Chapter 7
Further Properties of Analytic Functions

7.1 The Open Mapping Theorem; Schwarz' Lemma

The Uniqueness Theorem (6.9) states that a non-constant analytic function in a region cannot be constant on any open set. Similarly, according to Proposition 3.7, $|f|$ cannot be constant. Thus a non-constant analytic function cannot map an open set into a point or a circular arc. By applying the Maximum-Modulus Theorem, we can derive the following sharper result on the mapping properties of an analytic function.

7.1 Open Mapping Theorem

The image of an open set under a nonconstant analytic mapping is an open set.

Proof

(due to Carathéodory). We will show that if f is non-constant and analytic at α, the image under f of some (small) disc containing α will contain a disc about $f(\alpha)$. Without loss of generality, assume $f(\alpha) = 0$. (Otherwise, consider $f(z) - f(\alpha)$.) By the Uniqueness Theorem, there is a circle C around α such that $f(z) \neq 0$ for $z \in C$. Let $2\epsilon = \min_{z \in C} |f(z)|$. It will follow that the image of the disc bounded by C contains the disc $D(0; \epsilon)$. For assume that $\omega \in D(0; \epsilon)$ and consider $f(z) - \omega$.

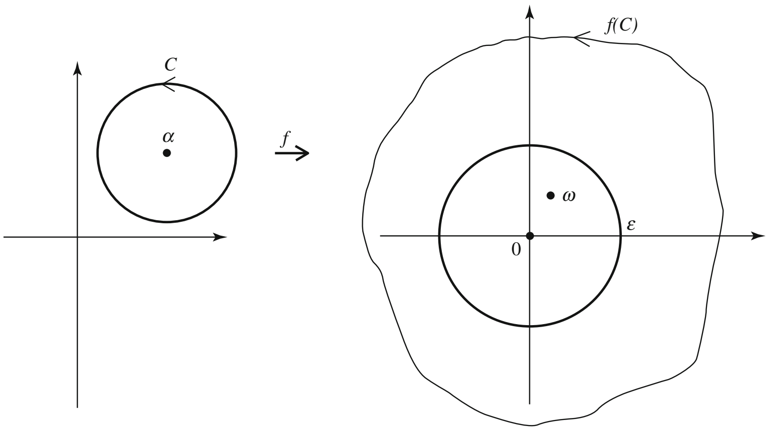
For $z \in C$

$$|f(z) - \omega| \geq |f(z)| - |\omega| \geq \epsilon,$$

while at α

$$|f(\alpha) - \omega| = |-\omega| < \epsilon.$$

J. Bak, D. J. Newman, *Complex Analysis,* DOI 10.1007/978-1-4419-7288-0_7



Hence $|f(z) - \omega|$ assumes its minimum somewhere inside C, and by the Minimum Modulus Theorem, $f(z) - \omega$ must equal zero somewhere inside C. Thus ω is in the range of f. □

The Maximum-Modulus Theorem can also be used in conjunction with other given information about a function to obtain stronger estimates for the modulus of f in its domain of analyticity. The following example is typical.

7.2 Schwarz' Lemma

Suppose that f is analytic in the unit disc, that $f \ll 1$ there and that $f(0) = 0$. Then

i. $|f(z)| \le |z|$
ii. $|f'(0)| \le 1$

with equality in either of the above if and only if $f(z) = e^{i\theta} z$.

Proof

We apply the Maximum-Modulus Theorem to the analytic function

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & 0 < |z| < 1 \\ f'(0) & z = 0. \end{cases}$$

(See Proposition 6.7.)

Since $g \ll 1/r$ on the circle of radius r, by letting $r \to 1$ and applying the Maximum-Modulus Theorem, we find that $|g(z)| \le 1$ throughout the unit disc, proving (i) and (ii). Furthermore, if $|g(z_0)| = 1$ for some z_0 such that $|z_0| < 1$, then by the Maximum-Modulus Theorem, g would be a constant (of modulus 1), and $f(z) = e^{i\theta} z$. □

A class of functions analytic in the unit disc and bounded there by 1 is given by the set of bilinear transformations

$$B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$$

where $|\alpha| < 1$. Note that

$$\left|\frac{1}{\bar{\alpha}}\right| > 1,$$

so that B_α is analytic throughout $|z| \le 1$. On $|z| = 1$

$$|B_\alpha|^2 = \left(\frac{z-\alpha}{1-\bar{\alpha}z}\right)\left(\frac{\bar{z}-\bar{\alpha}}{1-\alpha\bar{z}}\right) = \frac{|z|^2 - \alpha\bar{z} - \bar{\alpha}z + |\alpha|^2}{1-\alpha\bar{z}-\bar{\alpha}z+|\alpha|^2|z|^2} = 1$$

so that $|B_\alpha| \equiv 1$ on the boundary. For this reason, the functions B_α can be used, in variations of Schwarz' Lemma, to solve various extremal problems for analytic functions.

