Chapter 15
Maximum-Modulus Theorems for Unbounded Domains

15.1 A General Maximum-Modulus Theorem

The Maximum-Modulus Theorem (6.13) shows that a function which is C-analytic in a compact domain D assumes its maximum modulus on the boundary. In general, if we consider unbounded domains, the theorem no longer holds. For example, $f(z) = e^z$ is analytic and unbounded in the right half-plane despite the fact that on the boundary $|e^z| = |e^{iy}| = 1$. Nevertheless, given certain restrictions on the growth of the function, we can conclude that it attains its maximum modulus on the boundary. The most natural such condition is that the function remain bounded throughout D.

15.1 Theorem

Suppose f is C-analytic in a region D. If there are two constants M_1 and M_2 such that

$$|f(z)| \le M_1 \quad \text{for } z \in \partial D$$
$$|f(z)| \le M_2 \quad \text{for all } z \in D$$

then, in fact,

$$|f(z)| \le M_1 \quad \text{for all } z \in D.$$

Proof

Without loss of generality, we suppose $|f(z)| \le 1$ on ∂D. Assuming, then, that $|f(z)| \le M$ in D, we wish to prove $|f(z_0)| \le 1$ for every $z_0 \in D$. We will first prove the theorem in the special case where D is the right half-plane and then extend the proof to a general region.

In the case of the right half-plane, fix $z_0 \in D$ and consider the auxiliary function

$$h(z) = \frac{f^N(z)}{z+1}$$

where N is a positive integer. By the hypothesis on f, $|h(z)| \le 1$ on the imaginary axis and $|h(z)| \le M^N/R$ for all $z \in D$ such that $|z| = R$. Thus we have

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$|h(z)| \le \mathrm{Max}(1, M^N/R)$ on the boundary of the right semi-circle $D_R = \{z \in D : |z| \le R\}$. Choosing $R > M^N$ and large enough so that $z_0 \in D_R$, we conclude $|h(z)| \le 1$ along the boundary of the *compact* domain D_R and hence by the Maximum Modulus Theorem $|h(z_0)| \le 1$. Thus for each $z_0 \in D$

$$\left|\frac{f^N(z_0)}{z_0+1}\right| \le 1$$

or

$$|f(z_0)| \le |z_0+1|^{1/N}.$$

If we now let $N \to \infty$, we see $|f(z_0)| \le 1$ as desired.

In the more general case, where D is an arbitrary region, we must replace $1/(z+1)$ by a function g, analytic in D and such that $g(z) \to 0$ as $z \to \infty$. Such a function is given by

$$g(z) = \frac{f(z) - f(a)}{z - a}$$

where a is any fixed point in D. Clearly g, like f, is C-analytic in D (Proposition 6.7). The boundedness of f assures $g(z) \to 0$ as $z \to \infty$ and this, in turn, implies that $|g(z)| \le K$, some constant, throughout $\bar{D}$.

Again, we set $D_R = \{z \in D : |z| \le R\}$. Setting $h(z) = f^N(z)g(z)$, because $g \to 0$ as $z \to \infty$ we may take R large enough so that $|h(z)| \le K$ along the boundary of D_R. Hence, by the Maximum Modulus Theorem, $|h(z_0)| \le K$ for every $z_0 \in D$. Assuming, then, that $g(z_0) \ne 0$, we can write

$$|f(z_0)| \le \left|\frac{K}{g(z_0)}\right|^{1/N},$$

and letting $N \to \infty$ yields $|f(z_0)| \le 1$. Note, finally, that unless f is constant, the zeroes of g form a discrete set (Theorem 6.9); hence, by continuity,

$$|f(z_0)| \le 1 \text{ for every } z_0 \in D. \qquad \square$$

The above theorem may be used to derive the following stronger form of Liouville's Theorem.

