Chapter 5
Properties of Entire Functions

5.1 The Cauchy Integral Formula and Taylor Expansion for Entire Functions

We now show that if f is entire and if

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & z \neq a \\ f'(a) & z = a \end{cases}$$

then the Integral Theorem (4.15) and Closed Curve Theorem (4.16) apply to g as well as to f. (Note that since f is entire, g is continuous; however, it is not obvious that g is entire.) We begin by showing that the Rectangle Theorem applies to g.

5.1 Rectangle Theorem II

If f is entire and if

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & z \neq a \\ f'(a) & z = a \end{cases}$$

then $\int_\Gamma g(z)dz = 0$, where Γ is the boundary of a rectangle R.

Proof

We consider three cases.

I. $a \in \text{ext } R$.

In this case, g is analytic throughout R and the proof is exactly the same as that of Theorem 4.14. Note that the proof required only that the integrand be analytic throughout R and Γ.

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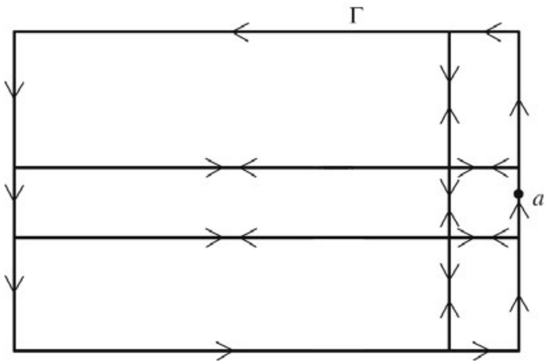
II. $a \in \Gamma$.

Divide R into six subrectangles as indicated and note that because of the cancellations involved

$$\int_\Gamma g = \sum_{k=1}^{6} \int_{\Gamma_k} g \tag{1}$$



where Γ_k, $1 \leq k \leq 6$, denote the boundaries of the subrectangles. Since g is continuous in the compact domain $\bar{R}$, $g \ll M$ for some constant M. If we take the boundary of the rectangle containing a (call it Γ_1) to have length less than ϵ,

$$\int_{\Gamma_1} g \ll M\epsilon \quad \text{by the } M\text{-}L \text{ formula}$$

while

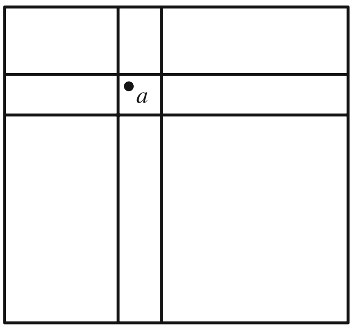
$$\int_{\Gamma_k} g = 0, \qquad k \neq 1$$

as in case (I). Hence by (1)

$$\int_\Gamma g \ll M\epsilon$$

for any $\epsilon > 0$ and the proof is complete.

III. $a \in \text{int } R$.



Here, as in the previous case, we subdivide R–this time into nine rectangles. Along the boundaries of the eight rectangles (not containing a)

$$\int_{\Gamma_k} g = 0,$$

while the integral along the boundary of the remaining subrectangle can be made arbitrarily small by choosing its length to be as small as required. As in the previous case, we conclude

$$\int_{\Gamma} g = \sum_{k=1}^{9} \int_{\Gamma_k} g = 0. \qquad \square$$

5.2 Corollary

Suppose g is as above. Then the Integral Theorem and the Closed Curve Theorem apply to g.

Proof

We observe that since g is continuous, the proofs of Theorems 4.15 and 4.16 apply, without any modification, to g. $\square$

5.3 Cauchy Integral Formula

Suppose that f is entire, that a is some complex number, and that C is the curve

$$C : Re^{i\theta}, \qquad 0 \le \theta \le 2\pi, \qquad \textit{with } R > |a|.$$

Then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz.$$

Proof

By Corollary 5.2

$$\int_C \frac{f(z) - f(a)}{z-a}\, dz = 0$$

so that

$$f(a) \int_C \frac{dz}{z-a} = \int_C \frac{f(z)}{z-a}\, dz$$

and the proof follows once we show that

$$\int_C \frac{dz}{z-a} = 2\pi i.$$

This lemma is proven below in somewhat greater generality. $\square$

5.4 Lemma

Suppose a is contained in the circle C_ρ: that is, C_ρ has center α, radius ρ, and $|a-\alpha| < \rho$. Then

$$\int_{C_\rho} \frac{dz}{z-a} = 2\pi i.$$

Proof

First we note that

$$\int_{C_\rho} \frac{dz}{z-\alpha} = \int_0^{2\pi} \frac{i\rho e^{i\theta}}{\rho e^{i\theta}} d\theta = 2\pi i,$$

while

$$\int_{C_\rho} \frac{dz}{(z-\alpha)^{k+1}} = 0 \qquad \text{for } k = 1, 2, 3, \ldots.$$

