Chapter 11
Controlling Transient Chaos and Applications

Besides the occurrence of chaos in a large variety of natural processes, chaos may also occur because one may wish to design a physical, biological, or chemical experiment, or to project an industrial plant to behave in a chaotic manner. That chaos may indeed be desirable is further evidenced by the fact that it can be controlled using small perturbation of some accessible parameter or dynamical variable of the system.

The key ingredient for the control of chaos is the observation that any chaotic set has embedded within it a large number of unstable periodic orbits of low periods. Because of ergodicity, the trajectory visits or accesses the neighborhood of each one of these periodic orbits. Some of these periodic orbits may correspond to desired system performance according to some criterion. The second ingredient is the realization that chaos, while signifying sensitive dependence on small changes to the current state, thereby rendering the system state unpredictable over long times, implies that the system's behavior can be altered using small perturbations. The accessibility of the chaotic system to many different periodic orbits combined with its sensitivity to small perturbations allows for the control and manipulation of the chaotic process. Specifically, the Ott–Grebogi–Yorke (OGY) approach [566] is as follows. One first determines some of the unstable low-period periodic orbits that are embedded in the chaotic attractor. One then examines the locations and the stabilities of these orbits and chooses one that yields the desired system performance. Finally, one applies small controls to stabilize this desired periodic orbit. A particularly appealing feature of the OGY approach is that control can be achieved based on data using nonlinear time series analysis for the observation and understanding of the system. This is important, since chaotic systems can be complicated and equations of the process are often unknown.

Since the seminal paper on the OGY paradigm there has been a tremendous amount of research on controlling chaos. The focus of this chapter is on *controlling transient* chaos. We shall present the basic idea and methodology of controlling the dynamics on nonattracting chaotic sets. The existence of transient chaos makes a new type of control possible, i.e., to convert *transient chaos into permanent chaos* via small and infrequent perturbations. The methods of maintaining chaos will be reviewed. We will then consider applications: voltage collapse and prevention,

Y.-C. Lai and T. Tél, *Transient Chaos: Complex Dynamics on Finite-Time Scales*, Applied Mathematical Sciences 173, DOI 10.1007/978-1-4419-6987-3_11,

and how to prevent species extinction. An algorithm for maintaining chaos in the presence of noise will also be presented, as well as a method of encoding digital information using transient chaos.

11.1 Controlling Transient Chaos: General Introduction

11.1.1 Basic Idea and Method

It is possible to control motion on a nonattracting chaotic set to convert transiently chaotic dynamics into periodic dynamics by stabilizing one of the infinite number of unstable periodic orbits embedded in the set. The feature of this type of control is that it stabilizes an orbit that is *not* on the actual attractor of the system. One thus selects an *atypical behavior* that cannot be revealed by a long-time observation of the unperturbed motion. This type of control is effectively *stabilizing a metastable state*. To be specific, we shall discuss the control method in the OGY paradigm [566] because it leads to an algorithm capable of carrying out the finest possible selection of the target orbit to be stabilized and applying the weakest possible perturbation. Other methods [79,123,667,694,785], e.g., the delayed feedback-control method of Pyragas [617], are also applicable. To be concrete, we focus on invertible dynamical systems.

To achieve control of transient chaos, one has to use an *ensemble* of trajectories [767] because any randomly chosen initial point leads to a trajectory that escapes any neighborhood of the saddle in finite time. This ensemble is typically chosen to start from a compact region having intersections with the stable manifold of the chaotic saddle. One also selects a target region I containing a predetermined unstable periodic orbit on the chaotic saddle. Then the ensemble of trajectories start to evolve, and one waits until a trajectory enters the target region to activate control. The control perturbation is adjusted with time so as to stabilize the periodic orbit. Only small local perturbations are allowed, smaller in size than some value δ, the maximum allowed perturbation. In general, δ is proportional to the linear extension of the target region [566].

To illustrate the OGY method, we shall use a two-dimensional map with p as an externally accessible control parameter [767]. We restrict parameter perturbations to be small, i.e., $|p-\overline{p}| < \delta$, where $\overline{p}$ is some nominal parameter value, and $\delta \ll 1$ defines the range of parameter variation. We wish to program the parameter p so that a chaotic trajectory is stabilized when it enters an ε-neighborhood of the target periodic orbit. Without loss of generality, we assume that the target orbit is an unstable fixed point embedded in the chaotic saddle, denoted by $\mathbf{x}_\mathrm{F}(\overline{p})$. The location of the fixed point in the phase space depends on the control parameter p. Upon application of a small perturbation Δp, we have $p = \overline{p} + \Delta p$. Since Δp is small, we expect $\mathbf{x}_\mathrm{F}(p)$ to be close to $\mathbf{x}_\mathrm{F}(\overline{p})$, and write

$$\mathbf{x}_\mathrm{F}(p) \approx \mathbf{x}_F(\overline{p}) + \mathbf{g}\Delta p, \tag{11.1}$$

where $\mathbf{g}$ is a vector given by

$$\mathbf{g} \equiv \left.\frac{\partial \mathbf{x}_\mathrm{F}}{\partial p}\right|_{p=\overline{p}} \approx \frac{\mathbf{x}_\mathrm{F}(p) - \mathbf{x}_\mathrm{F}(\overline{p})}{\Delta p}. \tag{11.2}$$

The vector $\mathbf{g}$ needs to be determined before a control law can be applied to stabilizing the fixed point $\mathbf{x}_\mathrm{F}(\overline{p})$.

To formulate a control law, we make use of the fact that the dynamics of any smooth nonlinear system is approximately linear in a small neighborhood of a fixed point. Thus, near $\mathbf{x}_\mathrm{F}(\overline{p})$, we have

$$\mathbf{x}_{n+1} - \mathbf{x}_\mathrm{F}(p) \approx \mathbf{J}[\mathbf{x}_\mathrm{F}(p)] \cdot (\mathbf{x}_n - \mathbf{x}_\mathrm{F}(p)), \tag{11.3}$$

where $\mathbf{J}[\mathbf{x}_\mathrm{F}(p)]$ is the 2×2 derivative matrix of the map $\mathbf{f}(\mathbf{x}, p)$ evaluated at the fixed point $\mathbf{x}_\mathrm{F}(p)$, defined as

$$\mathbf{J}[\mathbf{x}_\mathrm{F}(p)] = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_\mathrm{F}(p)} \approx \mathbf{J}[\mathbf{x}_\mathrm{F}(\overline{p})] + \left.\frac{\partial \mathbf{J}}{\partial p}\right|_{p=\overline{p}} \Delta p. \tag{11.4}$$

Note that $\Delta p \sim \varepsilon$ and $|\mathbf{x}_n - \mathbf{x}_\mathrm{F}(p)| \sim \varepsilon$, where ε is the size of the small neighborhood in which the linear approximation (11.3) is valid. Substituting (11.1) and (11.4) into (11.3), and keeping only terms that are of first order in ε, we obtain

$$\mathbf{x}_{n+1} - \mathbf{x}_\mathrm{F}(\overline{p}) \approx \mathbf{g}\Delta p + \mathbf{J}[\mathbf{x}_\mathrm{F}(\overline{p})] \cdot [\mathbf{x}_n - \mathbf{x}_\mathrm{F}(\overline{p}) - \mathbf{g}\Delta p]. \tag{11.5}$$

Since $\mathbf{x}_\mathrm{F}(\overline{p})$ is embedded in the chaotic saddle, it has one stable and one unstable direction. Let $\mathbf{e}_\mathrm{s}$ and $\mathbf{e}_\mathrm{u}$ be the stable and the unstable unit eigenvectors at $\mathbf{x}_\mathrm{F}(\overline{p})$, respectively, and let $\mathbf{f}_\mathrm{s}$ and $\mathbf{f}_\mathrm{u}$ be two unit vectors that satisfy $\mathbf{f}_\mathrm{s} \cdot \mathbf{e}_\mathrm{s} = \mathbf{f}_\mathrm{u} \cdot \mathbf{e}_\mathrm{u} = 1$ and $\mathbf{f}_\mathrm{s} \cdot \mathbf{e}_\mathrm{u} = \mathbf{f}_\mathrm{u} \cdot \mathbf{e}_\mathrm{s} = 0$ (relations by which the vectors $\mathbf{f}_\mathrm{s}$ and $\mathbf{f}_\mathrm{u}$ can be determined from the eigenvectors $\mathbf{e}_\mathrm{s}$ and $\mathbf{e}_\mathrm{u}$), which are the contravariant basis vectors associated with the eigenspaces $\mathbf{e}_\mathrm{s}$ and $\mathbf{e}_\mathrm{u}$ [566]. The derivative matrix $\mathbf{J}[\mathbf{x}_\mathrm{F}(\overline{p})]$ can then be written as

$$\mathbf{J}[\mathbf{x}_\mathrm{F}(\overline{p})] = \lambda_\mathrm{u}\mathbf{e}_\mathrm{u}\mathbf{f}_\mathrm{u} + \lambda_\mathrm{s}\mathbf{e}_\mathrm{s}\mathbf{f}_\mathrm{s}, \tag{11.6}$$

where λ_s and λ_u are the stable and the unstable eigenvalues in the eigendirections $\mathbf{e}_\mathrm{s}$ and $\mathbf{e}_\mathrm{u}$, respectively.