Example 1

Suppose that f is analytic and bounded by 1 in the unit disc and that $f(\frac{1}{2}) = 0$. We wish to estimate $|f(\frac{3}{4})|$. Since $f(\frac{1}{2}) = 0$,

$$g(z) = \begin{cases} f(z) \Big/ \left(\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\right) & z \ne \frac{1}{2} \\ \frac{3}{4} f'\left(\frac{1}{2}\right) & z = \frac{1}{2} \end{cases}$$

is likewise analytic in $|z| < 1$. Letting $|z| \to 1$, we find that $|g| \le 1$; so that

$$|f(z)| \le \left|\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\right|$$

throughout the disc. In particular,

$$\left|f\left(\frac{3}{4}\right)\right| \le \frac{2}{5}.$$

Note that the maximum value, $\frac{2}{5}$, is achieved by

$$B_{\frac{1}{2}}(z) = \frac{z-\frac{1}{2}}{1-\frac{1}{2}z}.$$

◊

Example 2

Next we show that among all functions f which are analytic and bounded by 1 in the unit disc, $\max|f'(\frac{1}{3})|$ is assumed when $f(\frac{1}{3}) = 0$.

Suppose $f(\frac{1}{3}) \neq 0$ and consider

$$g(z) = \frac{f(z) - f(\frac{1}{3})}{1 - \overline{f(\frac{1}{3})} f(z)}.$$

Again, since

$$\left| \frac{\omega - f(\frac{1}{3})}{1 - \overline{f(\frac{1}{3})}\omega} \right| = 1$$

when $|\omega| = 1$, while $|f| < 1$ in $|z| < 1$, the Maximum-Modulus Theorem assures us that g, like f, is bounded by 1. A direct calculations shows that

$$g'\left(\frac{1}{3}\right) = f'\left(\frac{1}{3}\right) \Big/ \left(1 - \left|f\left(\frac{1}{3}\right)\right|^2\right)$$

so that

$$\left|g'\left(\frac{1}{3}\right)\right| > \left|f'\left(\frac{1}{3}\right)\right|.$$

We note that max $|f'(\frac{1}{3})|$ is assumed by the function $B_{1/3}(z)$. [See Exercises 10 and 11.] ♢

Example 2 has an interesting physical interpretation. Given the constraint on f that it must map the unit disc into the unit disc, the way to maximize $|f'(\frac{1}{3})|$ is by

a. mapping $\frac{1}{3}$ into 0 and
b. mapping the boundary of the unit disc onto itself.

It is as though by thus allowing the maximum room for expansion around $f(\frac{1}{3})$, we obtain max $|f'(\frac{1}{3})|$. We will see a similar phenomenon when we study the Riemann Mapping Theorem.

Returning once again to entire functions, the Maximum-Modulus Theorem may be used to derive further extensions of Liouville's Theorem.

7.3 Proposition

If f is an entire function satisfying

$$|f(z)| \leq 1/|\text{Im}\, z|$$

for all z, then $f \equiv 0$.

Proof

By hypothesis $f \ll 1$ throughout $|\text{Im}\, z| > 1$ but f could be unbounded near the real axis. To estimate $|f|$ on the circle $|z| = R$, we introduce the auxiliary function

