Chapter 11
Applications of the Residue Theorem to the Evaluation of Integrals and Sums

Introduction

In the next section, we will see how various types of (real) definite integrals can be associated with integrals around closed curves in the complex plane, so that the Residue Theorem will become a handy tool for definite integration.

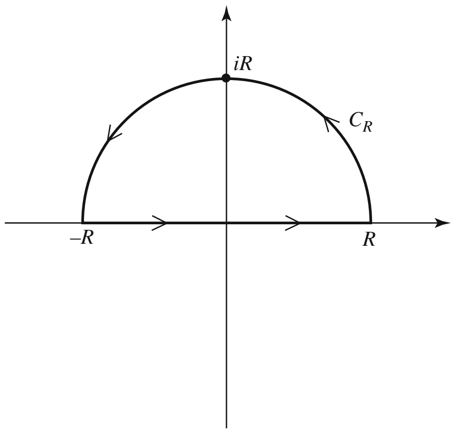
11.1 Evaluation of Definite Integrals by Contour Integral Techniques

I *Integrals of the Form* $\int_{-\infty}^{\infty}(P(x)/Q(x))dx$, *where* P *and* Q *are polynomials.* From real-variable calculus we know that an integral of this type will converge if $Q(x) \neq 0$ and $\deg Q - \deg P \geq 2$. Making these assumptions, we note that

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{P(x)}{Q(x)}dx,$$

and we seek to estimate the second integral for large values of R.

Let C_R be the closed contour consisting of the real line segment from $-R$ to R and the upper semi-circle Γ_R centered at the origin and of radius R large enough to enclose all zeroes of Q lying in the upper half-plane.



J. Bak, D. J. Newman, *Complex Analysis,* DOI 10.1007/978-1-4419-7288-0_11

By the Residue Theorem

$$\int_{C_R} \frac{P(z)}{Q(z)} dz = 2\pi i \sum_k \operatorname{Res}\left(\frac{P}{Q}; z_k\right)$$

where the points z_k are the zeroes of Q in the upper half-plane.

Thus

$$\int_{-R}^{R} \frac{P(x)}{Q(x)} dx + \int_{\Gamma_R} \frac{P(z)}{Q(z)} dz = 2\pi i \sum_k \operatorname{Res}\left(\frac{P}{Q}; z_k\right) \tag{1}$$

To estimate $\int_{\Gamma_R} P/Q$, note that since $\deg Q - \deg P \geq 2$, by the usual $M - L$ estimates

$$\int_{\Gamma_R} \frac{P}{Q} \ll \pi \cdot R \cdot \frac{A}{R^2}$$

and hence

$$\lim_{R\to\infty} \int_{\Gamma_R} \frac{P(z)}{Q(z)} dz = 0. \tag{2}$$

Combining (1) and (2) shows that

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_k \operatorname{Res}\left(\frac{P}{Q}; z_k\right)$$

EXAMPLE

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\pi i \sum_{k=1}^{2} \operatorname{Res}\left(\frac{1}{z^4+1}; z_k\right)$$

where $z_1 = e^{i\pi/4}$ and $z_2 = e^{3\pi i/4}$ represent the poles of $1/(z^4+1)$ in the upper half-plane. Since each is a simple pole, the residues are given by the values of $1/4z^3$ at the poles. Thus

$$\operatorname{Res}\left(\frac{1}{z^4+1}; e^{i\pi/4}\right) = \frac{1}{4z_1^3} = \frac{-z_1}{4} = -\frac{1}{8}(\sqrt{2} + i\sqrt{2})$$

and

$$\operatorname{Res}\left(\frac{1}{z^4+1}; e^{i3\pi/4}\right) = \frac{1}{8}(\sqrt{2} - i\sqrt{2}),$$

so that

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi\sqrt{2}}{2}. \qquad \diamond$$

II. *Integrals of the Form* $\int_{-\infty}^{\infty} \mathcal{R}(x) \cos x \, dx$ *or* $\int_{-\infty}^{\infty} \mathcal{R}(x) \sin x \, dx$. Assuming that

$$\mathcal{R}(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials and $Q(x) \neq 0$ (except perhaps at a zero of $\cos x$ or $\sin x$), the above integrals converge as long as $\deg Q > \deg P$.

