Chapter 10
The Residue Theorem

10.1 Winding Numbers and the Cauchy Residue Theorem

We now seek to generalize the Cauchy Closed Curve Theorem (8.6) to functions which have isolated singularities. Note that by 9.10 and 9.11, if γ is a circle surrounding a single isolated singularity z_0 and $f(z) = \sum_{-\infty}^{\infty} C_k(z-z_0)^k$ in a deleted neighborhood of z_0 that contains γ, then

$$\int_\gamma f = 2\pi i C_{-1}.$$

Thus the coefficient C_{-1} is of special significance in this context.

10.1 Definition

If $f(z) = \sum_{-\infty}^{\infty} C_k(z-z_0)^k$ in a deleted neighborhood of z_0, C_{-1} is called the *residue of* f *at* z_0. We use the notation $C_{-1} = \operatorname{Res}(f; z_0)$.

Evaluation of Residues

(i) If f has a simple pole at z_0; i.e., if

$$f(z) = \frac{A(z)}{B(z)}$$

where A and B are analytic at z_0, $A(z_0) \neq 0$ and B has a simple zero at z_0, then

$$C_{-1} = \lim_{z \to z_0} (z-z_0) f(z) = \frac{A(z_0)}{B'(z_0)}. \tag{1}$$

Proof

Since

$$f(z) = \frac{C_{-1}}{z-z_0} + C_0 + C_1(z-z_0) + \cdots,$$
$$(z-z_0)f(z) = C_{-1} + C_0(z-z_0) + C_1(z-z_0)^2 + \cdots,$$

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and

$$\lim_{z \to z_0} (z - z_0) f(z) = C_{-1}.$$

The second equality in (1) follows since

$$\begin{aligned} \lim_{z \to z_0} (z - z_0) f(z) &= \lim_{z \to z_0} (z - z_0) \frac{A(z)}{B(z)} \\ &= \lim_{z \to z_0} A(z) \Bigg/ \frac{B(z) - B(z_0)}{z - z_0} = \frac{A(z_0)}{B'(z_0)}. \end{aligned} \qquad \square$$

(ii) If f has a pole of order k at z_0,

$$C_{-1} = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z - z_0)^k f(z) \right] \text{ evaluated at } z_0.$$

Proof

Setting

$$\begin{aligned} f(z) &= C_{-k}(z - z_0)^{-k} + \cdots C_{-1}(z - z_0)^{-1} + C_0 + C_1(z - z_0) + \cdots \\ g(z) &= (z - z_0)^k f(z) = C_{-k} + \cdots C_{-1}(z - z_0)^{k-1} + C_0(z - z_0)^k + \cdots \\ \frac{d^{k-1} g(z)}{dz^{k-1}} &= (k-1)! C_{-1} + k! C_0 (z - z_0) + \cdots \end{aligned}$$

and the equality follows. $\square$

(iii) In most cases of higher-order poles, as with essential singularities, the most convenient way to determine the residue is directly from the Laurent expansion.

EXAMPLES

i. $\operatorname{Res}(\csc z; 0) = \dfrac{1}{\cos 0} = 1.$

ii. $\operatorname{Res}\left(\dfrac{1}{z^4 - 1}\ ;\ i\right) = \dfrac{1}{4i^3} = \dfrac{i}{4}.$

iii. $\operatorname{Res}\left(\dfrac{1}{z^3}\ ;\ 0\right) = 0.$

iv. $\operatorname{Res}\left(\sin \dfrac{1}{z-1}\ ;\ 1\right) = 1$, since

$$\sin \frac{1}{z-1} = \frac{1}{z-1} - \frac{1}{3!(z-1)^3} + \frac{1}{5!(z-1)^5} - + \cdots . \qquad \Diamond$$

Winding Number. To evaluate $\int_\gamma f$ when γ is a general closed curve (and when f may have isolated singularities), we introduce the following concept.

10.2 Definition

Suppose that γ is a closed curve and that $a \notin \gamma$. Then

$$n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a}$$

is called the *winding number of* γ *around* a.

Note that if γ represents the boundary of a circle (traversed counter-clockwise)

$$n(\gamma, a) = \begin{cases} 1 & \text{if } a \text{ is inside the circle} \\ 0 & \text{if } a \text{ is outside the circle.} \end{cases}$$

The first identity was proven in Lemma 5.4. The second was shown in Example 1 following the Cauchy Closed Curve Theorem. Also, if γ circles the point a k times—i.e., if $\gamma(\theta) = a + re^{i\theta}, 0 \le \theta \le 2k\pi$—then

$$n(\gamma,\ a) = \frac{1}{2\pi i} \int_0^{2k\pi} i d\theta = k,$$

which explains the terminology "winding number."

