Chapter 8
Discontinuous Dynamical Systems

8.1 Generalities

The book [39] edited by D.V. Anosov and V.I. Arnold considers two fundamentally different dynamical systems: flows and cascades. Roughly speaking, flows are dynamical systems with continuous time and cascades are dynamical systems with discrete time. One of the most important theoretical problems is to consider *Discontinuous Dynamical Systems* (DDS). That is, the systems whose trajectories are piecewise continuous curves. Analyzing the behavior of the trajectories, we can conclude that DDS combine features of vector fields and maps. They cannot be reduced to flows or cascades but are close to flows since time is continuous. That is why we propose to call them also as *Discontinuous Flows* (DF). One must emphasize that DF are not differential equations with discontinuous right side, which often have been accepted as DDS [68]. One should also agree that nonautonomous impulsive differential equations, which were thoroughly described in previous chapters are not discontinuous flows.

Let us remind the definition of a continuous dynamical system. Denote by X a complete metric space, with a countable base, and with ρ a metric function. A dynamical system on X is defined to be a mapping $\phi : \mathbb{R} \times X \to X$, such that

1. $\phi(0, x) = x$ for all $x \in X$, (Identical property);
2. $\phi(t + s, x) = \phi(t, \phi(s, x))$ for all $x \in X$, and $t, s \in \mathbb{R}$, (Group property);
3. $\phi(t, x)$ is a continuous function.

Definitely, one may expect that systems with similar properties can be defined for processes with discontinuities. Present chapter is devoted to the problem of identification of such kind of systems, one of the most interesting and difficult problems for impulsive differential equations.

To motivate the reader, we may propose the following simple example, where an autonomous system with even linear elements is not a dynamical system.

Example 8.1.1. Let us study the motion of the following system

$$\ddot{x} + \omega^2 x = 0,$$
$$\Delta \dot{x}|_{x=x_0} = k,$$

where ω, k, and x_0 are positive constants.

M. Akhmet, *Principles of Discontinuous Dynamical Systems*,
DOI 10.1007/978-1-4419-6581-3_8,

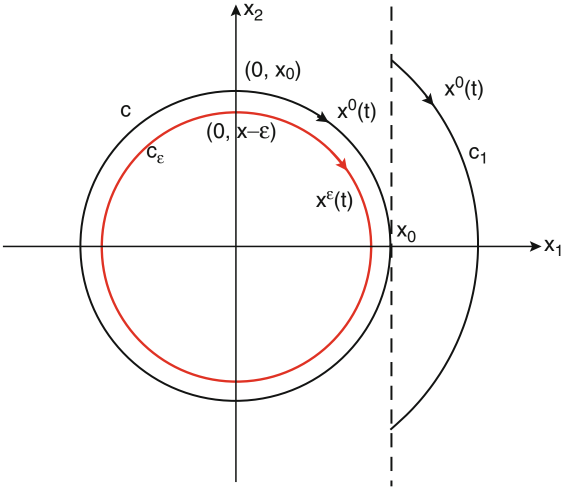

Fig. 8.1 Trajectories of the system (8.1), (8.2)

Denote $x_1 = x$ and $x_2 = \frac{1}{\omega}\dot{x}$. By using this substitution, the system can be rewritten in the form:

$$\begin{aligned} \dot{x_1} &= \omega x_2, \\ \dot{x_2} &= -\omega x_1, \quad x_1 \neq x_0, \end{aligned} \tag{8.1}$$

$$x_2^+ = k_1 + x_2^-, \quad x_1 = x_0, \tag{8.2}$$

where $k_1 = \frac{k}{\omega}$. A solution of the system (8.1) is $x_1(t) = r \sin \omega t$ and $x_2(t) = r \cos \omega t$, where r is a fixed real number.

Let us observe the behavior of solutions of the system (8.1), (8.2) in Fig. 8.1. Consider the solution $x^0(t) = x(t, 0, (0, x_0))$. The point moves along the circle **c** until it meets the line $x_1 = x_0$ at the point $(x_0, 0)$. Then it jumps and continues to move along the arc of the circle c_1. Then, it meets the line $x_1 = x_0$ again and jumps.

One may examine that the solution is not continuous in the initial value. Indeed, let us take another solution $x^\epsilon(t) = x(t, 0, (0, x_0 - \epsilon))$ of this system, which starts at the point $(0, x_0 - \epsilon)$, where ϵ is a fixed positive real number. The solution $x^0(t)$ jumps at the point $(x_0, 0)$ and continues along the arc c_1, as explained above. However, the solution $x^\epsilon(t)$ continues its motion along the circle c_ϵ without any jump. So, as it is seen in Fig. 8.1, the distance between these two trajectories cannot be less than $\sqrt{x_0^2 - k_1^2} - x_0$, despite the initial points of these two solutions can be chosen arbitrarily close. This example demonstrates that the solution $x^0(t)$ of the system (8.1), (8.2) does not depend continuously on the initial value. Obviously, we cannot accept the system as a dynamical system. We may remark that this type of "irregularity" in models with impacts causes many interesting phenomena, for instance, collision bifurcation [51, 118].

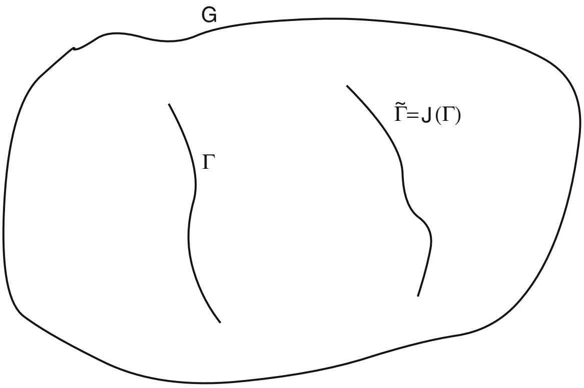

Fig. 8.2 A domain of a discontinuous dynamical system

Let $G = \bigcup G_j$ where G_j, $j = 1, 2, \ldots, m$, are disjoint open connected subsets of $\mathbb{R}^n$. Denote by G_r, an r-neighborhood of G in $\mathbb{R}^n$ for a fixed $r > 0$. Let $\Phi : G_r \longrightarrow \mathbb{R}$ be a function from $C^1(G_r)$ and assume that the surface $\Gamma = \Phi^{-1}(0)$ is a closed subset of $\bar{G}$, where $\bar{G}$ is the closure of G. Denote by Γ_r, the r-neighborhood of Γ in $\mathbb{R}^n$, and define a function $J : \Gamma_r \to G_r$, such that $J(\Gamma) \subset \bar{G}$ is a closed set. We shall need the following assumptions:

(C1) $\nabla\Phi(x) \neq 0$ for all $x \in \Gamma$;
(C2) $J \in C^1(\Gamma_r)$ and $\det[\frac{\partial J(x)}{\partial x}] \neq 0$, for all $x \in \Gamma_r$,

where $\nabla\Phi(x)$ denotes the gradient vector of Φ with respect to x. Let $\tilde{\Gamma} = J(\Gamma)$, (see Fig. 8.2), $\tilde{\Phi}(x) = \Phi(J^{-1}(x))$. One can verify that $\tilde{\Gamma} = \left\{x \in G \mid \tilde{\Phi}(x) = 0\right\}$. Condition (C1) implies that for every $x_0 \in \Gamma$ there exists a number j and a function $\varphi_{x_0}\left(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\right)$ such that Γ is the graph of the function $x_j = \varphi_{x_0}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ in a neighborhood of x_0. The same is true for every $x_0 \in \tilde{\Gamma}$.

Sets Γ and $\tilde{\Gamma}$ consist of disjoint manifolds. These manifolds are with or without boundaries. We shall denote unions of all these boundaries as $\partial\Gamma$ and $\partial\tilde{\Gamma}$. One may recommend to the reader books [58,73] to recall definitions of manifolds. It is easily seen that restrictions $J|_{\Gamma}$, $\tilde{J}|_{\tilde{\Gamma}}$ are one-to-one functions.

Remark 8.1.1. It is natural to consider domains of continuous dynamical systems as connected sets [157]. Otherwise, each region of a partition can be discussed as a domain of a continuous dynamical system. A trajectory of a discontinuous dynamical system may jump from one component to another, such that only the union of the disjoint regions is a domain.

Lemma 8.1.1. $\nabla\tilde{\Phi}(x) \neq 0$, $\forall x \in \tilde{\Gamma}$.

Proof. We can write that $\nabla\tilde{\Phi}(x) = \nabla\Phi\left(J^{-1}(x)\right)$, and then the equality

$$\nabla\Phi\left(J^{-1}(x)\right) = \frac{\partial\Phi(y)}{\partial y}\Big|_{y=J^{-1}(x)}\frac{\partial J^{-1}(x)}{\partial x}$$

implies that

$$\nabla\Phi\left(J^{-1}(x)\right) \neq 0.$$

The lemma is proved. □

We make the following assumptions which will be needed throughout the chapter.

(C3) $f \in C^1(G_r)$,
(C4) $\Gamma \cap \tilde{\Gamma} = \emptyset$,
(C5) $\langle\nabla\Phi(x), f(x)\rangle \neq 0$ if $x \in \Gamma$,
(C6) $\left\langle\nabla\tilde{\Phi}(x), f(x)\right\rangle \neq 0$ if $x \in \tilde{\Gamma}$.

Consider the following impulsive differential equation:

$$\begin{aligned} x' &= f(x), \\ \Delta x|_{x\in\Gamma} &= W(x), \end{aligned} \tag{8.3}$$

where $W(x) = J(x) - x$, in the domain $D = \left[G \cup \Gamma \cup \tilde{\Gamma}\right] \setminus \left[\partial\Gamma \cup \partial\tilde{\Gamma}\right]$.