Integrating $\mathcal{R}(z)\cos z$ along the same contour as in Type I is not appropriate since

$$\lim_{M\to\infty}\int_{\Gamma_M}\mathcal{R}(z)\cos z\,dz \neq 0.$$

If we consider

$$\int_{C_M}\mathcal{R}(z)e^{iz}dz,$$

however, we will be able to show that

$$\int_{\Gamma_M}\mathcal{R}(z)e^{iz}dz \to 0$$

so that

$$\int_{C_M}\mathcal{R}(z)e^{iz}dz \to \int_{-\infty}^{\infty}\mathcal{R}(x)e^{ix}dx. \tag{3}$$

$$\int_{-\infty}^{\infty}\mathcal{R}(x)\cos x\,dx \quad \text{and} \quad \int_{-\infty}^{\infty}\mathcal{R}(x)\sin x\,dx$$

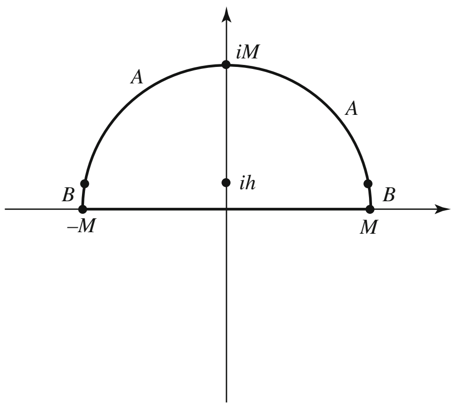
can then be determined as the real and imaginary parts of the limit in (3). Hence, applying the Residue Theorem in (3), we see that

$$\int_{-\infty}^{\infty}\mathcal{R}(x)\cos x dx = \operatorname{Re}\left[2\pi i\sum_k \operatorname{Res}(\mathcal{R}(z)e^{iz};\, z_k)\right]$$

and $\int_{-\infty}^{\infty}\mathcal{R}(x)\sin x dx = \operatorname{Im}\,[2\pi i\sum_k \operatorname{Res}(\mathcal{R}(z)e^{iz};\, z_k)]$, where the points z_k are the poles of $\mathcal{R}(z)$ in the upper half-plane.

To show that $\int_{\Gamma_M}\mathcal{R}(z)e^{iz}dz \to 0$, and complete the argument, we split Γ_M into two subsets:

$$A = \{z \in \Gamma_M : \operatorname{Im} z \geq h\}$$
$$B = \{z \in \Gamma_M : \operatorname{Im} z < h\}.$$



Using the facts that $\mathcal{R}(z) \ll K/|z|$ and $|e^z| = e^{\mathrm{Re}\, z}$, we obtain

$$\int_A \mathcal{R}(z)e^{iz}dz \ll K\frac{e^{-h}}{M} \cdot \pi M = C_1 e^{-h}.$$

But

$$\int_B \mathcal{R}(z)e^{iz}dz \ll \frac{K}{M}4h = C_2\frac{h}{M},$$

so

$$\int_{\Gamma_M} \mathcal{R}(z)e^{iz}dz \ll C_1 e^{-h} + C_2\frac{h}{M}.$$

If we now choose $h = \sqrt{M}$, for example, we find

$$\int_{\Gamma_M} \mathcal{R}(z)e^{iz}dz \ll C_1 e^{-\sqrt{M}} + \frac{C_2}{\sqrt{M}}$$

and

$$\lim_{M\to\infty}\int_{\Gamma_M} \mathcal{R}(z)e^{iz}dz = 0.$$

EXAMPLE

To evaluate $\int_{-\infty}^{\infty}(\sin x/x)dx$, we might write

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = \mathrm{Im}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx.$$

The pole of e^{ix}/x at $x = 0$ forces us to modify the technique slightly; we write instead:

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = \mathrm{Im}\int_{-\infty}^{\infty}\frac{e^{ix}-1}{x}dx.$$

Note that

$$\int_{C_M}\frac{e^{iz}-1}{z}dz = \int_{-M}^{M}\frac{e^{ix}-1}{x}dx + \int_{\Gamma_M}\frac{e^{iz}-1}{z}dz;$$

while, according to Cauchy's Theorem,

$$\int_{C_M}\frac{e^{iz}-1}{z}dz = 0$$

since the integrand has no poles! Thus

$$\int_{-M}^{M}\frac{e^{ix}-1}{x}dx = \int_{\Gamma_M}\frac{1-e^{iz}}{z}dz = \int_{\Gamma_M}\frac{1}{z}dz - \int_{\Gamma_M}\frac{e^{iz}}{z}dz$$
$$= \pi i - \int_{\Gamma_M}\frac{e^{iz}}{z}dz.$$

Since $\int_{\Gamma_M}(e^{iz}/z)dz$ approaches 0 as $M \to \infty$,

$$\int_{-\infty}^{\infty} \frac{e^{ix}-1}{x}dx = \pi i$$

and

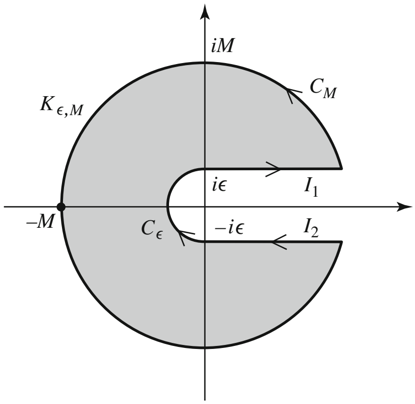
$$\int_{-\infty}^{\infty} \frac{\sin x}{x}dx = \pi.$$