10.3 Theorem

For any closed curve γ and $a \notin \gamma$, $n(\gamma, a)$ is an integer.

Proof

Suppose γ is given by $z(t),\ 0 \le t \le 1$, and set

$$F(s) = \int_0^s \frac{\dot{z}(t)}{z(t)-a} dt, \quad 0 \le s \le 1.$$

Then, as we saw in defining the logarithm function (Section 8.2) it follows from

$$\dot{F}(s) = \frac{\dot{z}(s)}{z(s)-a}$$

that

$$(z(s) - a)e^{-F(s)}$$

is a constant, and setting $s = 0$,

$$(z(s) - a)e^{-F(s)} = z(0) - a.$$

Hence

$$e^{F(s)} = \frac{z(s)-a}{z(0)-a}$$

and

$$e^{F(1)} = \frac{z(1) - a}{z(0) - a} = 1$$

since γ is closed; i.e., $z(1) = z(0)$. Thus

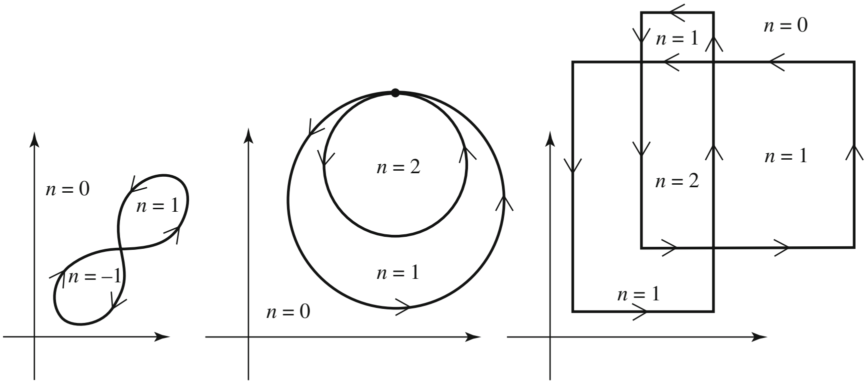
$$F(1) = 2\pi ki \quad \text{for some integer } k$$

and

$$n(\gamma, a) = \frac{1}{2\pi i} F(1) = k.$$

□

It follows from Definition 10.2 that if we fix γ and let a vary, $n(\gamma, a)$ is a continuous function of a (as long as $a \notin \gamma$). Since it is always integer-valued, we conclude that $n(\gamma, a)$ is constant in the connected components of the complement of γ. Moreover, $n(\gamma, a) \to 0$ as $a \to \infty$. Hence $n(\gamma, a) = 0$ in the unbounded component of γ (the set of points which can be connected to ∞ without intersecting γ). Some typical examples are illustrated below.



The Jordan Curve Theorem asserts that any simple closed curve divides the plane into exactly two components—one bounded, the other unbounded (here the curve need not necessarily be smooth)—so that if γ is such a "Jordan" curve (and is smooth besides),

$$n(\gamma, a) = \begin{cases} k & \text{if } a \in \text{ Bounded Component} \\ 0 & \text{if } a \in \text{ Unbounded Component.} \end{cases}$$

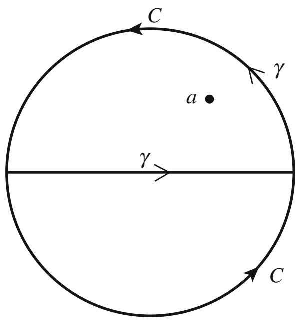
The proof of the Jordan Curve Theorem would lead us too far afield. Nevertheless, in all future examples when we deal with simple closed curves, we will be able to verify directly that $n(\gamma, a) = 0$ or ± 1 for all $a \notin \gamma$. In fact, by choosing the "proper" orientation, we will be able to show that $n(\gamma, a) = 0$ or 1 for all $a \notin \gamma$. (The "proper" orientation will be easily recognized as that one for which the bounded component of $\tilde{\gamma}$ lies to the left of γ.)

