamma(t) = \infty$.

A curve γ, satisfying (b) and (c), is said to "connect z_0 to ∞." (See Chapter 1.4.)

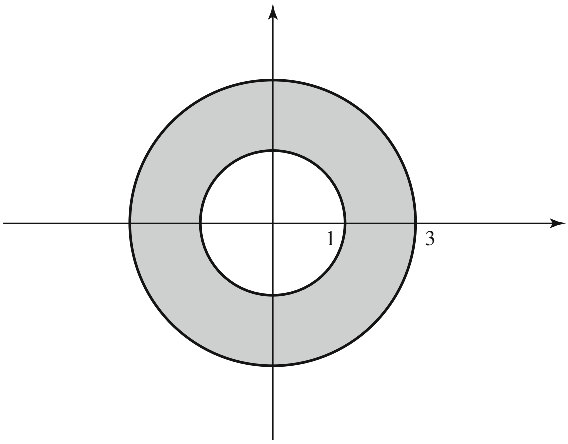
EXAMPLE 1
The plane minus the real axis is not simply connected since it is not a *region*; that is, a simply connected domain must be connected. ◊

J. Bak, D. J. Newman, *Complex Analysis,* DOI 10.1007/978-1-4419-7288-0_8

EXAMPLE 2
The annulus

$$A = \{z : 1 < |z| < 3\}$$

is not simply connected.

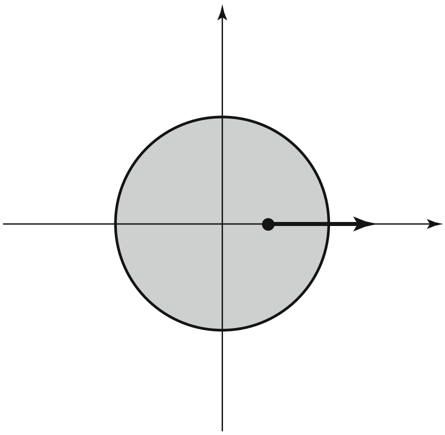


To prove this, note that $0 \in \tilde{A}$ and yet there is no γ which remains within $\epsilon = \frac{1}{2}$ of $\tilde{A}$ and connects 0 to ∞. If such a γ existed, by the continuity of $|\gamma(t)|$, there would have to be a point t_1 such that $|\gamma(t_1)| = 2$, but then $d(\gamma(t_1), \tilde{D}) = 1$. ◊

EXAMPLE 3
The unit disc minus the positive real axis is simply connected since for any z_0 in the complement

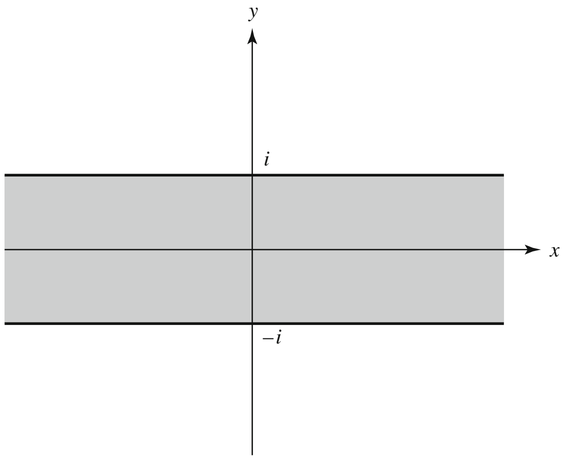
$$\gamma : \gamma(t) = (t + 1)z_0$$

connects z_0 to ∞ and is contained in the complement.



◊

EXAMPLE 4
The infinite strip $S = \{z : -1 < \operatorname{Im} z < 1\}$ is simply connected. Note that in this case, the complement $\tilde{S}$ is not connected.



