Chapter 2
Functions of the Complex Variable *z*

Introduction

We wish to examine the notion of a "function of z" where z is a complex variable. To be sure, a complex variable can be viewed as nothing but a pair of real variables so that in one sense a function of z is nothing but a function of two real variables. This was the point of view we took in the last section in discussing continuous functions. But somehow this point of view is too general. There are some functions which are "direct" functions of $z = x + iy$ and not simply functions of the separate pieces x and y.

Consider, for example, the function $x^2 - y^2 + 2ixy$. This is a direct function of $x + iy$ since $x^2 - y^2 + 2ixy = (x + iy)^2$; it is the function *squaring*. On the other hand, the only slightly different-looking function $x^2 + y^2 - 2ixy$ is not expressible as a polynomial in $x + iy$. Thus we are led to distinguish a special class of functions, those given by *direct* or *explicit* or *analytic* expressions in $x + iy$. When we finally do evolve a rigorous definition, these functions will be called the *analytic* functions. For now we restrict our attention to polynomials.

2.1 Analytic Polynomials

2.1 Definition

A polynomial $P(x, y)$ will be called an *analytic polynomial* if there exist (complex) constants α_k such that

$$P(x, y) = \alpha_0 + \alpha_1(x + iy) + \alpha_2(x + iy)^2 + \cdots + \alpha_N(x + iy)^N.$$

We will then say that P is a *polynomial in z* and write it as

$$P(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_N z^N.$$

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Indeed, $x^2 - y^2 + 2ixy$ is analytic. On the other hand, as we mentioned above, $x^2 + y^2 - 2ixy$ is *not* analytic, and we now prove this assertion. So suppose

$$x^2 + y^2 - 2ixy \equiv \sum_{k=0}^{N} \alpha_k (x + iy)^k.$$

Setting $y = 0$, we obtain

$$x^2 \equiv \sum_{k=0}^{N} \alpha_k x^k$$

or

$$\alpha_0 + \alpha_1 x + (\alpha_2 - 1)x^2 + \cdots + \alpha_N x^N \equiv 0.$$

Setting $x = 0$ gives $\alpha_0 = 0$; dividing out by x and again setting $x = 0$ shows $\alpha_1 = 0$, etc. We conclude that

$$\begin{aligned} &\alpha_1 = \alpha_3 = \alpha_4 = \cdots = \alpha_N = 0 \\ &\alpha_2 = 1, \end{aligned}$$

and so our assumption that

$$x^2 + y^2 - 2ixy \equiv \sum_{k=0}^{N} \alpha_k (x + iy)^k$$

has led us to

$$x^2 + y^2 - 2ixy \equiv (x + iy)^2 = x^2 - y^2 + 2ixy,$$

which is simply false!

A bit of experimentation, using the method described above (setting $y = 0$ and "comparing coefficients") will show how rare the analytic polynomials are. A randomly chosen polynomial, $P(x, y)$, will hardly ever be analytic.

EXAMPLE

$x^2 + iv(x, y)$ is not analytic for any choice of the real polynomial $v(x, y)$. For a polynomial in z can have a real part of degree 2 in x only if it is of the form $az^2 + bz + c$ with $a \neq 0$. In that case, however, the real part must contain a y^2 term as well. ♢

Another Way of Recognizing Analytic Polynomials We have seen, in our method of comparing coefficients, a perfectly adequate way of determining whether a given polynomial is or is not analytic. This method, we point out, can be condensed to the statement: $P(x, y)$ is analytic if and only if $P(x, y) = P(x+iy, 0)$. Looking ahead to the time we will try to extend the notion of "analytic" beyond the class of polynomials, however, we see that we can expect trouble! What is so simple for polynomials is totally intractable for more general functions. We can evaluate $P(x+iy,0)$ by simple arithmetic operations, but what does it mean to speak of $f(x+iy,0)$? For example, if $f(x,y) = \cos x + i\sin y$, we observe that $f(x,0) = \cos x$. But what shall we mean by $\cos(x+iy)$? What is needed is another means of recognizing the analytic polynomials, and for this we retreat to a familiar, real-variable situation. Suppose that we ask of a polynomial $P(x,y)$ whether it is a function of the single variable $x+2y$. Again the answer can be given in the spirit of our previous one, namely: $P(x,y)$ is a function of $x+2y$ if and only if $P(x,y) = P(x+2y,0)$. But it can also be given in terms of *partial derivatives*! A function of $x+2y$ undergoes the same change when x changes by ϵ as when y changes $\epsilon/2$ and this means exactly that its partial derivative with the respect to y is twice its partial derivative with respect to x. That is, $P(x,y)$ is a function of $x+2y$ if and only if $P_y = 2P_x$.