EXAMPLE
Let γ be a semicircle traversed counterclockwise. Then

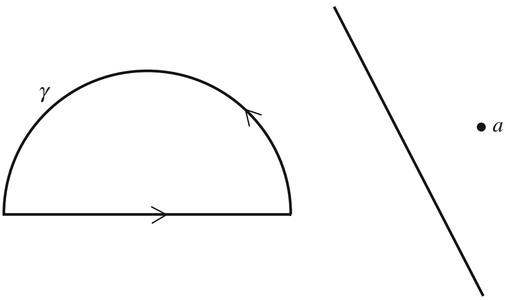
$$n(\gamma, a) = \begin{cases} 1 & \text{if } a \text{ is inside the semicircle} \\ 0 & \text{if } a \text{ is outside.} \end{cases}$$

The first assertion can be seen by citing the Closed Curve Theorem to show

$$\int_\gamma \frac{dz}{z-a} = \int_C \frac{dz}{z-a}$$



where C is the completed circle containing $z = a$. The second follows from the analyticity of $1/(z-a)$ in a half-plane containing γ but not a.



◊

To simplify our terminology, we introduce the following definition.

10.4 Definition

γ is called a *regular closed curve* if γ is a simple closed curve with $n(\gamma, a) = 0$ or 1 for all $a \notin \gamma$. In that case, we will call $\{a : n(\gamma, a) = 1\}$ the *inside* of γ. The set of points a where $n(\gamma, a) = 0$ is called the *outside* of γ.

10.5 Cauchy's Residue Theorem

Suppose f is analytic in a simply connected domain D except for isolated singularities at $z_1, z_2, \cdots, z_m$. Let γ be a closed curve not intersecting any of the singularities.

Then

$$\int_\gamma f = 2\pi i \sum_{k=1}^{m} n(\gamma, z_k)\mathrm{Res}(f; z_k).$$

Proof

(Note that since γ is a "general" curve, we cannot replace it by a finite union of "familiar" curves. Instead we proceed as in Section 9.2.)

If we subtract the principal parts

$$P_1\left(\frac{1}{z-z_1}\right), \cdots, P_m\left(\frac{1}{z-z_m}\right)$$

from f, the difference

$$g(z) = f(z) - P_1\left(\frac{1}{z-z_1}\right) - P_2\left(\frac{1}{z-z_2}\right) - \cdots - P_m\left(\frac{1}{z-z_m}\right)$$

(with the appropriate definitions at $z_1, \ldots, z_m$) is an analytic function in D. Hence, by the Closed Curve Theorem (8.6)

$$\int_\gamma g = 0$$

and

$$\int_\gamma f = \sum_{k=1}^{m} \int_\gamma P_k\left(\frac{1}{z-z_k}\right) dz. \tag{3}$$

Furthermore, for any k,

$$\int_\gamma \frac{1}{(z-z_k)^n} = 0, \quad \text{if } n \neq 1 \ \text{ since}$$

$(z-z_k)^{-n}$ is the derivative of

$$\frac{(z-z_k)^{1-n}}{1-n},$$

which is analytic along the closed curve γ. Hence if

$$P_k\left(\frac{1}{z-z_k}\right) = \frac{C_{-1}}{z-z_k} + \frac{C_{-2}}{(z-z_k)^2} + \cdots,$$

$$\int_\gamma P_k\left(\frac{1}{z-z_k}\right) dz = C_{-1}\int_\gamma \frac{dz}{z-z_k} = 2\pi i\ n(\gamma, z_k)\mathrm{Res}(f; z_k)$$

and by (3), the proof is complete. □

10.6 Corollary

If f is as above, and if γ is a regular closed curve in the domain of analyticity of f, then $\int_\gamma f = 2\pi i \sum_k \mathrm{Res}(f; z_k)$, where the sum is taken over all the singularities of f inside γ.

10.2 Applications of the Residue Theorem

10.7 Definition

We say f is *meromorphic* in a domain D if f is analytic there except at isolated poles.

