Chapter 1
The Complex Numbers

Introduction

Numbers of the form $a + b\sqrt{-1}$, where a and b are real numbers—what we call complex numbers—appeared as early as the 16th century. Cardan (1501–1576) worked with complex numbers in solving quadratic and cubic equations. In the 18th century, functions involving complex numbers were found by Euler to yield solutions to differential equations. As more manipulations involving complex numbers were tried, it became apparent that many problems in the theory of real-valued functions could be most easily solved using complex numbers and functions. For all their utility, however, complex numbers enjoyed a poor reputation and were not generally considered legitimate numbers until the middle of the 19th century. Descartes, for example, rejected complex roots of equations and coined the term "imaginary" for such roots. Euler, too, felt that complex numbers "exist only in the imagination" and considered complex roots of an equation useful only in showing that the equation actually has *no* solutions.

The wider acceptance of complex numbers is due largely to the geometric representation of complex numbers which was most fully developed and articulated by Gauss. He realized it was erroneous to assume "that there was some dark mystery in these numbers." In the geometric representation, he wrote, one finds the "intuitive meaning of complex numbers completely established and more is not needed to admit these quantities into the domain of arithmetic."

Gauss' work did, indeed, go far in establishing the complex number system on a firm basis. The first complete and formal definition, however, was given by his contemporary, William Hamilton. We begin with this definition, and then consider the geometry of complex numbers.

1.1 The Field of Complex Numbers

We will see that complex numbers can be written in the form $a + bi$, where a and b are real numbers and i is a square root of -1. This in itself is not a formal definition,

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however, since it presupposes a system in which a square root of -1 makes sense. The existence of such a system is precisely what we are trying to establish. Moreover, the operations of addition and multiplication that appear in the expression $a + bi$ have not been defined. The formal definition below gives these definitions in terms of ordered pairs.

1.1 Definition

The complex field $\mathbb{C}$ is the set of ordered pairs of real numbers (a, b) with addition and multiplication defined by

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d)\\ (a, b)(c, d) &= (ac - bd, ad + bc).\end{aligned}$$

The associative and commutative laws for addition and multiplication as well as the distributive law follow easily from the same properties of the real numbers. The additive identity, or *zero*, is given by $(0, 0)$, and hence the additive inverse of (a, b) is $(-a, -b)$. The multiplicative identity is $(1, 0)$. To find the multiplicative inverse of any nonzero (a, b) we set

$$(a, b)(x, y) = (1, 0),$$

which is equivalent to the system of equations:

$$\begin{aligned}ax - by &= 1\\ bx + ay &= 0\end{aligned}$$

and has the solution

$$x = \frac{a}{a^2 + b^2}, \quad y = \frac{-b}{a^2 + b^2}.$$

Thus the complex numbers form a field.

Suppose now that we associate complex numbers of the form $(a, 0)$ with the corresponding real numbers a. It follows that

$$(a_1, 0) + (a_2, 0) = (a_1 + a_2, 0) \text{ corresponds to } a_1 + a_2$$

and that

$$(a_1, 0)(a_2, 0) = (a_1 a_2, 0) \text{ corresponds to } a_1 a_2.$$

Thus the correspondence between $(a, 0)$ and a preserves all arithmetic operations and there can be no confusion in replacing $(a, 0)$ by a. In that sense, we say that the set of complex numbers of the form $(a, 0)$ is isomorphic with the set of real numbers, and we will no longer distinguish between them. In this manner we can now say that $(0, 1)$ is a square root of -1 since

$$(0, 1)(0, 1) = (-1, 0) = -1$$

and henceforth (0, 1) will be denoted i. Note also that

$$a(b, c) = (a, 0)(b, c) = (ab, ac),$$

so that we can rewrite any complex number in the following way:

$$(a, b) = (a, 0) + (0, b) = a + bi.$$

We will use the latter form throughout the text.

Returning to the question of square roots, there are in fact two complex square roots of -1: i and $-i$. Moreover, there are two square roots of any nonzero complex number $a + bi$. To solve

$$(x + iy)^2 = a + bi$$

we set

$$\begin{aligned} x^2 - y^2 &= a \\ 2xy &= b \end{aligned}$$

which is equivalent to

$$\begin{aligned} 4x^4 - 4ax^2 - b^2 &= 0 \\ y &= b/2x. \end{aligned}$$

Solving first for x^2, we find the two solutions are given by

$$\begin{aligned} x &= \pm\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \\ y &= \frac{b}{2x} = \pm\sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}} \cdot (\text{sign } b) \end{aligned}$$

where

$$\text{sign } b = \begin{cases} 1 & \text{if } b \geq 0 \\ -1 & \text{if } b < 0. \end{cases}$$

EXAMPLE

i. The two square roots of $2i$ are $1 + i$ and $-1 - i$.
ii. The square roots of $-5 - 12i$ are $2 - 3i$ and $-2 + 3i$. ◊

It follows that any quadratic equation with complex coefficients admits a solution in the complex field. For by the usual manipulations,

$$az^2 + bz + c = 0 \quad a, b, c \in \mathbb{C}, \quad a \neq 0$$

is seen to be equivalent to

$$\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2},$$

and hence has the solutions

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1}$$

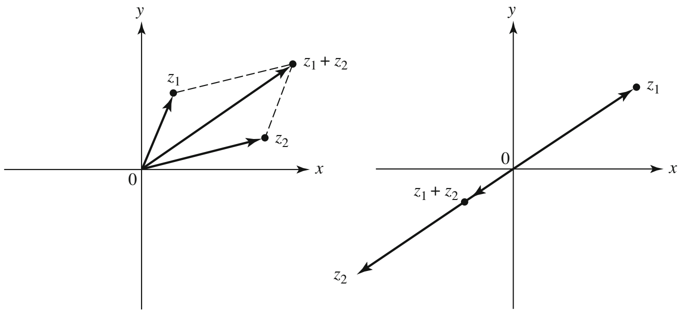
In Chapter 5, we will see that quadratic equations are not unique in this respect: every nonconstant polynomial with complex coefficients has a zero in the complex field.

