Chapter 9
Switched Kalman Filtering

In this chapter, the problem of Kalman filtering for a class of switched systems with state delays is investigated. Both discrete-time and continuous-time representations are treated. In both cases, attention is focused on the design of a stable filter guaranteeing a prescribed noise attenuation level in the $\mathcal{H}_\infty$ sense. By using an appropriate switched estimation scheme, sufficient conditions for the solvability of this problem are obtained in terms of algebraic Riccati equations (AREs), which, when solved, a desired $\mathcal{H}_\infty$ filter can be constructed.

9.1 Discrete Switched Delay System

The problem of optimal filtering has been well studied for more than three decades in various branches of science and engineering. Much focus has been directed to dynamical systems subject to stationary Gaussian input and measurement noise processes [3]. The celebrated Kalman filtering provides a solution to this problem. When the available plant model contains uncertain parameters, the robust state estimation problem comes into the scene for which several techniques have been proposed; see [326, 328–356, 360–372, 374–393, 399, 400] and the references cited therein. On another front of research, uncertain systems with state delay have received increasing interests in recent years [207, 208]. Most of the research efforts have been concentrated on robust stability and stabilization; see [54, 216] and the references cited therein. The problem of estimating the state of uncertain system with state delay has been overlooked despite its importance for control and signal processing.

We consider in this section the state estimation problem for linear switched discrete-time systems with norm-bounded parameter uncertainties and constant state delay. This delay factor arises naturally in different engineering fields [216]. It could result from constant processing delays as in digital systems, inherent gestation lags as in production systems, or finite transit time as in industrial mills. Indeed, this delay is among the main sources of instability in control systems. A related problem is the design of deterministic observers with unknown inputs [60] using algebraic methods. Here, we address the state estimator design problem such that the

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estimation error covariance has a guaranteed bound for all admissible uncertainties and state delay. The approach hinges upon the application of the multi-estimation structure depicted in Fig. 9.1 The main tool for solving the foregoing problem is the Riccati equation approach. It is shown that the stabilizing solution of robust Kalman filtering is given in terms of two algebraic Riccati equations. The existence of the solutions hinges on the quadratic stability of the uncertain system. In principle, all the developed results can be cast into the framework of linear matrix inequalities to yield a satisfactory solution (not necessarily stabilizing) [216].

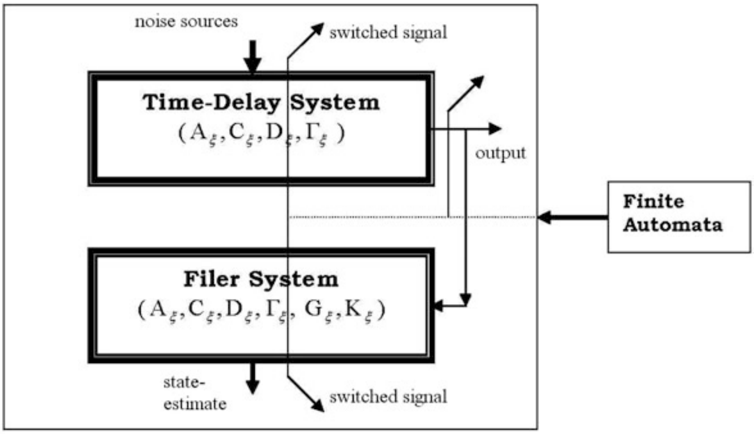

Fig. 9.1 Multi-estimator structure: discrete

9.1.1 Problem Formulation

We consider a class of switched time-delay systems represented by the discrete model:

$$\begin{aligned} x_{k+1} &= [A_{\xi(k)} + \Delta A_{\xi(k)}]\, x_k + D_{\xi(k)}\, x_{k-\tau\xi(k)} + \Gamma_{\xi(k)}\, w_k \\ &= A_{\xi(k),\Delta}\, x_k + D_{\xi(k)}\, x_{k-\tau\xi(k)} + \Gamma_{\xi(k)}\, w_k \end{aligned} \tag{9.1}$$

$$\begin{aligned} y_k &= [C_{\xi(k)} + \Delta C_{\xi(k)}]\, x_k + v_k \\ &= C_{\xi(k),\Delta}\, x_k + v_k \end{aligned} \tag{9.2}$$

$$z_k = C_{1,\xi(k)}\, x_k \tag{9.3}$$

$$\xi(k) \in \mathbf{S} = \{1, \ldots, s\} \tag{9.4}$$

where in (9.1), (9.2), (9.3), and (9.4), $x_k \in \Re^n$ is the state, $y_k \in \Re^p$ is the measured output, $z_k \in \Re^q$ is a linear combination of the state variables to be estimated, and $w_k \in \Re^r$ and $v_k \in \Re^p$ are, respectively, the process and measurement noise sequences. The matrices $A_{\xi(k)} \in \Re^{n\times n}$, $D_{\xi(k)} \in \Re^{n\times n}$, $\Gamma_{\xi(k)} \in \Re^{n\times r}$, and

$C_{\xi(k)} \in \Re^{p\times n}$ are the nominal plant matrices, which are allowed to depend on the system mode $\xi(k)$ with $A_{\xi(k)}$ being invertible for all modes. Here, $\tau_{\xi(k)}$ is a known constant scalar depending on the system mode and representing the amount of delay in the state. As in the previous chapters, the switching signal $\xi(k)$ defines the system mode at the discrete instant k, which determines the current system dynamics and the associated measurements and input vectors. The signal $\xi(k)$ can be generated by finite-discrete automata. The matrices $\Delta A_{\xi(k)}$ and $\Delta C_{\xi(k)}$ represent time-varying parametric uncertainties given by

$$\begin{bmatrix} \Delta A_{\xi(k)} \\ \Delta C_{\xi(k)} \end{bmatrix} = \begin{bmatrix} H_{1,\xi(k)} \\ H_{2,\xi(k)} \end{bmatrix} \Delta_{\xi(k)} \; E_{\xi(k)} \tag{9.5}$$

where $H_{1,\xi(k)} \in \Re^{n\times\alpha}$, $H_{2,\xi(k)} \in \Re^{m\times\alpha}$, and $E_{\xi(k)} \in \Re^{\beta\times n}$ are known matrices at every mode $\xi(k)$ and $\Delta_{\xi(k)} \in \Re^{\alpha\times\beta}$ is an unknown matrix satisfying

$$\Delta^t_{\xi(k)} \; \Delta_{\xi(k)} \quad \leq \quad I \quad k = 0, 1, 2... \tag{9.6}$$

The initial condition is specified as $\langle x_o, \phi(s)\rangle$, where $\phi(.) \in \ell_2[-\tau, 0]$. The vector x_o is assumed to be a zero-mean Gaussian random vector. The following standard assumptions on x_o and the noise sequences $\{w_{\xi(k)}\}$ and $\{v_{\xi(k)}\}$, are assumed:

$$\textbf{(a)} \;\; \mathbf{E}[w_{\xi(k)}] = 0, \;\; \mathbf{E}\left[w_{\xi(k)} w^t_{\xi(j)}\right] = W_{\xi(k)} \; \delta(k-j), W_k > 0 \; \forall k, j \tag{9.7}$$

$$\textbf{(b)} \;\; \mathbf{E}[v_{\xi(k)}] = 0, \;\; \mathbf{E}\left[v_{\xi(k)} v^t_{\xi(j)}\right] = V_{\xi(k)} \; \delta(k-j), V_k > 0 \; \forall k, j \tag{9.8}$$

$$\textbf{(c)} \;\; \mathbf{E}\left[w_{\xi(k)} v^t_{\xi(j)}\right] = 0, \;\; \mathbf{E}\left[x_o w^t_{\xi(k)}\right] = 0, \;\; \mathbf{E}\left[x_o v^t_{\xi(k)}\right] = 0 \; \forall k, j \tag{9.9}$$

$$\textbf{(d)} \;\; \mathbf{E}\left[x_o x^t_o\right] = R_o \tag{9.10}$$

where $\mathbf{E}[.]$ stands for the mathematical expectation and

$$\delta(s) = \begin{cases} 1 & s = 0 \\ 0 & \text{otherwise} \end{cases}$$

is the Dirac function.

It is interesting to observe that system (9.1), (9.2), (9.3), and (9.4) can be expressed as the composition of two subsystems: the first is a deterministic time-delay subsystem described by

$$\bar{x}_{k+1} = A_{\xi(k)} \; \bar{x}_k + B_{\xi(k)} \; u_{\xi(k),k} + D_{\xi(k)} \; \bar{x}_{k-\tau\xi(k)} \tag{9.11}$$

$$\bar{y}_k = C_{\xi(k)} \; \bar{x}_k, \quad \bar{x}(0) = 0 \tag{9.12}$$

and the other is a stochastic autonomous subsystem given by

$$\tilde{x}_{k+1} = A_{\xi(k)} \; \tilde{x}_k + D_{\xi(k)} \; \tilde{x}_{k-\tau\xi(k)} + \Gamma_{\xi(k)} \; w_k \tag{9.13}$$

$$\tilde{y}_k = C_{\xi(k)}\, \tilde{x}_k + v_{\xi(k),k}, \quad \bar{x}(0) = x_o \tag{9.14}$$

such that

$$x_{k+1} = \bar{x}_{k+1} + \tilde{x}_{k+1} \tag{9.15}$$

$$y_k = \bar{y}_k + \tilde{y}_k \tag{9.16}$$

Such an approach is very useful when $D_{\xi(k)} \equiv 0$ corresponding to the delay-free case [21]. We follow hereafter the approach developed in [287].

9.1.2 Switched State Estimation

Given the measurement sequences $y_{\xi(0)}, ..., y_{\xi(k)}$, the switched state estimation of interest is to determine at each discrete step k an unbiased linear estimation $\hat{x}_k$ of the unknown system state x_k with the following properties:

- The error model of the state estimation should be stable in the sense of Lyapunov.
- The variances of all components of the estimation error $x_k - \hat{x}_k$ should not exceed any finite bound.
- The estimation method must be applicable for all switching sequences $\xi(0), ..., \xi(k)$.

The switching state estimation under consideration could be solved by the block diagram depicted in Fig. 9.1. Observe that the current state estimate $\hat{x}_{k+1}$ is generated by

$$\hat{x}_{k+1} = G_{\xi(k)}\, \hat{x}_k + K_{\xi(k)} y_{\xi(k)} \tag{9.17}$$

where $G_{\xi(k)} \in \Re^{n \times n}$ and $K_{\xi(k)} \in \Re^{n \times p}$ are appropriately selected gain matrices such that there exists a matrix $\Psi \geq 0$ satisfying

$$\begin{aligned} \mathbf{E}[\{x_k - \hat{x}_k\}\{x_k - \hat{x}_k\}^t] &= \mathbf{E}\left[e_k e_k^t\right] \\ &\leq \Psi \end{aligned} \tag{9.18}$$

Note that (9.18) implies

$$\begin{aligned} \mathbf{E}[\{x_k - \hat{x}_k\}^t\{x_k - \hat{x}_k\}] &= \mathbf{E}\left[e_k^t e_k\right] \\ &\leq tr(\Psi) \end{aligned} \tag{9.19}$$

In this case, the estimator (9.17) is said to provide a guaranteed cost (GC) matrix Ψ.

It should be noted that if the signal $\xi(k)$ changes, the last state estimate $\hat{x}_k$ generated by the estimator for the system mode $\xi(k-1)$ is now utilized as initial state estimate for the estimator designed for the new system mode $\xi(k)$.

9.1.3 Robust Linear Filtering

We proceed to analyze the switched state estimator (9.17) by defining

$$G_{\xi(k)} = (A_{\xi(k)} + \delta A_{\xi(k)}) - K_{\xi(k)} C_{\xi(k)} \tag{9.20}$$

where $\delta A_{\xi(k)}$ and $K_{\xi(k)}$ are now the unknown matrices to be determined later on. Using (9.1), (9.2) and (9.20) to express the dynamics of the state estimator in the form

$$\begin{aligned}\hat{x}_{k+1} &= [(A_{\xi(k)} + \delta A_{\xi(k)}) - K_{\xi(k)} C_k]\hat{x}_k \\ &+ K_{\xi(k)}[C_{\xi(k)} x_k + v_k]\end{aligned} \tag{9.21}$$

Introducing the augmented state vector

$$\zeta_k = \begin{bmatrix} x_k \\ x_k - \hat{x}_k \end{bmatrix} = \begin{bmatrix} x_k \\ e_k \end{bmatrix} \in \Re^{2n} \tag{9.22}$$

Then, it follows from (9.1) and (9.21) that

$$\begin{aligned}\zeta_{k+1} &= [\mathsf{A}_{\xi(k)} + \mathsf{H}_{\xi(k)} \Delta_{\xi(k)} \mathsf{E}_{\xi(k)}]\, \zeta_k + \mathsf{D}_{\xi(k)}\, \xi_{k-\tau_{\xi(k)}} + \mathsf{B}_{\xi(k)}\, \eta_k \\ &= \mathsf{A}_{\xi(k),\Delta}\, \zeta_k + \mathsf{D}_{\xi(k)}\, \xi_{k-\tau_{\xi(k)}} + \mathsf{B}_{\xi(k)}\, \eta_k\end{aligned} \tag{9.23}$$

where η_k is a stationary zero-mean noise signal with identity covariance matrix and

$$\mathsf{A}_{\xi(k)} = \begin{bmatrix} A_{\xi(k)} & 0 \\ -\delta A_{\xi(k)} & (A_{\xi(k)} + \delta A_{\xi(k)}) - K_{\xi(k)} C_{\xi(k)} \end{bmatrix}$$

$$\mathsf{D}_{\xi(k)} = \begin{bmatrix} D_{\xi(k)} & 0 \\ D_{\xi(k)} & 0 \end{bmatrix}, \quad \eta_k = \begin{bmatrix} w_k \\ v_k \end{bmatrix} \tag{9.24}$$

$$\mathsf{B}_k \mathsf{B}_k^t = \begin{bmatrix} \Gamma_{\xi(k)} W_{\xi(k)} \Gamma_{\xi(k)}^t & 0 \\ 0 & W_{\xi(k)} + K_{\xi(k)} V_{\xi(k)} K_{\xi(k)}^t \end{bmatrix} \tag{9.25}$$

$$\mathsf{H}_{\xi(k)} = \begin{bmatrix} H_{1,\xi(k)} \\ H_{1,\xi(k)} - K_{\xi(k)} H_{2,\xi(k)} \end{bmatrix}, \quad \mathsf{E}_{\xi(k)} = [E_{\xi(k)} \quad 0] \tag{9.26}$$

Definition 9.1 Estimator (9.17) is said to be a **switched quadratic estimator (SQE)** at mode $\xi(k)$ associated with a sequence of matrices $\{\Omega_{\xi(k)}\} > 0$ for system (9.1) and (9.2) if there exist a sequence of scalars $\{\lambda_{\xi(k)}\} > 0$ and a sequence of matrices $\{\Omega_{\xi(k)}\}$ such that

$$0 < \Omega_{\xi(k)} = \begin{bmatrix} \Omega_{1,\xi(k)} & \Omega_{3,\xi(k)} \\ \Omega_{3,\xi(k)}^t & \Omega_{2,\xi(k)} \end{bmatrix} \tag{9.27}$$

satisfying the algebraic matrix inequality

$$(1+\lambda_{\xi(k)})\mathsf{A}_{\xi(k),\Delta}\ \Omega_{\xi(k)}\ \mathsf{A}^t_{\xi(k),\Delta} - \Omega_{\xi(k+1)} + \left(1+\lambda^{-1}_{\xi(k)}\right)\mathsf{D}_{\xi(k)}\ \Omega_{k-\tau_{\xi(k)}}\ \mathsf{D}^t_{\xi(k)} + \mathsf{B}_{\xi(k)}\mathsf{B}^t_{\xi(k)} \ \leq\ 0, \qquad k\ \geq\ 0 \tag{9.28}$$

for all admissible uncertainties satisfying (9.5) and (9.6)