10.8 Theorem

Suppose γ is a regular closed curve. If f is meromorphic inside and on γ and contains no zeroes or poles on γ, and if
$\mathbb{Z}$ = *number of zeroes of f inside γ (a zero of order k being counted k times),*
$\mathbb{P}$ = *number of poles of f inside γ (again with multiplicity),*
then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \mathbb{Z} - \mathbb{P}.$$

Proof

Note that f'/f is analytic except at the zeroes or poles of f. If f has a zero of order k at $z = a$, that is, if

$$f(z) = (z-a)^k g(z) \qquad \text{with } g(z) \neq 0,$$

then

$$f'(z) = (z-a)^{k-1}\left[kg(z) + (z-a)g'(z)\right]$$

has a zero of order $k-1$ at z, and

$$\frac{f'(z)}{f(z)} = \frac{k}{z-a} + \frac{g'(z)}{g(z)}$$

Hence, at each zero of f of order k, f'/f has a simple pole with residue k. Similarly, if

$$f(z) = (z-a)^{-k} g(z),$$

then

$$\frac{f'(z)}{f(z)} = \frac{-k}{z-a} + \frac{g'(z)}{g(z)},$$

so that at each pole of f, f'/f has a simple pole with residue $-k$. By Corollary 10.6, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \sum \text{Res}\left(\frac{f'}{f}\right) = \mathbb{Z} - \mathbb{P}.$$

$\square$

If we take f to be analytic, we obtain

10.9 Corollary (Argument Principle)

If f is analytic inside and on a regular closed curve γ (and is nonzero on γ) then

$$\mathbb{Z}(f) = \text{the number of zeroes of } f \text{ inside } \gamma = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}.$$

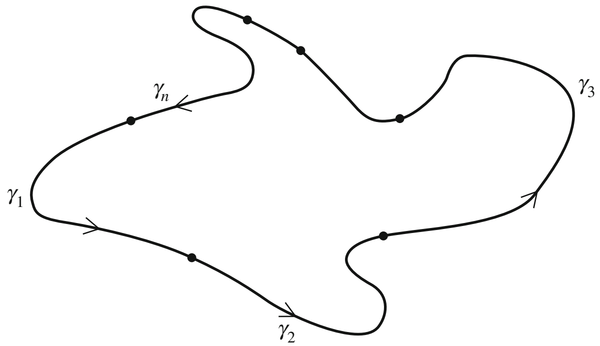
Remarks

1. The above is known as the "Argument Principle" because if γ is given by $z(t)$, $0 \le t \le 1$,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \frac{\log f(z(1)) - \log f(z(0))}{2\pi i} = \frac{1}{2\pi}\Delta \operatorname{Arg} f(z) \tag{1}$$

as z travels around γ from the starting point $z(0)$ to the terminal point $z(1) = z(0)$. To prove (1), we split γ into a finite number of simple arcs

$$\begin{aligned} &\gamma_1 \colon z(t), \quad 0 \le t \le t_1 \\ &\gamma_2 \colon z(t), \quad t_1 \le t \le t_2 \\ &\qquad\qquad \cdots \\ &\gamma_n \colon z(t), \quad t_{n-1} \le t \le t_n = 1. \end{aligned}$$



Since an analytic branch of $\log f$ can be defined in a simply connected domain containing γ_1,

$$\int_{\gamma_1} \frac{f'}{f} = \log f(z(t_1)) - \log f(z(0)).$$

Similarly

$$\int_{\gamma_k} \frac{f'}{f} = \log f(z(t_k)) - \log f(z(t_{k-1})), \qquad k = 2, 3, \ldots, n.$$

We note that

$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} + \cdots + \int_{\gamma_n},$$

and the first equality in (1) follows. Note, also, that since $z(0) = z(1)$ and since

$$\log w = \log |w| + i \text{ Arg } \omega,$$
$$\log f(z(1)) - \log f(z(0)) = i \left[\text{ Arg } f(z(1)) - \text{ Arg } f(z(0)) \right],$$

and the second equality follows.

2. We may also view $\int_\gamma f'/f$ as the winding number of the curve $f(\gamma(z))$ around $z = 0$. (See Definition 10.2.) Thus, if f is analytic inside and on γ, the number of zeroes of f inside γ is equal to the number of times that the curve $f(\gamma)$ winds around the origin. By considering $f(z) - a$, it follows that the number of times that $f = a$ inside γ equals the number of times that $f(\gamma)$ winds around the complex number a. As an example, consider the function described in Exercise 3b of Chapter 7.