One property of real numbers that does not carry over to the complex plane is the notion of *order*. We leave it as an exercise for those readers familiar with the axioms of order to check that the number i cannot be designated as either positive or negative without producing a contradiction.

1.2 The Complex Plane

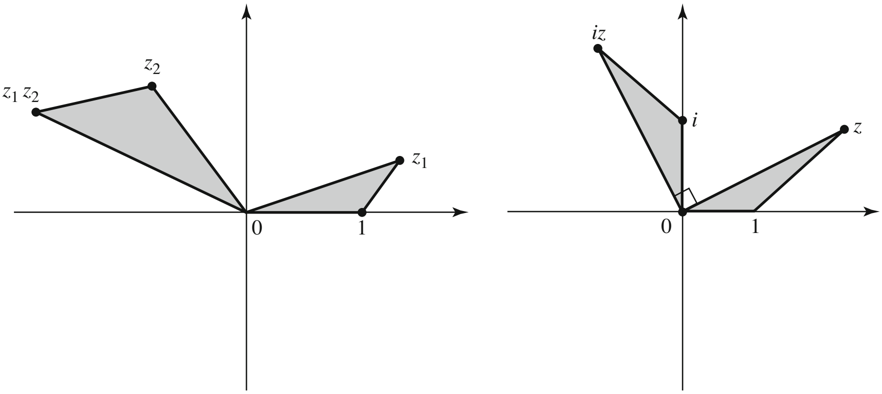
Thinking of complex numbers as ordered pairs of real numbers (a, b) is closely linked with the geometric interpretation of the complex field, discovered by Wallis, and later developed by Argand and by Gauss. To each complex number $a + bi$ we simply associate the point (a, b) in the Cartesian plane. Real numbers are thus associated with points on the x-axis, called the *real axis* while the purely imaginary numbers bi correspond to points on the y-axis, designated as the *imaginary axis*.

Addition and multiplication can also be given a geometric interpretation. The sum of z_1 and z_2 corresponds to the vector sum: If the vector from 0 to z_2 is shifted parallel to the x and y axes so that its initial point is z_1, the resulting terminal point is $z_1 + z_2$. If 0, z_1 and z_2 are not collinear this is the so-called parallelogram law; see below.



The geometric method for obtaining the product $z_1 z_2$ is somewhat more complicated. If we form a triangle with two sides given by the vectors (originating from 0 to) 1 and z_1 and then form a similar triangle with the same orientation and the vector z_2 corresponding to the vector 1, the vector which then corresponds to z_1 will be $z_1 z_2$.

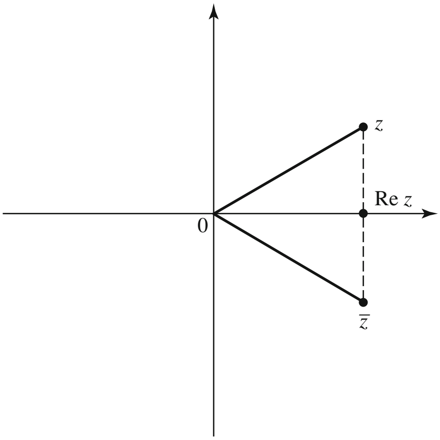
This can be verified geometrically but will be most transparent when we introduce polar coordinates later in this section. For the moment, we observe that multiplication by i is equivalent geometrically to a counterclockwise rotation of 90°.



With $z = x + iy$, the following terms are commonly used:

Re z, *the real part of* z, is x;
Im z, *the imaginary part of* z, is y (note that Im z is a real number);
$\bar{z}$, the *conjugate* of z, is $x - iy$.

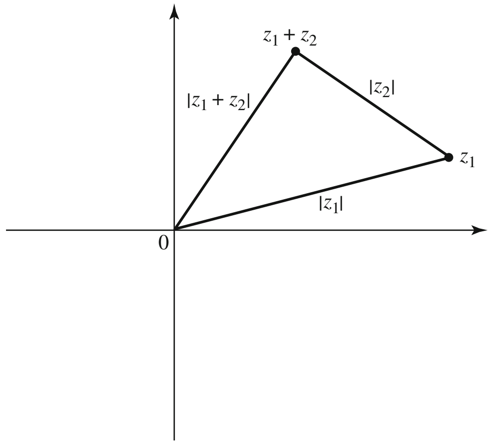
Geometrically, $\bar{z}$ is the mirror image of z reflected across the real axis.



$|z|$, the *absolute value* or *modulus* of z, is equal to $\sqrt{x^2 + y^2}$; that is, it is the length of the vector z. Note also that $|z_1 - z_2|$ is the (Euclidean) distance between z_1 and z_2. Hence we can think of $|z_2|$ as the distance between $z_1 + z_2$ and z_1 and thereby obtain a proof of the triangle inequality:

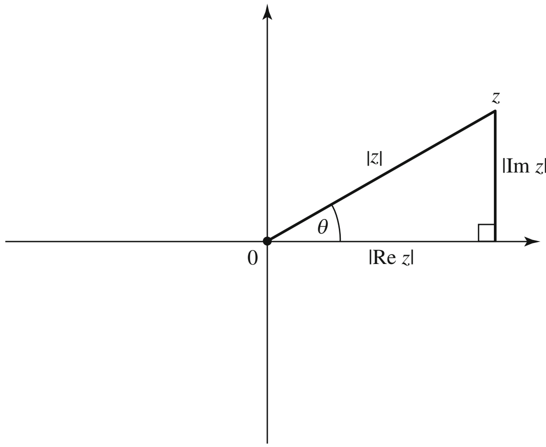
$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

An algebraic proof of the inequality is outlined in Exercise 8.