The second equality follows not only from a direct evaluation of the integral

$$\int_{C_\rho} \frac{dz}{(z-\alpha)^{k+1}} = \frac{i}{\rho^k} \int_0^{2\pi} e^{-ik\theta} d\theta = 0$$

but also from the fact that $1/(z-\alpha)^{k+1}$ is equal to the derivative of the analytic function $-1/k(z-\alpha)^k$.

To evaluate $\int_{C_\rho} (1/(z-a))dz$, write

$$\frac{1}{z-a} = \frac{1}{(z-\alpha)-(a-\alpha)} = \frac{1}{(z-\alpha)[1-(a-\alpha)/(z-\alpha)]}$$
$$= \frac{1}{(z-\alpha)} \cdot \frac{1}{1-\omega}$$

where

$$\omega = \frac{a-\alpha}{z-\alpha} \quad \text{has fixed modulus } \frac{|a-\alpha|}{\rho} < 1 \quad \text{throughout } C_\rho. \tag{1}$$

By (1) and the fact that $1/(1-\omega) = 1 + \omega + \omega^2 + \cdots$, we obtain

$$\frac{1}{z-a} = \frac{1}{z-\alpha}\left[1 + \frac{a-\alpha}{z-\alpha} + \frac{(a-\alpha)^2}{(z-\alpha)^2} + \cdots\right]$$
$$= \frac{1}{z-\alpha} + \frac{a-\alpha}{(z-\alpha)^2} + \frac{(a-\alpha)^2}{(z-\alpha)^3} + \cdots.$$

Since the convergence is uniform throughout C_ρ,

$$\int_{C_\rho} \frac{1}{z-a}\, dz = \int_{C_\rho} \frac{1}{z-\alpha}\, dz + \sum_{k=1}^{\infty} \int_{C_\rho} \frac{(a-\alpha)^k}{(z-\alpha)^{k+1}}\, dz = 2\pi i. \qquad \square$$

5.5 Taylor Expansion of an Entire Function

If f is entire, it has a power series representation. In fact, $f^{(k)}(0)$ *exists for* $k = 1, 2, 3, \ldots,$ *and*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

for all z.

Proof

Suppose $a \neq 0$, $R = |a| + 1$ and let C be the circle: $|\omega| = R$. By the Cauchy Integral Formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{\omega - z} d\omega$$

for all $z \ll a$.

As before, note that

$$\frac{1}{\omega - z} = \frac{1}{\omega\left(1 - \frac{z}{\omega}\right)} = \frac{1}{\omega} + \frac{z}{\omega^2} + \frac{z^2}{\omega^3} + \cdots,$$

and since the convergence is uniform throughout C

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_C f(\omega) \left[\frac{1}{\omega} + \frac{z}{\omega^2} + \frac{z^2}{\omega^3} + \cdots \right] d\omega \\ &= \frac{1}{2\pi i} \int_C \frac{f(\omega)}{\omega} d\omega + \left(\frac{1}{2\pi i} \int_C \frac{f(\omega)}{\omega^2} d\omega \right) z + \left(\frac{1}{2\pi i} \int_C \frac{f(\omega)}{\omega^3} d\omega \right) z^2 + \cdots \\ &= \sum_{k=0}^{\infty} C_k z^k \end{aligned}$$

where

$$C_k = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{\omega^{k+1}} d\omega. \tag{1}$$

Since for each z, there exists some $a \gg z$, the proof of the first part of the theorem appears to be complete. There is, however, one wrinkle. The contour C–and hence the coefficients of the power series–depended on a, for the radius R had to be chosen larger than $|a|$ to insure the uniform convergence of the power series for $1/(\omega - z)$. On the other hand, if we think of a as being fixed, we have shown that there exist coefficients $C_0(a), C_1(a), C_2(a), \ldots,$ such that

$$f(z) = \sum C_k(a) z^k \tag{2}$$

for all $z \ll a$. To see that this is sufficient we note that although, a priori, the coefficients could change as we consider complex numbers a of increasing magnitude, they are in fact constant.

