Chapter 17
Different Forms of Analytic Functions

Introduction

The analytic functions we have encountered so far have generally been defined either by power series or as a combination of the elementary polynomial, trigonometric and exponential functions, along with their inverse functions. In this chapter, we consider three different ways of representing analytic functions. We begin with infinite products and then take a closer look at functions defined by definite integrals, a topic touched upon earlier in Chapter 7 and in Chapter 12.2. Finally, we define Dirichlet series, which provide a link between analytic functions and number theory.

17.1 Infinite Products

17.1 Definition

a. Let $\{u_k\}_{k=1}^{\infty}$ be a sequence of nonzero complex numbers. The infinite product $\prod_{k=1}^{\infty} u_k$ is said to converge if the sequence of partial products $P_N = u_1 u_2 \ldots u_N$ converges to a nonzero limit as $N \to \infty$. If $P_N \to 0$, we say the infinite product *diverges* to 0.
b. If finitely many terms u_k are equal to zero, we will say the product converges to zero provided $\prod_{\substack{k=1 \\ u_k \neq 0}}^{\infty} u_k$ converges.

EXAMPLES

i. $\prod_{k=1}^{\infty}(1 + 1/k) = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \ldots$ diverges (to ∞) since $P_N = N + 1 \to \infty$.
ii. $\prod_{k=2}^{\infty}(1 - 1/k)$ diverges to zero.
iii. $\prod_{k=2}^{\infty}(1 - 1/k^2) = \prod_{k=2}^{\infty}(k - 1)(k + 1)/k^2$ converges.
We leave it as an exercise to prove this by finding an explicit formula for P_N.
iv. $\prod_{k=1}^{\infty}(1 - 1/k^2)$ converges to 0 since $\prod_{k=2}^{\infty}(1 - 1/k^2)$ converges. $\lozenge$

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Notes

1. If $P_{N-1} \neq 0$,

$$u_N = \frac{P_N}{P_{N-1}}.$$

Hence if $\prod_{k=1}^{\infty} u_k$ converges, $u_N \to 1$ as $N \to \infty$. For this reason, we will usually write infinite products in the form $\prod_k (1+z_k)$ with the understanding that $z_k \to 0$ if the product converges.

2. If $\{a_k\}_{k=1}^{\infty}$ is a sequence of positive real numbers, $\prod_{k=1}^{\infty}(1+a_k)$ converges if and only if $\sum_{k=1}^{\infty} a_k$ converges. This follows from the inequalities

$$a_1 + a_2 + \cdots + a_N \leq \prod_{k=1}^{N}(1+a_k) \leq e^{a_1+a_2+\cdots+a_N}.$$

The right-hand inequality is a direct consequence of the fact that $1 + x \leq e^x$ for all real x. It is not true for complex numbers z_k, however, that $\prod_{k=1}^{\infty}(1+z_k)$ converges if any only if $\sum_{k=1}^{\infty} z_k$ converges (see Exercise 5), but we do have the following theorem.

17.2 Proposition

Let $z_k \neq -1,\ k = 1, 2, \ldots$. $\prod_{k=1}^{\infty}(1+z_k)$ *converges if and only if* $\sum_{k=1}^{\infty} \log(1+z_k)$ *converges.* (log z *here denotes the principal branch of the logarithm; i.e.,* $-\pi <$ Im $\log z = \operatorname{Arg} z \leq \pi$.)

Proof

Let $S_N = \sum_{k=1}^{N} \log(1+z_k)$. Then $P_N = e^{S_N}$ and if $S_N \to S,\ P_N \to P = e^S$. Suppose, on the other hand, that $P_N \to P \neq 0$. Then, some branch of the logarithm (which we will denote $\log^*$) is continuous at P and $\log^* P_N \to \log^* P$ as $N \to \infty$. Suppose we inductively define integers n_k so that

$$\sum_{k=1}^{N}(\log(1+z_k) + 2\pi i n_k) = \log^* P_N.$$

Then since $\log^* P_N$ converges,

$$\sum_{k=1}^{N}(\log(1+z_k) + 2\pi i n_k)$$

converges; therefore $\log(1+z_k) + 2\pi i n_k \to 0$ as $k \to \infty$. Since $z_k \to 0$ and log denotes the principal branch, it follows that $n_k = 0$ for k sufficiently large.

