Chapter 3
Analytic Functions

3.1 Analyticity and the Cauchy-Riemann Equations

The direct functions of z which we have studied so far—polynomials and convergent power series—were shown to be differentiable functions of z. We now take a closer look at the property of differentiability and its relation to the Cauchy-Riemann equations.

As we mentioned earlier (after Definition 2.4), if f is differentiable,

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

must exist regardless of the manner in which h approaches 0 through complex values. An immediate consequence is that the partial derivatives of f must satisfy the Cauchy-Riemann equations.

3.1 Proposition

If $f = u + iv$ is differentiable at z, f_x and f_y exist there and satisfy the Cauchy-Riemann equation

$$f_y = i f_x$$

or, equivalently,

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

Proof

Suppose first that $h \to 0$ through real values. Then

$$\frac{f(z+h) - f(z)}{h} = \frac{f(x+h, y) - f(x, y)}{h} \to f_x.$$

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On the other hand, if h $\to$ 0 along the imaginary axis, $h = i\eta$ and

$$\frac{f(z+h)-f(z)}{h} = \frac{f(x,y+\eta)-f(x,y)}{i\eta} \to \frac{f_y}{i}.$$

(See Exercise 1.) Since the two limits must be equal,

$$f_y = i f_x.$$

As we mentioned in Chapter 2, setting $f = u + iv$, the equation $f_y = i f_x$ takes the form

$$u_y + i v_y = i(u_x + i v_x)$$

and hence

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$ □

The converse of the above proposition is not true. There are functions which are not differentiable at a point despite the fact that the partial derivatives exist and satisfy the Cauchy-Riemann equations there.

For example, consider

$$f(z) = f(x,y) = \begin{cases} \dfrac{xy(x+iy)}{x^2+y^2} & z \neq 0 \\ 0 & z = 0. \end{cases}$$

$f = 0$ on both axes so that $f_x = f_y = 0$ at the origin but

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

does not exist. For on the line $y = \alpha x$

$$\frac{f(z)-f(0)}{z} \equiv \frac{\alpha}{1+\alpha^2} \quad \text{for } z \neq 0$$

and hence the limit depends on α!

The following partial converse, however, is true.

3.2 Proposition

Suppose f_x and f_y exist in a neighborhood of z. Then if f_x and f_y are continuous at z and $f_y = i f_x$ there, f is differentiable at z.

Proof

Let $f = u + iv, h = \xi + i\eta$.

We will show that

$$\frac{f(z+h)-f(z)}{h} \to f_x(z) = u_x(z) + i v_x(z)$$

as $h \to 0$. By the Mean-Value Theorem (for real functions of a real variable)

$$\begin{aligned}\frac{u(z+h)-u(z)}{h} &= \frac{u(x+\xi, y+\eta)-u(x,y)}{\xi+i\eta} \\ &= \frac{u(x+\xi, y+\eta)-u(x+\xi,y)}{\xi+i\eta} \\ &\quad + \frac{u(x+\xi, y)-u(x,y)}{\xi+i\eta} \\ &= \frac{\eta}{\xi+i\eta} u_y(x+\xi, y+\theta_1\eta) \\ &\quad + \frac{\xi}{\xi+i\eta} u_x(x+\theta_2\xi, y),\end{aligned}$$

and

$$\begin{aligned}\frac{v(z+h)-v(z)}{h} &= \frac{\eta}{\xi+i\eta} v_y(x+\xi, y+\theta_3\eta) \\ &\quad + \frac{\xi}{\xi+i\eta} v_x(x+\theta_4\xi, y)\end{aligned}$$

for some θ_k,

$$0 < \theta_k < 1, \quad k = 1, 2, 3, 4.$$

Thus

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} &= \frac{\eta}{\xi+i\eta}[u_y(z_1) + i v_y(z_2)] \\ &\quad + \frac{\xi}{\xi+i\eta}[u_x(z_3) + i v_x(z_4)]\end{aligned}$$

