# Chapter 10
# Chaos and Shadowing

## 10.1 Introduction and Preliminaries

The proof of the existence of chaotic attractors remains an important and difficult problem, which is still not resolved fully, even for the Lorentz system [49, 72, 84, 150]. In this chapter, a multidimensional chaos is generated by a special initial value problem for the nonautonomous impulsive differential equation. The existence of a chaotic attractor is shown, where density of periodic solutions, sensitivity of solutions, and existence of a trajectory, which is dense in the set of all orbits are observed. That is, we concentrate on the topological ingredients of the version proposed by Devaney [62]. An appropriate example is constructed, where a chaotic attractor is indicated, and the intermittency is observed.

The discontinuous system consists of an impulsive differential equation and of a discrete equation, which generates the moments of impacts.

We suppose that the generator is chaotic while the impulsive system is dissipative for all possible sequences of moments of discontinuities, and we prove that the system has a similar chaotic nature. Similarly, if the generator function has a shadowing property [40,55,76,134], then the system admits an analogue of the property. The shadowing exists if the generator is uniformly hyperbolic on the invariant set of initial moments, or a nonhyperbolic map.

The results of this chapter illustrate that impulsive differential equations may play a special role in the investigation of the complex behavior of dynamical systems.

Finally, one must say that the $B$-equivalence method is used to obtain main results of this chapter. Thus, we will complete the integrity of the book.

Let us consider a continuous map $H : I \to \mathbb{R}, I = [0, 1]$, with a positively invariant compact set $\Lambda \subseteq I$. Let $\kappa_{i+1} = H(\kappa_i), \kappa_0 = t_0 \in \Lambda$, and the sequence $\zeta(t_0) = \{\zeta_i(t_0)\}$ be defined, where $\zeta_i(t_0) = i + \kappa_i(t_0), i \geq 0$.

One may consider the logistic map $h(t, \mu) = \mu t(1 - t), \mu > 0$, as an example of $H$. The main object of discussion in this chapter is the following special initial value problem,

$$z'(t) = Az(t) + f(z),$$

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$$\Delta|_{t=\zeta_i(t_0)} = Bz(\zeta_i(t_0)) + W(z(\zeta_i(t_0))),$$
$$z(t_0) = z_0,\ (t_0, z_0) \in \Lambda \times \mathbb{R}^n, \quad (10.1)$$

where $z \in \mathbb{R}^n, t_0 \in I, t \geq t_0$.

We shall need the following basic assumptions for the problem:

(C1) $A, B$ are $n \times n$ constant real valued matrices; $\det(\mathcal{I} + B) \neq 0$, where $\mathcal{I}$ is the identical matrix;
(C2) for all $x_1, x_2 \in \mathbb{R}^n$ functions $f(x) : \mathbb{R}^n \to \mathbb{R}^n, W : \mathbb{R}^n \to \mathbb{R}^n$, satisfy

$$||f(x_1) - f(x_2)|| + \|W(x_1) - W(x_2)\| \leq L||x_1 - x_2||,$$

where $L > 0$ is a constant;
(C3) $\sup_{x\in\mathbb{R}^n} ||f(x)|| + \sup_{x\in\mathbb{R}^n} ||W(x)|| = M_0 < \infty$;
(C4) the matrices $A$ and $B$ commute and the real parts of all eigenvalues of $A + \ln(\mathcal{I} + B)$ are negative.

From the previous chapters it implies that under these conditions a solution $z(t) = z(t, t_0, z_0), z_0 \in \mathbb{R}^n$ of (10.1) exists, and is unique on $[t_0, \infty)$.

Consider an unbounded and strictly increasing sequence $\theta$ with elements $\theta_i$, $i - 1 < \theta_i < i + 2, i \in \mathbb{Z}$. Let us denote by $Z(t, s)$ the transition matrix of the linear homogeneous system

$$z'(t) = Az(t),$$
$$\Delta z|_{t=\theta_i} = Bz(\theta_i). \quad (10.2)$$

Condition $(C4)$ and the result of Exercise 4.1.8 imply that there exist two positive numbers $N$ and $\omega$, which do not depend on $\theta$, such that $||Z(t, s)|| \leq N\mathrm{e}^{-\omega(t-s)}, t \geq$s. In what follows, we shall denote by $Z(t, s, \xi)$ the transition matrix $Z(t, s)$ if $\theta = \zeta(\xi)$.

We shall need the following additional assumptions:

(C6) $NL[\frac{2}{\omega} + \frac{\mathrm{e}^{\omega}}{1-\mathrm{e}^{-\omega}}] < 1$;
(C7) $-\omega + NL + \ln(1 + NL) < 0$.