If $\phi(t) : I \to \mathbb{R}^n$, where I is an interval, is a solution of (8.3), then it is required that it belongs to $\mathcal{PC}(I, \theta)$, where $\theta \subset I$ is a B-sequence. The solution must satisfy $\phi'(t) = f(\phi(t))$, if $t \notin \theta$, and $\phi(\theta_i+) = J(\phi(\theta_i)), \phi(\theta_i) \in \Gamma, \phi(\theta_i+) \in \tilde{\Gamma}$, for each $\theta_i \in \theta$. Sets Γ and $\tilde{\Gamma}$ may have common points with the boundary of the domain D, and the boundary points of these sets, Γ and $\tilde{\Gamma}$, do not belong to D, as they may cause a violence of the continuous dependence on initial value. If the boundary points are in the domain, then one needs specific additional conditions. For instance, if $x \in \partial\Gamma$, then we may request $J(x) = 0$.

Now, we continue with examples, where conditions (C1)–(C6) are satisfied.

Example 8.1.2. Let us consider the following system:

$$\begin{aligned} x_1' &= -x_1 - 3x_2, \\ x_2' &= 3x_1 - x_2, \\ \Delta x_1|_{x\in\Gamma} &= x_1 \\ \Delta x_2|_{x\in\Gamma} &= x_2, \end{aligned} \tag{8.4}$$

where $x = (x_1, x_2)$, and

$$\Gamma = \left\{(x_1, x_2)| \quad x_1^2 + x_2^2 = 1, \quad x_1, x_2 \in \mathbb{R}\right\},$$

$$\tilde{\Gamma} = \left\{(x_1, x_2)| \quad x_1^2 + x_2^2 = 4, \quad x_1, x_2 \in \mathbb{R}\right\}.$$

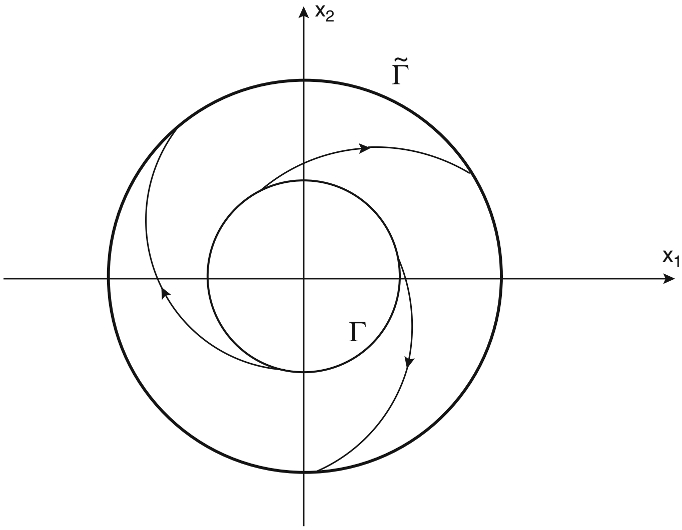

Fig. 8.3 The phase portrait of system (8.4)

Assume that $G = \{(x_1, x_2)| \quad 1 < x_1^2 + x_2^2 < 4, \quad x_1, x_2 \in \mathbb{R}\}$. A trajectory of the system is seen in Fig. 8.3. One can easily find that $\Phi(x) = x_1^2 + x_2^2 - 1, \tilde{\Phi}(x) = x_1^2 + x_2^2 - 4, f(x) = (-x_1 - 3x_2, 3x_1 - x_2), J(x) = (2x_1, 2x_2)$. Let us check conditions (C1)–(C6). We have that $\nabla\Phi(x) = (2x_1, 2x_2) \neq 0$. So, condition (C1) is satisfied. Moreover, J, f are continuously differentiable functions and $\det[\frac{\partial J(x)}{\partial x}] = det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4 \neq 0$, for all x. It is also obvious that $\Gamma \bigcap \tilde{\Gamma} = \emptyset$. Finally, $\langle \nabla\Phi(x), f(x)\rangle = \langle (2x_1, 2x_2), (-x_1 - 3x_2, 3x_1 - x_2)\rangle = 2\left(-x_1^2 - x_2^2\right) = -2 \neq 0$, for all $x \in \Gamma$, and $\left\langle \nabla\tilde{\Phi}(x), f(x)\right\rangle = \langle (2x_1, 2x_2), (-x_1 - 3x_2, 3x_1 - x_2)\rangle = 2\left(-x_1^2 - x_2^2\right) = -8 \neq 0$, for all $x \in \tilde{\Gamma}$. Thus, all conditions (C1)–(C6) are fulfilled.

Example 8.1.3. Let us consider the following system:

$$
\begin{aligned}
x_1' &= -\frac{1}{3}x_1 - 3x_2, \\
x_2' &= 3x_1 - \frac{1}{3}x_2, \\
\Delta x_1|_{x\in\Gamma} &= (2\cos\frac{\pi}{6} - 1)x_1 - 2\sin\frac{\pi}{6}x_2, \\
\Delta x_2|_{x\in\Gamma} &= 2\sin\frac{\pi}{6}x_1 + (2\cos\frac{\pi}{6} - 1)x_2. \qquad (8.5)
\end{aligned}
$$

where $G = \mathbb{R}^2$, and $\Gamma = \{(x_1, x_2)| \quad x_1 = x_2, \quad 0 < x_1\}$. Let us start to check conditions (C1)–(C6). One can easily find that $\tilde{\Gamma} = \{(x_1, x_2)| \sqrt{3}x_1 =$

$x_2, 0 < x_1\}, \Phi(x) = x_1 - x_2, \tilde{\Phi}(x) = \sqrt{3}x_1 - x_2,\ f(x) = (-\frac{1}{3}x_1 - 3x_2, 3x_1 - \frac{1}{3}x_2), J(x) = (2\cos\frac{\pi}{6}x_1 - 2\sin\frac{\pi}{6}x_2, 2\sin\frac{\pi}{6}x_1 + 2\cos\frac{\pi}{6}x_2)$. Consequently, we have that $\nabla\Phi(x) = (1, -1) \neq 0$, so, condition (C1) is satisfied. It is seen that J, f are continuously differentiable functions and $\det[\frac{\partial J(x)}{\partial x}] = \det\begin{pmatrix} 2\cos\frac{\pi}{6} & 2\sin\frac{\pi}{6} \\ 2\sin\frac{\pi}{6} & 2\cos\frac{\pi}{6} \end{pmatrix} = 4(\cos^2\frac{\pi}{6} + \sin^2\frac{\pi}{6}) = 4 \neq 0$, for all x. It is also obvious that $\Gamma \cap \tilde{\Gamma} = \emptyset$. Moreover,

$$\langle\nabla\Phi(x), f(x)\rangle = \left\langle(1, -1), (-\frac{1}{3}x_1 - 3x_2, 3x_1 - \frac{1}{3}x_2)\right\rangle = \left(\frac{-10}{3}x_1 - \frac{8}{3}x_2\right) \neq 0,$$

for all $x \in \Gamma$. The inequality $\langle\nabla\tilde{\Phi}(x), f(x)\rangle \neq 0$, for all $x \in \tilde{\Gamma}$, can be shown similarly. Thus, all conditions, (C1)–(C6) are fulfilled.

8.2 Local Existence and Uniqueness

Definition 8.2.1. A function $x(t) \in \mathcal{PC}^1(I, \theta)$, where $I \subset \mathbb{R}$ is an interval, $\theta \subset I$ is a B-sequence of discontinuity points, is said to be a solution of (8.3) if:

(i) the differential equation (8.3) is satisfied at each $t \in I \backslash \theta$ and $x'_-(\theta_i) = f(x(\theta_i)), \theta_i \in \theta$, where $x'_-(\theta_i-)$ is the left-sided derivative;
(ii) $\Delta x(\theta_i+) = W(x(\theta_i))$ for all $\theta_i \in \theta$.

Theorem 8.2.1. *Assume that conditions (C1)–(C4) hold. Then for every $x_0 \in D$ there exists an interval $(a, b) \subset \mathbb{R}, a < 0 < b$, such that a solution $x(t) = x(t, 0, x_0)$ of (8.3) exists and is unique on the interval.*

Proof. Consider the following alternative cases:

(a) Assume that $x_0 \notin \Gamma \cup \tilde{\Gamma}$. Then there exists a number $\epsilon > 0$ such that $B(x_0, \epsilon) \cap (\Gamma \cup \tilde{\Gamma}) = \emptyset$, and $B(x_0, \epsilon) \subset G$, where $B(x_0, \epsilon)$ is the ball with the center at x_0 and the radius ϵ. Therefore, by the existence and uniqueness theorem [59], $x(t)$ exists and is unique on an interval (a, b) as a solution of the system

$$y' = f(y). \tag{8.6}$$

(b) If $x_0 \in \Gamma$, then $x(0+) \in \tilde{\Gamma}$. There exists a number $\epsilon > 0$ such that $B(x(0+), \epsilon) \cap \Gamma \neq \emptyset$ and $B(x(0+), \epsilon) \subset G$. Hence, $x(t)$ can be continued at the right. Let us consider $t < 0$ now. By condition (C4), there exists a number $\epsilon > 0$ such that $B(x(0), \epsilon) \cap \tilde{\Gamma} \neq \emptyset$ and $x(t)$ can be proceeded at the left.
(c) We can discuss the case $x_0 \in \tilde{\Gamma}$ similarly to the previous one.

The uniqueness of the solution for all cases (a)–(c) follows the theorem on uniqueness of ordinary differential equations [77] and the invertability of J.

The theorem is proved. □

Since conditions (C1)–(C4) were verified in Examples 8.1.2 and 8.1.3, solutions of systems (8.4) and (8.5) locally exist and are unique.

8.3 Extension of Solutions

In this section, we will prove continuation theorems. The main results claim that every solution of (8.3) is continuable to ∞ and $-\infty$. In other words, $\mathbb{R}$ is a maximal interval of existence of each solution $x(t,0,x_0), x_0 \in D$ of (8.3). That is, $x(t,0,x_0) \in \mathcal{PC}(\mathbb{R})$. Illustrating examples are given, where solutions exist on $\mathbb{R}$.