Of course, the "2" can be replaced by any real number, and we obtain the more general statement: $P(x,y)$ is a function of $x+\lambda y$ if and only if $P_y = \lambda P_x$.

Indeed for polynomials, we can even ignore the limitation that λ be real, which yields the following proposition.

2.2 Definition

Let $f(x,y) = u(x,y) + iv(x,y)$ where u and v are real-valued functions. The partial derivatives f_x and f_y are defined by $u_x + iv_x$ and $u_y + iv_y$ respectively, provided the latter exist.

2.3 Proposition

A polynomial $P(x,y)$ is analytic if and only if $P_y = iP_x$.

Proof

The necessity of the condition can be proven in a straightforward manner. We leave the details as an exercise. To show that it is also sufficient, note that if

$$P_y = iP_x,$$

the condition must be met separately by the terms of any fixed degree. Suppose then that P has n-th degree terms of the form

$$Q(x,y) = C_0x^n + C_1x^{n-1}y + C_2x^{n-2}y^2 + \cdots + C_ny^n.$$

Since

$$Q_y = iQ_x,$$

$$\begin{aligned} &C_1x^{n-1} + 2C_2x^{n-2}y + \cdots + nC_ny^{n-1} \\ &\qquad = i[nC_0x^{n-1} + (n-1)C_1x^{n-2}y + \cdots + C_{n-1}y^{n-1}]. \end{aligned}$$

Comparing coefficients,

$$C_1 = inC_0 = i\binom{n}{1}C_0$$
$$C_2 = \frac{i(n-1)}{2}C_1 = i^2\frac{n(n-1)}{2}C_0 = i^2\binom{n}{2}C_0,$$

and in general

$$C_k = i^k\binom{n}{k}C_0$$

so that

$$Q(x, y) = \sum_{k=0}^{n} C_k x^{n-k} y^k = C_0 \sum_{k=0}^{n} \binom{n}{k} x^{n-k}(iy)^k = C_0(x+iy)^n.$$

Thus P is analytic. □

The condition $f_y = if_x$ is sometimes given in terms of the real and imaginary parts of f. That is, if $f = u + iv$, then

$$f_x = u_x + iv_x$$
$$f_y = u_y + iv_y$$

and the equation $f_y = if_x$ is equivalent to the twin equations

$$u_x = v_y; \qquad u_y = -v_x. \tag{1}$$

These are usually called the *Cauchy-Riemann equations*.

EXAMPLES

1. A non-constant analytic polynomial cannot be real-valued, for then both P_x and P_y would be real and the Cauchy-Riemann equations would not be satisfied.
2. Using the Cauchy-Riemann equations, one can verify that $x^2 - y^2 + 2ixy$ is analytic while $x^2 + y^2 - 2ixy$ is not. ♢

Finally, we note that polynomials in z have another property which distinguishes them as functions of z: they can be differentiated directly with respect to z. We will make this more precise below.

2.4 Definition

A complex-valued function f, defined in a neighborhood of z, is said to be *differentiable at z* if

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists. In that case, the limit is denoted $f'(z)$.

It is important to note that in Definition 2.4, h is not necessarily real. Hence the limit must exist irrespective of the manner in which h approaches 0 in the complex plane. For example, $f(z) = \bar{z}$ is not differentiable at any point z since

$$\frac{f(z+h) - f(z)}{h} = \frac{\bar{h}}{h}$$

which equals $+1$ if h is real and -1 if h is purely imaginary.