$\Diamond$

III *(A) Integrals of the Form* $\boldsymbol{\int_0^\infty (P(x)/Q(x))dx}$ As in **I**, to insure convergence of the integral, we assume that $\deg Q - \deg P \geq 2$ and that $Q(x) \neq 0$ for $x \geq 0$. Of course, if the integrand is an even function it can be evaluated as $\frac{1}{2}\int_{-\infty}^{\infty}(P(x)/Q(x))dx$. In other cases, set $\mathcal{R}(z) = P(z)/Q(z)$ and consider the integral of $\log z \cdot \mathcal{R}(z)$ around the keyhole-shaped contour $K_{\epsilon,M}$ consisting of

i. the horizontal line segment I_1 from $i\epsilon$ to $\sqrt{M^2-\epsilon^2}+i\epsilon$;
ii. the circular arc C_M of radius M traced counterclockwise from

$$\sqrt{M^2-\epsilon^2}+i\epsilon \text{ to } \sqrt{M^2-\epsilon^2}-i\epsilon;$$



iii. the horizontal line segment I_2 from

$$\sqrt{M^2-\epsilon^2}-i\epsilon \text{ to } -i\epsilon;$$

iv. the semi-circle C_ϵ of radius ϵ traced clockwise from $-i\epsilon$ to $i\epsilon$.

The inside of $K_{\epsilon,M}$ is a simply connected domain not containing 0 and hence $\log z$ may be defined there as an analytic function. (For simplicity, we choose $0 < \operatorname{Arg} z < 2\pi$.)

By the Residue Theorem

$$\lim_{\substack{\epsilon\to 0\\ M\to\infty}} \int_{K_{\epsilon,M}} \mathcal{R}(z)\log z\,dz = 2\pi i \sum_k \operatorname{Res}(\mathcal{R}(z)\log z; z_k). \tag{4}$$

Moreover, assuming ϵ is small enough and M large enough so that all the zeroes of Q lie inside $K_{\epsilon,M}$, the contour integral is related to $\int_0^\infty \mathcal{R}(x)\,dx$ as follows:

i. $\int_{C_\epsilon} \mathcal{R}(z) \log z\,dz \ll \pi\epsilon \max_{C_\epsilon} |\mathcal{R}(z)\log z| \ll A\epsilon|\log\epsilon|$ since $\mathcal{R}$ is continuous at 0 and $|\log z| < \log|z| + 2\pi$. Thus

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} \mathcal{R}(z)\log z\,dz = 0.$$

ii. $\int_{C_M} \mathcal{R}(z)\log z\,dz \ll 2\pi M \cdot \max_{C_M} |\log z||\mathcal{R}(z)| \leq AM\log M/M^2$ since $\mathcal{R}(z) \ll B/|z|^2$, and thus

$$\lim_{M\to\infty}\int_{C_M} \mathcal{R}(z)\log z\,dz = 0.$$

iii. $\lim_{\substack{\epsilon\to 0\\ M\to\infty}} \int_{I_1} \mathcal{R}(z)\log z\,dz = \int_0^\infty \mathcal{R}(x)\log x\,dx$ and

$$\lim_{\substack{\epsilon\to 0\\ M\to\infty}}\int_{I_2} \mathcal{R}(z)\log z\,dz = -\int_0^\infty \mathcal{R}(x)(\log x + 2\pi i)dx.$$

Combining all of the above results we find

$$\lim_{\substack{\epsilon\to 0\\ M\to\infty}}\int_{K_{\epsilon,M}} \mathcal{R}(z)\log z\,dz = -2\pi i\int_0^\infty \mathcal{R}(x)dx,$$

so that by (4)

$$\int_0^\infty \mathcal{R}(x)dx = -\sum_k \operatorname{Res}(\mathcal{R}(z)\log z; z_k)$$

where the sum is taken over all the poles of $\mathcal{R}$.