For, as we saw in Chapter 2 (Corollary 2.11), it follows from (2) that f is infinitely differentiable at 0 and that

$$C_k(a) = \frac{f^{(k)}(0)}{k!}.$$

Hence the coefficients are independent of a. Note, finally, that although the everywhere convergence of the series

$$\sum \frac{f^{(k)}(0)}{k!} z^k$$

is not proven explicitly, it is implicit in the fact that the series equals $f(z)$ for all z. □

5.6 Corollary

An entire function is infinitely differentiable.

Proof

Since f has a power series expansion, we may invoke Corollary 2.10–an everywhere convergent power series is infinitely differentiable. □

5.7 Corollary

If f is entire and if a is any complex number, then

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots \qquad \textit{for all } z.$$

Proof

Consider $g(\zeta) = f(\zeta + a)$ which is likewise entire. By 5.5

$$g(\zeta) = g(0) + g'(0)\zeta + \frac{g''(0)}{2!}\zeta^2 + \cdots,$$

so that

$$f(\zeta + a) = f(a) + f'(a)\zeta + \frac{f''(a)}{2!}\zeta^2 + \cdots.$$

Setting $\zeta = z - a$, the corollary follows. □

5.8 Proposition

If f is entire and if

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & z \neq a \\ f'(a) & z = a \end{cases}$$

then g is entire.

Proof

By the previous corollary, for $z \neq a$

$$g(z) = f'(a) + \frac{f''(a)}{2!}(z-a) + \frac{f^{(3)}(a)}{3!}(z-a)^2 + \cdots, \tag{1}$$

and by the definition of g, (1) is also valid at $z = a$. Since g is equal to an everywhere convergent power series, g is entire. □

5.9 Corollary

Suppose f is entire with zeroes at $a_1, a_2, \ldots, a_N$. Then if g is defined by

$$g(z) = \frac{f(z)}{(z-a_1)(z-a_2)\ldots(z-a_N)} \quad \textit{for } z \neq a_k,$$

$\lim_{z\to a_k} g(z)$ *exists for $k = 1, 2, \ldots, N$, and if $g(a_k)$ is defined by these limits, then g is entire.*

Proof

Let $f_0(z) = f(z)$ and let

$$f_k(z) = \frac{f_{k-1}(z) - f_{k-1}(a_k)}{z - a_k} = \frac{f_{k-1}(z)}{z - a_k}, \qquad z \neq a_k.$$

Assuming that f_{k-1} is entire, it follows from Proposition 5.8 that $f_k(z)$ has a limit as $z \to a_k$ and if we define $f_k(a_k)$ to be this limit, f_k is entire. Since f_0 is entire by hypothesis, the proof follows by induction. □

5.2 Liouville Theorems and the Fundamental Theorem of Algebra; The Gauss-Lucas Theorem

5.10 Liouville's Theorem

A bounded entire function is constant.

Proof

Let a and b represent any two complex numbers and let C be any positively oriented circle centered at 0 and with radius $R > \max(|a|, |b|)$. Then according to the

Cauchy Integral Formula (5.3)

$$f(b) - f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-b}dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)(b-a)}{(z-a)(z-b)}dz$$

$$\ll \frac{M|b-a|\cdot R}{(R-|a|)(R-|b|)} \tag{1}$$

using the usual estimate, where M represents the supposed upper bound for $|f|$. Since R may be taken as large as desired and since the expression in (1) approaches 0 as $R \to \infty$, $f(b) = f(a)$ and f is constant. □

5.11 The Extended Liouville Theorem

If f is entire and if, for some integer $k \geq 0$, there exist positive constants A and B such that

$$|f(z)| \leq A + B|z|^k,$$

then f is a polynomial of degree at most k.

Proof

Note that the case $k = 0$ is the original Liouville Theorem. The general case follows by induction. Thus, we consider

$$g(z) = \begin{cases} \dfrac{f(z) - f(0)}{z} & z \neq 0 \\ f'(0) & z = 0. \end{cases}$$

By 5.8, g is entire and by the hypothesis on f,

$$|g(z)| \leq C + D|z|^{k-1}.$$

Hence g is a polynomial of degree at most $k - 1$ and f is a polynomial of degree at most k. □

5.12 Fundamental Theorem of Algebra

Every non-constant polynomial with complex coefficients has a zero in $\mathbb{C}$.