The following result shows that if (9.17) is a **SQE** for system (9.1) and (9.2) with cost matrix $\Omega_{\xi(k)}$, then $\Omega_{\xi(k)}$ defines an upper bound for the filtering error covariance, that is,

$$\mathbf{E}\left[e_k\ e^t_k\right]\ \leq\ \Omega_{2,\xi(k)}, \quad \forall k\ \geq\ 0 \tag{9.29}$$

Theorem 9.2 *Consider the time-delay system (9.1) and (9.2) satisfying (9.5) and (9.6) and with known initial state. Suppose there exists a solution* $\Omega_{\xi(k)} = \Omega^t_{\xi(k)} \geq 0$ *to inequality (21) for some* $\lambda_k > 0$ *and for all admissible uncertainties. Then the estimator (9.17) provides an upper bound for the filtering error covariance, that is,*

$$\boldsymbol{E}\ \left[e_k\ e^t_k\right]\ \leq\ [0\quad I]\Omega_{\xi(k)}[0\quad I]^t \qquad \forall k\ \geq\ 0 \tag{9.30}$$

Proof Suppose that estimator (9.17) is a **QE** with cost matrix $\Omega_{\xi(k)}$. By evaluating the one-step ahead covariance matrix $\Sigma_{\zeta,\xi(k+1)} = \mathbf{E}\left[\zeta_{k+1}\ \zeta^t_{k+1}\right]$, we get

$$\begin{aligned}\Sigma_{\zeta,\xi(k+1)} &= \mathbf{E}[\mathsf{A}_{\xi(k),\Delta}\ \xi_k + \mathsf{D}_{\xi(k)}\zeta_{k-\tau} + \mathsf{B}_{\xi(k)}\eta_k]\times \\ &\quad [\mathsf{A}_{\xi(k),\Delta}\zeta_k + \mathsf{D}_k\zeta_{k-\tau_{\xi(k)}} + \mathsf{B}_{\xi(k)}\eta_k]^t \\ &= \mathbf{E}\left[\mathsf{A}_{\xi(k),\Delta}\zeta_k\zeta^t_k\mathsf{A}_{\xi(k),\Delta}\right] + \mathbf{E}\left[\mathsf{A}_{\xi(k),\Delta}\zeta_k\zeta^t_{k-\tau_{\xi(k)}}\mathsf{D}^t_{\xi(k)}\right] \\ &\quad +\mathbf{E}\left[\mathsf{D}_{\xi(k)}\zeta_{k-\tau_{\xi(k)}}\zeta^t_k\mathsf{A}_{\xi(k),\Delta}\right] \\ &\quad +\mathbf{E}\left[\mathsf{D}_{\xi(k)}\zeta_{k-\tau_{\xi(k)}}\zeta^t_{k-\tau_{\xi(k)}}\mathsf{D}^t_{\xi(k)}\right] + \mathbf{E}\left[\mathsf{B}_{\xi(k)}\eta_k\eta^t_k\mathsf{B}^t_{\xi(k)}\right]\end{aligned} \tag{9.31}$$

Note that

$$\begin{aligned}&\mathsf{D}_{\xi(k)}\mathbf{E}\left[\zeta_{k-\tau_{\xi(k)}}\zeta^t_{\xi(k)}\right]\mathsf{A}^t_{\xi(k),\Delta} + \mathsf{A}_{\xi(k),\Delta}\mathbf{E}\left[\zeta_k\zeta^t_{k-\tau_{\xi(k)}}\right]\ \mathsf{D}^t_{\xi(k)} \\ &\quad\leq\ \lambda_{\xi(k)}\ \mathsf{A}_{\xi(k),\Delta}\mathbf{E}\left[\zeta_k\ \zeta^t_k\right]\mathsf{A}^t_{\xi(k),\Delta} \\ &\qquad+\lambda^{-1}_{\xi(k)}\mathsf{D}_{\xi(k)}\mathbf{E}\left[\zeta_{k-\tau_{\xi(k)}}\ \zeta^t_{k-\tau_{\xi(k)}}\right]\mathsf{D}^t_{\xi(k)}\end{aligned} \tag{9.32}$$

Using inequality (9.32) into (9.31) and arranging terms, we get

$$\begin{aligned}\Sigma_{\zeta,\xi(k+1)}\ &\leq\ (1+\lambda_{\xi(k)})\mathsf{A}_{\xi(k),\Delta}\Sigma_{\zeta,\xi(k)}\mathsf{A}^t_{\xi(k),\Delta} \\ &\quad+\ \left(1+\lambda^{-1}_{\xi(k)}\right)\mathsf{D}_k\ \Sigma_{\zeta,k-\tau_{\xi(k)}}\ \mathsf{D}^t_k + \mathsf{B}_k\mathsf{B}^t_{\xi(k)}\end{aligned} \tag{9.33}$$

Letting $\Xi_{\xi(k)} = \Sigma_{\zeta,\xi(k)} - \Omega_{\xi(k)}$ with $e_k = x_k - \hat{x}_k$ and considering inequalities (21) and (25), we get

$$\Xi_{\xi(k+1)} \leq (1+\lambda_{\xi(k)})\mathsf{A}_{\xi(k)}\Xi_{\xi(k)}\mathsf{A}^t_{\xi(k)} + \left(1+\lambda^{-1}_{\xi(k)}\right)\mathsf{D}_k\Xi_{k-\tau_{\xi(k)}}\mathsf{D}^t_{\xi(k)} \qquad (9.34)$$

By considering that the state is known over the period $[-\tau_{\xi(k)}, 0]$, it justifies letting $\Sigma_{\zeta,\xi(k)} = 0 \; \forall k \in [-\tau_{\xi(k)}, 0]$. Then it follows from (9.34) that $\Xi_{\xi(k)} \leq 0$ for $k > 0$; that is, $\Sigma_{\zeta,\xi(k)} \leq \Omega_{\xi(k)}$ for $k > 0$. Hence, $\mathbf{E}\left[e_k e^t_k\right] \leq [0 \;\; I]\Omega_{\xi(k)}[0 \;\; I]^t \; \forall k \geq 0$. ■

9.1.4 A Design Approach

Motivated by the recent results of robust filtering theory [15, 399, 400], we employ hereafter a Riccati equation approach to solve the robust Kalman filtering for switched time-delay systems. To this end, we define matrices $0 < \mathsf{P}_{\xi(k)} = \mathsf{P}^t_{\xi(k)} \in \Re^{n\times n}$; $0 < \mathsf{S}_{\xi(k)} = \mathsf{S}^t_{\xi(k)} \in \Re^{n\times n}$ as the solutions of the Riccati difference equations (RDEs):

$$\begin{aligned}
\mathsf{P}_{\xi(k+1)} =\; & (1+\lambda_{\xi(k)})\{A_{\xi(k)}\left(I+\mu_{\xi(k)}\mathsf{P}_{\xi(k)}Y_{\xi(k)}\right)\mathsf{P}_{\xi(k)}A^t_{\xi(k)}\} \\
& + \left(1+\lambda^{-1}_{\xi(k)}\right)D_{\xi(k)}\mathsf{P}_{k-\tau_{\xi(k)}}D^t_{\xi(k)} + \hat{W}_{\xi(k)} \\
\mathsf{P}_{k-\tau_{\xi(k)}} =\; & 0 \quad \forall\, k \;\in [0, \tau_{\xi(k)}] \qquad (9.35) \\
\mathsf{S}_{\xi(k+1)} =\; & (1+\lambda_{\xi(k)})\hat{A}_{\xi(k)}\left(I+\mu_{\xi(k)}\mathsf{S}_{\xi(k)}Y_{\xi(k)}\right)\mathsf{S}_{\xi(k)}\hat{A}^t_{\xi(k)} \\
& +(1+\lambda_{\xi(k)})\delta A_{\xi(k)}\left(I+\mu_{\xi(k)}\mathsf{P}_{\xi(k)}Y_{\xi(k)}\right)\mathsf{P}_{\xi(k)}\delta A^t_{\xi(k)} \\
& + (1+\lambda_{\xi(k)})\mu_{\xi(k)}\hat{A}_{\xi(k)}\mathsf{P}_{\xi(k)}Y_{\xi(k)}\mathsf{S}_{\xi(k)}\hat{A}^t_{\xi(k)} \\
& +(1+\lambda_{\xi(k)})\hat{A}_{\xi(k)}\mu_{\xi(k)}\mathsf{S}_{\xi(k)}Y_{\xi(k)}\mathsf{P}_{\xi(k)}\delta A^t_{\xi(k)} \\
& -\hat{M}^t_{\xi(k)}\left(\hat{\Gamma}_{\xi(k)}+\hat{V}_{\xi(k)}\right)^{-1}\hat{M}_{\xi(k)} \\
& + \left(1+\lambda^{-1}_{\xi(k)}\right)D_{\xi(k)}P_{k-\tau_{\xi(k)}}D^t_{\xi(k)} + \hat{W}_{\xi(k)} \\
S_{k-\tau_{\xi(k)}} =\; & 0 \quad \forall\, k \;\in [0, \tau] \qquad (9.36)
\end{aligned}$$

where $\mu_{\xi(k)} > 0, \lambda_{\xi(k)} > 0$ are scaling parameters such that $P^{-1}_{\xi(k)} - \mu^{-1}_{\xi(k)}E_{\xi(k)}E^t_{\xi(k)} > 0$ and the matrices $\hat{A}_{\xi(k)}, \delta A_{\xi(k)}, \hat{V}_{\xi(k)}, \hat{W}_{\xi(k)}, \hat{\Gamma}_{\xi(k)}$, and $\hat{M}_{\xi(k)}$ are given by

$$Y_{\xi(k)} = E^t_{\xi(k)}\left(I-\mu_{\xi(k)}E_{\xi(k)}\mathsf{P}_{\xi(k)}E^t_{\xi(k)}\right)^{-1}E_{\xi(k)} \qquad (9.37)$$

$$\hat{W}_{\xi(k)} = W_{\xi(k)} + (1+\lambda_{\xi(k)})\mu^{-1}_{\xi(k)}H_{1,\xi(k)}H^t_{1,\xi(k)} \qquad (9.38)$$

$$\hat{V}_{\xi(k)} = V_{\xi(k)} + (1+\lambda_{\xi(k)})\mu^{-1}_{\xi(k)} H_{2,\xi(k)} H^t_{2,\xi(k)} \quad (9.39)$$

$$\mathcal{T}_{\xi(k)} = (1+\lambda_{\xi(k)})(\mathsf{P}_{\xi(k)} - \mathsf{S}_{\xi(k)}) \left(I + \mu_{\xi(k)} Y_{\xi(k)} \mathsf{P}_{\xi(k)}\right) A^t_{\xi(k)}$$

$$\begin{aligned} \mathcal{Z}_{\xi(k)} &= (1+\lambda_{\xi(k)})\hat{M}^t_{\xi(k)} \left(\hat{\Gamma}_{\xi(k)} + \hat{V}_{\xi(k)}\right)^{-1} \\ &\quad \times \left(C_k \mathsf{S}_k (I + \mu_{\xi(k)} Y_{\xi(k)} \mathsf{P}_{\xi(k)}) A^t_{\xi(k)} + \mu^{-1}_{\xi(k)} H_{2,\xi(k)} H^t_{1,\xi(k)}\right) \end{aligned} \quad (9.40)$$

$$\begin{aligned} \mathcal{X}_{\xi(k)} &= (1+\lambda_{\xi(k)})\mu_{\xi(k)} A_{\xi(k)} \mathsf{S}_{\xi(k)} Y_{\xi(k)} \mathsf{P}_{\xi(k)} A^t_{\xi(k)} \\ &\quad + (1+\lambda_{\xi(k)})\mu^{-1}_{\xi(k)} H_{1,\xi(k)} H^t_{1,\xi(k)} \\ &\quad + \left(1 + \lambda^{-1}_{\xi(k)}\right) D_{\xi(k)} \mathsf{P}_{k-\tau_{\xi(k)}} D^t_{\xi(k)} \end{aligned} \quad (9.41)$$

$$\hat{\Gamma}_{\xi(k)} = (1+\lambda_{\xi(k)}) C_{\xi(k)} \mathsf{S}_{\xi(k)} C^t_{\xi(k)} \quad (9.42)$$

$$\hat{A}_{\xi(k)} = A_{\xi(k)} + \delta A_{\xi(k)}, \quad \delta A_{\xi(k)} = \mathcal{T}^{-1}_{\xi(k)} \left(\mathcal{X}_{\xi(k)} + \mathcal{Z}_{\xi(k)}\right) \quad (9.43)$$

$$\begin{aligned} \hat{M}_{\xi(k)} &= (1+\lambda_{\xi(k)}) \Big[C_{\xi(k)} \mathsf{S}_{\xi(k)} A^t_{\xi(k)} + \mu_{\xi(k)} \mathsf{S}_{\xi(k)} Y_{\xi(k)} \mathsf{P}_{\xi(k)} \delta A^t_{\xi(k)} \\ &\quad + \mu_{\xi(k)} H_{2,\xi(k)} H^t_{1,\xi(k)} \Big] \end{aligned} \quad (9.44)$$

Note that the assumption that $A_{\xi(k)}$, being invertible for all k, is needed for the existence of $\mathcal{T}_{\xi(k)}$ and $\delta A_{\xi(k)}$. Let the $(\lambda, \mu)-$parametrized switched estimator be expressed as

$$\begin{aligned} \hat{x}_{k+1} &= \left(A_{\xi(k)} + \mathcal{T}^{-1}_{\xi(k)}(\mathcal{X}_{\xi(k)} + \mathcal{Z}_{\xi(k)})\right) \hat{x}_k \\ &\quad + K_{\xi(k)}[y_k - C_{\xi(k)} \hat{x}_k] \end{aligned} \quad (9.45)$$

where the Kalman gain matrix $K_{\xi(k)} \in \Re^{n \times m}$ is to be determined. The following theorem summarizes the main result:

Theorem 9.3 *Consider system (9.1) and (9.2) satisfying the uncertainty structure (9.5) and (9.6) with zero initial condition and* $A_{\xi(k)}$ *being invertible* $\forall \xi(k) \in \mathbf{S}$. *Suppose the process and measurement noises satisfy (9.7), (9.8), (9.9), and (9.10). For some* $\mu_{\xi(k)} > 0$, $\lambda_{\xi(k)} > 0$, $\xi(k) \in \mathbf{S}$ *let* $0 < \mathsf{P}_{\xi(k)} = P^t_{\xi(k)}$ *and* $0 < \mathsf{S}_{\xi(k)} = \mathsf{S}^t_{\xi(k)}$ *be the solutions of RDEs (9.35) and (9.36), respectively. Then the* $(\lambda, \mu)-$*parametrized estimator (9.45) is an* **SQE estimator with GC**

$$\boldsymbol{E}[\{\hat{x}_k - x_k\}^t \{\hat{x}_k - x_k\}] \;\le\; tr\ (\mathsf{S}_{\xi(k)}) \quad (9.46)$$

Moreover, the gain matrix K is given by

$$K_{\xi(k)} = \hat{M}^t_{\xi(k)} \left(\hat{\Gamma}_{\xi(k)} + \hat{V}_{\xi(k)}\right)^{-1} \quad (9.47)$$

Proof Let

$$\mathsf{X}_{\xi(k)} = \begin{bmatrix} \mathsf{P}_{\xi(k)} & \mathsf{S}_{\xi(k)} \\ \mathsf{S}_{\xi(k)} & \mathsf{S}_k \end{bmatrix} \tag{9.48}$$

where $\mathsf{P}_{\xi(k)}$ and $\mathsf{S}_{\xi(k)}$ are the positive-definite solutions to (9.35) and (9.36) at mode $\xi(k)$, respectively. By using the following standard inequalities (see the Appendix)

- For any real matrices Σ_1 , Σ_2, and Σ_3 with appropriate dimensions and $\Sigma_3^t\ \Sigma_3\ \leq\ I$, it follows that

$$\Sigma_1 \Sigma_3 \Sigma_2 + \Sigma_2^t \Sigma_3^t \Sigma_1^t \ \leq\ \alpha^{-1}\ \Sigma_1 \Sigma_1^t + \alpha\ \Sigma_2^t \Sigma_2 \qquad \forall \alpha > 0 \tag{9.49}$$