2.5 Proposition

If f and g are both differentiable at z, then so are

$$\begin{aligned} h_1 &= f + g \\ h_2 &= fg \end{aligned}$$

and, if $g(z) \neq 0$,

$$h_3 = \frac{f}{g}.$$

In the respective cases,

$$\begin{aligned} h_1'(z) &= f'(z) + g'(z) \\ h_2'(z) &= f'(z)g(z) + f(z)g'(z) \\ h_3'(z) &= [f'(z)g(z) - f(z)g'(z)]/g^2(z). \end{aligned}$$

Proof

Exercise 6. □

2.6 Proposition

If $P(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_N z^N$, then P is differentiable at all points z and $P'(z) = \alpha_1 + 2\alpha_2 z + \cdots + N\alpha_N z^{N-1}$.

Proof

See Exercise 7. □

2.2 Power Series

We now consider a wider class of direct functions of z–those given by infinite polynomials or “power series” in z.

2.7 Definition

A *power series* in z is an infinite series of the form $\sum_{k=0}^{\infty} C_k z^k$.

To study the convergence of a power series, we recall the notion of the $\overline{\lim}$ of a positive real-valued sequence. That is,

$$\overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty}\left(\sup_{k\geq n} a_k\right).$$

Since $\sup_{k\geq n} a_k$ is a non-increasing function of n, the limit always exists or equals $+\infty$. The properties of the $\overline{\lim}$ which will be of interest to us are the following.

If $\overline{\lim}_{n\to\infty} a_n = L$,

i. for each N and for each $\epsilon > 0$, there exists some $k > N$ such that $a_k \geq L - \epsilon$;
ii. for each $\epsilon > 0$, there is some N such that $a_k \leq L + \epsilon$ for all $k > N$.
iii. $\overline{\lim}\, c a_n = cL$ for any nonnegative constant c.

2.8 Theorem

Suppose $\overline{\lim}|C_k|^{1/k} = L$.

1. *If* $L = 0$, $\sum C_k z^k$ *converges for all* z.
2. *If* $L = \infty$, $\sum C_k z^k$ *converges for* $z = 0$ *only.*
3. *If* $0 < L < \infty$, *set* $R = 1/L$. *Then* $\sum C_k z^k$ *converges for* $|z| < R$ *and diverges for* $|z| > R$. *(*R *is called the radius of convergence of the power series.)*

Proof

1. $L = 0$.
   Since $\overline{\lim}|C_k|^{1/k} = 0$, $\overline{\lim}|C_k|^{1/k}|z| = 0$ for all z. Thus, for each z, there is some N such that $k > N$ implies
   $$|C_k z^k| \leq \frac{1}{2^k},$$
   so that $\sum |C_k z^k|$ converges; therefore, by the Absolute Convergence Test, $\sum C_k z^k$ converges.
2. $L = \infty$.
   For any $z \neq 0$,
   $$|C_k|^{1/k} \geq \frac{1}{|z|}$$
   for infinitely many values of k. Hence $|C_k z^k| \geq 1$, the terms of the series do not approach zero, and the series diverges. (The fact that the series converges for $z = 0$ is obvious.)

3. $0 < L < \infty$, $R = 1/L$.
   Assume first that $|z| < R$ and set $|z| = R(1 - 2\delta)$. Then since $\overline{\lim}|C_k|^{1/k}|z| = (1 - 2\delta)$, $|C_k|^{1/k}|z| < 1 - \delta$ for sufficiently large k and $\sum C_k z^k$ is absolutely convergent. On the order hand, if $|z| > R$, $\overline{\lim}|C_k|^{1/k}|z| > 1$, so that for infinitely many values of k, $C_k z^k$ has absolute value greater than 1 and $\sum C_k z^k$ diverges. □

Note that if $\sum_{k=0}^{\infty} C_k z^k$ has radius of convergence R, the series converges uniformly in any smaller disc: $|z| \le R - \delta$. For then

$$\sum_{k=0}^{\infty} |C_k z^k| \le \sum_{k=0}^{\infty} |C_k|(R - \delta)^k,$$

which also converges. Hence a power series is continuous throughout its domain of convergence. (See Theorem 1.9.)