Hence $\sum_{k=1}^{\infty} \log(1+z_k)$ converges. □

17.3 Proposition

If $\sum_{k=1}^{\infty} |z_k|$ *converges,* $\prod_{k=1}^{\infty}(1+z_k)$ *converges.*

Proof

Assume $\sum_{k=1}^{\infty} |z_k|$ converges and take N such that for $k > N$, $|z_k| < \frac{1}{2}$. Then, for $k > N$

$$|\log(1+z_k)| = |z_k - \frac{z_k^2}{2} + \frac{z_k^3}{3} - + \cdots| \le |z_k| \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) \le 2|z_k|.$$

Hence $\sum_{k=N+1}^{\infty} \log(1+z_k)$ is convergent and by the previous proposition so is $\prod_{k=1}^{\infty}(1+z_k)$. □

17.4 Definition

$\prod_{k=1}^{\infty}(1+z_k)$ is called *absolutely convergent* if

$$\prod_{k=1}^{\infty}(1+|z_k|) \text{ converges.}$$

17.5 Proposition

An absolutely convergent product is convergent.

Proof

According to Note (2) (following Definition 17.1), the convergence of $\prod_{k=1}^{\infty}(1+|z_k|)$ is equivalent to the convergence of $\sum_{k=1}^{\infty} |z_k|$. Hence if $\prod_{k=1}^{\infty}(1+|z_k|)$ converges so does $\sum_{k=1}^{\infty} |z_k|$ and by the previous proposition, so does $\prod_{k=1}^{\infty}(1+z_k)$. □

We wish to consider analytic functions defined by infinite products; i.e., functions of the form

$$f(z) = \prod_{k=1}^{\infty}(1+u_k(z)).$$

Recall that f is analytic if each function u_k, $k = 1, 2, \ldots$ is analytic and the partial products converge to their limit function uniformly on compacta (Theorem 7.6).

17.6 Theorem

Suppose that $u_k(z)$ is analytic in a region D for $k = 1, 2, \ldots$, and that $\sum_{k=1}^{\infty} |u_k(z)|$ converges uniformly on compacta. Then the product $\prod_{k=1}^{\infty}(1+u_k(z))$ converges uniformly on compacta and represents an analytic function in D.

Proof

Let A be a compact subset of D. Since $\sum_{k=1}^{\infty} |u_k(z)|$ converges uniformly on A, for sufficiently large k, $|u_k(z)| < 1$ there. Hence, we may assume that $1 + u_k \neq 0$ for all k. If we then take N large enough so that $\sum_{k=N+1}^{\infty} |u_k(z)| < \epsilon/2$, it follows, as in the proof of Proposition 17.3, that

$$\left|\sum_{k=N+1}^{\infty} \log(1+u_k(z))\right| \leq \epsilon \text{ throughout } A.$$

That is, $\sum_{k=1}^{\infty} \log(1+u_k(z))$ converges uniformly on A to a limit function $S(z)$. It follows that $S(A)$ is bounded. Finally, since the exponential function is uniformly continuous in any bounded domain,

$$P_N(z) = \exp\left(\sum_{k=1}^{N} \log(1+u_k(z))\right)$$

converges uniformly to its limit function $e^{S(z)}$. □

EXAMPLES

1. $\prod_{k=1}^{\infty}(1+z^k)$ converges uniformly on any compact subset of the unit disc since any compact subset is contained in a disc of radius $\delta < 1$. Hence

$$\sum_{k=1}^{\infty} |z^k| \leq \sum_{k=1}^{\infty} \delta^k = \frac{\delta}{1-\delta}$$

and, by the M-test, $\sum_{k=1}^{\infty} |z^k|$ is uniformly convergent.