- Let $\Sigma_1,\ \Sigma_2,\ \Sigma_3$ and $0 < R = R^t$ be real constant matrices of compatible dimensions and $H(t)$ be a real matrix function satisfying $H^t(t)H(t) \leq I$. Then for any $\rho > 0$ satisfying $\rho \Sigma_2^t \Sigma_2 < R$, the following matrix inequality holds:

$$\begin{aligned} &(\Sigma_3 + \Sigma_1 H(t) \Sigma_2) R^{-1} \left(\Sigma_3^t + \Sigma_2^t H^t(t) \Sigma_1^t\right) \ \leq \\ &\rho^{-1} \Sigma_1 \Sigma_1^t + \Sigma_3 \left(R - \rho \Sigma_2^t \Sigma_2\right)^{-1} \Sigma_3^t \end{aligned} \tag{9.50}$$

combining (9.35), (9.36), (9.37), (9.38), (9.39), (9.40), (9.41), (9.42), (9.43), and (9.44), it is a simple task to show that

$$\begin{aligned} &(1 + \lambda_{\xi(k)}) \Big[\mathsf{A}_{\xi(k)} \mathsf{X}_{\xi(k)} \mathsf{A}_{\xi(k)}^t + \mu_{\xi(k)} \mathsf{A}_{\xi(k)} \mathsf{X}_{\xi(k)} \mathsf{E}_k^t \\ &[I - \mu_{\xi(k)} \mathsf{E}_{\xi(k)} \mathsf{X}_{\xi(k)} \mathsf{E}_{\xi(k)}^t]^{-1} \mathsf{E}_{\xi(k)} \mathsf{X}_{\xi(k)} \mathsf{A}_{\xi(k)}^t \Big] \\ &\quad - \mathsf{X}_{k+1} + (1 + \lambda_{\xi(k)}) \mu_{\xi(k)}^{-1} \mathsf{H}_{\xi(k)} \mathsf{H}_{\xi(k)}^t + \mathsf{B}_{\xi(k)} \mathsf{B}_{\xi(k)}^t \\ &\quad + \left(1 + \lambda_{\xi(k)}^{-1}\right) \mathsf{D}_k\ \mathsf{X}_{k - \tau_{\xi(k)}}\ \mathsf{D}_{\xi(k)}^t \\ &= \begin{bmatrix} \Pi_{1,\xi(k)} & \Pi_{3,\xi(k)} \\ \bullet & \Pi_{2,\xi(k)} \end{bmatrix} = 0 \end{aligned} \tag{9.51}$$

where $\Pi_{1,\xi(k)} \in \Re^{n\times n}$, $\Pi_{2,\xi(k)} \in \Re^{n\times n}$, $\Pi_{3,\xi(k)} \in \Re^{n\times n}$ and $\mathsf{A}_k, \mathsf{B}_k, \mathsf{H}_k, \mathsf{D}_k$ are given by (9.24), (9.25), and (9.26). One way to verify this is to expand (9.51) using (9.24), (9.25), and (9.26) and (9.48) to yield

$$\begin{aligned} \Pi_{1,\xi(k)} &= (1 + \lambda_{\xi(k)}) \{ A_{\xi(k)} \left(I + \mu_{\xi(k)} \mathsf{P}_{\xi(k)} Y_{\xi(k)}\right) \mathsf{P}_{\xi(k)} A_{\xi(k)}^t \} \\ &\quad + \left(1 + \lambda_{\xi(k)}^{-1}\right) D_{\xi(k)} \mathsf{P}_{k - \tau_{\xi(k)}} D_{\xi(k)}^t + \hat{W}_{\xi(k)} - \mathsf{P}_{\xi(k+1)} \end{aligned} \tag{9.52}$$

$$\begin{aligned} \Pi_{2,\xi(k)} &= (1 + \lambda_{\xi(k)}) \hat{A}_{\xi(k)} \left(I + \mu_{\xi(k)} \mathsf{S}_{\xi(k)} Y_{\xi(k)}\right) \mathsf{S}_{\xi(k)} \hat{A}_{\xi(k)}^t \\ &\quad + (1 + \lambda_{\xi(k)}) \delta A_{\xi(k)} \left(I + \mu_{\xi(k)} \mathsf{P}_{\xi(k)} Y_{\xi(k)}\right) \mathsf{P}_{\xi(k)} \delta A_{\xi(k)}^t \end{aligned}$$

$$
\begin{aligned}
&+ (1+\lambda_{\xi(k)})\mu_{\xi(k)}\hat{A}_{\xi(k)}\mathsf{P}_{\xi(k)}E^t_{\xi(k)}\left(I-\mu_{\xi(k)}E_{\xi(k)}P_{\xi(k)}E^t_{\xi(k)}\right)^{-1}\\
&\times E_{\xi(k)}\mathsf{S}_{\xi(k)}\hat{A}^t_{\xi(k)} + (1+\lambda_{\xi(k)})\hat{A}_{\xi(k)}\mu_{\xi(k)}\mathsf{S}_{\xi(k)}Y_{\xi(k)}\mathsf{P}_{\xi(k)}\delta A^t_{\xi(k)}\\
&-\hat{M}^t_{\xi(k)}\left(\hat{\Gamma}_{\xi(k)}+\hat{V}_{\xi(k)}\right)^{-1}\hat{M}_{\xi(k)} - \mathsf{S}_{\xi(k+1)}\\
&+\left(1+\lambda^{-1}_{\xi(k)}\right)D_{\xi(k)}\mathsf{P}_{k-\tau_{\xi(k)}}D^t_{\xi(k)} + \hat{W}_{\xi(k)} \qquad (9.53)
\end{aligned}
$$

$$
\begin{aligned}
\Pi_{3,\xi(k)} = &-(1+\lambda_{\xi(k)})A_{\xi(k)}\mathsf{P}_{\xi(k)}\delta A^t_{\xi(k)}\\
&+(1+\lambda_{\xi(k)})A_{\xi(k)}\mathsf{S}_{\xi(k)}\left[A^t_{\xi(k)}+\delta A^t_{\xi(k)}-C^t_{\xi(k)}K^t_{\xi(k)}\right]\\
&-\mu_{\xi(k)}(1+\lambda_{\xi(k)})A_{\xi(k)}\mathsf{P}_{\xi(k)}E^t_{\xi(k)}\\
&\times\left(I-\mu_{\xi(k)}E_{\xi(k)}\mathsf{P}_{\xi(k)}E^t_{\xi(k)}\right)^{-1}E_{\xi(k)}\mathsf{P}_{\xi(k)}\delta A^t_{\xi(k)}\\
&+\left(1+\lambda^{-1}_{\xi(k)}\right)D_{\xi(k)}\mathsf{P}_{k-\tau_{\xi(k)}}D^t_{\xi(k)} - \mathsf{S}_{\xi(k+1)}\\
&+(1+\lambda_{\xi(k)})\mu^{-1}_{\xi(k)}H_{1,\xi(k)}\left[H^t_{1,\xi(k)}-H_{2,\xi(k)}K^t_{\xi(k)}\right] \qquad (9.54)
\end{aligned}
$$

By setting $\Pi_{1,\xi(k)} \equiv 0$ in (9.52) and using (9.37), (9.38), (9.39), (9.40), (9.41), (9.42), (9.43), and (9.44) we immediately obtain (9.35). Next, we enforce $\Pi_2 \equiv 0$ in (9.53). By using (9.39), (9.40), (9.41), (9.42), (9.43), and (9.44) with some standard matrix manipulations, we define $K_{\xi(k)}$ as in (9.47) to yield (9.36). Finally, by using (9.36), (9.37), (9.38), (9.39), (9.40), (9.41), and (9.42) in (9.54) and setting $\delta A_{\xi(k)}$ as in (9.43), we find that $\Pi_{3,\xi(k)} \equiv 0$.

Now, using the results of [45], it is easy to see on using inequality (9.50) with some algebraic manipulations that (9.51) implies that

$$
\begin{aligned}
&(1+\lambda_{\xi(k)})[\mathsf{A}_{\xi(k)}+\mathsf{H}_{\xi(k)}\Delta_{\xi(k)}\mathsf{E}_{\xi(k)}]\mathsf{X}_{\xi(k)}[\mathsf{A}_{\xi(k)}+\mathsf{H}_{\xi(k)}\Delta_{\xi(k)}\mathsf{E}_{\xi(k)}]^t\\
-\ &\mathsf{X}_{\xi(k+1)} + \left(1+\lambda^{-1}_{\xi(k)}\right)\mathsf{D}_{\xi(k)}\,\mathsf{X}_{k-\tau_{\xi(k)}}\,\mathsf{D}^t_{\xi(k)} + \mathsf{B}_{\xi(k)}\mathsf{B}^t_{\xi(k)}\\
=\ &\left(1+\lambda_{\xi(k)}\right)\mathsf{A}_{\xi(k),\Delta}\mathsf{X}_{\xi(k)}\mathsf{A}^t_{\xi(k),\Delta} - \mathsf{X}_{\xi(k+1)}\\
+\ &\left(1+\lambda^{-1}_{\xi(k)}\right)\mathsf{D}_{\xi(k)}\,\mathsf{X}_{k-\tau_{\xi(k)}}\,\mathsf{D}^t_{\xi(k)} + \mathsf{B}_{\xi(k)}\mathsf{B}^t_{\xi(k)} \le 0 \qquad (9.55)
\end{aligned}
$$

$\forall\ \Delta_{\xi(k)} : \Delta^t_{\xi(k)}\ \Delta_{\xi(k)} \le I \quad \forall\ \xi(k) \in \mathbf{S}$.

It follows from **Theorem 9.2** that (9.45) is a quadratic estimator and

$$
\mathbf{E}\left[e_k e^t_k\right] = \mathbf{E}[0\ \ I]\mathsf{X}_{\xi(k)}[0\ \ I]^t \le \mathsf{S}_{\xi(k)}
$$

which implies that $\mathbf{E}\left[e^t_k\ e_k\right] \le tr\ (\mathsf{S}_{\xi(k)})$. ■

Remark 9.4 From the foregoing analysis, it is seen that our results are independent of the size of the delay. This might be considered to yield a conservative design method. However, as shown in the simulation example, the developed switched estimation method works well for a wide range of the delay factor $\tau_{\xi(k)}$.

Remark 9.5 In the case of systems without uncertainties and delay factors, that is, $H_{1,\xi(k)} = 0,\; H_{2,\xi(k)} = 0,\; E_{\xi(k)} = 0,\; D_{\xi(k)} = 0$, it can be easily shown that

$$\begin{aligned} Y_{\xi(k)} &= 0; \quad \mathcal{X}_{\xi(k)} = 0; \quad \hat{W}_{\xi(k)} = W_{\xi(k)} \\ \mathcal{T}_{\xi(k)} &= (1+\lambda_{\xi(k)})(\mathsf{P}_{\xi(k)} - \mathsf{S}_{\xi(k)})A^t_{\xi(k)} \\ \mathcal{Z}_{\xi(k)} &= (1+\lambda_{\xi(k)})^2 A_{\xi(k)}\mathsf{S}_{\xi(k)}C^t_{\xi(k)} \\ &\quad \times \left[(1+\lambda_{\xi(k)})C_{\xi(k)}\mathsf{S}_{\xi(k)}C^t_{\xi(k)} + V_{\xi(k)}\right]^{-1} C_{\xi(k)}\mathsf{S}_{\xi(k)}A^t_{\xi(k)} \end{aligned}$$

Now, in terms of $\mathsf{L}_k = \mathsf{P}_k - \mathsf{S}_k$ and

$$\begin{aligned} \Psi_{\xi(k)} &= \mathsf{S}_{\xi(k)}C^t_{\xi(k)}\left[(1+\lambda_{\xi(k)})C_{\xi(k)}\mathsf{S}_{\xi(k)}C^t_{\xi(k)} + V_{\xi(k)}\right]^{-1} C_{\xi(k)}\mathsf{S}_{\xi(k)} \\ \Phi_{\xi(k)} &= A_{\xi(k)}\Psi_{\xi(k)}A^t_{\xi(k)} \\ \mathcal{R}_{\xi(k)} &= A^{-t}_{\xi(k)}(\mathsf{P}_{\xi(k)} - \mathsf{S}_{\xi(k)})^{-1}, \quad \hat{A}_{\xi(k)} = (1+\lambda_{\xi(k)})\mathcal{R}_{\xi(k)}\Phi_{\xi(k)} \end{aligned}$$

we manipulate (9.35) and (9.36) to reach

$$\begin{aligned} \mathsf{L}_{\xi(k+1)} &= (1+\lambda_{\xi(k)})\left[A_{\xi(k)}\mathsf{L}_{\xi(k)}A^t_{\xi(k)} + \Lambda_{\xi(k)}\right], \quad \mathsf{L}_{k-\tau_{\xi(k)}} = 0 \quad \forall k \in [0, \tau_{\xi(k)}] \\ \Lambda_{\xi(k)} &= \Phi_{\xi(k)} \\ &\quad - (1+\lambda_{\xi(k)})\Big[A_{\xi(k)}(\mathsf{P}_{\xi(k)} - \mathsf{L}_{\xi(k)})\Phi^t_{\xi(k)}\mathcal{R}^t_k\xi(k) \\ &\quad + \mathcal{R}_{\xi(k)}\Phi_{\xi(k)}(\mathsf{P}_{\xi(k)} - \mathsf{L}_{\xi(k)})A^t_{\xi(k)} \\ &\quad + (1+\lambda_{\xi(k)})\mathcal{R}_{\xi(k)}\Phi_{\xi(k)}(2\mathsf{P}_{\xi(k)} - \mathsf{L}_{\xi(k)})\Phi^t_{\xi(k)}\mathcal{R}^t_{\xi(k)}\Big] \end{aligned} \tag{9.56}$$

By iterating on (9.35) and (9.56), it follows that $\mathsf{L}_{\xi(k)} = \mathsf{P}_{\xi(k)} - \mathsf{S}_{\xi(k)} > 0\ \forall \xi(k) \in$ **S**.

It can be shown in the general case that manipulation of (9.35), (9.36), (9.37), (9.38), (9.39), (9.40), (9.41), (9.42), (9.43), (9.44), and (9.45) yields

$$\begin{aligned} \mathsf{L}_{\xi(k+1)} &= (1+\lambda_{\xi(k)})[A_{\xi(k)}(I + \mu_{\xi(k)}\mathsf{L}_{\xi(k)}Y_{\xi(k)})\mathsf{L}_{\xi(k)}A^t_{\xi(k)} + \Upsilon_{\xi(k)}], \\ \mathsf{L}_{k-\tau_{\xi(k)}} &= 0 \quad \forall k \in [0, \tau_{\xi(k)}] \end{aligned}$$

In this case, Υ_k depends on the system matrices

$$A_{\xi(k)}, H_{1,\xi(k)}, H_{2,\xi(k)}, D_{\xi(k)}, C_{\xi(k)}, \mathsf{P}_{\xi(k)}$$

Remark 9.6 Note that $\mathsf{P}_{\xi(k)}$ does not depend on the filter matrices, and the structure of $\mathsf{X}_{\xi(k)}$ is identical to that of the joint covariance matrix of the state of a certain system and its standard $\mathcal{H}_2$-optimal estimator. By similarity to the standard $\mathcal{H}_2-$optimal filter, an estimate of z_k in (9.3) will be given by $\hat{z}_k = C_{1,\xi(k)}\hat{x}_k$.

Remark 9.7 In the delay-free case $(D_{\xi(k)} \equiv 0)$, we suppress the parameter $\lambda_{\xi(k)}$ and observe that (9.45) reduces to the recursive Kalman filter for the system

$$\begin{aligned} x_{k+1} &= \hat{A}_{\xi(k)}\ x_k + \hat{w}_k \\ y_k &= C_{\xi(k)}\ x_k + \hat{v}_k \end{aligned} \tag{9.57}$$

where $\hat{w}_k$ and $\hat{v}_k$ are zero-mean white noise sequences with covariance matrices $\hat{W}_{\xi(k)}$ and $\hat{\Gamma}_{\xi(k)}$, respectively, and having cross-covariance matrix $\hat{M}_{\xi(k)}$. Hence, the approach to robust filtering in **Theorem 9.3** generalizes the results of [287, 399, 400] to switched systems and corresponds to designing a standard Kalman filter for a related discrete-time system which captures all admissible uncertainties and time delay, but does not involve parameter uncertainties. In this regard, the matrix $\delta A_{\xi(k)}$ reflects the effect of uncertainties $(\Delta A_{\xi(k)}, \Delta C_{\xi(k)})$ and time-delay factor $D_{\xi(k)}$ on the structure of the filter.