When the trajectory point $\mathbf{x}_n$ falls into the small ε-neighborhood of the desired fixed point $\mathbf{x}_\mathrm{F}(\overline{p})$, (11.3) becomes valid. A small parameter perturbation Δp_n can there be applied at time n to make the fixed point shift slightly so that at the next iteration $(n+1)$, $\mathbf{x}_{n+1}$ falls on the stable direction of $\mathbf{x}_\mathrm{F}(\overline{p})$:

$$\mathbf{f}_\mathrm{u} \cdot [\mathbf{x}_{n+1} - \mathbf{x}_\mathrm{F}(\overline{p})] = 0. \tag{11.7}$$

For sufficiently small $\mathbf{x}_n - \mathbf{x}_F(\overline{p})$ we can substitute (11.5) into (11.7) to obtain $\Delta p_n = c_n$, where c_n is given by

$$c_n = \frac{\lambda_u \mathbf{f}_u \cdot [\mathbf{x}_n - \mathbf{x}_F(\overline{p})]}{(\lambda_u - 1)\mathbf{f}_u \cdot \mathbf{g}} \equiv \mathbf{C} \cdot [\mathbf{x}_n - \mathbf{x}_F(\overline{p})] \,. \tag{11.8}$$

We assume in the above that the generic condition $\mathbf{g} \cdot \mathbf{f}_u \neq 0$ is satisfied, so c_n can be calculated. Once $\mathbf{x}_{n+1}$ falls on the stable direction of $\mathbf{x}_F(\overline{p})$, we can set the control perturbation to zero, and the trajectory for subsequent time will approach the fixed point at the geometrical rate λ_s.

The considerations above apply only to a local small neighborhood of $\mathbf{x}_F(\overline{p})$. Globally, one can specify the parameter perturbation Δp_n by setting $\Delta p_n = 0$ if $|c_n|$ is too large, since the range of the parameter perturbation is limited to be small. Thus, practically, we have

$$\Delta p_n = \begin{cases} c_n, & \text{if } |c_n| < \delta, \\ 0, & \text{if } |c_n| \geq \delta. \end{cases} \tag{11.9}$$

In this way, in the definition of c_n in (11.8), it is unnecessary to restrict the quantity $|\mathbf{x}_n - \mathbf{x}_F(\overline{p})|$ to be small.

Figure 11.1 shows an example [767] of controlling a fixed point on the Hénon chaotic saddle in comparison with the uncontrolled trajectory. We see that the controlled motion is not a part of the asymptotic dynamics [605, 606, 792].

11.1.2 Scaling Laws Associated with Control

There are scaling laws characterizing the ensemble of trajectories in the limit of a small allowed perturbation δ. Many of the trajectories approach the asymptotic attractor before entering the target region enclosing the periodic orbit to be stabilized on the chaotic saddle. Short transients are therefore irrelevant for the controlling process, but trajectories with lifetimes significantly larger than $1/\kappa$ are unprobable. As a result, the average time τ_c needed to achieve control is independent of δ and is limited from above by the chaotic lifetime $\tau \approx 1/\kappa$ for some values of δ:

$$\tau_c \leq 1/\kappa. \tag{11.10}$$

Because of the escape, only a small portion of all trajectories can be controlled. When the target region is a disk, the number of controlled trajectories $N(\delta)$ decreases with decreasing δ according to the power law [767]:

$$N(\delta) \sim \delta^{\gamma(\kappa)}, \tag{11.11}$$

where the exponent $\gamma(\kappa)$ depends on the escape rate of the saddle. The number of controlled trajectories is proportional to the c-measure $\mu_c(I)$ of the target region I. The c-measure is smooth along the unstable direction, so the measure μ_c of a region

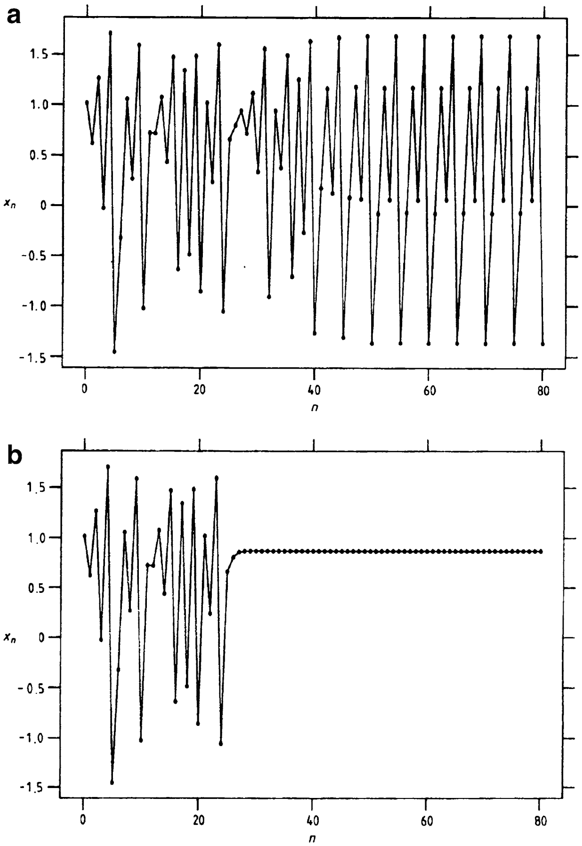

Fig. 11.1 For the Hénon map at parameters $a = 1.45$, $b = 0.2$, where the attractor is a period-5 cycle, (**a**) a transiently chaotic time series. The trajectory ceases to be chaotic at about the 38th time step, where it enters the neighborhood of the period-5 attractor. (**b**) Controlled signal started from the same initial point. The Hénon map has the form given in the caption of Fig. 5.9 with $a = 1.45 + p_n$, $J = -b$, where the maximum allowed perturbation is $\delta = 0.1$. The fixed point is at $x_F = y_F = 0.868858$. Control sets in at the 26th step, and the fixed point on the saddle is stabilized [767] (with kind permission from Institute of Physics)

of size l_1 and l_2 along the unstable and the stable direction, respectively, scales according to (2.89). For $l_1 \sim l_2 \sim \delta$, the scaling exponent is given by

$$\gamma(\kappa) = 1 + \alpha_2(\kappa), \tag{11.12}$$

where $\alpha_2(\kappa)$ is the crowding index along the stable direction. In the particular case in which the target region contains a fixed point, the exponent $\gamma(\kappa)$ is (2.91):

$$\gamma(\kappa) = 1 + \frac{\lambda_1^* - \kappa}{|\lambda_2^*|}, \tag{11.13}$$

where λ_i^* $(i = 1,2)$ are the local Lyapunov exponents of the fixed point to be stabilized ($\lambda_{1(2)}^* = \ln | \lambda_{u(s)} |$).

When applying the OGY method to controlling permanent chaos, the scaling properties of an ensemble of N_0 trajectories are different. The average time τ_c needed to achieve control is a function of the linear size of the target region, which is proportional to the maximum allowed perturbation δ. It was pointed out [566] that $\tau_c(\delta)$ increases algebraically as δ is decreased,

$$\tau_c(\delta) \sim \delta^{-\gamma}, \quad \text{for} \quad \delta \ll 1, \tag{11.14}$$

where $\gamma > 0$ is a characteristic exponent. This scaling law shows that the dynamics of reaching the target region is itself a kind of transient chaos. The process of control can be interpreted as leaking a closed chaotic system at the target region (Sect. 2.7). The time needed to achieve control is thus the average lifetime of the invariant chaotic saddle of the leaked system. For small leak sizes the lifetime scales as the inverse of the natural measure μ of the leak. We have

$$\gamma = 1 + \alpha_2, \tag{11.15}$$

where $\alpha_2 = \lambda_1^* / | \lambda_2^* |$ is the crowding index along the stable direction of the chaotic set in the closed system.[1] However, the number $N(\delta)$ of controlled trajectories is fixed, $N(\delta) = N_0$, since all N_0 trajectories of the ensemble are controlled sooner or later.

The scaling laws in the control of permanent and transient chaos thus appear to be the two extremes of a general process, where for the former, $N(\delta)$ is constant, but for the latter, $\tau_c(\delta)$ is constant. There exists a unifying relation between $N(\delta)$ and $\tau_c(\delta)$ that holds in both cases [769]. The key observation is that the number of controlled trajectories in the entire process is proportional to the average number of trajectories controlled *per unit time* multiplied by the average time needed to achieve control. The average number of trajectories controlled per time step is proportional to the probability of falling in the target region. For small regions this is proportional to the c-measure of the target region in the uncontrolled system. Since the latter scale as δ^{γ} or $\delta^{\gamma(\kappa)}$, the number of trajectories controlled per time step follows the scaling law

$$\frac{N(\delta)}{\tau_c(\delta)} \sim \delta^{\gamma(\kappa)}, \tag{11.16}$$

which is valid for both permanent ($\kappa = 0$) and transient ($\kappa > 0$) chaos.

[1] For a larger target region, the exponent γ depends strongly on the location of I even if $\mu(I)$ is kept constant [101, 103, 571, 572, 574]. From the general theory of leaked systems (cf. Sect. 2.7), this can be understood as being due to the complicated overlap of the leak with its preimages.

11.1.3 Remarks

11.1.3.1 Controlling Fractal Basin Boundaries

An immediate application is the control of motions on a fractal basin boundary [418, 434, 706], which contains a chaotic saddle whose stable manifold constitutes the boundary. One can then control a desirable periodic orbit in the saddle. By applying weak control perturbations, a hyperbolic orbit on the basin boundary can be converted into an attracting orbit [418]. The methodology is potentially important in applications where periodic driving can result in a catastrophic failure of the system. A particular example is ship capsizing, where the method of controlling motion on fractal basin boundaries was computationally tested to prevent chaos-induced ship capsizing even in cases where the driving due to environmental influences (e.g., waves) is not periodic but has a substantially irregular (chaotic) component [195].

An alternative method for steering most trajectories to a desirable attractor is to build a hierarchy of paths to it and then stabilize trajectories around one of the paths in the hierarchy [434]. A pronounced improvement in the probability for a random trajectory to approach a desirable attractor can be achieved when there are fractal basin boundaries or riddled basins.