15.2 Definition

Let γ be a path parameterized by $\gamma = \gamma(t)$, $0 \le t < \infty$. We will say that *f approaches infinity along γ* if, for any positive integer N, there exists a point t_0 such that

$$|f(\gamma(t))| \ge N \quad \text{for all } t \ge t_0.$$

15.3 Theorem

If f is a nonconstant entire function, there exists a curve along which f approaches infinity.

Note: An equivalent formulation of Liouville's Theorem (5.10) is that, for any non-constant entire function f, there exists a sequence of points $z_1, z_2 \ldots$ such that $f(z_n) \to \infty$ as $n \to \infty$. However, the existence of a curve along which $f \to \infty$ does not immediately follow. If we simply connect the points $z_1, z_2, \ldots$ successively, we have no control over the behavior of f at the intermediate points. The proof of Theorem 15.3 will depend on judiciously choosing the points z_k and the connecting lines so that we can guarantee that $f \to \infty$ along the path thus formed.

Proof of Theorem 15.3

Let $T_1 = \{z : |f(z)| > 1\}$ and fix S_1, a connected component of T_1. We will need the following facts about S_1:

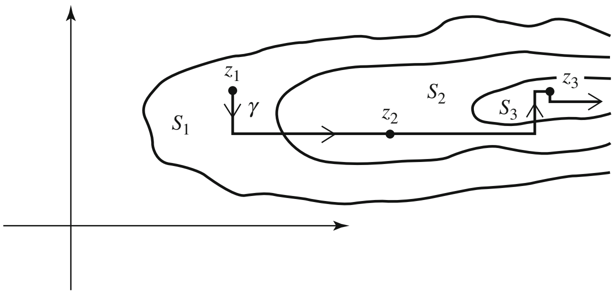
a. S_1 is an open set
b. $|f(z)| = 1$ for $z \in \partial S_1$
c. f is unbounded on S_1.

(a) is immediate. To prove (b), we first note that $|f(z)| \geq 1$ on the boundary of S_1 by continuity. If $|f(z)| > 1$ for some $z \in \partial S_1$, then $|f(w)| > 1$ for all w in a neighborhood of z and thus z would be an interior, rather than a boundary point of S_1. Finally, if f were bounded throughout S_1, we could apply (b) and Theorem 15.1 to show $|f(z)| \leq 1$ throughout S_1, contradicting its definition.

Now set $T_2 = \{z \in S_1 : |f(z)| > 2\}$ and choose a connected component S_2. (Note that, by (c), T_2 is non-empty.) As above, we can prove that f is unbounded on S_2. Proceeding inductively, we obtain a sequence of regions

$$S_1 \supset S_2 \supset S_3 \supset \cdots$$

such that $|f(z)| > k$ for all $z \in S_k$.



Finally, we choose a point $z_k \in S_k$ for $k = 1, 2, \ldots$. Since each set S_k is a region which contains all points z_n, $n \geq k$, we can connect z_k to z_{k+1} by a polygonal path γ_k contained in S_k. Thus $|f(z)| > k$ for all $z \in \gamma_k$. If we then form the path $\gamma = \bigcup_{k=1}^{\infty} \gamma_k$, it follows that f approaches ∞ along γ, proving the theorem. $\square$

15.2 The Phragmén-Lindelöf Theorem

We now return to theorems of maximum-modulus type.

Theorem 15.1 is rather general in that it applies to any region. On the other hand, if we restrict ourselves to various specific regions D, we will be able to derive the same type of conclusion under a much weaker hypothesis on f. We begin, as before, by considering the right half-plane. As previously noted, the function e^z is unbounded in this domain despite the fact that it is bounded by 1 on the imaginary axis. The same, of course, is true of the function $e^{\delta z}$ for any $\delta > 0$. However, if $f(z)$ has slower growth than $e^{\delta z}$, we have the following extension of Theorem 15.1.