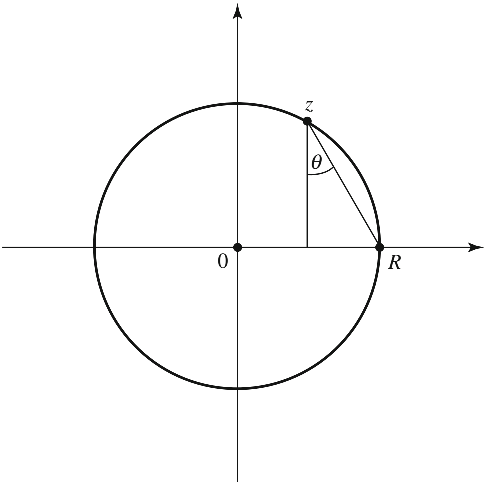
$$g(z) = (z^2 - R^2) f(z).$$

For any z such that $|z| = R$ and $\operatorname{Re} z \geq 0$

$$|(z - R)f(z)| \leq |z - R|/|\operatorname{Im} z| = \sec\theta$$

for some θ, $0 \leq \theta \leq \pi/4$ (see the following diagram), so that

$$|(z - R)f(z)| \leq \sqrt{2}.$$



Similarly, if $|z| = R$ and $\operatorname{Re} z \leq 0$, then

$$|(z + R)f(z)| \leq \sqrt{2}.$$

Thus

$$|g(z)| = |z + R||z - R||f(z)| \leq 3R$$

for all z with $|z| = R$. By the Maximum-Modulus Theorem, the same upper bound holds throughout $|z| < R$. Hence

$$|g(z)| = |z^2 - R^2||f(z)| \leq 3R$$

and

$$|f(z)| \leq \frac{3R}{|z^2 - R^2|}$$

as long as $R \gg z$. Letting $R \to \infty$, we see that $f(z) = 0$. Since this holds for all z, the theorem is proven. $\square$

7.2 The Converse of Cauchy's Theorem: Morera's Theorem; The Schwarz Reflection Principle and Analytic Arcs

The key result in our study of analytic functions so far has been the Rectangle Theorem (6.1). Thus, it may not come as a surprise that the property described there is almost equivalent to analyticity.

7.4 Morera's Theorem

Let f be a continuous function on an open set D. If

$$\int_\Gamma f(z)dz = 0$$

whenever Γ is the boundary of a closed rectangle in D, then f is analytic on D.

Since line integrals are unaffected by the value of the integrand at a single point, the continuity of f is a necessary hypothesis. Note also that in the proof, we actually require only that $\int_\Gamma f = 0$ for rectangles whose sides are parallel to the horizontal and vertical axes.

Proof

In a small disc about any point $z_0 \in D$, we can define a primitive

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta$$

where the path of integration is the horizontal followed by the vertical segments from z_0 to z. If we then consider a difference quotient of F and apply the fact that $\int_\Gamma f = 0$ around any rectangle, we may conclude (as in Theorems 4.15 and 6.2) that

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_{z}^{z+h} f(\zeta)d\zeta \to f(z)$$

as $h \to 0$. (Here we are using the continuity of f.) Hence F is analytic in a neighborhood of z_0. Since analytic functions are infinitely differentiable and $F'(z) = f(z)$, f is analytic at z_0. Finally, since z_0 was arbitrary, f is analytic in D. □

Morera's Theorem is often used to establish the analyticity of functions given in integral form. For example, consider

$$f(z) = \int_0^\infty \frac{e^{zt}}{t+1}dt.$$

If $\operatorname{Re} z = x < 0$,

$$\int_0^\infty \frac{|e^{zt}|}{t+1}dt < \int_0^\infty e^{xt}dt = -\frac{1}{x}$$

so that the integral is absolutely convergent and $|f(z)| \leq 1/|x|$. To show that f is analytic in the left half-plane $D : \operatorname{Re} z < 0$, we may consider

$$\int_\Gamma f(z)dz = \int_\Gamma \left(\int_0^\infty \frac{e^{zt}}{t+1} dt \right) dz,$$

where Γ is the boundary of some closed rectangle in D.

Since

$$\int_\Gamma \int_0^\infty \frac{|e^{zt}|}{t+1} dt\, dz$$

converges, we can interchange the order of integration; hence

$$\int_\Gamma f = \int_0^\infty \int_\Gamma \frac{e^{zt}}{t+1} dz\, dt = \int_0^\infty 0\, dt = 0$$

by the analyticity of $e^{zt}/(t+1)$ as a function of z. By Morera's Theorem, then, f is analytic in D.

7.5 Definition

Suppose $\{f_n\}$ and f are defined in D. We will say f_n *converges to* f *uniformly on compacta* if $f_n \to f$ uniformly on every compact subset $K \subset D$.