◊

EXAMPLE 5
Any open convex set is simply connected. See Exercises 1 and 2. ◊

Definition 8.1 requires some explanation. It may seem somewhat simpler to say a region D is simply connected if every point in its complement can be connected, by a curve in the complement, to ∞. However, although this is the case in all the above examples, it is still somewhat too restrictive. For example, suppose the complement is the (connected) set

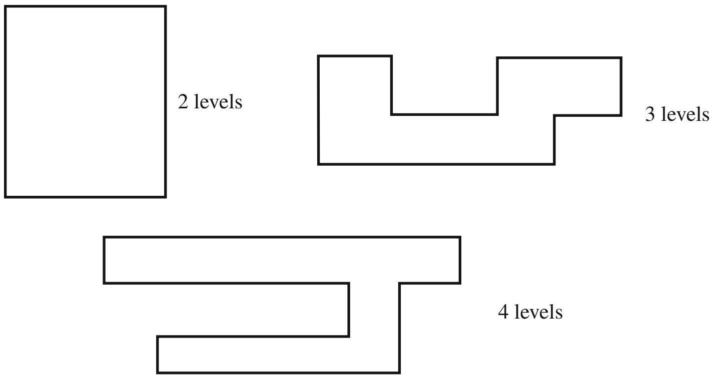
$$\tilde{D} = \left\{ x + iy : \begin{array}{c} 0 < x \le 1 \\ y = \sin \frac{1}{x} \end{array} \right\} \cup \{iy : -1 \le y < \infty\}.$$

By Definition 8.1, D would then be simply connected although the points on the curve $y = \sin(1/x)$ cannot be connected to ∞ by a curve in $\tilde{D}$. For a comparison of Definition 8.1 with other definitions of simple connectedness, see [Newman, pp. 164ff]. Also, see Appendix I.

Before proving the general closed curve theorem, we first prove an analogue for simple closed polygonal paths. Recall that a polygonal path is a finite chain of horizontal and vertical line segments.

8.2 Definition

Let Γ be a polygonal path. We define the *number of levels of* Γ as the number of *different* values y_0 for which the line $\operatorname{Im} z = y_0$ contains a horizontal segment of Γ.

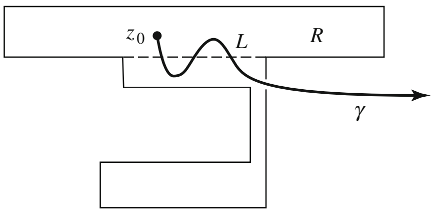


8.3 Lemma

Let Γ be a simple closed polygonal path contained in a simply connected domain D. Suppose the top level of Γ consists of the points $y = y_1, x \in X_1$ and the next level is given by $y = y_2, x \in X_2$. Then the set $R = \{z = x + iy : \substack{y_2 \le y \le y_1 \\ x \in X_1}\}$ is contained in D.

Proof

Note that R is a finite union of disjoint closed rectangles. We will show that for any $z_0 \in R$ and any curve γ connecting z_0 to ∞, $\gamma \cap \Gamma \neq \emptyset$. Then, since $\tilde{D}$ is closed and Γ is compact, $d(\Gamma, \tilde{D}) = \delta > 0$, and γ would not remain within $\epsilon = \delta/2$ of $\overline{D}$. Thus $z_0 \in D$.



To show $\gamma \cap \Gamma \neq \emptyset$, we proceed by induction on the number of levels of Γ. If Γ has only two levels, it is the boundary of a single rectangle and the proof is straightforward (the details are given in Exercise 5). Otherwise we consider

$$L = \{x + iy : y = y_2, x \in X_1 \backslash X_2\}.$$

Note that z_0 is contained in one of the rectangles of R, so γ must intersect the boundary of R. Thus, if γ doesn't meet $R \cap \Gamma$, it must meet L. Setting

$$t_0 = \sup\{t : \gamma(t) \in R\}$$

we note that for small enough $h > 0$, $\gamma(t_0 + h)$ would be between the top two levels of a simple closed polygonal curve which is a connected component of

$$\Gamma' = (\Gamma \cap \tilde{R}) \cup \bar{L}$$

and has one less level than Γ. But then, by induction $\gamma(t) \in \Gamma'$ for some $t > t_0 + h$. Finally, since $\gamma(t) \notin R$ for $t > t_0$ and since $L \subset R$, $\gamma(t) \in \Gamma$ and the proof is complete. □

8.4 Theorem

Suppose f is analytic in a simply connected region D and Γ is a simple closed polygonal path contained in D. Then $\int_\Gamma f = 0$.