9.1.5 Steady-State Robust Filter

In the sequel, we investigate the asymptotic properties of the recursive Kalman filter developed in the foregoing section. For this purpose, we consider that the switched signals are independent of the discrete instants. In this case, the uncertain switched time-delay system is given by

$$\begin{aligned} x_{k+1} &= [A_\xi + H_{1,\xi}\Delta_{k,\xi}E_\xi]\ x_k + D_\xi\ x_{k-\tau_\xi} +\ w_k \\ &= A_{\xi,\Delta}\ x_k + D_\xi\ x_{k-\tau_\xi} +\ w_k \end{aligned} \tag{9.58}$$

$$\begin{aligned} y_k &= [C_\xi + H_{2,\xi}\Delta_{k,\xi}E_\xi]\ x_k + v_k \\ &= C_{\xi,\Delta}\ x_k + v_k \end{aligned} \tag{9.59}$$

$$\xi \in \bar{\mathbf{S}} = \{1, \ldots, s\} \tag{9.60}$$

where now the switching signal ξ defines the system mode, which determines the current system dynamics and the associated measurements and input vectors. The signal ξ can be generated by finite-discrete automata but has a constant value independent of the discrete instant k. In addition, $\Delta_{k,\xi}$ satisfies (9.6). In the sequel, we assume that A_ξ is a Schur matrix; that is, $|\lambda(A_\xi)| < 1$. The matrices $A_\xi \in \Re^{n\times n}, C_\xi \in \Re^{m\times n}$ are mode-dependent constant matrices representing the nominal plant. The uncertain parameter matrix $\Delta_{k,\xi}$ is, however, time varying. In this regard, the objective is to design a switched shift-invariant a priori estimator of the form

$$\hat{x}_{k+1} = \hat{A}_\xi\hat{x}_k + K_\xi\ [y_k - C_\xi\hat{x}_k] \tag{9.61}$$

that achieves the following asymptotic performance bound

$$\lim_{k\to\infty} \mathbf{E}\left\{(\hat{x}_k - x_k)(\hat{x}_k - x_k)^t\right\} \;\leq\; S_\xi \tag{9.62}$$

Theorem 9.8 *Consider the uncertain time-delay system (9.58), (9.59), and (9.60) with A_ξ being invertible at every mode ξ. If for some scalars $\mu_\xi > 0, \lambda_\xi > 0,\ \xi \in \bar{\mathbf{S}}$, there exist stabilizing solutions $P_\xi \geq 0,\ S_\xi \geq 0$ for the AREs*

$$\begin{aligned} P_\xi &= (1+\lambda_\xi)\{A_\xi \left(I + \mu_\xi P_\xi Y_\xi\right) P_\xi A_\xi^t\} \\ &\quad +(1+\lambda_\xi^{-1}) D_\xi P_\xi D_\xi^t + \hat{W}_\xi \end{aligned} \tag{9.63}$$

$$\begin{aligned} S_\xi &= (1+\lambda_\xi)\hat{A}_\xi \left(I + \mu_\xi S_\xi Y_\xi\right) S_\xi \hat{A}_\xi^t \\ &\quad +(1+\lambda_\xi)\delta A_\xi \left(I + \mu_\xi P_\xi Y_\xi\right) P_\xi \delta A_\xi^t \\ &\quad + (1+\lambda_\xi)\mu_\xi \hat{A}_\xi P_\xi Y_\xi S_\xi \hat{A}_\xi^t + (1+\lambda_\xi)\hat{A}_\xi \mu_\xi S_\xi Y_\xi P_\xi \delta A_\xi^t \\ &\quad -\hat{M}_\xi^t \left(\hat{\Gamma}_\xi + \hat{V}_\xi\right)^{-1} \hat{M}_\xi \end{aligned} \tag{9.64}$$

$$Y_\xi = E_\xi^t \left(I - \mu_\xi E_\xi P_\xi E_\xi^t\right)^{-1} E_\xi$$

$$\hat{W}_\xi = W_\xi + (1+\lambda_\xi)\mu_\xi^{-1} H_{1,\xi} H_{1,\xi}^t \tag{9.65}$$

$$\hat{V}_\xi = V_\xi + (1+\lambda_\xi)\mu_\xi^{-1} H_{2,\xi} H_{2,\xi}^t, \quad \hat{\Gamma}_\xi = (1+\lambda_\xi) C_\xi S_\xi C_\xi^t \tag{9.66}$$

$$\hat{M}_\xi = (1+\lambda_\xi)[C_\xi S_\xi A_\xi^t + \mu_\xi S_\xi Y_\xi P_\xi \delta A_\xi^t + \mu H_{2,\xi} H_{1,\xi}^t] \tag{9.67}$$

Then the estimator (9.61) is a stable switched quadratic (SSQ) estimator and achieves (9.62) with

$$\hat{A}_\xi = A_\xi + \delta A_\xi, \quad \delta A_\xi = \mathcal{T}_\xi^{-1}\left(\mathcal{X}_\xi + \mathcal{Z}_\xi\right) \tag{9.68}$$

$$K_\xi = \hat{M}^t \left\{\hat{\Gamma} + \hat{V}\right\}^{-1}, \quad \mathcal{T}_\xi = (1+\lambda_\xi)(P_\xi - S_\xi)\left(I + \mu_\xi Y_\xi P_\xi\right) A_\xi^t \tag{9.69}$$

$$\begin{aligned} \mathcal{Z}_\xi &= (1+\lambda_\xi)\hat{M}_\xi^t \left(\hat{\Gamma}_\xi + \hat{V}_\xi\right)^{-1} \\ &\quad \times \left(C_\xi S_\xi (I + \mu_\xi Y_\xi P_\xi) A_\xi^t + \mu_\xi^{-1} H_{2,\xi} H_{1,\xi}^t\right) \end{aligned} \tag{9.70}$$

$$\begin{aligned} \mathcal{X}_\xi &= (1+\lambda_\xi)\mu_\xi A_\xi S_\xi Y_\xi P_\xi A_\xi^t + (1+\lambda_\xi)\mu_\xi^{-1} H_{1,\xi} H_{1,\xi}^t \\ &\quad + \left(1 + \lambda_\xi^{-1}\right) D_\xi S_\xi D_\xi^t \end{aligned} \tag{9.71}$$

Proof To examine the stability of the closed-loop system, we augment (9.58), (9.59), and (9.60) with ($w_k = 0,\ v_k = 0$) to obtain

$$\zeta_{k+1} = \mathsf{A}_{\xi,\Delta}\zeta_k + \mathsf{D}_\xi \zeta_{k-\tau_\xi}$$

$$= \begin{bmatrix} A_{\xi,\Delta} & 0 \\ A_{\xi,\Delta} - \hat{A}_\xi - K_\xi(C_{\xi,\Delta} - C_\xi) & \hat{A}_\xi - K_\xi C_\xi \end{bmatrix} \zeta_k + \begin{bmatrix} D_\xi & 0 \\ D_\xi & 0 \end{bmatrix} \zeta_{k-\tau_\xi} \tag{9.72}$$

Introduce a discrete Lyapunov – Krasovskii functional

$$V_{k,\xi} = \zeta_k^t X_\xi \zeta_k + \sum_{j=k-\tau_\xi}^{k-1} \zeta_j^t \left(1 + \lambda_\xi^{-1}\right) \mathsf{D}_\xi^t X_\xi \mathsf{D}_\xi \zeta_j \tag{9.73}$$

for some $\lambda_\xi > 0$. By evaluating the first-order difference $\Delta V_{k,\xi} = V_{k+1,\xi} - V_{k,\xi}$ along the trajectories of (9.72) and arranging terms, we get

$$\begin{aligned}
\Delta V_{k,\xi} &= \zeta_k^t \left[\mathsf{A}_{\xi,\Delta}^t X_\xi \mathsf{A}_{\xi,\Delta} - X_\xi\right] \zeta_k + \zeta_{k-\tau_\xi}^t \mathsf{D}_\xi^t X_\xi \mathsf{A}_{\xi,\Delta} \zeta_k + \zeta_k^t \mathsf{A}_{\xi,\Delta}^t X_\xi \mathsf{D}_\xi \zeta_{k-\tau_\xi} \\
&\quad + \zeta_{k-\tau_\xi}^t \mathsf{D}^t X_\xi \mathsf{D}_\xi \zeta_{k-\tau_\xi} + (1 + \lambda^{-1}) \zeta_k^t \mathsf{D}_\xi^t X_\xi \mathsf{D}_\xi \zeta_k \\
&\quad - (1 + \lambda^{-1}) \zeta_{k-\tau_\xi}^t \mathsf{D}_\xi^t X_\xi \mathsf{D}_\xi \zeta_{k-\tau_\xi} \\
&\leq \zeta_k^t \left[\mathsf{A}_{\xi,\Delta}^t X_\xi \mathsf{A}_{\xi,\Delta} - X_\xi + \left(1 + \lambda_\xi^{-1}\right) \mathsf{D}_\xi^t X_\xi \mathsf{D}_\xi\right] \zeta_k \\
&\quad + \lambda^{-1} \zeta_{k-\tau_\xi}^t \mathsf{D}_\xi^t X_\xi \mathsf{D}_\xi \zeta_{k-\tau_\xi} + \lambda \zeta_k^t \mathsf{A}_{\xi,\Delta}^t X_\xi \mathsf{A}_{\xi,\Delta} \zeta_k \\
&\quad + \zeta_{k-\tau}^t \mathsf{D}^t X \mathsf{D} \xi_{k-\tau} - \xi_{k-\tau}^t (1 + \lambda^{-1}) \mathsf{D}^t X \mathsf{D} \xi_{k-\tau} \\
&= \zeta_k^t \left[(1 + \lambda_\xi) \mathsf{A}_{\xi,\Delta}^t X_\xi \mathsf{A}_{\xi,\Delta} - X_\xi + \left(1 + \lambda_\xi^{-1}\right) \mathsf{D}_\xi^t X_\xi \mathsf{D}_\xi\right] \zeta_k
\end{aligned} \tag{9.74}$$

Sufficient condition of asymptotic stability $\Delta V_{k,\xi} < 0$, $\zeta_k \neq 0$ is implied by

$$(1 + \lambda_\xi) \mathsf{A}_{\xi,\Delta}^t X_\xi \mathsf{A}_{\xi,\Delta} - X_\xi + \left(1 + \lambda_\xi^{-1}\right) \mathsf{D}_\xi^t X_\xi \mathsf{D}_\xi < 0 \tag{9.75}$$

Now select X_ξ as

$$X_\xi = \begin{bmatrix} P_\xi & S_\xi \\ S_\xi & S_\xi \end{bmatrix} \tag{9.76}$$

with P_ξ and S_ξ being the stabilizing solutions of (9.63) and (9.64), respectively. Following a similar procedure as in the proof of **Theorem 9.3** and in view of **Definition 9.1**, it follows in the steady state as $k \to \infty$ that the augmented system (9.72) is asymptotically stable. The guaranteed performance $\mathbf{E}[e_k e_k^t] \leq S$ follows from similar lines of argument as in the proof of **Theorem 9.3**. ■

Remark 9.9 Note that the invertibility of A_ξ is needed for the existence of $\mathcal{T}_\xi$ *and* δA_ξ. In the switched delayless case $D_\xi \equiv 0$, it follows from (9.59) and (9.62) with $\hat{W}_\xi = B_\xi \bar{B}_\xi^{\,t}$ that

$$P_\xi = (1+\lambda_\xi)\{A_\xi P_\xi A_\xi^t + A_\xi P_\xi \left[\left(\mu_\xi^{-1} I + E_\xi P_\xi E_\xi^t \right)^{-1} P_\xi A_\xi^t \right\} + \hat{W}_\xi \quad (9.77)$$

which is a bounded real lemma equation for the system

$$\Sigma_\xi = \left(A_\xi \sqrt{1+\lambda_\xi},\, \bar{B}_\xi,\, E_\xi,\, 0 \right)$$

Suppose that for $\mu_\xi = \mu^+$, *the ARE* (9.77) *admits a solution* $P_\xi = P^+$. This implies that the $\mathcal{H}_\infty$-norm of Σ_ξ is less than $(\mu^+)^{-1/2}$. It then follows, given a λ_ξ, that system (9.58) and (9.59) is quadratically stable for some $\mu_\xi \le \mu^+$.

9.1.6 Simulation Example

Consider the following discrete time-delay system with two operational modes

Mode 1

$$\begin{aligned}
x_{k+1} &= \left(\begin{bmatrix} 0.2 & -0.1 & 0 \\ 0.004 & 0.4 & 0.1 \\ 0 & 0.1 & 0.6 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix} \Delta_k [0.5 \;\; 0.4 \;\; 0.2] \right) x_k \\
&+ \begin{bmatrix} -0.1 & 0 & 0 \\ 0.05 & -0.2 & 0.1 \\ 0 & 0 & -0.1 \end{bmatrix} x_{k-3} + w_k \\
y_k &= \left(\begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 1.0 & 0 \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix} \Delta_k [0.5 \;\; 0.4 \;\; 0.2] \right) x_k + v_k
\end{aligned}$$

Mode 2

$$\begin{aligned}
x_{k+1} &= \left(\begin{bmatrix} 0.3 & 0 & -0.1 \\ 0.002 & 0.5 & 0.2 \\ 0 & -0.1 & 0.7 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \end{bmatrix} \Delta_k [0.3 \;\; 0.2 \;\; 0.1] \right) x_k \\
&+ \begin{bmatrix} -0.3 & 0 & 0 \\ 0.02 & -0.1 & -0.1 \\ 0 & 0 & -0.2 \end{bmatrix} x_{k-3} + w_k \\
y_k &= \left(\begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 2.0 & 0 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix} \Delta_k [0.3 \;\; 0.5 \;\; 0.3] \right) x_k + v_k
\end{aligned}$$

which is of the type (9.1)-(9.2) with three units of delay. We further assume that $W_1 = I,\ W_2 = 0.8I,\ V_1 = 0.2I,\ V_2 = 0.3I$. To determine the Kalman gains, we solve (9.63) and (9.64) with the aid of (9.65), (9.66), (9.67), (9.68), (9.69), (9.70), and (9.71) for selected values of λ, μ. The numerical computation is basically of the form of iterative schemes and the results for a typical case of $\mu_1 = \mu_2 = 0.7, \lambda_1 = 0.3,\ \lambda_2 = 0.4$ are given by

$$P_1 = \begin{bmatrix} 0.141 & 0.005 & 0.003 \\ 0.005 & 0.255 & 0.175 \\ 0.003 & 0.175 & 0.501 \end{bmatrix}, \; S_1 = 10^{-4} \begin{bmatrix} 0.284 & -1.17 & -2.966 \\ -1.17 & 4.813 & 12.208 \\ -2.966 & 12.208 & 30.962 \end{bmatrix}$$

$$K_1 = 10^{-6} \begin{bmatrix} -0.841 & -1.309 \\ 3.463 & 5.388 \\ 8.782 & 13.665 \end{bmatrix}, \; \hat{A}_1 = \begin{bmatrix} 0.331 & -0.019 & -0.034 \\ -0.277 & 0.254 & 0.175 \\ -1.641 & -0.897 & 0.961 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0.137 & 0.004 & 0.003 \\ 0.004 & 0.215 & 0.166 \\ 0.003 & 0.166 & 0.498 \end{bmatrix}, \; S_2 = 10^{-4} \begin{bmatrix} 0.277 & -1.23 & -2.887 \\ -1.23 & 4.813 & 11.768 \\ -2.887 & 11.768 & 31.102 \end{bmatrix}$$

$$K_2 = 10^{-6} \begin{bmatrix} -0.798 & -1.287 \\ 4.114 & 4.987 \\ 7.978 & 14.121 \end{bmatrix}, \; \hat{A}_2 = \begin{bmatrix} 0.342 & -0.023 & -0.042 \\ -0.281 & 0.261 & 0.183 \\ -1.666 & -0.928 & 0.973 \end{bmatrix}$$

The developed estimator is indeed asymptotically stable since

$$\lambda(\hat{A})_1 = \{0.3020, 0.4800, 0.7650\} \in (0, 1)$$
$$\lambda(\hat{A})_1 = \{0.7844, 0.3052, 0.4864\} \in (0, 1)$$

The guaranteed cost over the two modes is 36.059×10^{-4}. Several simulation studies have been carried out to examine the performance of the steady-state Kalman filter. In Table 9.1, the guaranteed cost is presented for selected values of the scaling parameters (μ, λ). It is readily evident that the scaling parameters (μ, λ) have a crucial impact on the optimality of the guaranteed cost. This is equally true for specified μ while changing λ or given λ and varying μ.