11.1.3.2 Controlling Chaotic Scattering

A feature of chaotic saddles in Hamiltonian systems is that they typically contain a nonhyperbolic component where the local Lyapunov exponents are arbitrarily close to zero (Sect. 6.4). A problem is to investigate the influence of the nonhyperbolic component on the control process. If one selects a periodic orbit close to a KAM surface, the time to achieve control is usually long due to the stickiness effect. Numerical investigation showed [453] that the average time to achieve control could be an order of magnitude longer than the average chaotic lifetime on the hyperbolic component.

In general, controlling a collisional scattering process means *stabilizing the intermediate complexes* of a reaction that would otherwise be of finite lifetime. Although KAM surfaces can be important for the controlling process, the qualitative behavior of the controlled ensemble is similar to that of a fully hyperbolic system.

11.1.3.3 Improved Method of Controlling a Chaotic Saddle

As we have seen, a major difference between stabilizing unstable periodic orbits embedded in a chaotic attractor and in a chaotic saddle is that for the attractor, the probability that a chaotic trajectory enters the neighborhood of the desired unstable periodic orbit is one, while for transient chaos, only a small set of initial conditions can be controlled, since most trajectories will have already left the chaotic saddle before entering the neighborhood of the target periodic orbit. An issue is how to maximize this probability of control of transient chaos. A useful observation is that there exists a dense chaotic orbit in the saddle *that comes arbitrarily close to any* *target unstable periodic orbit*. Such a dense orbit is the complement of the set of all unstable periodic orbits in the saddle, and can be numerically obtained by the PIM-triple method (Sect. 1.2.2.4). The probability that a trajectory approaches this orbit can be significantly larger than the probability that the trajectory enters the neighborhood of the target unstable periodic orbit, if the reference orbit is long. By stabilizing a trajectory about the reference orbit first, and then switching to stabilize it about the target periodic orbit after the trajectory comes close to it, we can increase substantially the probability that a trajectory can be controlled [447, 461]. This can indeed be achieved, since there exist stable and unstable directions at each point of the reference orbit on the chaotic saddle. Hence in principle, controlling a trajectory near the reference orbit is equivalent to stabilizing a long unstable periodic orbit. The longer the length of the reference orbit, the larger the probability of controlling periodic orbits.

11.2 Maintaining Chaos: General Introduction

The conversion of transient chaos into permanent chaos is called chaos maintenance or preservation, and the basic ideas date back to the work of Yang et al. [840], Schwartz and Triandaf [697], and Kapitaniak and Brindley [383]. The term partial control of chaos is also in use [849], since the algorithms do not determine exactly where the trajectory goes around a nonattracting chaotic set. The practical relevance of this approach is due to the fact that there are systems that require chaos in order to function properly. Notable examples are mechanical systems in which the avoidance of resonance via chaos is desirable [697], advection in fluids where complete stirring can be achieved only via permanent chaos (cf. Chap. 10), and biological systems in which the disappearance of chaos may signal pathological phenomena (see point (2) of Sect. 4.4.3). Under certain conditions, simple regular attractors may appear, and it is then important to intervene in order to maintain chaos. Later in this chapter we shall investigate two examples in detail in which maintenance of chaos is useful: preventing voltage collapse (Sect. 11.3) and species extinction (Sect. 11.4).

11.2.1 Basic Idea

The aim is to intervene the dynamics in such a way as to keep chaotic behavior alive in situations in which it would naturally be absent. In fact, stabilizing a trajectory about a reference orbit on a chaotic saddle, which can enhance the probability of converting a transiently chaotic behavior into a periodic one, as described in the last subsection, can be considered as an attempt to maintain chaos if the reference orbit is long. Other types of algorithms are based on the observation that systems exhibiting transient chaos have special regions in their phase spaces, called loss regions or *escape regions*. They are identified by the property that after the orbit enters such a region, it immediately ceases its chaotic motion, i.e., it is rapidly drawn to some simple attractor. Examples of loss regions are the primary escape interval I_0 of open one-dimensional maps (see Fig. 2.1) and the area bounded by the outermost branch of the chaotic saddle's unstable manifold and the outermost branch of its stable manifold (cf. the shaded area AB in Fig. 3.12).

The strategy can be formulated straightforwardly for map $\mathbf{f}(\mathbf{x}_n, p)$, where p denotes the parameter whose temporal change will be used to maintain chaos. After identifying a loss region L, one considers the preimages L_m of this region under the unperturbed map $\mathbf{f}(\mathbf{x}_n, \bar{p})$, where $\bar{p}$ is the nominal parameter value. The set L_m is thus the set of points mapped onto the loss region in m iterates, and the width of L_m decreases exponentially along the unstable direction(s) as m increases. Yang et al. [840] suggested the following approach. Pick a large value M of m and consider the preimages of the loss region up to level $M+1$. If the unperturbed orbit lands in L_{M+1} on iterate n, one applies a control parameter p_n (different from $\bar{p}$) in order to kick the orbit out of L_M on the next iterate. Since L_M is thin, the required change $\Delta p_n = p_n - \bar{p}$ is small. After the orbit is kicked out of L_M, it is likely to execute a chaotic motion. Due to the fractal structure of the nonattracting set, the orbit falls with probability one outside this chaotic set, i.e., in a region L_r with $r > M$. Long chaotic sequences are expected if r happens to be much larger than M. After some time, the orbit falls again in L_{M+1} when a small control is activated, and so on.

The amount of the control parameter shift Δp_n at the nth step can be estimated by using the sensitivity vector (11.2) evaluated in the loss region and its preimages. By assuming that this vector is approximately a constant $\bar{\mathbf{g}}$ over these regions, and using the maximum width d_M of region L_M, one finds that

$$\Delta p_n \approx \frac{d_M}{|\,\bar{\mathbf{g}}\,|}, \tag{11.17}$$

which is a small number for $M \gg 1$. These ideas were successfully applied to maintaining chaos in different models [840], and also in an experiment in which the intermittent signal of a megnetomechanical ribbon [350] was converted into a nonintermittent chaotic signal. This means that chaos was maintained on a chaotic saddle lying outside a marginally stable periodic orbit.

11.2.2 Maintaining Chaos Using a Periodic Saddle Orbit

The method proposed by Schwartz and Triandaf [697] (see also [698]) can be applied to situations in which the chaotic attractor of an invertible system has been destroyed in a crisis (Chap. 3). (The parameter whose change leads to crisis is not necessarily the same as the parameter p that will be used in the control process.) The system may then exhibit transient chaos until it reaches the periodic saddle point mediating the crisis. If the trajectory happens to fall on one side of the stable manifold of this mediating orbit, it directly approaches a periodic attractor. On the other side of the stable manifold, however, it has chances to return to the chaotic saddle appearing there as the remnant of a former chaotic attractor. We shall call this side of the manifold the chaotic side. Once the trajectory enters a neighborhood of the saddle orbit, a small perturbation in parameter p is applied to ensure that the trajectory falls on the chaotic side in the next step.[2] To optimize chaos maintenance, one can use the distribution of lifetimes to select a target point $\mathbf{x}_{\text{tar}}$ lying close to the mediating orbit with a particularly long lifetime, which ensures that perturbations should be applied only rarely.

In a two-dimensional map, the local dynamics around a saddle point can be approximated by equations (11.5) and (11.6). Note, however, that the hyperbolic fixed point $\mathbf{x}_{\text{F}}$ is now the mediating orbit and not an unstable point inside the saddle as in Sect. 11.1. The required amount of control Δp_n at time instant n when the trajectory happens to be close to the mediating orbit can be obtained from these equations by requiring $\mathbf{x}_{n+1} = \mathbf{x}_{\text{tar}}$. After a multiplication of (11.5) by $\mathbf{f}_{\text{s}}$, one obtains

$$\Delta p_n = \frac{\mathbf{f}_{\text{s}} \cdot [\lambda_{\text{s}}(\mathbf{x}_n - \mathbf{x}_{\text{F}}(\overline{p})) - (\mathbf{x}_{\text{tar}} - \mathbf{x}_{\text{F}}(\overline{p}))]}{(\lambda_{\text{s}} - 1)\mathbf{f}_{\text{s}} \cdot \mathbf{g}}. \tag{11.18}$$

The required control is thus proportional to $\lambda_{\text{s}}[\mathbf{x}_n - \mathbf{x}_{\text{F}}(\overline{p}) - (\mathbf{x}_{\text{tar}} - \mathbf{x}_{\text{F}}(\overline{p}))]$.

The method can be extended to higher dimensions and has successfully been applied by In et al. [351] to maintain chaos in a magnetoelastic ribbon experiment, as shown in Fig. 1.22. The perturbation leads to permanent chaos in a system in which the natural attractor would be periodic.

11.2.3 Practical Method of Control

In [185], a practical method was suggested for converting transient chaos into sustained chaos, based on measured time series. In contrast to the situation of chaotic attractors, these time series consist of short segments of chaotic oscillations exhibiting a number of local maxima and minima. Let e_n ($n = 1,\ldots,L$) be the set of extrema (maxima or minima) from one measured segment of one dynamical variable $x(t)$. In order to detect the underlying dynamics, an ensemble of transient chaotic trajectories from a large number of random initial conditions can be used, each yielding a number of points in the e_{n+1} versus e_n plot. As a crude approximation, the dynamics of the underlying system can be represented by a map $e_{n+1} = M(e_n)$, where if the underlying dynamics is approximately one-dimensional, $M(e)$ is a one-dimensional smooth curve. For higher-dimensional dynamics, the plot $M(e)$ typically exhibits some complicated structure. It is possible to identify regions of the plane $\mathbf{e}_n = (e_n, e_{n+1})$ in which the chaotic saddle lies, and a loss region where

[2] If the perturbation is chosen so that the trajectory falls on the other side, one can speed up the escape process to the simple attractor and can *reduce* the average lifetime of chaos [383].

escape from the chaotic saddle occurs. Thus, by applying a small perturbation to an accessible set ($\{e_n\}$) of dynamical variables at a time when the trajectory is in the escape region, chaotic motion can also be maintained for a finite period of time. The difference between this approach and that of the previous subsections is that the region is identified here only in the two-dimensional plane (e_n, e_{n+1}), and not in the full phase space. Because of this, information about target points in this method is incomplete. The situation can be improved if more dynamical variables are experimentally accessible [185].