Proof of Theorem 8.4

The proof will again be by induction on the number of levels of Γ. Define L, R and Γ' as in the lemma. We can write

$$\int_\Gamma f = \int_{\partial R} f + \int_{\Gamma'} f$$

the integral over L being taken in opposite directions. Since ∂R consists of the boundaries of rectangles and since f is analytic throughout these rectangles (by the lemma), $\int_{\partial R} f = 0$ by the Rectangle Theorem (6.1).

Proceeding by induction on the number of levels of Γ, we may assume

$$\int_{\Gamma'} f = 0$$

since it has one less level than Γ. Hence $\int_\Gamma f = 0$ and the proof is complete. □

8.5 Theorem

If f is analytic in a simply connected region D, there exists a "primitive" F, analytic in D and such that $F' = f$.

Proof

Choose $z_0 \in D$ and define

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta,$$

where the path of integration is a polygonal path contained in D.

By the previous theorem, F is well-defined for if we take Γ_1 and Γ_2 to be two such polygonal paths from z_0 to z,

$$\int_{\Gamma_1} f - \int_{\Gamma_2} f = \int_\Gamma f$$

where Γ is a closed polygonal curve. We leave it as an exercise to show that any closed polygonal curve can be decomposed into a finite number of simple closed polygonal curves and line segments traversed twice in opposite directions. Thus it follows from Lemma 8.3 that $\int_\Gamma f = 0$ and $\int_{\Gamma_1} f = \int_{\Gamma_2} f$.

To show that $F' = f$, we consider

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_z^{z+h} f(\zeta)d\zeta$$

where now (by taking h small enough), we may take the simplest path of integration: horizontally and then vertically from z to $z+h$. It follows, then, as in Theorems 5.2 and 6.2, that $F'(z) = f(z)$. □

8.6 General Closed Curve Theorem

Suppose that f is analytic in a simply connected region D and that C is a smooth closed curve contained in D. Then

$$\int_C f = 0.$$

Proof

$$\int_C f = \int_C F'(z)dz$$

where F is the primitive function guaranteed by Theorem 8.5,

$$= F(z(b)) - F(z(a)) = 0$$

since the endpoints of the closed curve coincide. □

It might be noted that while Theorem 8.6 is stated for simply connected regions, it has implications for other domains as well. For example, if f is analytic in the punctured plane $z \neq 0$ and C is a closed curve in the upper half-plane, $\int_C f = 0$ since C may be viewed as a closed curve in the simply connected subset $\operatorname{Im} z > 0$, where f is analytic. In general, if f is analytic in D and if C is contained in a simply connected subset of D, then $\int_C f = 0$.

EXAMPLE 1

Suppose C is the circle $\alpha + re^{i\theta}, 0 \leq \theta \leq 2\pi$ and $|a - \alpha| > r$. Then

$$\int_C \frac{dz}{z-a} = 0$$

since $1/(z-a)$ is analytic in the simply connected disc: $|z - \alpha| < |a - \alpha|$ which contains C. (Compare with Lemma 5.4.) ◊

Cauchy's Theorem also allows us at times to switch an integral along one closed contour to another.