2.
$$\prod_{k=1}^{\infty}\left(1+\frac{1}{k^z}\right)$$

represents an analytic function in the half-plane $D : \operatorname{Re} z > 1$. In any compact subset of D, $\operatorname{Re} z \geq 1+\delta$ throughout so that

$$\left|\frac{1}{k^z}\right| = \frac{1}{k^{\operatorname{Re} z}} \leq \frac{1}{k^{1+\delta}}, \qquad k = 1, 2, \ldots .$$

Hence

$$\sum_{k=1}^{\infty}\left|\frac{1}{k^z}\right|$$

and, consequently,

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{k^z}\right)$$

are uniformly convergent. ◊

The Weierstrass Product Theorem. According to the Uniqueness Theorem (6.9), a nontrivial entire function cannot have an accumulation point of zeroes. That is, if $\{\lambda_k\} \to \lambda$ and if f is an entire function with zeroes at all the points λ_k, then $f \equiv 0$. On the other hand, an entire function may be zero at all the points of a sequence which diverges to ∞. For example $\sin z$ is zero at all integral multiples of π. Similarly, $e^z - 1$ is zero at all the integral multiples of $2\pi i$. The Weierstrass Product Theorem shows that these examples are in no way exceptional.

17.7 Theorem (Weierstrass)

Suppose $\{\lambda_k\}_{k=1}^{\infty} \to \infty$. *Then there exists an entire function* f *such that* $f(z) = 0$ *if and only if* $z = \lambda_k,\ k = 1, 2, \ldots$.

Note: To define an entire function with zeroes at the points λ_k, it would seem natural to write

$$f(z) = \prod_{k=1}^{\infty} (z - \lambda_k).$$

However, since $\lambda_k \to \infty$, the terms of the product would not approach 1 (for fixed z) and hence the product would diverge. Instead, we consider the infinite product of linear functions given by

$$f(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{\lambda_k}\right),$$

assuming for now that $\lambda_k \neq 0$. Indeed, if $\sum_{k=1}^{\infty} |1/\lambda_k|$ converges, $\sum_{k=1}^{\infty} |z/\lambda_k|$ converges uniformly on every compact set so that the product is uniformly convergent on compacta and gives the desired entire function. Moreover, if $\sum_{k=1}^{\infty} 1/|\lambda_k|$ diverges but $\sum_{k=1}^{\infty} 1/|\lambda_k|^2$ converges, we can modify the above construction by considering

$$f(z) = \prod_{k=1}^{\infty} \left[\left(1 - \frac{z}{\lambda_k}\right) e^{z/\lambda_k}\right].$$

With the "convergence factors" e^{z/λ_k}, the product is uniformly convergent on compacta since, for $|\lambda_k| > 2|z|$,

$$\begin{aligned}\left|\log\left[\left(1 - \frac{z}{\lambda_k}\right) e^{z/\lambda_k}\right]\right| &= \left|\left(-\frac{z}{\lambda_k} - \frac{z^2}{2\lambda_k^2} - \frac{z^3}{3\lambda_k^3} + \cdots\right) + \frac{z}{\lambda_k}\right| \\ &\leq \left|\frac{z^2}{\lambda_k^2}\right| \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots\right) = \left|\frac{z^2}{\lambda_k^2}\right|.\end{aligned}$$

Hence the series

$$\sum_{k=1}^{\infty} \log\left[(1 - z/\lambda_k) e^{z/\lambda_k}\right], \quad z \neq \lambda_k$$

is uniformly convergent and the product is uniformly convergent on compacta.

By the same reasoning, if $\sum_{k=1}^{\infty} 1/|\lambda_k|^{m+1}$ converges for some positive integer m and we consider the convergence factors

$$E_k(z) = \exp\left(z/\lambda_k + z^2/2\lambda_k^2 + \cdots + z^m/m\lambda_k^m\right),$$

it follows that the infinite product

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{\lambda_k}\right)E_k(z)$$

is uniformly convergent on compacta and represents an entire function with the desired zeroes. There are sequences $\{\lambda_k\}$, however, such that $\lambda_k \to \infty$ and yet $\sum_{k=1}^{\infty} 1/|\lambda_k|^N$ diverges for all N. (For example, $\{\lambda_k\} = \{\log k\}_{k=2}^{\infty}$.) Hence, for the general case we must introduce a slight variation.