Proof

Let $P(z)$ be any polynomial. If $P(z) \neq 0$ for all $z \in \mathbb{C}$, $f(z) = 1/P(z)$ is an entire function. Furthermore if P is non-constant, $P \to \infty$ as $z \to \infty$ and f is bounded. But then, by Liouville's Theorem, f is constant, and so is P, contrary to our assumption. □

Remarks

1. If α is a zero of an n-th degree polynomial P_n, $P_n(z) = (z-\alpha)P_{n-1}(z)$, where P_{n-1} is a polynomial of degree $n-1$. This can be seen by the usual Euclidean Algorithm or by noting that

$$\left|\frac{P_n(z)}{z-\alpha}\right| \leq A + B|z|^{n-1}$$

and hence is equal to an $(n-1)$-st degree polynomial by the Extended Liouville Theorem.
2. α is called a zero of *multiplicity* k (or order k) if $P(z) = (z-\alpha)^k Q(z)$, where Q is a polynomial with $Q(\alpha) \neq 0$. Equivalently, α is a zero of multiplicity k if $P(\alpha) = P'(\alpha) = \cdots = P^{(k-1)}(\alpha) = 0,\ P^{(k)}(\alpha) \neq 0$. The equivalence of the two definitions is easily established and is left as an exercise.
3. Although the Fundamental Theorem of Algebra only assures the existence of a single zero, an induction argument shows that an n-th degree polynomial has n zeroes (counting multiplicity). For, assuming every k-th degree polynomial can be written

$$P_k(z) = A(z-z_1)\cdots(z-z_k),$$

it follows that

$$P_{k+1}(z) = A(z-z_0)(z-z_1)\cdots(z-z_k).$$

By the above remark, any polynomial

$$P_n(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0 \tag{2}$$

can also be expressed as

$$P_n(z) = a_n(z-z_1)(z-z_2)\cdots(z-z_n), \tag{3}$$

where $z_1, z_2, ...z_n$ are the zeroes of P_n. A comparison of (2) and (3) yields the well-known relations between the zeroes of a polynomial and its coefficients. For example,

$$\sum z_k = -a_{n-1}/a_n. \tag{4}$$

There are many entire functions, such as $e^z - 1$, which have infinitely many zeroes, and whose derivatives are never zero. So there is no general analytic analogue of Rolle's Theorem. However, for polynomials, the Gauss-Lucas Theorem, below, offers a striking analogy and, in some ways a stronger form, of Rolle's Theorem.

Recall that a convex set is one that contains the entire line segment connecting any two of its points. Hence, if z_1 and z_2 belong to a convex set, so does every complex number of the form $tz_1 + (1-t)z_2$, for $0 \leq t \leq 1$. We leave it as an exercise to show that if $z_1, z_2, ..., z_n$ belong to a convex set, so does every "convex" combination of the form

$$a_1z_1 + a_2z_2 + \cdots + a_nz_n;\ a_i \geq 0 \text{ for all } i, \text{ and } \sum a_i = 1. \tag{5}$$

5.13 Definition

The convex hull of a set S of complex numbers is the smallest convex set containing S.

5.14 Gauss-Lucas Theorem

The zeroes of the derivative of any polynomial lie within the convex hull of the zeroes of the polynomial.

Proof

Assume that the zeroes of P are $z_1, z_2, ..., z_n$ and that α is a zero of P' but not a zero of P, Then

$$\frac{P'(\alpha)}{P(\alpha)} = \frac{1}{\alpha - z_1} + \frac{1}{\alpha - z_2} + \cdots + \frac{1}{\alpha - z_n} = 0 \tag{6}$$

Rewriting

$$\frac{1}{\alpha - z_i} = \frac{\overline{\alpha} - \overline{z_i}}{|\alpha - z_i|^2}$$

we can apply (6) to obtain

$$\overline{\alpha} = \sum a_i \overline{z_i}, \text{ with } a_i = \frac{1}{|\alpha - z_i|^2} \Big/ \sum \frac{1}{|\alpha - z_i|^2} . \tag{7}$$

Finally, by taking conjugates in (7), we obtain an identical expression for α in terms of $z_1, z_2, ..., z_n$. Hence α is in the convex hull of $\{z_1, z_2, ..., z_n\}$. □

A final remark

The Fundamental Theorem of Algebra can be considered a "nonexistence theorem" in the following sense. Recall that the complex numbers come into consideration when the reals are supplemented to include a solution of the equation $x^2 + 1 = 0$. One might have supposed that further extensions would arise as we sought zeroes of other polynomials with real or complex coefficients. By the Fundamental Theorem of Algebra, all such solutions are already contained in the field of complex numbers, and hence no such further extensions are possible. This is usually expressed by saying that the field of complex numbers is *algebraically closed*.