For the purpose of comparison, a standard Kalman filter was designed for the tw-mode nominal delayless system under consideration by setting $\Delta_k \equiv 0, x_{k-3} \equiv 0$. Then, we applied the developed robust Kalman filter and the standard Kalman filter with $\Delta_k = 0$, $\Delta_k = 0.2$, $\Delta_k = -0.2$ and retained the delayed state. The resulting filtering costs for both filtering schemes are provided in Table 9.2. Again, it is clearly shown that the developed robust Kalman filter outperforms the standard nominal Kalman filter in the presence of uncertainty and delay factor.

Together, Tables 9.1 and 9.2 demonstrate the superior performance of the developed robust Kalman filter.

In the next section, we look at the switched linear Kalman filter for a class of continuous-time state-delay systems with norm-bounded uncertain parameters. Essentially, this is the continuous analog of the foregoing section.

9.2 Continuous Switched Delay System

State estimation forms an integral part of control systems theory. Estimating the state variables of a dynamic model is important to help in improving our knowledge about different systems for the purpose of analysis and control design. The seminal Kalman filtering algorithm [3] is the optimal estimator over all

Table 9.1 The guaranteed cost (GC) vs. the scaling parameters (μ, λ)

Mode 1						
$\lambda = 0.2$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	219.241	141.706	37.029	27.583	65.742	120.333
$\lambda = 0.4$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	173.601	121.016	31.125	25.113	60.142	101.471
$\lambda = 0.8$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	96.711	101.315	29.451	24.881	51.371	89.116
$\lambda = 1.4$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	109.345	117.996	38.222	24.003	61.332	110.541
$\lambda = 2.7$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	165.124	122.236	46.113	35.723	72.119	121.171
$\lambda = 3.4$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	259.943	165.176	51.152	41.907	77.139	151.454
Mode 2						
$\lambda = 0.2$						
μ	0.21	0.43	0.71	1.1	1.48	2.31
$GC \times 10^{-4}$	219.241	141.706	37.029	27.583	65.742	120.333
$\lambda = 0.4$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	173.601	121.016	31.125	25.113	60.142	101.471
$\lambda = 0.8$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	96.711	101.315	29.451	24.881	51.371	89.116
$\lambda = 1.4$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	109.345	117.996	38.222	24.003	61.332	110.541
$\lambda = 2.7$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	165.124	122.236	46.113	35.723	72.119	121.171
$\lambda = 3.4$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	259.943	165.176	51.152	41.907	77.139	151.454

Table 9.2 Comparison between the nominal kalman filter

	Actual cost		
Filter	$\Delta_k = -0.8$	$\Delta_k = 0$	$\Delta_k = 0.8$
Nominal kalman filter	72.034×10^{-2}	45.147×10^{-4}	235.654×10^{-2}
Robust kalman filter	66.802×10^{-4}	61.147×10^{-4}	67.113×10^{-4}

possible linear ones and gives unbiased estimates of the unknown state vectors under the conditions that the system and measurement noise processes are mutually independent Gaussian distributions. Robust state estimation arises out of the desire to estimate unmeasurable state variables when the plant model has uncertain parameters. In [16], a Kalman filtering approach has been studied with an $\mathcal{H}_\infty - norm$ constraint. For linear systems with norm-bounded parameter uncertainty, the robust estimation problem has been addressed in [329, 398, 399] and the references cited therein, where $\mathcal{H}_\infty$-estimators have been constructed. On another front of research, uncertain systems with state delay have received increasing interests in recent years [206, 344]. When dealing with continuous-time systems with state delay, there have been three basic approaches [206]:

- Infinite-dimensional systems theory, which is based on embedding the class of TLS into a larger class of dynamical systems for which the state evolution is described by appropriate operators in infinite-dimensional spaces;
- Algebraic systems theory, in which the evolution of delay-differential systems is provided in terms of linear systems over rings; and
- Functional differential systems, by incorporating the influence of the hereditary effects of system dynamics on the rate of change of the system and it provides an appropriate mathematical structure in which the system state evolves either in finite-dimensional space or in function space.

In this section, we follow the third approach for convenient representation and numerical compatibility.

The purpose of this section is to consider the state-estimation problem for a class of linear continuous-state delay systems with norm-bounded parameter uncertainties. Specifically, we address the state-estimator design problem such that the estimation error covariance has a guaranteed bound for all admissible uncertainties. The approach hinges upon the application of the multi-estimation structure depicted in Fig. 9.2. The main tool for solving the foregoing problem is

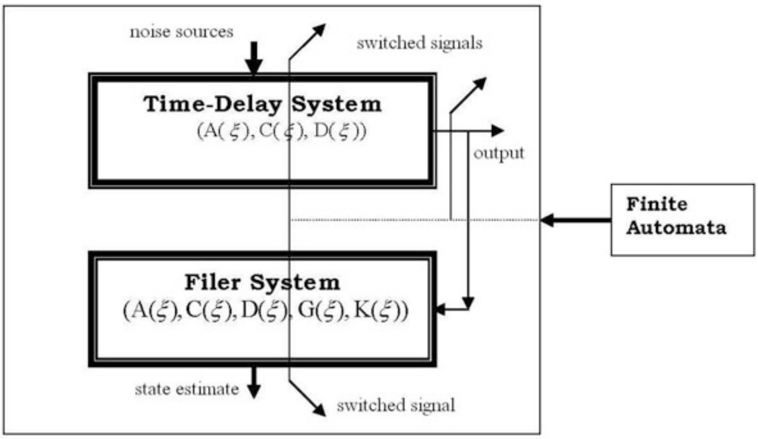

Fig. 9.2 Multi-estimator structure: continuous

the Riccati equation approach. Both time-varying and steady-state robust Kalman filtering are considered.

9.2.1 Problem Formulation

We consider a class of uncertain switched time-delay systems represented by

$$\begin{aligned}\dot{x}(t) &= [A(\xi(t)) + \Delta A(\xi(t))]x(t) + A_d(\xi(t))x(t - \tau_{\xi(t)}) + w(t)\\ &= A_{\Delta,\xi(t)}x(t) + A_d(\xi(t))x(t - \tau_{\xi(t)}) + w(t) \qquad (9.78)\\ y(t) &= [C(\xi(t)) + \Delta C(\xi(t))]x(t) + v(t)\\ &= C_{\Delta,\xi(t)}x(t) + v(t) \qquad (9.79)\end{aligned}$$

where $x(t) \in \Re^n$ is the state, $y(t) \in \Re^m$ is the measured output, and $w(t) \in \Re^n$ and $v(t) \in \Re^m$ are, respectively, the process and measurement noises. In (9.78) and (9.79), $A(\xi(t)) \in \Re^{n\times n}$, $A_d(\xi(t)) \in \Re^{n\times n}$, and $C(\xi(t)) \in \Re^{m\times n}$ are mode-dependent piecewise-continuous matrix functions. The switching rule $\xi(t)$ is not known a priori but we assume its instantaneous value is available in real time for practical implementations. Define the indicator function

$$\begin{aligned}\xi(t) &= [\xi_1(t), ..., \alpha_s(t)]^t, \quad \forall\, j \in \mathbf{S}\\ \xi_j(t) &= \begin{cases} = 1 & \text{when system (9.78) is in the jth mode,}\\ = 0 & \text{otherwise}\end{cases} \qquad (9.80)\end{aligned}$$

Here, $\tau_{\xi(t)}$ is a mode-dependent constant scalar representing the amount of time lag in the state. The matrices $\Delta A(\xi(t))$ and $\Delta C(\xi(t))$ represent time-varying parametric uncertainties, which are of the form

$$\begin{bmatrix}\Delta A(\xi(t))\\ \Delta C(\xi(t))\end{bmatrix} = \begin{bmatrix}H(\xi(t))\\ H_c(\xi(t))\end{bmatrix}\Delta(\xi(t))\; E(\xi(t)) \qquad (9.81)$$

where $H(\xi(t)) \in \Re^{n\times\alpha}$, $H_c(\xi(t)) \in \Re^{m\times\alpha}$, and $E(\xi(t)) \in \Re^{\beta\times n}$ are known piecewise-continuous matrix functions and $\Delta(\xi(t)) \in \Re^{\alpha\times\beta}$ is an unknown matrix with Lebesgue measurable elements satisfying

$$\Delta^t(\xi(t))\Delta(\xi(t)) \quad \leq \quad I \quad\quad \forall\, t \qquad (9.82)$$

The initial condition is specified as $\langle x(0), x(s)\rangle = \langle x_o, \phi(s)\rangle$, where $\phi(\cdot) \in \mathcal{L}_2[-\tau_{\xi(t)}, 0]$ which is assumed to be a zero-mean Gaussian random vector. The following standard assumptions on noise statistics are recalled:

Assumption 9.10 $\forall t\ ,\ s \geq 0$

$$(\mathbf{a})\mathbf{E}[w(t)] = 0; \quad \mathbf{E}[w(t)w^t(s)] = W(t)\delta(t-s);\ W(t) > 0 \tag{9.83}$$

$$(\mathbf{b})\mathbf{E}[v(t)] = 0; \quad \mathbf{E}[v(t)v^t(s)] = V(t)\delta(t-s);\ V(t) > 0 \tag{9.84}$$

$$(\mathbf{c})\mathbf{E}[x(0)w^t(t)] = 0; \quad \mathbf{E}[x(0)v^t(t)] = 0 \tag{9.85}$$

$$(\mathbf{d})\mathbf{E}[w(t)v^t(s)] = 0; \quad \mathbf{E}[x(0)x^t(0)] = R_o \tag{9.86}$$

where, as before, $\boldsymbol{E}[\cdot]$ *stands for the mathematical expectation and* $\delta(\cdot)$ *is the Dirac function.*

9.2.2 Robust Linear Filtering

Our objective is to design a stable switched-state estimator of the form

$$\dot{\hat{x}}(t) = G(\xi(t))\ \hat{x}(t) + K(\xi(t))\ y(t), \qquad \hat{x}(0) = 0 \tag{9.87}$$

where $G(\xi(t)) \in \Re^{n\times n}$ and $K(\xi(t)) \in \Re^{n\times m}$ are mode-dependent piecewise-continuous matrices to be determined such that there exists a matrix $\Psi \geq 0$ satisfying

$$\mathbf{E}[(x-\hat{x})(x-\hat{x})^t] \ \leq\ \Psi(\xi(t)),\ \forall \Delta : \Delta^t(\xi(t))\Delta(\xi(t)) \leq I \tag{9.88}$$

Note that (10) implies

$$\mathbf{E}[(x-\hat{x})^t(x-\hat{x})] \ \leq tr(\Psi)(\xi(t)),\ \forall \Delta(\xi(t)) : \Delta^t(\xi(t))\Delta(\xi(t)) \leq I \tag{9.89}$$

In this case, the switched estimator (9.87) is said to provide a guaranteed cost (GC) matrix Ψ.

Examination of the switched estimator proceeds by analyzing the estimation error

$$e(t) = x(t) - \hat{x}(t) \tag{9.90}$$

Substituting (9.78) and (9.87) into (9.90), we express the dynamics of the error in the form:

$$\begin{aligned}\dot{e}(\xi(t)) = G(\xi(t))e(t) &+ [A(\xi(t)) - G(\xi(t)) - K(\xi(t))C(\xi(t))]x(t)\\ &+[\Delta A(\xi(t)) - K(\xi(t))\Delta C(\xi(t))]x(t)\\ &+A_d(\xi(t))x(t-\tau_{\xi(t)}) + [w(t) - K(\xi(t))v(t)]\end{aligned} \tag{9.91}$$

By introducing the extended state vector

$$\zeta(t) \ = \ \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \ \in \Re^{2n} \tag{9.92}$$

it follows from (9.78), (9.79), and (9.91) that

$$\dot{\zeta}(t) = [\mathsf{A}(\xi(t)) + \mathsf{H}(\xi(t))\Delta(\xi(t))\mathsf{E}(\xi(t))]\zeta(t) + \mathsf{D}(\xi(t))\zeta(t-\tau_{\xi(t)}) + \mathsf{B}(\xi(t))\eta(t)$$
$$= \mathsf{A}(\xi(t), \Delta)\zeta(t) + \mathsf{D}(\xi(t))\zeta(t-\tau_{\xi(t)}) + \mathsf{B}(\xi(t))\eta(t) \quad (9.93)$$

where $\eta(t)$ is a stationary zero-mean noise signal with identity covariance matrix and

$$\mathsf{A}(\xi(t)) = \begin{bmatrix} A(\xi(t)) & 0 \\ A(\xi(t)) - G(\xi(t)) - K(\xi(t))C(\xi(t)) & G(\xi(t)) \end{bmatrix} \quad (9.94)$$

$$\mathsf{H}(\xi(t)) = \begin{bmatrix} H(\xi(t)) \\ H(\xi(t)) - K(\xi(t))H_c(\xi(t)) \end{bmatrix}$$
$$\mathsf{E}(\xi(t)) = [E(\xi(t)) \quad 0] \quad (9.95)$$

$$\mathsf{B}\mathsf{B}^t(\xi(t)) = \begin{bmatrix} W(\xi(t)) & W(\xi(t)) \\ W(\xi(t)) & W(\xi(t)) + K(\xi(t))V(\xi(t))K^t(\xi(t)) \end{bmatrix} \quad (9.96)$$

$$\mathsf{D}(\xi(t)) = \begin{bmatrix} A_d(\xi(t)) & 0 \\ A_d(\xi(t)) & 0 \end{bmatrix}, \quad \eta = \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \quad (9.97)$$

Definition 9.11 *Estimator (9.87) is said to be a switched quadratic estimator (SQE) associated with a matrix $\Omega(\xi(t)) > 0$ for system (9.78) if there exists a scalar $\lambda(\xi(t)) > 0$ and a matrix*

$$0 < \Omega(\xi(t)) = \begin{bmatrix} \Omega_{1,\xi(t)} & \Omega_{3,\xi(t)} \\ \bullet & \Omega_{2,\xi(t)} \end{bmatrix} \quad (9.98)$$

satisfying the algebraic inequality

$$-\dot{\Omega}(\xi(t)) + \mathsf{A}_{\xi(t),\Delta}\Omega(\xi(t)) + \Omega(\xi(t))\mathsf{A}^t_{\xi(t),\Delta} + \lambda(\xi(t))\ \Omega(t-\tau_{\xi(t)})$$
$$+\lambda^{-1}(\xi(t))\mathsf{D}(\xi(t))\Omega(t-\tau_{\xi(t)})\mathsf{D}^t(\xi(t)) + \mathsf{B}(\xi(t))\mathsf{B}^t(\xi(t)) \le 0 \quad (9.99)$$

The next result shows that if (9.87) is an SQE for system (9.78) and (9.79) with cost matrix $\Omega(\xi(t))$, then $\Omega(\xi(t))$ defines an upper bound for the filtering error covariance, that is,

$$\mathbf{E}[e(t)e^t(t)] \le \Omega_{2,\xi(t)} \quad \forall\ (t, \xi(t)) \quad (9.100)$$

for all admissible uncertainties satisfying (9.80) and (9.81).