where $|z_k - z| \to 0$ as $h \to 0$, $k = 1, 2, 3, 4$. Since $f_y = i f_x$ at z we can subtract $f_x(z)$ in the form of

$$\frac{\eta}{\xi+i\eta} f_y + \frac{\xi}{\xi+i\eta} f_x$$

to obtain

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} - f_x(z) &= \frac{\eta}{\xi+i\eta}[(u_y(z_1)-u_y(z)) + i(v_y(z_2)-v_y(z))] \\ &\quad + \frac{\xi}{\xi+i\eta}[(u_x(z_3)-u_x(z)) + i(v_x(z_4)-v_x(z))].\end{aligned}$$

Finally, since

$$\left|\frac{\eta}{\xi + i\eta}\right|, \left|\frac{\xi}{\xi + i\eta}\right| \leq 1,$$

while each of the bracketed expressions approaches 0 as $h \to 0$,

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = f_x(z).$$

□

EXAMPLE

Let $f(z) = |z|^2 = x^2 + y^2$. Then $f_x = 2x$, $f_y = 2y$ so that f has continuous partial derivatives for all z. By the previous proposition, then f is differentiable if and only if $f_y = i f_x$. Hence f is differentiable only at the point $z = 0$. ◊

To avoid pathologies such as that given in the example above, we adopt the following definition.

3.3 Definition

f is *analytic at* z if f is differentiable in a neighborhood of z. Similarly, f is *analytic on a set* S if f is differentiable at all points of some open set containing S.

Note that this definition is consistent with Definition 2.1 for analytic polynomials. For we have already noted (Proposition 2.6) that "polynomials in z" are everywhere differentiable. Conversely, if a polynomial P is analytic at a point z, its partial derivatives must satisfy the Cauchy-Riemann equations throughout a neighborhood of z. Hence, as in Proposition 2.3, it follows that P must be a "polynomial in z."

Functions, such as polynomials or everywhere convergent power series, that are everywhere differentiable are called *entire* functions.

As we saw in Propositions 2.5 and 2.6, many of the properties of differentiability are analogous to those of differentiable functions of a real variable. Similarly, the composition of differentiable functions is differentiable (see Exercise 3). As in the "real" case, the inverse of a differentiable function need not even be continuous. Under the appropriate hypothesis, however, we can establish the differentiability of inverse functions.

3.4 Definition

Suppose that S and T are open sets and that f is 1-1 on S with $f(S) = T$. g is the *inverse of* f *on* T if $f(g(z)) = z$ for $z \in T$. g is the *inverse of* f *at* z_0 if g is the inverse of f in some neighborhood of z_0.

Note that an inverse function must be 1-1 for if $f^{-1}(z) = f^{-1}(z_0)$, $f(f^{-1}(z)) = f(f^{-1}(z_0))$; *i.e.*, $z = z_0$.

3.5 Proposition

Suppose that g is the inverse of f at z_0 and that g is continuous there. If f is differentiable at $g(z_0)$ and if $f'(g(z_0)) \neq 0$, then g is differentiable at z_0 and

$$g'(z_0) = \frac{1}{f'(g(z_0))}.$$

Proof

$$\frac{g(z) - g(z_0)}{z - z_0} = \frac{1}{\dfrac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)}}$$

for all $z \neq z_0$ in a neighborhood of z_0. Since g is continuous at z_0, $g(z) \to g(z_0)$ as $z \to z_0$, and by the differentiability of f,

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \frac{1}{f'(g(z_0))}.$$

□

As we shall see in the coming chapters, the property of analyticity is a very far-reaching one. Some immediate consequences are proven below.

3.6 Proposition

If $f = u + iv$ is analytic in a region D and u is constant, then f is constant.

Proof

Since u is constant, $u_x = u_y = 0$; therefore, by the Cauchy-Riemann equations, $v_x = v_y = 0$. According to Theorem 1.10, u and v are each constant in D; hence f is constant. □

3.7 Proposition

If f is analytic in a region and if $|f|$ is constant there, then f is constant.