5.3 Newton's Method and Its Application to Polynomial Equations

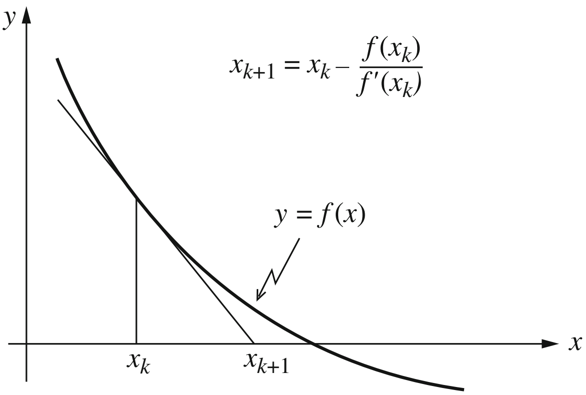
I. *Introduction* We saw in Chapter 1 that solutions of quadratic and cubic equations can be found in terms of square roots and cube roots of various expressions involving the coefficients. A similar formula is also available for fourth degree polynomial equations. On the other hand, one of the highlights of modern mathematics is the famous theorem that no such solution, in terms of n-th roots, can be given for the general polynomial equation of degree five or higher. In spite of this, there are many graphing calculators that allow the user to input the coefficients of a polynomial of any degree and then almost immediately output all of its zeroes, correct to eight or nine decimal places. The explanation for this magic is that, although there are no formulas for solving all polynomial equations, there are many algorithms which can be used to find arbitrarily good approximations to the solutions.

One extremely popular and effective method for approximating solutions to equations of the form $f(z) = 0$, variations of which are incorporated in many calculators, is known as Newton's Method. It can be informally described as follows:

i) Choose a point z_0 "sufficiently close" to a solution of the equation, which we will call s.
ii) Define $z_1 = z_0 - f(z_0)/f'(z_0)$ and continue recursively, defining $z_{n+1} = z_n - f(z_n)/f'(z_n)$.

Then, if z_0 is sufficiently close to the root s, the sequence $\{z_n\}$ will converge to s. In fact, the convergence is usually extremely rapid.

If we are trying to approximate a real solution s to the "real" equation $f(x) = 0$, the algorithm has a very nice geometric interpretation. That is, suppose $(x_0, f(x_0))$ is a point P on the graph of the function $y = f(x)$.Then the tangent to the graph at point P is given by the equation $L(x) = f(x_0) + f'(x_0)(x - x_0)$. Hence $x_1 = x_0 - f(x_0)/f'(x_0)$ is precisely the point where the tangent line crosses the x-axis.



Similarly, x_{n+1} is the zero of the tangent to $y = f(x)$ at the point $(x_n, f(x_n))$. Thus, there is a very clear visual insight into the nature of the sequence generated by the algorithm and it is easy to convince oneself that the sequence converges to the solution s in most cases. However, the geometric argument leaves many questions unanswered. For example, how do we know if x_0 is sufficiently close to the root s? Furthermore, if the sequence does converge, how quickly does it converge? Experimenting with simple examples will verify the assertion made earlier that the convergence is, in fact, very quick, but why is it? Finally, and of special interest to us, why does the method work in the complex plane, where the geometric interpretation is no longer applicable? The answer to all these questions can be found by taking a slight detour into the topic of fixed-point iteration.

II. *Fixed-Point Iteration* Suppose we are given an equation in the form $z = g(z)$. Then a solution s is a "fixed-point" of the function g. As we will see below, under the proper conditions, approximating such a fixed point can often be accomplished by recursively defining $z_{n+1} = g(z_n)$, a process known as fixed point iteration.

5.15 Lemma

Let s denote a root of the equation $z = g(z)$, for some analytic function g. Suppose that z_0 belongs to a disc of the form $D(s; r)$ throughout which $|g'(z)| \le K$, and let $z_1 = g(z_0)$. Then $|z_1 - s| \le K|z_0 - s|$.

Proof

Note that $|z_1 - s| = |g(z_0) - g(s)|$. Using the complex version of the Fundamental Theorem of Calculus,

$$g(z_0) - g(s) = \int_s^{z_0} g'(z)dz$$

where we choose the path of integration to be the straight line from s to z_0. The result then follows immediately from the $M - L$ formula. □

5.16 Theorem

Let s denote a root of the equation $z = g(z)$,for some analytic function g. Suppose that z_0 belongs to a disc of the form $D(s; r)$ throughout which $|g'(z)| \le K < 1$ and define the sequence $\{z_n\}$ recursively as: $z_{n+1} = g(z_n)$; $n = 0, 1, 2, \ldots$. Then $\{z_n\} \to s$ as $n \to \infty$.