Definition 8.3.1. A solution $x(t) = x(t,0,x_0)$ of (8.3) is said to be continuable to a set $S \subset \mathbb{R}^n$ as time decreases (increases) if there exists a moment $\xi \in \mathbb{R}$, such that $\xi \leq 0$ $(\xi \geq 0)$ and $x(\xi) \in S$.

The following theorems provide sufficient conditions for the continuation of solutions of (8.3).

Theorem 8.3.1. *Assume that:*

(a) every solution $y(t,0,x_0), x_0 \in D$, of (8.6) is continuable to either ∞ or Γ, as time increases;
(b) there exists a positive number $\bar{\theta}$ such that

$$\frac{\epsilon_x}{\sup_{B(x,\epsilon_x)} \|f(x)\|} \geq \bar{\theta},$$

for every $x \in \tilde{\Gamma}$ and all $\epsilon_x > 0$ with $B(x,\epsilon_x) \cap \Gamma = \emptyset$.

Then every solution $x(t) = x(t,0,x_0), x_0 \in D$, of (8.3) is continuable to ∞.

Proof. Fix $x_0 \in D$ and let $x(t) = x(t,0,x_0)$ be a solution of (8.3). Consider the following two cases.

(A) If x(t) is a continuous solution of (8.3), then it is a solution of (8.6) and is continuable to ∞.
(B) Let $x(\theta_i+) \in \tilde{\Gamma}$ for a fixed i. We set $M_x = \sup_{B(x,\epsilon_x)} \|f(x)\|$. Assume that there exists a number $\xi > \theta_i$, such that $||x(\xi) - x(\theta_i+)|| = \epsilon_{x(\theta_i+)}$ (otherwise $x(t)$ is continuable to ∞). Then

$$x(\xi) = x(\theta_i+) + \int_{\theta_i}^{\xi} f(x(s))ds,$$

and $\epsilon_{x(\theta_i+)} \leq M_{x(\theta_i+)}(\xi - \theta_i) \leq M_{x(\theta_i+)}(\theta_{i+1} - \theta_i)$, where $M_{x(\theta_i+)} > 0$ (Why?). The last inequality implies that $\theta_{i+1} - \theta_i \geq \bar{\theta}$ for all i. That is, θ_i is a sequence of β-type if $\theta_i \geq 0$. The proof is complete. □

In a similar manner, one can prove that the following theorem is valid.

Theorem 8.3.2. *Assume that:*

(a) every solution $y(t, 0, x_0)$, $x_0 \in D$ of (8.6) is continuable to either $-\infty$ or $\tilde{\Gamma}$, as time decreases;
(b) there exists a positive number $\bar{\theta}$ such that

$$\frac{\epsilon_x}{\sup_{B(x,\epsilon_x)} \|f(x)\|} \geq \bar{\theta},$$

for every $x \in \Gamma$ and all $\epsilon_x > 0$ with $B(x, \epsilon_x) \cap \tilde{\Gamma} = \emptyset$.

Then, every solution $x(t) = x(t, 0, x_0)$, $x_0 \in D$, of (8.3) is continuable to $-\infty$.

Theorems 8.3.1 and 8.3.2 imply that the following assertion is valid.

Theorem 8.3.3. *Assume that*

(a) every solution $y(t, 0, x_0)$, $x_0 \in D$, of (8.6) satisfies the following conditions:

(a1) it is continuable to either ∞ or Γ, as time increases,
(a2) it is continuable to either $-\infty$ or $\tilde{\Gamma}$, as time decreases;

(b) there exists a positive number $\bar{\theta}$ such that

$$\frac{\epsilon_x}{\sup_{B(x,\epsilon_x)} \|f(x)\|} \geq \bar{\theta},$$

for every $x \in \tilde{\Gamma}$ and all $\epsilon_x > 0$ with $B(x, \epsilon_x) \cap \Gamma = \emptyset$.
(c) there exists a positive number $\tilde{\theta}$ such that

$$\frac{\tilde{\epsilon}_x}{\sup_{B(x,\tilde{\epsilon}_x)} \|f(x)\|} \geq \tilde{\theta},$$

for every $x \in \Gamma$ and all $\tilde{\epsilon}_x > 0$ with $B(x, \tilde{\epsilon}_x) \cap \tilde{\Gamma} = \emptyset$.

Then, every solution $x(t) = x(t, 0, x_0)$, $x_0 \in D$, of (8.3) is continuable on $\mathbb{R}$.

Other sufficient conditions for the continuation of solutions of (8.3) are provided by the following theorems.

Theorem 8.3.4. *Assume that*

(a) every solution $y(t, 0, x_0)$, $x_0 \in D$, of (8.6) satisfies the following conditions:

(a1) it is continuable either to ∞ or Γ, as t increases;
(a2) it is continuable either to $-\infty$ or $\tilde{\Gamma}$, as t decreases;

(b) $\sup_D \|f(x)\| < +\infty$.
(c) $\text{dist}(\Gamma, \tilde{\Gamma}) > 0$.

Then every solution $x(t, 0, x_0)$, $x_0 \in D$, of (8.3) is continuable on $\mathbb{R}$.

Proof. Fix $x_0 \in D$ and let $x(t) = x(t, 0, x_0)$ be a solution of (8.3). According to Definition 2.1.1, we shall consider the following three cases:

(A) If x(t) is a continuous solution of (8.3), then it is a solution of (8.6) and, thus is continuable on $\mathbb{R}$.

(B) Denote by $\theta_{\max}$ and $\theta_{\min}$ the maximal and minimal elements of the set $\{\theta_i\}$, respectively. Consider $t \geq \theta_{\max}$. By the condition on J we have that $x(\theta_{\max}+) = J(x(\theta_{\max}-)) \in D$ and the solution $x(t) = y\left(t, \theta_{\max}, x(\theta_{\max}+)\right)$, where y is a solution of (8.6) and is continuable to ∞. For $t \leq \theta_{\min}$, one can apply the same arguments to show that x(t) is continuable to $-\infty$.

(C) Consider the following three alternatives.

(c_1) If the sequence $\{\theta_i\}$ has a maximal element $\theta_{\max} \in \mathbb{R}$, but does not have a minimal one, then by using (B), it is easy to prove that $x(t)$ is continuable to ∞. Let t be decreasing. We have that

$$x(\theta_i+) = x(\theta_{i+1}) + \int_{\theta_{i+1}}^{\theta_i} f(x(s))ds. \tag{8.7}$$

Denote $\sup_D \|f(x)\| = M$ and $\operatorname{dist}(\Gamma, \tilde{\Gamma}) = \alpha$. Then (8.7) implies that $\frac{\alpha}{M} \leq (\theta_{i+1} - \theta_i)$. Hence, $\frac{\alpha}{M}(i - i_0) \geq \left(\theta_i - \theta_{i_0}\right)$, where i_0 is fixed. The last inequality shows that $\theta_i \to -\infty$ as $i \to -\infty$. Thus, $x(t)$ is continuable to $-\infty$.

(c_2) Assume that the sequence $\{\theta_i\}$ has a minimal element $\theta_{\min}$, and does not have a maximal one. Then by the arguments of (B) $x(t)$ is continuable to $-\infty$. For increasing t we have that

$$x(\theta_{i+1}) = x(\theta_i+) + \int_{\theta_i}^{\theta_{i+1}} f(x(s))ds, \tag{8.8}$$

$\frac{\alpha}{M} \leq (\theta_{i+1} - \theta_i)$ or $\frac{\alpha}{M}(i - i_0) \leq \left(\theta_i - \theta_{i_0}\right)$, where i_0 is fixed. Hence, $\theta_i \to \infty$ as $i \to \infty$. That is, $x(t)$ is continuable to ∞.

(c_3) Assume that $\{\theta_i\}$ has neither a minimal nor a maximal element. The result for this case follows (c_1) and (c_2). The proof is complete. □

Theorem 8.3.5. *Assume that*

(a) every solution $y(t, 0, x_0), x_0 \in D$, of (8.6) is continuable to either ∞ or Γ, as time increases;

(b) there exists a neighborhood S of Γ in D such that

(b1) $\operatorname{dist}(\Gamma, \partial S) > 0$;

(b2) $\sup_S \|f(x)\| < \infty$;

(b3) $\tilde{\Gamma} \cap S = \emptyset$.

Then every solution $x(t) = x(t, 0, x_0), x_0 \in D$, of (8.3) is continuable to ∞.

Proof. Denote $d = \mathrm{dist}(\Gamma, \partial \mathrm{S})$ and $M = \sup_S \|f(x)\|$. For a fixed i one can see that

$$x(\theta_{i+1}) = x(\theta_i+) + \int_{\theta_i}^{\theta_{i+1}} f(x(s))ds.$$

Condition (b3) implies that $d < \|x(\theta_{i+1}) - x(\theta_i+)\| \le M(\theta_{i+1} - \theta_i)$. Thus $\theta_{i+1} - \theta_i \ge \frac{d}{M} > 0$ for all i. Further discussion is fully analogous to that of the last theorem. □

Exercise 8.3.1. Prove the following theorem.

Theorem 8.3.6. *Assume that:*

(a) every solution $y(t, 0, x_0), x_0 \in D$, of (8.6) is continuable to either $-\infty$ or $\tilde{\Gamma}$, as time decreases,
(b) there exists a neighborhood $\tilde{S}$ of $\tilde{\Gamma}$ in D such that:

  (b1) $\mathrm{dist}(\tilde{\Gamma}, \partial\tilde{\mathrm{S}}) > 0$;
  (b2) $\sup_{\tilde{S}} \|f(x)\| < \infty$;
  (b3) $\Gamma \cap \tilde{S} = \emptyset$.

Then, every solution $x(t) = x(t, 0, x_0), x_0 \in D$, of (8.3) is continuable to $-\infty$.

Using the conditions of both Theorems 8.3.5 and 8.3.6, one can formulate the following assertion.