The following theorem asserts that analyticity is preserved under uniform limits, in marked contrast to the property of differentiability on the real line. There, the uniform limit of differentiable functions may be *nowhere* differentiable.

7.6 Theorem

Suppose $\{f_n\}$ *represents a sequence of functions, analytic in an open domain D and such that* $f_n \to f$ *uniformly on compacta. Then* f *is analytic in D.*

Proof

In some compact neighborhood K of each point z_0, f is the uniform limit of continuous functions; hence f is continuous in D. Furthermore, for every rectangle $\Gamma \subset K$

$$\int_\Gamma f = \int_\Gamma \lim f_n = \lim_n \int_\Gamma f_n = 0,$$

since $f_n \to f$ uniformly on Γ. Hence, by Morera's theorem, f is analytic in D. □

7.7 Theorem

Suppose f *is continuous in an open set D and analytic there except possibly at the points of a line segment L. Then* f *is analytic throughout D.*

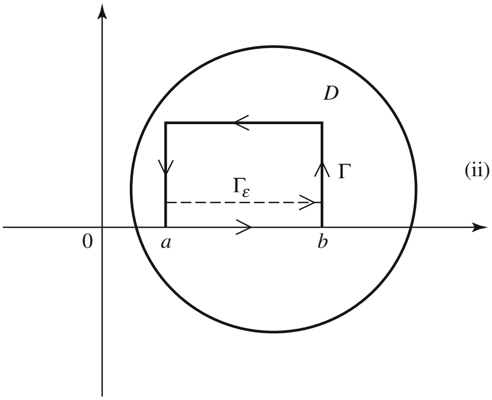
Proof

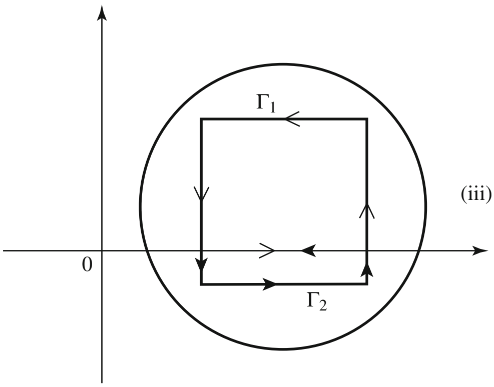
Without loss of generality, we may assume the exceptional points lie on the real axis. Otherwise, we could begin by considering $g(z) = f(Az + B)$ where $Az + B$ maps the real axis onto the line containing L. (See Exercise 15.) Of course, the analyticity of f on D is equivalent to the analyticity of g on the corresponding region. Moreover, since analyticity is a local property, we may assume D is a disc.

To show $\int_\Gamma f = 0$ for every closed rectangle in D with boundary Γ (and with sides parallel to the real and imaginary axes), we consider three cases.

i. L doesn't meet the rectangle bounded by Γ.
Here $\int_\Gamma f = 0$ by the analyticity of f throughout the interior of Γ (Theorem 6.1).

ii. One side of Γ coincides with L.
In this case, we let Γ_ϵ be the rectangle composed of the sides of Γ with the bottom (or top) side shifted up (or down) by ϵ in the positive





(or negative) y-direction. Then

$$\int_\Gamma f = \lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} f,$$

since

$$\int_a^b f(x+i\epsilon)dx \to \int_a^b f(x)dx$$

by the continuity of f. Hence

$$\int_\Gamma f = 0,$$

iii. If Γ surrounds L, we write

$$\int_\Gamma f = \int_{\Gamma_1} f + \int_{\Gamma_2} f$$

where Γ_1 and Γ_2 are as in (ii). Again we conclude

$$\int_\Gamma f = 0.$$

Finally, By Morera's Theorem, f is analytic in D. □

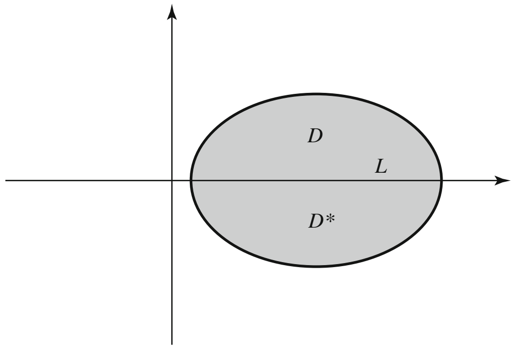
A wide range of results, all of which are known as the Schwarz Reflection Principle, are typified by the following theorem.