11.3 Voltage Collapse and Prevention

We present an example of application of maintaining chaos: voltage collapse in electrical power systems and prevention. We shall describe a model system and demonstrate that voltage collapse is typically preceded by transient chaos. A practical control method will then be discussed to convert transient chaos into sustained chaos, thereby preventing voltage collapse while at the same time *preserving the natural dynamics* of the system.

11.3.1 Modeling Voltage Collapse in Electrical Power Systems

Electrical power systems are essentially nonlinear dynamical systems. Most major power-system failures in the past were reported to be caused by the dynamic response of the system to disturbances [132, 200]. Voltage collapse occurs when the system is heavily loaded. In such a case, dynamical variables of the system, such as various voltages, fluctuate randomly for a period of time before collapsing to zero suddenly, leading to a complete blackout of the system. Due to an ever-increasing demand for electrical power, there is an interest in operating the power system near the edge of its stability boundary. As a consequence, the system becomes highly nonlinear and can exhibit chaotic behaviors. One possible mechanism for voltage collapse is then as follows. The system operates in a parameter region where there is a chaotic attractor. A disturbance or a temporal overload causes a shift in a system parameter so that a boundary crisis occurs, after which the system exhibits transient chaos, leading to a voltage collapse. To understand the phenomenon of voltage collapse, Dobson and Chiang [132, 200] introduced a model power system consisting of a generator, an infinite bus, a nonlinear load, and a capacitor in parallel with the nonlinear load. Subsequently, Wang and Abed pointed out that the presence of the capacitor could cause an increase in the reactive power demand of the load to almost practically unreachable values even in normally encountered parameter regimes. A modified model was proposed [818, 819], which is mathematically described by the following set of differential equations:

$$\begin{aligned}
\dot{\delta}_m &= \omega, \\
0.01464\dot{\omega} &= -0.05\omega + 1.0 - 5.25V\sin(\delta_m - \delta), \\
-0.03\dot{\delta} &= -2.1V^2 + 2.8V + Q(\delta_m, \delta, V) - 0.3 - Q_1, \\
-0.0765\dot{V} &= 0.84V^2 - 1.204V - 0.03\,[P(\delta_m, \delta, V) - 0.6] \\
&\quad -0.4[Q(\delta_m, \delta, V) - 0.3 - Q_1],
\end{aligned} \tag{11.19}$$

where the dynamical variables δ_m, ω, δ, and V are from circuit analysis, Q_1, the load, is a bifurcation parameter, and $P(\delta_m, \delta, V)$ and $Q(\delta_m, \delta, V)$ are the real and reactive powers supplied to the load by the network, which are nonlinear functions of their variables [818, 819]. A bifurcation analysis indicated [185] that there is a period-doubling cascade to chaos, and a crisis occurs at $Q_{1c} \approx 2.56037833$, after which the chaotic attractor is converted into a chaotic saddle. The range for the attractor is relatively small. Suppose the system operates at some value of Q_1 before the crisis. A small change in Q_1 can push the system over the crisis where there is transient chaos. A voltage collapse can then occur. Figure 11.2 shows a time series $V(t)$ for $Q_1 = 2.5603784 > Q_{1c}$, where $V(t)$ goes to zero suddenly after about 80 time units.

How to prevent voltage collapse? A possible approach is to reduce the load Q_1 to bring the system back into the parameter regime where there is an attractor. In a practical situation, however, it may not be feasible to change the load of an electrical power system in a relatively short time. One viable strategy is then to control transient chaos.

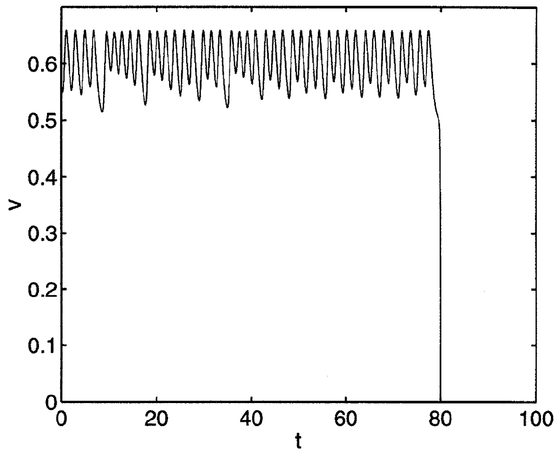

Fig. 11.2 A typical example of voltage collapse in the power system (11.19) [185] (copyright 1999, the American Physical Society)

11.3.2 Example of Control

Figure 11.3a shows the return map obtained from the local minima of $V(t)$ for $Q_1 = 2.5603784 > Q_{1c}$. There is a primary escape interval below which $V(t)$ goes to zero quickly, as shown in Fig. 11.3b. The vertical lines denote the regions from which target points are chosen. The escape interval corresponds to an escape region on the chaotic saddle. In contrast, before the crisis, there is no such gap in the return map. To achieve control in the regime of transient chaos, a set of 3,000 target points was selected [185] in the vicinity of the escape interval with long lifetime. Figure 11.4 shows the lifetime versus the value of local minima. The plot is not smooth and contains an infinite number of singularities corresponding to points on the stable manifold of the chaotic saddle. This singular structure renders selection of desired target points possible. Each target point contains the

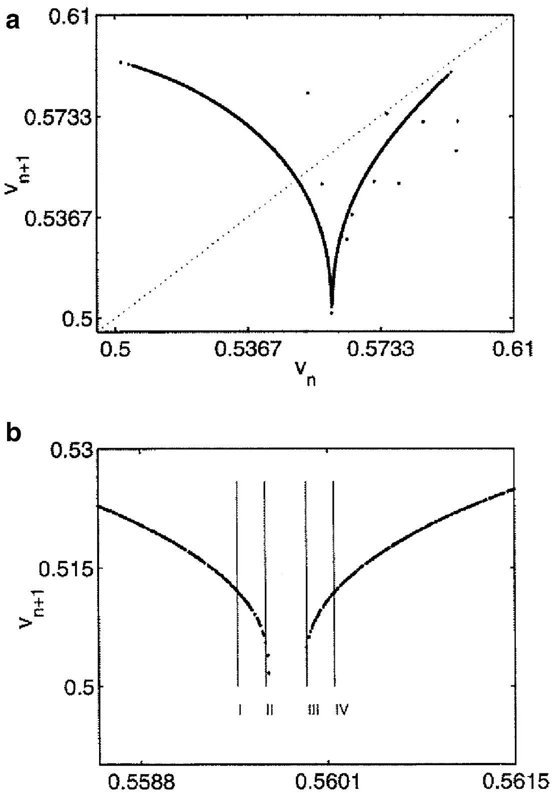

Fig. 11.3 Return map constructed from the local minima of $V(t)$: (**a**) after the crisis for $Q_1 = 2.5603784$; and (**b**) a magnification of part of (**a**) near the cusp. There is a primary escape interval, enclosed between lines II and III, through which a trajectory approaches asymptotically the state with $V = 0$ (voltage collapse). Two regions to the *left* (I – II) and to the *right* (III – IV) of the gap are the regions from which target points can be chosen for control [185] (copyright 1999, the American Physical Society)

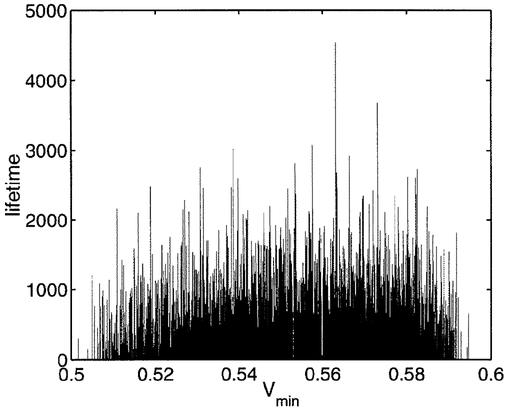

Fig. 11.4 Lifetime versus the local minima of $V(t)$ in the return map. The plot contains an infinite number of singularities corresponding to points on the stable manifold of the chaotic saddle [185] (copyright 1999, the American Physical Society)

values of the four dynamical variables in (11.19), although $V(t)$ is always at a local minimum. The set of target points is then stored for computing the control perturbation. When an actual trajectory falls into the escape interval, the computer selects a target point such that the required perturbation to kick the trajectory onto the target point is minimal. Perturbations can be applied to the dynamical variables $\mathbf{x}$ directly. Or if there is an accessible system parameter p that can be adjusted, perturbations can be applied to the parameter based on the difference between the trajectory point in the escaping window and the target point: $\Delta p = (\partial \mathbf{x}/\partial p)|_{\text{target}}\Delta \mathbf{x}$. In the power-system model (11.19), since all four dynamical variables can be perturbed, it is convenient to apply control directly to these variables. An example of control is shown in Fig. 11.5a, a controlled voltage signal $V(t)$. The required control perturbations are shown in Fig. 11.5b. In the time interval shown, only four small perturbations are required to sustain transient chaos. In general, the average time interval for applying perturbations is approximately the average lifetime of the chaotic saddle. Perturbations are required only when the system drifts into the regime of transient chaos, since transient chaos is the culprit of voltage collapse.