The solution $z(t) = z(t, t_0, z_0)$ of (10.1) satisfies the following integral equation:

$$z(t) = Z(t, t_0, t_0)z_0 + \int_{t_0}^{t} Z(t, s, t_0) f(z(s))ds + \sum_{t_0 \leq \zeta_i < t} Z(t, \zeta_i(t_0), t_0)W(z(\zeta_i(t_0))).$$

Using the last formula and technique of Chap. 7 (see Theorem 7.1.5), one can verify that all solutions eventually, as $t$ increases, enter the tube with the radius $M = NM_0[\frac{1}{\omega} + \frac{\mathrm{e}^{\omega}}{1-\mathrm{e}^{-\omega}}], t \in \mathbb{R}$. That is, the discussion of this chapter can be made assuming that all solutions are inside the tube. Moreover, if the sequence $\kappa(t_0)$ is periodic with a period $p \in \mathbb{N}$, then there is a solution of (10.1) with the same period, and its integral curve is placed in the tube.

We assume that:

(C8) $Bx + W(x) \neq 0$, if $\|x\| \leq M$.

The last condition implies that periodic solutions are different for different $p$.

Denote by $\mathcal{PC}$ the set of all solutions $z(t) = z(t, t_0, z_0), t_0 \in \Lambda, z_0 \in \mathbb{R}^n, t \geq t_0$ of (10.1), and denote $\mathcal{PCA} = \{z \in \mathcal{PC} : \|z(t_0)\| < M, t_0 \in \Lambda\}$. In the next section, we define conditions with which $\mathcal{PCA}$ is a chaotic attractor.

## 10.2 The Devaney's Chaos

Let us assume that the map $H$ admits all Devaney's ingredients of chaos on the set $\Lambda$, that is:

1. there exists a positive $\delta_0$ such that for each $t \in \Lambda$ and $\epsilon > 0$ there is a point $\tilde{t} \in \Lambda$ with $|t - \tilde{t}| < \epsilon$ and $|H^i(t) - H^i(\tilde{t})| \geq \delta_0$, for some positive integer $i$ (sensitivity);
2. there exists an element $t^* \in \Lambda$ such that the set $H^i(t^*), i \geq 0$, is dense in $\Lambda$ (transitivity);
3. the set of period$-p$ points, $p \geq 1$, is dense in $\Lambda$ (density of periodic points).

Let us define the chaos for the discontinuous dynamics of (10.1).

**Definition 10.2.1.** We say that (10.1) is sensitive on $\Lambda$ if there exist positive real numbers $\epsilon_0, \epsilon_1$ such that for each $t_0 \in \Lambda$, and $\delta > 0$ one can find a number $t_1 \in \Lambda, |t_0 - t_1| < \delta$, such that for each couple of solutions $z(t) = z(t, t_0, z_0), z_1(t) = z(t, t_1, z_1), z_0, z_1 \in \mathbb{R}^n$, there exists an interval $Q \subset [t_0, \infty)$ with the length not less than $\epsilon_1$ such that $||z(t) - z_1(t)|| \geq \epsilon_0, t \in Q$, and there are no points of discontinuity of $z(t), z_1(t)$ in $Q$.

We shall denote $z(t)(\epsilon, J)z_1(t)$, if solutions $z(t)$ and $z_1(t)$ of (10.1), $z(t) = z(t, t_0, z_0), z_1(t) = z(t, t_1, z_1), t_0, t_1 \in \Lambda$, are $\epsilon$-equivalent on $J$. The concept of the equivalence is described in Sect. 5.4.

**Definition 10.2.2.** The set of all periodic solutions $\phi(t) = \phi(t, t_0), t_0 \in \Lambda$, of (10.1) is called dense in $\mathcal{PC}$ if for every solution $z(t) \in \mathcal{PC}$ and each $\epsilon > 0, E > 0$, there exist a periodic solution $\phi(t, t^*), t^* \in \Lambda$, and an interval $J \subset [t_0, \infty)$ with the length $E$ such that $\phi(t)(\epsilon, J)z(t)$.

**Definition 10.2.3.** A solution $z_*(t) \in \mathcal{PC}$ of (10.1) is called dense in the set of all orbits of $\mathcal{PC}$ if for every solution $z(t) \in \mathcal{PC}$ of (10.1), and each $\epsilon > 0, E > 0$, there exist an interval $J \subset [0, \infty)$ with the length $E$ and a real number $\xi$ such that $z_*(t + \xi)(\epsilon, J)z(t)$.

**Definition 10.2.4.** The problem (10.1) is chaotic if: ($i$) it is sensitive; ($ii$) the set of all periodic solutions $\phi(t, t_0), t_0 \in \Lambda$, is dense in $\mathcal{PC}$; ($iii$) there exists a solution $z_*(t)$, which is dense in $\mathcal{PC}$.