Proof

If $|f| = 0$, the proof is immediate. Otherwise

$$u^2 + v^2 \equiv C \neq 0.$$

Taking the partial derivatives with respect to x and y, we see that

$$\begin{aligned} uu_x + vv_x &\equiv 0 \\ uu_y + vv_y &\equiv 0. \end{aligned}$$

Making use of the Cauchy-Riemann equations, we obtain

$$\begin{aligned} uu_x - vu_y &\equiv 0 \\ vu_x + uu_y &\equiv 0, \end{aligned}$$

so that

$$(u^2 + v^2)u_x \equiv 0$$

and $u_x = v_y \equiv 0$. Similarly, u_y and v_x are identically zero, hence f is constant. $\square$

3.2 The Functions e^z, sin z, cos z

We wish to define an exponential function of the complex variable z; that is, we seek an analytic function f such that

$$f(z_1 + z_2) = f(z_1)f(z_2), \tag{1}$$

$$f(x) = e^x \quad \text{for all real } x. \tag{2}$$

According to (1) and (2) we must have

$$f(z) = f(x + iy) = f(x)f(iy) = e^x f(iy).$$

Setting $f(iy) = A(y) + iB(y)$, it follows that

$$f(z) = e^x A(y) + ie^x B(y).$$

For f to be analytic, the Cauchy-Riemann equations must be satisfied; therefore $A(y) = B'(y)$ and $A'(y) = -B(y)$, so that $A'' = -A$. Thus we consider

$$\begin{aligned} A(y) &= \alpha \cos y + \beta \sin y \\ B(y) &= -A'(y) = -\beta \cos y + \alpha \sin y. \end{aligned}$$

Since $f(x) = e^x$, however, $A(0) = \alpha = 1$ and $B(0) = -\beta = 0$, so that, finally, we are led to examine

$$f(z) = e^x \cos y + ie^x \sin y.$$

Indeed, it is easy to verify that f is an entire function with the desired properties (1) and (2). (See Exercise 11.) Hence f is an entire "extension" of the real exponential function and we write $f(z) = e^z$.

The following properties of e^z are easily proven:

i. $|e^z| = e^x$.

ii. $e^z \neq 0$.

This follows from (i) since $e^x \neq 0$. Also, according to (1), above, $e^z e^{-z} = e^0 = 1$.

iii. $e^{iy} = \text{cis } y$.

iv. $e^z = \alpha$ has infinitely many solutions for any $\alpha \neq 0$.

Proof

Set $\alpha = r \operatorname{cis} \theta = re^{i\theta}$, $r > 0$. Since $e^z = e^x e^{iy}$, we will have $e^z = \alpha$ if $x = \log r$ and $e^{iy} = e^{i\theta}$. Hence $e^z = \alpha$ for all points $z = x + iy$ with $x = \log r$, $y = \operatorname{Arg} \alpha = \theta \pm 2k\pi, k = 0, 1, 2 \ldots.$ □

v. $(e^z)' = e^z$.
   Recall that $(e^z)' = (e^z)_x = e^z$.

To define $\sin z$ and $\cos z$, note that for real y

$$e^{iy} = \cos y + i \sin y$$
$$e^{-iy} = \cos y - i \sin y$$

so that

$$\sin y = \frac{1}{2i}(e^{iy} - e^{-iy})$$

and

$$\cos y = \frac{1}{2}(e^{iy} + e^{-iy}).$$

Thus we can define entire extensions of $\sin x$ and $\cos x$ by setting

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$
$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}).$$

Many of the familiar properties of the sin and cos functions remain valid in the larger setting of the complex plane. For example,

$$\sin 2z = 2 \sin z \cos z$$
$$\sin^2 z + cos^2 z = 1$$
$$(\sin z)' = \cos z.$$

These identities are easily verified and are left as an exercise. Moreover, in Section 6.3, we will see that, in general, functional equations of the above form, known to be true on the real axis, remain valid throughout the complex plane.

On the other hand, unlike $\sin x$, $\sin z$ is not bounded in modulus by 1. For example, $|\sin 10i| = \frac{1}{2}(e^{10} - e^{-10}) > 10,000$.