15.4 Phragmén-Lindelöf Theorem

Let D denote the right half-plane and suppose f is C-analytic in D. If

$$|f(z)| \leq 1 \tag{1}$$

on the imaginary axis and if, for each $\epsilon > 0$, there exists a constant A_ϵ such that

$$|f(z)| \leq A_\epsilon e^{\epsilon|z|} \tag{2}$$

throughout D, then (1) *holds for all $z \in D$.*

Before proceeding with the proof, we will need the following lemma, which is a slightly weaker form of the theorem.

Lemma 1

Suppose f is C-analytic in the right half-plane D. If

$$|f(z)| \leq 1 \tag{3}$$

on the imaginary axis, and if for some $\delta > 0$, there exist constants A and B such that

$$|f(z)| \leq A \exp(B|z|^{1-\delta}) \tag{4}$$

for all $z \in D$, then (3) *holds throughout D.*

Proof of Lemma 1

Here we use the auxiliary function

$$h(z) = \frac{f^N(z)}{\exp(z^{1-\delta/2})}$$

and wish to show $|h(z_0)| \leq 1$ for each $z_0 \in D$. Let us first analyze the denominator $g(z) = \exp(z^{1-\delta/2})$. In the open right half-plane $z^{1-\delta/2}$ may be defined as an analytic function (see the comments following Theorem 8.8). To fix its value, we take it to be positive on the positive real axis. Then, for $z = re^{i\theta}$, $-\pi/2 < \theta < \pi/2$,

$$z^{1-\delta/2} = r^{1-\delta/2} e^{i\theta(1-\delta/2)}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

which is also continuous to the boundary.

Finally,

$$g(z) = \exp(z^{1-\delta/2}) = \exp(r^{1-\delta/2} e^{i\theta(1-\delta/2)}).$$

Thus, for $z = iy$

$$|g(z)| = \exp\left(|y|^{1-\delta/2} \cos\left(1 - \frac{\delta}{2}\right)\frac{\pi}{2}\right) \geq e^0 = 1 \tag{5}$$

and for $|z| = R,\ z \in D$,

$$|g(z)| = \exp\left(R^{1-\delta/2} \cos\left(1 - \frac{\delta}{2}\right)\theta\right) \geq \exp(R^{1-\delta/2} m). \tag{6}$$

where m is the minimum value of

$$\cos\left(1 - \frac{\delta}{2}\right)\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Now consider $|h(z)|$ on the boundary of D_R. On the imaginary axis, by (3) and (5), $|h(z)| \leq 1$. For $|z| = R$, by (4) and (6),

$$|h(z)| \leq A^N \exp(NBR^{1-\delta} - mR^{1-\delta/2}).$$

Since the expression in parenthesis approaches $-\infty$ as $R \to \infty$, we have for R large enough, $|h(z)| \leq 1$ on the boundary of D_R. Once again, invoking the maximum modulus theorem,

$$|h(z_0)| \leq 1$$

for every $z_0 \in D$ and thus

$$|f(z_0)| \leq |\exp(z_0^{1-\delta/2})|^{1/N}.$$

Finally, letting $N \to \infty$ gives the desired result. □

Note: While the lemma was stated in the right half-plane, it is obviously true in any other half-plane as well. For example, if f satisfies the growth conditions (3) and (4) in the upper half-plane, $g(z) = f(-iz)$ would satisfy the hypotheses of the lemma. Hence, $g \ll 1$ in the right half-plane and $f \ll 1$ in the upper half-plane.

Similarly, by mapping other regions analytically onto the right half-plane, we can derive results similar to Lemma 1 for functions which are C-analytic in the given regions. We record one example which will serve as another lemma to Theorem 15.4.

Lemma 2

Suppose f is C-analytic in a quadrant. If $|f(z)| \leq 1$ on the boundary and if for some $\delta > 0$, there exist constants A and B such that

$$|f(z)| \leq A \exp(B|z|^{2-\delta}) \text{ for every } z \text{ in the quadrant,}$$

then $|f(z)| \leq 1$ throughout the quadrant.