Proof

Assume for the moment that $\lambda_k \neq 0$ and set

$$E_k(z) = \exp\left(\frac{z}{\lambda_k} + \frac{z^2}{2\lambda_k^2} + \cdots + \frac{z^k}{k\lambda_k^k}\right).$$

Suppose, moreover, that $|z| < M$. Then since $\lambda_k \to \infty$, for sufficiently large k, $|\lambda_k| > 2|z|$ and

$$\left|\log\left[\left(1-\frac{z}{\lambda_k}\right)E_k(z)\right]\right| \leq \sum_{j=k+1}^{\infty}\left|\frac{z^j}{j\lambda_k^j}\right| \leq \left|\frac{z}{\lambda_k}\right|^k \leq \frac{1}{2^k}.$$

Hence both

$$\sum_{k=1}^{\infty}\log\left[\left(1-\frac{z}{\lambda_k}\right)E_k(z)\right] \text{ and } \prod_{k=1}^{\infty}\left[\left(1-\frac{z}{\lambda_k}\right)E_k(z)\right]$$

are uniformly convergent on compacta. Note also that the individual factors are zero only at the points λ_k, and by the definition of convergence the infinite product is zero at those points only. Finally, if we seek an entire function with zeroes at the origin as well, we need only set

$$f(z) = z^P\prod_{k=1}^{\infty}\left[\left(1-\frac{z}{\lambda_k}\right)E_k(z)\right].$$

□

EXAMPLES

1. To find an entire function f with a single zero at every negative integer $\lambda_k = -k$, note that $\sum_{k=1}^{\infty} 1/|\lambda_k|$ diverges but $\sum_{k=1}^{\infty} 1/|\lambda_k|^2$ converges so that we can define

$$f(z) = \prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-z/k}.$$

2. An entire function with zeroes at all the points $\lambda_k = \log k,\ k = 1, 2, \ldots$, is given by

$$f(z) = z \prod_{k=2}^{\infty} \left[\left(1 - \frac{z}{\log k}\right) \exp\left(\frac{z}{\log k} + \frac{z^2}{2\log^2 k} + \cdots + \frac{z^k}{k \log^k k} \right) \right].$$

3. An entire function with a single zero at every integer is given by

$$f(z) = z \prod_{k=1}^{\infty} \left[\left(1 - \frac{z}{k}\right) e^{z/k} \left(1 + \frac{z}{k}\right) e^{-z/k} \right] = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right). \qquad \diamondsuit$$

17.8 Proposition

Let

$$f(z) = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Then $f(z) = (\sin \pi z)/\pi$.

Proof

Consider the quotient

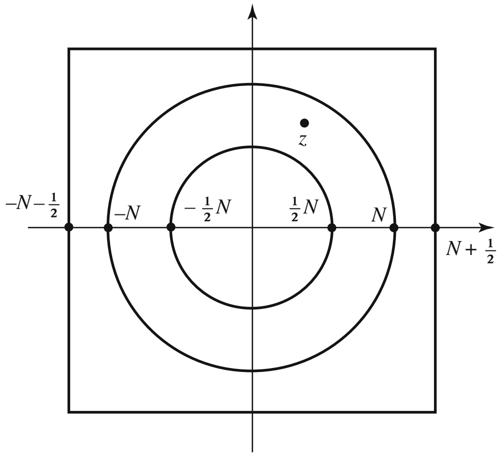
$$Q(z) = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) \Big/ \sin \pi z.$$

Q is entire and zero-free. To show that Q is constant we seek estimates on its growth for large z. Assume then that $\frac{1}{2}N \le |z| \le N$. Then $|Q(z)|$ is bounded by the maximum value assumed by Q on the square of side $2N + 1$ centered at the origin (Theorem 6.13). We have already proved, however, (see Chapter 11.2) that along this square (which avoids the zeroes of $\sin \pi z$), $|1/\sin \pi z| \le 4$. Moreover,

$$\begin{aligned}
\left| \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) \right| &= \left| \prod_{k=1}^{N} \left(1 - \frac{z}{k}\right)\left(1 + \frac{z}{k}\right) \prod_{k=N+1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) \right| \\
&\le \prod_{k=1}^{N} e^{2|z/k|} \prod_{k=N+1}^{\infty} e^{|z^2/k^2|} \\
&\le \exp\left(2|z|(1 + \log N) + \frac{|z^2|}{N} \right)
\end{aligned}$$

since

$$\sum_{k=1}^{N} \frac{1}{k} < 1 + \log N \text{ and } \sum_{k=N+1}^{\infty} \frac{1}{k^2} < \frac{1}{N}.$$



Noting again that for large N, $2(1+\log N) < \sqrt{N/2} \leq |z|^{1/2}$ while $|z^2|/N \leq |z|$, it follows that

$$|Q(z)| = \left| \frac{z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)}{\sin \pi z} \right| \leq A \exp(|z|^{3/2}).$$