Theorem 8.3.7. *Assume that:*

(a) every solution $y(t, 0, x_0), x_0 \in D$, of (8.6) satisfies the following conditions:

  (a1) it is continuable to either ∞ or Γ, as time increases;
  (a2) it is continuable to either $-\infty$ or $\tilde{\Gamma}$, as time decreases;

(b) there exist neighborhoods S and $\tilde{S}$ of Γ and $\tilde{\Gamma}$ in D, respectively, such that:

  (b1) $\mathrm{dist}(\Gamma, \partial\mathrm{S}) > 0, \mathrm{dist}(\tilde{\Gamma}, \partial\tilde{\mathrm{S}}) > 0$;
  (b2) $\sup_{S \cup \tilde{S}} \|f(x)\| < \infty$;
  (b3) $\tilde{\Gamma} \cap S = \emptyset, \Gamma \cap \tilde{S} = \emptyset$.

Then, every solution $x(t) = x(t, 0, x_0), x_0 \in D$, of (8.3) is continuable on $\mathbb{R}$.

Example 8.3.1. Let us consider system (8.5) and study the extension property. The differential equation in the system is a linear one, consequently, each solution of this equation is continuable to ∞, since maximal interval of existence is $\mathbb{R}$. The first condition of Theorem 8.3.1 is satisfied. Let us fix an initial value $x_0 = (x_1^0, x_2^0) \in \tilde{\Gamma}$, that is $\sqrt{3}x_1^0 = x_2^0$. Then, one can easily evaluate the distance between Γ and x_0

$$\mathrm{dist}(\mathrm{x}_0, \Gamma) = \frac{\left|\mathrm{x}_1^0 - \mathrm{x}_2^0\right|}{\sqrt{2}} = \frac{\sqrt{3}-1}{\sqrt{2}} \left|\mathrm{x}_1^0\right| = \frac{\sqrt{3}-1}{2\sqrt{2}} \|\mathrm{x}_0\| .$$

Fix

$$\epsilon_{x_0} = \frac{\sqrt{3}-1}{2\sqrt{2}} \|x_0\| \tag{8.9}$$

and take any $x \in B\left(x_0, \epsilon_{x_0}\right)$, then

$$\|x\| < \epsilon_{x_0} + \|x_0\| . \tag{8.10}$$

Substituting (8.9) into (8.10), one can conclude that

$$\|x\| < \left[\frac{\sqrt{3}-1+2\sqrt{2}}{2\sqrt{2}}\right] \|x_0\| .$$

Computing the norm of the function f in this ball, we get that

$$\|f(x)\| \le \frac{\sqrt{41}}{6}\left[\sqrt{3}-1+2\sqrt{2}\right] \|x_0\| = M_{x_0}.$$

By easy calculation,

$$\inf \frac{\epsilon_x}{M_x} = \frac{\frac{\sqrt{3}-1}{2\sqrt{2}} \|x_0\|}{\frac{\sqrt{41}}{6}\left[\sqrt{3}-1+2\sqrt{2}\right] \|x_0\|} = \frac{3\left(\sqrt{3}-1\right)}{\sqrt{82}\left(\sqrt{3}-1+2\sqrt{2}\right)} > 0.$$

We can see, now, that condition (b) is valid. Thus, all conditions of Theorem 8.3.1 are satisfied, and every solution of system (8.5) is continuable to ∞. The continuation of solutions for decreasing t can be shown by using Theorem 8.3.2.

Example 8.3.2. Let us examine system (8.4). The domain of this system is $D = \{(x_1, x_2)| \quad 1 \le x_1^2 + x_2^2 \le 4, \quad x_1, x_2 \in \mathbb{R}\}$. Manifolds Γ and $\tilde{\Gamma}$ are boundaries of this ring. They are circles with radii 1 and 2, and $\mathrm{dist}(\tilde{\Gamma}, \Gamma) = 1$, respectively.

The differential equation in (8.4) is a linear system with constant coefficients, and one can determine that all solutions are continuable to Γ, as time increases, and are continuable to $\tilde{\Gamma}$, as time decreases. Hence, the first condition of Theorem 8.3.4 is satisfied.

Moreover,

$$\|f(x)\| = \sqrt{(-x_1-3x_2)^2+(3x_1-x_2)^2} = \sqrt{10}\sqrt{x_1^2+x_2^2}, \tag{8.11}$$

and

$$\sup_D \|f(x)\| = 2\sqrt{10} < \infty.$$

Since, all conditions of Theorem 8.3.4 are satisfied, every solution of system (8.4) is continuable on $\mathbb{R}$.

Example 8.3.3. Consider the following impulsive autonomous system:

$$\begin{aligned} x_1' &= -2x_1 - 3x_2, \\ x_2' &= 3x_1 - 2x_2, \\ \Delta x_1|_{x\in\Gamma} &= (2\cos\frac{\pi}{6} - 1)x_1 - 2\sin\frac{\pi}{6}x_2, \\ \Delta x_2|_{x\in\Gamma} &= 2\sin\frac{\pi}{6}x_1 + (2\cos\frac{\pi}{6} - 1)x_2, \end{aligned} \tag{8.12}$$

where manifolds of discontinuity are

$$\Gamma = \left\{(x_1, x_2)| \quad x_1 = \sqrt{3}x_2, \quad \frac{1}{2} < x_2 < \frac{3}{2}\right\}$$

and

$$\tilde{\Gamma} = \left\{(x_1, x_2)| \quad \sqrt{3}x_1 = x_2, \quad 1 < x_1 < 3\right\}.$$

Domain $D = \mathbb{R}^2 \setminus \left\{\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right), \left(1, \sqrt{3}\right), \left(3, 3\sqrt{3}\right)\right\}$. Let us look for sufficient conditions of Theorem 8.3.5 to indicate continuation of solutions of the system (8.12) for increasing t. The differential equation in (8.12) is a linear system and maximal interval of existence is $\mathbb{R}$, so each solution of the differential equation is continuable to ∞ as time increases. Hence, the first condition is satisfied. While dealing with other conditions, we prefer to use both polar and Cartesian coordinates. First, let us define an auxiliary set S in polar coordinates (see Fig. 8.4),

$$S = \left\{(\rho, \theta) \mid \quad \frac{9}{10} < \rho < \frac{21}{10}, \quad \frac{\pi}{12} < \theta < \frac{\pi}{4}\right\}.$$

One can easily see that $\Gamma \subset S$ and $\tilde{\Gamma} \cap S = \emptyset$. The distance between Γ and ∂S, is the minimum of the following two numbers: the distance between Γ and the arc $\gamma = \{(\rho, \theta)|\rho = \frac{9}{10}, \frac{\pi}{12} < \theta < \frac{\pi}{4}\}$; the distance between Γ and the line $\ell = \{(\rho, \theta)|\frac{9}{10} < \rho < \frac{21}{10}, \theta = \frac{\pi}{4}\}$. One can find that $dist(\Gamma, \gamma) = \frac{1}{10}$. Next, let us write the equation of the line in Cartesian coordinates as

$$\ell = \left\{(x_1, x_2) \mid \quad x_1 = x_2, \; x_1, x_2 \in \mathbb{R}^+\right\}.$$

To find $dist(\Gamma, \ell)$, it is sufficient to find out the distance between the line ℓ and the points $A\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $B\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$,

$$dist\,(\ell, A) = \frac{\left|\frac{\sqrt{3}}{2} - \frac{1}{2}\right|}{\sqrt{1+1}} = \frac{\sqrt{3}-1}{2\sqrt{2}}, \qquad dist\,(\ell, B) = \frac{\left|\frac{3\sqrt{3}}{2} - \frac{3}{2}\right|}{\sqrt{1+1}} = \frac{3\sqrt{3}-3}{2\sqrt{2}}.$$

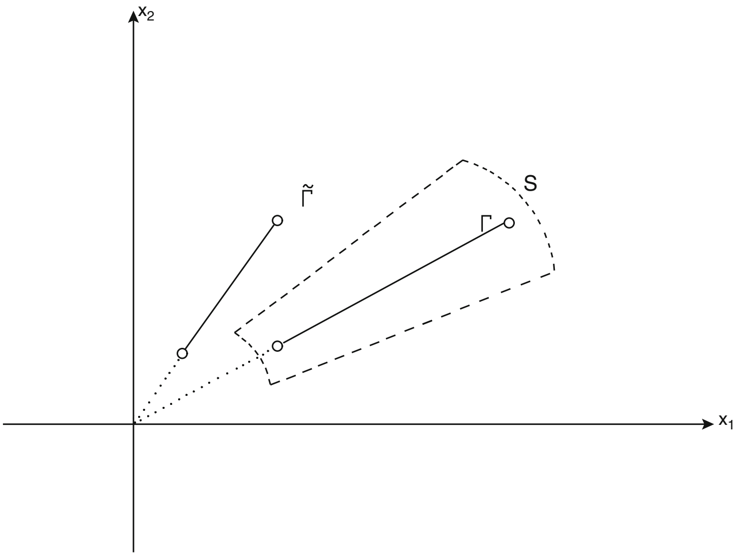

Fig. 8.4 Manifolds Γ, $\tilde{\Gamma}$, and an auxiliary set S

Then, distance between Γ and the surface ∂S is

$$dist(\Gamma, \partial S) = \frac{1}{10}.$$

Now, we take the norm of the function $f(x)$,

$$\|f(x)\| = \sqrt{4+9}\sqrt{x_1^2 + x_2^2}.$$

Since

$$\frac{9}{10} < \sqrt{x_1^2 + x_2^2} < \frac{21}{10},$$

$$\sup_S \|f(x)\| = \frac{21\sqrt{13}}{10}.$$

Thus, all conditions of Theorem 8.3.5 are satisfied, and every solution of system (8.12) is continuable to ∞.

Exercise 8.3.2. Prove, by using Theorem 8.3.6, that all solutions of system (8.12) are continuable to $-\infty$.