All three cases above can be combined by noting that a power series always converges inside a disc of radius

$$R = 1/\overline{\lim}|C_k|^{1/k}.$$

Here $R = 0$ means that the series converges at $z = 0$ only and $R = \infty$ means that the series converges for all z. In the cases where $0 < R < \infty$, while the theorem assures us that the series diverges for $|z| > R$, it says nothing about the behavior of the power series on the circle of convergence $|z| = R$. As the following examples demonstrate, the series may converge for all or some or none of the points on the circle of convergence.

EXAMPLES

1. Since $n^{1/n} \to 1$, $\sum_{n=1}^{\infty} nz^n$ converges for $|z| < 1$ and diverges for $|z| > 1$. The series also diverges for $|z| = 1$ for then $|nz^n| = n \to \infty$. (See Exercise 8.)
2. $\sum_{n=1}^{\infty}(z^n/n^2)$ also has radius of convergence equal to 1. In this case, however, the series converges for all points z on the unit circle since
$$\left|\frac{z^n}{n^2}\right| = \frac{1}{n^2} \quad \text{for } |z| = 1.$$
3. $\sum_{n=1}^{\infty}(z^n/n)$ has radius of convergence equal to 1. In this case, the series converges at all points of the unit circle except $z = 1$. (See Exercise 12.)
4. $\sum_{n=0}^{\infty}(z^n/n!)$ converges for all z since
$$\frac{1}{(n!)^{1/n}} \to 0.$$
   (See Exercise 13.)
5. $\sum_{n=0}^{\infty}[1 + (-1)^n]^n z^n$ has radius of convergence $\frac{1}{2}$ since $\overline{\lim}[1 + (-1)^n] = \lim 2 = 2$.

6. $\sum_{n=0}^{\infty} z^{n^2} = 1 + z + z^4 + z^9 + z^{16} + \cdots$ has radius of convergence 1. In this case $\overline{\lim}|C_n|^{1/n} = \lim 1 = 1$.
7. Any series of the form $\sum C_n z^n$ with $C_n = \pm 1$ for all n has radius of convergence equal to 1. ◊

It is easily seen that the sum of two power series is convergent wherever both of the original two power series are convergent. In fact, it follows directly from the definition of infinite series that

$$\sum_{n=0}^{\infty}(a_n + b_n)z^n = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n.$$

Similarly if $\sum_{n=0}^{\infty} a_n z^n = A$ and $\sum_{n=0}^{\infty} b_n z^n = B$, the Cauchy product $\sum_{n=0}^{\infty} c_n z^n$ defined by $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ converges for appropriate values of z to the product AB. The proof is the same as that for "real" power series and is outlined in Exercises 17 and 18.

2.3 Differentiability and Uniqueness of Power Series

We now show that power series, like polynomials, are differentiable functions of z. Suppose then that $\sum C_n z^n$ converges in some disc $D(0; R)$, $R > 0$. Then the series $\sum nC_n z^{n-1}$ obtained by differentiating $\sum C_n z^n$ term by term is convergent in $D(0; R)$, since

$$\overline{\lim}|nC_n|^{1/(n-1)} = \overline{\lim}(|nC_n|^{1/n})^{n/(n-1)} = \overline{\lim}|C_n|^{1/n}.$$

2.9 Theorem

Suppose $f(z) = \sum_{n=0}^{\infty} C_n z^n$ converges for $|z| < R$. Then $f'(z)$ exists and equals $\sum_{n=0}^{\infty} nC_n z^{n-1}$ throughout $|z| < R$.

Proof

We will prove the theorem in two stages. First, we will assume that $R = \infty$, then we will consider the more general situation. Of course, the second case contains the first, so the eager reader may skip the first proof. We include it since it contains the key ideas with less cumbersome details.