By Theorem 16.12, then, we must have

$$\frac{z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)}{\sin \pi z} = Ae^{Bz}.$$

However, Q is an even function so that $B = 0$, and the constant A can be determined by noting that

$$A = Q(0) = \lim_{z \to 0} \frac{z}{\sin \pi z} = \frac{1}{\pi}.$$

□

Some consequences of the above proposition:

i. Setting $z = \frac{1}{2}$, we have

$$1 = \frac{\pi}{2} \prod_{k=1}^{\infty} \left[1 - \frac{1}{(2k)^2}\right]$$

so that

$$\frac{2}{\pi} = \left(\frac{1 \cdot 3}{2 \cdot 2}\right) \left(\frac{3 \cdot 5}{4 \cdot 4}\right) \left(\frac{5 \cdot 7}{6 \cdot 6}\right) \cdots$$

or

$$\pi = 2 \cdot \left(\frac{2 \cdot 2}{1 \cdot 3}\right) \left(\frac{4 \cdot 4}{3 \cdot 5}\right) \left(\frac{6 \cdot 6}{5 \cdot 7}\right) \cdots .$$

ii. Suppose we expand the terms in the product to obtain an infinite series. Then we will have

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$
$$= \pi z \left[1 - \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right) z^2 + \left(\sum_{k<j} \frac{1}{k^2 j^2}\right) z^4 - + \cdots \right].$$

A comparison with the familiar power series

$$\sin \pi z = \pi z - \frac{\pi^3 z^3}{6} + \frac{\pi^5 z^5}{120} - + \cdots$$

shows that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

(See 11.2 for an earlier proof of this identity.)

17.2 Analytic Functions Defined by Definite Integrals

We noted previously that Morera's Theorem (7.4) can be used to prove the analyticity of certain functions given in integral form. We now examine this notion in somewhat greater detail.

17.9 Theorem

Suppose $\varphi(z, t)$ is a continuous function of t, $a \le t \le b$, for fixed z and an analytic function of $z \in D$ for fixed t. Then

$$f(z) = \int_a^b \varphi(z, t)dt$$

is analytic in D and

$$f'(z) = \int_a^b \frac{\partial}{\partial z}(\varphi(z, t))dt. \tag{1}$$

Proof

Since f is a continuous function of z, according to Morera's Theorem (7.4), we need only prove that $\int_\Gamma f(z)dz = 0$ for any rectangle $\Gamma \subset D$. We can reverse the order of integration, however, and write

$$\int_\Gamma f(z)dz = \int_\Gamma \left(\int_a^b \varphi(z, t)dt\right) dz = \int_a^b \left(\int_\Gamma \varphi(z, t)dz\right) dt$$

since φ is continuous in t and in z. Thus, since φ is analytic in z,

$$\int_\Gamma f(z)dz = \int_a^b 0\,dt = 0$$

We leave it as an exercise to show that f' is given by the formula in (1). □

EXAMPLES

1. $f(z) = \int_0^1 dt/(t-z)$ is analytic in $D = \mathbb{C}\setminus[0,1]$.
   In fact, direct integration shows that $f(z) = \log(1-1/z)$, and we can use Theorem 10.8 to show that f is analytic in D. Recall then that $\Delta\text{Arg}(1-1/z)$, as z traverses a closed curve, gives the number of zeroes minus the number of poles of $1-1/z$ that lie inside the curve. Yet if the curve is a simple closed curve encircling the interval $[0,1]$, because $1-1/z$ has one zero and one pole inside, $\Delta\text{Arg}(1-1/z) = 0$. The same argument shows that f has a jump discontinuity of $2\pi i$ as z crosses through any point x, $0 < x < 1$ from the upper to the lower half-plane.
2. $g(z) = \int_0^\infty dt/(e^t - z)$ is analytic in $\mathbb{C}\setminus[1,\infty)$. Although g is given by an improper integral, it is the uniform limit of

$$g_N(z) = \int_0^N \frac{dt}{e^t - z}$$

   on any compact subset of $\mathbb{C}\setminus[1,\infty)$, and hence g is analytic. As we shall see below, g has a "jump" of $2\pi i/x$ as z crosses from the upper half-plane to the lower half-plane through any point $x > 1$. ◊