8.4 The Group Property

In the previous sections of the chapter, we have dealt with existence and uniqueness of solutions of the system (8.3), and furthermore, we have given the conditions that are sufficient for all solutions of (8.3) to be continuable on $\mathbb{R}$.

Now, we may discuss the group property, which is one of the most significant properties of dynamical systems and one of the most difficult for the present discussion. Next example shows that even in a simple case the group property can be violated.

Example 8.4.1. Let us consider the system (8.4), where we only replace the set G by a new one $G = \{(x_1, x_2) | \quad x_1^2 + x_2^2 > 1, \quad x_1, x_2 \in \mathbb{R}\}$. To demonstrate that the group property is not valid for all solutions, we use Fig. 8.5. Consider a trajectory, which starts at x_0 and reaches the point P at some positive moment t. Moving back it could not return to x_0, for decreasing t, because of the discontinuity set $\tilde{\Gamma}$. That is, equality $x(-t, 0, x(t, 0, x_0)) = x_0$, which is a consequence of the property is not true for all moments of time. Hence, the property is not valid for the system. It is obvious, also, that uniqueness of solutions is not true in this case, and it is not surprising, as it is known that the group property and the uniqueness are strongly related to each other.

The last example shows that specific conditions to guarantee the group property should be found.

The following condition is one of the most needed in this chapter.

(C7) (a) for every $x \in \Gamma$ there exists $\epsilon_x > 0$ such that $\text{sign}\Phi(x)$ is a constant function in $[B(x, \epsilon_x) \cap G] \backslash \Gamma$;
(b) for every $x \in \tilde{\Gamma}$ there exists $\epsilon_x > 0$ such that $\text{sign}\tilde{\Phi}(x)$ is a constant function in $[B(x, \epsilon_x) \cap G] \backslash \tilde{\Gamma}$.

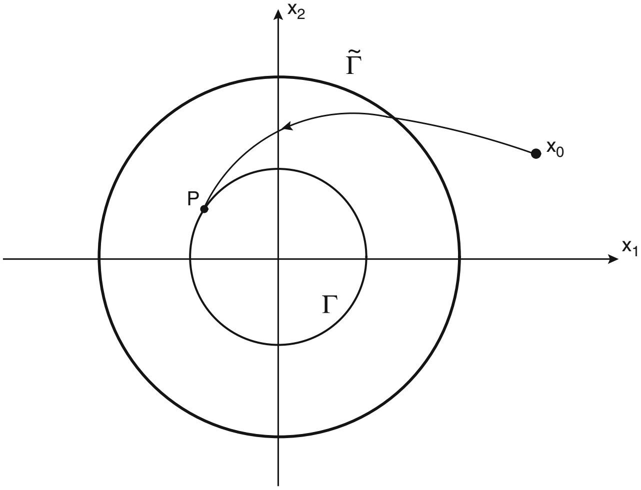

Fig. 8.5 The trajectory of Example 8.4.1

Lemma 8.4.1. *Assume that (C1)–(C7) hold and* $y(t) : (-\alpha, \alpha) \to \mathbb{R}^n$, $\alpha > 0$, *is a solution of (8.6). Then* $y(0) \notin \Gamma$ *and* $y(0) \notin \tilde{\Gamma}$.

Proof. Assume, on the contrary, that $y(0) = y_0 \in \Gamma$. We have that

$$\Phi(y(t)) = \Phi(y(t)) - \Phi(y_0) = \langle \nabla\Phi(y_0), y(t) - y_0\rangle + o(||y(t) - y_0||) =$$

$$\langle \nabla\Phi(y_0), f(y_0)t + o(|t|)\rangle + o(||f(y_0)||t + o(|t|)) = \langle \nabla\Phi(y(0)), f(y(0))\rangle\, t + o(|t|).$$

By condition (C7) function $\operatorname{sign}\Phi(y(t))$ has a constant value for sufficiently small $|t|$, and by condition (C4) the value of $\langle \nabla\Phi(y(0)), f(y(0))\rangle$ is not zero. This contradiction proves our lemma for Γ. The proof for $\tilde{\Gamma}$ is similar. □

Lemma 8.4.2. *Assume that (C1)–(C7) hold. Then* $x(-t, 0, x(t, 0, x_0)) = x_0$ *for all* $x_0 \in D, t \in \mathbb{R}$.

Proof. Consider $t > 0$. If the set $\{\theta_i\}$ is empty, then the proof follows immediately the assertion for continuous dynamical systems [39]. One can see that it remains to check the validity of $x(-\theta_i, 0, x(\theta_i+)) = x(\theta_i)$ for all i, and the condition $x(-\theta_1, 0, x(\theta_1, 0, x_0)) = x_0$. The first one is obvious because of invertability of J. Let us consider the second one. Denote $x(t) = x(t, 0, x_0)$, $\tilde{x}(t) = x(t, 0, x(\theta_1))$. Since $x(\theta_1) \in \Gamma$, then by (C4), the solution $\tilde{x}$ moves along the trajectory of (8.6) for decreasing t, and it cannot meet $\tilde{\Gamma}$ if $t > -\theta_1$. Indeed, assume on the contrary that there exists moment θ, $-\theta_1 < \theta < 0$, where $\tilde{x}$ intersects $\tilde{\Gamma}$. Then $\tilde{x}(\theta+) = x(\theta + \theta_1)$. We have obtained a contradiction to Lemma 8.4.1 since $x(t + \theta + \theta_1)$ is the solution of (8.6) in a neighborhood of $t = 0$. If $t < 0$, the proof is very similar to that of $t > 0$, and the proof with $t = 0$ is primitive. The lemma is proved. □

Let us continue with the following auxiliary result.

Consider a solution $x(t) : \mathbb{R} \to \mathbb{R}^n$ of (8.3). Let $\{\theta_i\}$ be the sequence of discontinuity points of $x(t)$. Fix $\bar{\theta} \in \mathbb{R}$ and introduce a function $\psi(t) = x(t + \bar{\theta})$.

Lemma 8.4.3. *The sequence* $\{\theta_i - \bar{\theta}\}$ *is a set of all solutions of the equation*

$$\Phi(\psi(t)) = 0. \tag{8.13}$$

Proof. We have $\Phi(\psi((\theta_i - \bar{\theta})) = \Phi(x((\theta_i - \bar{\theta}) + \bar{\theta})) = \Phi(x(\theta_i)) = 0$. Assume that $t = \varphi$ is a solution of (8.13), then $\Phi(x(\varphi + \bar{\theta})) = \Phi(\psi(\varphi)) = 0$. That is, $\varphi + \bar{\theta}$ is one of the numbers $\{\theta_i\}$. Let $\varphi + \bar{\theta} = \theta_j$, then $\varphi = \theta_j - \bar{\theta}$. The lemma is proved. □

Lemma 8.4.4. *If* $x(t) : \mathbb{R} \to \mathbb{R}^n$ *is a solution of (8.3), then* $x(t + \bar{\theta})$, $\bar{\theta} \in \mathbb{R}$, *is also a solution of (8.3).*

Proof. From Lemma 8.4.3, it follows that $\psi = x(t + \bar{\theta})$ is a continuous function on the interval $(\theta_i - \bar{\theta}, \theta_{i+1} - \bar{\theta}]$, $i \in \mathbb{Z}$. Fix $i \in \mathbb{Z}$, and consider $t \in \left(\theta_i - \bar{\theta}, \theta_{i+1} - \bar{\theta}\right]$.

We have that $t + \bar{\theta} \in (\theta_i, \theta_{i+1}]$ and one can verify that $\psi'(t) = f(\psi(t))$. That is, (8.3) is satisfied by $x(t + \theta)$.

For fixed i, we have that $\psi((\theta_i - \bar{\theta})+) = x(\theta_i+) = J(x(\theta_i)) = J(\psi(\theta_i - \bar{\theta}))$. Thus, one can see that the jump equations in (8.3) are also satisfied by $x(t + \bar{\theta})$, and this completes the proof. □

Lemmas 8.4.2 and 8.4.4 imply that the following theorem is valid. The proof of this theorem is similar to that of continuous dynamical systems [157].

Theorem 8.4.1. *Assume that conditions (C1)–(C7) are fulfilled. Then*

$$x(t_2, x(t_1, x_0)) = x(t_2 + t_1, x_0), \tag{8.14}$$

for all $t_1, t_2 \in \mathbb{R}$.

Remark 8.4.1. Since $x(0, x_0) = x_0$, one can conclude on the basis of Theorem 8.4.1 that $x(t, x_0), t \in \mathbb{R}, x_0 \in D$, defines a one-parameter group of transformations of D into itself.

Exercise 8.4.1. Verify that condition (C7) is fulfilled in Example 8.1.2, and it is not correct in Example 8.4.1.

8.5 Continuity Properties

A dependence of solutions on initial values is a very effective method to investigate various problems of dynamical systems, and we deal with the continuous dependence in this section. It is assumed that all considered solutions are continuable on $\mathbb{R}$. The next example demonstrates that the continuity property should be discussed very carefully when one is busy with nonfixed moments of discontinuity.

Example 8.5.1. Consider the autonomous system

$$\begin{aligned} x_1' &= 0, \\ x_2' &= 0, \\ \Delta x_1'|_{x\in\Gamma} &= 0, \\ \Delta x_2'|_{x\in\Gamma} &= -1, \end{aligned} \tag{8.15}$$

where $\Gamma = \{x \in \mathbb{R}^2 : x_1 = x_2\}$. Take solutions $x_0(t) = x(t, 0, (3, 3))$, and $x(t) = x(t, 0, (x_0^1, x_0^2)), x_0^1 > 3, x_0^2 < 3$, and consider them for increasing t. One can easily see that the more points (x_0^1, x_0^2) and $(3, 3)$ are close, the more the distance $||x_0(t) - x(t)||, t > 0$, is close to 1.