10.10 Rouché's Theorem

Suppose that f and g are analytic inside and on a regular closed curve γ and that $|f(z)| > |g(z)|$ for all $z \in \gamma$. Then

$$\mathbb{Z}(f+g) = \mathbb{Z}(f) \qquad \text{inside } \gamma.$$

Proof

Note first that if $f(z) = A(z)B(z)$

$$\frac{f'}{f} = \frac{A'}{A} + \frac{B'}{B}$$

so that

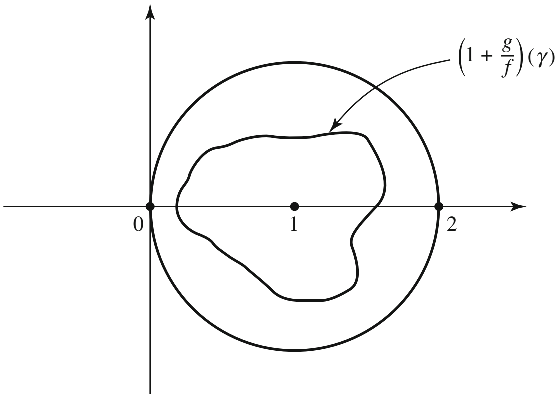
$$\int_\gamma \frac{f'}{f} = \int_\gamma \frac{A'}{A} + \int_\gamma \frac{B'}{B}.$$

Thus, if we write

$$f + g = f\left(1 + \frac{g}{f}\right),$$

$$\mathbb{Z}(f+g) = \frac{1}{2\pi i}\int_\gamma \frac{(f+g)'}{f+g} = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} + \frac{1}{2\pi i}\int_\gamma \frac{\left(1+\frac{g}{f}\right)'}{1+\frac{g}{f}}$$
$$= \mathbb{Z}(f) + \frac{1}{2\pi i}\int_\gamma \frac{\left(1+\frac{g}{f}\right)'}{1+\frac{g}{f}}.$$

But this last integral is zero since, by hypothesis, $(1 + g/f)(\gamma)$ remains within a disc of radius 1 around $z = 1$. Hence the winding number of $(1 + g/f)(\gamma)$ around 0 is 0 [i.e., setting $\omega = 1 + g/f$ it follows that $\omega(z)$ remains in the right half-plane for $z \in \gamma$ and hence that $\int_{\gamma *} \frac{d\omega}{\omega} = 0$.] □



EXAMPLE
Since $|4z^2| > |2z^{10} + 1|$ on $|z| = 1$, each of the polynomials

$$2z^{10} + 4z^2 + 1 \text{ and } 2z^{10} - 4z^2 + 1$$

has exactly two zeroes in $|z| < 1$. ◊

Recall that according to the Cauchy Integral Formula (6.4)

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{\omega - z} d\omega$$

where C is a circle containing z. By application of the Residue Theorem, we can extend the result as follows.

10.11 Generalized Cauchy Integral Formula

Suppose that f is analytic in a simply connected domain D and that γ is a regular closed curve contained in D. Then for each z inside γ and $k = 0, 1, 2 \ldots$,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(\omega)}{(\omega - z)^{k+1}} d\omega.$$

Proof

Note that since

$$f(\omega) = f(z) + f'(z)(\omega - z) + \cdots + \frac{f^{(k)}(z)}{k!}(\omega - z)^k + \cdots$$

throughout a neighborhood of z,

$$\text{Res}\left(\frac{f(\omega)}{(\omega - z)^{k+1}}; z\right) = \frac{f^{(k)}(z)}{k!}.$$

Since $f(\omega)/(\omega - z)^{k+1}$ has no other singularities in D, the result follows from Corollary 10.6. □

We now derive an extension of Theorem 7.6 for the limit of analytic functions.

10.12 Theorem

Suppose a sequence of functions f_n, analytic *in a region D, converges to f uniformly on compacta of D. Then f is* analytic, $f_n' \to f'$ *in D and the convergence of f_n' is also uniform on compacta of D.*

Proof

We proved the analyticity of f in Theorem 7.6. By the Integral Formula 10.11, if we pick any $z_0 \in D$ and let $C = C(z_0; r)$ for some $r < 1$,

$$f_n'(z) - f'(z) = \frac{1}{2\pi i}\int_C \frac{f_n(\omega) - f(\omega)}{(\omega - z)^2} d\omega$$

for all z in $D(z_0; r)$. Moreover, if we take n large enough so that $|f_n - f| < \epsilon r^2/4$ throughout the compact $\overline{D(z_0; r)}$, it follows that

$$|f_n'(z) - f'(z)| < \epsilon$$

for all z in $D(z_0; r/2)$. Thus, to see that the convergence is uniform on compacta, we need only note that any compact subset D can be covered by finitely many discs of the form: $|z - z_0| < r/2$. □

10.13 Hurwitz's Theorem

Let $\{f_n\}$ be a sequence of non-vanishing analytic functions in a region D and suppose $f_n \to f$ uniformly on compacta of D. Then either $f \equiv 0$ in D or $f(z) \neq 0$ for all $z \in D$.