Theorem 9.12 *Consider the time delay (9.78) and (9.79) satisfying (9.80) and (9.81) and with known initial state. Suppose there exists a solution $\Omega(\xi(t)) \ge 0$ to inequality (9.99) for some $\lambda(\xi(t)) > 0$ and for all admissible uncertainties. Then the switched estimator (9.87) provides an upper bound for the filtering error covariance, that is,*

$$\boldsymbol{E}[e(t)e^t(t)] \;\leq\; \Omega(\xi(t)) \tag{9.101}$$

Proof Suppose that the estimator (9.87) is an SQE with cost matrix $\Omega(\xi(t))$. By evaluating the derivative of the covariance matrix $\Sigma(\xi(t)) = \mathbf{E}[\zeta(t)\ \zeta^t(t)]$, we get

$$\begin{aligned}\dot{\Sigma}(\xi(t)) &= \mathsf{A}_{\xi(t),\Delta}\ \Sigma(\xi(t)) + \Sigma(\xi(t))\mathsf{A}^t_{\xi(t),\Delta} + \mathsf{D}(\xi(t))\mathbf{E}[\zeta(t-\tau_{\xi(t)})\zeta^t(t)] \\ &+ \mathbf{E}[\zeta(t)\ \zeta^t(t-\tau_{\xi(t)})]\ \mathsf{D}^t(\xi(t)) \\ &+ \mathbf{E}[\eta(t)\ \zeta^t(t)] + \mathbf{E}[\zeta(t)\ \eta^t(t)]\end{aligned} \tag{9.102}$$

Using inequality (9.49), we get the following inequality:

$$\begin{aligned}\mathsf{D}(\xi(t))\mathbf{E}[\zeta(t-\tau_{\xi(t)})\ \zeta^t(t)] + \mathbf{E}[\zeta(t)\ \zeta^t(t-\tau_{\xi(t)})]\ \mathsf{D}^t(\xi(t)) = \\ \mathsf{D}(\xi(t))\Sigma(t-\tau_{\xi(t)}) + \Sigma(t-\tau_{\xi(t)})\mathsf{D}^t(\xi(t)) \leq \\ \lambda(\xi(t))\ \Sigma(t-\tau_{\xi(t)}) + \lambda^{-1}(\xi(t))\mathsf{D}(\xi(t))\Sigma(t-\tau_{\xi(t)})\mathsf{D}^t((\xi(t))\end{aligned} \tag{9.103}$$

Substituting (9.103) into (9.102) and arranging the terms, we obtain

$$\begin{aligned}\dot{\Sigma}(\xi(t)) \;&\leq\; \mathsf{A}_{\Delta,\xi(t)}(\xi(t))\ \Sigma(\xi(t)) + \Sigma(\xi(t))\mathsf{A}^t_{\Delta}(t) + \lambda(\xi(t))\ \Sigma(t-\tau_{\xi(t)}) \\ &+ \lambda^{-1}(\xi(t))\mathsf{D}(\xi(t))\ \Sigma(t-\tau_{\xi(t)})\ \mathsf{D}^t(\xi(t)) +\ \mathsf{B}(\xi(t))\mathsf{B}^t(\xi(t))\end{aligned} \tag{9.104}$$

Combining (9.99) and (9.104) and letting $\Xi(\xi(t)) = \Sigma(\xi(t)) - \Omega(\xi(t))$, we obtain

$$\begin{aligned}\dot{\Xi}(\xi(t)) \;&\leq\; \mathsf{A}_{\xi(t),\Delta}\ \Xi(\xi(t)) + \Xi(\xi(t))\mathsf{A}^t_{\xi(t),\Delta} + \lambda(\xi(t))\ \Xi(t-\tau_{\xi(t)}) \\ &+ \lambda^{-1}(\xi(t))\mathsf{D}(\xi(t))\ \Xi(t-\tau_{\xi(t)})\ \mathsf{D}^t(\xi(t))\end{aligned} \tag{9.105}$$

On considering that the state is known over the period $[-\tau_{\xi(t)}, 0]$ justifies letting $\Sigma(\xi(t))|_{t=0} = 0$. Hence, inequality (9.105) implies that $\Xi(\xi(t)) \;\leq\; 0 \quad \forall\, t \;>\; 0$, that is, $\Sigma(\xi(t)) \;\leq\; \Omega(\xi(t)) \quad \forall\, t \;>\; 0$. Finally, it is obvious that

$$\mathbf{E}[e(t)e^t(t)] = [0 \quad I]\Sigma(\xi(t))\begin{bmatrix} 0 \\ I \end{bmatrix} \;\leq\; \Omega_{2,\xi(t)} \tag{9.106}$$

9.2.3 A Design Approach

In line of the discrete-time case, we employ hereafter a Riccati equation approach to solve the robust Kalman filtering for switched continuous time-delay systems. To this end, we define switched matrices $P(\xi(t)) = P^t(\xi(t)) \in \Re^{n\times n}$; $L(\xi(t)) = L^t(\xi(t)) \in \Re^{n\times n}$ as the solutions of the Riccati differential equations (RDEs):

$$\begin{aligned}
\dot{P}(\xi(t)) &= A(\xi(t))P(\xi(t)) + P(\xi(t))A^t(\xi(t)) + \lambda(\xi(t))P(t-\tau_{\xi(t)}) \\
&\quad + \lambda^{-1}(\xi(t))A_d(\xi(t))P(t-\tau_{\xi(t)})A_d^t(\xi(t)) + \hat{W}(\xi(t)) \\
&\quad + \mu(\xi(t))P(\xi(t))E^t(\xi(t))E(\xi(t))P(\xi(t)) \\
P(t-\tau_{(\xi(t))}) &= 0 \quad \forall\, t \ \in [0, \tau_{\xi(t)}] \qquad (9.107) \\
\dot{L}(\xi(t)) &= A(\xi(t))L(\xi(t)) + L(\xi(t))A^t(\xi(t)) + \lambda(\xi(t))L(t-\tau_{\xi(t)}) \\
&\quad + \lambda^{-1}(\xi(t))A_d(\xi(t))P(t-\tau_{\xi(t)})A_d^t(\xi(t)) + \hat{W}(\xi(t)) \\
&\quad + \mu(\xi(t))L(\xi(t))E^t(\xi(t))E(\xi(t))L(\xi(t)) \\
&\quad - \Big[L(\xi(t))C^t(\xi(t)) + \mu^{-1}(\xi(t))H(\xi(t))H_c^t(\xi(t))\Big]\hat{V}^{-1}(\xi(t)) \\
&\quad [C(\xi(t))L(\xi(t)) + \mu^{-1}(\xi(t))H_c(\xi(t))H^t(\xi(t))] \\
L(t-\tau_{\xi(t)}) &= 0 \ \ \forall t \ \in [0, \tau_{\xi(t)}] \qquad (9.108)
\end{aligned}$$

where $\lambda(\xi(t)) > 0, \mu(\xi(t)) > 0 \forall t$ are scaling parameters and the matrices $\hat{A}(\xi(t))$, $\hat{V}(\xi(t))$, and $\hat{W}(\xi(t))$ are given by

$$\hat{W}(\xi(t)) = W(\xi(t)) + \mu^{-1}(\xi(t))H(\xi(t))H^t(\xi(t)) \qquad (9.109)$$

$$\hat{V}(\xi(t)) = V(\xi(t)) + \mu^{-1}(\xi(t))H_c(\xi(t))H_c^t(\xi(t)) \qquad (9.110)$$

$$\begin{aligned}
\hat{A}(\xi(t)) &= A(\xi(t)) + \delta A(\xi(t)) \\
&= A(\xi(t)) + \mu^{-1}(\xi(t))L^t(\xi(t))E^t(\xi(t))E(\xi(t)) \qquad (9.111)
\end{aligned}$$

Let the $(\lambda, \mu)-$parameterized switched estimator be expressed as

$$\begin{aligned}
\dot{\hat{x}}(t) &= \Big\{A(\xi(t)) + \mu^{-1}(\xi(t))L^t(\xi(t))E^t(\xi(t))E(\xi(t))\Big\}\hat{x}(t) \\
&\quad + K(\xi(t))\big\{y(t) - C(\xi(t))\hat{x}(t)\big\} \qquad (9.112)
\end{aligned}$$

where the gain matrix $K(\xi(t)) \in \Re^{n\times m}$ is to be determined. The following theorem summarizes the main result:

Theorem 9.13 *Consider system (9.78) and (9.79) satisfying the uncertain structure (9.80) and (9.81) with zero initial condition. Suppose the process and measurement noises satisfy* **Assumption 9.10**. *For some $\mu(\xi(t)) > 0$, $\lambda(\xi(t)) > 0$, let $P(\xi(t)) = P^t(\xi(t))$ and $L(\xi(t)) = L^t(\xi(t))$ be the solutions of RDEs (9.107) and (9.108), respectively. Then the $(\lambda, \mu)-$parametrized estimator (31) is the SQE estimator with GC such that*

$$\boldsymbol{E}[\{x(t) - \hat{x}(t)\}^t\ \{x(t) - \hat{x}(t)\}] \ \le \ tr[L(\xi(t))] \qquad (9.113)$$

Moreover, the gain matrix $K(\xi(t))$ is given by

$$K(\xi(t)) = \Big\{L(\xi(t))C^t(\xi(t)) + \mu^{-1}(\xi(t))H(\xi(t))H_c^t(\xi(t))\Big\}\ \hat{V}^{-1}(\xi(t)) \qquad (9.114)$$

Proof Let

$$X(\xi(t)) = \begin{bmatrix} P(\xi(t)) & L(\xi(t)) \\ L(\xi(t)) & L(\xi(t)) \end{bmatrix} \tag{9.115}$$

where $P(\xi(t))$ and $L(\xi(t))$ are the positive-definite solutions to (9.107) and (9.108), respectively. By combining (9.107), (9.108), (9.109), (9.110), (9.111), (9.112), and (9.113) with some standard matrix manipulations, it is easy to see that

$$\begin{aligned} &-\dot{X}(\xi(t)) + \mathsf{A}(\xi(t))X(\xi(t)) + X(\xi(t))\mathsf{A}^t(\xi(t)) + \lambda(\xi(t))X(t-\tau_{\xi(t)}) \\ &+\mu^{-1}(\xi(t))\mathsf{H}(\xi(t))\mathsf{H}^t(\xi(t)) + \mu(\xi(t))X(\xi(t))\mathsf{E}^t(\xi(t))\mathsf{E}(\xi(t))X(\xi(t)) \\ &+\lambda^{-1}(\xi(t))\mathsf{D}(\xi(t))\ X(t-\tau_{\xi(t)})\ \mathsf{D}^t(\xi(t)) + \mathsf{B}(\xi(t))\mathsf{B}^t(\xi(t)) \ = 0 \end{aligned} \tag{9.116}$$

where $\mathsf{A}(\xi(t))$, $\mathsf{B}(\xi(t))$, $\mathsf{H}(\xi(t))$, $\mathsf{D}(\xi(t))$ are given by (9.94), (9.96), and (9.97). A simple comparison of (9.87) and (9.114) taking into consideration (9.111), (9.112), (9.113), and (9.114) and (9.116) shows that

$$G(\xi(t)) = \hat{A}(\xi(t)) - K(\xi(t))\ C(\xi(t))$$

By making use of a version of inequality (9.49) that for some $\mu(\xi(t)) > 0$, we have

$$\begin{aligned} &\mathsf{H}(\xi(t))\Delta(\xi(t))\mathsf{E}(\xi(t))X(\xi(t)) + X(\xi(t))\mathsf{E}^t(\xi(t))\Delta^t(\xi(t))\mathsf{H}^t(\xi(t)) \ \le \\ &\mu(\xi(t))\ X(\xi(t))\mathsf{E}^t(\xi(t))\mathsf{E}(\xi(t))X(\xi(t)) + \mu^{-1}(\xi(t))\mathsf{H}(\xi(t))\mathsf{H}^t(\xi(t)) \end{aligned} \tag{9.117}$$

Using (9.117), it is now a simple task to verify that (9.116) becomes

$$\begin{aligned} &-\dot{X}(\xi(t)) + \mathsf{A}_{\xi(t),\Delta}X(\xi(t)) + X(\xi(t))\mathsf{A}^t_{\xi(t),\Delta} + \lambda(\xi(t))X(t-\tau_{\xi(t)}) \\ &+\lambda^{-1}(\xi(t))\mathsf{D}(\xi(t))X(t-\tau_{\xi(t)})\mathsf{D}^t(\xi(t)) + \mathsf{B}(\xi(t))\mathsf{B}^t(\xi(t)) \ \le \ 0 \end{aligned} \tag{9.118}$$

$$\forall\Delta(\xi(t)):\ \Delta^t(\xi(t))\ \Delta(\xi(t)) \ \le \ I \quad \forall(\xi(t)) \in \bar{\mathbf{S}} \tag{9.119}$$

By **Theorem 9.12**, it follows that for some $\mu(\xi(t)) > 0$, $\lambda(\xi(t)) > 0$, that (9.112) is a switched quadratic estimator and $\mathbf{E}[e(t)e^t(t)] \ \le \ L(\xi(t))$. This implies that $\mathbf{E}[e^t(t)e(t)] \ \le \ tr[L(\xi(t))]$

Remark 9.14 It is known that the uncertainty representation (9.80) and (9.81) is not unique. We note that $H(\xi(t))$, $H_c(\xi(t))$ may be postmultiplied and $E(\xi(t))$ may be premultiplied by any unitary matrix since eventually this unitary matrix may be absorbed in $\Delta(\xi(t))$. It is significant to observe that such unitary multiplication does not affect the solution developed in this section.

Remark 9.15 Had we defined

$$X(\xi(t)) = \begin{bmatrix} P^{-1}(\xi(t)) & 0 \\ 0 & L(\xi(t)) \end{bmatrix} \tag{9.120}$$

we would have obtained

$$\begin{aligned}
\dot{P}(\xi(t)) &= P(\xi(t))A(\xi(t)) + A^t(\xi(t))P(\xi(t)) + \lambda(\xi(t))P(t-\tau_{\xi(t)}) \\
&+ P(\xi(t))\hat{W}(\xi(t))P(\xi(t)) \\
&+ \lambda^{-1}(\xi(t))P(t-\tau_{\xi(t)})A_d(\xi(t))P^{-1}(t-\tau_{\xi(t)})A_d^t(\xi(t))P(t-\tau_{\xi(t)}) \\
&+ \mu(\xi(t))E^t(\xi(t))E(\xi(t)) \\
P(t-\tau_{\xi(t)}) &= 0 \quad \forall\, t \in [0, \tau_{\xi(t)}]
\end{aligned} \tag{9.121}$$

$$\begin{aligned}
\dot{L}(\xi(t)) &= A(\xi(t))L(\xi(t)) + L(\xi(t))A^t(\xi(t)) + \lambda(\xi(t))L(t-\tau_{\xi(t)}) \\
&+ \hat{W}(\xi(t)) + \lambda^{-1}(\xi(t))A_d(\xi(t))P(t-\tau_{\xi(t)})A_d^t(\xi(t)) \\
&+ \mu(\xi(t))L(\xi(t))E^t(\xi(t))E(\xi(t))L(\xi(t)) \\
&- \left[L(\xi(t))C^t(\xi(t)) + \mu^{-1}(\xi(t))H(\xi(t))H_c^t(\xi(t))\right]\hat{V}^{-1}(\xi(t)) \\
&\quad [C(\xi(t))L(\xi(t)) + \mu^{-1}(\xi(t))H_c(\xi(t))H^t(\xi(t))] \\
L(t-\tau_{\xi(t)}) &= 0 \quad \forall\, t \in [0, \tau_{\xi(t)}]
\end{aligned} \tag{9.122}$$

We note that (9.121), which is of a nonstandard form, although X(t) in (9.120) is frequently used in similar situations for the nonswitched delayless systems [329, 398, 399]. Indeed, the difficulty comes from the delay-term

$$\lambda^{-1}(\xi(t))P(t-\tau_{\xi(t)})A_d(\xi(t))P^{-1}(t-\tau_{\xi(t)})A_d^t(\xi(t))P(t-\tau_{\xi(t)})$$

This point emphasizes the fact that not every result of delayless systems are straightforwardly transformable to time delay systems.