*Remark 10.2.1.* Definitions of the chaotic ingredients have been worked out in detail issuing from the two reasons: the considered system is nonautonomous and consequently we analyze integral curves, but not trajectories; the system is impulsive and different solutions have different points of discontinuity that necessitates the $B$-topology.

**Theorem 10.2.1.** *Assume that conditions (C1)–(C6) are fulfilled. Then the set of all periodic solutions $\phi(t,t_0), t_0 \in \Lambda$, of (10.1) is dense in $\mathcal{PC}$.*

*Proof.* Fix $t_1 \in \Lambda$ and $E, \epsilon > 0$. The density of periodic points of $H$ and uniform continuity of this map imply that for an arbitrary large number $\tilde{T}$ there exists a sequence $\zeta(t_0)$, defined by a periodic sequence $\kappa(t_0)$, such that $\|\zeta(t_1) - \zeta(t_0)\|_Q < \epsilon$, where $Q = (t_1, t_1 + \tilde{T} + E)$. We shall find the number $\tilde{T}$ so large that solution $z(t) = z(t, t_1, z_1)$, $\|z_1\| < M$, is $\epsilon$-equivalent to $\phi(t,t_0)$ on $J = (t_1 + \tilde{T}, t_1 + \tilde{T} + E)$.

Denote by $Z_1(t,s) = Z(t,s,t_1)$ and $Z_2(t,s) = Z(t,s,t_0), t \geq s$, the transition matrices. We have that

$$z(t) = Z_1(t,1)z(1) + \int_{c1}^{t} Z_1(t,s) f(z(s))ds + \sum_{1 \leq \zeta_i < t} Z_1(t, \zeta_i(t_1)) W(z(\zeta_i(t_1))),$$

$$\phi(t) = Z_2(t,1)\phi(1) + \int_{c1}^{t} Z_2(t,s) f(\phi(s))ds + \sum_{1 \leq \zeta_i < t} Z_2(t, \zeta_i(t_0)) W(\phi(\zeta_i(t_0))).$$

The difference between $z(t)$ and $\phi(t)$ cannot be evaluated by using the last two expressions since the moments of discontinuities do not coincide. The method of $B$-equivalence is helpful here. Introduce the following $B$-maps

$$W_i^1(z) = (\mathcal{I} + B)\Big[(\mathrm{e}^{A(\zeta_i(t_1) - \zeta_j(t_0))} - \mathcal{I})z + \int_{\zeta_j(t_0)}^{\zeta_i(t_1)} \mathrm{e}^{A(\zeta_i(t_1) - s)} f(z(s))ds\Big] +$$

$$W((\mathcal{I} + B)[\mathrm{e}^{A(\zeta_i(t_1) - \zeta_j(t_0))} z + \int_{\zeta_j(t_0)}^{\zeta_i(t_1)} \mathrm{e}^{A(\zeta_i(t_1) - s)} f(z(s))ds]) -$$

$$\int_{\zeta_j(t_0)}^{\zeta_i(t_1)} \mathrm{e}^{A(\zeta_i(t_1) - s)} f(z_1(s))ds - W(z),$$

where $z(t), z_1(t), z(\zeta_i(t_0)) = z, z_1(\zeta_i(t_1)) = z(\zeta_i(t_1)+)$, are solutions of the equation $z' = Az$. One can easily verify that $M_1 = \sup_{\|z\| \leq M, i \in \mathbb{Z}} \|W_i^1(z)\| < \infty$. Consider the following system:

$$\begin{aligned} &v'(t) = Av(t) + f(v), t \neq \zeta_i(t_0), \\ &\Delta v|_{t = \zeta_i(t_0)} = Bv(\zeta_i(t_0)) + W(v(\zeta_i(t_0))) + W_i^1(v(\zeta_i(t_0))), \end{aligned} \qquad (10.3)$$

together with the system

$$z'(t) = Az(t) + f(z), t \neq \zeta_i(t_1),$$
$$\Delta|_{t=\zeta_i(t_1)} = Bz(\zeta_i(t_1)) + W(\zeta_i(t_1)), \tag{10.4}$$

where $t_0, t_1$ are the numbers under discussion.