17.10 Proposition

Suppose that f and g are continuous real-valued functions on $[a,b]$ and that $f' > 0$ is also continuous. Then

$$F(z) = \int_a^b \frac{g(t)}{f(t) - z}dt$$

is analytic outside the interval $[\alpha,\beta]$ where $\alpha = f(a)$, $\beta = f(b)$ and

$$\lim_{y\to 0^+}[F(x+iy) - F(x-iy)] = 2\pi i \frac{g(f^{-1}(x))}{f'(f^{-1}(x))} \text{ for all } x \in (\alpha,\ \beta).$$

Proof

The analyticity of F is proven in Theorem 17.9. By rationalizing the denominator, we obtain

$$F(x+iy) = \int_a^b \frac{[f(t)-x]\,g(t)}{[f(t)-x]^2 + y^2}dt + iy\int_a^b \frac{g(t)dt}{[f(t)-x]^2 + y^2}.$$

Hence

$$F(x+iy)-F(x-iy)=2iy\int_a^b \frac{g(t)dt}{[f(t)-x]^2+y^2},$$

and setting $t = f^{-1}(u),\ \alpha = f(a), \beta = f(b)$

$$F(x+iy)-F(x-iy)=2i\int_\alpha^\beta \frac{yg(f^{-1}(u))du}{f'(f^{-1}(u))\left[(u-x)^2+y^2\right]}.$$

We leave it as an exercise to complete the proof by showing that

$$\int_\alpha^\beta \frac{h(u)y}{(u-x)^2+y^2}du \to \pi h(x)$$

as $y \to 0$ for any continuous function h on $[\alpha,\ \beta]$ and $\alpha < x < \beta$. □

17.3 Analytic Functions Defined by Dirichlet Series

Series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z}$$

are known as Dirichlet Series. Note that $n^{-z} = \exp(-z \log n)$ represents an entire function for every positive integer n. ($\log n$ is chosen as the principal value; i.e., $\log n$ is real-valued, so n^{-z} is positive for all real z. The coefficients a_n, of course, can be any complex constants.) Since the partial sums are entire, a function $f(z)$, defined by a Dirichlet series, is analytic in any region where the series converges uniformly. According to the theorems below, the natural regions of convergence for Dirichlet series are half-planes of the form $\operatorname{Re} z > x_0$, much as discs centered at the origin are the natural regions associated with power series.

17.11 Theorem

If $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ *converges for* $z = z_0$*, then it converges for all* z *in the half-plane* $H = \{z : \operatorname{Re} z > \operatorname{Re} z_0\}$. *Moreover, the convergence is uniform in any compact subset of* H.

Proof

To show that $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ converges, we will show that the partial sums form a Cauchy sequence. That is, we will show that

$$\left|\sum_{n=M}^{N} \frac{a_n}{n^z}\right| = \left|\frac{a_M}{M^z} + \cdots \frac{a_N}{N^z}\right|$$

is arbitrarily small for sufficiently large values of M.

Our proof is based on "summation by parts" and the following two observations:

(i) Since

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{z_0}}$$

converges, there exists a positive constant A with

$$\left| \sum_{n=1}^{T} \frac{a_n}{n^{z_0}} \right| < A \tag{2}$$

for all positive integers T.

(ii)

$$\left| \frac{1}{n^w} - \frac{1}{(n+1)^w} \right| < \frac{|w|}{n^{\operatorname{Re} w+1}}$$

The above inequality follows easily from the usual $M - L$ formula, since

$$\frac{1}{n^w} - \frac{1}{(n+1)^w} = \int_n^{n+1} wt^{-w-1} dt.$$

To complete the proof, let

$$A_n = \sum_{k=1}^{n} \frac{a_k}{k^{z_0}}, \qquad b_n = \frac{1}{n^w}, \text{ with } w = z - z_0.$$

Then

$$\begin{aligned} \frac{a_M}{M^z} + \cdots + \frac{a_N}{N^z} &= (A_M - A_{M-1})b_M + \cdots + (A_N - A_{N-1})b_N \\ &= -A_{M-1}b_M + \sum_{k=M}^{N-1} A_k(b_k - b_{k+1}) + A_N b_N. \end{aligned} \tag{3}$$