Proof of Lemma 2

Without loss of generality, we consider the first quadrant. Set $g(z) = f(\sqrt{z})$. Then g is C-analytic in the upper half-plane. Furthermore, by the hypothesis on f, $|g(z)| \leq 1$ on the boundary and

$$|g(z)| \leq A \exp(B|z|^{1-\delta/2})$$

throughout the half-plane. By Lemma 1, $|g(z)| \leq 1$ throughout the half-plane and thus $|f(z)| \leq 1$ for all points z in the quadrant. □

Proof of Theorem 15.4

We consider

$$h(z) = \frac{f^N(z)}{e^z}$$

and, as before, the proof will follow if we can show $|h(z)| \leq 1$ throughout the right half-plane. To do this, we consider the first and fourth quadrants separately. To estimate $h(z)$ on the boundary of the first quadrant, note that $|e^{iy}| = 1$ and hence, by (1)

$$|h(z)| \leq 1 \quad \text{on the positive imaginary axis.}$$

Also, by (2), $|f(z)| \leq A_{1/N} e^{(1/N)|z|}$ so that setting $B_N = (A_{1/N})^N$, $|f^N(z)| \leq B_N e^{|z|}$ throughout the half-plane. On the positive x-axis, though, $|e^z| = e^{|z|}$ and hence $|h(z)| \leq B_N$ for $z > 0$. Thus $|h(z)| \leq \text{Max}(1, B_N)$ along the boundary of the first quadrant. Furthermore, throughout the first quadrant

$$|h(z)| \leq |f^N(z)| \leq B_N e^{|z|}$$

so that we can apply Lemma 2 to conclude

$$|h(z)| \leq \text{Max}(1, B_N)$$

in the first quadrant. By the exact same reasoning,

$$|h(z)| \leq \text{Max}(1, B_N)$$

in the fourth quadrant. Hence $h(z)$ is a bounded C-analytic function in the right half-plane and is bounded by 1 on the imaginary axis. By Theorem 15.1, $|h(z)| \leq 1$ throughout the right half-plane, and the proof is complete. □

By mapping a wedge of angle α onto the right half-plane, we derive the following corollary.

15.5 Corollary

Let

$$D = \left\{ z : -\frac{\alpha}{2} < \operatorname{Arg} z < \frac{\alpha}{2} \right\}, \textit{ where } 0 < \alpha \le 2\pi,$$

and suppose f is C-analytic in D. If

$$|f(z)| \le 1 \tag{1}$$

on ∂D and if, for each $\epsilon > 0$, there exists a constant A_ϵ such that

$$|f(z)| \le A_\epsilon \exp(\epsilon |z|^{\pi/\alpha}), \tag{2}$$

then (1) *holds throughout D.*

Proof

Given f as above, consider $g(z) = f(z^{\alpha/\pi})$ in the right half-plane and apply Theorem 15.4. □

An interesting special case of the corollary arises if we take a wedge of angle 2π (the whole plane slit along one ray). In that case, the boundary is a single ray and, by the above corollary, if f is bounded on that ray and has slower growth than $e^{\epsilon\sqrt{|z|}}$ for each $\epsilon > 0$, it is in fact bounded throughout the wedge. Now, we may view an entire function as a C-analytic function in every wedge of angle 2π. This leads to the following theorem.

15.6 Theorem

If f is a non-constant entire function and for each $\epsilon > 0$ there exists a constant A_ϵ such that

$$|f(z)| \le A_\epsilon e^{\epsilon\sqrt{|z|}}$$

then $f(z)$ is unbounded on every ray!