Example

To evaluate $\int_0^\infty dx/(1+x^3)$, note that at $z_1 = e^{i\pi/3}$,

$$\operatorname{Res}\left(\frac{\log z}{1+z^3}; z_1\right) = -\frac{i\pi}{9}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right);$$

at $z_2 = -1 = e^{i\pi}$,

$$\operatorname{Res}\left(\frac{\log z}{1+z^3}; z_2\right) = \frac{i\pi}{3};$$

and at $z_3 = e^{i5\pi/3}$,

$$\operatorname{Res}\left(\frac{\log z}{1+z^3}; z_3\right) = \frac{-5\pi i}{9}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right);$$

so that

$$\sum_k \operatorname{Res}\left(\frac{\log z}{1+z^3}; z_k\right) = -\frac{2\pi}{9}\sqrt{3}$$

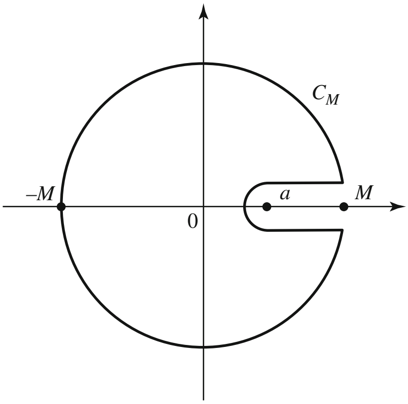
and

$$\int_0^\infty \frac{dx}{1+x^3} = \frac{2}{9}\pi\sqrt{3}.$$

$\Diamond$

(B) Integrals of the form $\int_a^\infty (P(x)/Q(x))dx$ can be evaluated in a similar manner by considering

$$\int_{C_M} \log(z-a)\frac{P(z)}{Q(z)}dz$$



with C_M as indicated. In fact, since

$$\int_0^\infty - \int_a^\infty = \int_0^a,$$

this method can be used to find *indefinite* integrals of rational functions.

(C) Integration around the "keyhole" contour can also be used to evaluate integrals of the form

$$\int_0^\infty \frac{x^{\alpha-1}}{P(x)}dx$$

where $0 < \alpha < 1$ and P is a polynomial with $\deg P \geq 1$.

Throughout the inside of the contour $K_{\epsilon,M}$, $z^{\alpha-1} = \exp[(\alpha-1)\log z]$ can be defined as an analytic function (again, with $0 < \operatorname{Arg} z < 2\pi$, for example).

As we integrate along I_1 (as $\epsilon \to 0$)

$$z^{\alpha-1} = \exp((\alpha-1)\log x) = x^{\alpha-1}$$

while, throughout I_2

$$z^{\alpha-1} = e^{(\alpha-1)(\log x + 2\pi i)} = x^{\alpha-1}e^{2\pi i(\alpha-1)}.$$

Since the integrals along the two circular segments approach zero as before, the integral around $K_{\epsilon,M}$ is given by the integrals along I_1 and I_2 and hence

$$\left[1 - e^{2\pi i(\alpha-1)}\right]\int_0^\infty \frac{x^{\alpha-1}}{P(x)}dx = 2\pi i \sum_k \operatorname{Res}\left(\frac{z^{\alpha-1}}{P(z)}; z_k\right),$$

the sum being taken over the zeroes of P.

EXAMPLE
To evaluate $\int_0^\infty dx/\sqrt{x}(1+x)$, note that

$$\operatorname{Res}\left(\frac{1}{\sqrt{z}(1+z)}; -1\right) = -i$$

and

$$\left(1 - e^{-\pi i}\right)\int_0^\infty \frac{dx}{\sqrt{x}(1+x)} = 2\pi$$

so that

$$\int_0^\infty \frac{dx}{\sqrt{x}(1+x)} = \pi.$$

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IV $\int_0^{2\pi} \mathcal{R}(\cos\theta, \sin\theta)d\theta$ *where $\mathcal{R}$ Represents a Rational Function* Here we take a slightly different point of view. Previously, we viewed the definite integrals as integrals along real line segments which were then supplemented into closed contours in the complex plane. In this case, we think of the real integral itself as the parametric representation of a line integral taken around the unit circle.

Recall that

$$\int_{|z|=1} f(z)dz$$

becomes

$$\int_0^{2\pi} f(e^{i\theta})ie^{i\theta}d\theta$$

on setting $z = e^{i\theta},\ 0 \le \theta \le 2\pi$.

More specifically, the integral $\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta$ is equal to

$$\int_{|z|=1} \mathcal{R}\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{dz}{iz} \tag{5}$$

since with $z = e^{i\theta}$

$$d\theta = \frac{dz}{iz},$$
$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

and
$$\sin\theta = \frac{\left(e^{i\theta} - e^{-i\theta}\right)}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$
The contour integral (5), as always, can be evaluated by the Residue Theorem.

EXAMPLE
$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} &= \frac{2}{i}\int_{|z|=1} \frac{dz}{z^2+4z+1}\\ &= 4\pi \operatorname{Res}\left(\frac{1}{z^2+4z+1}; \sqrt{3}-2\right)\\ &= \frac{2}{3}\pi\sqrt{3}.\end{aligned}$$
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11.2 Application of Contour Integral Methods to Evaluation and Estimation of Sums

I To evaluate sums of the form $\sum_{n=-\infty}^{\infty} f(n)$, we seek a function g whose residues are given by $\{f(n) : n = 0, \pm 1, \pm 2, \ldots\}$.