A key question in any scheme of controlling transient chaos concerns the probability of a typical trajectory being controlled.[3] Since the system performs normally before the collapse and since control is activated only when $V(t)$ falls into the escape

[3] We address initial conditions only in the original basin of the attractor because, before the collapse, the system performs normally and operates in the precrisis regime. We are not concerned with initial conditions outside the basin, although they usually yield trajectories leading to $V = 0$. A voltage collapse can thus be regarded as a catastrophic event. Our control method is applicable to preventing this type of catastrophe.

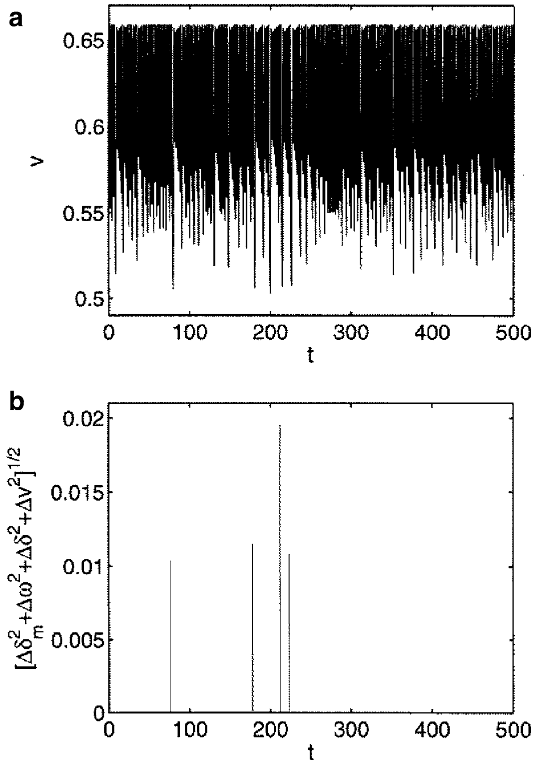

Fig. 11.5 An example of controlling transient chaos to prevent voltage collapse: (**a**) a controlled time series $V(t)$; and (**b**) required control perturbations. Apparently, only infrequent perturbations are needed to prevent voltage collapse [185] (copyright 1999, the American Physical Society)

interval, almost all trajectories can be controlled [185]. This implies that voltage collapse can be effectively prevented by controlling transient chaos.

11.4 Maintaining Chaos to Prevent Species Extinction

We consider the problem of species extinction in ecological systems, which can occur as a consequence of deterministic transient chaos even in the absence of external disturbances. Controlling transient chaos by applying small, ecologically feasible perturbations to the populations at appropriate but rare times thus provides a possibility for preventing species extinction.

11.4.1 Food-Chain Model

Extinction of species has been a mystery in nature [604]. A common belief about local extinction is that it is typically caused by external environmental factors such as sudden changes in climate. For a species of small population size, small random changes in the population (known as "demographic stochasticity") can also lead to extinction. How species extinction occurs is extremely complex, since each species typically lives in an environment that involves interactions with many other species (e.g., through competition for common food sources, predator–prey interactions, etc.) as well as physical factors such as weather and environmental disturbances. From a mathematical point of view, a dynamical model for the population size of a species is complex, involving temporal and spatial variations, external driving, and random perturbations. Such a system should, in general, be modeled by nonlinear partial differential equations with random and/or regular external driving forces. A difficulty associated with this approach is that the analysis and numerical solution of stochastic and/or driven nonlinear partial differential equations present a challenging problem.

Nonetheless, in certain situations the mathematical model for species extinction can become simpler. For example, it was suggested by McCann and Yodzis [510] that deterministic chaos in simple but plausible ecosystem models, mathematically described by coupled ordinary differential equations, can provide a hint as to how local species extinction can arise without the necessity of considering temporal or spatial variations and external factors. The key observation is that the population dynamics of a large class of ecosystems can be effectively modeled by deterministic chaotic systems [317, 339, 507, 508]. It was shown [510] that transient chaotic behavior responsible for species extinction can indeed occur in a simple three-species food-chain model that incorporates biologically reasonable assumptions about species interactions [316]. The model involves a resource species, a prey (consumer), and a predator [510], and is given by

$$
\begin{aligned}
\frac{\mathrm{d}R}{\mathrm{d}t} &= R\left(1-\frac{R}{K}\right) - \frac{x_C y_C C R}{R+R_0}, \\
\frac{\mathrm{d}C}{\mathrm{d}t} &= x_C C\left(\frac{y_C R}{R+R_0} - 1\right) - \frac{x_P y_P P C}{C+C_0}, \\
\frac{\mathrm{d}P}{\mathrm{d}t} &= x_P P\left(-1+\frac{y_P C}{C+C_0}\right),
\end{aligned}
\tag{11.20}
$$

where R, C, and P are the population densities of the resource, the consumer, and the predator, respectively; K is the resource carrying capacity; and x_C, y_C, x_P, y_P, R_0, and C_0 are parameters.

The biological assumptions of the model are as follows: (1) the life history of each species involves continuous growth and overlapping generations, with no age structure, permitting the use of differential equations; (2) the resource population R grows logistically; (3) each consumer species (immediate consumer C, predator P) without food dies off exponentially; (4) each consumer's feeding rate, e.g., $x_C y_C R/(R+R_0)$, saturates at high food levels. The resource population R, growing alone, equilibrates at its carrying capacity K. The resource population and the intermediate consumer, without the predator, either settles to a stable equilibrium or a stable limit cycle, a kind of "biological oscillator." The oscillations are generated by the saturating feeding response, which permits the resource to periodically "escape" control by the consumer. With the top predator, there are in a sense two coupled oscillators in the food chain. A system of coupled oscillators can typically give rise to chaotic dynamics.

11.4.2 Dynamical Mechanism of Species Extinction

How species extinction can occur in the model can be revealed by a bifurcation analysis [510]. In particular, a chaotic attractor can arise via the period-doubling route and is then destroyed through boundary crisis, say at $K = K_c$. None of the populations corresponding to trajectories on the chaotic attractor is extinct, because the attractor is located in a phase-space region away from the origin, $(R,C,P) = (0,0,0)$. In this parameter range, however, there is also a limit-cycle attractor, located in the plane $P = 0$, which coexists with the chaotic attractor. Trajectories on the limit-cycle attractor correspond to the situation in which the predator population is extinct. For K slightly less than K_c, depending on the choice of the initial condition, the system either approaches the chaotic attractor or the limit cycle with $P = 0$. For K slightly below K_c, there is still a finite distance from the tip of the chaotic attractor to the basin boundary. Thus, for any initial condition chosen in the basin of the chaotic attractor, the population of the predator $P(t)$ behaves chaotically in time but never decreases to zero, because the attractor lives in a region where $P(t) \neq 0$. In this case, the predator never becomes extinct.

As the carrying capacity K increases through the critical value K_c, the predator will eventually become extinct for almost all initial conditions. This is quite counterintuitive, but it can be understood from the dynamics. At $K = K_c$, the tip of the chaotic attractor touches the basin boundary (as in Fig. 3.2), creating "holes" on the basin boundary through which trajectories can now leak and enter the basin of the limit-cycle attractor with $P = 0$. Species extinction can thus occur as the result of transient chaos.

11.4.3 Control to Prevent Species Extinction

One way to prevent extinction is to decrease the resource carrying capacity K so that the sustained chaotic motion on the attractor is restored. But ecologically, it may not be feasible to adjust the carrying capacity of an environment, and even if this can be done, it may take some time to accomplish after detecting that the predator population is in danger. The predator may already have become extinct before the carrying capacity can be changed. An alternative approach was proposed to restore sustained chaotic motions without the need to vary the carrying capacity of the environment but instead, by making use of the idea of maintaining chaos via small feedback controls (Sect. 11.2).

One can identify the "dangerous" escape regions surrounding the collision points between the chaotic attractor and the basin boundary by monitoring the populations of R, C, and P. If it is determined that the populations are close to a dangerous region, small but judiciously chosen perturbations to the populations are applied to guarantee that no immediate exit from the hole occurs. By targeting a set of points in the escape region for which the trajectory maps back to the region of recurrent chaotic motion, one can compute the required perturbations. Usually the perturbations need to be applied only rarely. This technique may be of practical use: by applying small but occasional adjustments to the population at appropriate times estimated from time series, species extinction can be prevented. From an ecological point of view, it may be more feasible to make small adjustments to the local *populations* than to change the carrying capacity of the environment.