Case (1): Assume $\sum_{n=0}^{\infty} C_n z^n$ converges for all z. Then

$$\frac{f(z+h) - f(z)}{h} = \sum_{n=0}^{\infty} C_n \frac{[(z+h)^n - z^n]}{h}$$

and

$$\frac{f(z+h) - f(z)}{h} - \sum_{n=0}^{\infty} nC_n z^{n-1} = \sum_{n=2}^{\infty} C_n b_n$$

where

$$\begin{aligned} b_n &= \frac{(z+h)^n - z^n}{h} - nz^{n-1} \\ &= \sum_{k=2}^{n} \binom{n}{k} h^{k-1} z^{n-k} \le |h| \sum_{k=0}^{n} \binom{n}{k} |z|^{n-k} = |h|(|z|+1)^n \end{aligned}$$

for $|h| \le 1$. Hence, for $|h| \le 1$,

$$\left| \frac{f(z+h) - f(z)}{h} - \sum_{n=0}^{\infty} nC_n z^{n-1} \right| \le |h| \sum_{n=0}^{\infty} |C_n|(|z|+1)^n \le A|h|$$

since $\sum_{n=0}^{\infty} |C_n| z^n$ converges for all z. Letting $h \to 0$, we conclude that

$$f'(z) = \sum nC_n z^{n-1}.$$

Case (2): $0 < R < \infty$.

Let $|z| = R - 2\delta, \delta > 0$, and assume $|h| < \delta$. Then $|z+h| < R$ and, as in the previous case. we can write

$$\frac{f(z+h) - f(z)}{h} - \sum_{n=0}^{\infty} nC_n z^{n-1} = \sum_{n=2}^{\infty} C_n b_n,$$

where

$$b_n = \sum_{k=2}^{n} \binom{n}{k} h^{k-1} z^{n-k}.$$

If $z = 0, b_n = h^{n-1}$ and the proof follows easily. Otherwise, to obtain a useful estimate for b_n we must be a little more careful. Note then that

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \le n^2 \binom{n}{k-2} \text{ for } k \ge 2.$$

Hence, for $z \ne 0$,

$$\begin{aligned} |b_n| &\le \frac{n^2|h|}{|z|^2} \sum_{k=2}^{n} \binom{n}{k-2} |h|^{k-2} |z|^{n-(k-2)} \\ &\le \frac{n^2|h|}{|z|^2} \sum_{j=0}^{n} \binom{n}{j} |h|^j |z|^{n-j} \\ &= \frac{n^2|h|}{|z|^2} (|z|+|h|)^n \\ &\le \frac{n^2|h|}{|z|^2} (R-\delta)^n \end{aligned}$$

and

$$\left| \frac{f(z+h) - f(z)}{h} - \sum_{n=0}^{\infty} nC_n z^{n-1} \right| \leq \frac{|h|}{|z|^2} \sum_{n=0}^{\infty} n^2 |C_n| (R-\delta)^n \leq A|h|,$$

since $z \neq 0$ is fixed and since $\sum_{n=0}^{\infty} n^2 |C_n| z^n$ also converges for $|z| < R$. Again, letting $h \to 0$, we conclude that $f'(z) = \sum_{n=0}^{\infty} nC_n z^{n-1}$. □

Example
$f(z) = \sum_{n=0}^{\infty}(z^n/n!)$ is convergent for all z and, according to Theorem 2.9,

$$f'(z) = \sum_{n=0}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = f(z).$$

◊

2.10 Corollary

Power series are infinitely differentiable within their domain of convergence.

Proof

Applying the above results to $f'(z) = \sum_{n=0}^{\infty} nC_n z^{n-1}$ which has the same radius of convergence as f, we see that f is twice differentiable. By induction, $f^{(n)}$ is differentiable for all n. □

2.11 Corollary

If $f(z) = \sum_{n=0}^{\infty} C_n z^n$ has a nonzero radius of convergence,

$$C_n = \frac{f^{(n)}(0)}{n!} \quad \textit{for all } n.$$

Proof

By definition $f(0) = C_0$. Differentiating the power series term-by-term gives

$$f'(z) = C_1 + 2C_2 z + 3C_3 z^2 + \cdots$$

so that

$$f'(0) = C_1.$$

Similarly

$$f^{(n)}(z) = n!C_n + (n+1)!C_{n+1}z + \frac{(n+2)!}{2!}C_{n+2}z^2 + \cdots,$$

and the result follows by setting $z = 0$. □

According to Corollary 2.11, if a power series is equal to zero throughout a neighborhood of the origin, it must be identically zero. For then all its derivatives at the origin–and hence all the coefficients of the power series–would equal 0. By the same reasoning, if a power series were equal to zero throughout an interval containing the origin, it would be identically zero. An even stronger result is proven below.