Exercises

1. Show that

$$f_x = \lim_{\substack{h \to 0 \\ h \text{ real}}} \frac{f(z+h) - f(z)}{h}; \; f_y = \lim_{\substack{h \to 0 \\ h \text{ real}}} \frac{f(z+ih) - f(z)}{h},$$

provided the limits exist.

2. a. Show that $f(z) = x^2 + iy^2$ is differentiable at all points on the line $y = x$.
   b. Show that it is nowhere analytic.

3. Prove that the composition of differentiable functions is differentiable. That is, if f is differentiable at z, and if g is differentiable at $f(z)$, then $g \circ f$ is differentiable at z. [*Hint*: Begin by noting

$$g(f(z+h)) - g(f(z)) = [g'(f(z)) + \epsilon][f(z+h) - f(z)]$$

where $\epsilon \to 0$ as $h \to 0$.]

4. Suppose that g is a continuous "$\sqrt{z}$" (i.e., $g^2(z) = z$) in some neighborhood of z. Verify that $g'(z) = 1/(2\sqrt{z})$. [*Hint*: Use

$$1 = \frac{g^2(z) - g^2(z_0)}{z - z_0}$$

to evaluate

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}.]$$

5. Suppose f is analytic in a region and $f' \equiv 0$ there. Show that f is constant.

6. Assume that f is analytic in a region and that at every point, either $f = 0$ or $f' = 0$. Show that f is constant. [*Hint*: Consider f^2.]

7. Show that a nonconstant analytic function cannot map a region into a straight line or into a circular arc.

8. Find all analytic functions $f = u + iv$ with $u(x, y) = x^2 - y^2$.

9. Show that there are no analytic functions $f = u + iv$ with $u(x, y) = x^2 + y^2$.

10. Suppose f is an entire function of the form

$$f(x, y) = u(x) + iv(y).$$

Show that f is a linear polynomial.

11. a. Show that e^z is entire by verifying the Cauchy-Riemann equations for its real and imaginary parts.
    b. Prove:

$$e^{z_1 + z_2} = e^{z_1} e^{z_2}.$$

12. Show: $|e^z| = e^x$.

13. Discuss the behavior of e^z as $z \to \infty$ along the various rays from the origin.

14. Find all solutions of
    a. $e^z = 1$, b. $e^z = i$,
    c. $e^z = -3$, d. $e^z = 1 + i$.

15. Verify the identities

    a. $\sin 2z = 2 \sin z \cos z$,
    b. $\sin^2 z + \cos^2 z = 1$,
    c. $(\sin z)' = \cos z$.

16.* Show that

    a. $\sin(\frac{\pi}{2} + iy) = \frac{1}{2}(e^y + e^{-y}) = \cosh y$
    b. $|\sin z| \geq 1$ at all points on the square with vertices $\pm(N + \frac{1}{2})\pi \pm (N + \frac{1}{2})\pi i$, for any positive integer N.
    c. $|\sin z| \to \infty$, as $\text{Im} z = y \to \pm\infty$.

17. Find $(\cos z)'$.

18. Find $\sin^{-1}(2)$–that is, find the solutions of $\sin z = 2$. [*Hint*: First set $w = e^{iz}$ and solve for ω.]

19.* Find all solutions of the equation:

$$e^{e^z} = 1.$$

20. Show that $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$.

21. Show that the power series

$$f(z) = 1 + z + \frac{z^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is equal to e^z. [*Hint*: First show that $f(z)f(w) = f(z + w)$, then show

$$f(x) = e^x$$
$$f(iy) = \cos y + i \sin y$$

using the power series representations for the real functions

$$e^x, \cos x, \sin x.]$$

22. Show:

$$g(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \ldots$$

is equal to $\sin z$. [*Hint*: Use the power series representation for e^z given in (21) to show that

$$g(z) = \frac{1}{2i}(e^{iz} - e^{-iz}).]$$

23. Find a power series representation for $\cos z$.