A potential problem in designing the control algorithm based on the map derived from a Poincaré surface of section is that a substantial fraction of trajectories escape and approach the limit cycle at $P = 0$ without even being controlled. The reason is that it usually takes a long time for a trajectory to return to the surface of section. In the case of transient chaos, a trajectory, because of its finite lifetime, may never pierce through the surface of section before exiting. The following approach was proposed [716] to maintain sustained chaotic motion for almost all transient chaotic trajectories. A critical two-dimensional plane in the three-dimensional phase space (R,C,P) is identified: $P = P_{\rm crit} =$ constant, which separates the region of recurrent chaotic motions from the region in which the dynamics is such that the population $P(t)$ goes directly to zero. This plane need not be the basin boundary, nor is it a Poincaré surface of section. The criteria for choosing this plane are these: (1) ecologically, it is chosen with respect to the population that can become extinct; and (2) dynamically, it should be sufficiently close to the originally recurrent chaotic region. *The plane $P = P_{\rm crit}$ thus represents a critical level of the endangered population at which human intervention must be introduced to prevent the extinction of the species P.* The concept of a "threshold population size" may provide a useful rule of thumb for manipulating the dynamics, and similar ideas were actually used in conservation theory [270]. That the critical plane is chosen close to the recurrent chaotic region indicates that arbitrarily close to but above the critical plane, there exists an infinite number of points in the phase space, trajectories starting from which can resume recurrent chaotic motions for at least a finite amount of time. These considerations are illustrated [716], for example, in Fig. 11.6, where the lifetime span is plotted for trajectories resulting from a grid of 500×500 points chosen from a two-dimensional region in the (R,P) plane at $C = 0.5$. Here, the lifetime is defined to be the time that the trajectory spends in the phase-space region with $P(t) > P_{\rm crit}$. For this example, a simple search procedure leads to the choice of a critical plane at $P_{\rm crit} = 0.57$. In Fig. 11.6, the yellow and red spots represent points with greater

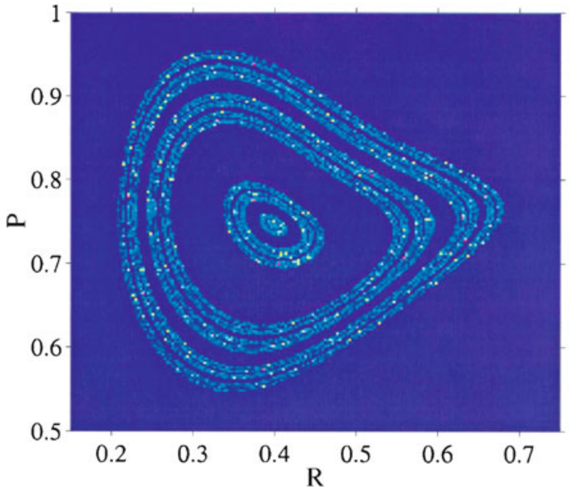

Fig. 11.6 Lifetime plot of trajectories resulting from a grid of 500×500 points chosen from a two-dimensional region in the (R,P) plane at $C = 0.5$, where the lifetime is defined to be the time that the trajectory spends in the phase-space region with $P(t) > P_{\rm crit} = 0.57$. *Brighter colors* indicate longer lifetime. Model parameters are $x_C = 0.4$, $y_C = 2.009$, $x_P = 0.08$, $y_P = 2.876$, $R_0 = 0.16129$, and $C_0 = 0.5$ [510]. The bifurcation parameter is set to be $K = 1.02 > K_{\rm c}$ so that there is transient chaos [716] (with kind permission from Elsevier Science)

lifetimes than the blue spots. It can be seen that the distribution of the lifetime is highly nonuniform, due to the fractal structure of the natural measure of the chaotic saddle.

The setting of a critical plane and the fact that there exists an infinite number of "hot" spots with long chaotic lifetimes provide us with a feasible way to design intervention or control. Say the population $P(t)$ falls slightly below the critical level at time t. Let (R_-, C_-, P_-) be the values of the state variables at this time, where P_- is slightly less than $P_{\rm crit}$, and let (R_+, C_+, P_+) be the values of the state variables a little before t, where P_+ is slightly above $P_{\rm crit}$. At time t, arbitrarily small random adjustments $[\delta R(t), \delta C(t), \delta P(t)]$ are made to *all* the populations in the phase space within a small neighborhood centered at (R_+, C_+, P_+), so that the trajectory collapses to a point. With a nonzero probability, the trajectory will be close to one of the hot spots contained in the neighborhood so that chaotic motion can occur for a finite amount of time. Note that it is not meaningful to kick the trajectory back directly to the point (R_+, C_+, P_+), since this point maps to (R_-, C_-, P_-) immediately. Figure 11.7a shows a controlled population $P(t)$ for $K = 1.02$, which indicates a sustained, sizable population of the predator through a long time. Figure 11.7b shows the magnitude of the applied perturbations $\delta X(t) \equiv \sqrt{[\delta R(t)]^2 + [\delta C(t)]^2 + [\delta P(t)]^2}$ versus time. It can be seen that the required perturbations $[\delta R(t), \delta C(t), \delta P(t)]$ are indeed small ($\delta X(t) < 0.04$, compared with the size of the population, which is about one) and rare (only about 100 perturbations

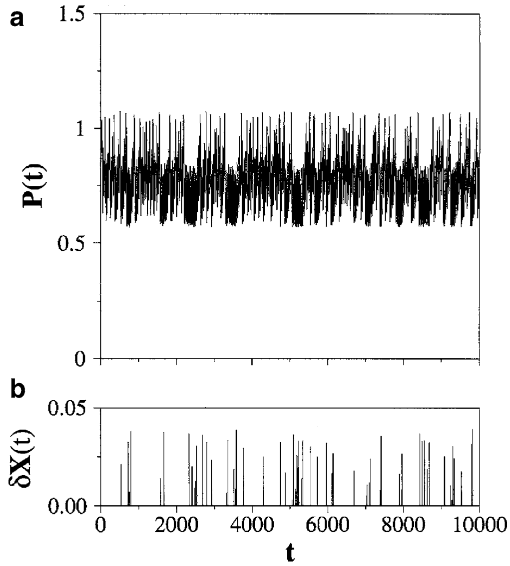

Fig. 11.7 (**a**) A controlled population $P(t)$ for $K = 1.02$, which indicates a sustained, sizable population of the predator through a long time. (**b**) Magnitude of the applied perturbations $\delta X(t)$ versus time [716] (with kind permission from Elsevier Science)

are applied in a time interval of $(0, 10000)$). Numerical computations reveal [716] that the chaotic population $P(t)$ can be maintained practically indefinitely through the use of occasional and small adjustments to all the populations, for almost all initial conditions chosen in the original basin of the chaotic attractor.

An issue of practical interest is how often small adjustments need to be applied so that finite species populations can be maintained. To address this question, we observe that the time intervals for successive adjustments of the populations are in fact the times that the trajectory stays in the region where $P > P_{\text{crit}}$. Their average is the average lifetime τ of the chaotic saddle. For the parameter setting in Fig. 11.6, it was found [716] that $\tau \approx 209$, which means that roughly 50 adjustments to the populations need to be made in a time interval of length of $10,000$. This estimate is consistent with the result in Fig. 11.7b.

The model discussed has incorporated within itself biologically and ecologically reasonable assumptions [510]. Even then, the neglected degrees of freedom would show up as small corrections, and there is always random noise present in any environment. It thus becomes important to assess the influence of random noise. The simplicity embedded in the control method makes it evident that control is robust against the influence of weak noise. The reason is that in the algorithm, deliberate effort is made to avoid the need to utilize detailed and more accurate information about the dynamics, such as the derivative matrices and the stable and the unstable eigenvalues associated with target points. As such, if deterministic transient chaos is the main culprit for the extinction of a species for a particular system, it is possible to control transient chaos to effectively prevent extinction even in noisy environments, regardless of the details of the system dynamics. This may be of value to the important environmental problem of species preservation.

11.5 Maintaining Chaos in the Presence of Noise, Safe Sets

The presence of weak environmental noise may drastically decrease the efficiency of the algorithms used to maintain chaos. The dynamics is then described by the stochastic map $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, p_n) + \sigma\boldsymbol{\xi}_n$, where σ is the noise amplitude, and we assume that p_n is chosen according to one of the chaos-maintaining scenarios described in the previous sections. The destructive effect of noise can be weakened or even eliminated by introducing an *additional* control variable $\mathbf{r}_n$, applied under the influence of noise. The overall dynamics is then described as

$$\begin{aligned} \mathbf{x}'_{n+1} &= \mathbf{f}(\mathbf{x}_n, p_n) + \sigma\boldsymbol{\xi}_n, \\ \mathbf{x}_{n+1} &= \mathbf{x}'_{n+1} + \mathbf{r}_n. \end{aligned} \tag{11.21}$$

We assume that the noise is bounded, $|\ \boldsymbol{\xi}_n\ | \leq 1$, and weak, $\sigma \ll 1$. An interesting question is how the magnitude r of the control variable should be chosen in order to ensure maintenance of chaos despite the presence of noise.

The presence of noise implies that trajectories fall, in general, a distance of order σ away from points with long-lived chaotic transients. The amount of control needed to compensate this shift is therefore at least $r = \sigma$. This strategy therefore does not work for a control weaker than the noise amplitude: $r < \sigma$. A remarkable recent observation of Sanjuán, Yorke, and coworkers [4,661,848,849] was that there is a strategy for maintaining chaos even if the original control parameter p is unchanged (it is kept at its nominal value $\bar{p}$) and even if noise is *stronger* than control.

The problem can also be considered as a mathematical game between two players called the "protagonist" and the "adversary." The adversary chooses the amount $\boldsymbol{\xi}_n$ of noise, knowing x_n and the map $\mathbf{f}(\mathbf{x}) \equiv \mathbf{f}(\mathbf{x}, \bar{p})$. The protagonist's goal is to survive around the chaotic saddle, and he/she can choose the response to the adversary's action, namely the amount $\mathbf{r}_n$ of control. The initial condition can also be chosen by the protagonist. This game was also called Yorke's game of survival [4]. The probability that the protagonist will survive in the vicinity of the chaotic saddle is zero, even without noise, because of escape. This fact makes the survival of the protagonist nontrivial, in particular if the adversary is allowed to act more strongly than the protagonist: $r < \sigma$.

The idea ensuring survival is based not directly on the chaotic saddle, but rather on a related concept, the existence of a horseshoe map (cf. Sect. 1.2.2.1) around it. The action of this map implies that there is a particular set of points, the *safe set*, that lies outside but close to the saddle, and the strategy ensures that points of the map (11.21) remain on the safe set forever.