Proof

Note that, as in Lemma 5.15,

$$|z_{n+1} - s| \le K|z_n - s|$$

and hence, by induction, $z_n \in D(s; r)$ for all n and $|z_n - s| \le K^n|z_0 - s|$. Since $K < 1$, the result follows immediately. □

5.17 Corollary

Let s denote a root of the equation $z = g(z)$, for some analytic function g and assume that $|g'(s)| < 1$. Then there exists a disc of the form $D(s; r)$ such that if $z_0 \in D(s{:}r)$

and if we define the sequence $\{z_n\}$ *recursively as:* $z_{n+1} = g(z_n)$; $n = 0, 1, 2, ...$, $\{z_n\} \to s$ *as* $n \to \infty$.

Proof

Since $|g'(s)| < 1$, there exists a constant K with $|g'(s)| < K < 1$. But then, since g' is analytic, there must exist exist a disc $D(s; r)$ throughout which $|g'(z)| < K$. □

Suppose we let $\varepsilon_n = |z_n - s|$ denote the n-th error, i.e. the absolute value of the difference between the n-th approximation z_n and the desired solution, s. Then the above results show that, with an appropriate starting value z_0, the sequence of errors satisfies the inequality

$$\varepsilon_{n+1} \le K\varepsilon_n \tag{1}$$

If, e.g. $K = \frac{1}{2}$, the error will be reduced by a factor of $\frac{1}{10}$ for every 3 or 4 iterations. An iteration scheme which satisfies inequality (1) for any value of K, $0 < K < 1$, is said to converge linearly. In that case, the number of iterations required to obtain n decimal place accuracy is roughly proportional to n.

Corollary 5.17 shows that an important condition for the convergence of fixed-point iteration is that $|g'(s)| < 1$ This raises the following practical problem. An equation in the familiar form $f(z) = 0$ can certainly be rewritten as an equivalent equation in the fixed point form $z = g(z)$. For example, one could simply add the monomial z to both sides of the equation. But how can we rewrite $f(z) = 0$ in the form $z = g(z)$ with the additional condition that $|g'(s)| < 1$ at the *unknown* solution s ? One answer to this problem will provide the insight to Newton's method that we are looking for. That is, suppose the equation $f(z) = 0$ is rewritten in the form $z = g(z) = z - f(z)/f'(z)$. Then the fixed point iteration algorithm is precisely Newton's Method. Moreover, we can find the exact value of $g'(s)$!!

5.18 Lemma

If f *is analytic and has a zero of order* k *at* $z = s$, *and if* $g(z) = z - f(z)/f'(z)$, *then* g *is also analytic at* s *and* $g'(s) = 1 - \frac{1}{k}$.

Proof

By hypothesis, $f(z) = (z - s)^k h(z)$, with $h(s) \neq 0$. Hence

$$f(z)/f'(z) = \frac{(z - s)h(z)}{kh(z) + (z - s)h'(z)}$$

Thus f/f' is analytic at s (with the appropriate value of 0 at s), and its power series expansion about the point s is of the form $\frac{1}{k}(z - s) + a_2(z - s)^2 + \cdots$. Hence $g'(s) = 1 - \frac{1}{k}$ □

Applying Corollary 5.17 then yields

5.19 Theorem

Let s denote a root of the equation $f(z) = 0$. Let $g(z) = z - f(z)/f'(z)$, and define the sequence $\{z_n\}$ recursively as: $z_{n+1} = g(z_n)$; $n = 0, 1, 2, ..$ Then there exists a disc of the form $D(s; r)$ such that $z_0 \in D(s; r)$ guarantees that $\{z_n\} \to s$ as $n \to \infty$. □

If $f(z)$ has a simple zero at s, according to Lemma 5.18, $g(z) = z - f(z)/f'(z)$ will have $g'(s) = 0$. In this case, the iteration scheme will converge especially rapidly.

5.20 Lemma

Let s denote a root of the equation $z = g(z)$, for some analytic function g such that $g'(s) = 0$. Suppose that z_0 belongs to a disc of the form $D(s; r)$ throughout which

$$|g''(z)| \leq M$$

and let $z_1 = g(z_0)$. Then $|z_1 - s| \leq \frac{1}{2}M|z_0 - s|^2$.