$-A_{M-1}b_M$ and $A_N b_N$ both go to zero for sufficiently large values of M and N, since

$$|A_k| < A \text{ for all } k, \quad \text{and} \quad |b_n| = 1/n^{\operatorname{Re}(z-z_0)}$$

The remaining sum on the right side of (3) is also arbitrarily small for sufficiently large M since, according to (i) and (ii), it is bounded in absolute value by

$$\sum_{k=M}^{\infty} \frac{A|z - z_0|}{k^{1+\delta}}, \text{ where } \delta = \operatorname{Re}(z - z_0),$$

which is the "tail" of a convergent series. Hence

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z}$$

converges.

Finally, note that if K is a compact subset of H, there is a positive value of δ, with $\operatorname{Re}(z - z_0) > \delta$ for all z in K, as well as a positive constant B with $|z| < B$ throughout K. Hence the expression in (3) will have a uniformly small absolute value for all z in K, once M is sufficiently large. So the series converges to its limit function uniformly in K. □

Note that in the proof of Theorem 17.11, we never actually used the convergence of the Dirichlet series at z_0. The only actual requirement for the conclusion was that there was a finite upper bound for the absolute value of its partial sums.

EXAMPLE

Suppose $a_n = (-1)^n$. Then $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ has bounded partial sums (although it diverges) at $z = 0$. According to Theorem 17.11, then, it converges and represents an analytic function in the right half-plane: $\operatorname{Re} z > 0$. The fact that it diverges at $z = 0$ also implies that its partial sums are not bounded for any value of z with a negative real part. ◊

17.12 Theorem

If $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ converges for some, but not all, values of z, there exists a real constant x_0 (called the abscissa of convergence) such that $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ converges if $\operatorname{Re} z > x_0$ and diverges if $\operatorname{Re} z < x_0$.

Proof

Let x_0 be the greatest lower bound of the real parts of all the complex numbers z for which $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ converges. By Theorem 17.11, if $x_0 = -\infty$, the series converges for all z. If the series neither converges for all z nor diverges for all z, $-\infty < x_0 < \infty$ and the theorem follows from Theorem 17.11 □

The abscissa of convergence of the Dirichlet series bears an obvious analogy to the radius of convergence of a power series. However, the analogy does not extend to the idea of absolute convergence. Power series converge absolutely in any compact subset of their region of convergence. On the other hand, consider the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^z}$$

As we mentioned above, the series converges (and represents an analytic function) in the right half-plane: $\operatorname{Re} z > 0$. However, it converges absolutely only if $\operatorname{Re} z > 1$.

This is the general situation with Dirichlet series. In addition to the half-plane of convergence H, there is a half-plane of absolute convergence H_1, which may be a proper subset of H.

17.13 Theorem

Suppose $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ *converges absolutely for some, but not all, values of* z*. Then there exists a constant* x_1 *(called the abscissa of absolute convergence) such that* $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ *converges absolutely if* $\operatorname{Re} z > x_1$ *and does not converge absolutely if* $\operatorname{Re} z < x_1$*.*

Proof

$$\left|\frac{a_n}{n^z}\right| = \frac{|a_n|}{n^x}$$

So if $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ is absolutely convergent at z_0, it is also absolutely convergent at all points z with $\operatorname{Re} z \geq \operatorname{Re} z_0$. The theorem follows with x_1 equal to the greatest lower bound of the real parts of all complex z for which $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ converges absolutely. □

Note that if the coefficients a_n are all positive, the abscissas of convergence and absolute convergence must be identical. Otherwise there would be a real number x between them where the Dirichlet series is convergent but not absolutely convergent. But this is obviously impossible since the terms in the Dirichlet series, for real values of z, are all positive.