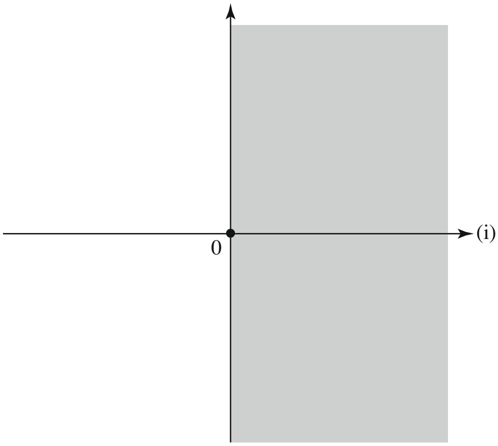
Arg z, the *argument* of z, defined for $z \neq 0$, is the angle which the vector (originating from 0) to z makes with the positive x-axis. Thus Arg z is defined (modulo 2π) as that number θ for which

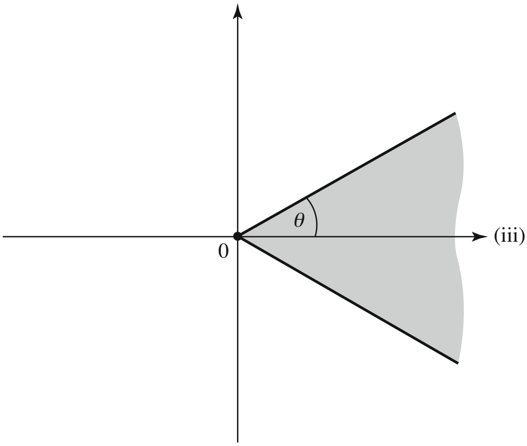
$$\cos\theta = \frac{\operatorname{Re} z}{|z|}; \quad \sin\theta = \frac{\operatorname{Im} z}{|z|}.$$

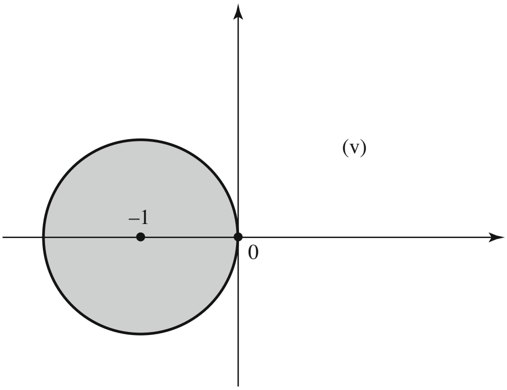


EXAMPLES

i. The set of points given by the equation $\operatorname{Re} z > 0$ is represented geometrically by the right half-plane.
ii. $\{z : z = \bar{z}\}$ is the real line.
iii. $\{z : -\theta < \operatorname{Arg} z < \theta\}$ is an angular sector (wedge) of angle 2θ.
iv. $\{z : |\operatorname{Arg} z - \pi/2| < \pi/2\} = \{z : \operatorname{Im} z > 0\}$.
v. $\{z : |z + 1| < 1\}$ is the disc of radius 1 centered at -1. ◊


0
(i)


θ
0
(iii)


(v)
−1
0

A nonzero complex number is completely determined by its modulus and argument. If $z = x + iy$ with $|z| = r$ and $\operatorname{Arg} z = \theta$, it follows that $x = r\cos\theta$, $y = r\sin\theta$ and

$$z = r(\cos\theta + i\sin\theta).$$

We abbreviate $\cos\theta + i\sin\theta$ as $\operatorname{cis}\theta$. In this context, r and θ are called the polar coordinates of z and $r\operatorname{cis}\theta$ is called the polar form of the complex number z. This form is especially handy for multiplication. Let $z_1 = r_1\operatorname{cis}\theta_1$, $z_2 = r_2\operatorname{cis}\theta_2$. Then

$$z_1z_2 = r_1r_2\operatorname{cis}\theta_1\operatorname{cis}\theta_2 = r_1r_2\operatorname{cis}(\theta_1+\theta_2),$$

since

$$\begin{aligned}
&(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\\
&\quad= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\\
&\quad= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\\
&\quad= \operatorname{cis}(\theta_1+\theta_2).
\end{aligned}$$

Thus, if z is the product of two complex numbers, $|z|$ is the product of their moduli and $\operatorname{Arg} z$ is the sum of their arguments (modulo 2π). (This can be used to verify the geometric construction for z_1z_2 given at the beginning of this section.) Similarly z_1/z_2 can be obtained by dividing the moduli and subtracting the arguments:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1-\theta_2).$$

It follows by induction that if $z = r\operatorname{cis}\theta$ and n is any integer,

$$z^n = r^n\operatorname{cis} n\theta. \tag{1}$$

Identity (1) is especially handy for solving "pure" equations of the form $z^n = z_0$.

EXAMPLE

To find the cube roots of 1, we write $z^3 = 1$ in the polar form

$$r^3\operatorname{cis}3\theta = 1\operatorname{cis}0,$$

which is satisfied if and only if

$$r = 1, 3\theta = 0 \text{ (modulo } 2\pi).$$

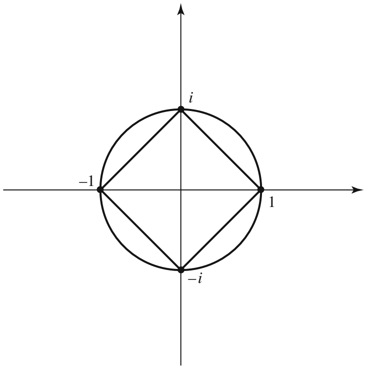
Hence the three solutions are given by

$$z_1 = \operatorname{cis}0, \quad z_2 = \operatorname{cis}\frac{2\pi}{3}, \quad z_3 = \operatorname{cis}\frac{4\pi}{3},$$

or in rectangular (x, y) coordinates

$$z_1 = 1, \quad z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad z_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

The polar form of the three cube roots reveals that they are the vertices of an equilateral triangle inscribed in the unit circle. Similarly the n-th roots of 1 are located at the vertices of the regular n-gon inscribed in the unit circle with one vertex at $z = 1$. For example, the fourth roots of 1 are ± 1 and $\pm i$. ◊



1.3 The Solution of the Cubic Equation

As we mentioned at the beginning of this chapter, complex numbers were applied to the solution of quadratic and cubic equations as far back as the 16th century. While neither of these applications was sufficient to gain a wide acceptance of complex numbers, there was a fundamental difference between the two. In the case of quadratic equations, it may have seemed interesting that solutions could always be found among the complex numbers, but this was generally viewed as nothing more than an oddity at best. After all, if a quadratic equation (with real coefficients) had no real solutions, it seemed just as reasonable to simply say that there were no solutions as to describe so-called solutions in terms of some imaginary number.