2.12 Uniqueness Theorem for Power Series

Suppose $\sum_{n=0}^{\infty} C_n z^n$ *is zero at all points of a nonzero sequence* $\{z_k\}$ *which converges to zero. Then the power series is identically zero.*

[Note: If we set $f(z) = \sum C_n z^n$, it follows from the continuity of power series that $f(0) = 0$. We can show by a similar argument that $f'(0) = 0$; however, a slightly different argument is needed to show that the higher coefficients are also 0.]

Proof

Let

$$f(z) = C_0 + C_1 z + C_2 z^2 + \cdots .$$

By the continuity of f at the origin

$$C_0 = f(0) = \lim_{z \to 0} f(z) = \lim_{k \to \infty} f(z_k) = 0.$$

But then

$$g(z) = \frac{f(z)}{z} = C_1 + C_2 z + C_3 z^2 + \cdots$$

is also continuous at the origin and

$$C_1 = g(0) = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{k \to \infty} \frac{f(z_k)}{z_k} = 0.$$

Similarly, if $C_j = 0$ for $0 \le j < n$, then

$$C_n = \lim_{z \to 0} \frac{f(z)}{z^n} = \lim_{k \to \infty} \frac{f(z_k)}{z_k^n} = 0,$$

so that the power series is identically zero. □

2.13 Corollary

If a power series equals zero at all the points of a set with an accumulation point at the origin, the power series is identically zero.

Proof

Exercise 20. □

The Uniqueness Theorem derives its name from the following corollary.

2.14 Corollary

If $\sum a_n z^n$ and $\sum b_n z^n$ converge and agree on a set of points with an accumulation point at the origin, then $a_n = b_n$ for all n.

Proof

Apply 2.13 to the difference:

$$\sum (a_n - b_n) z^n.$$

□

Power Series Expansion about $z = \alpha$ All of the previous results on power series are easily adapted to power series of the form

$$\sum C_n (z - \alpha)^n.$$

By the simple substitution $w = z - \alpha$, we see, for example, that series of the above form converge in a disc of radius R about $z = \alpha$ and are differentiable throughout $|z - \alpha| < R$ where $R = 1/\overline{\lim}|C_n|^{1/n}$. (See Exercises 22 and 23.)

Exercises

1. Complete the proof of Proposition 2.3 by showing that for an analytic polynomial P, $P_y = iP_x$. [*Hint*: Prove it first for the monomials.]

2.* a. Suppose $f(z)$ is real-valued and differentiable for all real z. Show that $f'(z)$ is also real-valued for real z.
   b. Suppose $f(z)$ is real-valued and differentiable for all imaginary points z. Show that $f'(z)$ is imaginary at all imaginary points z.

3. By comparing coefficients or by use of the Cauchy-Riemann equations, determine which of the following polynomials are analytic.
   a. $P(x + iy) = x^3 - 3xy^2 - x + i(3x^2y - y^3 - y)$.
   b. $P(x + iy) = x^2 + iy^2$.
   c. $P(x + iy) = 2xy + i(y^2 - x^2)$.

4. Show that no nonconstant analytic polynomial can take imaginary values only.

5. Find the derivative $P'(z)$ of the analytic polynomials in (3). Show that in each case $P'(z) = P_x$. Explain.

6. Prove Proposition 2.5 by arguments analogous to those of real-variable calculus.

7. Prove Proposition 2.6. [*Hint*: Prove it for monomials and apply Proposition 2.5.]

8. Show $S_n = n^{1/n} \to 1$ as $n \to \infty$ by considering $\log S_n$.

9. Find the radius of convergence of the following power series:
   a. $\sum_{n=0}^{\infty} z^{n!}$, b. $\sum_{n=0}^{\infty} (n + 2^n) z^n$.

10. Suppose $\sum c_n z^n$ has radius of convergence R. Find the radius of convergence of
    a. $\sum n^p c_n z^n$, b. $\sum |c_n| z^n$,
    c. $\sum c_n^2 z^n$.