Systems (10.3) and (10.4) are $B$-equivalent. That is, their solutions with the same initial condition coincide on the common domain if only $t \notin (\zeta_i(\widehat{t_0), \zeta_i}(t_1)], i \in \mathbb{Z}$. So, if $v(t), v(1) = z(1)$, is the solution of (10.3), then $v(t) = z(t)$ for all $t \notin (\zeta_i(\widehat{t_0), \zeta_i}(t_1)], i \in \mathbb{Z}$. For $v(t)$ we have that

$$v(t) = Z_2(t,1)v(1) + \int_{c1}^{t} Z_2(t,s)f(v(s))ds +$$
$$\sum_{1\le\zeta_i<t} Z_2(t,\zeta_i(t_0))[W(v(\zeta_i(t_0)) + W_1(v(\zeta_i(t_0)))].$$

Thus,

$$\|\phi(t) - v(t)\| \le \|\phi(1) - v(1)\|\|Z_2(t,1)\| + \int_{c1}^{t} \|Z_2(t,s)\|L\|\phi(s) - v(s)\|ds +$$
$$\sum_{1\le\zeta_j(t_0)<t} \|Z_2(t,\zeta_j(t_0))\|L\|\phi(\zeta_j(t_0)) - v(\zeta_j(t_0))\| +$$
$$\sum_{1\le\zeta_j(t_0)<t} \|Z_2(t,\zeta_j(t_0))\|\|W_1(v(\zeta_i(t_0))\| \le$$
$$2MN + M_1\frac{e^{\omega}}{1-e^{-\omega}} + \int_{c1}^{t} Ne^{-\omega(t-s)}L\|z(s) - v(s)\|ds +$$
$$\sum_{1\le\zeta_j<t} Ne^{-\omega(t-\zeta_j(t_0))}L\|v(\zeta_j(t_0)) - v(\zeta_j(t_0))\|.$$

Now, applying Lemma 2.5.1, we can find that

$$\|z(t) - v(t)\| \le (2MN + M_1\frac{e^{\omega}}{1-e^{-\omega}})e^{(-\omega+NL+\ln(1+NL))(t-1)}.$$

The last inequality implies that $\|z(t)-v(t)\| < \epsilon$ if $t > \tilde{T}, t \notin [\zeta_i(\widehat{t_0), \zeta_i}(t_1)], i \ge 0$, where $\tilde{T} = 1 + \ln(\frac{\epsilon}{2MN+M_1e^{\omega}(1-e^{-\omega})^{-1}})(-\omega + NL + \ln(1+NL))^{-1}$, (we may assume that $\epsilon < 2M$ ). That is why, $z(t)(\epsilon, J)\phi(t)$ if $J = (t_1 + \tilde{T}, t_1 + \tilde{T} + E)$. The theorem is proved. □

**Theorem 10.2.2.** *Assume that conditions (C1)–(C6) are fulfilled. Then there exists a solution of (10.1), which is dense in $\mathcal{PC}$.*

*Proof.* Fix positive $E, \epsilon$, and $t^* \in \Lambda$ such that the orbit of $t^*$ is dense in $\Lambda$. Set $z_*(t) = z(t, t^*, z^*)$, $\|z^*\| < M$. Let us prove that $z_*(t)$ is the dense solution.

Consider an arbitrary solution $z(t) = z(t, t_0, z_0) \in \mathcal{PC}$. Consider an interval $J_1 = (0, E_1)$, where $E_1$ is an arbitrarily large positive number. By density of the orbit of $t^*$ and uniform continuity of $H$, there exists a natural $m$ such that

$$\|\zeta(t_1) - \zeta(t^*, m)\|_{J_1} < \epsilon, \tag{10.5}$$

where $\zeta(t^*, m) = \{\zeta_{i+m}(t^*)\}$.

We have

$$z_*(t+m) = Z_*(t+m, 1+m) z_*(1+m) + \int_{1+m}^{t+m} Z_*(t+m, u) f(z_*(u)) du +$$

$$\sum_{1+m \le \zeta_i(t_0) < t+m} Z_*(t+m, \zeta_i(t_0)) W(z_*(\zeta_i(t_0))) = Z_*(t+m, 1+m) z_*(1+m) +$$

$$\int_1^t Z_*(t, u) f(z_*(u+m)) du + \sum_{1+m \le \zeta_i(t_0) < t+m} Z_*(t+m, \zeta_i(t_0)) W(z_*(\zeta_i(t_0))),$$

and

$$z_1(t) = Z_1(t, 1) z_1(1) + \int_1^t Z_1(t, u) f(z_1(u)) du + \sum_{1 \le \zeta_i(t_1) < t} Z_1(t, \zeta_i(t_1)) W(z_1(\zeta_i(t_1))),$$

where $Z_*$ and $Z_1$ are fundamental matrices corresponding to points $t_*$ and $t_1$, respectively. Now, using the last two formulas, similarly to proof of Theorem 10.2.1, using (10.5) and the $B$-equivalence technique, we can find a sufficiently large number $E_1 > 2E$, and a natural number $m$ such that $z_*(t+m)$ and $z_1(t)$ are $\epsilon$-equivalent on $J = (E_1/2, E_1)$. The theorem is proved. □

Let $\overline{m} = \max_{|u| \le 1} \|e^{Au}\|, \underline{m} = \min_{|u| \le 1} \|e^{Au}\|$.

Condition $(C7)$ implies that $\eta = \min_{\|x\| \le M} (Bx + W(x)) > 0$.