7.8 Schwarz Reflection Principle

Suppose f is C-analytic in a region D that is contained in either the upper or lower half plane and whose boundary contains a segment L on the real axis, and suppose f is real for real z. Then we can define an analytic "extension" g of f to the region $D \cup L \cup D^*$ that is symmetric with respect to the real axis by setting

$$g(z) = \begin{cases} f(z) & z \in D \cup L \\ \overline{f(\bar{z})} & z \in D^* \end{cases}$$

where $D^* = \{z : \bar{z} \in D\}$.



Proof

At points in D, $g = f$ and hence g is analytic there. If $z \in D^*$ and h is small enough so that $z + h \in D^*$

$$\frac{g(z+h) - g(z)}{h} = \frac{\overline{f(\bar{z} + \bar{h}) - f(\bar{z})}}{h} = \overline{\left[\frac{f(\bar{z} + \bar{h}) - f(\bar{z})}{\bar{h}}\right]}$$

which approaches $\overline{f'(\bar{z})}$ as h approaches 0. Hence g is analytic in D^*. Since f is continuous on the real axis, so is g and we can apply Theorem 7.7 to conclude that g is analytic throughout the region $D \cup L \cup D^*$. □

By invoking the Uniqueness Theorem, we obtain the following immediate corollary:

7.9 Corollary

If f is analytic in a region symmetric with respect to the real axis and if f is real for real z, then

$$f(z) = \overline{f(\bar{z})}.$$

The Schwarz reflection principle can be applied in more general situations. The key is to extend the concept of reflection across other curves.

7.10 Definition

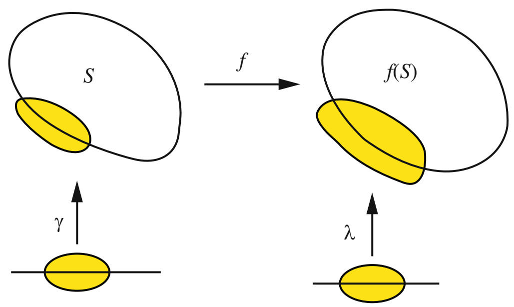
A curve $\gamma : [a, b] \to \mathbb{C}$ will be called a regular analytic arc if γ is an analytic, one-to-one function on $[a, b]$ with $\gamma' \neq 0$.

Note that, by the definition of analyticity, γ is the restriction to $[a, b]$ of a function $\gamma(z)$ which is analytic in an open set S containing $[a, b]$. Moreover, if all points of S are sufficiently close to $[a, b]$, $\gamma' \neq 0$ and γ will remain one-to-one throughout S. (Otherwise, the original curve would fail to be one-to-one or γ' would be zero at some point of $[a, b]$.) So assume that $\gamma(z)$ is analytic and one-to-one in such an open set S which is also symmetric with respect to the interval $[a, b]$. Then we can define the reflection w^* of a point w in $\gamma(S)$, across the curve γ, as $\gamma(\overline{\gamma^{-1}(w)})$. That is, if $w = \gamma(z)$, $w^* = \gamma(\bar{z})$. It follows immediately that $(w^*)^* = w$, that points on the original curve are reflected into themselves, and that points not on the curve γ are reflected onto other points not on γ. In fact, the arc formed by taking the image under γ of the vertical line from any nonreal z to its conjugate $\bar{z}$ must intersect the original curve γ (i.e. $\gamma(t)$, $a \leq t \leq b$) orthogonally, by the conformality of γ. Hence w and w^* are on opposite sides of γ.

For example, if $\gamma(t) = it$, $-\infty < t < \infty$, and $w = u + iv$, then $w^* = \gamma\left(\overline{v - iu}\right) = -u + iv = -\bar{w}$, which is the reflection of w across the imaginary axis. Similarly, suppose γ is an arc of the circle $\gamma(t) = \mathrm{R}e^{it}$. Then $\gamma(z) = \mathrm{R}e^{iz} = \mathrm{R}e^{-y}e^{ix}$. If $w = \gamma(z) = re^{i\theta}$, $\mathrm{R}e^{-y} = r$ and x = θ, so that $w^* = \gamma(x - iy) =$

$\frac{R^2}{r}e^{i\theta} = \frac{R^2}{\bar{w}}$. Note that w^* is on the same ray as w, and $|ww^*| = R^2$, so that w and w^* are on opposite sides of the circle of radius R.