11. Suppose $\sum a_n z^n$ and $\sum b_n z^n$ have radii of convergence R_1 and R_2, respectively. What can be said about the radius of convergence of $\sum (a_n + b_n) z^n$? Show, by example, that the radius of convergence of the latter may be greater than R_1 and R_2.

12. Show that $\sum_{n=1}^{\infty} (z^n/n)$ converges at all points on the unit circle except $z = 1$. [*Hint*: Let $z = \operatorname{cis}\theta$ and analyze the real and imaginary parts of the series separately.]

13. a. Suppose $\{a_n\}$ is a sequence of positive real numbers and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$

Show then that $\lim_{n\to\infty} a_n^{1/n} = L$.

b. Use the result above to prove

$$\left(\frac{1}{n!}\right)^{1/n} \to 0.$$

14. Use Exercise (13a) to find the radius of convergence of

a. $\sum_{n=0}^{\infty} \dfrac{(-1)^n z^n}{n!}$,

b. $\sum_{n=0}^{\infty} \dfrac{z^{2n+1}}{(2n+1)!}$,

c. $\sum_{n=1}^{\infty} \dfrac{n! z^n}{n^n}$,

d. $\sum_{n=0}^{\infty} \dfrac{2^n z^n}{n!}$.

15.* Find the radius of convergence of

a. $\sum \sin n\, z^n$,

b. $\sum e^{-n^2} z^n$,

16.* Find the radius of convergence of $\sum c_n z^n$ if $c_{2k} = 2^k$; $c_{2k-1} = (1 + 1/k)^{k^2}$, $k = 1, 2, \ldots$

17. Suppose $\sum_{k=0}^{\infty} a_k = A$ and $\sum_{k=0}^{\infty} b_k = B$. Suppose further that each of the series is absolutely convergent. Show that if

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}$$

then

$$\sum_{k=0}^{\infty} c_k = AB.$$

Outline: Use the fact that $\sum |a_k|$ and $\sum |b_k|$ converge to show that $\sum d_k$ converges where

$$d_k = \sum_{j=0}^{k} |a_j||b_{k-j}|.$$

In particular,

$$d_{n+1} + d_{n+2} + \cdots \to 0 \text{ as } n \to \infty.$$

Note then that if

$$\begin{aligned} A_n &= a_0 + a_1 + \cdots + a_n \\ B_n &= b_0 + b_1 + \cdots + b_n \\ C_n &= c_0 + c_1 + \cdots + c_n, \end{aligned}$$

$A_n B_n = C_n + R_n$, where $|R_n| \le d_{n+1} + d_{n+2} + \cdots + d_{2n}$, and the result follows by letting $n \to \infty$.

18. Suppose $\sum a_n z^n$ and $\sum b_n z^n$ have radii of convergence R_1 and R_2 respectively. Show that the Cauchy product $\sum c_n z^n$ converges for $|z| < \min(R_1, R_2)$.

19. a. Using the identity

$$(1-z)(1+z+z^2+\cdots+z^N) = 1-z^{N+1}$$

show that

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \text{ for } |z| < 1.$$

b. By taking the Cauchy product of $\sum_{n=0}^{\infty} z^n$ with itself, find a closed form for $\sum_{n=0}^{\infty} nz^n$.

20. Prove Corollary 2.13 by showing that if a set S has an accumulation point at 0, it contains a sequence of nonzero terms which converge to 0.

21. Show that there is no power series $f(z) = \sum_{n=0}^{\infty} C_n z^n$ such that

   i. $f(z) = 1$ for $z = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ and

   ii. $f'(0) > 0$.

22. Assume $\overline{\lim}|C_n|^{1/n} < \infty$. Show that if we set

$$f(z) = \sum_{n=0}^{\infty} C_n (z-\alpha)^n,$$

then

$$C_n = \frac{f^{(n)}(\alpha)}{n!}.$$

23. Find the domain of convergence of

   a. $\sum_{n=0}^{\infty} n(z-1)^n$,

   b. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(z+1)^n$,

   c. $\sum_{n=0}^{\infty} n^2(2z-1)^n$.