Cubic equations presented a much more tantalizing situation. For one thing, every cubic equation with real coefficients has a real solution. The fact that such a real solution could be found through the use of complex numbers showed that the complex numbers were at least useful, even if somewhat illegitimate. In fact, the solution of the cubic equation was followed by a string of other applications which demonstrated the uncanny ability of complex numbers to play a role in the solution of problems involving real numbers and functions.

Let's see how complex numbers were first applied to cubic equations. There is obviously no loss in assuming that the general cubic equation:

$$ax^3 + bx^2 + cx + d = 0$$

has leading coefficient $a = 1$. The equation can then be further reduced to the simpler form:

$$x^3 + px + q = 0 \tag{1}$$

if we change x into $x - \frac{b}{3}$. The first recorded solution for cubic equations involved a method for finding the real solution of the above "reduced" or "depressed" cubic in the form:

$$x^3 + px = q \tag{2}$$

To the modern reader, of course, equation (2) is, for all practical purposes, identical to equation (1). But in the early 16th century, mathematicians were not entirely comfortable with negative numbers either, and it was assumed that the coefficients p and q in equation (2) denoted positive real numbers. In fact, in that case, $f(x) = x^3 + px$ is a monotonically increasing function, so that equation (2) has exactly one positive real solution. To find that solution, del Ferro (1465–1526) suggested setting $x = u + v$, so that (2) could be rewritten as:

$$u^3 + v^3 + (3uv + p)(u + v) = q \tag{3}$$

The solution to (3) can be found, then, by solving the pair of equations: $3uv+p = 0$ and $u^3 + v^3 = q$. Using the first equation to express v in terms of u, and substituting into the second equation leads to:

$$u^6 - u^3 q - \frac{p^3}{27} = 0$$

which is a quadratic equation for u^3 and has the solutions

$$u^3 = \frac{q \pm \sqrt{q^2 + 4p^3/27}}{2}.$$

The identical formula can be obtained for v^3, and since $u^3 + v^3 = q$,

$$x = u + v = \sqrt[3]{\frac{q + \sqrt{q^2 + 4p^3/27}}{2}} + \sqrt[3]{\frac{q - \sqrt{q^2 + 4p^3/27}}{2}}. \tag{4}$$

or, as del Ferro would have written it to avoid the cube root of a negative number,

$$x = u + v = \sqrt[3]{\frac{\sqrt{q^2 + 4p^3/27} + q}{2}} - \sqrt[3]{\frac{\sqrt{q^2 + 4p^3/27} - q}{2}}$$

For example, if $p = 6$ and $q = 20$, we find $x = \sqrt[3]{6\sqrt{3} + 10} - \sqrt[3]{6\sqrt{3} - 10}$ or (check this!) $x = 2$.

Although (4) was originally intended to be applied with $p, q > 0$, it can obviously be applied equally well for any values of p and q. Changing q into $-q$ would simply cause the same change in x. For example, the unique real solution of the equation $x^3 + 6x = -20$ is $x = -2$. Changing p into a negative number, however, can introduce complex values. To be precise, if $q^2 + 4p^3/27 < 0$; i.e., if $4p^3 < -27q^2$, equation (4) gives the solution as the sum of the cube roots of two complex conjugates. For example, if we apply (4) to the equation $x^3 - 6x = 4$, we obtain

$$x = \sqrt[3]{2 + 2i} + \sqrt[3]{2 - 2i} \tag{5}$$

Since we saw (in the last section) that we can calculate the three cube roots of any complex number using its polar form, and since the cube roots of a conjugate of any complex number are the conjugates of its cube roots, we realize that (5) actually does give the three real roots of $x^3 - 6x = 4$.

To Cardan, however, who published formula (4) in his *Ars Magna* (1545), the case: $4p^3 < -27q^2$ presented a dilemma. We leave it as an exercise to verify that equation (2) has three real roots if and only if $4p^3 < -27q^2$. Ironically, then, precisely in the case when all three solutions are real, if formula (4) is applicable at all, it gives the solutions in terms of cube roots of complex numbers! Moreover, Cardan was willing to try a direct approach to finding the cube roots of a complex number (as we found the square roots of any complex number in section 1), but solving the equation $(x + iy)^3 = a + bi$ by equating real and imaginary parts led to an equation no less complicated than the original cubic. Cardan, therefore, labeled this situation the "irreducible" case of the depressed cubic equation.

Fortunately, however, the idea of applying (4) even in the "irreducible" case, was never laid to rest. Bombelli's *Algebra* (1574) included the equation $x^3 = 15x + 4$, which led to the mysterious solution

$$x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} \tag{6}$$

By a direct approach, combined with the assumption that the cube roots in (6) would involve integral real and imaginary parts, Bombelli was able to show that formula (6) did "contain" the solution $x = 4$ in the form of $(2 + i) + (2 - i)$. He did not suggest that (6) might also contain the other two real roots nor did he generalize the method. In fact, over a hundred years later, the issue was still not resolved. Thus Leibniz (1646–1716) continued to question how "a quantity could be real when imaginary or impossible numbers were used to express it". But he too could not let the matter go. Among unpublished papers found after his death, there were several identities similar to

$$\sqrt[3]{36 + \sqrt{-2000}} + \sqrt[3]{36 - \sqrt{-2000}} = -6$$

which he found by applying (4) to: $x^3 - 48x - 72 = 0$.

So complex numbers maintained their presence, albeit as second-class citizens, in the world of numbers until the early 19th century when the spread of their geometric interpretation began the process of their acceptance as first-class citizens.

1.4 Topological Aspects of the Complex Plane

I. *Sequences and Series* The concept of absolute value can be used to define the notion of a limit of a sequence of complex numbers.