Suppose then that f is analytic in a region S and continuous to the boundary, which includes the regular analytic curve γ, and assume that $f(\gamma) \subset \lambda$, another regular analytic curve. Let z^* denote the reflection of z across γ, and let w^ denote the reflection of w across λ. Then f can be extended to S^* by defining $f(z)$ at a point $z \in S^*$ as $(f(z^*))^\wedge$. This defines an analytic extension of f to S^* since it is equal to the composition: $\lambda \circ \overline{\lambda^{-1}} \circ f \circ \gamma \circ \overline{\gamma^{-1}}$. As in our proof of the original form of the Schwarz reflection principle, the analyticity of f follows from the fact that $\overline{h(\overline{z})}$ is analytic at z (and has a derivative equal to $\overline{h'(\overline{z})}$) whenever h is analytic at $\overline{z}$.



Example 1: Suppose f is analytic in the unit disc and continuous to the boundary, which it maps into itself. Then f can be extended by defining $f(z) = 1/\overline{f(1/\overline{z})}$ at points z outside the unit circle. Note that the extended function is analytic everywhere except at the reflections of the zeroes of f inside the unit circle, which the extended function would map into ∞. Thus if we were looking for a bilinear function f mapping the unit circle into itself, with $f(\alpha) = 0$, it would follow that $f(1/\overline{\alpha}) = \infty$, so that we might consider $f(z) = (z - \alpha)/(z - 1/\overline{\alpha})$. However, in its current form f does not map the unit circle into itself. In particular, $|f(1)| = |\alpha|$, so we must multiply our function by a constant of magnitude $1/|\alpha|$, which leads us to consider functions of the form $f(z) = (z - \alpha)/(1 - \overline{\alpha}z)$. As we saw in the last section, these bilinear functions do, in fact, map the unit circle into itself. ◊

Example 2: Suppose f is an analytic map of a rectangle R onto another rectangle S, which maps each side of R onto a side of S. Then f can be extended analytically across the sides of R, mapping rectangles adjacent to R onto rectangles adjacent to S. Continuing in this manner, f can be extended to an entire function! It is easily seen, moreover, that the extended entire function has "linear growth"; i.e. $|f(z)| \leq A|z| + B$, for some positive constants A and B. Hence, according to the Extended Liouville Theorem, f must be a linear polynomial. ◊

Exercises

1. Show that if f is analytic and non-constant on a compact domain, Re f and Im f assume their maxima and minima on the boundary.

2. Prove that the image of a region under a non-constant analytic function is also a region.

3. a. Suppose f is nonconstant and analytic on S and $f(S) = T$. Show that if $f(z)$ is a boundary point of T, z is a boundary point of S.
   b. Let $f(z) = z^2$ on the set S which is the union of the semi-discs $S_1 = \{z : |z| \le 2; \operatorname{Re} z \le 0\}$ and $S_2 = \{z : |z| \le 1; \operatorname{Re} z \ge 0\}$. Show that there are points z on the boundary of S for which $f(z)$ is an interior point of $f(S)$.

4. Suppose f is C-analytic in $D(0; 1)$ and maps the unit circle into itself. Show then that f maps the entire disc onto itself. [*Hint:* Use the Maximum-Modulus Theorem to show that f maps $D(0; 1)$ into itself. Then apply the previous exercise to conclude that the mapping is onto.]

5. Suppose f is *entire* and $|f| = 1$ on $|z| = 1$. Prove $f(z) = Cz^n$. [*Hint:* First use the maximum and minimum modulus theorem to show

$$f(z) = C \prod_{i=1}^{n} \frac{z - \alpha_i}{1 - \bar{\alpha}_i z}.]$$

6.* Show that for any given rational function $f(z)$, with poles in the unit disc, it is possible to find another rational function $g(z)$, with no poles in the unit disc, and such that $|f(z)| = |g(z)|$ if $|z| = 1$.