Suppose we set $g(z) = f(z)\varphi(z)$. Then the function φ should have a simple pole with residue 1 at every integer. Such a function is given by
$$\varphi(z) = \pi\frac{\cos\pi z}{\sin\pi z},$$
since $\sin\pi z$ has a simple zero at every integer and
$$\operatorname{Res}\left(\frac{\pi\cos\pi z}{\sin\pi z}; n\right) = \frac{\pi\cos\pi n}{\pi\cos\pi n} = 1.$$
(Note that $\sin z$ has no other zeroes in the complex plane.)

We first apply the Residue Theorem to the integral
$$\int_{C_N} f(z)\cdot\pi\cot\pi z dz \tag{1}$$
where C_N is a simple closed contour enclosing the integers $n = 0, \pm 1, \pm 2, \ldots, \pm N$ and the poles of f (which we assume to be finite in number). Thus
$$\int_{C_N} \pi f(z)\cot\pi z dz = 2\pi i\left[\sum_{\substack{n=-N\\ n\neq z_k}}^{N} f(n) + \sum_k \operatorname{Res}(f(z)\pi\cot\pi z; z_k)\right] \tag{2}$$
where $\{z_k\}$ are the poles of f.

Furthermore, to ensure convergence of $\sum_{n=-\infty}^{\infty} f(n)$, we will assume that $|f(z)| \leq \frac{A}{z^2}$ so that

$$\lim_{z\to\infty} zf(z) = 0, \tag{3}$$

and by a proper choice of C_N, we will be able to show that

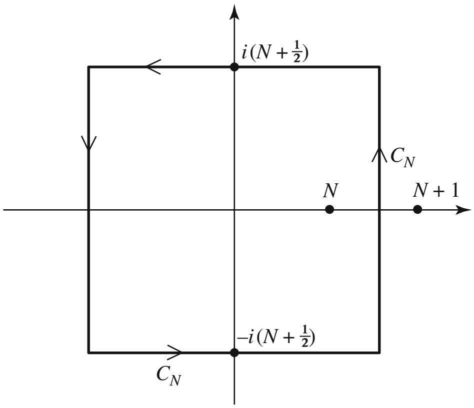
$$\lim_{N\to\infty} \int_{C_N} f(z)\pi \cot \pi z \, dz = 0. \tag{4}$$

Then by (2)

$$\sum_{\substack{n=-\infty \\ n\neq z_k}}^{\infty} f(n) = -\sum_k \operatorname{Res}(f(z)\pi \cot \pi z; z_k). \tag{5}$$

To demonstrate the existence of a contour C_N satisfying (4), we will let C_N be the square with vertices $\pm(N+\frac{1}{2}) \pm (N+\frac{1}{2})i$. Having thus avoided the poles of $\cot \pi z$, we can show that $|\cot \pi z| < 2$ on C_N. For example, if $\operatorname{Re} z = x = N + \frac{1}{2}$ and $\operatorname{Im} z = y$ then

$$\cot \pi z = i\frac{e^{2\pi i z}+1}{e^{2\pi i z}-1} = i\frac{e^{\pi i - 2\pi y}+1}{e^{\pi i - 2\pi y}-1}$$



and

$$|\cot \pi z| = \left|\frac{1-e^{-2\pi y}}{1+e^{-2\pi y}}\right| < 1.$$

Similarly, if $\operatorname{Im} z = y = N + \frac{1}{2}$

$$|\cot \pi z| \leq \frac{1+e^{-\pi(2N+1)}}{1-e^{-\pi(2N+1)}} < 2$$

since the latter expression is maximized at $N = 0$. (The same bounds apply to the other sides of C_N as well, since $\cot z$ is an odd function.)

Since the length of C_N is $8N+4$, by the usual estimates,

$$\int_{C_N} f(z)\pi \cot \pi z \ll (8N+4)2\pi \max_{z\in C_N} |f(z)|$$
$$\ll A \max_{c_N} |zf(z)|;$$

thus

$$\int_{C_N} f(z)\pi \cot \pi z\, dz \to 0 \text{ by (3).}$$

EXAMPLE
To find

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

note that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^2}$$

and hence, by (5),

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2}\mathrm{Res}\left(\frac{\pi \cot \pi z}{z^2}; 0\right).$$

The residue can be determined by using the Laurent expansion for $\cot z$; *i.e.*,

$$\cot z = \frac{1}{z} - \frac{z}{3} - \frac{1}{45}z^3 + \cdots$$

so that

$$\frac{\pi \cot \pi z}{z^2} = \frac{1}{z^3} - \frac{\pi^2}{3z} - \frac{\pi^4 z}{45} - \cdots.$$

Thus

$$\mathrm{Res}\left(\frac{\pi \cot \pi z}{z^2}; 0\right) = \frac{-\pi^2}{3}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \qquad \diamondsuit$$

II To evaluate sums of the form $\sum_{n=-\infty}^{\infty} (-1)^n f(n)$, where $f(z)$ has a finite number of poles, we integrate again around the square C_N, this time using the auxiliary function $\pi f(z) \csc \pi z$.