Proof

Suppose $f(z) = 0$ for some $z \in D$. If $f \not\equiv 0$, there is some circle C centered at z and such that $f(z) \neq 0$ on C; hence

$$\frac{f_n'}{f_n} \to \frac{f'}{f} \quad \text{uniformly on } C.$$

However

$$\frac{1}{2\pi i}\int_C \frac{f'}{f} = \mathbb{Z}(f) \geq 1,$$

while

$$\frac{1}{2\pi i}\int_C \frac{f_n'}{f_n} = \mathbb{Z}(f_n) = 0.$$

Hence

$$f \equiv 0. \qquad \square$$

[Note that it is possible to have $f \equiv 0$ despite the fact that $f_n(z) \neq 0$ for all n. Consider, for example, $f_n(z) = (1/n)e^z$.]

EXAMPLE

Since $\sin \pi = 0$, there must be some n_0 such that

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots \frac{z^{2n+1}}{(2n+1)!}$$

has a zero in $|z - \pi| < 1$ for all $n > n_0$. $\diamondsuit$

10.14 Corollary

Suppose that f_n is a sequence of analytic functions in a region D, that $f_n \to f$ *uniformly on compacta in D, and that $f_n \neq a$. Then either $f \equiv a$ or $f \neq a$ in D.*

Proof

Consider $g_n(z) = f_n(z) - a$, etc. $\square$

10.15 Theorem

Suppose that f_n is a sequence of analytic functions, and that $f_n \to f$ uniformly on compacta in a region D. If f_n is 1-1 in *D for all n, then either f is constant or f is* 1-1 *in D.*

Proof

Assume $z_1 \neq z_2$, $f(z_1) = f(z_2) = a$ and take disjoint discs D_1 and D_2 (in D) surrounding z_1 and z_2, respectively. If $f \not\equiv a$, by 10.13, $f_n(z) = a$ must have a solution in D_1 once n is large enough. (Otherwise we could find a subsequence $f_{n_k} \to f$ with no a-values D_1.) But then since f_n is 1-1, $f_n(z) \neq a$ throughout D_2 for all large n and hence $f(z_2) \neq a$, contradicting our hypothesis. $\square$

Exercises

1. Determine the singularities and associated residues of

 a. $\dfrac{1}{z^4+z^2}$ b. $\cot z$

 c. $\csc z$ d. $\dfrac{\exp(1/z^2)}{z-1}$

 e. $\dfrac{1}{z^2+3z+2}$ f. $\sin\dfrac{1}{z}$

 g. $ze^{3/z}$ h. $\dfrac{1}{az^2+bz+c}$, $a \neq 0$.

2. Use the Residue Theorem to evaluate

 a. $\int_{|z|=1} \cot z\, dz$ b. $\int_{|z|=2} \dfrac{dz}{(z-4)(z^3-1)}$

 c. $\int_{|z|=1} \sin\dfrac{1}{z} dz$ d. $\int_{|z|=2} ze^{3/z} dz$.

3. Prove that for any positive integer n, $\text{Res}((1-e^{-z})^{-n};\ 0) = 1$. [*Hint*: Consider

$$\int_C \frac{dz}{(1-e^{-z})^n}$$

 and make the change of variables $\omega = 1 - e^{-z}$ to show

$$\text{Res}((1-e^{-z})^{-n}; 0) = \text{Res}\left(\frac{1}{\omega^n(1-\omega)}; 0\right).]$$

4.* Show that $\int_{|z|=1} (z+1/z)^{2m+1} dz = 2\pi i \binom{2m+1}{m}$, for any nonnegative integer m.

5.* Let C be a regular curve enclosing the distinct points $\omega_1, \omega_2, ...\omega_n$ and let $p(\omega) = (\omega-\omega_1)(\omega-\omega_2)\cdots(\omega-\omega_n)$. Suppose that $f(\omega)$ is analytic in a region that includes C. Show that

$$P(z) = \frac{1}{2\pi i}\int_C \frac{f(\omega)}{p(\omega)} \cdot \frac{p(\omega)-p(z)}{\omega - z} d\omega$$

 is a polynomial of degree $n-1$, with $P(\omega_i) = f(\omega_i)$, $i = 1, 2, ...n$.