Remark 9.16 It is interesting to observe that the switched estimator (9.113) is independent of the delay factor $\tau_{\xi(t)}$ and it reduces to the standard Kalman filtering algorithm in the case of systems without uncertainties and delay factor $H(\xi(t)) \equiv 0,\ H_c(\xi(t)) \equiv 0,\ E(\xi(t)) \equiv 0,\ A_d(\xi(t)) \equiv 0,\ \lambda(\xi(t)) \equiv 0.$

Remark 9.17 In the delay-free case $(A_d(\xi(t)) \equiv 0,\ \lambda(\xi(t)) \equiv 0)$, we observe that (9.114) reduces to the Kalman filter for the system

$$\dot{x}(t) = \hat{A}(\xi(t))\,x(t) + \hat{w}(t) \tag{9.123}$$

$$y(t) = C(\xi(t))\,x(t) + \hat{v}(t) \tag{9.124}$$

where $\hat{w}(t)$ and $\hat{v}(t)$ are zero-mean white noise sequences with covariance matrices $\hat{W}(t)$ and $\hat{V}(t)$, respectively, and having cross-covariance matrix $[\mu^{-1}(\xi(t))H(\xi(t))H_c^t(\xi(t))]$. Looked at in this light, the approach developed here before to robust filtering in **Theorem 9.13** corresponds to designing a standard Kalman filter for a related continuous-time system, which captures all admissible uncertainties and time delay, but does not involve parameter uncertainties. Indeed, the robust filter (9.113) using (9.108), (9.109), (9.110), (9.111), and (9.112) can be rewritten as

$$\dot{\hat{x}}(t) = [A(\xi(t)) + \delta A(\xi(t))]\,\hat{x}(t) + K(\xi(t))\,\{y(t) - C(\xi(t))\,\hat{x}(t)\} \qquad (9.125)$$

where $\delta A(\xi(t))$ is defined in (9.108) and it reflects the effect of uncertainties $\{\Delta A(\xi(t)), \Delta C(\xi(t))\}$ and time-delay factor $A_d((\xi(t))$ on the structure of the filter.

Remark 9.18 In view of the foregoing notes, it is readily evident that the results achieved by **Theorems 9.12** and **9.13** extend some of the existing robust estimation results to linear systems with state delay.

9.2.4 Steady-State Robust Filter

Now, we investigate the asymptotic properties of the Kalman filter developed previously, where the switching signal ξ is now independent of time. For this purpose, we consider the uncertain time-delay system

$$\begin{aligned} \dot{x}(t) &= [A(\xi) + H(\xi)\Delta(\xi)E(\xi)]x(t) + A_d(\xi)x(t-\tau_\xi) + w(t) \\ &= A_{\xi,\Delta}x(t) + A_d(\xi)x(t-\tau_\xi) + w(t) \qquad (9.126) \\ y(t) &= [C(\xi) + H_c(\xi)\Delta(\xi)E(\xi)]x(t) + v(t) \\ &= C_{\xi,\Delta}x(t) + v(t) \qquad (9.127) \end{aligned}$$

where $\Delta(\xi)$ satisfies (9.81). The matrices $A(\xi) \in \Re^{n\times n}$, $C(\xi) \in \Re^{m\times n}$ are mode-dependent constant matrices representing the nominal plant. It is assumed that $A(\xi),\ \xi \in \bar{\mathbf{S}}$ is Hurwitz. Our objective now is to design a switched time-invariant a priori estimator of the form:

$$\dot{\hat{x}}(t) = \hat{A}(\xi)\,\hat{x}(t) + K(\xi)\,[y(t) - C(\xi)\hat{x}(t)] \quad \hat{x}(0) = 0 \qquad (9.128)$$

that achieves the following asymptotic performance bound

$$\lim_{t\to\infty} \mathbf{E}\left\{[\hat{x}(t) - x(t)][\hat{x}(t) - x(t)]^t\right\} \quad \le \quad L(\xi) \qquad (9.129)$$

Theorem 9.19 *Consider the uncertain time-delay system (9.126) and (9.127) with $A(\xi)$ being Hurwitz. If for some scalars $\mu(\xi) > 0,\ \lambda(\xi) > 0,$ there exist stabilizing solutions for the AREs*

$$\begin{aligned} &A(\xi)P(\xi) + P(\xi)A^t(\xi) + \lambda(\xi)P(\xi) + \hat{W}(\xi) + \lambda^{-1}(\xi)A_d(\xi)P(\xi)A_d^t(\xi) \\ &+ \mu(\xi)P(\xi)E^t(\xi)E(\xi)P(\xi) = 0 \qquad (9.130) \end{aligned}$$

$$\begin{aligned} & A(\xi)L(\xi) + L(\xi)A^t(\xi) + \lambda(\xi)L(\xi) + \hat{W}(\xi) + \lambda^{-1}(\xi)A_d(\xi)P(\xi)A_d^t(\xi) \\ + & \mu(\xi)L(\xi)E^t(\xi)E(\xi)L(\xi) - \\ & \left[L(\xi)C^t(\xi) + \mu^{-1}(\xi)H(\xi)H_c^t(\xi)\right]\hat{V}^{-1}(\xi) \\ & [C(\xi)L(\xi) + \mu^{-1}(\xi)H_c(\xi)H^t(\xi)] = 0 \end{aligned} \tag{9.131}$$

then the estimator (9.125) is a stable switched quadratic (SSQ) and achieves (9.129) with

$$\begin{aligned} \hat{W}(\xi) &= W + \mu^{-1}(\xi)H(\xi)H^t(\xi) \\ \hat{V}(\xi) &= V + \mu^{-1}(\xi)H_c(\xi)H_c^t(\xi) \end{aligned} \tag{9.132}$$

$$\begin{aligned} \hat{A}(\xi) &= A(\xi) + \delta A(\xi) \\ &= A(\xi) + \mu^{-1}(\xi)L^t(\xi)E^t(\xi)E(\xi) \\ K(\xi) &= \left\{L(\xi)C^t(\xi) + \mu^{-1}(\xi)H(\xi)H_c^t(\xi)\right\}\ \hat{V}^{-1}(\xi) \end{aligned} \tag{9.133}$$

for some $L(\xi)\ \geq\ 0$.

Proof To examine the stability of the closed-loop system, we augment (9.126), (9.127) and (9.128) with $(w(t) = 0, v(t) = 0)$, to obtain

$$\begin{aligned} \dot{\xi}(t) &= \mathsf{A}_{\xi,\Delta}\xi(t) + \mathsf{D}(\xi)\xi(t-\tau_\xi) \\ &= \begin{bmatrix} A_{\xi,\Delta} & 0 \\ A_{\xi,\Delta} - G(\xi) - K(\xi)C_{\xi,\Delta} & G(\xi) \end{bmatrix}\xi(t) \\ &+ \begin{bmatrix} A_d(\xi) & 0 \\ A_d(\xi) & 0 \end{bmatrix}\xi(t-\tau_\xi) \end{aligned} \tag{9.134}$$

By a similar argument as in the proof of **Theorem 9.13**, it is easy to see that

$$X(\xi)\mathsf{A}_{\xi,\Delta} + \mathsf{A}_{\xi,\Delta}^t X(\xi) - \lambda(\xi)X(\xi) + \lambda^{-1}(\xi)\mathsf{D}(\xi)X(\xi)\mathsf{D}^t(\xi)\ <\ 0 \tag{9.135}$$

where

$$X(\xi) = \begin{bmatrix} P(\xi) & L(\xi) \\ L(\xi) & L(\xi) \end{bmatrix} \tag{9.136}$$

Introducing a Lyapunov – Krasovskii functional

$$V(\xi) = \zeta^t(t)X(\xi)\zeta(t) + \int_{t-\tau_\xi}^{t} \zeta^t(\alpha)\lambda^{-1}(\xi)\mathsf{D}(\xi)X(\xi)\mathsf{D}^t(\xi)\zeta(\alpha)\ d\alpha \tag{9.137}$$

and observe that $V(\xi)\ >\ 0$, for $\zeta(t)\ \neq 0$, for some $\lambda(\xi) > 0$ and $V(\xi) = 0$ when $\zeta\ = 0$. By differentiating the Lyapunov – Krasovskii functional (9.137) along the trajectories of system (9.134), we get

$$\begin{aligned}
\dot{V}(\xi) &= \zeta^t(t)\Big[X(\xi)\mathsf{A}_{\xi,\Delta} + \mathsf{A}^t_{\xi,\Delta}X(\xi) + \lambda^{-1}(\xi)\mathsf{D}(\xi)X(\xi)\mathsf{D}^t(\xi)\Big]\zeta(t) \\
&\quad +\zeta^t(t)X(\xi)\mathsf{D}(\xi)\zeta(t-\tau_\xi) + \zeta^t(t-\tau_\xi)\mathsf{D}^t(\xi)X(\xi)\zeta(t) \\
&\quad -\lambda^{-1}(\xi)\zeta^t(t-\tau_\xi)\mathsf{D}(\xi)X(\xi)\mathsf{D}^t(\xi)\zeta(t-\tau_\xi) \\
&\le \zeta^t(t)\Big[X(\xi)\mathsf{A}_{\xi,\Delta} + \mathsf{A}^t_{\xi,\Delta}X(\xi) + \lambda X(\xi) + \lambda^{-1}(\xi)\mathsf{D}(\xi)X(\xi)\mathsf{D}^t(\xi)\Big]\zeta(t) \\
&\quad +\lambda^{-1}(\xi)\zeta^t(t-\tau_\xi)\mathsf{D}(\xi)X(\xi)\mathsf{D}^t(\xi)\zeta(t-\tau_\xi) \\
&\quad -\lambda^{-1}(\xi)\zeta^t(t-\tau_\xi)\mathsf{D}(\xi)X(\xi)\mathsf{D}^t(\xi)\zeta(t-\tau_\xi) \\
&= \zeta^t(t)\Big[X(\xi)\mathsf{A}_{\xi,\Delta} + \mathsf{A}^t_{\xi,\Delta}X(\xi) + \lambda(\xi)X(\xi) + \lambda^{-1}(\xi)\mathsf{D}(\xi)X\mathsf{D}^t(\xi)\Big]\zeta(t) \\
&< \ 0
\end{aligned} \tag{9.138}$$

which means that the augmented system (9.132) is asymptotically stable. In turn, this implies that (9.125) is SSQ. The guaranteed performance

$$\mathbf{E}[e(t)e^t(t)] \ \le \ L(\xi) \tag{9.139}$$

follows from similar lines of argument as in the proof of **Theorem 9.13**.

The next theorem provides LMI-based solution to the steady-state robust Kalman filter.

Theorem 9.20 *Consider the uncertain switched time-delay system (9.126) and (9.127) with $A(\xi)$ being Hurwitz. The estimator*

$$\begin{aligned}
\dot{\hat{x}}(t) &= [A(\xi) + \mu^{-1}(\xi)L^t(\xi)E^t(\xi)E(\xi)]\hat{x}(t) \\
&\quad + \Big[L(\xi)C^t(\xi) + \mu^{-1}(\xi)H(\xi)H_c^t(\xi)\Big]\hat{V}^{-1}(\xi)[y(t) - C(\xi)\hat{x}(t)]
\end{aligned} \tag{9.140}$$

where

$$\hat{V}(\xi) = V + \mu^{-1}(\xi)H_c(\xi)H_c^t(\xi) \tag{9.141}$$

is a stable switched quadratic and achieves (9.139) for some $L(\xi) \ge 0$ if for some scalars $\mu(\xi) > 0,\ \lambda(\xi) > 0,$ there exist matrices $0 < Y(\xi) = Y^t(\xi)$ and $0 < X(\xi) = X^t(\xi)$ satisfying the LMIs

$$\begin{bmatrix} A(\xi)Y(\xi) + Y(\xi)A^t(\xi) + Q_y(Y,\lambda,(\xi)) & A_d(\xi)Y(\xi) & Y(\xi)E^t(\xi) \\ Y(\xi)A_d^t(\xi) & -\lambda(\xi)I & 0 \\ E(\xi)Y(\xi) & 0 & -\mu^{-1}(\xi)I \end{bmatrix} < 0 \tag{9.142}$$

$$\begin{bmatrix} A(\xi)X(\xi) + X(\xi)A^t(\xi) + Q_x(X,\lambda,(\xi)) & A_d(\xi)Y(\xi) & X(\xi)E^t(\xi) \\ Y(\xi)A_d^t(\xi) & -\lambda(\xi)I & 0 \\ E(\xi)X(\xi) & 0 & -\mu^{-1}(\xi)I \end{bmatrix} < 0 \tag{9.143}$$

where

$$
\begin{aligned}
Q_y(Y, \lambda, (\xi)) &= \lambda(\xi)Y(\xi) + W + \mu^{-1}(\xi)H(\xi)H^t(\xi), \\
Q_x(X, \lambda, (\xi)) &= \lambda(\xi)X(\xi) + W + \mu^{-1}(\xi)H(\xi)H^t(\xi) \\
&\quad - \left[X(\xi)C^t(\xi) + \mu^{-1}(\xi)H(\xi)H_c^t(\xi)\right]\hat{V}^{-1}(\xi) \\
&\quad [C(\xi)X(\xi) + \mu^{-1}(\xi)H_c(\xi)H^t(\xi)]
\end{aligned} \tag{9.144}
$$

Proof By inequality (9.50) and (9.130) and (9.131), it follows that there exist matrices $0 < Y(\xi) = Y^t(\xi)$ and $0 < X(\xi) = X^t(\xi)$ satisfying the algebraic Riccati inequalities (ARIs):

$$
\begin{aligned}
& A(\xi)Y(\xi) + Y(\xi)A^t(\xi) + \lambda(\xi)Y(\xi) + \hat{W}(\xi) + \lambda^{-1}(\xi)A_d(\xi)Y(\xi)A_d^t(\xi) \\
&+ \mu(\xi)Y(\xi)E^t(\xi)E(\xi)Y(\xi) \; < \; 0
\end{aligned} \tag{9.145}
$$

$$
\begin{aligned}
& A(\xi)X(\xi) + X(\xi)A^t(\xi) + \lambda(\xi)X(\xi) + \hat{W}(\xi) + \lambda^{-1}(\xi)A_d(\xi)Y(\xi)A_d^t(\xi) \\
&+ \mu(\xi)X(\xi)E^t(\xi)E(\xi)X(\xi) - \\
& \left[X(\xi)C^t(\xi) + \mu^{-1}(\xi)H(\xi)H_c^t(\xi)\right]\hat{V}^{-1}(\xi) \\
& [C(\xi)X(\xi) + \mu^{-1}(\xi)H_c(\xi)H^t(\xi)] \; < \; 0
\end{aligned} \tag{9.146}
$$

such that $Y(\xi) > P(\xi)$, $X(\xi) > L(\xi)$. Application of (9.50) to the ARIs (9.145) and (9.146) yields the LMIs (9.142) and (9.143).

Remark 9.21 It should be emphasized the AREs (9.130) and (9.131) do not have clear-cut monotonicity properties enjoyed by standard AREs. The main reason for this is the presence of the term $A_d(\xi)P(\xi)A_d^t(\xi)$.

Extending on the results [329], given τ_ξ, it follows that the uncertain time-delay system

$$
\dot{x}(t) = [A(\xi) + H(\xi)\Delta(\xi)E(\xi)]x(t) + A_d(\xi)x(t - \tau_\xi) \tag{9.147}
$$

is switched quadratically stable (SQS) if there exist matrices $0 < \bar{P}(\xi) = \bar{P}^t(\xi) \in \Re^{n\times n}$, $0 < \bar{R}(\xi) = \bar{R}^t(\xi) \in \Re^{n\times n}$ satisfying the ARI:

$$
\begin{aligned}
& \bar{P}(\xi)A(\xi) + A^t(\xi)\bar{P}(\xi) + E^t(\xi)E(\xi) + \bar{P}(\xi)H(\xi)H^t(\xi)\bar{P}(\xi) \\
&\quad + \bar{P}(\xi)A_d(\xi)\bar{R}^{-1}(\xi)A_d^t(\xi)\bar{P}(\xi) + \bar{R}(\xi) < 0
\end{aligned} \tag{9.148}
$$

The next theorem examines further properties of the positive-definite solution of the ARE (9.130).