EXAMPLE

The function $\zeta(z)$ is defined by the Dirichlet series $\sum_{n=1}^{\infty} \frac{1}{n^z}$. This series converges absolutely for $\operatorname{Re} z > 1$, and diverges if $\operatorname{Re} z < 1$. ◊

Since Dirichlet series converge uniformly within their half-plane of convergence, they can be differentiated term-by-term. So if $f(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z}$, then

$$f'(z) = \sum_{n=1}^{\infty} \frac{-a_n \log n}{n^z}.$$

For any value of z within the half-planes of convergence for two Dirichlet series, we have :

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z} + \sum_{n=1}^{\infty} \frac{b_n}{n^z} = \sum_{n=1}^{\infty} \frac{a_n + b_n}{n^z}.$$

We can also multiply two Dirichlet series. Rewriting the product as another Dirichlet series involves a rearrangement of the terms, which is justified if the two series are absolutely convergent. Hence, within the half-planes of *absolute* convergence, we have

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z} \sum_{n=1}^{\infty} \frac{b_n}{n^z} = \sum_{n=1}^{\infty} \frac{c_n}{n^z}$$

with c_n defined as the "convolution" of a_n and b_n. That is,

$$c_n = \sum_{d|n} a_d b_{n/d}$$

where the sum is taken over all the positive divisors of n.

EXAMPLE

$$\zeta^2(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{d(n)}{n^z}$$

where $d(n)$ equals the number of positive divisors of n.

Exercises

1. Prove

$$\prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right)$$

converges by finding an explicit formula for P_N.

2. As above, prove

$$\prod_{k=2}^{\infty} \left[1 + \frac{(-1)^k}{k}\right]$$

converges.

3.* Prove that $\prod_n (1 + \frac{i}{n})$ diverges, but $\prod_n |1 + \frac{i}{n}|$ converges.

4. Show that if $\sum_{k=1}^{\infty} z_k$ converges and $\sum_{k=1}^{\infty} |z_k|^2$ converges, then $\prod_{k=1}^{\infty}(1 + z_k)$ converges.

5. Show that

$$\prod_{k=2}^{\infty} \left[1 + \frac{(-1)^k}{\sqrt{k}}\right]$$

diverges even though

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{\sqrt{k}}$$

converges.

6. Prove that $(1+z)(1+z^2)(1+z^4)\ldots = \prod_{k=0}^{\infty}(1+z^{2^k})$ converges uniformly on compacta to $1/(1-z)$ in $|z| < 1$. [*Hint*: Find P_N.]

7. Define an entire function g with single zeroes at and only at all the "squares" $\lambda_k = k^2$; $k = 1, 2, \ldots$.

8. Show that one solution to (7) is given by $\sin \pi\sqrt{z}/\pi\sqrt{z}$.

9. Prove that

$$\cos \pi z = \prod_{k=0}^{\infty} \left[1 - \frac{4z^2}{(2k+1)^2}\right].$$

10. a. Define a function f, analytic in $|z| < 1$ and such that

$$f(z) = 0 \quad \text{if and only if } z = 1 - \frac{1}{k};\ k = 1, 2, \ldots.$$

[*Hint*: Find an entire function g with zeroes at $\lambda_k = k, k = 1,\ 2, \ldots$ and consider $f(z) = g(1/(1-z))$.]

b. Generalize the above results.

11. Given $F(z) = \int_a^b \varphi(z,t)dt$. Derive the formula for $F'(z)$ by writing

$$F'(z) = \frac{1}{2\pi i}\int_C \frac{F(\zeta)}{(\zeta - z)^2}d\zeta = \frac{1}{2\pi i}\int_C \left(\int_a^b \frac{\varphi(\zeta,t)}{(\zeta - z)^2}dt\right)d\zeta$$

and switching the order of integration.

12. Complete Proposition 17.10 by splitting

$$\int_\alpha^\beta \frac{h(u)ydu}{(u-x)^2 + y^2} \text{ into } \int_\alpha^{x-\epsilon} + \int_{x-\epsilon}^{x+\epsilon} + \int_{x+\epsilon}^\beta .$$

.

13. Show that

$$f(z) = \int_0^1 \frac{dt}{1 - zt}$$

is analytic outside $[1, \infty]$. Find the discontinuity of f as z "crosses" a point $x > 1$.

14.* a. Let $\phi(n)$ be the Euler totient function; i.e., the number of positive integers not exceeding n, which are relatively prime to n. Prove that

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^z}$$

is absolutely convergent for $\text{Re}\, z > 2$.

b. It can be shown that $\sum_{d|n} \phi(d) = n$, for all $n \geq 1$ [Apostol, p.26]. Show that $\zeta(z) \sum_{n=1}^{\infty} \dfrac{\phi(n)}{n^z} = \zeta(z-1)$, for $\text{Re}\, z > 2$.