1.2 Definition

The sequence $z_1, z_2, z_3, \ldots$ converges to z if the sequence of real numbers $|z_n - z|$ converges to 0. That is, $z_n \to z$ if $|z_n - z| \to 0$.

Geometrically, $z_n \to z$ if each disc about z contains all but finitely many of the members of the sequence $\{z_n\}$.

Since

$$|\mathrm{Re}\, z|, |\mathrm{Im}\, z| \le |z| \le |\mathrm{Re}\, z| + |\mathrm{Im}\, z|,$$

$z_n \to z$ if and only if $\mathrm{Re}\, z_n \to \mathrm{Re}\, z$ and $\mathrm{Im}\, z_n \to \mathrm{Im}\, z$.

EXAMPLES

1. $z^n \to 0$ if $|z| < 1$ since $|z^n - 0| = |z|^n \to 0$.
2. $\dfrac{n}{n+i} \to 1$ since $\left|\dfrac{n}{n+i} - 1\right| = \left|\dfrac{-i}{n+i}\right| = \dfrac{1}{\sqrt{n^2+1}} \to 0$. ◊

1.3 Definition

$\{z_n\}$ is called a *Cauchy sequence* if for each $\epsilon > 0$ there exists an integer N such that $n, m > N$ implies $|z_n - z_m| < \epsilon$.

1.4 Proposition

$\{z_n\}$ converges if and only if $\{z_n\}$ is a Cauchy sequence.

Proof

If $z_n \to z$, then $\mathrm{Re}\, z_n \to \mathrm{Re}\, z$, $\mathrm{Im}\, z_n \to \mathrm{Im}\, z$ and hence $\{\mathrm{Re}\, z_n\}$ and $\{\mathrm{Im}\, z_n\}$ are Cauchy sequences. But since

$$\begin{aligned} |z_n - z_m| &\le |\mathrm{Re}(z_n - z_m)| + |\mathrm{Im}\,(z_n - z_m)| \\ &= |\mathrm{Re}\, z_n - \mathrm{Re}\, z_m| + |\mathrm{Im}\, z_n - \mathrm{Im}\, z_m|, \end{aligned}$$

$\{z_n\}$ is also a Cauchy sequence.

Conversely, if $\{z_n\}$ is a Cauchy sequence so are the real sequences $\{\mathrm{Re}\, z_n\}$ and $\{\mathrm{Im}\, z_n\}$. Hence both $\{\mathrm{Re}\, z_n\}$ and $\{\mathrm{Im}\, z_n\}$ converge, and thus $\{z_n\}$ converges. □

An infinite series $\sum_{k=1}^{\infty} z_k$ is said to converge if the sequence $\{s_n\}$ of partial sums, defined by $s_n = z_1 + z_2 + \cdots z_n$, converges. If so, the limit of the sequence is called the sum of the series. The basic properties of infinite series listed below will be familiar from the theory of real series.

i. The sum and the difference of two convergent series are convergent.
ii. A necessary condition for $\sum_{k=1}^{\infty} z_k$ to converge is that $z_n \to 0$ as $n \to \infty$.
iii. A sufficient condition for $\sum_{k=1}^{\infty} z_k$ to converge is that $\sum_{k=1}^{\infty} |z_k|$ converges. When $\sum_{k=1}^{\infty} |z_k|$ converges, we will say $\sum_{k=1}^{\infty} z_k$ is *absolutely convergent*.

Property (iii), which will be important in later chapters, follows from Proposition 1.4. For if $\sum_{k=1}^{\infty} |z_k|$ converges and $t_n = |z_1| + |z_2| + \cdots + |z_n|$ then $\{t_n\}$ is a Cauchy sequence. But then so is the sequence $\{s_n\}$ given by $s_n = z_1 + z_2 + \cdots + z_n$, since

$$|s_m - s_n| = |z_{n+1} + z_{n+2} + \cdots + z_m| \\ \leq |z_{n+1}| + |z_{n+2}| + \cdots + |z_m| = |t_m - t_n|$$

by the triangle inequality. Hence $\sum_{k=1}^{\infty} z_k$ converges.

EXAMPLES

1. $\sum_{k=1}^{\infty} \dfrac{i^k}{k^2+i}$ converges since

$$\left|\frac{i^k}{k^2+i}\right| = \frac{1}{\sqrt{k^4+1}} \text{ and since } \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^4+1}} \text{ converges.}$$

2. $\sum_{k=1}^{\infty} \dfrac{1}{k+i}$ diverges, since

$$\frac{1}{k+i} = \frac{k-i}{k^2+1}, \text{ which implies that } \sum_{k=1}^{\infty} \operatorname{Re}\left(\frac{1}{k+i}\right) \text{ diverges.}$$ ◊

II. *Classification of Sets in the Complex Plane* We give some definitions relating to planar sets.

1.5 Definitions

$D(z_0; r)$ denotes the open disc of radius $r > 0$ centered at z_0; i.e., $D(z_0; r) = \{z : |z - z_0| < r\}$.

$D(z_0; r)$ is also called a *neighborhood* (or r-neighborhood) of z_0.

$C(z_0; r)$ is the circle of radius $r > 0$ centered at z_0.

A set S is said to be *open* if for any $z \in S$, there exists $\delta > 0$ such that $D(z; \delta) \subset S$.

For any set S, $\tilde{S} = \mathbb{C}\backslash S$ denotes the complement of S; i.e., $\tilde{S} = \{z \in \mathbb{C} : z \notin S\}$.

A set is *closed set* if its complement is open. Equivalently, S is closed if $\{z_n\} \subset S$ and $z_n \to z$ imply $z \in S$.

∂S, the *boundary* of S, is defined as the set of points whose δ-neighborhoods have a nonempty intersection with both S and $\tilde{S}$, for every $\delta > 0$.

$\bar{S}$, the *closure* of S, is given by $\bar{S} = S \cup \partial S$.

S is *bounded* if it is contained in $D(0; M)$ for some $M > 0$.