From now on we make the assumption:

(C8) $L < \frac{\underline{m}\eta}{2\overline{m}M} \min\left(1, \frac{\underline{m}\overline{m}}{\overline{m} + \underline{m}}\right)$.

**Theorem 10.2.3.** *Assume that conditions (C1)–(C8) are fulfilled. Then (10.1) is sensitive on $\mathcal{PC}$.*

*Proof.* Fix a solution $z(t) = z(t, t_0, z_0), t_0 \in \Lambda, z_0 \in \mathbb{R}^n$, and a positive $\delta$. By sensitivity of $H$ there exist $t_1 \in \Lambda, k > 0$, such that $|t_0 - t_1| < \delta, |\zeta_k(t_0) - \zeta_k(t_1)| \ge \delta_0$. Consequently, by uniform continuity of $H$, there exist numbers $\delta_1, \delta_2$, which do not depend on $k$ and $t_0, t_1 \in \Lambda$, such that $|\zeta_{k-1}(t_0) - \zeta_{k-1}(t_1)| \ge \delta_1$,

$|\zeta_{k-2}(t_0) - \zeta_{k-2}(t_1)| \geq \delta_2$. Obviously, one can assume that $k > 3$. Moreover, uniform continuity of $H$ implies that $k$ can be an arbitrarily large number. Take arbitrary $z_1 \in \mathbb{R}^n$ and solution $z_1(t) = z(t, t_1, z_1)$.

Now, let us prove the sensitiveness through the solution $z_1(t)$.

Condition $(C8)$ implies that there exists a positive number $\nu$ such that $\frac{2\overline{m}M}{\underline{m}\eta} < \nu < \frac{\underline{m}\eta - 2\overline{m}ML}{\overline{m}}$.

We shall show that constants $\epsilon_0, \epsilon_1$ for Definition 10.2.1 can be taken equal to $\epsilon_0 = \min(\underline{m}\eta - \overline{m}(\nu + 2LM), \underline{m}\nu - \overline{m}2LM), \epsilon_1 = \min(\underline{\delta}, \frac{1}{2}(1 - \overline{\delta})$, where $\overline{\delta} = \max(\delta_0, \delta_1, \delta_2)), \underline{\delta} = \min(\delta_0, \delta_1, \delta_2)$. One can easily see that among numbers $k$ and $k-1$ there exists one , let us say $k$ itself, such that $|\zeta_k(t_0) - \zeta_k(t_1)| \geq \epsilon_1$ and interval $[\zeta_k(t_0) - \epsilon_1, \zeta_k(t_0))$ does not have points of discontinuity from $\zeta(t_0)$ and $\zeta(t_1)$.

Assume that $\|z(\zeta_k(t_0)) - z_1(\zeta_k(t_0))\| < \nu$. Then, for $t \in [\zeta_k(t_0), \zeta_k(t_1)]$,

$$z(t) = \mathrm{e}^{A(t-\zeta_k(t_0))}(\mathcal{I} + B)z((\zeta_k(t_0)) + \int_{\zeta_k(t_0)}^{t} \mathrm{e}^{A(t-s)} f(z(s))ds +$$

$$\mathrm{e}^{A(t-\zeta_k(t_0))} W(z((\zeta_k(t_0))),$$

$$z_1(t) = \mathrm{e}^{A(t-\zeta_k(t_0))} z_1((\zeta_k(t_0)) + \int_{\zeta_k(t_0)}^{t} \mathrm{e}^{A(t-s)} f(z_1(s))ds.$$

We have that

$$\|z(t) - z_1(t)\| = \|\mathrm{e}^{A(t-\zeta_k(t_0))}[Bz(\zeta_k(t_0)) + W(z(\zeta_k(t_0)))] + \mathrm{e}^{A(t-\zeta_k(t_0))}[z((\zeta_k(t_0)) -$$

$$z_1((\zeta_k(t_0))] + \int_{\zeta_k(t_0)}^{t} \mathrm{e}^{A(t-s)}(f(z(s)) - f(z_1(s)))ds\| \geq \underline{m}\eta - \overline{m}(\nu + 2LM) \geq \epsilon_0.$$

If $\|z(\zeta_k(t_0)) - z_1(\zeta_k(t_0))\| > \nu$, then, for $t \in [\zeta_k(t_0) - \epsilon_1, \zeta_k(t_0))$,

$$z(t) = \mathrm{e}^{A(t-\zeta_k(t_0))} z((\zeta_k(t_0)) + \int_{\zeta_k(t_0)}^{t} \mathrm{e}^{A(t-s)} f(z(s))ds,$$

$$z_1(t) = \mathrm{e}^{A(t-\zeta_k(t_0))} z_1((\zeta_k(t_0)) + \int_{\zeta_k(t_0)}^{t} \mathrm{e}^{A(t-s)} f(z_1(s))ds.$$

and $\|z(t) - z_1(t)\| \geq \underline{m}\nu - \overline{m}2LM \geq \epsilon_0$. The theorem is proved. □

On the basis of Theorems 10.2.1–10.2.3, we can conclude that (10.1) admits the Devaney's chaos.