Proof

If f were bounded on some ray R, by Corollary 15.5 it would also be bounded on the wedge $\mathbb{C}\backslash R$; that is, f would be bounded in the entire plane. But, then, by Liouville's Theorem f would reduce to a constant, contradicting the hypothesis of the theorem. □

EXAMPLE

$\cos z$ has a power series involving only even terms, hence $\cos\sqrt{z}$ is an entire function that is bounded on the positive x-axis. Hence, by the above theorem, it must grow as fast as $e^{\epsilon\sqrt{|z|}}$ for some $\epsilon > 0$. Setting

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

shows that this is in fact the case. (Consider points z along the imaginary axis.) $\Diamond$

An Application of Theorem 15.6: The differential equation $f'(z) = -f(z)$ has the explicit solution $f(z) = Ae^{-z}$. However, if we consider the very similar equation

$$f'(z) = -f\left(\frac{z}{2}\right) \tag{1}$$

no such explicit solution can be found. Nevertheless, one may seek to study the behavior of a solution $f(z)$ as $z \to \infty$ along the positive x-axis. To accomplish this, we will find the solution in the form of a power series which is, in fact, an entire function. Furthermore, we will show that the solution is of "small" growth, so that Theorem 15.6 is applicable, and f is unbounded on every ray. Thus, unlike Ae^{-z}, the solution to (1) has no limit as $z \to +\infty$. The details are as follows:

15.7 Proposition

Let f be a solution of the differential equation $f'(z) = -f(z/2)$, analytic at $z = 0$. Then f is entire and is unbounded on every ray.

Proof

Let f have the power series representation

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Because of (1), we must have

$$\sum_{k=1}^{\infty} k a_k z^{k-1} = -\sum_{k=1}^{\infty} a_k \left(\frac{z}{2}\right)^k$$

or

$$a_k = -\frac{a_{k-1}}{2^{k-1}k}.$$

By induction, then,

$$a_n = \frac{(-1)^n a_0}{n!2^{n(n-1)/2}}.$$

Hence $f(z)$ is given by

$$f(z) = A\sum_k b_k z^k \quad \text{where } b_k = \frac{(-1)^k}{k!2^{k(k-1)/2}}, \tag{2}$$

and a simple check shows that (2) does in fact represent an entire solution of (1).

We now show that f satisfies the hypothesis of Theorem 15.6. For this we fix $\epsilon > 0$ and show that, for z sufficiently large,

$$|f(z)| < \exp(|z|^{\epsilon}).$$

Assume then that $|z| = R = 2^N,\ N > 2$, and let

$$M(R) = \operatorname*{Max}_{|z|=R} |f(z)|, \quad M = M(1).$$

According to (1),

$$\begin{aligned} f(z) - f(0) = \int_0^z f'(\rho)d\rho = \int_0^z -f\left(\frac{\rho}{2}\right) d\rho \\ \ll RM\left(\frac{R}{2}\right) \end{aligned}$$

so that

$$|f(z)| \leq 2RM\left(\frac{R}{2}\right).$$

Setting $M(R/2) = |f(z_1)|$ for some $z_1 \in D(0; R/2)$ and proceeding inductively, we obtain

$$|f(z)| \leq MR^N = M|z|^{|\log z|/\log 2}. \tag{3}$$

The right-hand side of (3) is bounded above by $\exp(|z|^{\epsilon})$ for *all* z sufficiently large; therefore, we get $R_0 = R_0(\epsilon)$ such that

$$|f(z)| < \exp(|z|^{\epsilon})$$

for all z with $|z| \geq R_0$, as desired. □

Exercises

1.* Show that the conclusion of Theorem 15.1 would hold if we insisted only that $f \ll 1$ along the boundary and $f(z) \ll \log z$ throughout the domain. How could the hypothesis be further relaxed?

2. What is the "smallest" non-constant analytic function in the quadrant $D = \{x + iy : x, y < 0\}$, which is bounded along the boundary?

3. Show that $e^{e^z} \ll 1$ throughout the boundary of the region

$$D = \left\{x + iy : -\frac{\pi}{2} < y < \frac{\pi}{2}\right\}.$$

Show that it is the "smallest" such analytic function.

4.* Suppose g is a non-constant entire function which is bounded on every ray. (See 12.2). Show that for any A and B, there must exist some point z with $|g(z)| > A\exp(|z|^B)$. [*Hint*: If not, divide the plane into a finite number of very small wedges and apply 15.5 and Liouville's Theorem to conclude that g is constant.]