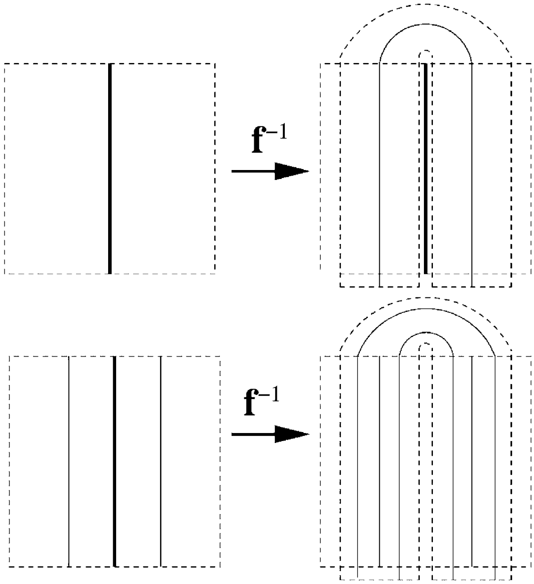

Fig. 11.8 Schematic diagram showing the construction of safe sets S^k. The set S^0 (*thick line*) consists of a vertical segment that divides Q into two halves and lies outside the horseshoe shape of $\mathbf{f}^{-1}(Q)$, marked by *dashed lines*. Safe sets S^1 (*thin black line*) and S^2 (*thin gray line*) are the preimages of S^0 and S^1, respectively [849] (copyright 2008, the American Physical Society)

Different safe sets are needed for different values of σ. Therefore, a family of safe sets $\{S^j\}$ is defined based on the horseshoe construction. Figure 11.8 shows how these sets are generated on a topological square, denoted by Q, containing the chaotic saddle. The action of the inverse map $\mathbf{f}^{-1}$ deforms this square into a horseshoe. The safe set S^0 of level 0 is chosen as a vertical line segment that divides the square Q into two halves. Points of S^0 are in the primary escape region of Q, i.e., they leave the square in one iteration. The preimage of S^0 within Q contains two vertical segments. They form the safe set S^1 of level 1. Following this procedure, one defines the set S^k for any $k > 1$ as the preimage of S^{k-1} in Q (see Fig. 11.8).

Thus the safe set S^k of level k has the following properties:

- S^k consists of 2^k vertical segments.
- Any vertical segment of S^k has two adjacent segments of S^{k+1} that are closer to it than any other segments of S^k.
- The maximum distance, denoted by δ_k, between any of the 2^k segments of S^k and its adjacent segments of S^{k+1} goes to zero as $k \to \infty$.

The safe set is thus always outside the chaotic saddle, but it is close to the saddle for $k \gg 1$. The key idea of the strategy for ensuring survival, and thus maintaining chaos, is to place the initial condition on one segment of an adequate safe set S^k. Then one just has to apply the control $\mathbf{r}_n$ to make point $\mathbf{x}_{n+1}$ of (11.21) lie on the original segment or another segment of S^k.

The adequate safe set S^k corresponds to a level k for which $\delta_{k-1} < \sigma$. For small σ, k is always large. If the initial condition $\mathbf{x}_0$ is on such an S^k, its unperturbed image $\mathbf{f}(\mathbf{x}_0)$ lies on a segment of S^{k-1} that has two adjacent segments of S^k. The perturbation $\boldsymbol{\xi}_0$ due to noise leads to a point $\mathbf{x}_1' = \mathbf{f}(\mathbf{x}_0) + \sigma\boldsymbol{\xi}_0$. If this point lies in the region between the aforementioned two segments of S^k, there exists a control variable $\mathbf{r}_0$, smaller than or equal to $\delta_{k-1} < \sigma$ in modulus, which puts the trajectory on a segment of S^k. If point $\mathbf{x}_1'$ is outside the region between the two curves of S^k, its distance from the segment of S^{k-1} is at most σ, and a perturbation smaller than σ can put the point on the closest segment of S^k. Thus, the image point $\mathbf{x}_1$ lies on a segment of S^k. The same strategy can be applied at any iteration step. One thus always finds a constant r such that with $| \mathbf{r}_n | \leq r < \sigma$ the trajectory $\mathbf{x}_n$ for any n lies on S^k (with the same k as the initial condition), and the system is maintained to remain close to the chaotic saddle forever. The lower bound for the ratio r/σ was shown [4, 848, 849] to be $1/2$, which means that in some cases, chaos can be maintained with a control as weak as half of the strength of noise. Whether this optimal limit can be reached depends on the noise amplitude and the properties of the original map $\mathbf{f}$.

11.6 Encoding Digital Information Using Transient Chaos

Developments in nonlinear dynamics and chaos have led to the idea of encoding digital information using chaos [88–90, 318, 319, 354, 650]. In particular, it was demonstrated both theoretically and experimentally by Hayes et al. [318, 319] that a chaotic system can be manipulated, via arbitrarily small time-dependent perturbations, to generate controlled chaotic orbits whose symbolic representations correspond to the digital representation of a desirable message. Imagine a chaotic oscillator that generates a large-amplitude signal consisting of an apparently random sequence of positive and negative peaks. A possible way to assign a symbolic representation to the signal is to associate a positive peak with a one, and a negative peak with a zero, thereby generating a binary sequence. The use of small perturbations to an accessible system parameter or variable can then cause the signal to follow the orbit whose binary sequence encodes a desirable message that one wishes to transmit. One advantage of this type of message-encoding strategy is that the nonlinear chaotic oscillator that generates the waveform for transmission can remain simple and efficient, while all the necessary electronics controlling the encoding of the signal can remain at a low-powered microelectronic level.

The basic principle that makes the above scheme of digital encoding with chaos possible lies in the link between chaos and information (Sects. 2.6.3, 8.2.1). The fundamental unpredictability of chaos implies that chaotic systems can be regarded as sources that naturally generate digital communication signals. By manipulating a chaotic system in an intelligent way, digital information can be encoded. A central issue in any digital communication scheme is *channel capacity* [71, 708], a quantity that measures the amount of information that can be encoded. For a chaotic system, channel capacity is equivalent to the *topological entropy* (Sect. 1.2.3.3) because it defines the rate at which information is generated by the system.

In a digital communication scheme, it is desirable to have the channel capacity as large as possible to maximize the amount of information that can be encoded. For nonlinear digital communication, it is generally advantageous to use transient chaos as information sources from the standpoint of channel capacity. The orbital complexity associated with trajectories on a chaotic saddle can be greater than that of trajectories on a chaotic attractor, because crisis is generally a complexity-increasing event (Sect. 3.1.1). For a symbolic dynamics of two symbols, the maximally allowed value of the topological entropy, $\ln 2$, is often realized in a parameter regime in which there is transient chaos (see, e.g., Fig. 3.10). Thus, it is desirable to design a chaotic system operating in a transient chaotic regime for digital encoding.

11.6.1 The Channel Capacity

For illustrative purpose, we demonstrate how transient chaos can be utilized to encode digital information using the one-dimensional logistic map $x_{n+1} = f(x_n) = rx_n(1-x_n)$. A symbolic dynamics for the logistic map can be defined by setting the symbolic partition at the critical point $x_c = 0.5$. A trajectory point x bears the symbol **0** if $x < x_c$ and the symbol **1** if $x > x_c$. A trajectory in the phase space thus corresponds to a sequence in the symbolic space. The topological entropy K_0 quantifies how random such a symbol sequence can be. Its value is obtained from the number Ω_m of possible symbol sequences of length m as given by (1.25). In practice, one can plot $\ln \Omega_m$ versus m for, say, $1 \leq m \leq 16$. The slope of such a plot is approximately K_0.

As r is increased toward $r_{\mathrm{c}} = 4$, the topological entropy K_0 continuously increases from zero to $\ln 2$ except when r falls in one of the infinite number of periodic windows. The topological entropy of the chaotic repeller remains constant in the window, where the constant is the value of K_0 at the beginning of the window. Since $\ln 2$ is the maximally realizable value of the topological entropy for a symbolic dynamics of two symbols, and since a crisis occurs at r_{c}, the entropy remains at $\ln 2$ for $r > r_{\mathrm{c}}$, as shown in Fig. 11.9. This can be advantageous because message encoding becomes quite straightforward for hyperbolic transient chaos, since there are no forbidden words associated with the symbolic dynamics. In the communication terminology, such a channel is unconstrained.

11.6.2 Message Encoding, Control Scheme, and Noise Immunity

To encode an arbitrary binary message into a trajectory that lives on a nonattracting chaotic set, it is necessary to use small perturbations to an accessible system parameter or a dynamical variable. For the logistic map we choose to perturb the state

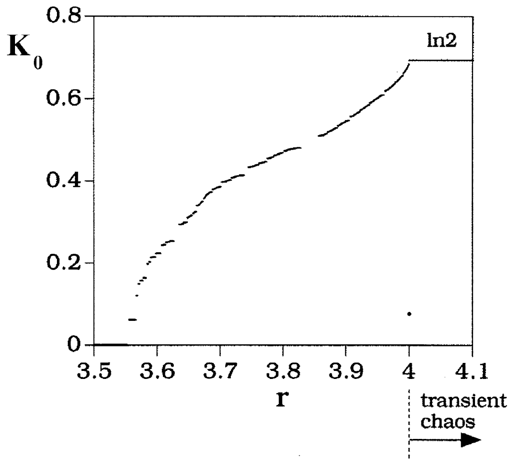

Fig. 11.9 For the logistic map, the topological entropy K_0 versus r for $3.5 < r < 4.1$. After the devil-staircase, for $r > 4$, K_0 remains at $\ln 2$, the maximum possible value for a symbolic dynamics of two symbols [439] (with kind permission from World Scientific Publishing Co)

variable x. Say we wish to apply only small perturbations of order 2^{-m}. A viable procedure is as follows. First, we convert the message into a binary sequence using the ASCII code and store the sequence in a symbol register. Next, we choose an initial condition whose trajectory stays near the chaotic repeller for a certain number n_c ($n_c > m$) of iterations. This is practically feasible, since one can run the system and predetermine the phase-space regions where initial conditions yield trajectories whose lifetimes are at least n_c. We then determine all m symbols corresponding to m points on the trajectory starting from x_0 and check to see whether the mth symbol agrees with the first message bit in the symbol register. If yes, we iterate x_0 once to get x_1 and determine the mth symbol from x_1 (equivalently, the $(m+1)$th symbol from x_0) to see whether it matches the second message bit in the symbol register. If not, we apply a small perturbation to x_0 so that the mth symbol from it matches the first message bit. This process continues until all the message bits in the symbol register are encoded into the chaotic trajectory.