Fix a point $x_0 \in \Gamma \backslash \partial\Gamma$, and denote by $B(x_0, r)$ an open ball with the center at x_0 and the radius $r > 0$. By condition (C5), if r is sufficiently small, the ball is divided by the surface Γ into two connected and open regions. Denote by $b^+(x_0, r)$ the region, which $x(t, 0, x_0)$ enters as time decreases. Let $c^+(x_0, r) = (\Gamma \cap B(x_0, r)) \cup b^+(x_0, r)$. If $x_0 \notin \Gamma$, then $c^+(x_0, r) = B(x_0, r)$, where the radius r so small that $B(x_0, r) \cap \Gamma = \emptyset$. Similarly, if $x_0 \in \tilde{\Gamma} \setminus \partial\tilde{\Gamma}$ denote by $b^-(x_0, r)$ the region, which $x(t, 0, x_0)$ enters as time increases. Then write $c^-(x_0, r) = (\tilde{\Gamma} \cap B(x_0, r)) \cup b^-(x_0, r)$. We set also $c^-(x_0, r)$ equals $B(x_0, r)$, where the radius r so small that $B(x_0, r) \cap \tilde{\Gamma} = \emptyset$, if $x_0 \notin \tilde{\Gamma}$.

Let $x^0(t) : \mathbb{R} \to \mathbb{R}^n, x^0(t) = x(t, 0, x_0)$, be a solution of (8.3).

Definition 8.5.1. The solution $x^0(t)$ of (8.3) B-continuously depends on x_0 for increasing t, if to any $\epsilon > 0$ and finite interval $[0, b], 0 < b$, there corresponds $\delta > 0$ such that any other solution $x(t) = x(t, 0, \bar{x})$ of (8.3) lies in the ϵ-neighborhood of $x^0(t)$ on $[0, b]$, if $\bar{x} \in c^+(x_0, \delta)$.

Definition 8.5.2. The solution $x^0(t) : \mathbb{R} \to \mathbb{R}^n, x^0(t) = x(t, 0, x_0)$, of (8.3) B-continuously depends on x_0 for decreasing t, if to any $\epsilon > 0$ and finite interval $[a, 0], a < 0$, there corresponds $\delta > 0$ such that any other solution $x(t) = x(t, 0, \bar{x})$ of (8.3) lies in the ϵ-neighborhood of $x^0(t)$ on $[a, 0]$, if $\bar{x} \in c^-(x_0, \delta)$.

Definition 8.5.3. The solution $x^0(t) : \mathbb{R} \to \mathbb{R}^n, x^0(t) = x(t, 0, x_0)$, of (8.3) B-continuously depends on x_0 if it continuously depends on the initial value for both decreasing and increasing t.

Theorem 8.5.1. *Assume that conditions (C1)–(C6) are satisfied. Then each solution $x^0(t) = x(t, 0, x_0), x_0 \in D$, of (8.3) continuously depends on x_0.*

Proof. We consider a particular case with a finite interval $[0, b]$, and the points of discontinuity $\theta_i, i = 1, \ldots, m$, of the solution $x^0(t)$ in the interval such that $0 < \theta_1 < \cdots < \theta_m < b$. Moreover, we assume that $t = 0$, and $t = b$ are not the moments of discontinuity. All other cases can be considered similarly.

Fix a positive number α. Let $F_\alpha = \{(t, x) | t \in [0, b], \left\| x - x^0(t) \right\| < \alpha\}, G_i(\alpha)$, $i = 0, 1, 2, \ldots, m + 1$, be α-neighborhoods of points $(0, x_0)$, $(\theta_i, x(\theta_i))$, $i = 1, 2, \ldots, m$, $\left(b, x^0(b)\right)$ in $\mathbb{R} \times \mathbb{R}^n$, respectively, and $\bar{G}_i(\alpha), i = 1, 2, \ldots, k$, be an α-neighborhood of the point $\left(\theta_i, x^0(\theta_i+)\right)$. Write

$$G^\alpha = F_\alpha \cup \left(\cup_{i=0}^{m+1} G_i(\alpha)\right) \cup \left(\cup_{i=1}^{m} \bar{G}_i(\alpha)\right).$$

Take α sufficiently small so that $G^\alpha \subset \mathbb{R} \times D$. Fix $\epsilon, 0 < \epsilon < \alpha$.

1. In view of the continuity of solutions [77], there exists $\bar{\delta}_m, 0 < \bar{\delta}_m < \epsilon$, such that every solution $x_m(t)$ of (8.6), which starts in $\bar{G}_m(\bar{\delta}_m)$, is continuable to $t = b$, does not intersect Γ, and

$$\left\| x_m(t) - x^0(t) \right\| < \epsilon,$$

for all t from the common domain of $x_m(t)$ and $x^0(t)$.
2. By continuity of J there exists $0 < \delta_m < \epsilon$, such that $(\kappa, x) \in G_m(\delta_m)$ implies $(\kappa, x + J(x)) \in \bar{G}_m(\bar{\delta}_m) \cap D$.

3. The continuity theorem yields that there exists $\bar{\delta}_{m-1}, 0 < \bar{\delta}_{m-1} < \epsilon$, such that a solution $x_{m-1}(t)$ of (8.6), which starts in $\bar{G}_{m-1}(\bar{\delta}_{m-1})$, intersects Γ in $G_m(\delta_m)$ (we continue the solution $x_{m-1}(t)$ only to the moment of the intersection) and $\left\| x_{m-1}(t) - x^0(t) \right\| < \epsilon$ for all t from the common domain of $x_{m-1}(t)$ and $x^0(t)$.

Continuing the process for $m-2, m-3, \ldots, 1$, one can obtain a sequence of families of solutions of (8.6) $x_i(t)$, and corresponding numbers $\delta_i, \bar{\delta}_i, i = 1, 2, \ldots, m$. Finally, we find a number $\delta, 0 < \delta < \epsilon$, such that each solution $x_0(t)$, which starts in $G_0(\delta)$ intersects Γ in $G_1(\delta_1)$, if t increases, and satisfies $||x_0(t) - x^0(t)|| < \epsilon$ if t is from the common domain of $x_0(t)$ and $x^0(t)$. Thus, if one chooses a solution $x(t) = x(t, 0, \bar{x}), \bar{x} \in G_0(\delta)$, of (8.3), then it coincides over the first interval of continuity, except possibly, the δ_1-neighborhood of θ_1, with one of the solutions $x_0(t)$. Then on the interval $[\theta_1, \theta_2]$ it coincides with one of the solutions $x_1(t)$, except possibly, the δ_1-neighborhood of θ_1 and the δ_2-neighborhood of θ_2, etc. Finally, one can see that the integral curve of $x(t)$ belongs to G^{ϵ}, it has exactly m meeting points with Γ, $\theta_i^1, i = 1, 2, \ldots, m, \left| \theta_i^1 - \theta_i \right| < \epsilon$ for all i, and it is continuable to $t = b$. The theorem is proved. □

8.6 B-Equivalence

In this section, we construct an auxiliary system of differential equations with impulses at fixed moments, a B-equivalent system, for equations (8.3). One have to emphasize that B-equivalence plays less general role for autonomous impulsive systems than for nonautonomous equations. In this part of the manuscript, we specify a B-equivalent system around a solution of equations (8.3).

First, we need to introduce two maps, which will be used throughout the rest of the chapter. Fix $\kappa \in \mathbb{R}$. Denote by $x(t) = x(t, \kappa, x)$ a solution of (8.6), $\tau = \tau(x)$ the moment of the meeting of $x(t)$ with the surface Γ.

Lemma 8.6.1. $\tau(x) \in C^1$.

Proof. Differentiating $\Phi\left(x\left(\tau, \kappa, x\right)\right) = 0$, and using (C5) one can get that

$$\frac{\partial \Phi\left(x\left(\tau, \kappa, x\right)\right)}{\partial \tau} = \frac{\partial \Phi\left(x\left(\tau, \kappa, x\right)\right)}{\partial x} \frac{dx(t)}{dt}\bigg|_{t=\tau} = \frac{\partial \Phi\left(x\left(\tau, \kappa, x\right)\right)}{\partial x} f\left(x\left(\tau, \kappa, x\right)\right) \neq 0$$

Now, the proof follows immediately the implicit function theorem. □

Corollary 8.6.1. $\tau(x)$ *is a continuous function.*

Let $x_1 = x(t, \tau, x(\tau)) + J(x(\tau))$ be another solution of (8.6). Define the map $\Psi(x) = x_1(\kappa)$.

Similarly to Lemma 8.6.1, one can show that the following assertion is valid.

Lemma 8.6.2. $\Psi(x) \in C^1$

Consider a solution $x^0(t) : [a,b] \to R^n, a \le 0 \le b$, of (8.3). Assume that all discontinuity points $\theta_i, i = -k, \ldots, -1, 1, \ldots, m$, are interior points of $[a,b]$. That is, $a < \theta_{-k}$ and $\theta_m < b$.

The following system of differential equations with impulses at fixed moments, which are points of discontinuity of $x^0(t)$, is very important in the sequel:

$$y' = f(y),$$
$$\Delta y|_{t=\theta_i} = W_i(y(\theta_i)). \tag{8.16}$$

The function f is the same as in (8.3) and maps $W_i, -k \le i \le m$, will be defined below. There exists a positive number r, such that r-neighborhoods $G_i(r)$ of $(\theta_i, x^0(\theta_i))$ do not intersect each other. In view of (C5), one can suppose that r is sufficiently small so that every solution of (8.6) which starts in $G_i(r)$ intersects Γ in $G_i(r)$ as t increases or decreases.

Fix $i = -k, \ldots, m$ and let $\xi(t) = x(t, \theta_i, x), (\theta_i, x) \in G_i(r)$, be a solution of (8.6), $\tau_i = \tau_i(x)$ the meeting time of $\xi(t)$ with Γ and $\psi(t) = x(t, \tau_i, \xi(\tau_i) + J(\xi(\tau_i))$ another solution of (8.6). One should mention that $|\tau_i(x) - \theta_i| = O(r)$. Denote $W_i(x) = \psi(\theta_i) - x$. One can see that

$$W_i(x) = \int_{\theta_i}^{\tau_i} f(\xi(s))ds + J(x + \int_{\theta_i}^{\tau_i} f(\xi(s))ds) + \int_{\tau_i}^{\theta_i} f(\psi(s))ds \tag{8.17}$$

is a map of an intersection of the plane $t = \theta_i$ with $G_i(r)$ into the plane $t = \theta_i$. The functions $W_i, -k \le i \le m$, are obtained by using the map Ψ, which has been defined above in this section. Hence, Lemma 8.6.2 implies that all W_i are continuously differentiable maps.