EXAMPLE 2

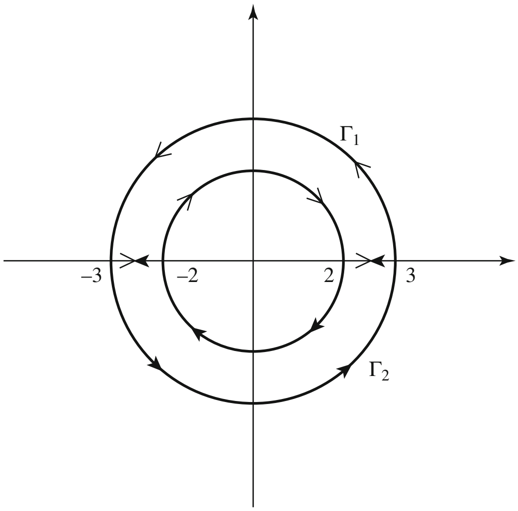
Suppose f is analytic in the annulus: $1 \le |z| \le 4$. Then

$$\int_{|z|=2} f(z)dz = \int_{|z|=3} f(z)dz$$

since, by adding the integrals along the real axis from 2 to 3 and from -2 to -3 in both directions, we can write

$$\int_{|z|=3} f(z)dz - \int_{|z|=2} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz$$

where Γ_1 and Γ_2 are closed curves contained in simply connected subsets of the annulus. (See below.)



◊

8.2 The Analytic Function log z

8.7 Definition

We will say f is an *analytic branch of log z* in a domain D if

(1) f is analytic in D, and
(2) f is an inverse of the exponential function there; i.e., $\exp(f(z)) = z$.

Of course if f is an analytic branch of log z then so is

$$g(z) = f(z) + 2\pi k i$$

for any fixed integer k.

Since $e^{\omega} \neq 0$ for any ω, log 0 is not defined. However, for any $z = Re^{i\theta}$, $R > 0$, if we set

$$f(z) = \log z = u(z) + iv(z)$$

condition (2) above becomes

$$\exp(f(z)) = e^{u(z)} \cdot e^{iv(z)} = Re^{i\theta}$$

which is possible if and only if

(3) $e^{u(z)} = |z| = R$

and

(4) $v(z) = \operatorname{Arg} z = \theta + 2k\pi$.

Hence a function f satisfying (2) can always be found by setting

(5) $f(z) = u(z) + iv(z) = \log|z| + i \operatorname{Arg} z$.

However, $\operatorname{Arg} z$ is not a well-defined function [see Chapter 1.2] and even if we adopt a particular convention for $\operatorname{Arg} z$, it is not clear that the function defined in (5) is analytic (or even continuous) in D. However, if D is a simply connected domain not containing 0, we may define an analytic branch of $\log z$ there. (Recall that according to Theorem 3.5 if an analytic inverse of e^z exists, its derivative must be $1/z$. Thus we proceed as follows.)

8.8 Theorem

Suppose that D is simply connected and that $0 \notin D$. Choose $z_0 \in D$, fix a value of $\log z_0$ *and set*

(6) $f(z) = \displaystyle\int_{z_0}^{z} \frac{d\zeta}{\zeta} + \log z_0$.

Then f is an analytic branch of $\log z$ *in D.*

Proof

f is well-defined since $1/\zeta$ is an analytic function of ζ in D and hence the integral along any two paths from z_0 to z yield the same value (Theorem 8.5). Furthermore, $f'(z) = 1/z$, so f is analytic in D.

To show that $\exp(f(z)) = z$, we consider

$$g(z) = ze^{-f(z)}.$$

Since $g'(z) = e^{-f(z)} - zf'(z)e^{-f(z)} = 0$, g is constant and

$$g(z) = g(z_0) = z_0 e^{-f(z_0)} = 1.$$

Hence

$$e^{f(z)} = z.$$

□

In an analogous manner, we can define an analytic branch of $\log f(z)$ in any simply connected domain where f is analytic and unequal to 0. We simply fix z_0 and a value of $\log f(z_0)$ and set

$$\log f(z) = \int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)} d\zeta + \log f(z_0).$$

In a typical situation, suppose D represents the whole plane minus the non-positive real axis: $x \leq 0$. If we choose $z_0 = 1$ and $\log 1 = 0$ in (6) the resulting function,

$$f(z) = \int_{1}^{z} \frac{d\zeta}{\zeta},$$

is an analytic branch of $\log z$ with

$$-\pi < \operatorname{Im}(\log z) = \operatorname{Arg} z < \pi.$$

(This latter inequality can be seen by integrating from 1 to $|z|$ and from $|z|$ to z.)