The required parameter perturbation can be computed using the *coding function* [318, 319]. First divide the unit interval in x into N bins of size $\delta x = 1/N$, where $\delta x \ll 1/2^m$ and $1/2^m$ is the maximally allowed perturbation. We then choose a point from each bin, iterate it m times, and determine the corresponding symbol sequence of length m: $S_1 S_2 \ldots S_m$, where S_i can be either zero or one. Any point leaving the unit interval in fewer than m iterations is disregarded. For those points x for which a symbol sequence of length m can be defined, the following is computed:

$$R = \sum_{i=1}^{m} S_i/2^i, \tag{11.22}$$

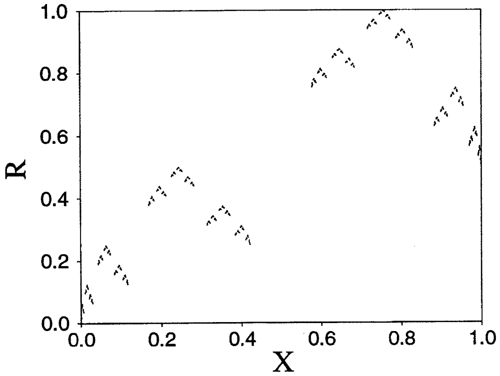

Fig. 11.10 Coding function $R(x)$ for the logistic map at $r = 4.1$, where R can assume any value between 0 and 1, but there are many gaps on the x-axis, due to the fact that the chaotic repeller is a fractal Cantor set [439] (with kind permission from World Scientific Publishing Co)

where $0 \le R \le 1$. This gives the value of the coding function $R(x)$ for points on the chaotic repeller. Since the chaotic repeller has topological entropy $\ln 2$, R can in principle have any value between 0 and 1. Figure 11.10 shows the coding function for the logistic map at $r = 4.1$, where $\delta x = 2 \times 10^{-4}$.

With the coding function, the determination of the state perturbations becomes straightforward. Let the natural m-bit symbol sequence from x_0 be $a_1 a_2 \ldots a_{m-1} a_m$ (produced by iterating the map directly) and let the first message bit to be encoded be b_1. We compare the natural symbol sequence $a_1 a_2 \ldots a_{m-1} a_m$ with the desirable symbol sequence $a_1 a_2 \ldots a_{m-1} b_1$ and compute $\delta R = (a_m - b_1)/2^m$. From the coding function $R(x)$, we can then compute the perturbation δx. This is done by locating pairs of points with the same value of δR in the computer representation of the coding function $R(x)$ and choosing the one that yields the smallest value of δx. Thus, by applying δx to the initial condition x_0, the trajectory point after m iterations is associated with the symbol that is the first message bit. Note that if a_m is identical to the message bit b_1, no perturbation is necessary. To encode the next message bit, we iterate the perturbed initial condition once to obtain x_1. Let $x'_0 = x_1$. The natural m-bit symbol sequence of x'_0 is $a'_1 a'_2 \ldots b_1 a'_m$, where $a'_1 = a_2$, $a'_2 = a_3$, ..., and a'_m is the binary symbol corresponding to the trajectory point $f^{(m)}(x'_0)$. We now compare a'_m and b_2 to determine the next perturbation to be applied to x'_0. Continuing this procedure, we can encode an arbitrary message into the chaotic trajectory $\{x_n\}$.

An example of encoding a specific piece of information [439] is shown in Fig. 11.11a, where the English word "TIGER" is encoded into a trajectory on the chaotic repeller of the logistic map for $r = 4.1$. The binary (ASCII) representation of the word is shown at the top of the figure. Assuming that perturbations of magnitude 2^{-8} are to be applied, we generate a set of initial conditions whose lifetimes in the unit interval under the map are at least 8. Shown in Fig. 11.11a is a time series for which the first binary bit of the message is encoded into the trajectory at $n = 8$. Time-dependent perturbations are applied at subsequent iterations so that the entire

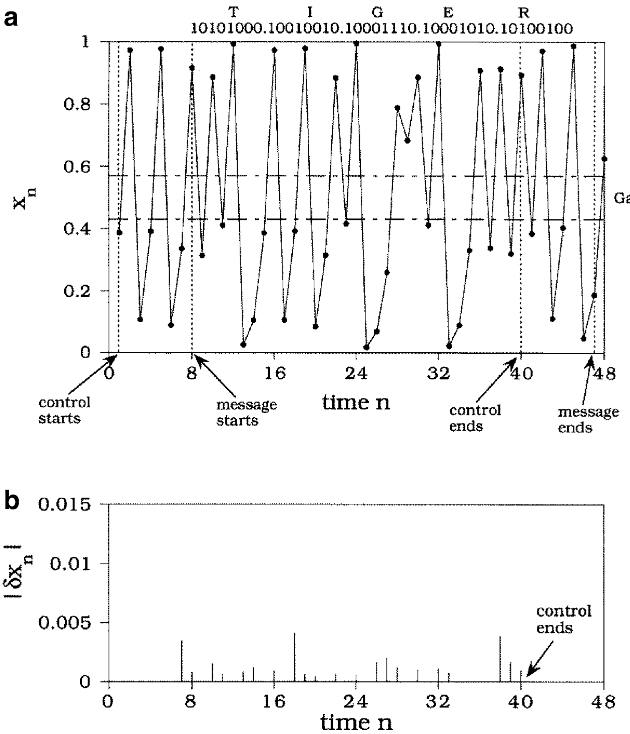

Fig. 11.11 Encoding the English word "TIGER" into a trajectory on the chaotic repeller at $r = 4.1$ for the logistic map. The binary representation of the word is shown at the top of the figure. Shown in (**a**) is a time series where small control is initiated at $n = 1$ and the first binary bit of the message is encoded into the trajectory at $n = 8$. The *dashed–dotted lines* represent the endpoints of the primary escape interval. Time-dependent perturbations are applied at subsequent iterations so that the entire message "TIGER" can be encoded into the trajectory. The magnitudes of the control perturbations required are shown in (**b**) [439] (with kind permission from World Scientific Publishing Co)

message "TIGER" can be encoded into the trajectory. Figure 11.11b shows the magnitude of the control perturbations applied at different time steps. We see that the perturbations required are small. No control perturbation is required for the first six time steps because for this initial condition, the natural symbols corresponding to the trajectory points from $n = 8$ to $n = 13$ happen to coincide with the first six bits of the message.

Some features of the control scheme are as follows. Since the channel capacity of the chaotic repeller is $\ln 2$, there are no forbidden symbol sequences. Thus, in the encoding scheme, any binary sequences can be produced by a typical trajectory near the chaotic repeller. Since we use the coding function $R(x)$ to compute the perturbation δx, once the perturbation is turned on, the trajectory is automatically confined in the vicinity of the chaotic repeller because the coding function is defined with respect to trajectories on the repeller. Suppose that small perturbations on the order of 2^{-m} are to be applied. To encode a message, we need only identify a set of initial conditions that can stay near the chaotic repeller for m iterations. Since the typical value of m is, say, 10, it is fairly straightforward to identify a large number of such initial conditions. In practice, before encoding, we can run the system to produce a set of initial conditions whose lifetimes are greater than m. Together with the coding function that also needs to be determined beforehand, one can in principle encode any binary sequence into a dynamical trajectory on the chaotic repeller.

Besides possessing the maximum topological entropy $\ln 2$, the chaotic repellers of the logistic map for $r > 4$ also have the property of strong noise immunity. To see this, we contrast a chaotic repeller with the chaotic attractor at $r = 4$. For the chaotic repeller, we see that there is a primary escape interval of size $\sim \sqrt{s}$, where $s = r/4 - 1$, about the partition point $x_c = 1/2$. For the chaotic attractor there is no such gap. A trajectory on the chaotic attractor can then come arbitrarily close to the partition point. In a noisy environment, this may cause a bit error. Say the trajectory point is to the immediate right of x_c. This point thus has the symbol 1. Due to noise, the trajectory can be kicked through x_c, and it thus assumes the wrong symbol 0. For a trajectory on the chaotic repeller, this situation is improved. Insofar as the noise amplitude is smaller than the size of the primary escape interval across the partition point x_c, the symbolic dynamics is immune to noise. This may be of value to practical implementation of communication with chaos [89, 90].[4]

Since all chaotic repellers for $r > 4$ in the logistic map have the same topological entropy $\ln 2$, it appears that it is more advantageous to use chaotic repellers at large r because they possess larger gaps across x_c, and thus their corresponding symbolic dynamics are more robust against noise. However, as r increases, the average lifetime of transient chaos decreases. In general, in choosing an optimal chaotic repeller for digital encoding, there is a trade-off between the ease of generating a trajectory near the chaotic repeller and the noise immunity [89, 90].

Although our discussion has been focused on one-dimensional maps, similar ideas apply to transient chaos in two-dimensional maps [443].

[4] The stability of transient chaos against noise has been discussed in Chap. 4.