Theorem 9.22 *If system (9.147) is (SQS), then there exist some $\mu(\xi) > 0, \lambda(\xi) > 0$ such that the ARE (9.130) admits a positive-definite solution $P(\xi) > 0$ for some $0 < R(\xi) = R^t(\xi)$. Furthermore, for a given $\mu(\xi) > 0, \lambda(\xi) > 0$ and $R(\xi) > 0$, if there exist $\bar{\mu}(\xi) > 0$, $\bar{\lambda}(\xi) > 0$ such that (9.130) admits a positive-definite solution $0 < \bar{P}(\xi) = \bar{P}^t(\xi)$ for some $0 < \bar{R}(\xi) = \bar{R}^t(\xi)$, then for any $\mu(\xi) \in$*

$(0, \bar{\mu}(\xi)]$, $\lambda(\xi) \in (0, \bar{\lambda}(\xi)]$, *the solution of (9.131)* $P(\xi) > 0$ *satisfies* $0 < P(\xi) \leq (\mu(\xi)/\bar{\mu}(\xi))\bar{P}(\xi)$ *for some* $0 < R(\xi) \leq (\lambda(\xi)/\bar{\lambda}(\xi))\bar{R}(\xi)$.

Proof Using (9.148), it follows for some $\mu(\xi) > 0$ that

$$\bar{P}(\xi)A(\xi) + A^t(\xi)\bar{P}(\xi) + E^t(\xi)E(\xi) + \bar{P}(\xi)[\mu(\xi)W + H(\xi)H^t(\xi)]\bar{P}(\xi) + \bar{P}(\xi)A_d(\xi)\bar{R}^{-1}(\xi)A_d^t(\xi)\bar{P}(\xi) + \bar{R}(\xi) < 0 \quad (9.149)$$

By setting $\widehat{P}(\xi) = \mu(\xi)\bar{P}(\xi)$, we get

$$\widehat{P}(\xi)A(\xi) + A^t(\xi)\widehat{P}(\xi) + \mu(\xi)E^t(\xi)E(\xi) + \widehat{P}(\xi)[W + \mu^{-1}(\xi)H(\xi)H^t(\xi)]\widehat{P}(\xi) + \mu^{-1}(\xi)\widehat{P}(\xi)A_d(\xi)\bar{R}^{-1}(\xi)A_d^t(\xi)\widehat{P}(\xi) + \mu(\xi)\bar{R}(\xi) < 0 \quad (9.150)$$

This implies that

$$A(\xi)\widehat{P}(\xi) + \widehat{P}(\xi)A^t(\xi) + \mu(\xi)\widehat{P}(\xi)E^t(\xi)E(\xi)\widehat{P}(\xi) + [W + \mu^{-1}(\xi)H(\xi)H^t(\xi)] + \mu^{-1}(\xi)A_d(\xi)\bar{R}^{-1}(\xi)A_d^t(\xi) + \mu(\xi)\widehat{P}(\xi)\bar{R}(\xi)\widehat{P}(\xi) \; < \; 0 \quad (9.151)$$

By letting $\bar{R}(\xi) = (\lambda(\xi)/\mu(\xi))\widehat{P}^{-1}(\xi)$ for some $\lambda(\xi) > 0$, it follows that there exist a positive-definite solution $P(\xi) > 0$ to the ARE

$$A(\xi)P(\xi) + P(\xi)A^t(\xi) + \lambda(\xi)P(\xi) + \mu(\xi)P(\xi)E^t(\xi)E(\xi)P(\xi) + [W + \mu^{-1}(\xi)H(\xi)H^t(\xi)] + \lambda^{-1}(\xi)A_d(\xi)P(\xi)A_d^t(\xi) \; = \; 0 \quad (9.152)$$

The remaining part regarding the monotonicity of $\mu(\xi)$ and that $0 < P(\xi) < (\mu(\xi)/\bar{\mu}(\xi))\bar{P}(\xi)$ for some $0 < R(\xi) \leq (\lambda(\xi)/\bar{\lambda}(\xi))\bar{R}(\xi)$ follows by applying the results of [398].

Remark 9.23 For any pairs $(\lambda_1(\xi), \mu_1(\xi))$, $(\lambda_2(\xi), \mu_2(\xi)) \in (0, \bar{\lambda}(\xi)] \times (0, \bar{\mu}(\xi)]$,
$\lambda_1(\xi) \leq \lambda_2(\xi), \mu_1(\xi) \leq \mu_2(\xi)$, it follows from **Theorem 9.22** that

$$P(\xi, \mu_1)/\mu_1(\xi) \; \leq \; P(\xi, \mu_2)/\mu_2(\xi)$$

for some $R(\xi, \lambda_1) \leq R(\xi, \lambda_2)$. Thus

$$d^2P(\xi)/d\mu^2(\xi) \; \geq \; 0$$

This can also be justified by differentiating (9.130) using (9.132) and (9.133) to yield

$$A(\xi)\frac{d^2P(\xi)}{d\mu^2(\xi)} + \frac{d^2P(\xi)}{d\mu^2(\xi)}A^t(\xi) + \lambda(\xi)\frac{d^2P(\xi)}{d\mu^2(\xi)}$$
$$+\lambda^{-1}(\xi)D(\xi)\frac{d^2P(\xi)}{d\mu^2(\xi)}D^t(\xi) + \frac{2}{\mu^3(\xi)}H_1(\xi)H_1^t(\xi) = 0 \qquad (9.153)$$

Since $A(\xi)$ is Hurwitz, it is obvious from (9.153) that

$$\frac{d^2P(\xi)}{d\mu^2(\xi)} \ \geq \ 0$$

By the same arguments, we have

$$A(\xi)\frac{d^2P(\xi)}{d\lambda^2(\xi)} + \frac{d^2P(\xi)}{d\lambda^2(\xi)}A^t(\xi) + \lambda(\xi)\frac{d^2P(\xi)}{d\lambda^2(\xi)} + 2\frac{dP(\xi)}{d\lambda(\xi)}$$
$$+D(\xi)\left\{\frac{2P(\xi)}{\lambda^3(\xi)} - \frac{2P(\xi)}{\lambda^2(\xi)}\frac{dP(\xi)}{\lambda(\xi)} + \frac{1}{\lambda(\xi)}\frac{d^2P(\xi)}{d\lambda^2(\xi)}\right\}D^t(\xi)$$
$$\mu(\xi)P(\xi)E^t(\xi)E(\xi)P(\xi)\frac{d^2P(\xi)}{d\lambda^2(\xi)} + \mu(\xi)\frac{d^2P(\xi)}{d\lambda^2(\xi)}E^t(\xi)E(\xi)P(\xi)$$
$$+2\mu(\xi)\frac{dP(\xi)}{d\lambda(\xi)}E^t(\xi)E(\xi)\frac{dP(\xi)}{d\lambda(\xi)} = 0 \qquad (9.154)$$

which leads to

$$\frac{d^2P(\xi)}{d\lambda^2(\xi)} \ \geq \ 0$$

Following a similar procedure, it can be shown that

$$\frac{d^2L(\xi)}{d\mu^2(\xi)} \ \geq \ 0, \ \frac{d^2L(\xi)}{d\lambda^2(\xi)} \ \geq \ 0$$

Thus we conclude that $tr\ (L)(\xi)$ is a convex function over the region $(0, \bar{\lambda}(\xi)] \times (0, \bar{\mu}(\xi)]$. This indicates that a suboptimal robust Kalman filter can be obtained via convex optimization approach.

9.2.5 Numerical Simulation

For the purpose of illustrating the developed theory, we focus on the steady-state Kalman filtering and proceed to determine the estimator gains. Essentially, seek to solve (9.130), (9.131), (9.132), and (9.133) when $\lambda(\xi) \in [\lambda_1 \rightarrow \lambda_2]$, $\mu(\xi) \in [\mu_1 \rightarrow \mu_2]$, where $\lambda_1, \lambda_2, \mu_1, \mu_2$ are given constants and $\xi \in \{1, 2, 3\}$. Initially, we observe that (9.130) depends on $P(\xi)$ only and it is not of the standard-forms of

AREs. On the contrary, (9.131) depends on both $L(\xi)$ and $P(\xi)$ and it can be put into the standard ARE form. For numerical simulation, we employ a Kronecker Product-like technique to reduce (9.130) into a system of nonlinear algebraic equations of the form

$$f(\alpha) = G\,\alpha + h(\alpha) + q \tag{9.155}$$

where $\alpha \in \Re^{n(n+1)/2}$ is a vector of the unknown elements of the P matrix. The algebraic equation (9.135) can then be solved using an iterative Newton – Raphson technique according to the rule

$$\alpha_{(i+1)} = \alpha_{(i)} - \gamma_{(i)}[G + \nabla_\alpha h(\alpha_{(i)})]^{-1} f(\alpha_{(i)}) \tag{9.156}$$

where i is the iteration index, $\alpha_{(o)} = 0$, $\nabla_\alpha h(\alpha)$ is the Jacobian of $h(\alpha)$, and the step-size $\gamma_{(i)}$ is given by $\gamma_{(i)} = 1/[||f(\alpha(i))|| + 1]$.

Given the solution of (9.130), we proceed to solve (9.131) using a standard hamiltonian/eigenvector method. All the computations are carried out using the Linear Algebra and System (L-A-S) software [5]. As a typical case, consider a time-delay system of the type (9.78)-(9.79) with

Mode 1

$$\begin{aligned} A &= \begin{bmatrix} -2 & 0.5 \\ 1 & -3 \end{bmatrix},\; A_d = \begin{bmatrix} -0.2 & -0.1 \\ 0.1 & 0.4 \end{bmatrix} \\ W &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\; H = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \\ C &= [1 \quad -3],\; E = [0.5 \quad 1],\; H_c = 2,\; V = 1 \end{aligned}$$

Mode 2

$$\begin{aligned} A &= \begin{bmatrix} -3 & 0 \\ 0.5 & -4 \end{bmatrix},\; A_d = \begin{bmatrix} -0.3 & -0.2 \\ 0 & 0.3 \end{bmatrix} \\ W &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},\; H = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix} \\ C &= [0.7 \quad -2],\; E = [0.4 \quad 0.7],\; H_c = 1,\; V = 1 \end{aligned}$$

Mode 3

$$\begin{aligned} A &= \begin{bmatrix} -4 & 1 \\ 0 & -5 \end{bmatrix},\; A_d = \begin{bmatrix} -0.2 & -0.1 \\ 0.1 & 0.4 \end{bmatrix} \\ W &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\; H = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \\ C &= [0.6 \quad -1],\; E = [0.3 \quad 0.8],\; H_c = 2,\; V = 0.7 \end{aligned}$$

A summary of the computational results is presented in Table 9.3 and from which we observe the following:

Table 9.3 Summary of some of the computational results

Mode	λ	μ	P		L		δA		K^t		$\lambda(\hat{A})$	
1	0.1	0.6	0.574	0.175	0.566	0.176	0.382	0.784	0.314	0.301	−3.227	−0.472
			0.175	0.457	0.176	0.464	0.459	0.919				
2	0.1	0.6	0.535	0.156	0.500	0.136	0.242	0.483	0.314	0.301	−3.249	−0.897
			0.156	0.431	0.136	0.421	0.306	0.612				
3	0.1	0.6	0.525	0.151	0.480	0.124	0.202	0.404	0.315	0.302	−3.258	−1.017
			0.151	0.425	0.124	0.409	0.261	0.523				
1	0.2	0.7	0.564	0.206	0.554	0.210	0.406	0.811	0.317	0.295	−3.279	−0.498
			0.206	0.374	0.210	0.386	0.409	0.818				
2	0.2	0.7	0.525	0.183	0.488	0.167	0.257	0.514	0.318	0.301	−3.293	−0.913
			0.183	0.350	0.167	0.346	0.269	0.537				
3	0.2	0.7	0.515	0.177	0.468	0.153	0.215	0.430	0.318	0.294	−3.298	−1.031
			0.177	0.344	0.153	0.334	0.228	0.456				
1	0.4	0.8	0.603	0.242	0.576	0.238	0.438	0.877	0.324	0.296	−3.302	−0.450
			0.242	0.357	0.238	0.367	0.405	0.810				
2	0.4	0.8	0.564	0.217	0.508	0.192	0.278	0.557	0.325	0.301	−3.311	−0.880
			0.217	0.335	0.192	0.328	0.265	0.530				
3	0.4	0.8	0.555	0.210	0.486	0.177	0.233	0.466	0.326	0.295	−3.415	−1.002
			0.210	0.330	0.177	0.316	0.225	0.449				

Table 9.3 (continued)

Mode	λ	μ	P		L		δA		K^t		$\lambda(\hat{A})$	
1	0.6	0.6	0.665	0.276	0.612	0.259	0.471	0.942	0.331	0.300	−3.311	−0.382
			0.276	0.369	0.259	0.372	0.418	0.836				
2	0.6	0.6	0.629	0.252	0.540	0.210	0.300	0.600	0.333	0.299	−3.318	−0.834
			0.252	0.348	0.210	0.333	0.274	0.548				
3	0.6	0.6	0.622	0.247	0.517	0.195	0.252	0.503	0.335	0.299	−3.321	−0.962
			0.247	0.344	0.195	0.321	0.232	0.464				
1	0.8	0.7	0.751	0.319	0.656	0.281	0.507	1.015	0.340	0.304	−3.318	−0.300
			0.319	0.392	0.281	0.384	0.437	0.874				
2	0.8	0.7	0.723	0.298	0.580	0.229	0.324	0.649	0.343	0.304	−3.324	−0.779
			0.298	0.374	0.229	0.344	0.287	0.574				
3	0.8	0.7	0.726	0.298	0.556	0.213	0.273	0.545	0.345	0.304	−3.326	−0.914
			0.298	0.372	0.213	0.332	0.244	0.487				
1	0.9	0.8	0.806	0.346	0.681	0.292	0.527	1.055	0.344	0.307	−3.321	−0.255
			0.346	0.407	0.292	0.392	0.448	0.897				
2	0.9	0.8	0.790	0.330	0.603	0.239	0.338	0.676	0.349	0.307	−3.326	−0.747
			0.330	0.392	0.239	0.352	0.294	0.589				
3	0.9	0.8	0.805	0.336	0.579	0.222	0.284	0.569	0.351	0.307	−3.328	−0.887
			0.336	0.393	0.222	0.339	0.250	0.501				

(1) For a given $\lambda \in [0.1 - 0.9]$, increasing μ by 50% results in 0.3% increase in $||K||$ (for small λ) and about 1.12% increase in $||K||$ when λ is relatively large.
(2) For a given μ, increasing λ from 0.1 to 0.9 causes $||K||$ to increase by about 5.35%.
(3) For $\mu < 0.6$, $\lambda in [0.1, 0.9]$, the estimator is unstable.
(4) Increasing (λ, μ) beyond $(1, 1)$ yields unstable estimator.

Therefore, we conclude that:

(1) The stable-estimator gains are practically insensitive to the (λ, μ) parameters.
(2) There is a finite range for (λ, μ) that guarantees stable performance of the developed Kalman filter.

9.3 Notes and References

We have considered in this chapter a robust Kalman filter for a class of switched continuous-time systems with norm-bounded uncertainties and unknown constant state delay. Both time-varying and steady-state filtering algorithms have been examined. The main results are contained in two parts: Part 1 includes **Theorems 9.2** and **9.3** that deal with time-varying problems on a finite horizon and Part 2 includes **Theorems 9.8–9.13** that treat the steady-state problem and its related properties. It has been established that the Kalman filter algorithm is related to solutions of two Riccati equations involving scalar parameters. Important properties of the robust filter have been delineated. It has been further shown that the guaranteed cost is a convex function of the scaling parameters. A numerical simulation is provided to illustrate the developed theory. The research results in the literature are few and hence researchers are encouraged to develop pertinent results.