It seems natural to consider the chaos only for uniformly bounded solutions on $[0, \infty)$, since the domain of chaos is always assumed to be a compact set, but we consider chaotic properties of all solutions, since the chaotic scenario for these unbounded solutions starts at the moment they reach the region where solutions from $\mathcal{PCA}$ are placed. This set is a chaotic attractor as it is easily seen that $\mathcal{PCA}$ admits defined above all ingredients of Devaney's chaos.

## 10.3 Shadowing Property

In this part of the chapter, we give definitions of shadowing property for the flow of system (10.1) and prove it for this system if the generator map has the property. A corollary of the result for a map $H$ with the hyperbolic set $\Lambda$ is obtained.

Assume that the generator map, $H(t)$, is defined in a neighborhood of the unit interval $I$.

The following definitions are from [55, 122, 131, 134] and are adapted for our system.

A sequence $\{\kappa_i\}_0^N, N \leq \infty$, is said to be a *true* trajectory of $H$, if $\kappa_0 \in \Lambda$ and $\kappa_{i+1} = H(\kappa_i), 0 \leq i < N$.

A sequence $\{\pi_i\}_0^N, N \leq \infty$, is said to be a $\kappa$*-pseudo-orbit*, $\kappa > 0$, of $H$, if $|\pi_{i+1} - H(\pi_i)| < \kappa$, and $|p_i - \lambda| < \kappa$ for all $0 \leq i < N$, and $\lambda \in \Lambda$.

The true orbit $\{\kappa_i\}_0^N$ $\delta$-shadows the pseudo-orbit $\{\pi_i\}_0^N$ if $|\kappa_i - \pi_i| < \delta$ for all $i$.

A sequence $\{z_i\}_0^N$ is said to be a *true discrete orbit* of (10.1) if $z_{i+1} = z(\zeta_{i+1}, \zeta_i, z_i)$, where $\zeta_i = i + \kappa_i$ for all $0 \leq i < N$. Let $\delta$ be a positive number, and $k$ a positive integer. A sequence $y_{ik}$ such that $0 \leq ik \leq N$ if $N < \infty$, and $i \geq 0$, if $N = \infty$, is said to be a *discrete* $\delta$*-pseudo-orbit* for the problem (10.1) with associated sequence $\{p_i\}_0^N$ if $\|y_{(i+1)k} - w(p_{(i+1)k})\| < \delta$ for all admissible $i$, and the solution $w(t)$ of the initial value problem

$$
\begin{aligned}
&w'(t) = Aw(t) + f(w),\\
&\Delta|_{t=p_i} w = Bw(p_i) + W(w(p_i)),\\
&w(p_{ik}) = y_{ik}.
\end{aligned}
\tag{10.6}
$$

A discrete $\delta$-pseudo-orbit $y_{ik}$ of problem (10.1) is said to be $\epsilon$-shadowed by a true orbit $\{z_i\}_0^N$ of (10.1) if $\|z_{ik} - y_{ik}\| < \epsilon$, and $|\zeta_{ik} - p_{ik}| < \epsilon$ for all $i$ such that $0 \leq ik \leq N$ if $N < \infty$, and $i \geq 0$, if $N = \infty$. Consider the logistic function $h(x, \mu) \equiv \mu x(1 - x)$ with coefficient $\mu = 3.8$. It is proved in [76] that for $\epsilon = 10^{-8}, N = 10^7, p_0 = 0.4$, the pseudo-orbit $p_i, i = 0$ to $N$, is $\epsilon$-shadowed by a true orbit, if $\delta = 3 \times 10^{-14}$. Several values of $\mu$ were claimed to be proper for the shadowing. Taking into account this result as well as results from [40, 55, 61, 120, 134] the following assertion is very useful.

**Theorem 10.3.1.** *Assume that conditions (C1)–(C6) are fulfilled. Then, given* $\epsilon > 0$, *there exists* $0 < \delta < \epsilon$ *and a positive integer* $k$ *such that a* $\delta$*-pseudo-orbit* $y_{ik}$ *of problem (10.1) is* $\epsilon$*-shadowed by a true orbit* $\{z_i\}_0^N$ *of (10.1) if* $p_i = i + \pi_i$, *and* $\pi_i$ *is* $\delta$*-shadowed by* $\{\kappa_i\}_0^N$.