6. Suppose f is defined by

$$f(z) = \int_\gamma \frac{\phi(\omega)d\omega}{\omega - z},$$

 where γ is a compact curve, ϕ is continuous on γ, and $z \notin \gamma$. Show that

$$f'(z) = \int_\gamma \frac{\phi(\omega)d\omega}{(\omega-z)^2}$$

 directly by considering

$$\frac{f(z+h)-f(z)}{h}.$$

 Give an alternate proof of Theorem 10.11.

7. Suppose that f is entire and that $f(z)$ is real if and only if z is real. Use the Argument Principle to show that f can have at most one zero. (Compare this with Exercise 13 of Chapter 5.)

8.* a. Show that Rouche's Theorem remains valid if the condition: $|f| > |g|$ on γ is replaced by: $|f| \geq |g|$ and $f + g \neq 0$ on γ.
 b. Find the number of zeroes of $z^5 + 2z^4 + 1$ in the unit disc.

9. Find the number of zeroes of

   a. $f_1(z) = 3e^z - z$ in $|z| \le 1$

   b. $f_2(z) = \frac{1}{3}e^z - z$ in $|z| \le 1$

   c. $f_3(z) = z^4 - 5z + 1$ in $1 \le |z| \le 2$

   d. $f_4(z) = z^6 - 5z^4 + 3z^2 - 1$ in $|z| \le 1$.

10.* Suppose $\lambda > 1$. Show that $\lambda - z - e^{-z} = 0$ has exactly one root (which is a real number) in the right half-plane.

11. Suppose f is analytic inside and on a regular closed curve γ and has no zeroes on γ. Show that if m is a positive integer then

$$\frac{1}{2\pi i}\int_\gamma z^m \frac{f'(z)}{f(z)}dz = \sum_k (z_k)^m$$

where the sum is taken over all the zeroes of f inside γ.

12. Show that for each $R > 0$, if n is large enough,

$$P_n(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} \text{ has } \textit{no} \text{ zeroes in } |z| \le R.$$

13.* a. Let $P(z)$ be any polynomial of the form: $a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, with all a_i real and $0 \le a_0 \le a_1 \le \cdots \le a_n$. Prove that all the zeroes of $P(z)$ lie inside the unit disc by applying Rouche's Theorem to $(1 - z)P(z)$.

   b. Prove that, for any $\rho < 1$, the polynomial $P_n(z) = 1 + 2z + 3z^2 + \cdots + (n+1)z^n$ has no zeroes inside the circle $|z| < \rho$ if n is sufficiently large.

14. Derive the Fundamental Theorem of Algebra as a corollary of Rouché's Theorem.

15. Supply the details of the following proof of Rouché's Theorem (due to Carathéodory). Set

$$J(\lambda) = \frac{1}{2\pi i}\int_\gamma \frac{(f + \lambda g)'}{f + \lambda g}, \qquad 0 \le \lambda \le 1.$$

Note that $J(\lambda)$ is defined for all $\lambda, 0 \le \lambda \le 1$. Furthermore $J(\lambda)$ is a continuous function of λ and is always integer-valued. Hence J is constant; in particular, $J(0) = J(1)$ so that

$$\mathbb{Z}(f) = \mathbb{Z}(f + g).$$

16. Recall, as in 8.2, that

$$\log(z^2 - 1) = \int_{\sqrt{2}}^{z} \frac{2\zeta}{\zeta^2 - 1}d\zeta$$

is analytic in the plane minus the interval $(-\infty, 1]$. Hence, so is

$$\sqrt{z^2 - 1} = \exp\left(\frac{1}{2}\log(z^2 - 1)\right). \tag{1}$$

Show that $\sqrt{z^2 - 1}$ (as defined in (1)) is analytic in the entire plane minus the interval $[-1, 1]$. [*Hint*: Use the Argument Principle to show that $\sqrt{z^2 - 1}$ is continuous along the interval $(-\infty, -1)$ and then apply Morera's Theorem.]

17. Show that an analytic $\sqrt[3]{(z-1)(z-2)(z-3)}$ can be defined in the entire plane minus $[1, 3]$.