Note that

$$\mathrm{Res}\left(\frac{\pi}{\sin \pi z}; n\right) = \frac{1}{\cos \pi n} = (-1)^n,$$

and since

$$\csc^2 \pi z = 1 + \cot^2 \pi z,$$

$\csc \pi z$ (like $\cot \pi z$) is bounded on C_N. Thus we may conclude that

$$\lim_{N\to\infty} \int_{C_N} \pi f(z) \csc \pi z \, dz = 0$$

and, by the Residue Theorem, that

$$\sum_{\substack{n=-\infty \\ n\neq z_k}}^{\infty} (-1)^n f(n) = -\sum_k \operatorname{Res}(\pi f(z) \csc \pi z; z_k)$$

where the z_k are the poles of f.

EXAMPLE

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + - \cdots = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{1}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n^2}$$

$$= \frac{1}{2} \operatorname{Res}\left(\frac{\pi \csc \pi z}{z^2}; 0\right) = \frac{\pi^2}{12}$$

since

$$\frac{\pi \csc \pi z}{z^2} = \frac{1}{z^3} + \frac{\pi^2}{6z} + \frac{7\pi^4 z}{360} + \cdots . \qquad \Diamond$$

III *Sums Involving Binomial coefficients* The connection between binomial coefficients and contour integration is an immediate corollary of the Residue Theorem since

$$\binom{n}{k} = \text{coefficient of } z^k \text{ in } (1 + z)^n$$

and hence

$$\binom{n}{k} = \frac{1}{2\pi i} \int_C \frac{(1+z)^n}{z^{k+1}} dz \tag{6}$$

where C is any simple closed contour surrounding the origin. The identity (6) has some immediate consequences. For example,

$$\binom{2n}{n} = \frac{1}{2\pi i} \int_C \frac{(1+z)^{2n}}{z^{n+1}} dz$$

and if we choose C to be the unit circle, we find

$$\binom{2n}{n} \leq 4^n.$$

This same identify (6) can be used to evaluate (or estimate) sums involving binomial coefficients.

EXAMPLE 1
To find

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n} = 1 + \frac{2}{5} + \frac{6}{25} + \cdots$$

we set

$$\binom{2n}{n} = \frac{1}{2\pi i} \int_C \frac{(1+z)^{2n}}{z^{n+1}} dz$$

where C is any simple contour surrounding the origin so that

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_C \frac{(1+z)^{2n}}{(5z)^n} \frac{dz}{z}. \tag{7}$$

If we could then interchange the order of summation and integration, we would conclude

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n} = \frac{5}{2\pi i} \int_C \frac{dz}{3z - 1 - z^2}.$$

However, we must indicate a contour C (surrounding 0) on which summation under the integral sign is justified. [Without this caution, C could be an arbitrary circle centered at 0 and if we let the radius R be large enough, we would conclude erroneously that

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n} = 0. \,]$$

One way to assure the legitimacy of the interchange is to obtain uniform convergence of the series $\sum_{n=0}^{\infty} [(1+z)^2/5z]^n$ throughout C. Thus we pick C to be the unit circle so that

$$\left| \frac{(1+z)^2}{5z} \right| \leq \frac{4}{5}$$

throughout C and the convergence is uniform. Hence

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n} &= \frac{5}{2\pi i} \int_{|z|=1} \frac{dz}{3z - 1 - z^2} \\ &= 5 \operatorname{Res}\left(\frac{1}{3z - 1 - z^2}; \frac{3 - \sqrt{5}}{2} \right) = \sqrt{5}. \end{aligned}$$

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EXAMPLE 2
To evaluate

$$\sum_{k=0}^{n} \binom{n}{k}^2, \text{ we cast } \binom{n}{k} \text{ in two roles:}$$

a. $\binom{n}{k}$ = coefficient of z^k in $(1+z)^n$

b. $\binom{n}{k}$ = coefficient of z^{-k} in $(1+1/z)^n$

so that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \text{constant term in } (1+z)^n \left(1+\frac{1}{z}\right)^n.$$

Thus

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k}^2 &= \frac{1}{2\pi i}\int_C (1+z)^n \left(1+\frac{1}{z}\right)^n \frac{dz}{z} \\ &= \frac{1}{2\pi i}\int_C \frac{(1+z)^{2n}}{z^{n+1}} dz \\ &= \text{coefficient of } z^n \text{ in } (1+z)^{2n} \\ &= \binom{2n}{n}.\end{aligned}$$