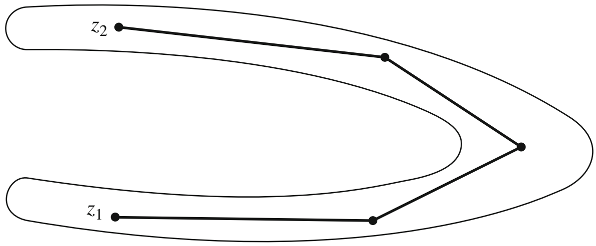
Sets that are closed and bounded are called *compact*.

S is said to be *disconnected* if there exist two disjoint open sets A and B whose union contains S while neither A nor B alone contains S. If S is not disconnected, it is called *connected*.

$[z_1, z_2]$ denotes the line segment with endpoints z_1 and z_2.

A *polygonal line* is a finite union of line segments of the form $[z_0, z_1] \cup [z_1, z_2] \cup [z_2, z_3] \ldots \cup [z_{n-1}, z_n]$.

If any two points of S can be connected by a polygonal line contained in S, S is said to be *polygonally connected*.



It can be shown that a polygonally connected set is connected. The converse, however, is false. For example, the set of points $z = x + iy$ with $y = x^2$ is clearly connected but is not polygonally connected since the set contains no straight line segments. In fact there are even connected sets whose points cannot be connected to one another by any curve in the set (see Exercise 23). On the other hand, for open sets, connectedness and polygonal connectedness are equivalent.

1.6 Definition

An open connected set will be called a *region*.

1.7 Proposition

A region S is polygonally connected.

Proof

Suppose $z_0 \in S$. Let A be the set of points of S which can be polygonally connected to z_0 in S and let B represent the set of points in S which cannot. Since any point z can be connected to any other point in $D(z; \delta)$, it follows that A is open. Similarly B is open. For if any point in a disc about z could be connected to z_0, then z could be connected to z_0. Now S is connected, $S = A \cup B$ and A is nonempty; hence we must conclude that B is empty. Finally, since every point in S can be connected to z_0, every pair of points can be connected to each other by a polygonal line in S. □

III *Continuous Functions*

1.8 Definition

A complex valued function $f(z)$ defined in a neighborhood of z_0 is *continuous at* z_0 if $z_n \to z_0$ implies that $f(z_n) \to f(z_0)$. Alternatively, f is continuous at z_0 if for each $\epsilon > 0$ there is some $\delta > 0$ such that $|z - z_0| < \delta$ implies that $|f(z) - f(z_0)| < \epsilon$. f is *continuous in a domain* D if for each sequence $\{z_n\} \subset D$ and $z \in D$ such that $z_n \to z$, we have $f(z_n) \to f(z)$.

If we split f into its real and imaginary parts

$$f(z) = f(x, y) = u(x, y) + iv(x, y),$$

where u and v are real-valued, it is clear that f is continuous if and only if u and v are continuous functions of (x, y). Thus, for example, any polynomial

$$P(x, y) = \sum_{j=1}^{m} \sum_{k=1}^{n} a_{kj} x^k y^j$$

is continuous in the whole plane. Similarly

$$\frac{1}{z} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$$

is continuous in the "punctured plane" $\{z : z \neq 0\}$. It follows also that the sum, product, and quotient (with nonzero denominator) of continuous functions are continuous.

We say $f \in C^n$ if the real and imaginary parts of f both have continuous partial derivatives of the n-th order.

A sequence of functions $\{f_n\}$ *converges to* f *uniformly in* D if for each $\epsilon > 0$, there is an $N > 0$ such that $n > N$ implies $|f_n(z) - f(z)| < \epsilon$ for all $z \in D$. Again, by referring to the real and imaginary parts of $\{f_n\}$, it is clear that the uniform limit of continuous functions is continuous.

1.9 *M*-Test

Suppose f_k *is continuous in* D, $k = 1, 2, \ldots$. *If* $|f_k(z)| \leq M_k$ *throughout* D *and if* $\sum_{k=1}^{\infty} M_k$ *converges, then* $\sum_{k=1}^{\infty} f_k(z)$ *converges to a function* f *which is continuous in* D.

Proof

The convergence of $\sum_{k=1}^{\infty} f_k(z)$ is immediate. Moreover, for each $\epsilon > 0$, we can choose N so that

$$\left| f(z) - \sum_{k=1}^{n} f_k(z) \right| = \left| \sum_{n+1}^{\infty} f_k(z) \right| \leq M_{n+1} + M_{n+2} + \cdots < \epsilon$$

for $n \geq N$. Hence the convergence is uniform and f is continuous. □

EXAMPLE
$f(z) = \sum_{k=1}^{\infty} kz^k$ is continuous in $D : |z| \leq \frac{1}{2}$ since

$$|kz^k| \leq \frac{k}{2^k} \text{ in } D \text{ and } \sum_{k=1}^{\infty} \frac{k}{2^k}$$

converges. (See Exercise 21.) ◊

Recall that a continuous function maps compact/connected sets into compact/connected sets. None of the other properties listed above, though, are preserved under continuous mappings. For example $f(z) = \operatorname{Re} z$ maps the open set $\mathbb{C}$ into the real line which is not open. The function $g(z) = 1/z$ maps the bounded set: $0 < |z| < 1$ onto the unbounded set: $|z| > 1$.

Most of the key results in subsequent chapters will concern properties of (a certain class of) functions defined on a region. We note that, arguing as in the proof of Proposition 1.7, we could show that any two points in a region can be connected by a polygonal line containing only horizontal and vertical line segments. For future reference we will introduce the term *polygonal path* to denote such a polygonal line.

One important result regarding real-valued functions on a region is given below.

1.10 Theorem

Suppose $u(x, y)$ has partial derivatives u_x and u_y that vanish at every point of a region D. Then u is constant in D.