7.* a. Suppose $|\alpha| < R$. Show that

$$\left| \frac{R(z - \alpha)}{R^2 - \overline{\alpha} z} \right|$$

is analytic for $|z| \le R$, and maps the circle $|z| = R$ into the unit circle.

b. Suppose $|\alpha_k| < R$ for $k = 1, 2, \ldots n$. Prove that (unless $|\alpha_k| = 0$ for all k)

$$\sqrt[n]{|z - \alpha_1| \cdot |z - \alpha_2| \cdots |z - \alpha_n|}$$

assumes a maximum value greater than R, and a minimum value less than R, at some points z on $|z| = R$. [Hint: Apply the maximum and minimum modulus theorems to $\prod_{k=1}^{n} (R^2 - \overline{\alpha_k} z)$.]

8. Suppose that f is analytic in the annulus: $1 \le |z| \le 2$, that $|f| \le 1$ for $|z| = 1$ and that $|f| \le 4$ for $|z| = 2$. Prove $|f(z)| \le |z|^2$ throughout the annulus.

9. Given f analytic in $|z| < 2$, bounded there by 10, and such that $f(1) = 0$. Find the best possible upper bound for $|f(\frac{1}{2})|$.

10. Suppose that f is analytic and bounded by 1 in the unit disc with $f(\alpha) \ne 0$ for some $\alpha \ll 1$. Show that there exists a function g, analytic and bounded by 1 in the unit disc, with $|g'(\alpha)| > |f'(\alpha)|$.

11. Find $\max_f |f'(\alpha)|$ where f ranges over the class of analytic functions bounded by 1 in the unit disc, and α is a fixed point of $|z| < 1$. [*Hint:* By the previous exercise, you may assume $f(\alpha) = 0$.] Show that

$$f'(\alpha) = \lim_{z \to \alpha} \frac{f(z)}{z - \alpha} \ll \lim_{z \to \alpha} \frac{B_\alpha(z)}{z - \alpha} = B'_\alpha(\alpha).$$

12. Suppose f is entire and $|f(z)| \le 1/|\operatorname{Re} z|^2$ for all z. Show that $f \equiv 0$.

13. Show that

$$f(z) = \int_0^1 \frac{\sin zt}{t} dt$$

is an entire function.

a. by applying Morera's Theorem,
b. by obtaining a power series expansion for f.

14. With f as in (13) show that

$$f'(z) = \int_0^1 \cos zt dt$$

a. by writing

$$\begin{aligned} f(z) &= \int_0^1 \int_0^z \cos zt \, dz \, dt \\ &= \int_0^z \left(\int_0^1 \cos zt \, dt \right) dz, \quad \text{etc.,} \end{aligned}$$

b. by using the power series for f.

15. Show that $g(z) = z_0 + e^{i\theta} z$, $\theta = \text{Arg}(z_1 - z_0)$, maps the real axis onto the line L through z_0 and z_1.

16. Suppose f is bounded and analytic in $\text{Im}\, z \geq 0$ and real on the real axis. Prove that f is constant.

17. Given an entire function which is real on the real axis and imaginary on the imaginary axis, prove that it is an odd function: i.e., $f(z) = -f(-z)$.

18.* Show that $v + iu$ is the reflection of the point $u + iv$ across the line $u = v$.

19. Suppose f is analytic in the semi-disc: $|z| \leq 1, \text{Im}\, z > 0$ and real on the semi-circle $|z| = 1, \text{Im}\, z > 0$. Show that if we set

$$g(z) = \begin{cases} f(z) & |z| \leq 1, \quad \text{Im}\, z > 0 \\ \overline{f\left(\frac{1}{\bar{z}}\right)} & |z| > 1, \quad \text{Im}\, z > 0 \end{cases}$$

then g is analytic in the upper half-plane.

20. Show that there is no non-constant analytic function in the unit disc which is real-valued on the unit circle.

21. Suppose f is analytic in the upper semi-disc: $|z| \leq 1, \text{Im}\, z > 0$ and is continuous to the boundary. Explain why it is not possible that $f(x) = |x|$ for all real values of x.

22.* Suppose an entire function maps two horizontal lines onto two other horizontal lines. Prove that its derivative is periodic. [Hint: Assume $f = u + iv$ maps the lines $y = y_1$ and $y = y_2$ onto $v = v_1$ and $v = v_2$ with $y_2 - y_1 = c$ and $v_2 - v_1 = d$. Show then that $f(z + 2ci) = f(z) + 2di$, for all z.]

23.* Prove that an entire function which maps a parallelogram onto another parallelogram, and maps each side of the original parallelogram onto a side of its image, must be a linear polynomial. [Hint: Use Exercise 22 to prove that f' is constant.]