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EXAMPLE 3
To estimate

$$1 - \binom{n}{1}\binom{2n}{1} + \binom{n}{2}\binom{2n}{2} - + \cdots \binom{n}{n}\binom{2n}{n}$$

we again note that since

$$\binom{n}{k} = \text{coefficient of } z^k \text{ in } (1+z)^n$$

and since

$$(-1)^k \binom{2n}{k} = \text{coefficient of } \frac{1}{z^k} \text{ in } \left(1-\frac{1}{z}\right)^{2n},$$

the sum is equal to the constant term in the product and is given by

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k}\binom{2n}{k} = \frac{1}{2\pi i}\int_C \frac{[(z-1)^2(z+1)]^n}{z^{2n+1}} dz.$$

In this case, however, there is no simple technique for evaluating the integral and instead we seek to estimate it. If we let C be the unit circle, then throughout C,

$$\left|(z-1)^2(z+1)\right| \leq \frac{16}{9}\sqrt{3}$$

[see Exercise 15] and hence

$$\left|\sum_{k=1}^{n}(-1)^k \binom{n}{k}\binom{2n}{k}\right| \leq \left(\frac{16}{9}\sqrt{3}\right)^n.$$

Note that this estimate is much smaller than one might guess by estimating the size of the various terms–the last term of the series alone is of the order of magnitude of 4^n. (See Exercise 16.) ♢

A more familiar series whose sum is of much smaller magnitude than its individual terms is

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots.$$

The fact that $e^{-x} \to 0$ as $x \to \infty$ is in sharp contrast to the growth of its individual terms. By employing our contour integral technique, we can demonstrate similar behavior for the series

$$B(x) = 1 - \frac{x}{1} + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + - \cdots$$

that is related to the Bessel Function. Since

$$\frac{1}{n!} = \text{coefficient of } z^n \text{ the expansion of } e^z$$

and

$$\frac{(-x)^n}{n!} = \text{coefficient of } z^{-n} \text{ in } e^{-x/z}$$

$$B(x) = \frac{1}{2\pi i}\int_C \frac{e^z e^{-x/z}}{z}dz$$

where C is any simple contour surrounding 0.

We seek a contour C on which

$$|e^{z-x/z}| = e^{\operatorname{Re}(z-x/z)}$$

is small. Setting $z = Re^{i\theta}$, we find

$$\operatorname{Re}(z - \frac{x}{z}) = R\cos\theta - \frac{x}{R}\cos\theta;$$

hence $R = \sqrt{x}$ seems a good choice, and we pick C to be the circle: $|z| = \sqrt{x}$.

Then

$$B(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{2i\sqrt{x}\sin\theta} d\theta$$

and since the integrand is bounded by 1 for all θ, we conclude

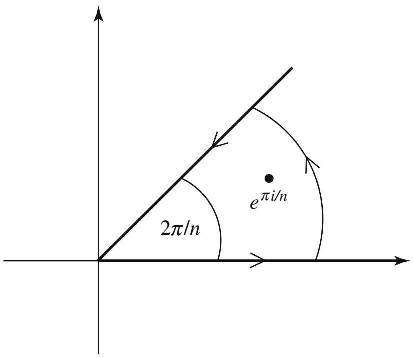
$$|B(x)| \leq 1$$

for all $x \geq 0$.

(In fact, a closer analysis would show that $B(x) \to 0$ as $x \to \infty$, but this would take us too far afield at this point.)

Exercises

1. Evaluate the following definite integrals
   a. $\int_{-\infty}^{\infty} \frac{x^2 dx}{(1+x^2)^2}$, b. $\int_0^{\infty} \frac{x^2 dx}{(x^2+4)^2(x^2+9)}$,
   c. $\int_0^{\infty} \frac{dx}{x^4+x^2+1}$, d. $\int_0^{\infty} \frac{\sin x dx}{x(1+x^2)}$,
   e. $\int_0^{\infty} \frac{\cos x dx}{1+x^2}$, f. $\int_0^{\infty} \frac{dx}{x^3+8}$,
   g. $\int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx,\ 0 < \alpha < 1$, h. $\int_0^{2\pi} \frac{dx}{(2+\cos x)^2}$,
   i. $\int_0^{2\pi} \frac{\sin^2 x dx}{5+3\cos x}$, j. $\int_0^{2\pi} \frac{dx}{a+\cos x}$, ($a$ real), $|a| > 1$.
2. Evaluate
$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx.$$
   [*Hint*: Integrate $(e^{2iz} - 1 - 2iz)/z^2$ around a large semi-circle.]
3. Evaluate
$$\int_0^{\infty} \frac{dx}{1+x^n}$$
   where $n \geq 2$ is a positive integer. [*Hint*: Consider the following contour.]