Proof

As in lemma 5.15, we begin by noting that $z_1 - s = g(z_0) - g(s) = \int_s^{z_0} g'(z)dz$. But for any value of z on the line segment $[s, z_0]$, we can write:

$$|g'(z)| = |g'(z) - g'(s)| = |\int_s^z g''(z)dz| \leq M|z - s| \tag{2}$$

Let $\Delta z = (z_0 - s)/n$ and write

$$\int_s^{z_0} g'(z)dz = \int_s^{s+\Delta z} g' + \int_{s+\Delta z}^{s+2\Delta z} g' + ... + \int_{z_0-\Delta z}^{z_0} g' \tag{3}$$

Then applying the *M-L* formula to each of the integrals in (3) and using the estimates for g' given by (2) show that $\int_s^{z_0} g'(z)dz$ is bounded by

$$\sum_{k=1}^{n} Mk(\Delta z)^2 = M\frac{n(n+1)}{2}\frac{|z_0 - s|^2}{n^2}$$

and the lemma follows by letting $n \to \infty$. □

5.21 Definition

If $\varepsilon_n = |z_n - s|$ satisfies $\varepsilon_{n+1} \leq K\varepsilon_n^2$, we say that the sequence $\{z_n\}$ converges quadratically to s.

Note that in the case of quadratic convergence, once the sequence of iterations is close to its limit, each iteration virtually doubles the number of decimal places which are accurate. If, for example, at some point the error is in the 10th decimal place, then at that point, ε_n is approximately 10^{-10}, so that $\varepsilon_{n+1} = K\varepsilon_n^2$ will be approximately 10^{-20}.

Lemmas 5.18 and 5.20 combine then to give us

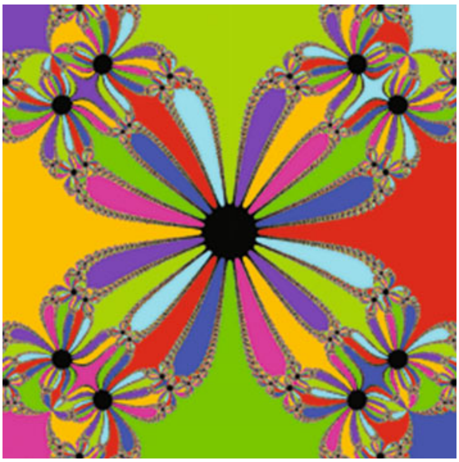
5.22 Theorem

If $f(z)$ has a simple zero at a point s, and if z_0 is sufficiently close to s, Newton's Method will produce a sequence which converges quadratically to s. □

III. *Newton's Method Applied to Polynomial Equations* While Newton's Method can be (and is) applied to all sorts of equations, it works especially well for polynomial equations. For one thing, we don't have to worry about the existence of solutions; they are guaranteed by the Fundamental Theorem of Algebra. That may be one reason why Newton himself applied his method only to polynomial equations. According to Theorems 5.19 and 5.22, as long as the initial approximation z_0 is sufficiently close to one of the roots, Newton's Method will converge to it. If we are looking for a simple zero of a polynomial, the method will actually converge quadratically. Of course, there are starting points which will not yield a convergent sequence. For example, if z_0 is a zero of the derivative of the polynomial, z_1 will not be defined! On the other hand, the set of "successful" starting points is surprisingly robust.

Modern technology has been applied to identifying what have been labeled "Newton basins", the distinct regions in the complex plane from which a starting value will yield a sequence converging to the distinct zeroes of a polynomial. If these regions are shaded in different colors, they yield remarkably interesting sketches. Aside from the example below, interested readers can generate their own sketches of the Newton basins for various polynomials at http://aleph0.clarku.edu/~djoyce/newton/technical.html

The sketch below shows the Newton basins for the eight zeroes of the polynomial $P(z) = (z^4 - 1)(z^4 + 4)$. The eight roots: $\pm 1, \pm i, \pm(1 + i), \pm(1 - i)$ are at the corners and the midpoints of the sides of the displayed square. The black regions contain the starting points which do not yield a convergent sequence.