Let us introduce the following sets: $F_r = \{(t,x) | t \in [a,b], \left\| x - x^0(t) \right\| < r\}$, and $\bar{G}_i(r), i = -k, \ldots, m$, an r-neighborhood of the point $\left(\theta_i, x^0(\theta_i+)\right)$. Write

$$G^r = F_r \cup \left(\cup_{i=-k}^{m} G_i(r)\right) \cup \left(\cup_{i=-k}^{m} \bar{G}_i(r)\right).$$

Take r sufficiently small so that $G^r \subset \mathbb{R} \times D$. Denote by $G(h)$ a h-neighborhood of $x^0(0)$.

Definition 8.6.1. Systems (8.3) and (8.16) are said to be B-equivalent in G^r if there exists $h > 0$, such that:

1. for every solution $x(t)$ of (8.3) such that $x(0) \in G(h)$, the integral curve of $x(t)$ belongs to G^r and there exists a solution $y(t) = y(t, 0, x(0))$ of (8.16) which satisfies

$$x(t) = y(t), t \in [a,b] \setminus \cup_{i=-k}^{m} \widehat{(\tau_i, \theta_i]}, \tag{8.18}$$

where τ_i are moments of discontinuity of $x(t)$. Particularly:

$$x(\theta_i) = \begin{cases} y(\theta_i), & \text{if } \theta_i \le \tau_i, \\ y(\theta_i^+), & \text{otherwise}, \end{cases}$$
$$y(\tau_i) = \begin{cases} x(\tau_i), & \text{if } \theta_i \ge \tau_i, \\ x(\tau_i^+), & \text{otherwise}. \end{cases} \tag{8.19}$$

2. Conversely, if (8.16) has a solution $y(t) = y(t, 0, x(0)), x(0) \in G(h)$, then there exists a solution $x(t) = x(t, 0, x(0))$ of (8.3) which has an integral curve in G^r, and (8.18) holds.

The following assertion follows immediately (8.17).

Lemma 8.6.3. *$x^0(t)$ is a solution of (8.3) and (8.16) simultaneously.*

Theorem 8.6.1. *Assume that conditions (C1)–(C6) are fulfilled. Then systems (8.3) and (8.16) are B-equivalent in G^r if r is sufficiently small.*

Proof. Assume that $r > 0$ is small so that W_i, $i = -k, \ldots, -1, 1, \ldots, m$, are defined. Let us check only the first condition of Definition 8.6.1 because that of the second one is analogous. Theorem 8.5.1 implies that there exists a small $h, 0 < h < r$, such that if $\|\bar{x} - x_0\| < h$ and $\bar{x} \in D$, then the solution $x(t) = x(t, 0, \bar{x})$ belongs to G^r. Assume that h is sufficiently small so that $x(t)$ has exactly $m + k$ moments of discontinuity $t = \tau_i, i = -k, \ldots, -1, 1, \ldots, m$. Without loss of generality, we suppose that $\theta_i > \tau_i$ for all i. It is obvious that we need only to prove the theorem for $[0, b]$, because for $[a, 0]$, the proof is similar. Consider the solution $y(t) = y(t, 0, x(0))$ of (8.16). By the theorem on existence and uniqueness [77] the equality

$$x(t) = y(t) \tag{8.20}$$

is valid on $[0, \tau_1]$. Since $(\tau_1, x(\tau_1)) \in G^r$ we see that

$$y(\theta_1+) = y(\tau_1) + \int_{\tau_1}^{\theta_1} f(y(s))ds + W_i(y(\theta_1)) \tag{8.21}$$

is defined and

$$x(\theta_1) = x(\tau_1) + J(x(\tau_1)) + \int_{\tau_1}^{\theta_1} f(x(s))ds. \tag{8.22}$$

Using (8.20)–(8.22) one can obtain that

$$y(\theta_1+) = x(\tau_1) + \int_{\tau_1}^{\theta_1} f(y(s))ds + \int_{\theta_1}^{\tau_1} f(y(s))ds +$$

$$J(y(\tau_1)) + \int_{\tau_1}^{\theta_1} f(x(s))ds = x(\theta_1).$$

Now, defining $x(t)$ and $y(t)$ as solutions of (8.6) with a common initial value $x(\theta_1)$, one can see that $x(t) = y(t), t \in (\theta_1, \tau_2]$. Continuing in the same manner for all $t \in [0, b]$ one can show that $y(t)$ is continuable to $t = b$ and (8.18) holds. Moreover, it is easily seen that for sufficiently small h, the integral curve of $y(t)$ belongs to G_r. The theorem is proved. □

8.7 Differentiability Properties

Let us consider derivatives of functions $\tau_i(x)$, $W_i(x), i = -k, \ldots, -1, 1 \ldots, m$, which were described in Sect. 8.6. We start with derivatives of $\tau_i(x)$. One should emphasize that $\tau_i, i = -k, \ldots, -1, 1 \ldots, m$ are maps, which are defined by the map τ in Sect. 8.6 with $\kappa = \theta_i, i = -k, \ldots, -1, 1 \ldots, m$. The equalities $\Phi(x(\tau_i(x))) = 0$ imply that

$$\Phi_x(x^0(\theta_i))f(x^0(\theta_i))d\tau_i + \sum_{k=1}^{n} \Phi_x(x^0(\theta_i))\frac{\partial x^0(\theta_i)}{\partial x_k}dx_k = 0.$$

Using the last expression, one can obtain that

$$\frac{\partial \tau_i(x^0(\theta_i))}{\partial x_j} = -\frac{\Phi_x(x^0(\theta_i))\frac{\partial x^0(\theta_i)}{\partial x_j}}{\Phi_x(x^0(\theta_i))f(x^0(\theta_i))}. \tag{8.23}$$

Similarly, for W_i the following expression is valid:

$$\frac{\partial W_i(x^0(\theta_i))}{\partial x_j} = f\frac{\partial \tau_i}{\partial x_j} + \frac{\partial J}{\partial x}(e_j + f\frac{\partial \tau_i}{\partial x_j}) - f^+\frac{\partial \tau_i}{\partial x_j}. \tag{8.24}$$

Thus, formulas (8.23) and (8.24) provide evaluations of the derivatives.

It is known that $x^0(t) : [a, b] \to \mathbb{R}^n$ is the solution of (8.3) and (8.16). Moreover, systems (8.3) and (8.16) are B-equivalent in G^r and there exists $\delta \in R, \delta > 0$, such that every solution which starts in $c^+(x_0, r)$ is continuable to $t = b$. Without loss of generality, assume that all points of discontinuity of $x^0(t)$ are interior points of $[a, b]$. Denote by $x^j(t), j = 1, 2, \ldots, n$, solutions of (8.3) such that $x^j(0) = x_0 + \xi e_j = (x_1^0, x_2^0, \ldots, x_{j-1}^0, x_j^0 + \xi, x_{j+1}^0, \ldots, x_n^0), \xi \in \mathbb{R}$, and let θ_i^j be the moments of discontinuity of $x^j(t)$. By Theorem 8.5.1, a solution $x^j(t), j = 1, 2, \ldots, n$, is defined on $[a, b]$ if $x_0 + \xi e_j$ belongs to $c^+(x_0, \delta)$ and $c^-(x_0, \delta)$ with sufficiently small δ.

Definition 8.7.1. The solution $x^0(t)$ is B-differentiable with respect to $x_j^0, j = 1, 2, \ldots, n$, on $[a, b]$ if for all $x_0 + \xi e_j$, which belong to $c^+(x_0, \delta)$ and $c^-(x_0, \delta)$ with sufficiently small δ it is true that:

A) there exist constants $\nu_{ij}, i = -k, \ldots, -1, 1, \ldots, m$, such that

$$\theta_i^j - \theta_i = \nu_{ij}\xi + o(|\xi|); \tag{8.25}$$

B) for all $t \in [a,b] \setminus \cup_{i=-k}^{m} \widehat{(\theta_i, \theta_i^j]}$, the following equality is satisfied:

$$x^j(t) - x^0(t) = u_j(t)\xi + o(|\xi|), \tag{8.26}$$

where $u_j(t) \in \mathcal{PC}([a,b], \theta])$.
The pair $\{u_j, \{\nu_{ij}\}_i\}$ is said to be a B-derivative of $x^0(t)$ with respect to x_j^0 on $[a,b]$.

Lemma 8.7.1. *Assume that conditions (C1)–(C6) hold. Then the solution $x^0(t)$ of (8.16) has B-derivatives with respect to $x_j^0, j = 1, 2, \ldots, n$, on $[a,b]$. Moreover, u_j is a solution of the linear system*

$$\begin{aligned} \frac{du}{dt} &= f_x(x^0(t))u, \\ \Delta u|_{t=\theta_i} &= W_{ix}(x^0(\theta_i))u(\theta_i), \end{aligned} \tag{8.27}$$

with $u(0) = e_j$, and constants $\nu_{ij} = 0$, for all i.