4.* Evaluate:

a. $\int_0^\infty \frac{\cos ax}{(x^2+1)^2} dx; \quad a \ge 0$

b. $\int_0^\infty \frac{x^2}{x^{10}+1} dx$ (See the hint for exercise 3.)

c.
$$\int_0^{2\pi} e^{e^{i\theta}} d\theta$$

5.* Show that
$$\int_0^{2\pi} (\cos x)^{2m} dx = \frac{2\pi}{4^m}\binom{2m}{m}$$
for any positive integer m.

6.* Show that
$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\pi}{2 \cdot 4 \cdot 6 \cdots 2n}$$

7. a. Show that $\int_{\Gamma_R} e^{iz^2} dz \to 0$ as $R \to \infty$ where Γ_R is the circular segment: $z = Re^{i\theta}$, $0 \le \theta \le \pi/4$.
   b. Evaluate $\int_0^\infty \cos x^2 dx$, $\int_0^\infty \sin x^2 dx$. *Note*: The convergence of the above integrals can be proven for example by making the substitution $u = x^2$ and applying Dirichlet's Test.

8. Suppose f is a rational function of the form P/Q with $\deg Q - \deg P \ge 2$. Show that the sum of the residues of f is zero.

9. Evaluate

a. $\sum_{n=1}^\infty \frac{1}{n^2+1}$,

b. $\sum_{n=1}^\infty \frac{1}{n^4}$,

c. $\sum_{n=1}^\infty \frac{(-1)^n}{n^2+1}$.

10.* a. Show that $\int_{C_n} \frac{1}{z \sin \pi z} dz \to 0$ as $N \to \infty$, where C_N is the square with vertices $\pm(N+\frac{1}{2}) \pm (N+\frac{1}{2})i$. (See Chapter 3, exercise 16.)
   b. By integrating $\frac{1}{(2z-1)\sin \pi z}$ around a suitable contour, show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.

11.* Show that
$$\int_{-\infty}^{\infty} \frac{e^{kx}}{1+e^x} dx$$
converges if $0 < k < 1$. Find its value by integrating around the rectangle with vertices at $\pm R$ and at $\pm R + 2\pi i$.

12.* Suppose f is analytic for $|z| \le 1$, and let $\log z$ be defined so that $\operatorname{Im} \log z = \arg z \in [0, 2\pi)$. Prove that $\frac{1}{2\pi i} \int_{|z|=1} f(z) \log z \, dz = \int_0^1 f(x) dx$

13. Evaluate
$$\sum_{n=0}^\infty \binom{3n}{n} \frac{1}{8^n}.$$

14. Show that

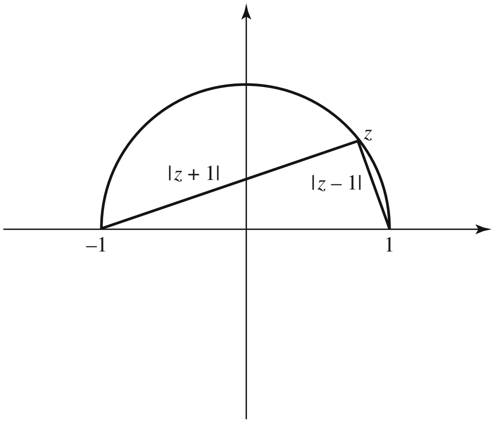
$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}} \text{ as long as} |x| < \frac{1}{4}.$$

Note: This is the sum of the middle column in Pascal's Triangle for powers of $1 + x$.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & x & & & \\
 & & 1 & & 2x & & x^2 & & \\
 & 1 & & 3x & & 3x^2 & & x^3 & \\
1 & & 4x & & 6x^2 & & 4x^3 & & x^4 \\
 & & & \cdot\ \cdot & \cdot & \cdot\ \cdot & & &
\end{array}$$

The equation can also be verified by applying the binomial expansion for $(1 \; - 4x)^{-1/2}$.

15. Complete Example 3 of Section 2-III by showing $|(z-1)^2(z+1)| \le (16/9)\sqrt{3}$ throughout $|z| = 1$. [*Hint*: Maximize a^2b given $a^2 + b^2 = 4$.]



16. a. Show that

$$\left| \frac{(z-1)^2(z+1)}{z^2} \right| \le 2\sqrt{2} \quad \textit{for} \quad |z| = \sqrt{2}$$

and thereby obtain an improved estimate for the example cited in (15).

b. Show that

$$\max_{|z|=R} \left| \frac{(z-1)^2(z+1)}{z^2} \right| \ge 2\sqrt{2} \text{ for any } R > 0.$$

(Thus, in a sense, the estimate in (a) is the best possible.)

17.* a. Express

$$\sum_{k=0}^{n} (-1)^k \binom{3n}{k} \binom{n}{k}$$

as a contour integral.

b. Use the integral above to prove that $\left| \sum_{k=0}^{n} (-1)^k \binom{3n}{k} \binom{n}{k} \right| \le 4^n$.