Exercises

1. Find the power series expansion of $f(z) = z^2$ around $z = 2$.
2. Find the power series expansion for e^z about any point a.
3. f is called an *odd* function if $f(z) = -f(-z)$ for all z; f is called *even* if $f(z) = f(-z)$.
   a. Show that an odd entire function has only odd terms in its power series expansion about $z = 0$. [*Hint*: show f odd $\Rightarrow$ f' even, etc., or use the identity
   $$f(z) = \frac{f(z) - f(-z)}{2}.]$$
   b. Prove an analogous result for even functions.
4. By comparing the different expressions for the power series expansion of an entire function f, prove that
   $$f^{(k)}(0) = \frac{k!}{2\pi i}\int_C \frac{f(\omega)}{\omega^{k+1}}d\omega, \qquad k = 0, 1, 2, \ldots$$
   for any circle C surrounding the origin.
5. (A Generalization of the Cauchy Integral Formula). Show that
   $$f^{(k)}(a) = \frac{k!}{2\pi i}\int_C \frac{f(\omega)}{(\omega - a)^{k+1}}d\omega, \qquad k = 1, 2, \ldots$$
   where C surrounds the point a and f is entire.
6. a. Suppose an entire function f is bounded by M along $|z| = R$. Show that the coefficients C_k in its power series expansion about 0 satisfy
   $$|C_k| \le \frac{M}{R^k}.$$
   b. Suppose a polynomial is bounded by 1 in the unit disc. Show that all its coefficients are bounded by 1.
7. (An alternate proof of Liouville's Theorem). Suppose that $|f(z)| \le A + B|z|^k$ and that f is entire. Show then that all the coefficients C_j, $j > k$, in its power series expansion are 0. (See Exercise 6a.)
8. Suppose f is entire and $|f(z)| \le A + B|z|^{3/2}$. Show that f is a linear polynomial.
9. Suppose f is entire and $|f'(z)| \le |z|$ for all z. Show that $f(z) = a + bz^2$ with $|b| \le \frac{1}{2}$.
10. Prove that a nonconstant entire function cannot satisfy the two equations
    i. $f(z + 1) = f(z)$
    ii. $f(z + i) = f(z)$

    for all z. [*Hint*: Show that a function satisfying both equalities would be bounded.]
11. A *real polynomial* is a polynomial whose coefficients are all real. Prove that a real polynomial of odd degree must have a real zero. (See Exercise 5 of Chapter 1.)
12. Show that every real polynomial is equal to a product of real linear and quadratic polynomials.
13. Suppose P is a polynomial such that $P(z)$ is real if and only if z is real. Prove that P is linear. [*Hint*: Set $P = u + iv$, $z = x + iy$ and note that $v = 0$ if and only if $y = 0$.

    Conclude that:

    a. either $v_y \ge 0$ throughout the real axis or $v_y \le 0$ throughout the real axis;
    b. either $u_x \ge 0$ or $u_x \le 0$ for all real values and hence u is monotonic along the real-axis;
    c. $P(z) = \alpha$ has only one solution for real-valued α.]

14. Show that α is a zero of multiplicity k if and only if

$$P(\alpha) = P'(\alpha) = \cdots = P^{(k-1)}(\alpha) = 0,$$
$$\text{and } P^{(k)}(\alpha) \neq 0.$$

15. Suppose that f is entire and that for each z, *either* $|f(z)| \leq 1$ or $|f'(z)| \leq 1$. Prove that f is a linear polynomial. [*Hint*: Use a line integral to show

$$|f(z)| \leq A + |z| \quad \text{where } A = \max(1, |f(0)|).]$$

16.* Let $(z_1 + z_2 + \cdots + z_n)/n$ denote the *centroid* of the complex numbers $z_1, z_2, ..., z_n$. Use formula (4) in section 5.2 to show that the centroid of the zeroes of a polynomial is the same as the centroid of the zeroes of its derivative.

17.* Use induction to show that if $z_1, z_2, ..., z_n$ belong to a convex set, so does every "convex" combination of the form

$$a_1 z_1 + a_2 z_2 + \cdots + a_n z_n;\ a_i \geq 0 \text{ for all } i, \text{ and } \sum a_i = 1.$$

18.* Let $P_k(z) = 1 + z + z^2/2! + \cdots + z^k/k!$, the kth partial sum of e^z.

a. Show that, for all values of $k \geq 1$,the centroid of the zeroes of P_k is -1.
b. Let z_k be a zero of P_k with maximal possible absolute value. Prove that $\{|z_k|\}$ is an increasing sequence.

19.* Let $P(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$. Use the Gauss-Lucas theorem to show that all the zeroes of $P(z)$ are inside the unit disc. (See exercise 20 of Chapter 1 for a more direct proof.)

20.* Find estimates for $\sqrt{i}$ by applying Newton's method to the polynomial equation $z^2 = i$, with $z_0 = 1$.