*Proof.* Fix positive $\epsilon$ and nonnegative integer $i$. We assume that $\|z_{ik} - y_{ik}\| < \epsilon$, and we will find $\delta$ and $k$, such that $\|z_{(i+1)k} - y_{(i+1)k}\| < \epsilon$. Assume, without loss of generality, that $\zeta_{ik} < p_{ik}$, and let $z(t) = z(t, \zeta_{ik}, z_{ik})$. We have that

$$\|z(p_{ik}) - y_{ik}\| \leq \|z(p_{ik}) - z_{ik}\| + \|z_{ik} - y_{ik}\| = \|\mathrm{e}^{A(p_{ik}-\zeta_{ik})}z_{ik}$$
$$+ \int_{\zeta_{ik}}^{p_{ik}} \mathrm{e}^{A(t-s)} f(z(s))ds\| + \|z_{ik} - y_{ik}\| \leq$$
$$\|[\mathcal{I} - \mathrm{e}^{A(p_{ik}-\zeta_{ik})}]\|\|z_{ik}\| + \delta N M_0 + \epsilon = \delta\phi(\delta) + \epsilon,$$

where $\phi(s)$ is a bounded function.

Similarly to the proof of Theorem 10.2.1, we find that (10.1) is $B$-equivalent to the following system:

$$\begin{aligned} v'(t) &= Av(t) + f(v), \\ \Delta v|_{t=p_i} &= Bv(p_i) + W(v(p_i)) + \tilde{W}_i^1(v(p_i)), \\ v(t_0) &= z_0,\ (t_0, z_0) \in \Lambda \times \mathbb{R}^n, \end{aligned} \tag{10.7}$$

with $M_2 = \sup_{\|z\|\leq M, i\in\mathbb{Z}} \|\tilde{W}_i^1(z)\| < \infty$.

Then we can obtain that

$$\|z(t) - w(t)\| \leq [N(\delta\phi(\delta) + \epsilon) + M_2 \frac{\mathrm{e}^{\omega}}{1 - \mathrm{e}^{-\omega}}]\mathrm{e}^{(-\omega + NL + \ln(1+NL))(t-1)},$$

if $t \notin \widehat{[p_i, \zeta_i)}$. Now, choose $k$ sufficiently large, and $\delta$ small for the right-hand side of the last inequality to be less than $\frac{\epsilon}{3}$ at $t = (i+1)k - 1$, and $\delta \max(1, \phi(\delta)) < \frac{\epsilon}{3}$. Then $\|z_{(i+1)k} - y_{(i+1)k}\| < \|z_{(i+1)k} - z(p_{(i+1)k})\| + \|z(p_{(i+1)k}) - w(p_{(i+1)k})\| + \|y_{(i+1)k} - w(p_{(i+1)k})\| < \epsilon$. The theorem is proved. □

Now, by using the Shadowing Theorem [55, 122, 131] one can easily prove that the following assertion is true.

**Theorem 10.3.2.** *Assume that conditions (C1)–(C6) are fulfilled and $H$ has a compact positively invariant hyperbolic set $\Lambda \subset I$. Then, given $\epsilon > 0$, there exist $0 < \delta < \epsilon$, and a positive integer $k$ such that a $\delta$-pseudo-orbit $\{y_{ik}\}_0^\infty$, of problem (10.1) is $\epsilon$-shadowed by a true orbit $\{z_i\}_0^\infty$ of (10.1) if $\pi_i = p_i - i, i \geq 0$, is a $\delta$-pseudo-orbit of $H$.*

## 10.4 Simulations

Consider the following initial value problem

$$\begin{aligned} x_1' &= 2/5 x_2 + l \sin^2 x_2, \\ x_2' &= 2/5 x_1 + l \sin^2 x_1, t \neq \zeta_i(t_0), \\ \Delta x_1|_{t=\zeta_i(t_0)} &= -\frac{4}{3}x_1, \\ \Delta x_2|_{t=\zeta_i(t_0)} &= -\frac{4}{3}x_2 + W(x_2), \end{aligned} \tag{10.8}$$

where $W(s) = 1 + s^2$, if $|s| \leq l, l$ is a positive constant, and $W(s) = 1 + l^2$, if $|s| > l$. One can easily see that all the functions are lipschitzian with a constant proportional to $l$. The matrices of coefficients

$$A = \begin{pmatrix} 0 & 2/5 \\ 2/5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -4/3 & 0 \\ 0 & -4/3 \end{pmatrix}$$

commute, and the eigenvalues of the matrix

$$A + \mathrm{L}n(\mathcal{I} + B) = \begin{pmatrix} -\ln 3 & 2/5 \\ 2/5 & -\ln 3 \end{pmatrix}$$

are negative: $\lambda_{1,2} = -\ln 3 \pm 2/5 < 0$.

The results of the last section make possible the following appropriate simulations.