Similarly, if D is the plane slit along the non-negative real axis and we choose that branch of $\log z$ for which $\log(-1) = \pi i$, we will have defined an analytic branch of $\log z$ with $0 < \operatorname{Arg} z < 2\pi$. [See Exercise 8.]

By proper application of the logarithm, we can also define analytic branches of $\sqrt{z}$, $z^{1/3}$, etc., in the appropriate domains.

For example, $\sqrt{z}$ may be defined, in any domain where $\log z$ is defined, as

(7) $\sqrt{z} = \exp(\frac{1}{2} \log z)$.

Since

$$\left(\exp\left(\frac{1}{2} \log z\right)\right)^2 = \exp(\log z) = z,$$

this does define a "$\sqrt{z}$" and it is analytic where the logarithm is. Note that different branches of $\log z$ may yield different branches of $\sqrt{z}$. Unlike $\log z$, however, which has infinitely many different branches

$$\log z + 2\pi ki$$

for any integer k, there are only two different branches of $\sqrt{z}$. This follows from the fact that the equation $w^2 = z$ has exactly two different solutions for any $z \neq 0$. It also follows from (7) since

$$\exp\left(\frac{1}{2} \log z\right) = \exp\left(\frac{1}{2}[\log z + 2\pi ki]\right)$$

if k is even.

The same technique may be used to define arbitrary powers of any nonzero complex number. For example,

$$i^i \equiv e^{i \log i} = \{\ldots e^{3\pi/2},\ e^{-\pi/2},\ e^{-5\pi/2},\ \ldots\}.$$

Exercises

1. A set S is called *star-like* if there exists a point $\alpha \in S$ such that the line segment connecting α and z is contained in S for all $z \in S$. Show that a star-like region is simply connected. [*Hint*: Show that $\gamma : \gamma(t) = tz + (1-t)\alpha, t \geq 1$ is contained in the complement for any z in the complement.]

2.* Prove that every convex region is simply connected.

3. Suppose a region S is simply connected and contains the circle $C = \{z : |z-\alpha| = r\}$. Show then that S contains the entire disc $D = \{z : |z-\alpha| \leq r\}$. [*Hint*: Show that since S is open (by definition) and C is compact, S contains the annulus $B = \{z : r - \delta \leq |z-\alpha| \leq r + \delta\}$ for some $\delta > 0$.]

4. Show that if

$$\tilde{s} = \left\{x + iy : \begin{matrix} 0 < x \leq 1 \\ y = \sin\frac{1}{x} \end{matrix}\right\} \cup \{iy : -1 \leq y < \infty\},$$

S is simply connected.

5.* Show that a polygonal line γ connecting z to ∞ intersects the boundary of every rectangle R containing z. [*Hint*: Consider $t_0 = \sup\{t : \gamma(t) \in R\}$.]

6. Define the "inside" of a simple closed polygonal path. Show that if such a path is contained in a simply connected domain, so is its "inside."

7. Show that any closed polygonal path can be decomposed into a finite union of simple closed polygonal paths and line segments traversed twice in opposite directions.

8. Show that $\pi i + \int_{-1}^{z} d\zeta/\zeta$ defines an analytic branch of $\log z$ in the plane slit along the non-negative axis with $0 < \operatorname{Im}\log z = \operatorname{Arg} z < 2\pi$.

9.* Define a function f analytic in the plane minus the non-positive real axis and such that $f(x) = x^x$ on the positive axis. Find $f(i)$, $f(-i)$. Show that $f(\bar{z}) = \overline{f(z)}$ for all z.