Proof

Let (x_1, y_1) and (x_2, y_2) be two points of D. Then, as noted above, they can be connected by a polygonal path that is contained in D. Any two successive vertices of the path represent the end-points of a horizontal or vertical segment. Hence, by the Mean-Value Theorem for one real variable, the change in u between these vertices is given by the value of a partial derivative of u somewhere between the end-points times the difference in the non-identical coordinates of the endpoints. Since, however, u_x and u_y vanish identically in D, the change in u is 0 between each pair of successive vertices; hence $u(x_1, y_1) = u(x_2, y_2)$. □

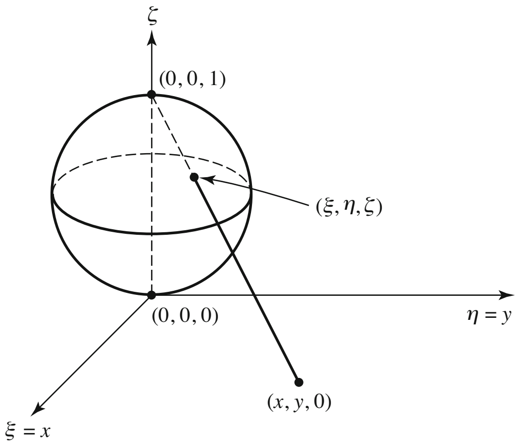
1.5 Stereographic Projection; The Point at Infinity

The complex numbers can also be represented by the points on the surface of a punctured sphere. Let

$$\sum = \left\{ (\xi, \eta, \zeta) : \xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 = \frac{1}{4} \right\}; \tag{1}$$

that is, let $\sum$ be the sphere in Euclidean (ξ, η, ζ) space with distance $\frac{1}{2}$ from $(0, 0, \frac{1}{2})$. Suppose, moreover, that the plane $\zeta = 0$ coincides with the complex plane $\mathbb{C}$, and that the ξ and η axes are the x and y axes, respectively. To each $(\xi, \eta, \zeta) \in \sum$ we associate the complex number z where the ray from $(0, 0, 1)$ through (ξ, η, ζ) intersects $\mathbb{C}$. This establishes a 1-1 correspondence, known as stereographic projection, between $\mathbb{C}$ and the points of $\sum$ other than $(0, 0, 1)$. Formulas governing this correspondence can be derived as follows. Since $(0, 0, 1)$, (ξ, η, ζ) and $(x, y, 0)$ are collinear,

$$\frac{x}{\xi} = \frac{y}{\eta} = \frac{1}{1-\zeta}$$



so that

$$x = \frac{\xi}{1-\zeta}; \quad y = \frac{\eta}{1-\zeta}. \tag{2}$$

We leave it as an exercise to show that the equations (1) and (2) can be solved for ξ, η, ζ in terms of x, y as

$$\xi = \frac{x}{x^2+y^2+1}; \quad \eta = \frac{y}{x^2+y^2+1}; \quad \zeta = \frac{x^2+y^2}{x^2+y^2+1}. \tag{3}$$

Now suppose that $\{\sigma_k\} = \{(\xi_k, \eta_k, \zeta_k)\}$ is a sequence of points of $\sum$ which converges to $(0, 0, 1)$ and let $\{z_k\}$ be the corresponding sequence in $\mathbb{C}$. By (2),

$$x^2 + y^2 = \frac{\xi^2+\eta^2}{(1-\zeta)^2} = \frac{\zeta}{1-\zeta},$$

so that as $\sigma_k \to (0, 0, 1)$, $|z_k| \to \infty$. Conversely, it follows from (3) that if $|z_k| \to \infty$, $\sigma_k \to (0, 0, 1)$. Loosely speaking, this suggests that the point $(0, 0, 1)$ on $\sum$ corresponds to ∞ in the complex plane. We can make this more precise by formally adjoining to $\mathbb{C}$ a "point at infinity" and defining its neighborhoods as the sets in $\mathbb{C}$ corresponding to the spherical neighborhoods of $(0, 0, 1)$. (See Exercise 24.)

While we will not examine the resulting "extended plane" in greater detail, we will adopt the following convention.

1.11 Definition

We say $\{z_k\} \to \infty$ if $|z_k| \to \infty$; i.e., $|z_k| \to \infty$ if for any $M > 0$, there exists an integer N such that $k > N$ implies $|z_k| > M$. Similarly, we say $f(z) \to \infty$ if $|f(z)| \to \infty$.

For future reference, we note the connection between circles on $\sum$ and circles in $\mathbb{C}$. By a circle on $\sum$, we mean the intersection of $\sum$ with a plane of the form $A\xi + B\eta + C\zeta = D$. According to (3), if S is such a circle and T is the corresponding set in $\mathbb{C}$,

$$(C - D)(x^2 + y^2) + Ax + By = D \tag{4}$$

for $(x, y) \in T$. Note that if $C \neq D$, (4) is the equation of a circle. If $C = D$, (4) represents a line. Since $C = D$ if and only if S intersects $(0, 0, 1)$, we have the following proposition.

1.12 Proposition

Let S be a circle on $\sum$ and let T be its projection on $\mathbb{C}$. Then

a. *if S contains $(0, 0, 1)$, T is a line;*
b. *if S doesn't contain $(0, 0, 1)$, T is a circle.*

The converse of Proposition 1.12 is also valid. We leave its proof as an exercise. (See Exercise 25.)

Exercises

1. Express in the form $a + bi$:
   a. $\dfrac{1}{6 + 2i}$
   b. $\dfrac{(2 + i)(3 + 2i)}{1 - i}$
   c. $\left(-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right)^4$
   d. $i^2, i^3, i^4, i^5, \ldots$
2. Find (in rectangular form) the two values of $\sqrt{-8 + 6i}$.
3. Solve the equation $z^2 + \sqrt{32}\, iz - 6i = 0$.
4. Prove the following identities:
   a. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
   b. $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$.
   c. $\overline{P(z)} = P(\bar{z})$, for any polynomial P with *real* coefficients.
   d. $\bar{\bar{z}} = z$.
5. Suppose P is a polynomial with real coefficients. Show that $P(z) = 0$ if and only if $P(\bar{z}) = 0$ [i.e., zeroes of "real" polynomials come in conjugate pairs].
6. Verify that $|z^2| = |z|^2$ using rectangular coordinates and then using polar coordinates.