Proof. We shall prove the lemma with respect to x_1^0. Let $y_1(t) = y(t, 0, x_0 + \xi e_1)$. By the theorem on differentiability with respect to parameters [77] we have that $y_1(t) - x^0(t) = u_1(t)\xi + \rho(\xi), \quad \rho(\xi) = o(|\xi|)$, for all $t \in [0, \theta_1]$. Particularly, $y_1(\theta_1) - x^0(\theta_1) = u_1(\theta_1)\xi + \rho(|\xi|)$. Then $y_1(\theta_1+) - x^0(\theta_1+) = W_1(y_1(\theta_1)) - W_1(x^0(\theta_1)) = W_{1x}(x^0(\theta_1))[u_1(\theta_1)\xi + \rho(\xi)] + \bar{\rho}_1(\xi)$. Since $\bar{\rho}_1 = o(|\xi|)$, we have that $y_1(\theta_1+) - x^0(\theta_1+) = u_1(\theta_1+)\xi + \tilde{\rho}_1(\xi)$, where $\tilde{\rho}_1 = o(|\xi|)$. Denote by $U(t), U(\theta_1) = \mathcal{I}$, the fundamental matrix of the system $u'(t) = f_x(x^0(t))$. Using the theorem from [60, 77] one can obtain that for all $t \in (\theta_1, \theta_2]$ the following relation is true $y_1(t) - x^0(t) = U(t)(y_1(\theta_1+) - x^0(\theta_1+)) + \rho(y_1(\theta_1+) - x^0(\theta_1+)) = U(t)u_1(\theta_m+)\xi + \rho_2(\xi) = u_1(t)\xi + \rho_2(\xi)$, where $\rho_2 = o(|\xi|)$. Continuing the process we can prove that (8.26) is valid. Formula (8.25) is trivial. The lemma is proved. □

Theorem 8.7.1. *Assume that conditions (C1)–(C6) are satisfied. Then the solution $x^0(t)$ of (8.3) has the B-derivative with respect to $x_j^0, j = 1, 2, \ldots, n$, on $[a,b]$. Moreover, the derivative $(u_j(t), \{\nu_{ij}\})$ is a solution of the variational system*

$$\begin{aligned} \frac{du}{dt} &= f_x(x^0(t))u, \\ \Delta u|_{t=\theta_i} &= W_{ix}(x^0(\theta_i))u(\theta_i), \\ \nu_{ij} &= -\frac{\Phi_x u(\theta_i)}{\Phi_x f}, \end{aligned} \tag{8.28}$$

with $u(0) = e_j$.

The last theorem follows immediately Theorem 8.6.1, Lemma 8.7.1, and formulas (8.23), (8.24).

Remark 8.7.1. Higher order differentiability of DDS is considered in [3].

8.8 Conclusion

Let $D \subset \mathbb{R}^n$ be a set, which is described for system (8.3) in the introductory part of this chapter.

Definition 8.8.1. A B-smooth discontinuous flow is a map $\phi : \mathbb{R} \times D \to D$, which satisfies the following properties:

(I) The *group property*:

(i) $\phi(0, x) : D \to D$ is the identity;
(ii) $\phi(t, \phi(s, x)) = \phi(t + s, x)$ is valid for all $t, s \in \mathbb{R}$ and $x \in D$.

(II) $\phi(t, x) \in \mathcal{PC}^1(\mathbb{R})$ for each fixed $x \in D$.
(III) $\phi(t, x)$ is B-differentiable in $x \in D$ on $[a, b] \subset \mathbb{R}$ for each a, b such that the discontinuity points of $\phi(t, x)$ are interior points of $[a, b]$.

Remark 8.8.1. One can see that system (8.3) defines a B-smooth discontinuous flow provided that (C1)–(C7) and the conditions of one of the extension theorems are fulfilled.

Let us weaken the smoothness condition to obtain the definition of a discontinuous flow.

Definition 8.8.2. A B-flow is a map $\phi : \mathbb{R} \times D \to D$, which satisfies the property (I) of Definition 8.8.1 and the following conditions:

(IV) $\phi(t, x) \in \mathcal{PC}(\mathbb{R})$, for each fixed $x \in D$, and $\phi(\theta_i, x) \in \Gamma, \quad \phi(\theta_i+, x) \in \tilde{\Gamma}$ for every discontinuity point.
(V) $\phi(t, x)$ is B-continuous in x on each finite and closed interval.

Remark 8.8.2. Comparing definitions of the B-differentiability and the B-continuity, one can conclude that every B-smooth discontinuous flow is a B-flow.

Exercise 8.8.1. Use the discontinuous dynamics to arrange a partition of D.

8.9 Examples

Example 8.9.1. Consider the following impulsive differential system:

$$\begin{aligned} x_1' &= \alpha x_1 - \beta x_2, \\ x_2' &= \beta x_1 + \alpha x_2, \end{aligned}$$

$$\Delta x_1|_{x\in\Gamma} = (\sqrt{3}+1)x_1 - x_2$$
$$\Delta x_2|_{x\in\Gamma} = x_1 + (\sqrt{3}+1)x_2, \tag{8.29}$$

where $\Gamma = \{(x_1,x_2)|\, x_2 = \frac{1}{2}x_1, x_1 > 0\}$, $\tilde{\Gamma} = \{(x_1,x_2)|x_2 = \frac{\sqrt{3}}{2}x_1, x_1 > 0\}$, constants α, β are positive. One can see that $\Phi(x) = x_2 - \frac{1}{2}x_1$, $f(x) = (\alpha x_1 - \beta x_2, \beta x_1 + \alpha x_2)$, $J(x) = (\sqrt{3}x_1 - x_2, x_1 + \sqrt{3}x_2)$. We assume that

$$D = \mathbb{R}^2\setminus\left[\left\{(x_1,x_2)|\quad \frac{1}{2}x_1 < x_2 < \frac{\sqrt{3}}{2}x_1, \quad x_1 > 0\right\} \cup (0,0)\right].$$

One can verify that the functions and the sets satisfy (C1)–(C7). Let us check if the conditions of Theorem 8.3.3 are fulfilled. Fix $x \in \tilde{\Gamma}$. Then dist(x, Γ) $= \frac{1}{2}$||x|| and

$$||f(x)|| = \sqrt{(\alpha x_1 - \beta x_2)^2 + (\beta x_1 + \alpha x_2)^2} = \sqrt{\alpha^2+\beta^2}||x||.$$

Thus

$$\sup_{B(x,\epsilon_x)} ||f|| = \sqrt{\alpha^2+\beta^2}(||x|| + \frac{1}{2}||x||) = \frac{3}{2}\sqrt{\alpha^2+\beta^2}||x||,$$

and

$$\inf_{\tilde{\Gamma}\times(0,\infty)} \frac{\epsilon_x}{\sup_{B(x,\epsilon_x)}||f||} = \frac{2}{3\sqrt{\alpha^2+\beta^2}} > 0.$$

Hence, all conditions of a discontinuous flow are fulfilled.

Example 8.9.2. Consider the following model of a simple neural nets from [123]. We have modified it according to the system (8.3).

$$x_1' = x_2,$$
$$x_2' = -\beta^2 x_1,$$
$$p' = -\gamma p + x_1 + B_0,$$
$$\Delta p|_{(x,p)\in\Gamma} = -p, \tag{8.30}$$

where $\Gamma = \{(x_1,x_2,p)|\, p = r,\, x_1^2 + \frac{x_2^2}{\beta^4} < 1\}$, $\tilde{\Gamma} = \{(x_1,x_2,p)| \quad p = 0,\, x_1^2 + \frac{x_2^2}{\beta^4} < 1\}$, $\Phi(x) = p - r, f(x) = (x_2, -\beta^2 x_1, -\gamma p + x_1 + B_0)$, $J(x) = (x_1, x_2, 0)$, $\beta,\gamma,r > 0$, are constants and $B_0 > \gamma r + 1$. We assume that $D = \{(x_1,x_2,p)|0 \le p \le r, x_1^2 + \frac{x_2^2}{\beta^4} < 1\}$. The variable p is a scalar input of a neural trigger and x_1, x_2, are other variables. The value of r is the threshold. One can verify that the functions and the sets satisfy (C1)–(C7) and the conditions of Theorem 8.3.4. That is, the system defines a B-smooth discontinuous flow.

Example 8.9.3. Let us consider the following system

$$\begin{aligned}
x_1' &= \alpha x_1 - \beta x_2,\\
x_2' &= \beta x_1 + \alpha x_2,\\
\Delta x_1|_{x\in\Gamma} &= (1+k)x_1,\\
\Delta x_2|_{x\in\Gamma} &= (1+k)x_2,
\end{aligned}\tag{8.31}$$

where $\Gamma = \{(x_1, x_2)|\, x_1^2 + x_2^2 = r\}$, $\tilde{\Gamma} = \{(x_1, x_2)|\, x_1^2 + x_2^2 = kr\}$, α, β, k are constants such that $\alpha,\ \beta < 0, 1 < k$. Assume that $D = \mathbb{R}^2$. One can see that all conditions (C1)–(C6) are valid for the system. But (C7) is not fulfilled. It is easy to see that a solution $x(t, 0, x_0)$ of (8.31), which starts outside of $\tilde{\Gamma}$, does not satisfy the condition $x(-t, 0, x(t, 0, x_0)) = x_0$ for all t. Thus (8.31) does not determine a discontinuous flow.

Notes

Apparently, T. Pavlidis [123,124], was the first, who formulated the problem of conditions for autonomous equations with discontinuities, which guarantee properties of dynamical systems. Papers [123, 124, 135, 136] contain interesting practical and theoretical ideas concerning discontinuous flows. These authors formulated some important conditions on differential equations, but not all of them were used to prove basic properties of discontinuous flows. Some ideas on the dynamical properties can be found also in [54, 87, 95, 111].

The chapter embodies results that provide conditions for the existence of a *discontinuous flow* and a *differentiable discontinuous flow*. Concepts of B-continuous and B-differentiable dependence of solutions on initial values are applied to describe DDS and to obtain conditions for the extension of solutions and the group property. Since DF have specific smoothness of solutions we call these systems *B-differentiable discontinuous flows.* The results are due to [1]. Since some conditions of the chapter are sufficient, but not necessary, one can develop them, but we are confident that B-continuity and B-differentiability of a motion cannot be ignored in the future investigations. It is obvious that results of the chapter can be extended for smooth of higher order and analytic discontinuous dynamics.