Choose $\mu = 3.8$ and $l = 10^{-2}$ in (10.8) and consider the solution $x(t) = (x_1, x_2)$ with initial moments $t_0 = 7/9$ and the initial value $x(t_0) = (0.005, 0.002)$.

If one consider the sequence $(x_1(n), x_2(n)), n = 1, 2, 3, \ldots, 75000$, in $x_1, x_2$-plane, then the attractor can be seen, Fig. 10.1. To approve that the attractor is chaotic, we verify the conditions of the chaotic theorems in the following way. If $|s| \leq l$, then $-\frac{4}{3}s + W(s) = s^2 - \frac{4}{3}s + 1$, and it is never equal to zero. If $|s| > l$, then $-\frac{4}{3}s + W(s) = l^2 - \frac{4}{3}s + 1$. For the last expression to be zero, we need, $s = \frac{3}{4}(1 + l^2)$. From the Figure, it is seen that the second coordinate takes values between 0.32 and 0.42. This is the region where $-\frac{4}{3}s + W(s)$ does not have zeros. All the other conditions required by theorems of this chapter could be easily checked with sufficiently small coefficient $l$.

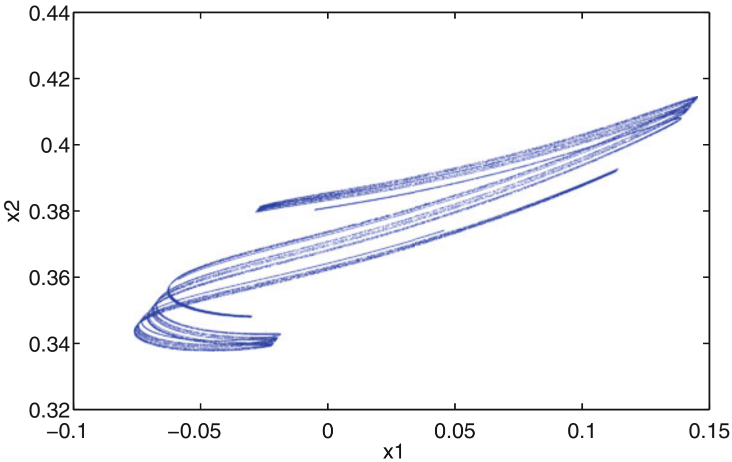


**Fig. 10.1** The chaotic attractor by a stroboscopic sequence $(x_1(n), x_2(n)), 1 \leq n \leq 75{,}000$

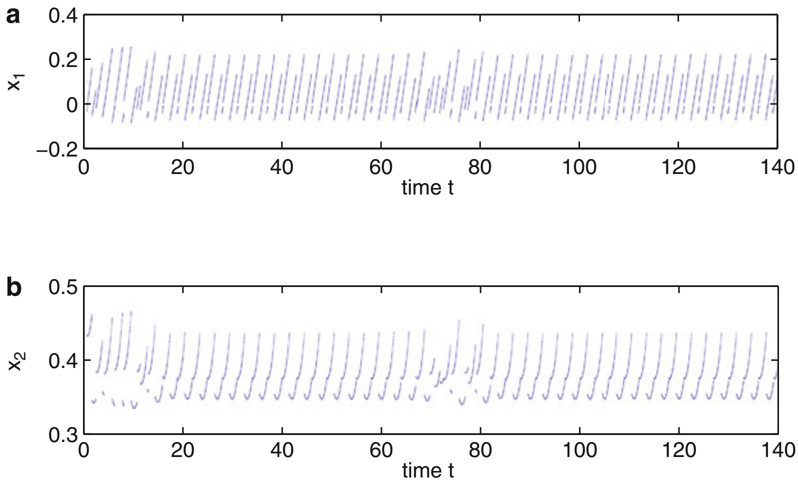


**Fig. 10.2** The intermittency of the both coordinates $x_1(t), x_2(t)$ is observable

Now, consider (10.8) with $\mu = 3.8282$. Then the phenomenon of intermittency, i.e., irregular switching between periodic and chaotic behavior, for the solution $x(t)$ can be observed in Fig.10.2. The coefficient's value is such that the logistic map admits intermittency [62].

## Notes

The investigation of the last chapter is inspired by the discontinuous dynamics of the neural information processing in the brain, information communication, and population dynamics [70,88,99,100,103,108,160]. While there are many interesting papers concerned with the complex behavior generated by impulses, the rigorous theory of chaotic impulsive systems remains far from being complete. Our goal is to develop further the theoretical foundations of this area of research. The complex dynamics is obtained using Devaney's definition for guidance. The main results of this chapter are published in [11]. There simulations for a pendulum are given. Applications of the present approach to the analysis of the cardiovascular system were considered in [12, 16]. More of our results on chaos excitability can be found in [8–10].