7. Show
   a. $|z^n| = |z|^n$.
   b. $|z|^2 = z\bar{z}$.
   c. $|\text{Re}\, z|, |\text{Im}\, z| \leq |z| \leq |\text{Re}\, z| + |\text{Im}\, z|$.
   (When is equality possible?)

8. a. Fill in the details of the following proof of the triangle inequality:

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\ &= |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2 \\ &= |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1\overline{z_2}) \\ &\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\ &= (|z_1| + |z_2|)^2. \end{aligned}$$

   b. When can equality occur?
   c. Show: $|z_1 - z_2| \geq |z_1| - |z_2|$.

9.* It is an interesting fact that a product of two sums of squares is itself a sum of squares. For example,

$$(1^2 + 2^2)(3^2 + 4^2) = 125 = 5^2 + 10^2 = 2^2 + 11^2.$$

   a. Prove the result using complex algebra. That is, show that for any two pairs of integers $\{a, b\}$ and $\{c, d\}$, we can find integers u, v with

$$(a^2 + b^2)(c^2 + d^2) = u^2 + v^2$$

   b. Show that, if a, b, c, d are all nonzero and *at least one* of the sets $\{a^2, b^2\}$ and $\{c^2, d^2\}$ consists of distinct positive integers, then we can find u^2, v^2 as above with u^2 and v^2 both nonzero.
   c. Show that, if a, b, c, d are all nonzero and *both* of the sets $\{a^2, b^2\}$ and $\{c^2, d^2\}$ consist of distinct positive integers, then there are two *different* sets $\{u^2, v^2\}$ and $\{s^2, t^2\}$ with

$$(a^2 + b^2)(c^2 + d^2) = u^2 + v^2 = s^2 + t^2.$$

   d. Give a geometric interpretation and proof of the results in b) and c), above.

10.* Prove: $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$ and interpret the result geometrically.

11. Let $z = x + iy$. Explain the connection between Arg z and $\tan^{-1}(y/x)$. (Warning: they are not identical.)

12. Solve the following equations in polar form and locate the roots in the complex plane:
    a. $z^6 = 1$.
    b. $z^4 = -1$.
    c. $z^4 = -1 + \sqrt{3}i$.

13. Show that the n-th roots of 1 (aside from 1) satisfy the "cyclotomic" equation $z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$. [*Hint*: Use the identity $z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \cdots + 1)$.]

14. Suppose we consider the $n - 1$ diagonals of a regular n-gon inscribed in a unit circle obtained by connecting one vertex with all the others. Show that the product of their lengths is n. [*Hint*: Let the vertices all be connected to 1 and apply the previous exercise.]

15. Describe the sets whose points satisfy the following relations. Which of the sets are regions?
   a. $|z - i| \le 1$.
   b. $\left|\dfrac{z-1}{z+1}\right| = 1$.
   c. $|z - 2| > |z - 3|$.
   d. $|z| < 1$ and $\operatorname{Im} z > 0$.
   e. $\dfrac{1}{z} = \bar{z}$.
   f. $|z|^2 = \operatorname{Im} z$.
   g. $|z^2 - 1| < 1$. [*Hint*: Use polar coordinates.]

16.* Identify the set of points which satisfy
   a. $|z| = \operatorname{Re} z + 1$
   b. $|z - 1| + |z + 1| = 4$
   c. $z^{n-1} = \bar{z}$, where n is an integer.

17. Let Arg w denote that value of the argument between $-\pi$ and π (inclusive). Show that

$$\operatorname{Arg}\left(\frac{z-1}{z+1}\right) = \begin{cases} \pi/2 & \text{if } \operatorname{Im} z > 0 \\ -\pi/2 & \text{if } \operatorname{Im} z < 0 \end{cases}$$

where z is a point on the unit circle $|z| = 1$.

18.* Find the three roots of $x^3 - 6x = 4$ by finding the three real-valued possibilities for $\sqrt[3]{2+2i} + \sqrt[3]{2-2i}$.

19.* Prove that $x^3 + px = q$ has three real roots if and only if $4p^3 < -27q^2$. (Hint: Find the local minimum and local maximum values of $x^3 + px - q$.)

20.* a. Let $P(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$. By considering $(1 - z)P(z)$, show that all the zeroes of $P(z)$ are inside the unit disc.
   b. Show that the same conclusion applies to any polynomial of the form: $a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, with all a_i real and $0 \le a_0 \le a_1 \le \cdots \le a_n$

21. Show that
   a. $f(z) = \sum_{k=0}^{\infty} kz^k$ is continuous in $|z| < 1$.
   b. $g(z) = \sum_{k=1}^{\infty} 1/(k^2 + z)$ is continuous in the right half-plane $\operatorname{Re} z > 0$.

22. Prove that a polygonally connected set is connected.

23. Let

$$S = \left\{x + iy : x = 0 \text{ or } x > 0,\ y = \sin\frac{1}{x}\right\}.$$

Show that S is connected, even though there are points in S that cannot be connected by any curve in S.

24. Let $S = \{(\xi, \eta, \zeta) \in \sum : \zeta \ge \zeta_0\}$, where $0 < \zeta_0 < 1$ and let T be the corresponding set in $\mathbb{C}$. Show that T is the exterior of a circle centered at 0.

25. Suppose $T \subset \mathbb{C}$. Show that the corresponding set $S \subset \sum$ is
   a. a circle if T is a circle.
   b. a circle minus $(0, 0, 1)$ if T is a line.

26. Let P be a nonconstant polynomial in z. Show that $P(z) \to \infty$ as $z \to \infty$.

27. Suppose that z is the stereographic projection of (ξ, η, ζ) and $1/z$ is the projection of (ξ', η', ζ').
   a. Show that $(\xi', \eta', \zeta') = (\xi, -\eta, 1 - \zeta)$.
   b. Show that the function $1/z$, $z \in \mathbb{C}$, is represented on $\sum$ by a $180°$ rotation about the diameter with endpoints $(-\frac{1}{2}, 0, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2})$.

28. Use exercise (27) to show that $f(z) = 1/z$ maps circles and lines in $\mathbb{C}$ onto other circles and lines.